mathematics

MDPI

Article

Contact-Complex Riemannian Submersions

Cornelia-Livia Bejan [1,2,*], Şemsi Eken Meriç [3] and Erol Kılıç [4]

1 Department of Mathematics, Technical University Gheorghe Asachi Iasi, 700050 Iasi, Romania
2 Seminar Matematic, Alexandru Ioan Cuza University, Bd. Carol I No. 11, 700506 Iasi, Romania
3 Department of Mathematics, Faculty of Science and Arts, Mersin University, Mersin 33343, Turkey; semsieken@mersin.edu.tr
4 Department of Mathematics, Faculty of Science and Arts, İnönü University, Malatya 42280, Turkey; erol.kilic@inonu.edu.tr
* Correspondence: cornelia-livia.bejan@academic.tuiasi.ro

Abstract: A submersion from an almost contact Riemannian manifold to an almost Hermitian manifold, acting on the horizontal distribution by preserving both the metric and the structure, is, roughly speaking a contact-complex Riemannian submersion. This paper deals mainly with a contact-complex Riemannian submersion from an η-Ricci soliton; it studies when the base manifold is Einstein on one side and when the fibres are η-Einstein submanifolds on the other side. Some results concerning the potential are also obtained here.

Keywords: Riemannian submersion; submanifold; almost-contact metric manifold; Ricci soliton

MSC: 53C40; 32Q15; 53D10

Citation: Bejan, C.-L.; Eken Meriç, Ş.; Kılıç, E. Contact-Complex Riemannian Submersions. *Mathematics* **2021**, *9*, 2996. https://doi.org/10.3390/math9232996

Academic Editor: Cristina-Elena Hretcanu

Received: 14 October 2021
Accepted: 15 November 2021
Published: 23 November 2021

1. Introduction

The notion of Ricci flow was introduced by R. S. Hamilton in 1892 to find a desired metric on a Riemannian manifold. For the metrics on a Riemannian manifold, the Ricci flow is an evolution equation that is given by

$$\frac{\partial}{\partial t} g(t) = -2Ric,$$

and it is a heat equation. Moreover, he showed that the self-similar solutions of Ricci flows are Ricci solitons and that they are natural generalizations of Einstein metrics [1]).

Let (M, g) be a Riemannian manifold. If there exists a smooth vector field (so-called potential field) ν and it satisfies

$$\frac{1}{2}(\mathscr{L}_\nu g) + Ric + \lambda g \quad = \quad 0,$$

then (M, g) is said to be a Ricci soliton. Here, $\mathscr{L}_\nu g$ is the Lie-derivative of the metric tensor g with respect to ν, Ric is the Ricci tensor of M, and λ is a constant. A Ricci soliton is denoted by (M, g, ν, λ), and it is called or shrinking, steady, expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

In 2009, J.T. Cho and M. Kimura introduced a more general notion called the η-Ricci soliton. According to this definition, a Riemannian manifold (M, g) is an η-Ricci soliton if it satisfies

$$\frac{1}{2}(\mathscr{L}_\nu g) + Ric + \lambda g + \mu \eta \otimes \eta \quad = \quad 0, \tag{1}$$

where λ, μ are functions and η is a 1-form. It is clear that if μ is zero, then the η-Ricci soliton becomes a Ricci soliton (see [2]).

Due to the geometric importance of Ricci solitons and their wide applications in theoretical physics, they have become a popular topic studied in the literature. So, the notion of the Ricci soliton has been studied on manifolds that are endowed with many different geometric structures, such as contact, complex, warped product, etc. (see [3–6]).

On the other hand, the concept of Riemannian submersion between Riemannian manifolds is very popular in theoretical physics, as well as in differential geometry, and particularly in general relativity and Kaluza–Klein theory. For this reason, Riemannian submersions have been studied intensively (see [7–13]).

In this paper, we consider a contact-complex Riemannian submersion π from an almost-contact metric manifold M onto an almost Hermitian manifold such that M admits an η-Ricci soliton. Firstly, we calculate the Ricci tensor of the almost-contact metric manifold M, and using it, we present some necessary conditions for which any fibre of π or base manifold B admits a Ricci soliton, η-Ricci soliton, Einstein, or η-Einstein. Moreover, we study a contact-complex Riemannian submersion with totally umbilical fibres whose total space M admits an η-Ricci soliton. Depending on whether the potential field ν of the η-Ricci soliton is vertical or horizontal, we obtain some new results.

Now, we briefly describe the content of the paper. The purpose of the Preliminaries is to review some basic notions, such as almost contact metric structure, Riemannian submersion, some properties of the vertical and horizontal distributions, and of the fundamental tensor fields. Then the main notion of our paper, namely the contact-complex Riemannian submersion, from an almost contact metric manifold, onto an almost Hermitian manifold, is described in Section 3. Then, the main results of the paper are contained in Section 4, which deals with contact-complex Riemannian submersions from manifolds admitting an η-Ricci soliton. Here we obtain conditions under which the base manifold is Einstein, the fibres are η-Einstein, the base manifold admits a Ricci soliton, and some other related facts.

2. Preliminaries

The authors recall the following notations from [13,14].

A Riemannian manifold M of dimension $(2m+1)$ has an almost-contact structure (ϕ, ξ, η) if it admits a vector field ξ (the so-called characteristic vector field), a $(1,1)-$tensor field ϕ, and a $1-$form η satisfying:

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi. \tag{2}$$

As a consequence of (2), we note that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$. If M is endowed with an almost-contact structure (ϕ, ξ, η), then it is called an almost-contact manifold. Moreover, if a Riemannian metric g on M satisfies

$$g(\phi X, \phi Y) \quad = \quad g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for any vector fields X, Y, then the metric g is said to be compatible with the almost-contact structure (ϕ, ξ, η). In this case, the manifold M is said to be endowed with the almost-contact metric structure (ϕ, ξ, η, g), and (M, ϕ, ξ, η, g) is called an almost-contact metric manifold.

Now, we recall the following concepts.

Let $\pi : (M^m, g) \to (B^n, g^{'})$ be a submersion between two Riemannian manifolds and let $r = m - n$ denote the dimension of any closed fibre $\pi^{-1}(x)$ for any $x \in B$. For any $p \in M$, putting $\mathcal{V}_p = ker\pi_{*p}$, we have an integrable distribution $\mathcal{V}$ that corresponds to the foliation of M determined by the fibres of π. Therefore, one has $\mathcal{V}_p = T_p\pi^{-1}(x)$, and $\mathcal{V}$ is called the vertical distribution. Let $\mathcal{H}$ be the horizontal distribution, which means that $\mathcal{H}$ is the orthogonal distribution of $\mathcal{V}$ with respect to g, i.e., $T_p(M) = \mathcal{V}_p \oplus \mathcal{H}_p$, $p \in M$. We note that for any $X' \in \Gamma(TB)$, the basic vector field π-related to X' is named the horizontal lift of X'. Here, $\pi_* X$ is denoted by the vector field X' to which X is π-related.

A map π between Riemannian manifolds M and B is called a Riemannian submersion if the following conditions hold:

(i) π has a maximal rank;
(ii) The differential π_{*p} preserves the length of the horizontal vector fields at each point of M.

For any $E \in \Gamma(TM)$, we denote vE and hE as the vertical and horizontal components of E, respectively.

Proposition 1. *Let $\pi : (M, g) \to (B, g^{'})$ be a Riemannian submersion. If X, Y are the basic vector fields, which are π-related to $X^{'}, Y^{'}$, one has*

(i) $g(X, Y) = g^{'}(X^{'}, Y^{'}) \circ \pi$;
(ii) *$h[X, Y]$ is the basic vector field π-related to $[X^{'}, Y^{'}]$;*
(iii) *$h(\nabla_X Y)$ is the basic vector field π-related to $\nabla^{'}_{X^{'}} Y^{'}$;*
(iv) *for any vertical vector field V, $[X, V]$ is vertical,*

where ∇ and $\nabla^{'}$ denote the Levi–Civita connections of M and B, respectively (see [13]).

The tensor fields $\mathcal{T}$ and $\mathcal{A}$ are said to be the fundamental tensor fields on the manifold M that are defined by

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h(\nabla_{vE} vF) + v(\nabla_{vE} hF),$$
$$\mathcal{A}(E, F) = \mathcal{A}_E F = v(\nabla_{hE} hF) + h(\nabla_{hE} vF),$$

for any $E, F \in \Gamma(TM)$.

The fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M satisfy the following properties:

$$g(\mathcal{T}_E F, G) \quad = \quad -g(\mathcal{T}_E G, F) \tag{4}$$

and

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{5}$$

for any $E, F, G \in \Gamma(TM)$, $V, W \in \Gamma(\mathcal{V})$.

Note the fact that the vanishing of the tensor field $\mathcal{T}$ or $\mathcal{A}$ has some geometric meanings. For instance, if the tensor $\mathcal{A}$ vanishes identically on M, the horizontal distribution $\mathcal{H}$ is integrable. If the tensor $\mathcal{T}$ vanishes identically, any fibre of π is a totally geodesic submanifold of M.

Using the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, one can see that

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{6}$$
$$\nabla_V X = h(\nabla_V X) + \mathcal{T}_V X, \tag{7}$$
$$\nabla_X V = \mathcal{A}_X V + v(\nabla_X V), \tag{8}$$
$$\nabla_X Y = h(\nabla_X Y) + \mathcal{A}_X Y, \tag{9}$$

where ∇ and $\hat{\nabla}$ are the Levi–Civita connections of M and any fibre of π, respectively, for any $V, W \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$.

We recall the following from [11].

Definition 1. *A distribution D on a Riemannian manifold (M, g) is called parallel if $\nabla_X Y \in \Gamma(D)$ for any vector field X on M and any $Y \in \Gamma(D)$, where ∇ is the Levi–Civita connection of g.*

On the other hand, the mean curvature vector field H on any fibre of the Riemannian submersion π is given by

$$\mathcal{N} = rH, \tag{10}$$

such that

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j \tag{11}$$

where r denotes the dimension of any fibre of π and $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$.

Using the equality (11), we get

$$g(\nabla_E \mathcal{N}, X) = \sum_{j=1}^{r} g\big((\nabla_E \mathcal{T})(U_j, U_j), X\big)$$

for any $E \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{H})$. We denote the horizontal divergence of the horizontal vector field X by $\check{\delta}(X)$, which is given by

$$\check{\delta}(X) \quad = \quad \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i), \tag{12}$$

where $\{X_i\}_{1 \le i \le n}$ is an orthonormal frame of $\mathcal{H}$, where n is also the dimension of B. On the other hand, any fibre of π is totally umbilical if

$$\mathcal{T}_U W \quad = \quad g(U, W) H, \tag{13}$$

is satisfied. Here, H is the mean curvature vector field of π in M for any $U, W \in \Gamma(\mathcal{V})$.

Furthermore, the Ricci tensor Ric on M satisfies

$$\begin{aligned} Ric(X,Y) \quad &= \quad Ric^{'}(X^{'}, Y^{'}) \circ \pi - \frac{1}{2}\{g(\nabla_X \mathcal{N}, Y) + g(\nabla_Y \mathcal{N}, X)\} \\ &\quad +2\sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) \end{aligned} \tag{14}$$

$$\begin{aligned} Ric(U,W) \quad &= \quad \hat{Ric}(U,W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_{i=1}^{n} g\big((\nabla_{X_i} \mathcal{T})(U,W), X_i\big) \\ &\quad - \sum_{i=1}^{n} g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W), \end{aligned} \tag{15}$$

for any $X, Y \in \Gamma(\mathcal{H})$ and $U, W \in \Gamma(\mathcal{V})$, where $Ric^{'}$ and $\hat{Ric}$ are the Ricci tensors of the base manifold B and any fibre of π, and $\{X_i\}$, $\{U_j\}$ are some orthonormal bases of $\mathcal{H}$ and $\mathcal{V}$, respectively.

3. Contact-Complex Riemannian Submersions

We recall some notations of [13] in the following.

Let $(M^{2m+1}, \phi, \eta, g, \xi)$ be an almost-contact metric manifold and let $(B^{2n}, \mathcal{J}, g^{'})$ be an almost-Hermitian manifold. A Riemannian submersion $\pi : M \to B$ is called a contact-complex Riemannian submersion if

$$\mathcal{J} \circ \pi_* \;=\; \pi_* \circ \phi.$$

We note here that the vertical distribution $\mathcal{V}$ and horizontal distribution $\mathcal{H}$ are of dimensions $2r+1$ and $2n$, respectively, where $r = m - n$.

For the contact-complex Riemannian submersion $\pi : (M^{2m+1}, g) \to (B^{2n}, g^{'})$, the following properties are satisfied:

(i) The distributions $\mathcal{V}$ and $\mathcal{H}$ are ϕ-invariant,
(ii) The characteristic vector field ξ is vertical,
(iii) $\mathcal{H} \subset ker\eta$, i.e., $\eta(X) = 0$, for any horizontal vector field X.

Example 1. *Let $\pi : S^{2n+1} \to P^n(C)$ be a projection from the total space of a principal fibre bundle S^{2n+1} onto an n-dimensional complex projective space $P^n(C)$. Then, $\pi : S^{2n+1} \to P^n(C)$ is a contact-complex Riemannian submersion with respect to the canonical metric g on S^{2n+1} and the Kaehler metric on $P^n(C)$ (for details, see [13]).*

4. Contact-Complex Riemannian Submersions Whose Total Space Admits an η-Ricci Soliton

Now, we recall the following lemma from [15].

Lemma 1. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds.The following statements are equivalent to each other:*

(i) the vertical distribution $\mathcal{V}$ is parallel;
(ii) the horizontal distribution $\mathcal{H}$ is parallel;
(iii) the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ vanish identically.

Throughout this paper, we make the following assumptions.

Assumption: A contact-complex Riemannian submersion $\pi : M \to B$ is defined from an almost-contact metric manifold (M, ϕ, ξ, η, g) onto an almost-Hermitian manifold $(B, \mathcal{J}, g')$.

Using (14) and (15), for any local orthonormal frames $\{X_i\}_{1\le i\le 2n}$ and $\{U_j, \xi\}_{1\le j\le 2r}$ of $\mathcal{H}$ and $\mathcal{V}$, respectively, we give the following:

Lemma 2. *Let $\pi : M \to B$ be a contact-complex Riemannian submersion between manifolds. For any $U, W \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$ that are π-related to X', Y', the Ricci tensor of M satisfies*

$$Ric(U,W) = \hat{Ric}(U,W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum i = 1^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U,W), X_i) + g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}W)\Big\}, \quad (16)$$

$$Ric(U,\xi) = \hat{Ric}(U,\xi) + g(\mathcal{N}, \mathcal{T}_U \xi) - \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U,\xi), X_i) + g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}\xi)\Big\}, \quad (17)$$

$$Ric(\xi,\xi) = \hat{Ric}(\xi,\xi) + g(\mathcal{N}, \mathcal{T}_\xi \xi) - \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(\xi,\xi), X_i) + g(\mathcal{A}_{X_i}\xi, \mathcal{A}_{X_i}\xi)\Big\}, \quad (18)$$

$$Ric(X,Y) = Ric'(X',Y')\circ\pi - \frac{1}{2}(\mathscr{L}_{\mathcal{N}}g)(X,Y) + 2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y) + g(\mathcal{T}_\xi X, \mathcal{T}_\xi Y). \quad (19)$$

Definition 2. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton and let $\pi : M \to B$ be a contact-complex Riemannian submersion. If v is vertical, we say that v is a vertical potential field. Similarly, if v is horizontal, we say that v is a horizontal potential field.*

Using equalities (16)–(18) in Lemma 2, we have the following theorem.

Theorem 1. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton with vertical potential field v and let $\pi : M \to B$ be a contact-complex Riemannian submersion. If one of the conditions in Lemma 1 is satisfied, then we have the following:*

(i) The base manifold B is Einstein.
(ii) Any fibre of π admits an η-Ricci soliton with potential field v.

Proof. (i) Since M admits an η-Ricci soliton, one has

$$\frac{1}{2}\{g(\nabla_X \nu, Y) + g(\nabla_Y \nu, X)\} + Ric(X,Y) + \lambda g(X,Y) \\ +\mu\eta(X)\eta(Y) = 0, \tag{20}$$

for any horizontal vector fields X, Y. Using (8) in (20) gives

$$\frac{1}{2}\{g(\mathcal{A}_X \nu, Y) + g(\mathcal{A}_Y \nu, X)\} + Ric(X,Y) + \lambda g(X,Y) \\ +\mu\eta(X)\eta(Y) = 0.$$

Applying (19) to the last equality, we get

$$\frac{1}{2}\{g(\mathcal{A}_X \nu, Y) + g(\mathcal{A}_Y \nu, X)\} + Ric^{'}(X^{'}, Y^{'}) \circ \pi - \frac{1}{2}(\mathscr{L}_{\mathcal{N}} g)(X,Y) \\ +2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) + g(\mathcal{T}_\xi X, \mathcal{T}_\xi Y) + \lambda g(X,Y) \\ +\mu\eta(X)\eta(Y) = 0. \tag{21}$$

Since $\eta(X) = 0$, for any horizontal vector field X and if one of the conditions of Lemma 1 is satisfied, Equation (21) gives

$$Ric^{'}(X^{'}, Y^{'}) \circ \pi + \lambda g(X,Y) = 0$$

which is equivalent to

$$\left(Ric^{'}(X^{'}, Y^{'}) + \lambda g^{'}(X^{'}, Y^{'})\right) \circ \pi = 0$$

for any vector fields $X^{'}, Y^{'}$ on $\Gamma(TB)$. Hence,

$$Ric^{'} + \lambda g^{'} = 0$$

is satisfied, which means that (i) is obtained.

(ii) One proof is provided in the following.

Since the total space M admits an η-Ricci soliton with vertical potential field ν, from (1), we can write

$$\frac{1}{2}\{g(\nabla_U \nu, W) + g(\nabla_W \nu, U)\} + Ric(U,W) + \lambda g(U,W) + \mu\eta(U)\eta(W) = 0, \tag{22}$$

for any $U, W \in \Gamma(\mathcal{V})$. Using (6) in (22), it follows that

$$\frac{1}{2}\{g(\hat{\nabla}_U \nu, W) + g(\hat{\nabla}_W \nu, U)\} + Ric(U,W) + \lambda g(U,W) + \mu\eta(U)\eta(W) = 0. \tag{23}$$

Applying (16) to Equation (23) gives

$$\frac{1}{2}(\mathscr{L}_\nu \hat{g})(U,W) + \hat{Ric}(U,W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U,W), X_i) \\ +g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W)\Big\} - g((\nabla_\xi \mathcal{T})(U,W), \xi) - g(\mathcal{A}_\xi U, \mathcal{A}_\xi W) \\ +\lambda g(U,W) + \mu\eta(U)\eta(W) = 0. \tag{24}$$

Since one of the conditions in Lemma 1 is satisfied, Equation (24) is equivalent to

$$\frac{1}{2}(\mathscr{L}_\nu \hat{g})(U,W) + \hat{Ric}(U,W) + \lambda \hat{g}(U,W) + \mu\eta(U)\eta(W) = 0,$$

which means that any fibre of π is an η-Ricci soliton, and the proof is complete. □

Using Lemma 2, we give the following theorem.

Theorem 2. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton with horizontal potential field ν and let $\pi : M \to B$ be a contact-complex Riemannian submersion. If one of the conditions in Lemma 1 is satisfied, then any fibre of π is η-Einstein.*

Proof. Case I. For any vertical vector fields $U, W \neq \xi$, we can write

$$\frac{1}{2}(\mathscr{L}_\nu g)(U,W) + Ric(U,W) + \lambda g(U,W) + \mu\eta(U)\eta(W) = 0, \tag{25}$$

for any vertical vector fields U, W. Using (7) in the Lie-derivative of (25), one has

$$\begin{aligned}\frac{1}{2}(\mathscr{L}_\nu g)(U,W) &= \frac{1}{2}\{g(\nabla_U \nu, W) + g(\nabla_W \nu, U)\}\\ &= \frac{1}{2}\{g(\mathcal{T}_U \nu, W) + g(\mathcal{T}_W \nu, U)\}\\ &= 0,\end{aligned}$$

and since Lemma 1 is satisfied, the tensor field $\mathcal{T} \equiv 0$. In addition, putting (16) into Equation (25) gives

$$\begin{aligned}&\hat{Ric}(U,W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_i \Big\{g((\nabla_{X_i}\mathcal{T})(U,W), X_i) + g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}W)\Big\}\\ &+\lambda g(U,W) + \mu\eta(U)\eta(W) = 0,\end{aligned}$$

which means that

$$\hat{Ric}(U,W) + \lambda \hat{g}(U,W) + \mu\eta(U)\eta(W) = 0. \tag{26}$$

Case II. For any vertical vector field $U \neq \xi$, Equation (1) gives

$$\frac{1}{2}\Big\{g(\nabla_U \nu, \xi) + g(\nabla_\xi \nu, U)\Big\} + Ric(U,\xi) + \lambda g(U,\xi) + \mu\eta(U)\eta(\xi) = 0.$$

Then, it follows that

$$Ric(U,\xi) + \lambda g(U,\xi) + \mu\eta(U)\eta(\xi) = 0. \tag{27}$$

Using (19) in (27) gives

$$\begin{aligned}&\hat{Ric}(U,\xi) + g(\mathcal{N}, \mathcal{T}_U \xi) - \sum_i \Big\{g((\nabla_{X_i}\mathcal{T})(U,\xi), X_i) + g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}\xi)\Big\}\\ &+\lambda g(U,\xi) + \mu\eta(U)\eta(\xi) = 0.\end{aligned}$$

Since $\mathcal{T} \equiv 0$, the last equality is equivalent to

$$\hat{Ric}(U,\xi) + \lambda g(U,\xi) + \mu\eta(U)\eta(\xi) = 0. \tag{28}$$

Case III. Finally, choosing $U = W = \xi$, Equation (1) gives

$$g(\nabla_\xi \nu, \xi) + Ric(\xi,\xi) + \lambda g(\xi,\xi) + \mu\eta(\xi)\eta(\xi) = 0.$$

Through similar calculations, we have

$$Ric(\xi,\xi) + \lambda g(\xi,\xi) + \mu\eta(\xi)\eta(\xi) = 0.$$

Applying (18) to the last equality and using the vanishing of the tensor field $\mathcal{T}$ gives

$$\hat{Ric}(\xi,\xi) + \lambda \hat{g}(\xi,\xi) + \mu\eta(\xi)\eta(\xi) = 0 \quad (29)$$

is obtained.

As a result of equalities (26), (28), and (29), we obtain that any fibre of π is η-Einstein. □

Considering Equation (29), we can give the following corollary.

Corollary 1. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton with horizontal potential field ν and let $\pi : M \to B$ be a contact-complex Riemannian submersion. If one of the conditions in Lemma 1 is satisfied, then the Ricci tensor of the distribution $Span\{\xi\}$ is given by*

$$\hat{Ric}(\xi,\xi) = -(\lambda + \mu).$$

Theorem 3. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton with a horizontal potential field ν and let $\pi : M \to B$ be a contact-complex Riemannian submersion. If one of the conditions in Lemma 1 is satisfied, then the base manifold B admits a Ricci soliton with potential field ν' such that $\pi_*\nu = \nu'$.*

Proof. For any horizontal vector fields X, Y, we can write

$$\frac{1}{2}\{g(\nabla_X \nu, Y) + g(\nabla_Y \nu, X)\} + Ric(X,Y) + \lambda g(X,Y) + \mu\eta(X)\eta(Y) = 0. \quad (30)$$

Since the vector fields X, Y are horizontal, we get $\eta(X) = \eta(Y) = 0$. Then, it follows that

$$\frac{1}{2}\{g(\nabla_X \nu, Y) + g(\nabla_Y \nu, X)\} + Ric(X,Y) + \lambda g(X,Y) = 0. \quad (31)$$

In addition, using (19) in (31), one has

$$\frac{1}{2}\{g(\nabla_X \nu, Y) + g(\nabla_Y \nu, X)\} + Ric'(X', Y')\circ\pi + 2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i)$$
$$+ \sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) + g(\mathcal{T}_\xi X, \mathcal{T}_\xi Y) + \lambda g(X,Y) = 0.$$

Since Lemma 1 is satisfied, it follows that

$$\frac{1}{2}\{g(\nabla_X \nu, Y) + g(\nabla_Y \nu, X)\} + Ric'(X', Y')\circ\pi + \lambda g(X,Y) = 0. \quad (32)$$

Moreover, considering Proposition 1, Equation (32) gives

$$\frac{1}{2}\{g'(\nabla'_{X'}\nu', Y')\circ\pi + g'(\nabla'_{Y'}\nu', X')\circ\pi\} + Ric'(X', Y')\circ\pi$$
$$+\lambda g'(X', Y')\circ\pi = 0,$$

for any $X', Y' \in \Gamma(TB)$. Then, the last equation is equivalent to

$$\frac{1}{2}\left(\mathcal{L}_{\nu'} g'\right)(X', Y') + Ric'(X', Y') + \lambda g'(X', Y') = 0,$$

where the vector field ν on M is π-related to ν' on B. Therefore, the base manifold B admits a Ricci soliton with potential field ν'. □

Theorem 4. *Let $(M, g, \xi, \lambda, \eta)$ be an η-Ricci soliton with horizontal potential field ν and let $\pi : M \to B$ be a contact-complex Riemannian submersion with totally umbilical fibres. If the horizontal distribution $\mathcal{H}$ is integrable, then any fibre of π is η-Einstein.*

Proof. Since the total space M admits an η-Ricci soliton, one has

$$\frac{1}{2}\Big(\mathscr{L}_\nu g\Big)(U, W) + Ric(U, W) + \lambda g(U, W) + \mu\eta(U)\eta(W) = 0, \tag{33}$$

for any vertical vector fields U, W. Putting (16) into the last equation gives

$$\begin{aligned}&\frac{1}{2}\Big(g(\nabla_U \nu, W) + g(\nabla_W U)\Big) + \hat{Ric}(U, W) + g(\mathcal{N}, \mathcal{T}_U W)\\ &- \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U, W), X_i) + g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}W)\Big\} + \lambda g(U, W)\\ &+\mu\eta(U)\eta(W) = 0.\end{aligned}$$

In addition, the horizontal distribution $\mathcal{H}$ is integrable, and it follows that

$$\begin{aligned}&\frac{1}{2}\Big(g(\nabla_U \nu, W) + g(\nabla_W U)\Big) + \hat{Ric}(U, W) + g(\mathcal{N}, \mathcal{T}_U W)\\ &- \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U, W), X_i) + \lambda g(U, W) + \mu\eta(U)\eta(W) = 0.\end{aligned}$$

Since any fibre of π is totally umbilical and by using Equations (6), (12), and (13) in the last equality, one has

$$\begin{aligned}&\frac{1}{2}\Big(g(\mathcal{T}_U \nu, W) + g(\mathcal{T}_W \nu, U)\Big) + \hat{Ric}(U, W) + (2r+1)\|H\|^2 g(U, W)\\ &-\check{\delta}(H)g(U, W) + \lambda g(U, W) + \mu\eta(U)\eta(W) = 0.\end{aligned}$$

Using (4) and (5), we get

$$\begin{aligned}&\hat{Ric}(U, W) + \Big\{(2r+1)\|H\|^2 + \check{\delta}(H) - g(H, \nu) + \lambda\Big\}g(U, W)\\ &+\mu\eta(U)\eta(W) = 0,\end{aligned}$$

which gives that any fibre of π is η-Einstein. □

5. Conclusions

The paper deals with an interesting concept, of a contact-complex Riemannian submersion, which puts in relation the almost contact metric structure from the domain manifold, to the almost Hermitian structure of the target manifold. The fundamental properties of the Riemannian submersions are used here to link some geometric feature on the domain manifold, with the ones on fibres and with those on the base manifold. We provide several new results, showing mainly when the base manifold admits a Ricci soliton, when it is Einstein, when the fibres are η-Ricci solitons, and when they are η-Einstein. Our future study will be developed on certain well known manifolds on which we may apply the above theory.

Author Contributions: Conceptualization, C.-L.B., Ş.E.M. and E.K.; methodology, C.-L.B., Ş.E.M. and E.K.; software, C.-L.B. and Ş.E.M.; validation, C.-L.B., Ş.E.M. and E.K.; formal analysis, C.-L.B. and Ş.E.M.; investigation, C.-L.B., Ş.E.M. and E.K.; resources, C.-L.B., Ş.E.M. and E.K.; data curation, C.-L.B., Ş.E.M. and E.K.; writing—original draft preparation, C.-L.B. and Ş.E.M.; writing—review and editing, C.-L.B., Ş.E.M. and E.K.; visualization,C.-L.B. and Ş.E.M.; supervision, E.K.; project administration, E.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors deeply thank all three referees for valuable suggestions, that are used here. To the memory of Aurel Bejancu (1946–2020).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hamilton, R.S. The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986). *Contemp. Math. Am. Math. Soc.* **1988**, *71*, 237–262.
2. Cho, J.T.; Kimura, A. Ricci Solitons and Lagrangian Submanifolds in Kaehler Manifolds. *Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci.* **2010**, *43*, 27–32.
3. Chen, B.-Y.; Deshmukh, S. Ricci solitons and concurrent vector fields. *Balkan J. Geom. Appl.* **2015**, *20*, 14–25.
4. Chen, B.-Y. Concircular vector fields and pseudo-Kähler manifolds. *Kragujev. J. Math.* **2016**, *40*, 7–14. [CrossRef]
5. Yüksel Perktaş, S.; Keleş, S. Ricci solitons in 3-dimensional normal almost paracontact metric manifolds. *Int. Electron. J. Geom.* **2015**, *8*, 34–45. [CrossRef]
6. Yoldaş, H.I.; Eken Meriç, Ş.; Yaşar, E. On generic submanifold of Sasakian manifold with concurrent vector field. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2019**, *68*, 1983–1994. [CrossRef]
7. Bejan, C.-L.; Eken Meriç, Ş. Conformality on Semi-Riemannian Manifolds. *Mediterr. J. Math.* **2016**, *13*, 2185–2198. [CrossRef]
8. Bejan, C.-L.; Eken Meriç, Ş.; Kiliç, E. Gradient Weyl-Ricci soliton. *Turk. J. Math.* **2020**, *44*, 1137–1145. [CrossRef]
9. Gray, A. Pseudo-Riemannian almost product manifolds and submersions. *J. Math. Mech.* **1967**, *16*, 715–737.
10. O'Neill, B. The fundamental equations of a Riemannian submersions. *Mich. Math. J.* **1966**, *13*, 459–469.
11. Şahin, B. *Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications*; Elsevier Academic: Amsterdam, The Netherlands, 2017.
12. Yadav, A.; Meena, K. Riemannian maps whose total manifolds admit a Ricci soliton. *J. Geom. Phys.* **2021**, *168*, 104317. [CrossRef]
13. Falcitelli, M.; Ianus, S.; Pastore, A.M. *Riemannian Submersions and Related Topics*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2004.
14. Blair, D.E. *Contact Manifolds in Riemannian Geometry*; Lecture Notes in Mathematics, 509; Springer: Berlin, Germany, 1976.
15. Eken Meriç, Ş.; Kılıç, E. Riemannian submersions whose total manifolds admit a Ricci soliton. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950196. [CrossRef]