

# Supplement to: Generators and relations for $n$ -qubit Clifford operators

Peter Selinger, Dalhousie University

## 1 Lemmas

**Lemma 1.1.** *Each of the following properties is a consequence of the equations from Figure 8(a)–(b).*

$$\omega^8 = 1 \tag{1}$$

$$HH = 1 \tag{2}$$

$$SSSS = 1 \tag{3}$$

$$SHSH = HSSS \cdot \omega \tag{4}$$

$$HSHS = SSSH \cdot \omega \tag{5}$$

$$HSSSH = SHS \cdot \omega^{-1} \tag{6}$$

$$SHSSHS = HSSH \cdot \omega^2 \tag{7}$$

$$HSSHSS = SSHSSH \cdot \omega^4 \tag{8}$$

$$SSSHSSH = HSSHS \cdot \omega^{-2} \tag{9}$$

$$SSSHSSS = HSH \cdot \omega^{-1} \tag{10}$$

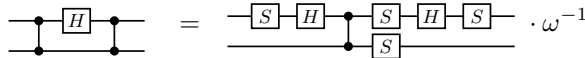
**Lemma 1.2.** *Each of the following properties, as well as its “upside-down” version, is a consequence of the equations from Figure 8(a)–(c).*



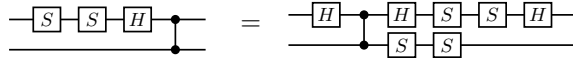
$$= \tag{11}$$



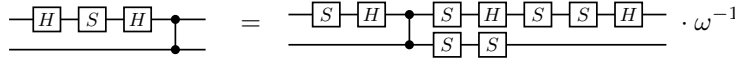
$$= \tag{12}$$



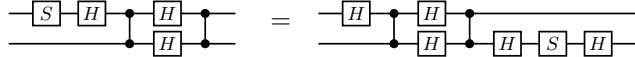
$$= \cdot \omega^{-1} \tag{13}$$



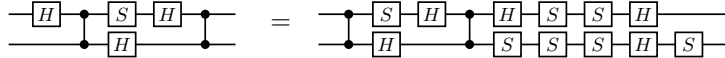
$$= \tag{14}$$



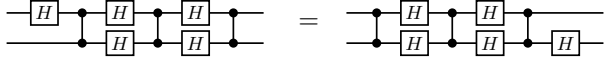
$$= \cdot \omega^{-1} \tag{15}$$



$$= \tag{16}$$



$$= \tag{17}$$

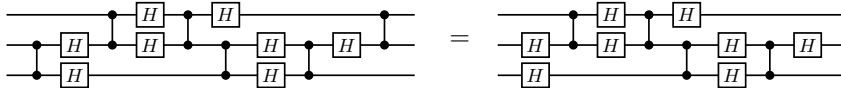


$$= \tag{18}$$

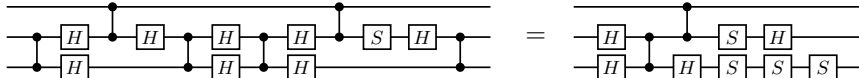
**Lemma 1.3.** *Each of the following properties, as well as its “upside-down” version, is a consequence of the equations from Figure 8(a)–(d).*



$$= \tag{19}$$



$$= \tag{20}$$



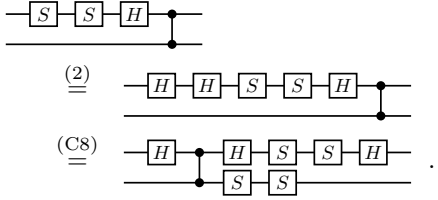
$$= \tag{21}$$

## 2 Proofs of lemmas

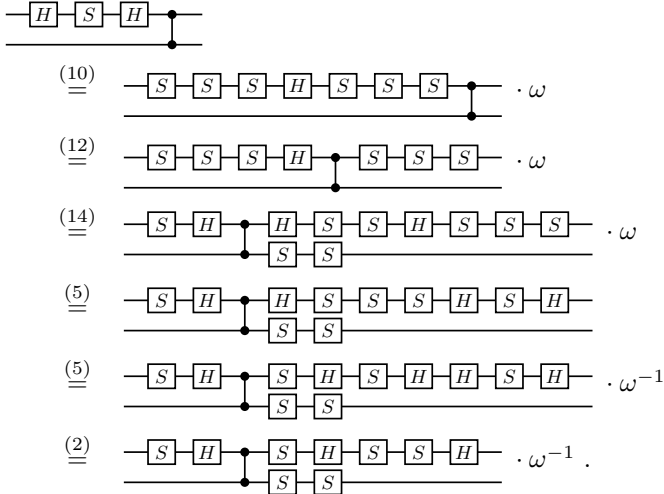
We write (CN) for axiom number  $N$  from Figure 8.

*Proof of Lemma 1.1.* Property (1) is (C1), property (2) is (C2), and property (3) is (C3). For (4), we have  $SHSH \stackrel{(3)}{=} SHSHSSSS \stackrel{(2)}{=} SHSHSHSSSS \stackrel{(C4)}{=} HSSS \cdot \omega$ . For (5), we have  $HSHS \stackrel{(2)}{=} HSHSHH \stackrel{(4)}{=} HHSSSH \cdot \omega \stackrel{(2)}{=} SSSH \cdot \omega$ . For (6), we have  $HSSSH \stackrel{(4)}{=} SHSHH \stackrel{(2)}{=} SHS$ . For (6), we have  $HSSSH \stackrel{(4)}{=} SHSHH \cdot \omega^{-1} \stackrel{(2)}{=} SHS \cdot \omega^{-1}$ . For (7), we have  $SHSSHS \stackrel{(6)}{=} HSSSHSSSH \cdot \omega^2 \stackrel{(2)}{=} HSSSSSSH \cdot \omega^2 \stackrel{(3)}{=} HSSH \cdot \omega^2$ . For (8), we have  $HSSHSS \stackrel{(6)}{=} HSHSSSHS \cdot \omega \stackrel{(5)}{=} SSSHSSHS \cdot \omega^2 \stackrel{(7)}{=} SSHSSH \cdot \omega^4$ . For (9), we have  $SSSHSSH \stackrel{(5)}{=} HSHSSSH \cdot \omega^{-1} \stackrel{(5)}{=} HSHHSHS \cdot \omega^{-2} \stackrel{(2)}{=} HSSHHS \cdot \omega^{-2}$ . For (10), we have  $SSSHSSS \stackrel{(4)}{=} SSSSHSH \cdot \omega^{-1} \stackrel{(3)}{=} HSH \cdot \omega^{-1}$ .  $\square$

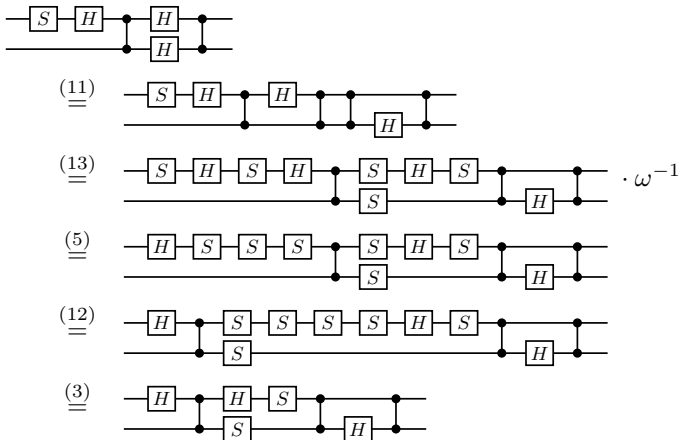
*Proof of Lemma 1.2.* We first note that the “upside-down” version of each axiom in Figure 8 is itself an axiom. Therefore, for every property we prove, the upside-down version is also valid. Property (11) is (C5), property (12) is (C6), and property (13) is (C10). For (14), we have

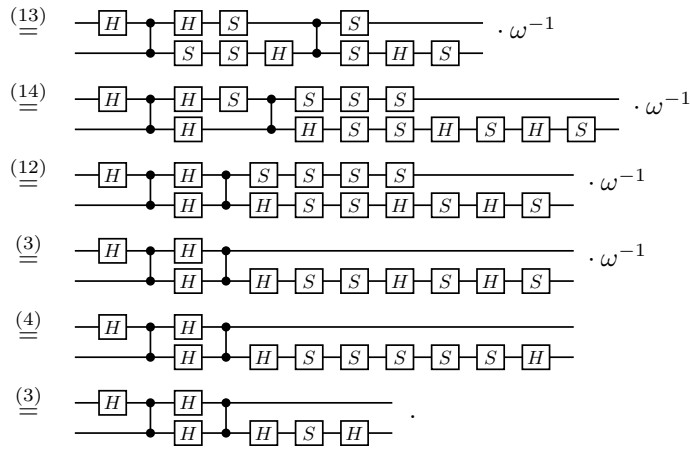


For (15), we have

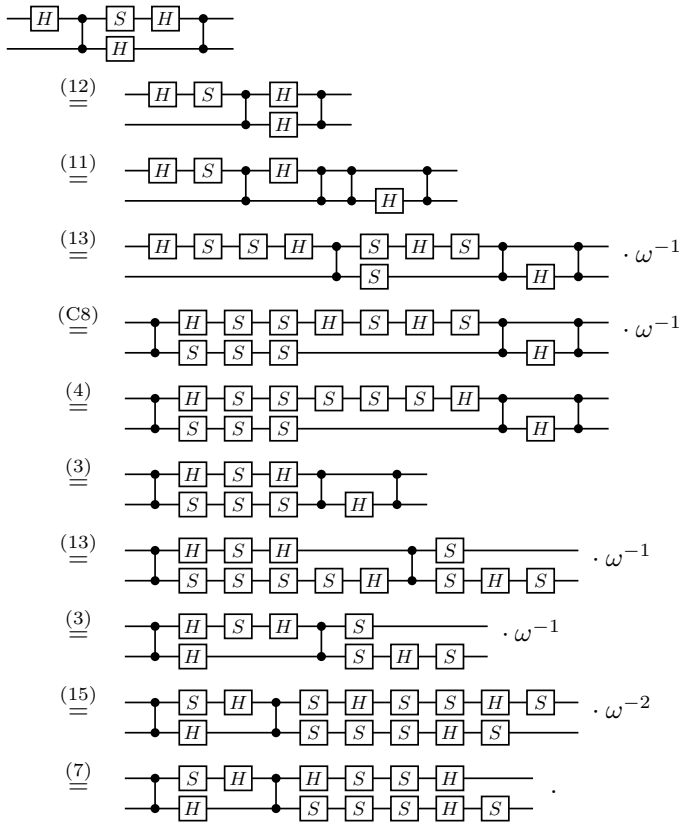


For (16), we have

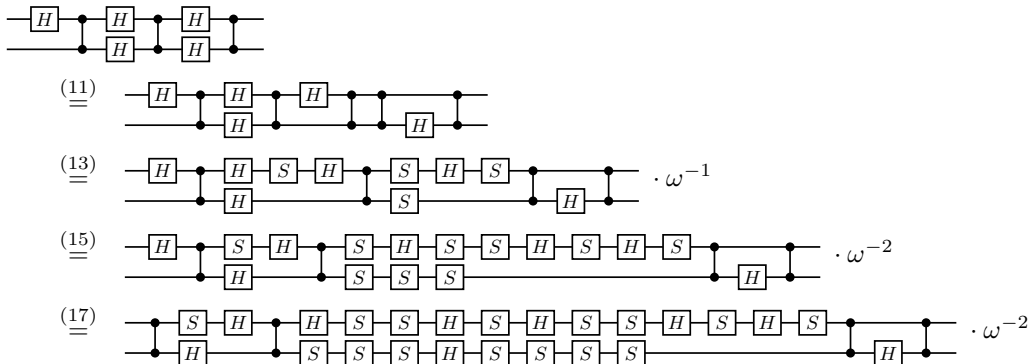




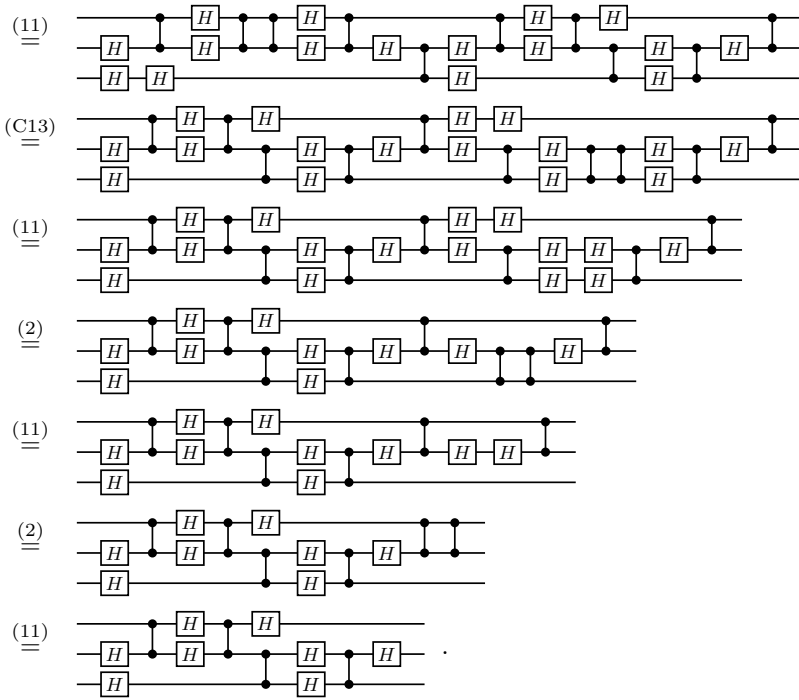
For (17), we have



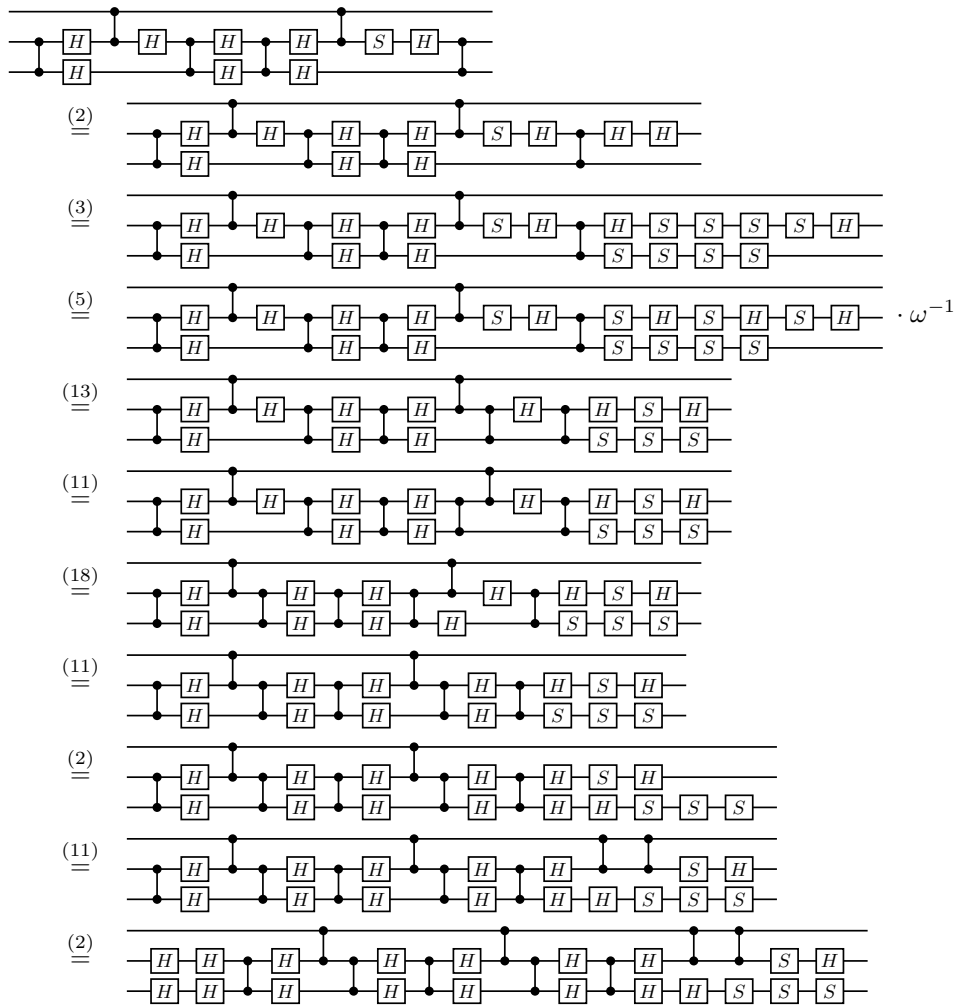
For (18), we have

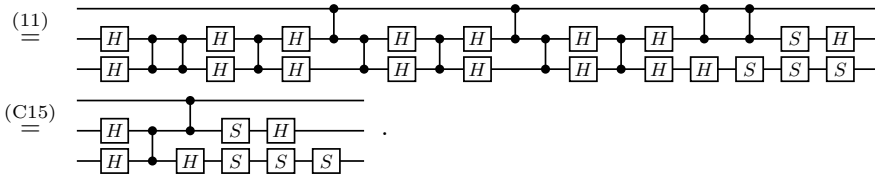






For (21), we have





□

### 3 Proofs of the equations from Proposition 7.1

In the following proofs, we write (inv) to denote some number of application of the properties (2), (3), and (11), usually to insert or delete a circuit together with its own inverse inside a larger circuit. We write (def) to denote a definition from Figure 1.

#### 3.1 Equation

$$\text{---} = \text{---} \boxed{A_1} \boxed{C_1} \boxed{E_1} \text{---}$$

**Proof:**  $\text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{A_1} \boxed{C_1} \boxed{E_1} \text{---}$

#### 3.2 Equation

$$\text{---} \boxed{C_1} \text{---} = \text{---} \boxed{B_1} \boxed{C_1} \boxed{D_1} \text{---}$$

**Proof:**  $\text{---} \boxed{C_1} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---} \boxed{H} \boxed{H} \text{---}$   
 $\stackrel{\text{(11)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \boxed{H} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \boxed{H} \boxed{H} \text{---}$   
 $\stackrel{\text{(11)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(11)}}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{B_1} \boxed{C_1} \boxed{D_1} \text{---}$

#### 3.3 Equation

$$\text{---} \boxed{H} \boxed{A_1} \text{---} = \text{---} \boxed{A_2} \text{---}$$

**Proof:**  $\text{---} \boxed{H} \boxed{A_1} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{H} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{A_2} \text{---}$

#### 3.4 Equation

$$\text{---} \boxed{H} \boxed{A_2} \text{---} = \text{---} \boxed{A_1} \text{---}$$

**Proof:**  $\text{---} \boxed{H} \boxed{A_2} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{H} \boxed{H} \text{---}$   
 $\stackrel{\text{(2)}}{=} \text{---}$   
 $\stackrel{\text{(def)}}{=} \text{---} \boxed{A_1} \text{---}$

### 3.5 Equation

$$\boxed{H} \boxed{A_3} = \boxed{A_3} \boxed{X} \boxed{S} \boxed{S} \boxed{S} \cdot \omega$$

**Proof:**

$$\begin{aligned} & \boxed{H} \boxed{A_3} \\ \stackrel{(\text{def})}{=} & \boxed{H} \boxed{H} \boxed{S} \boxed{H} \\ \stackrel{(2)}{=} & \boxed{S} \boxed{H} \\ \stackrel{(3)}{=} & \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \boxed{S} \\ \stackrel{(6)}{=} & \boxed{H} \boxed{S} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \\ \stackrel{(2)}{=} & \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \\ \stackrel{(\text{def})}{=} & \boxed{A_3} \boxed{X} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \end{aligned}$$

### 3.6 Equation

$$\boxed{S} \boxed{A_1} = \boxed{A_1} \boxed{S}$$

**Proof:**

$$\begin{aligned} & \boxed{S} \boxed{A_1} \\ \stackrel{(\text{def})}{=} & \boxed{S} \\ \stackrel{(\text{def})}{=} & \boxed{A_1} \boxed{S} \end{aligned}$$

### 3.7 Equation

$$\boxed{S} \boxed{A_2} = \boxed{A_3} \boxed{X} \boxed{S} \boxed{S} \boxed{S} \cdot \omega$$

**Proof:**

$$\begin{aligned} & \boxed{S} \boxed{A_2} \\ \stackrel{(\text{def})}{=} & \boxed{S} \boxed{H} \\ \stackrel{(3)}{=} & \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \boxed{S} \\ \stackrel{(6)}{=} & \boxed{H} \boxed{S} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \\ \stackrel{(2)}{=} & \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \\ \stackrel{(\text{def})}{=} & \boxed{A_3} \boxed{X} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \end{aligned}$$

### 3.8 Equation

$$\boxed{S} \boxed{A_3} = \boxed{A_2} \boxed{S} \boxed{S} \boxed{S} \cdot \omega$$

**Proof:**

$$\begin{aligned} & \boxed{S} \boxed{A_3} \\ \stackrel{(\text{def})}{=} & \boxed{S} \boxed{H} \boxed{S} \boxed{H} \\ \stackrel{(5)}{=} & \boxed{H} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \\ \stackrel{(\text{def})}{=} & \boxed{A_2} \boxed{S} \boxed{S} \boxed{S} \cdot \omega \end{aligned}$$

### 3.9 Equation

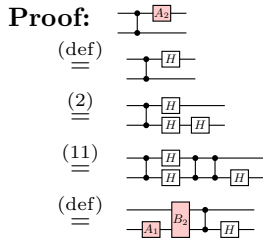
$$\begin{array}{c} \bullet \\ | \\ \boxed{A_1} \\ | \\ \bullet \\ \hline \end{array} = \begin{array}{c} \boxed{A_1} \\ | \\ \bullet \\ | \\ \bullet \\ \hline \end{array}$$

**Proof:**

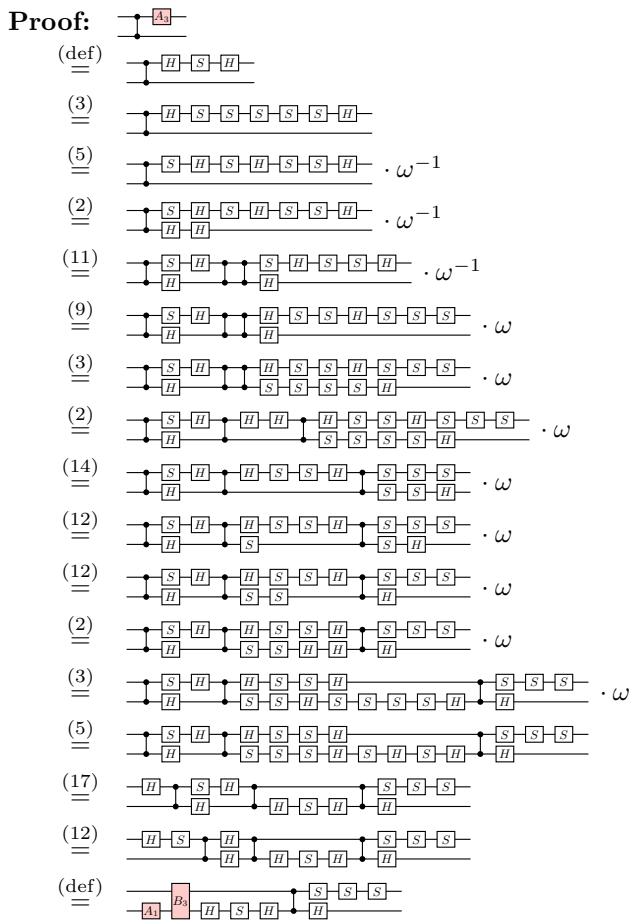
$$\begin{aligned} & \begin{array}{c} \bullet \\ | \\ \boxed{A_1} \\ | \\ \bullet \\ \hline \end{array} \\ \stackrel{(\text{def})}{=} & \begin{array}{c} \bullet \\ | \\ \bullet \\ \hline \end{array} \\ \stackrel{(\text{def})}{=} & \begin{array}{c} \boxed{A_1} \\ | \\ \bullet \\ | \\ \bullet \\ \hline \end{array} \end{aligned}$$



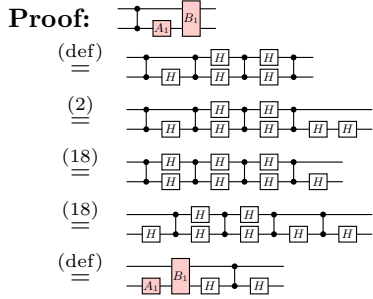
### 3.10 Equation



### 3.11 Equation

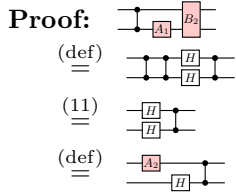


### 3.12 Equation



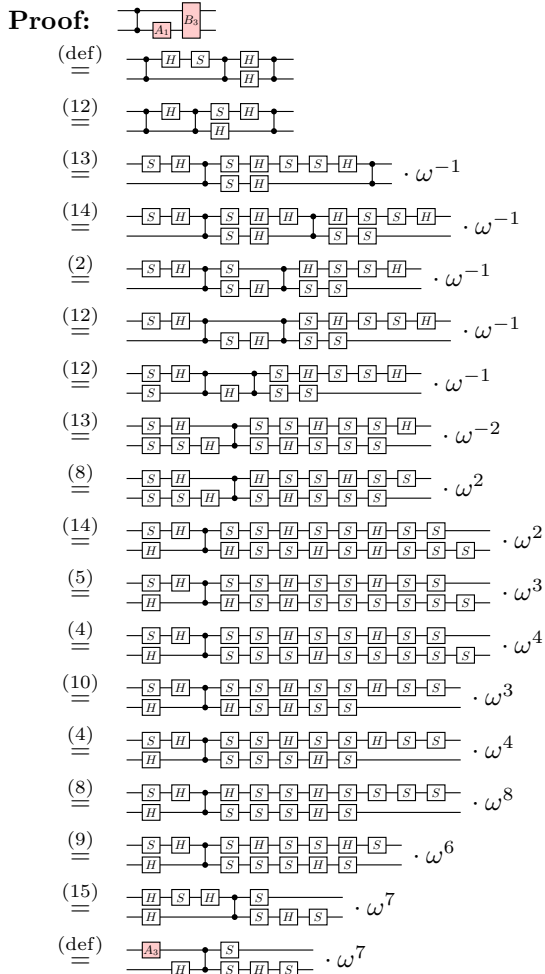
### 3.13 Equation

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{A_1} \boxed{B_2} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{A_2} \\ \boxed{H} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}$$



### 3.14 Equation

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{A_1} \boxed{B_3} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{A_3} \\ \boxed{H} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{S} \\ \boxed{S} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{H} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{S} \\ \boxed{S} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \cdot \omega^7$$



### 3.15 Equation

**Proof:**

### 3.16 Equation

**Proof:**

### 3.17 Equation

**Proof:**

(def)  $\equiv$

(13)  $\equiv$   $\cdot \omega^{-1}$

(5)  $\equiv$

(12)  $\equiv$

(12)  $\equiv$

(12)  $\equiv$

(2)  $\equiv$

(3)  $\equiv$

(2)  $\equiv$

(14)  $\equiv$

(12)  $\equiv$

(3)  $\equiv$

(12)  $\equiv$

(12)  $\equiv$

(5)  $\equiv$   $\cdot \omega^{-1}$

(13)  $\equiv$

(12)  $\equiv$

(2)  $\equiv$

(3)  $\equiv$

(5)  $\equiv$   $\cdot \omega^{-1}$

(17)  $\equiv$   $\cdot \omega^{-1}$

(3)  $\equiv$   $\cdot \omega^{-1}$

(12)  $\equiv$   $\cdot \omega^{-1}$

(10)  $\equiv$

(4)  $\equiv$   $\cdot \omega^{-1}$

(5)  $\equiv$   $\cdot \omega^{-2}$

(14)  $\equiv$   $\cdot \omega^{-2}$

(12)  $\equiv$   $\cdot \omega^{-2}$

(3)  $\equiv$   $\cdot \omega^{-2}$

(12)  $\equiv$   $\cdot \omega^{-2}$

(12)  $\equiv$   $\cdot \omega^{-2}$

(6)  $\equiv$   $\cdot \omega^{-1}$

(2)  $\equiv$   $\cdot \omega^{-1}$

(5)  $\equiv$   $\cdot \omega^{-2}$

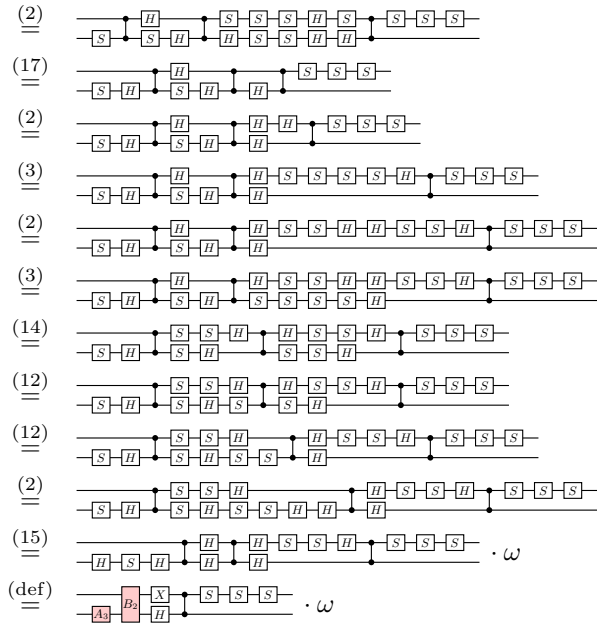
$$\begin{aligned}
& \underline{\underline{(17)}} \quad \text{Diagram 1} \cdot \omega^{-2} \\
& \underline{\underline{(10)}} \quad \text{Diagram 2} \cdot \omega^{-1} \\
& \underline{\underline{(3)}} \quad \text{Diagram 3} \cdot \omega^{-1} \\
& \underline{\underline{(2)}} \quad \text{Diagram 4} \cdot \omega^{-1} \\
& \underline{\underline{(14)}} \quad \text{Diagram 5} \cdot \omega^{-1} \\
& \underline{\underline{(12)}} \quad \text{Diagram 6} \cdot \omega^{-1} \\
& \underline{\underline{(12)}} \quad \text{Diagram 7} \cdot \omega^{-1} \\
& \underline{\underline{(5)}} \quad \text{Diagram 8} \cdot \omega^{-2} \\
& \underline{\underline{(3)}} \quad \text{Diagram 9} \cdot \omega^{-2} \\
& \underline{\underline{(5)}} \quad \text{Diagram 10} \cdot \omega^{-3} \\
& \underline{\underline{(2)}} \quad \text{Diagram 11} \cdot \omega^{-3} \\
& \underline{\underline{(17)}} \quad \text{Diagram 12} \cdot \omega^{-3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 13} \cdot \omega^{-3} \\
& \underline{\underline{(3)}} \quad \text{Diagram 14} \cdot \omega^{-3} \\
& \underline{\underline{(14)}} \quad \text{Diagram 15} \cdot \omega^{-3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 16} \cdot \omega^{-3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 17} \cdot \omega^{-3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 18} \cdot \omega^{-3} \\
& \underline{\underline{(2)}} \quad \text{Diagram 19} \cdot \omega^{-3} \\
& \underline{\underline{(15)}} \quad \text{Diagram 20} \cdot \omega^{-2} \\
& \underline{\underline{(1)}} \quad \text{Diagram 21} \cdot \omega^6 \\
& \underline{\underline{\text{(def)}}} \quad \text{Diagram 22} \cdot \omega^6
\end{aligned}$$

### 3.18 Equation

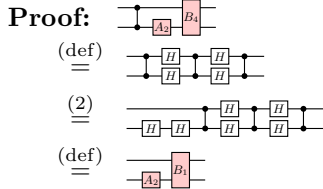
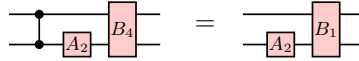
$$\text{Diagram with } A_2, B_3 \text{ boxes} = \text{Diagram with } A_3, B_2, X, H \text{ boxes and } S \text{ gates} \cdot \omega$$

**Proof:**

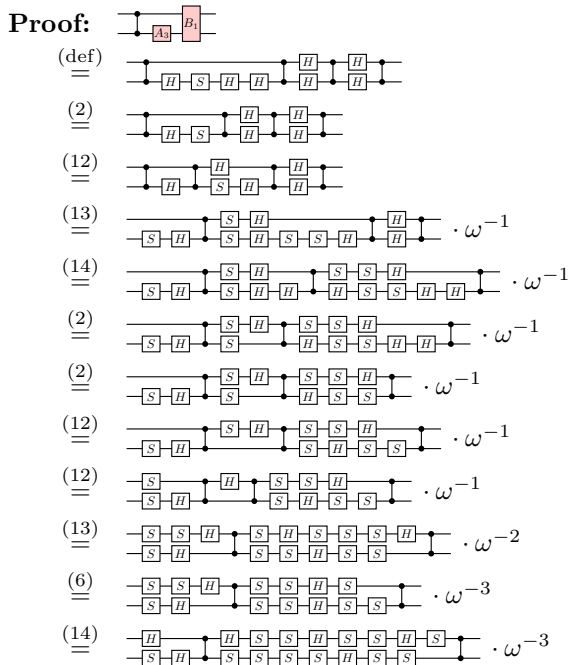
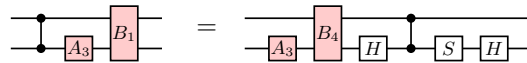
$$\begin{aligned}
& \underline{\underline{\text{(def)}}} \quad \text{Diagram 1} \\
& \underline{\underline{(12)}} \quad \text{Diagram 2} \\
& \underline{\underline{(3)}} \quad \text{Diagram 3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 4} \\
& \underline{\underline{(3)}} \quad \text{Diagram 5} \\
& \underline{\underline{(3)}} \quad \text{Diagram 6} \\
& \underline{\underline{(2)}} \quad \text{Diagram 7} \\
& \underline{\underline{(14)}} \quad \text{Diagram 8} \\
& \underline{\underline{(12)}} \quad \text{Diagram 9}
\end{aligned}$$



### 3.19 Equation



### 3.20 Equation









$$\begin{aligned}
& \stackrel{(12)}{=} \text{Diagram 1} \cdot \omega \\
& \stackrel{(2)}{=} \text{Diagram 2} \cdot \omega \\
& \stackrel{(3)}{=} \text{Diagram 3} \cdot \omega \\
& \stackrel{(2)}{=} \text{Diagram 4} \cdot \omega \\
& \stackrel{(17)}{=} \text{Diagram 5} \cdot \omega \\
& \stackrel{(12)}{=} \text{Diagram 6} \cdot \omega \\
& \stackrel{(\text{def})}{=} \text{Diagram 7} \cdot \omega
\end{aligned}$$

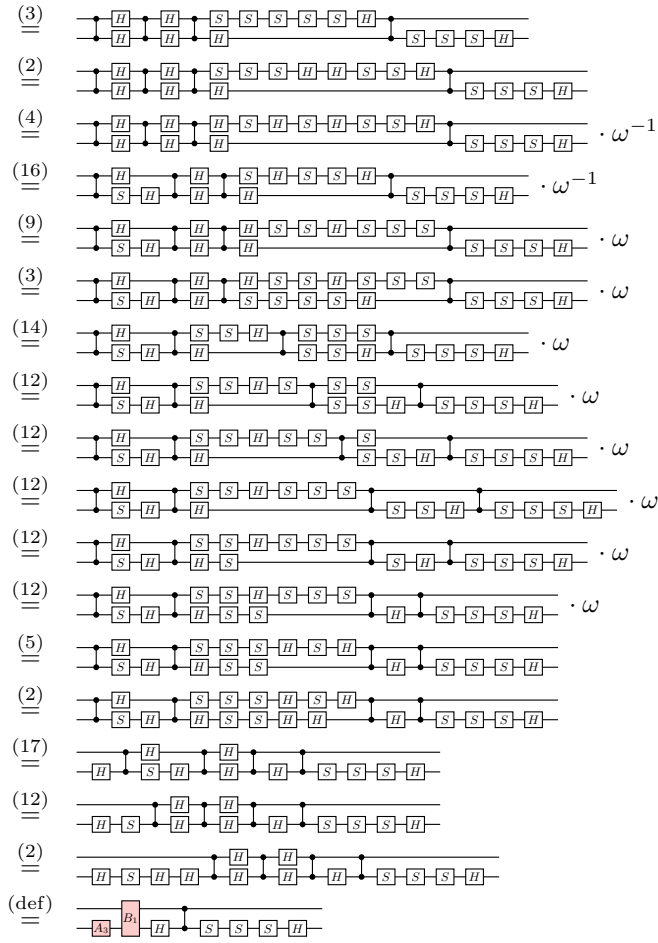
### 3.22 Equation

$$\text{Diagram 8} = \text{Diagram 9} \cdot \omega$$

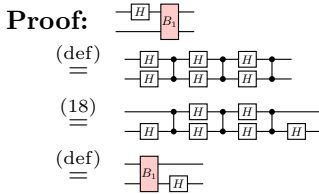
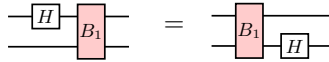
**Proof:**

$$\begin{aligned}
& \stackrel{(\text{def})}{=} \text{Diagram 10} \\
& \stackrel{(12)}{=} \text{Diagram 11} \\
& \stackrel{(17)}{=} \text{Diagram 12} \\
& \stackrel{(15)}{=} \text{Diagram 13} \cdot \omega^{-1} \\
& \stackrel{(2)}{=} \text{Diagram 14} \cdot \omega^{-1} \\
& \stackrel{(14)}{=} \text{Diagram 15} \cdot \omega^{-1} \\
& \stackrel{(3)}{=} \text{Diagram 16} \cdot \omega^{-1} \\
& \stackrel{(2)}{=} \text{Diagram 17} \cdot \omega^{-1} \\
& \stackrel{(3)}{=} \text{Diagram 18} \cdot \omega^{-1} \\
& \stackrel{(2)}{=} \text{Diagram 19} \cdot \omega^{-1} \\
& \stackrel{(12)}{=} \text{Diagram 20} \cdot \omega^{-1} \\
& \stackrel{(13)}{=} \text{Diagram 21} \cdot \omega^{-2} \\
& \stackrel{(7)}{=} \text{Diagram 22} \\
& \stackrel{(14)}{=} \text{Diagram 23} \\
& \stackrel{(14)}{=} \text{Diagram 24} \\
& \stackrel{(2)}{=} \text{Diagram 25} \\
& \stackrel{(3)}{=} \text{Diagram 26} \\
& \stackrel{(2)}{=} \text{Diagram 27} \\
& \stackrel{(14)}{=} \text{Diagram 28} \\
& \stackrel{(3)}{=} \text{Diagram 29} \\
& \stackrel{(2)}{=} \text{Diagram 30} \\
& \stackrel{(3)}{=} \text{Diagram 31} \\
& \stackrel{(2)}{=} \text{Diagram 32} \\
& \stackrel{(12)}{=} \text{Diagram 33}
\end{aligned}$$

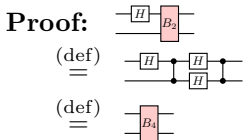
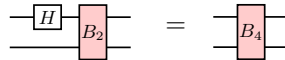




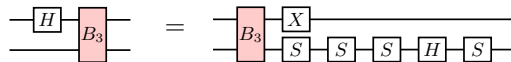
### 3.24 Equation

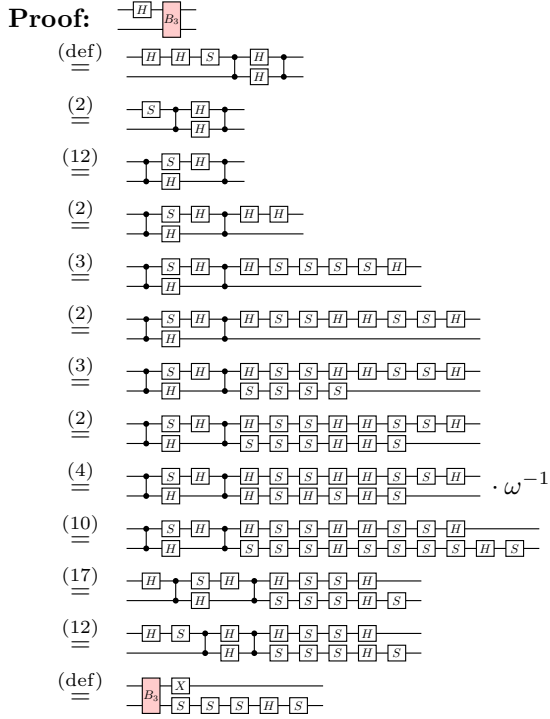


### 3.25 Equation

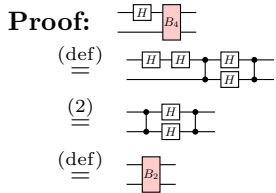
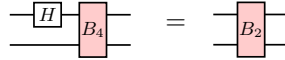


### 3.26 Equation

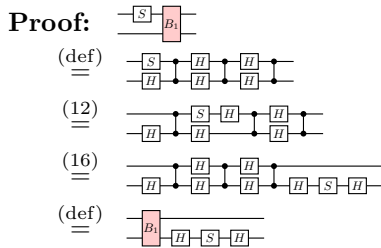
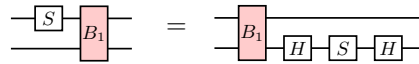




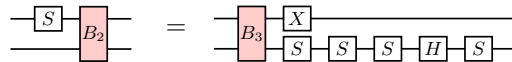
### 3.27 Equation

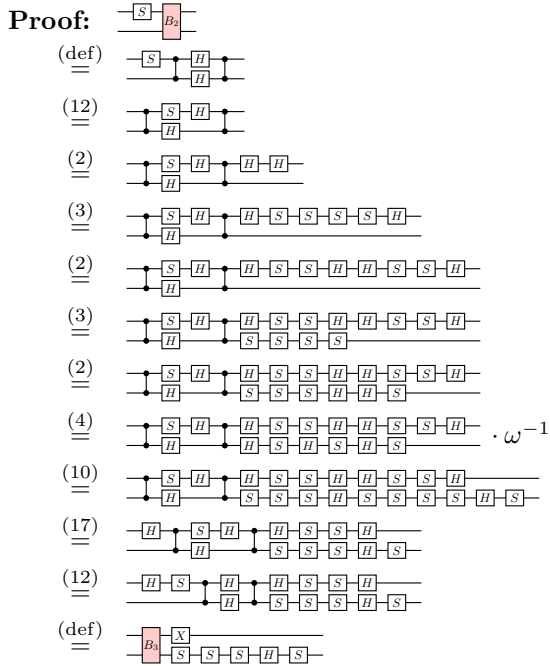


### 3.28 Equation



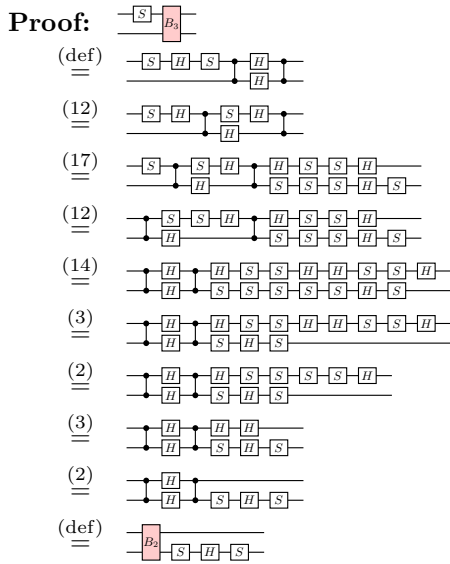
### 3.29 Equation





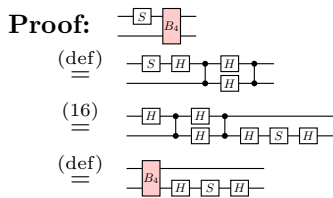
### 3.30 Equation

$$\begin{array}{c} \boxed{S} \\ \hline \boxed{B_3} \end{array} = \begin{array}{c} \boxed{B_2} \\ \hline \boxed{S} \boxed{H} \boxed{S} \end{array}$$

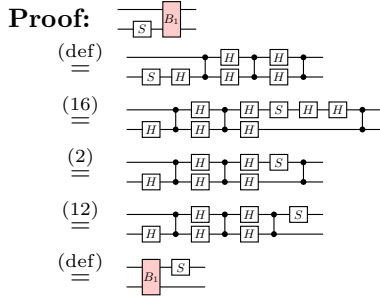


### 3.31 Equation

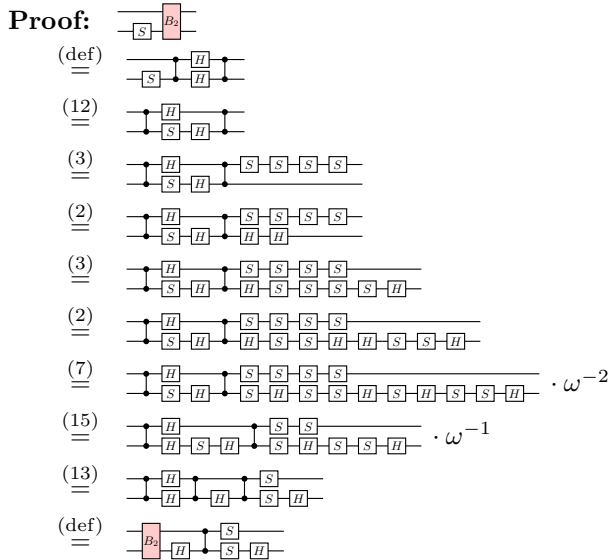
$$\begin{array}{c} \boxed{S} \\ \hline \boxed{B_4} \end{array} = \begin{array}{c} \boxed{B_4} \\ \hline \boxed{H} \boxed{S} \boxed{H} \end{array}$$



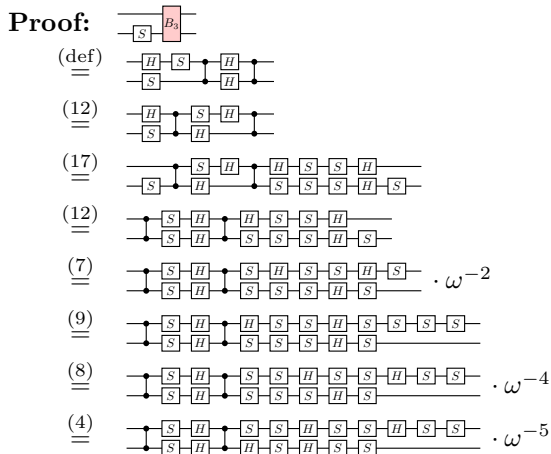
### 3.32 Equation



### 3.33 Equation



### 3.34 Equation



(10) · ω<sup>-4</sup>

(4) · ω<sup>-5</sup>

(5) · ω<sup>-6</sup>

(7) · ω<sup>-8</sup>

(15) · ω<sup>-7</sup>

(8) · ω<sup>-11</sup>

(6) · ω<sup>-10</sup>

(13) · ω<sup>-9</sup>

(9) · ω<sup>-7</sup>

(2) · ω<sup>-7</sup>

(14) · ω<sup>-7</sup>

(2) · ω<sup>-7</sup>

(14) · ω<sup>-7</sup>

(3) · ω<sup>-7</sup>

(12) · ω<sup>-7</sup>

(12) · ω<sup>-7</sup>

(12) · ω<sup>-7</sup>

(5) · ω<sup>-8</sup>

(17) · ω<sup>-8</sup>

(12) · ω<sup>-8</sup>

(1) · ω<sup>-8</sup>

(def) · ω<sup>-8</sup>

### 3.35 Equation

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \boxed{B_4} \\ | \\ \text{---} \\ | \\ \boxed{S} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \boxed{B_4} \\ | \\ \text{---} \\ | \\ \boxed{H} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \boxed{S} \\ | \\ \text{---} \\ | \\ \boxed{H} \end{array}$$

**Proof:** · ω<sup>-8</sup>

(def) · ω<sup>-8</sup>

(12) · ω<sup>-8</sup>

(3) · ω<sup>-8</sup>

(2) · ω<sup>-8</sup>

(3) · ω<sup>-8</sup>

(2) · ω<sup>-8</sup>

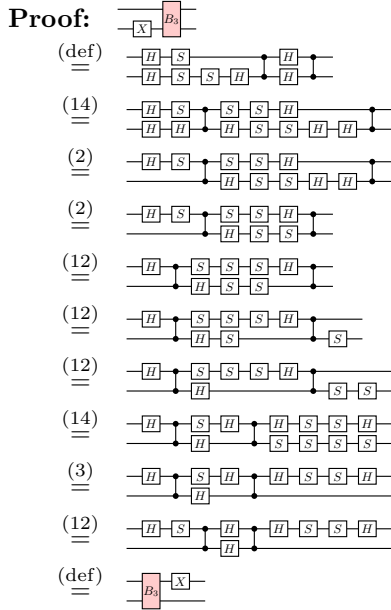
(7) · ω<sup>-2</sup>

(15) · ω<sup>-1</sup>

(13) · ω<sup>-1</sup>

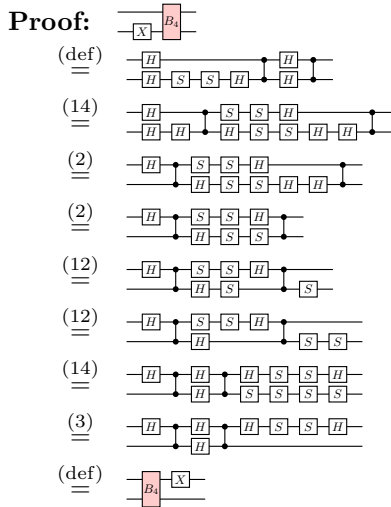






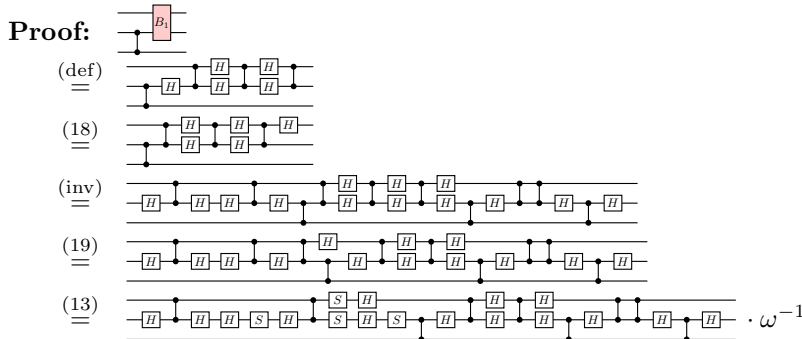
### 3.39 Equation

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{B_4} \\ \boxed{X} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{B_4} \\ \boxed{X} \end{array}$$

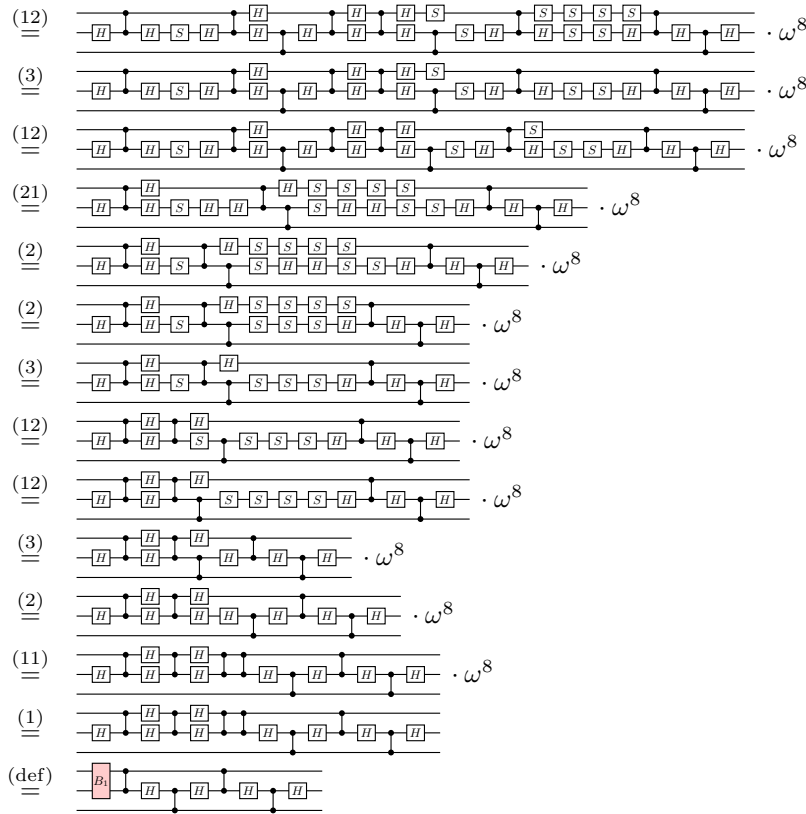


### 3.40 Equation

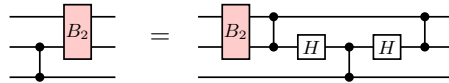
$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{B_1} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{B_1} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{H} \\ \boxed{H} \\ \boxed{H} \end{array}$$



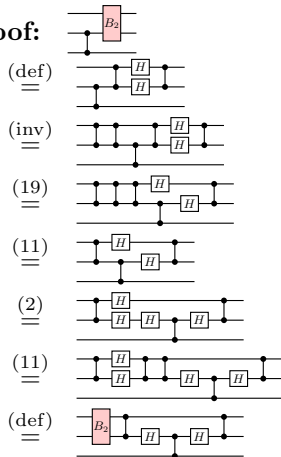
$$\begin{aligned}
& \underline{\underline{(15)}} \quad \text{Diagram 1} \cdot \omega^{-2} \\
& \underline{\underline{(7)}} \quad \text{Diagram 2} \\
& \underline{\underline{(2)}} \quad \text{Diagram 3} \\
& \underline{\underline{(12)}} \quad \text{Diagram 4} \\
& \underline{\underline{(12)}} \quad \text{Diagram 5} \\
& \underline{\underline{(12)}} \quad \text{Diagram 6} \\
& \underline{\underline{(14)}} \quad \text{Diagram 7} \\
& \underline{\underline{(2)}} \quad \text{Diagram 8} \\
& \underline{\underline{(14)}} \quad \text{Diagram 9} \\
& \underline{\underline{(8)}} \quad \text{Diagram 10} \cdot \omega^4 \\
& \underline{\underline{(2)}} \quad \text{Diagram 11} \cdot \omega^4 \\
& \underline{\underline{(12)}} \quad \text{Diagram 12} \cdot \omega^4 \\
& \underline{\underline{(12)}} \quad \text{Diagram 13} \cdot \omega^4 \\
& \underline{\underline{(14)}} \quad \text{Diagram 14} \cdot \omega^4 \\
& \underline{\underline{(8)}} \quad \text{Diagram 15} \cdot \omega^8 \\
& \underline{\underline{(2)}} \quad \text{Diagram 16} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 17} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 18} \cdot \omega^8 \\
& \underline{\underline{(3)}} \quad \text{Diagram 19} \cdot \omega^8 \\
& \underline{\underline{(14)}} \quad \text{Diagram 20} \cdot \omega^8 \\
& \underline{\underline{(3)}} \quad \text{Diagram 21} \cdot \omega^8 \\
& \underline{\underline{(2)}} \quad \text{Diagram 22} \cdot \omega^8 \\
& \underline{\underline{(16)}} \quad \text{Diagram 23} \cdot \omega^8 \\
& \underline{\underline{(2)}} \quad \text{Diagram 24} \cdot \omega^8 \\
& \underline{\underline{(16)}} \quad \text{Diagram 25} \cdot \omega^8 \\
& \underline{\underline{(2)}} \quad \text{Diagram 26} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 27} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 28} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 29} \cdot \omega^8 \\
& \underline{\underline{(14)}} \quad \text{Diagram 30} \cdot \omega^8 \\
& \underline{\underline{(12)}} \quad \text{Diagram 31} \cdot \omega^8
\end{aligned}$$



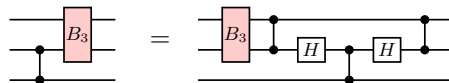
### 3.41 Equation



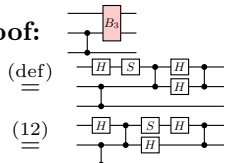
**Proof:**

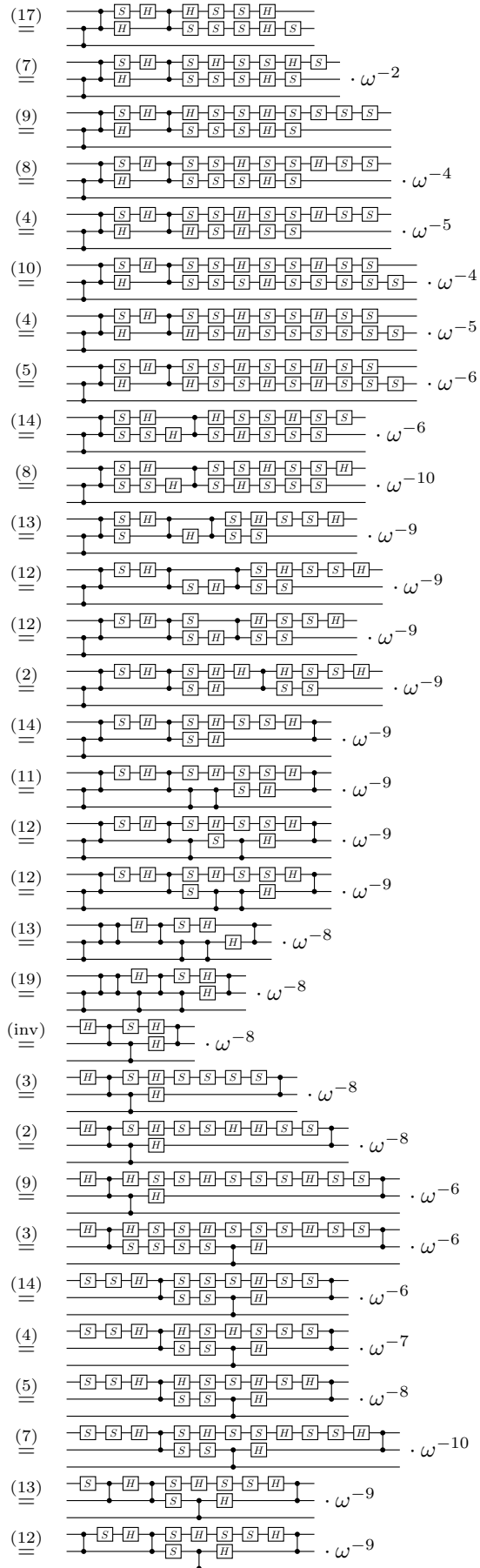


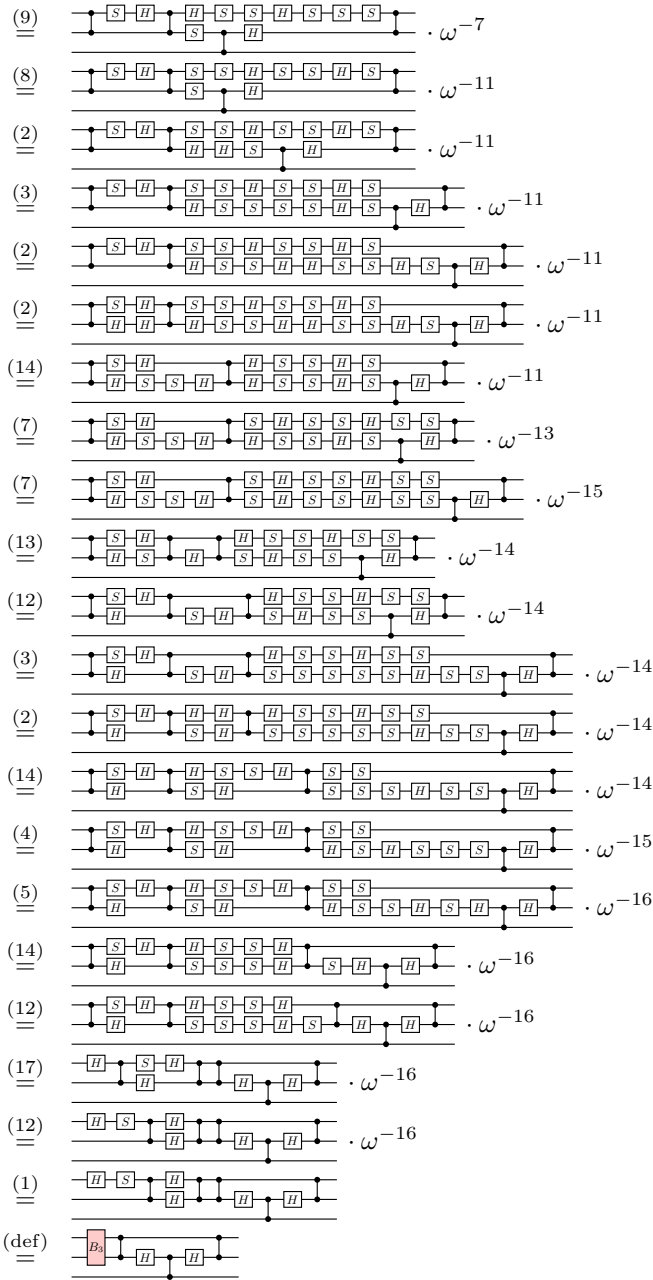
### 3.42 Equation



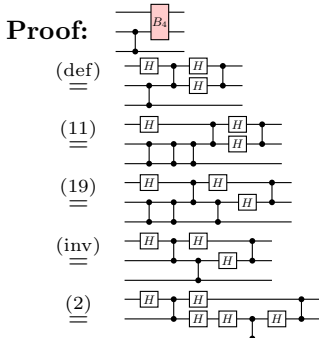
**Proof:**







### 3.43 Equation



$$\begin{aligned} & \underline{\underline{(11)}} \quad \begin{array}{c} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \boxed{H} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \\ & \underline{\underline{(\text{def})}} \quad \begin{array}{c} \text{---} \boxed{B} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \boxed{H} \text{---} \text{---} \text{---} \end{array} \end{aligned}$$

### 3.44 Equation

$$\text{---} \boxed{X} \boxed{C_1} \text{---} = \text{---} \boxed{C_2} \text{---}$$

**Proof:**

$$\begin{aligned} & \text{---} \boxed{X} \boxed{C_1} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{C_2} \text{---} \end{aligned}$$

### 3.45 Equation

$$\text{---} \boxed{X} \boxed{C_2} \text{---} = \text{---} \boxed{C_1} \text{---}$$

**Proof:**

$$\begin{aligned} & \text{---} \boxed{X} \boxed{C_2} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \text{---} \\ & \underline{\underline{(2)}} \quad \text{---} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \boxed{S} \boxed{H} \text{---} \\ & \underline{\underline{(3)}} \quad \text{---} \boxed{H} \boxed{H} \text{---} \\ & \underline{\underline{(2)}} \quad \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{C_1} \text{---} \end{aligned}$$

### 3.46 Equation

$$\text{---} \boxed{S} \boxed{C_1} \text{---} = \text{---} \boxed{C_1} \boxed{S} \text{---}$$

**Proof:**

$$\begin{aligned} & \text{---} \boxed{S} \boxed{C_1} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{S} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{C_1} \boxed{S} \text{---} \end{aligned}$$

### 3.47 Equation

$$\text{---} \boxed{S} \boxed{C_2} \text{---} = \text{---} \boxed{C_2} \boxed{S} \boxed{S} \boxed{S} \text{---} \cdot \omega^2$$

**Proof:**

$$\begin{aligned} & \text{---} \boxed{S} \boxed{C_2} \text{---} \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \text{---} \\ & \underline{\underline{(9)}} \quad \text{---} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{S} \text{---} \cdot \omega^2 \\ & \underline{\underline{(\text{def})}} \quad \text{---} \boxed{C_2} \boxed{S} \boxed{S} \boxed{S} \text{---} \cdot \omega^2 \end{aligned}$$

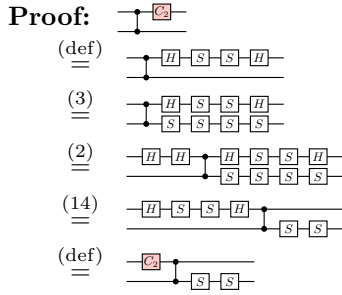
### 3.48 Equation

$$\begin{array}{c} \bullet \\ \text{---} \boxed{C_1} \text{---} \\ \bullet \end{array} = \begin{array}{c} \text{---} \boxed{C_1} \text{---} \\ \bullet \\ \bullet \end{array}$$

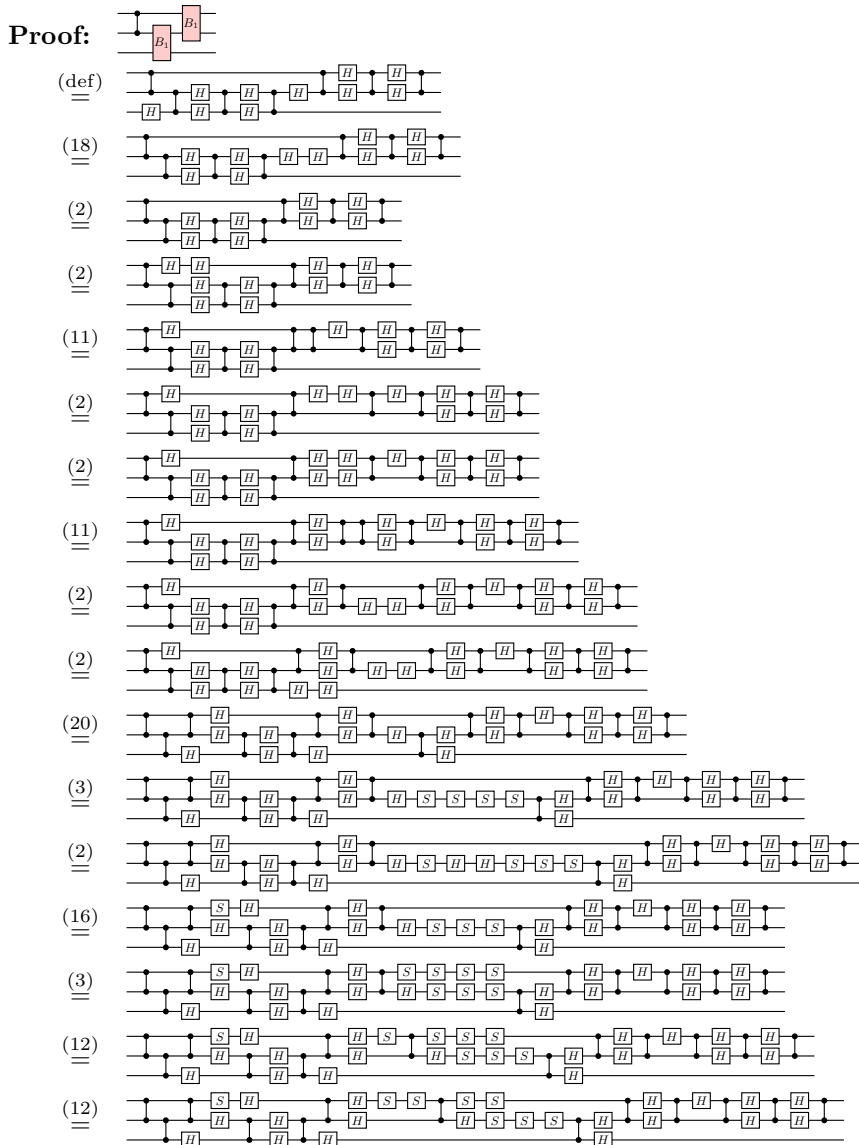
**Proof:**

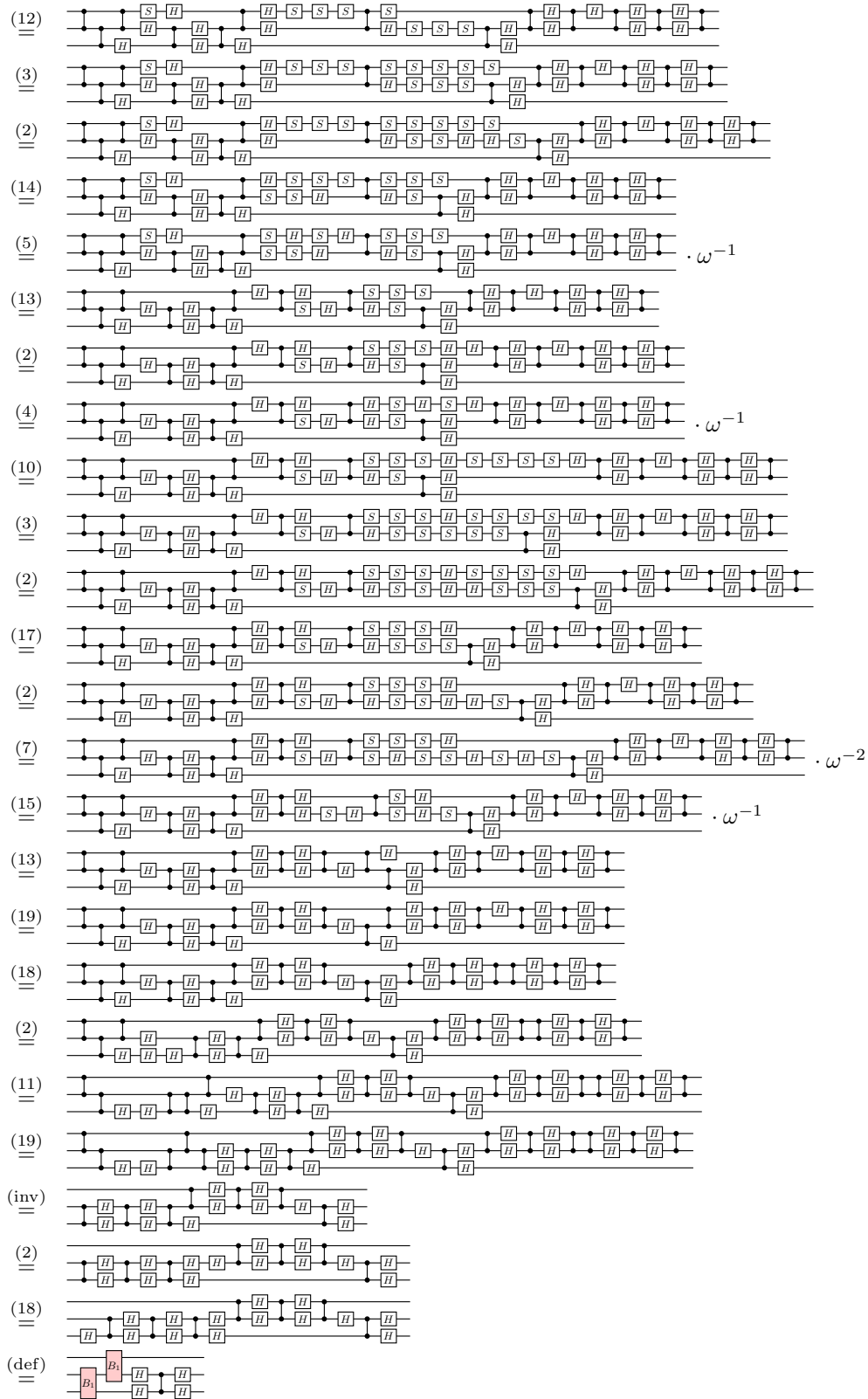
$$\begin{aligned} & \begin{array}{c} \bullet \\ \text{---} \boxed{C_1} \text{---} \\ \bullet \end{array} \\ & \underline{\underline{(\text{def})}} \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \\ & \underline{\underline{(\text{def})}} \quad \begin{array}{c} \text{---} \boxed{C_1} \text{---} \\ \bullet \\ \bullet \end{array} \end{aligned}$$

### 3.49 Equation

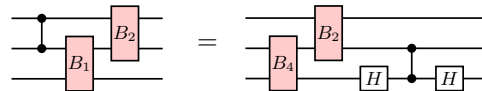


### 3.50 Equation



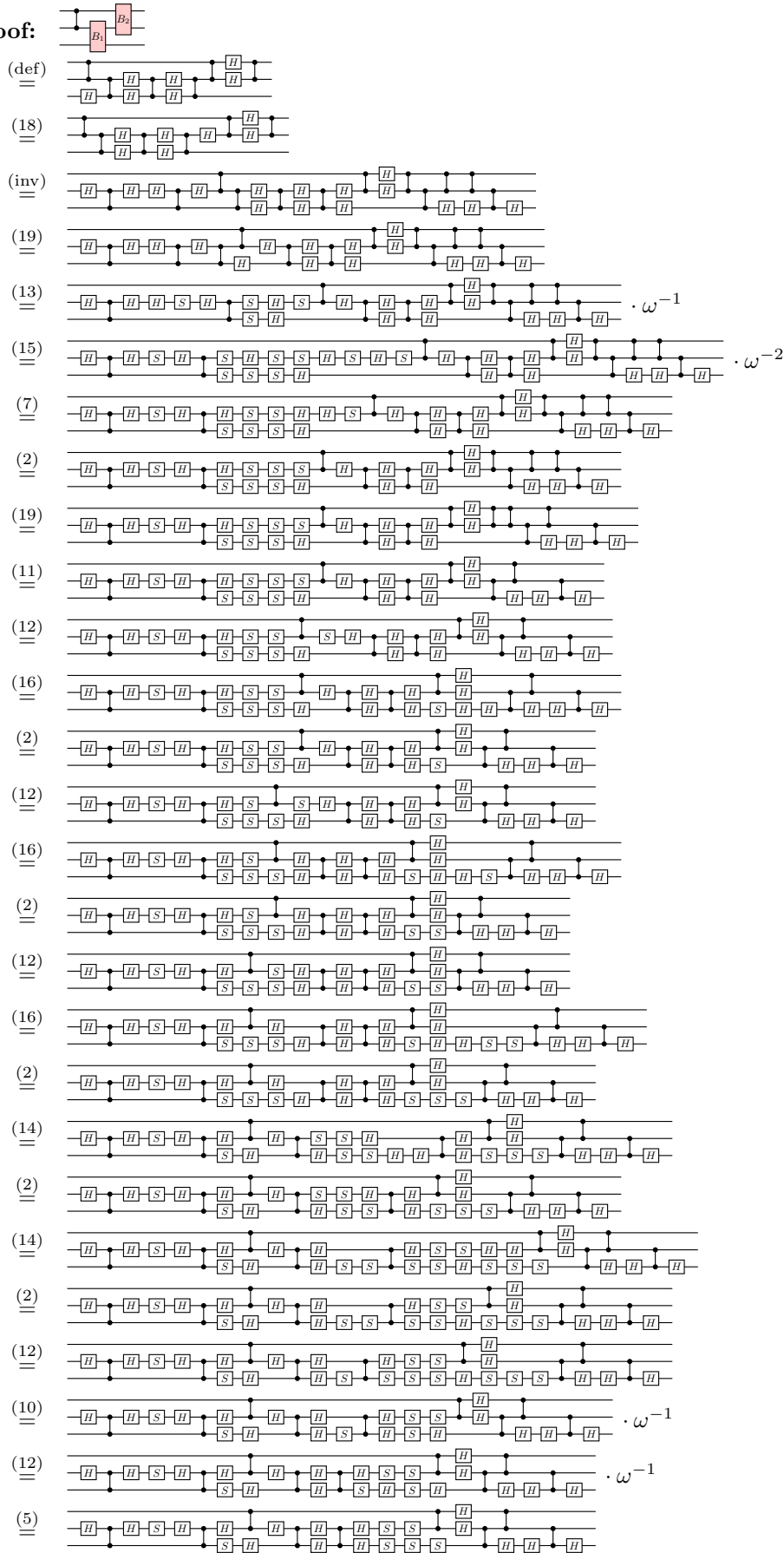


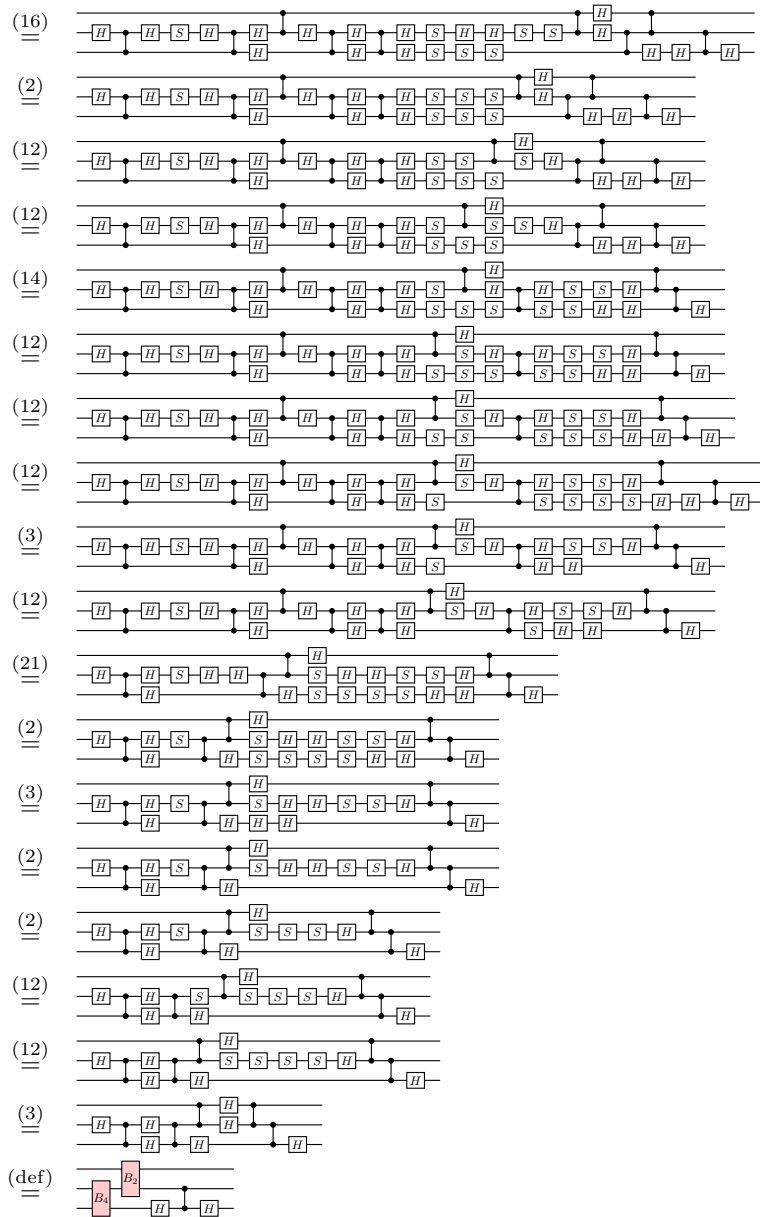
### 3.51 Equation





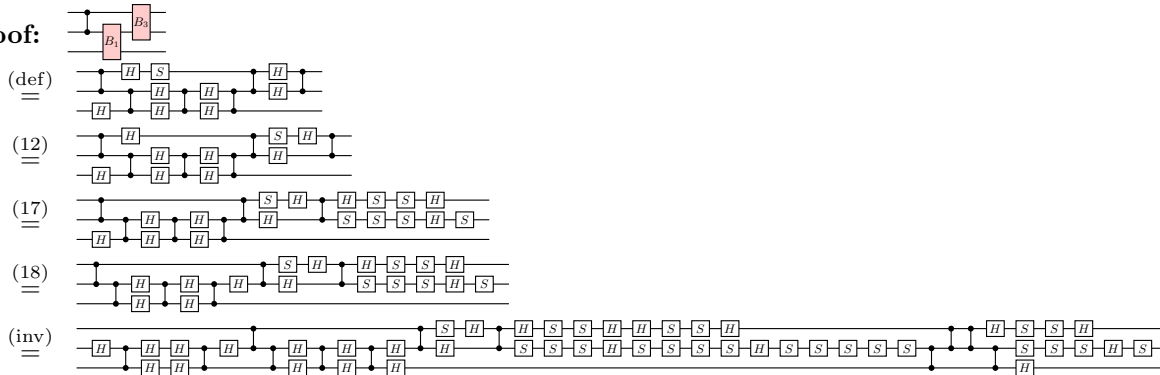
**Proof:**

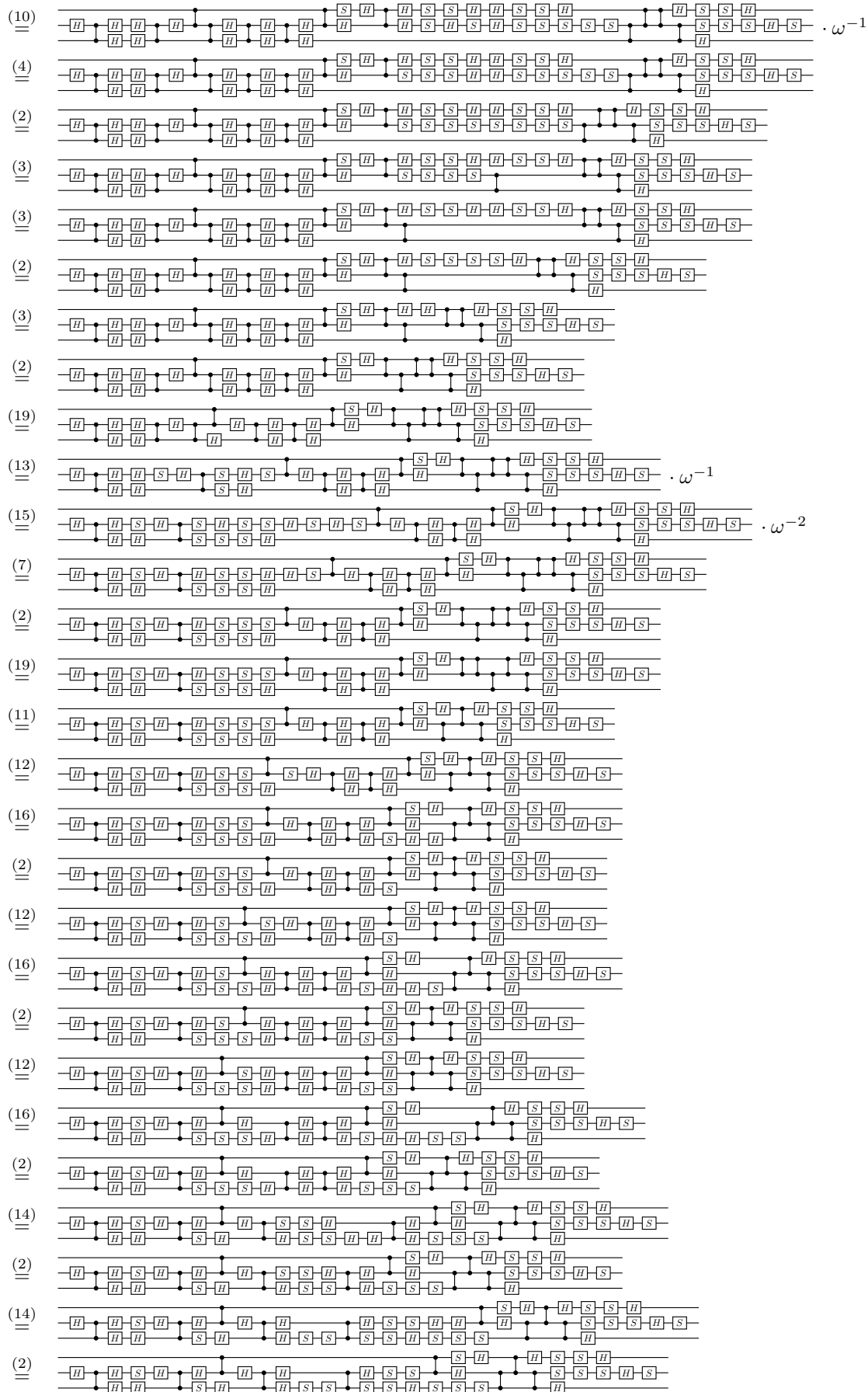


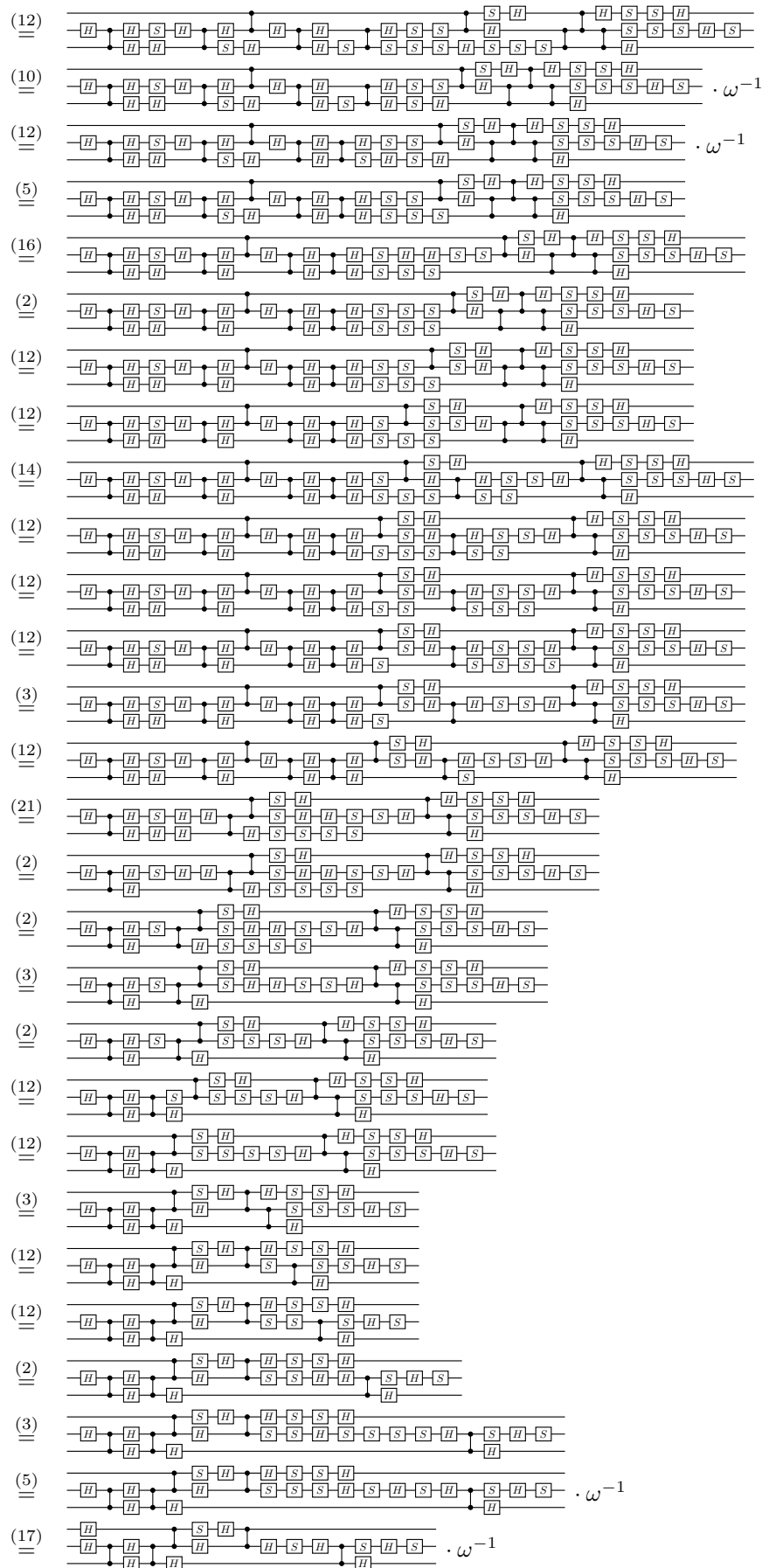


### 3.52 Equation

**Proof:**







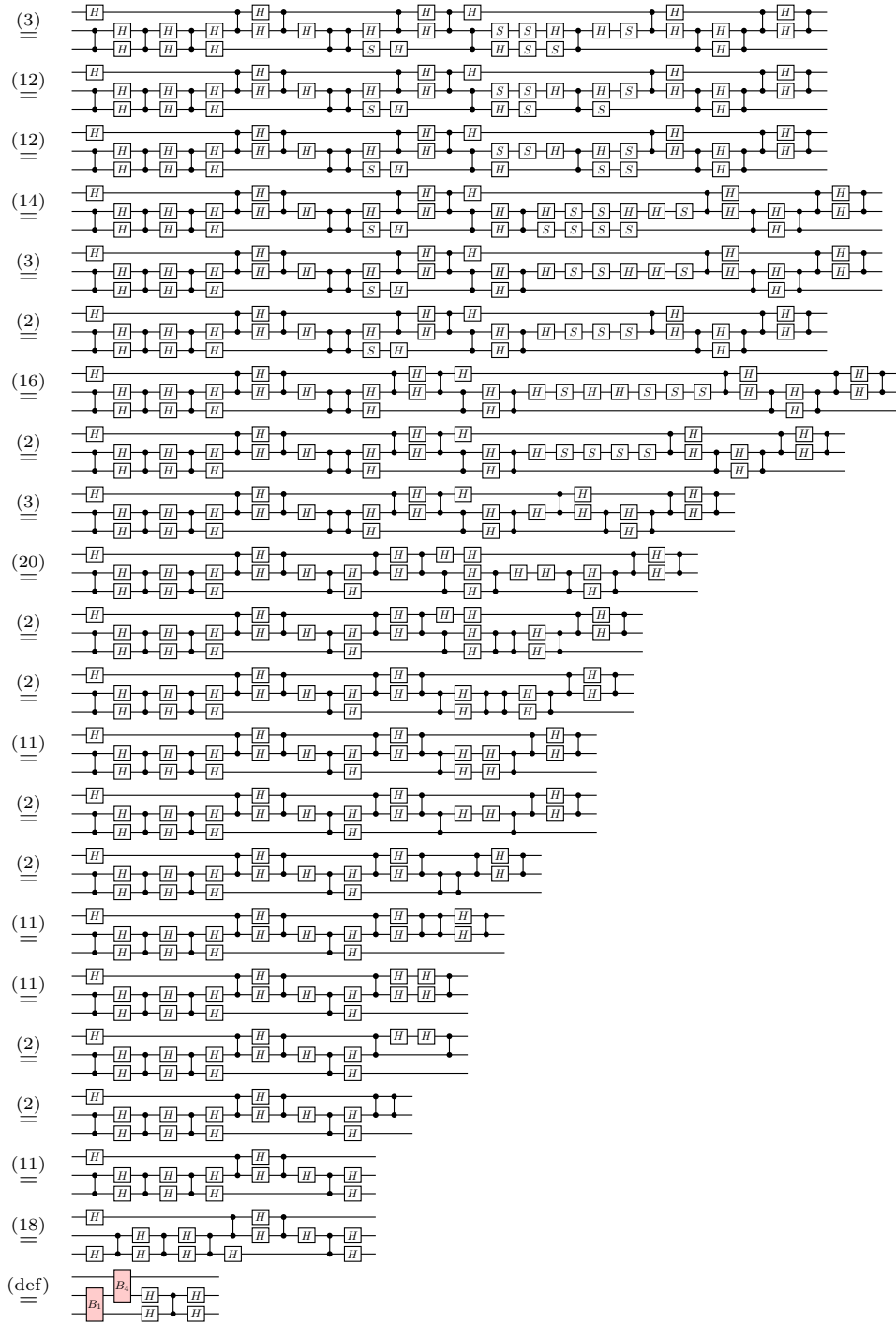
$$\begin{aligned}
 \underline{\underline{(12)}} \quad & \begin{array}{c} \text{H} \text{---} \text{S} \\ \text{H} \text{---} \text{H} \end{array} \cdot \omega^{-1} \\
 \underline{\underline{(1)}} \quad & \begin{array}{c} \text{H} \text{---} \text{S} \\ \text{H} \text{---} \text{H} \end{array} \cdot \omega^7 \\
 \underline{\underline{(\text{def})}} \quad & \begin{array}{c} \text{B}_3 \\ \text{B}_4 \end{array} \cdot \omega^7
 \end{aligned}$$

### 3.53 Equation

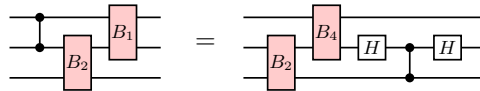
$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{B}_4 \\ \text{B}_1 \end{array} = \begin{array}{c} \text{B}_4 \\ \text{B}_1 \end{array} \begin{array}{c} \text{H} \\ \text{H} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}$$

**Proof:**

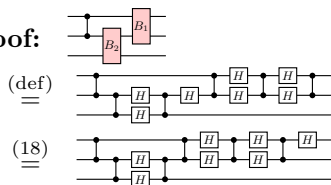
$$\begin{aligned}
 \underline{\underline{(\text{def})}} \quad & \begin{array}{c} \text{B}_4 \\ \text{B}_1 \end{array} \\
 \underline{\underline{(18)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(\text{inv})}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(18)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(18)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(18)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(2)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(19)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(13)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \cdot \omega^{-1} \\
 \underline{\underline{(15)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \cdot \omega^{-2} \\
 \underline{\underline{(7)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(2)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(17)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(10)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \cdot \omega^{-1} \\
 \underline{\underline{(4)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(2)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(2)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(3)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(13)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \cdot \omega^{-1} \\
 \underline{\underline{(5)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \\
 \underline{\underline{(12)}} \quad & \begin{array}{c} \text{H} \\ \text{H} \end{array} \cdot \omega^{-1}
 \end{aligned}$$

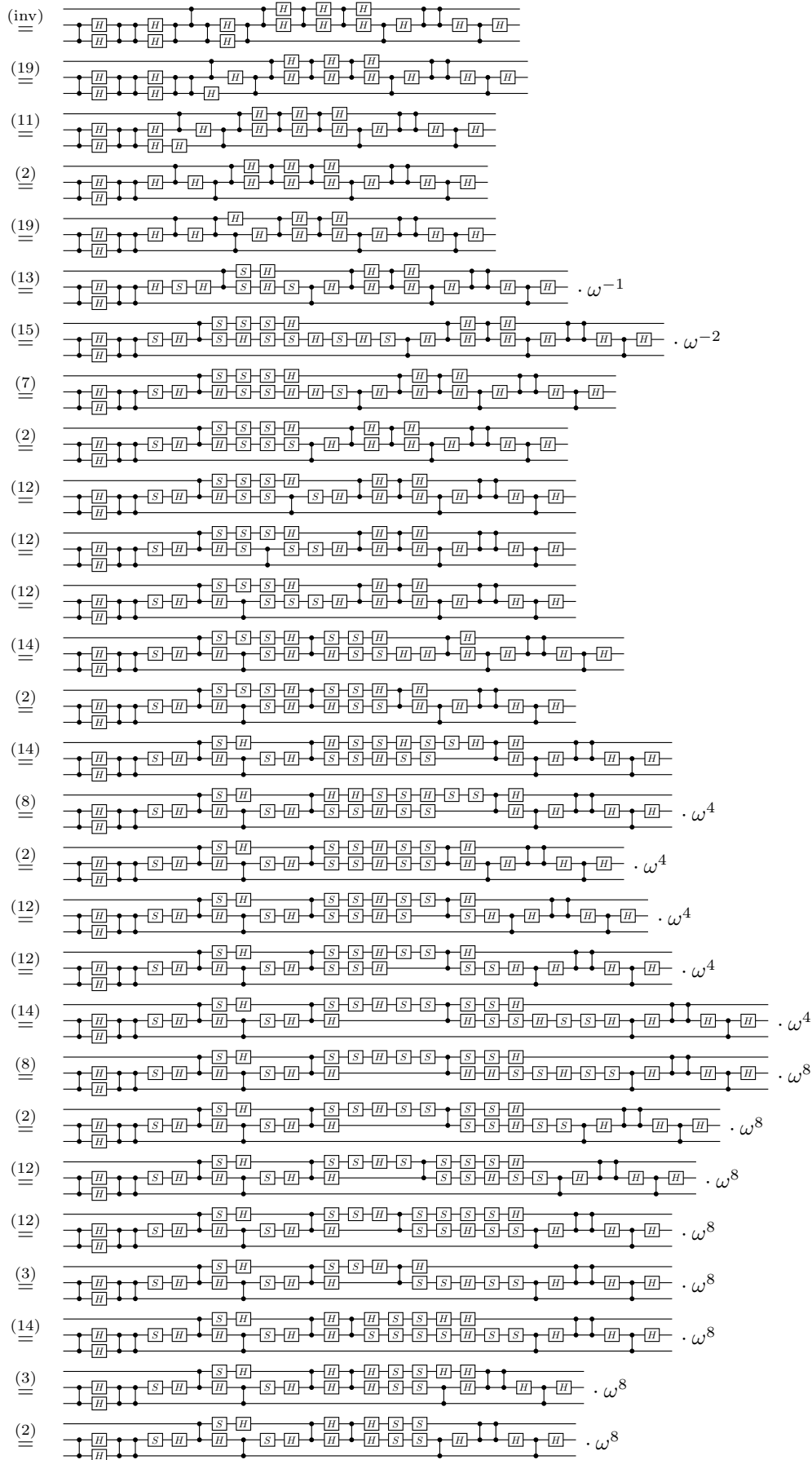


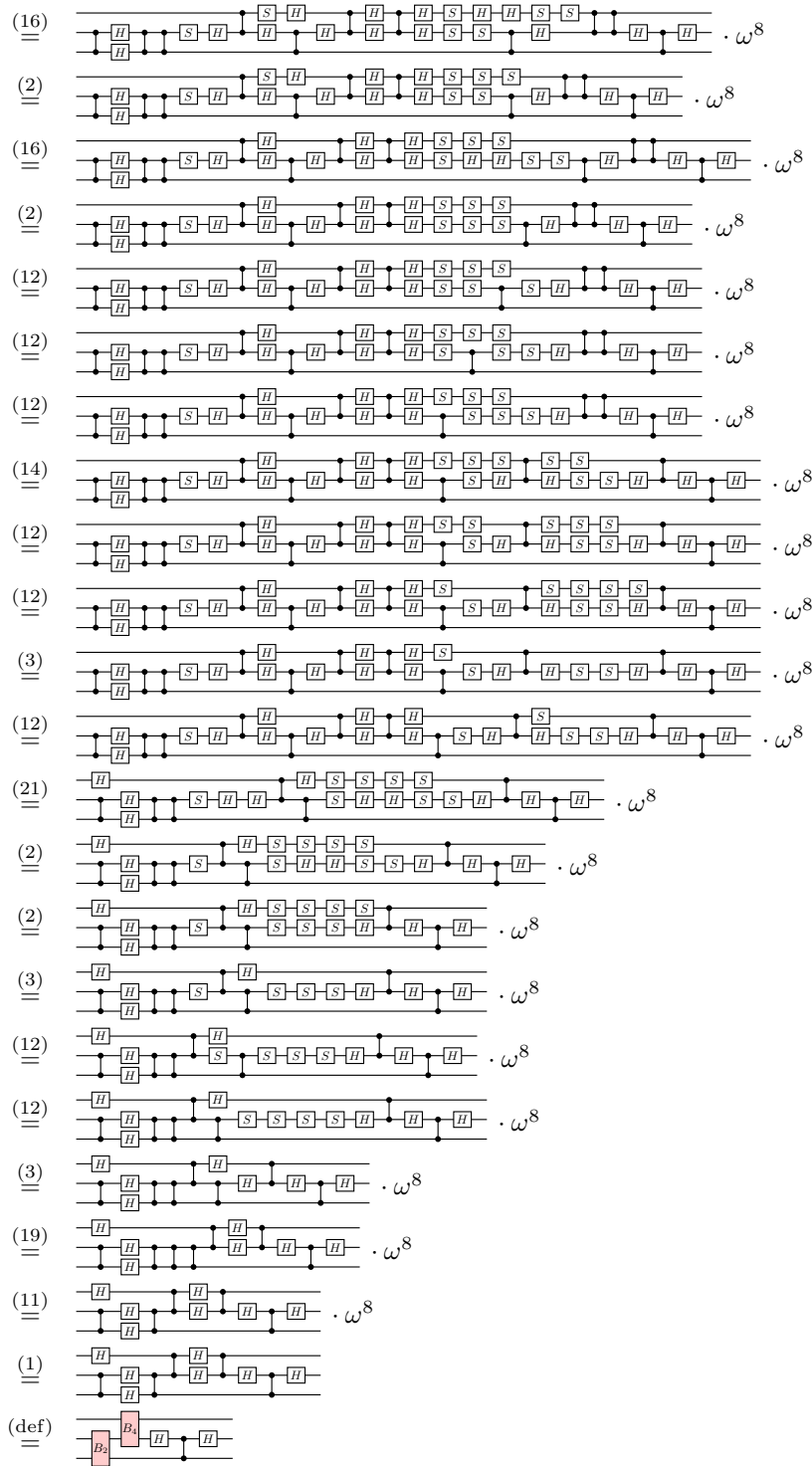
### 3.54 Equation



**Proof:**

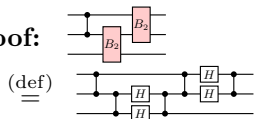




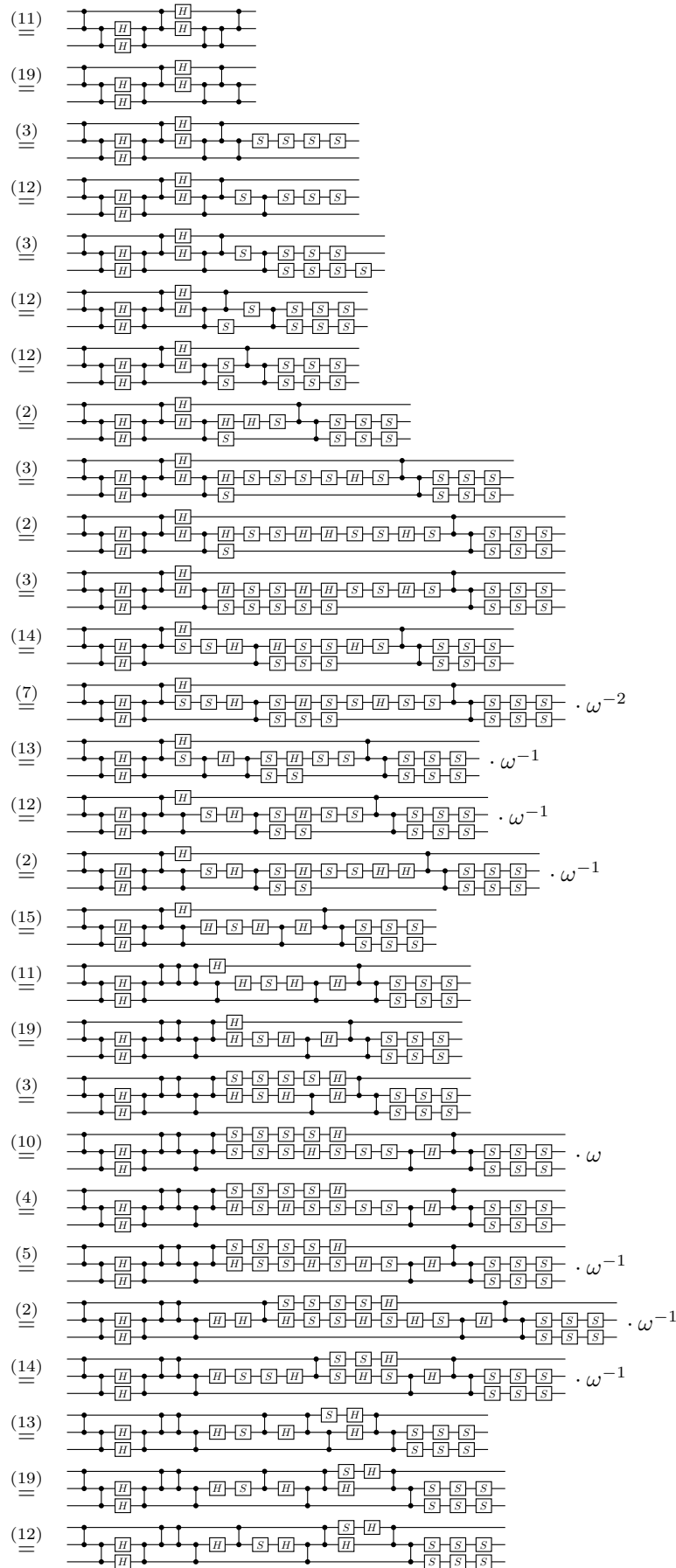


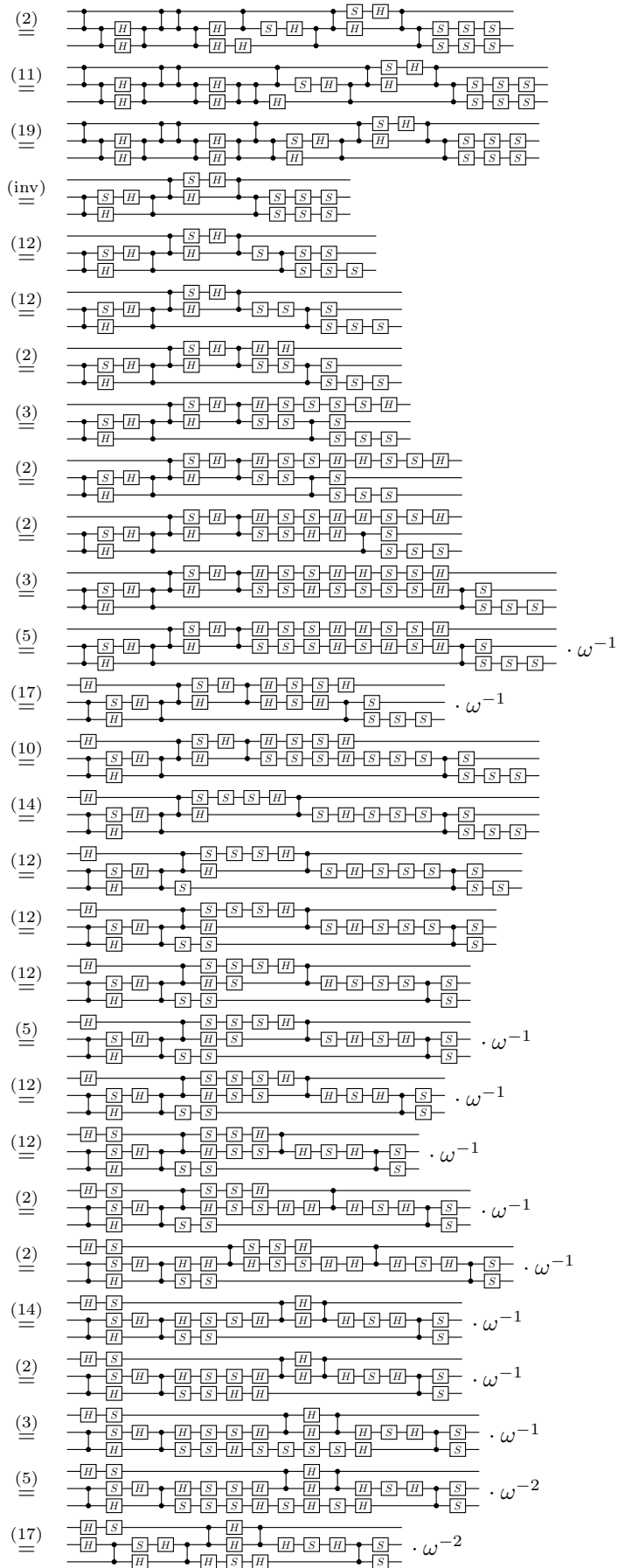
### 3.55 Equation

**Proof:**

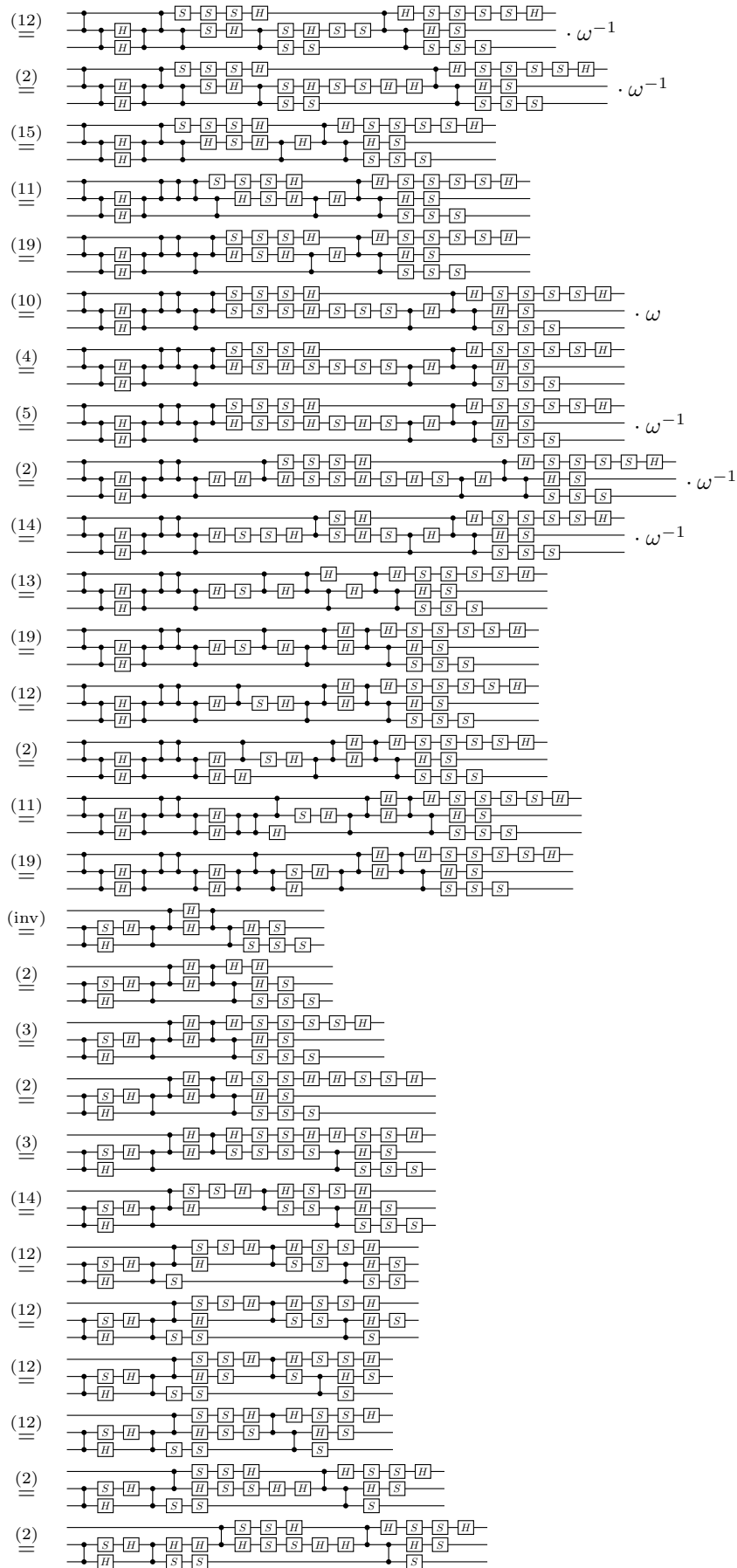












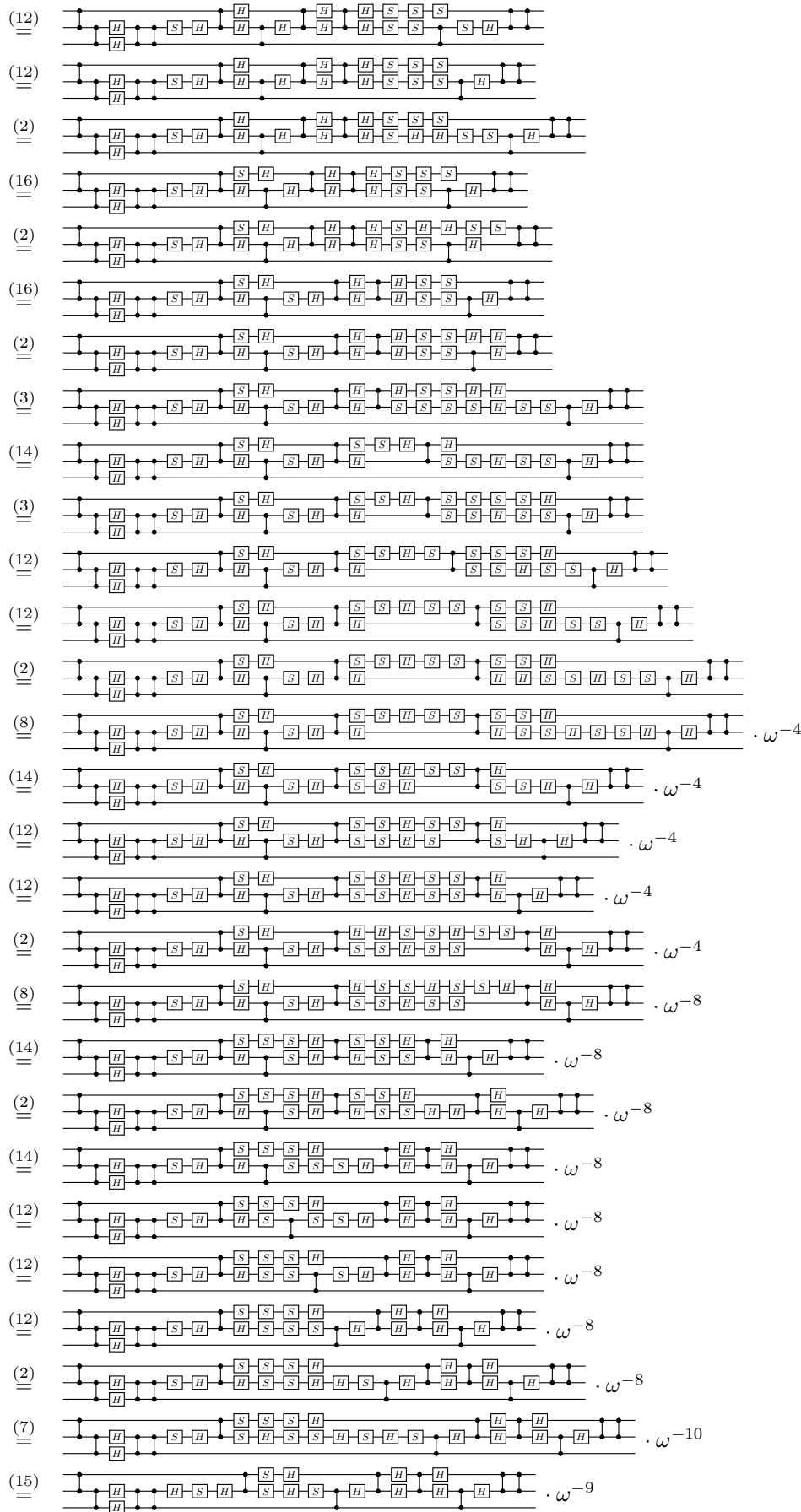
$$\begin{aligned}
& \underline{(14)} \quad \text{Diagram 1} \\
& \underline{(2)} \quad \text{Diagram 2} \\
& \underline{(3)} \quad \text{Diagram 3} \\
& \underline{(5)} \quad \text{Diagram 4} \cdot \omega^{-1} \\
& \underline{(17)} \quad \text{Diagram 5} \cdot \omega^{-1} \\
& \underline{(12)} \quad \text{Diagram 6} \cdot \omega^{-1} \\
& \underline{(1)} \quad \text{Diagram 7} \cdot \omega^7 \\
& \underline{\text{(def)}} \quad \text{Diagram 8} \cdot \omega^7
\end{aligned}$$

### 3.57 Equation

$$\text{Diagram A} = \text{Diagram B}$$

**Proof:**

$$\begin{aligned}
& \underline{\text{(def)}} \quad \text{Diagram 1} \\
& \underline{(11)} \quad \text{Diagram 2} \\
& \underline{(19)} \quad \text{Diagram 3} \\
& \underline{(3)} \quad \text{Diagram 4} \\
& \underline{(12)} \quad \text{Diagram 5} \\
& \underline{(12)} \quad \text{Diagram 6} \\
& \underline{(3)} \quad \text{Diagram 7} \\
& \underline{(2)} \quad \text{Diagram 8} \\
& \underline{(2)} \quad \text{Diagram 9} \\
& \underline{(21)} \quad \text{Diagram 10} \\
& \underline{(12)} \quad \text{Diagram 11} \\
& \underline{(3)} \quad \text{Diagram 12} \\
& \underline{(12)} \quad \text{Diagram 13} \\
& \underline{(12)} \quad \text{Diagram 14} \\
& \underline{(14)} \quad \text{Diagram 15} \\
& \underline{(12)} \quad \text{Diagram 16}
\end{aligned}$$



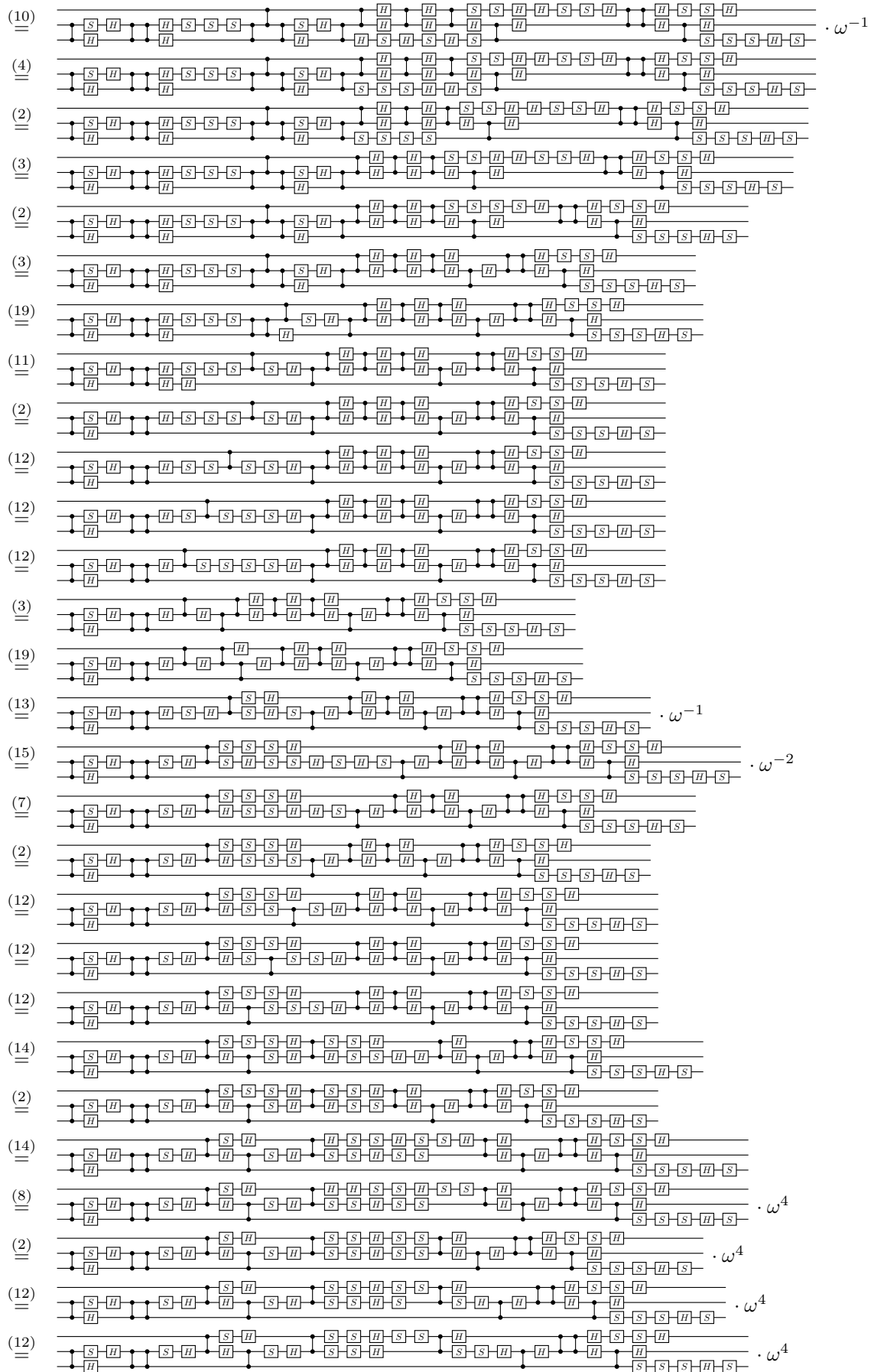
$$\begin{aligned}
(13) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(19) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(2) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(11) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(19) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(\text{inv}) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(18) \quad & \text{Diagram with 8 H gates} \cdot \omega^{-8} \\
(1) \quad & \text{Diagram with 8 H gates} \\
(\text{def}) \quad & \text{Diagram with } B_2, B_1, \text{ and } H \text{ gates}
\end{aligned}$$

### 3.58 Equation

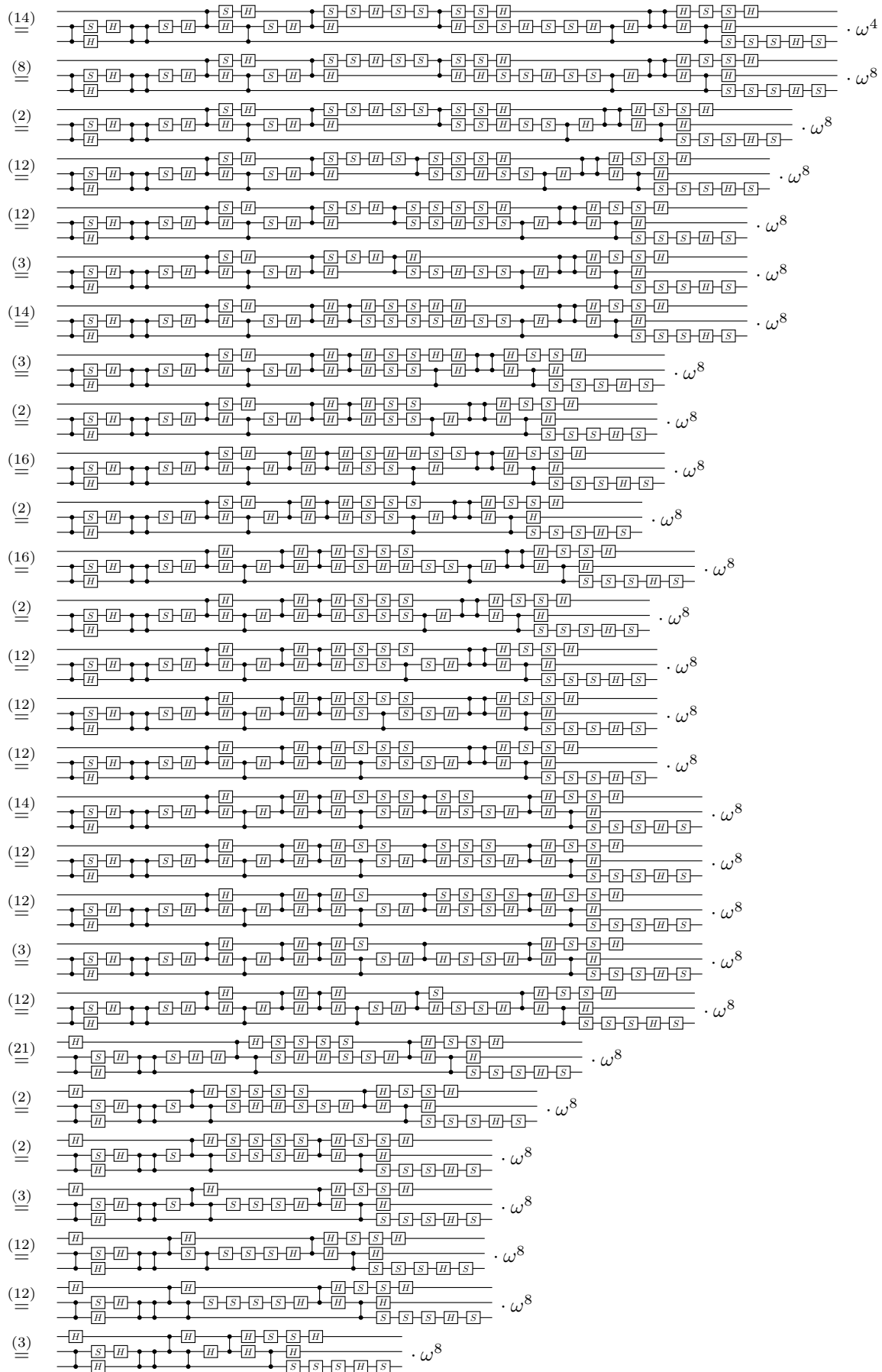
$$\text{Diagram with } B_3, B_1 \text{ gates} = \text{Diagram with } B_3, B_1, H, S \text{ gates} \cdot \omega^7$$

**Proof:**

$$\begin{aligned}
& \text{Diagram with } B_3, B_1 \text{ gates} \\
(\text{def}) \quad & \text{Diagram with } H, S \text{ gates} \\
(12) \quad & \text{Diagram with } H, S \text{ gates} \\
(17) \quad & \text{Diagram with } H, S \text{ gates} \\
(2) \quad & \text{Diagram with } H, S \text{ gates} \\
(12) \quad & \text{Diagram with } H, S \text{ gates} \\
(12) \quad & \text{Diagram with } H, S \text{ gates} \\
(14) \quad & \text{Diagram with } H, S \text{ gates} \\
(2) \quad & \text{Diagram with } H, S \text{ gates} \\
(12) \quad & \text{Diagram with } H, S \text{ gates} \\
(12) \quad & \text{Diagram with } H, S \text{ gates} \\
(14) \quad & \text{Diagram with } H, S \text{ gates} \\
(3) \quad & \text{Diagram with } H, S \text{ gates} \\
(18) \quad & \text{Diagram with } H, S \text{ gates} \\
(2) \quad & \text{Diagram with } H, S \text{ gates} \\
(\text{inv}) \quad & \text{Diagram with } H, S \text{ gates}
\end{aligned}$$





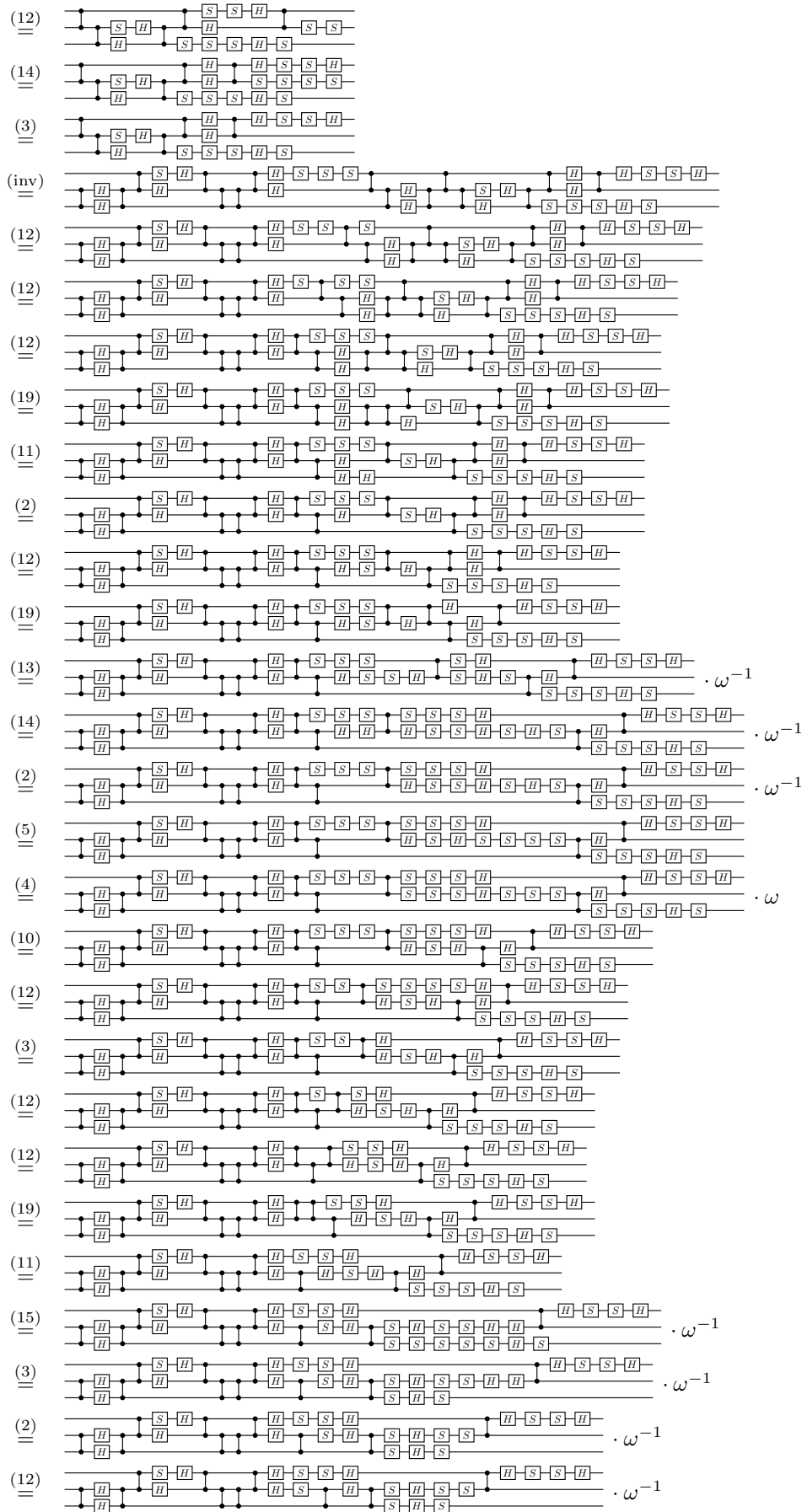


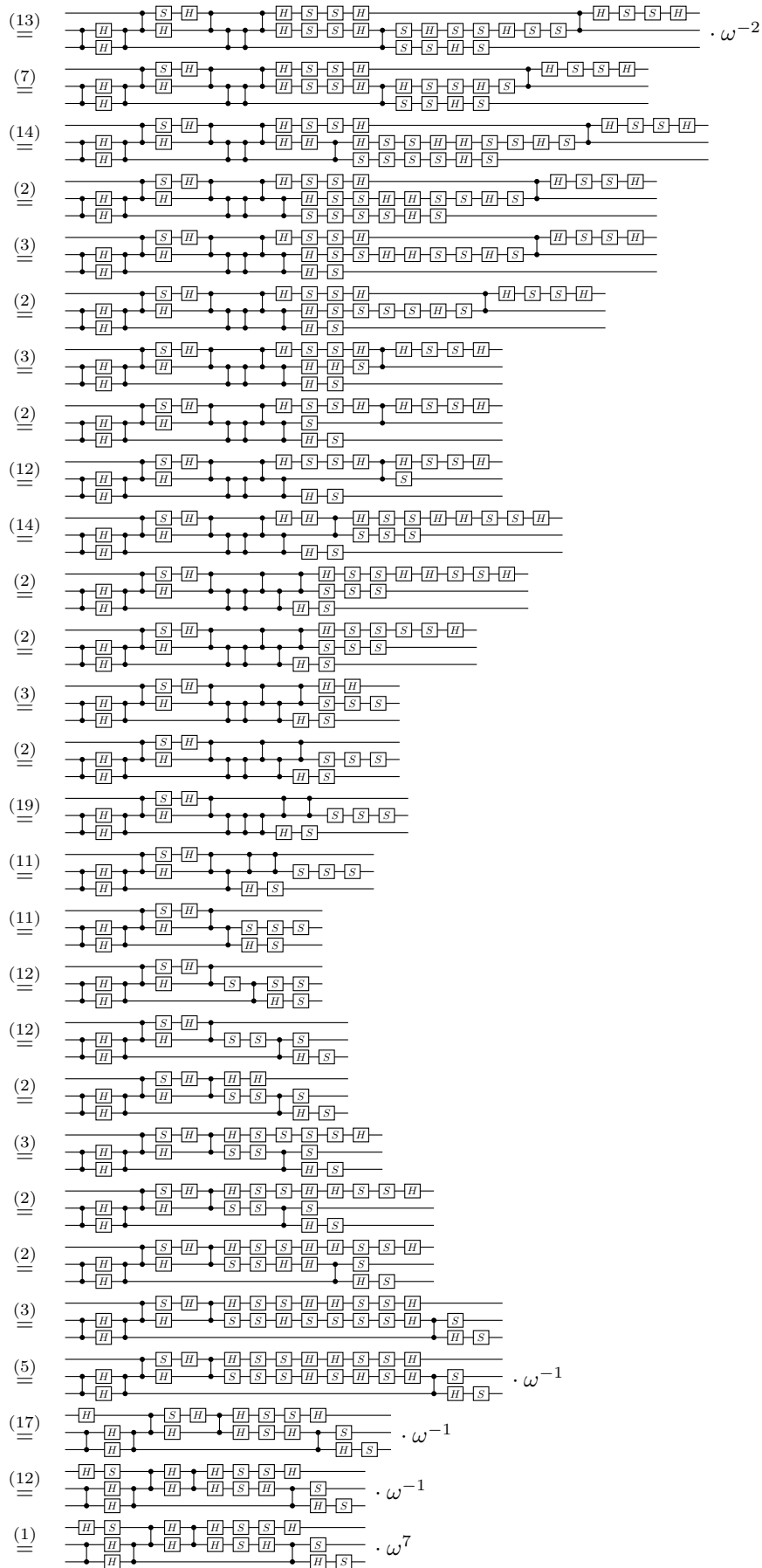
$$\begin{array}{l}
(19) \\
(11) \\
(3) \\
(14) \\
(12) \\
(12) \\
(12) \\
(12) \\
(2) \\
(2) \\
(14) \\
(2) \\
(3) \\
(5) \\
(17) \\
(12) \\
(\text{def})
\end{array}
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^8
\cdot \omega^7
\cdot \omega^7
\cdot \omega^7
\cdot \omega^7$$

### 3.59 Equation

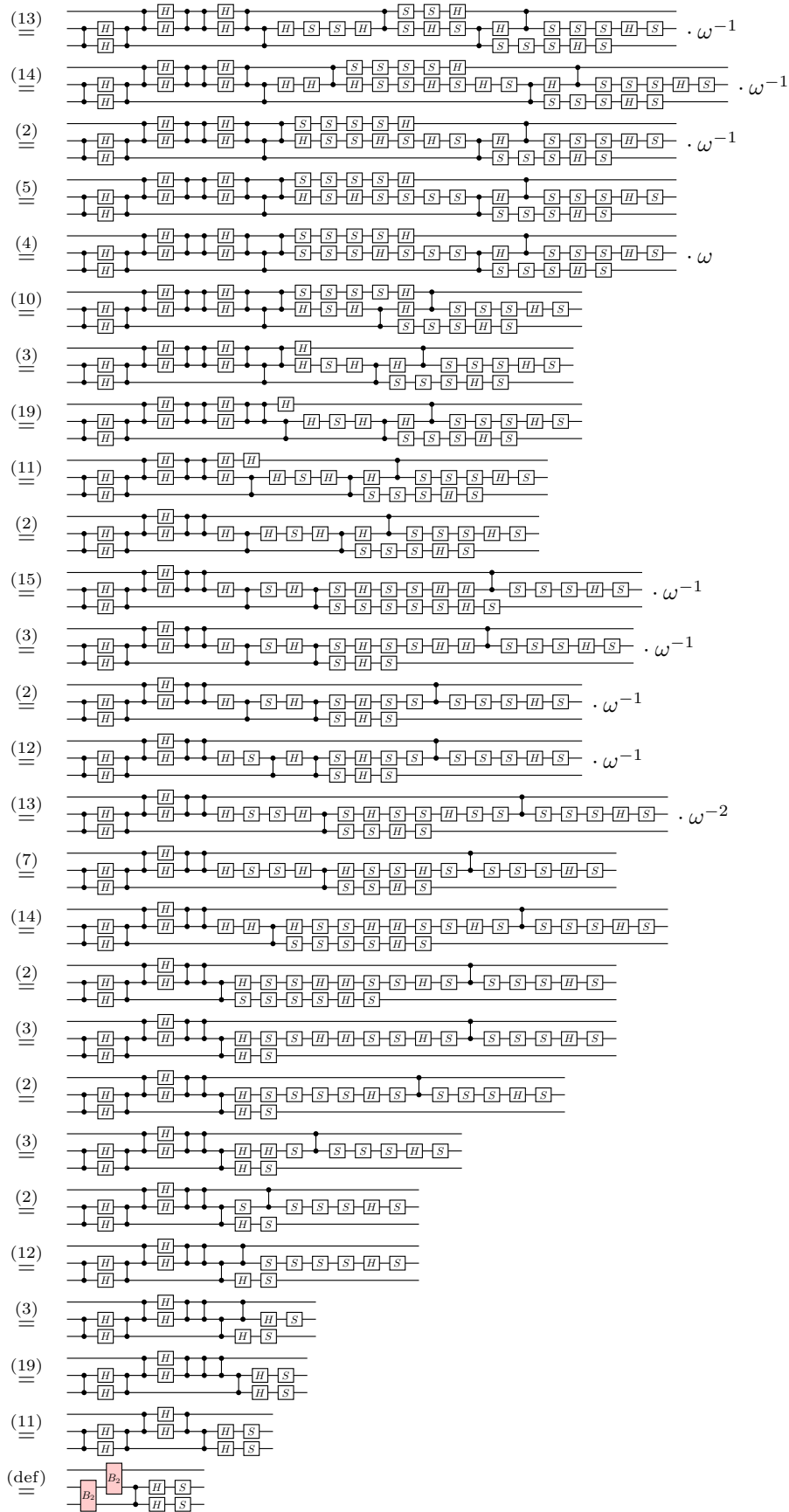
**Proof:**

$$\begin{array}{l}
(\text{def}) \\
(12) \\
(17) \\
(14) \\
(2) \\
(2) \\
(12)
\end{array}$$



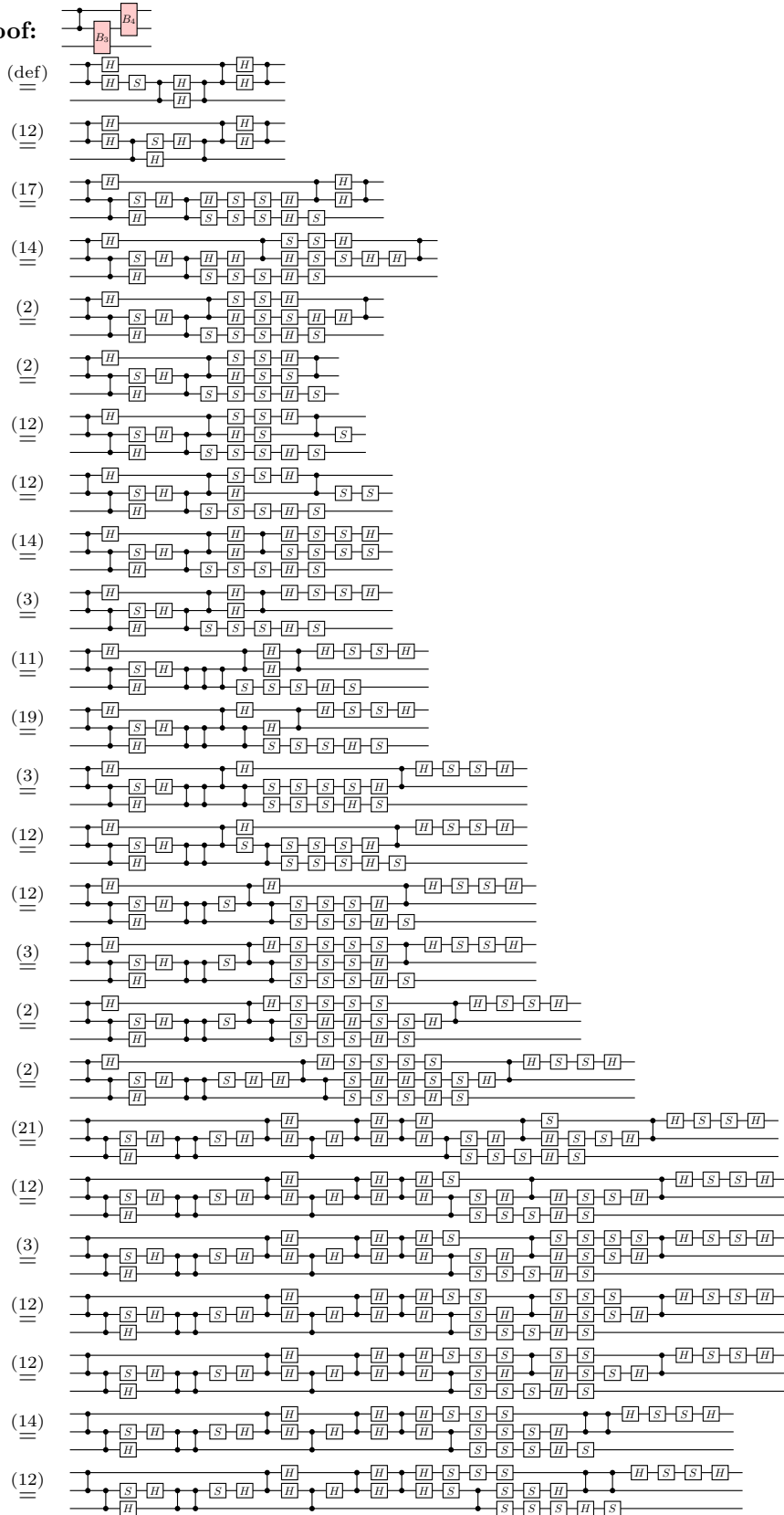


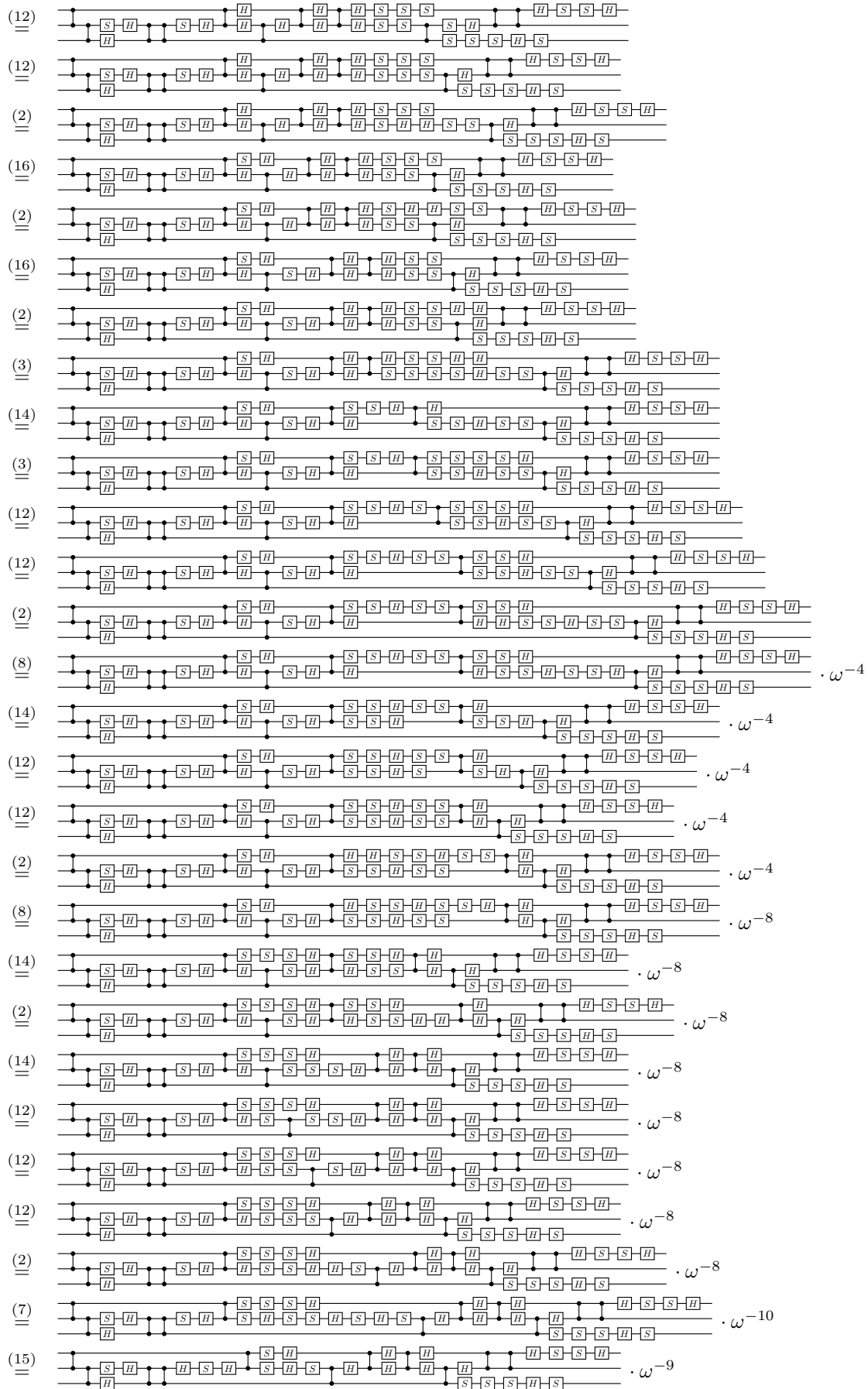




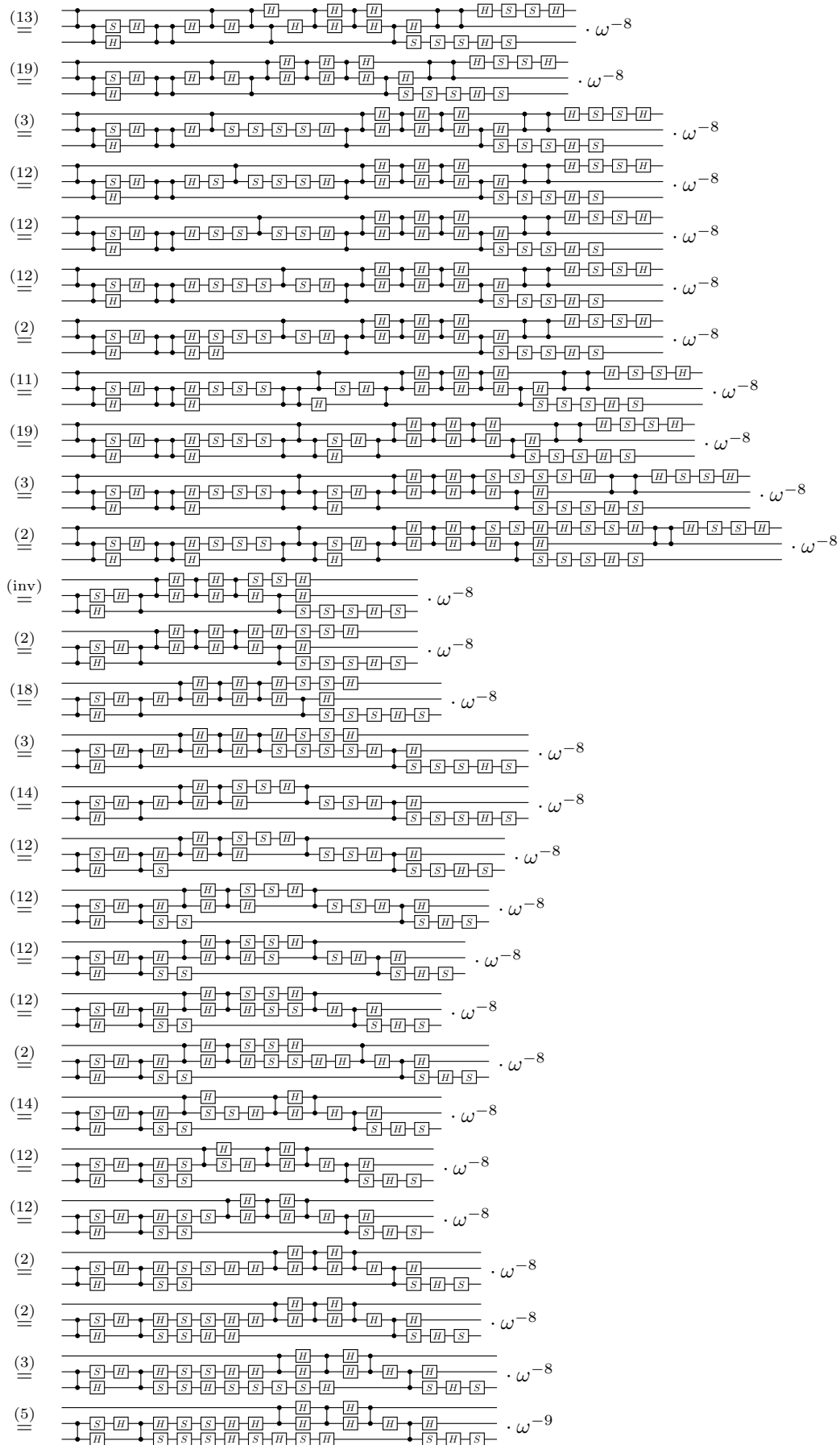
### 3.61 Equation

**Proof:**





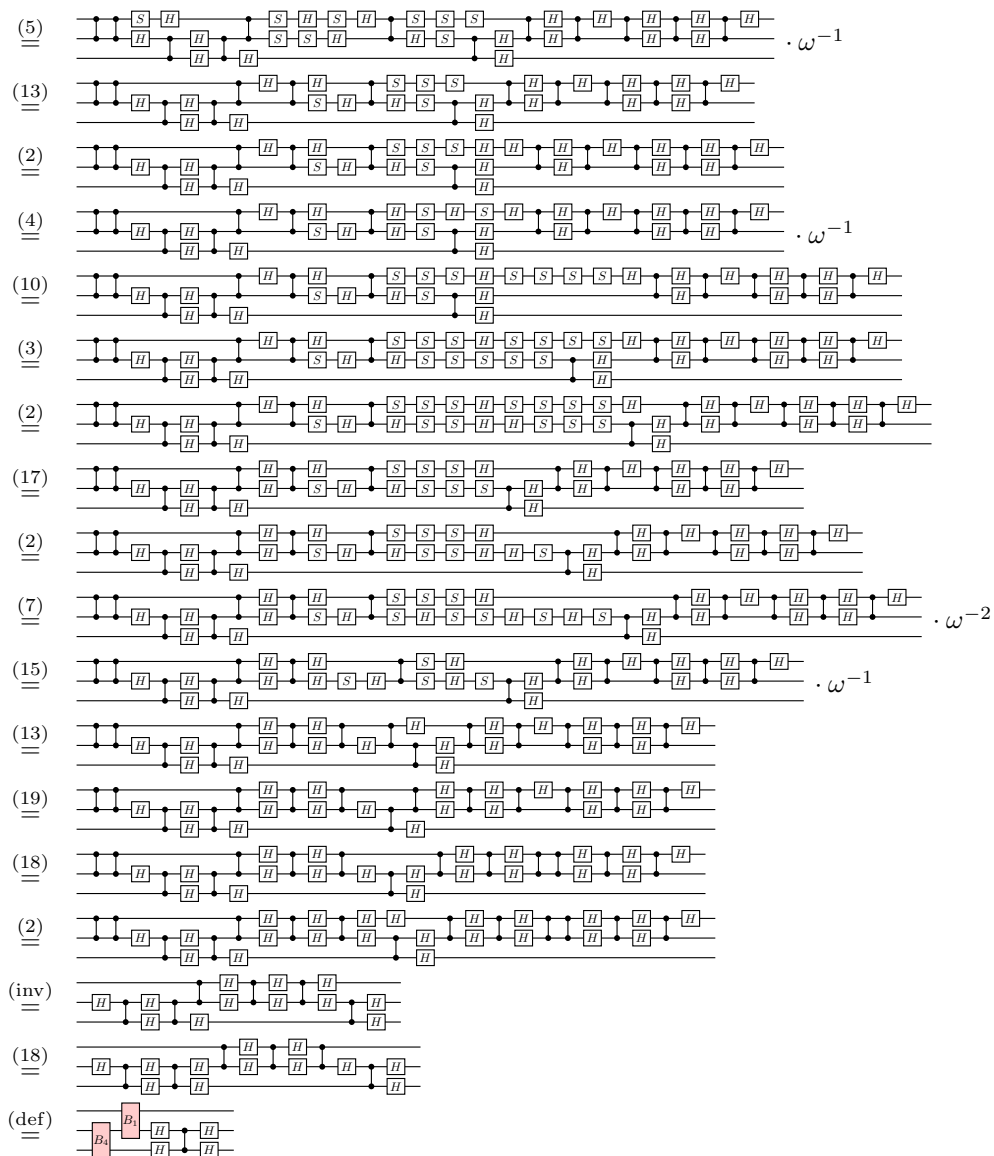




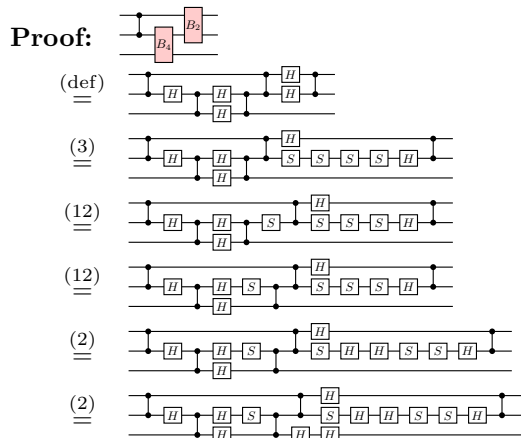
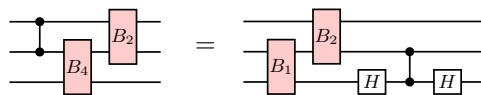
$$\begin{aligned}
(17) \quad & \text{---} \omega^{-9} \\
(12) \quad & \text{---} \omega^{-9} \\
(1) \quad & \text{---} \omega^7 \\
(\text{def}) \quad & \text{---} \omega^7
\end{aligned}$$

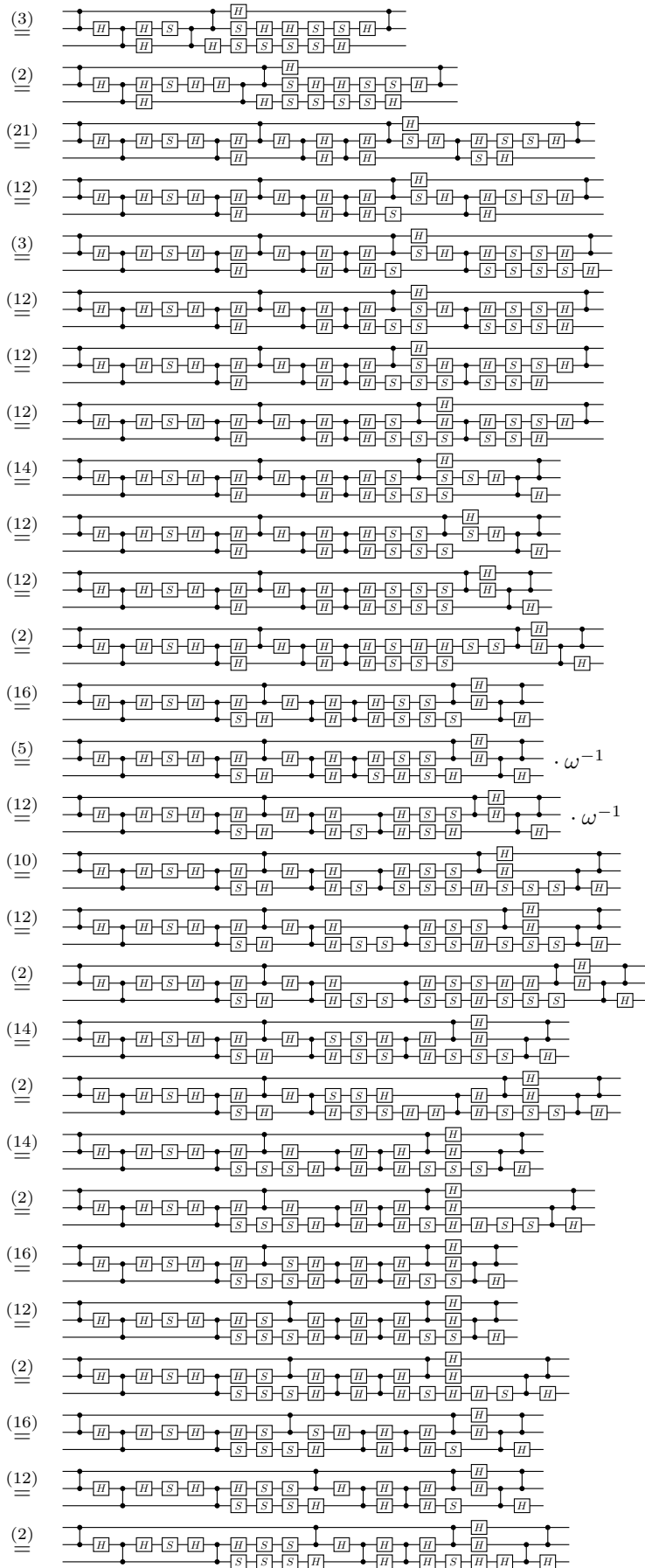
### 3.62 Equation

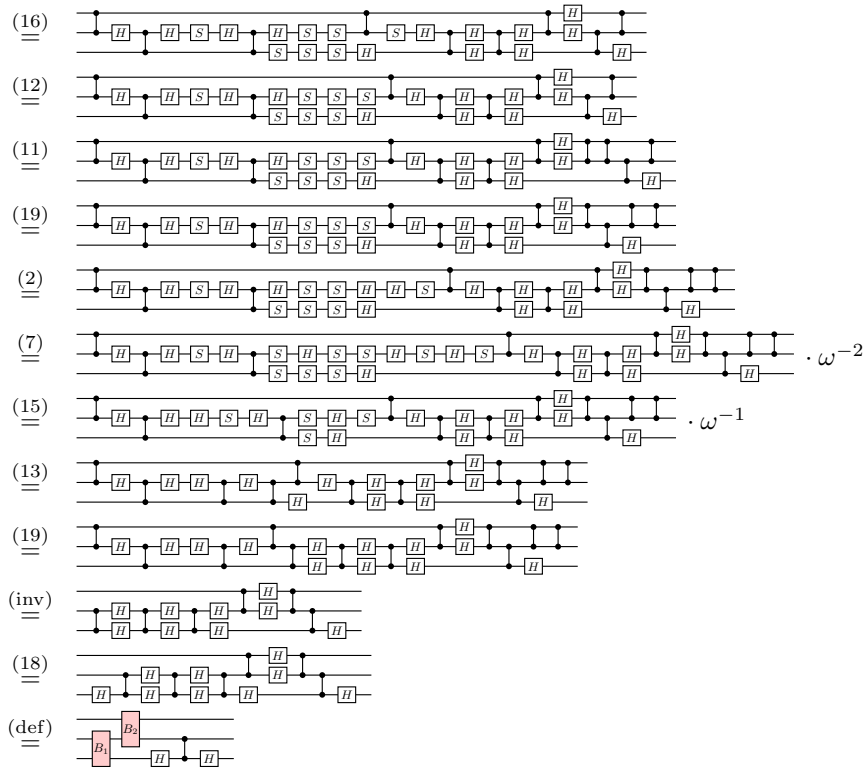
**Proof:**



### 3.63 Equation

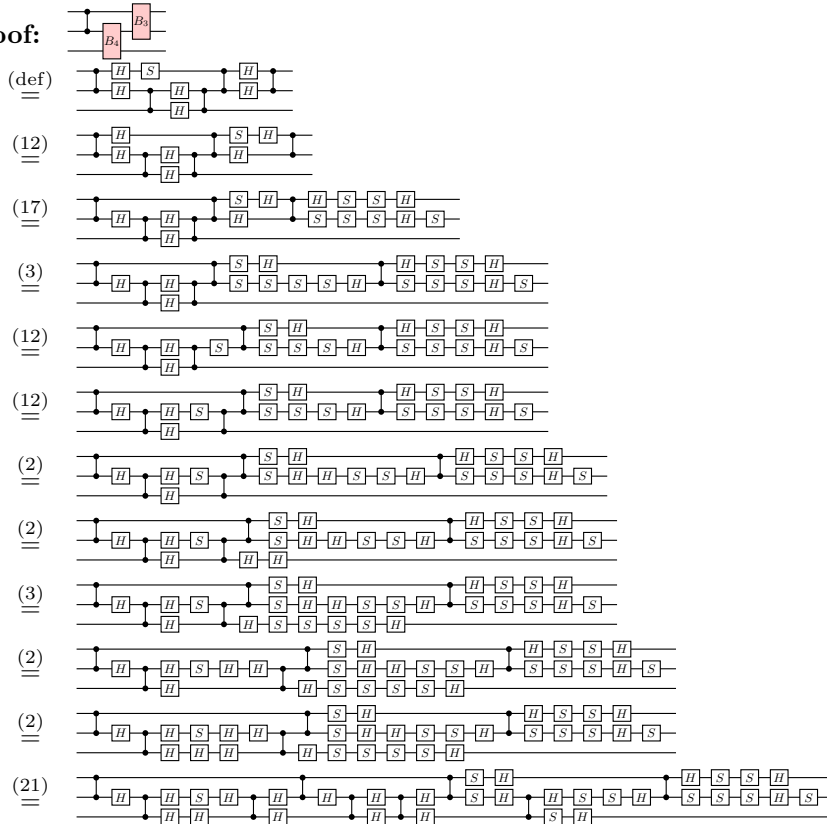


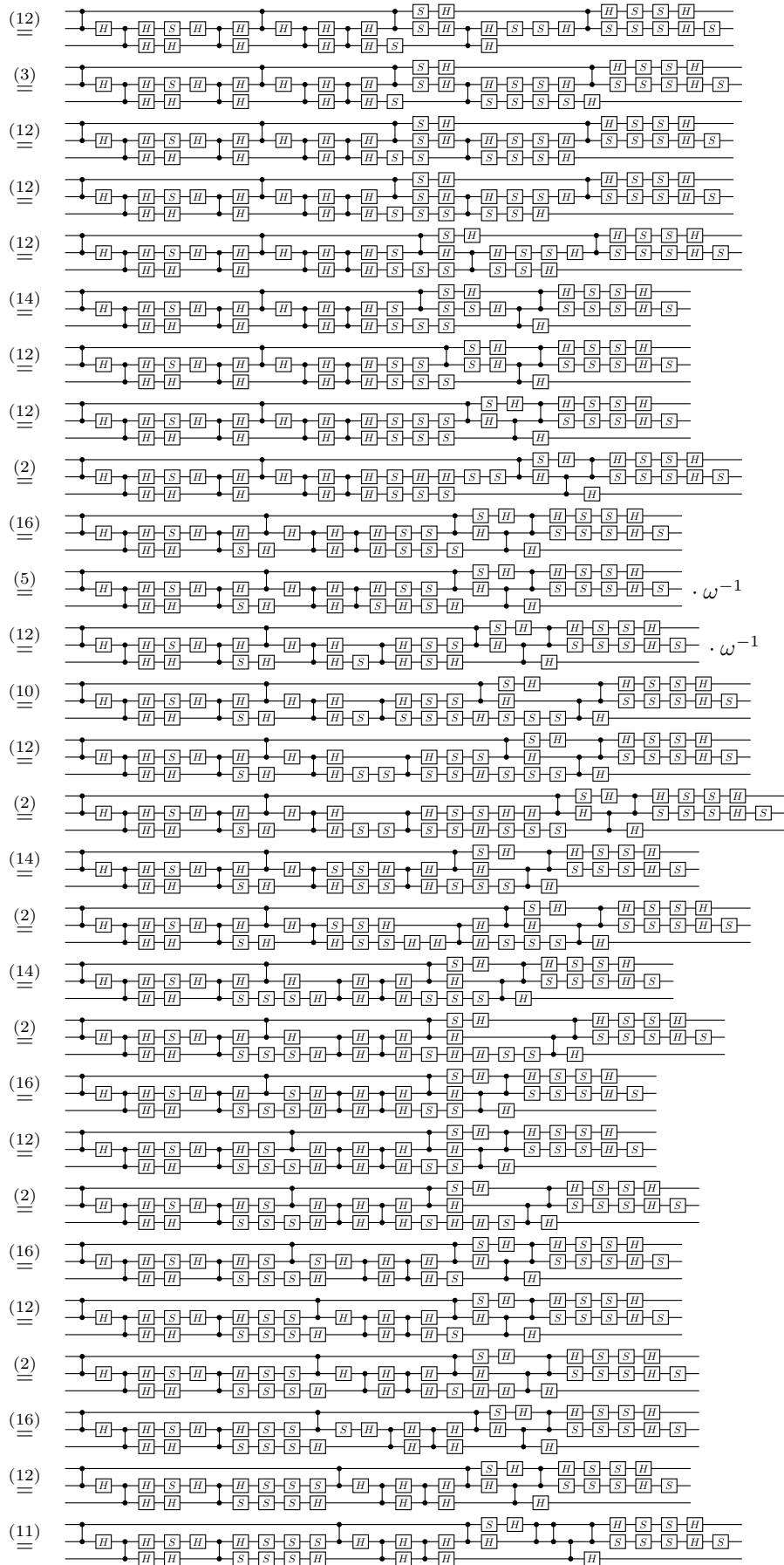


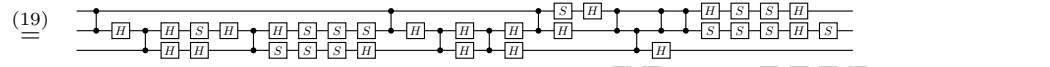


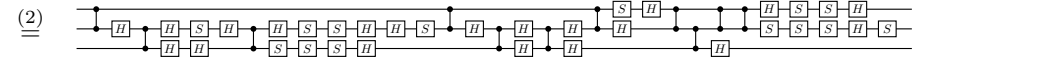
### 3.64 Equation

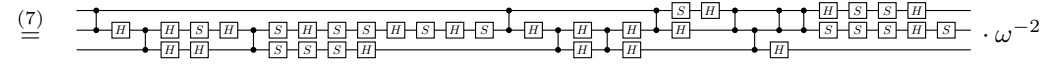
**Proof:**

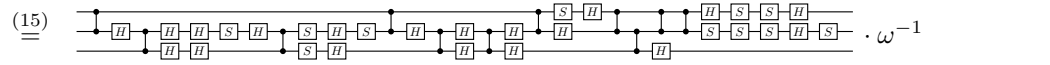


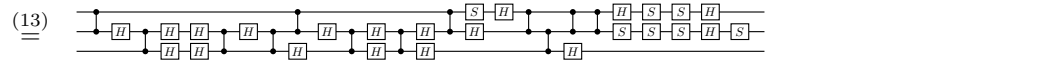


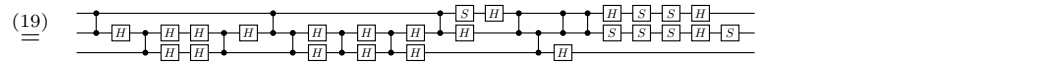
(19)   $\cdot \omega^{-2}$

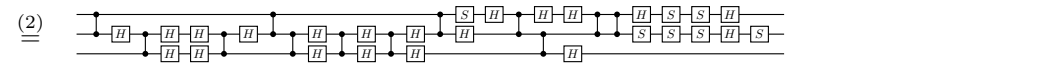
(2) 

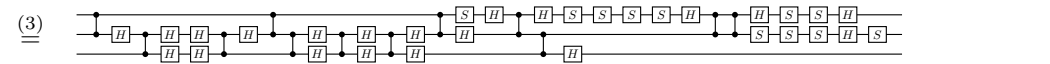
(7)   $\cdot \omega^{-2}$

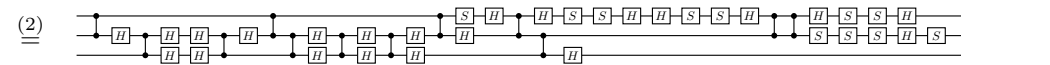
(15)   $\cdot \omega^{-1}$

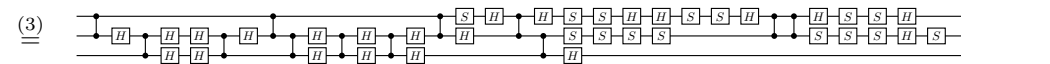
(13) 

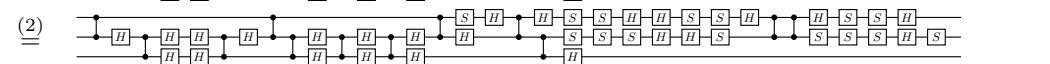
(19) 

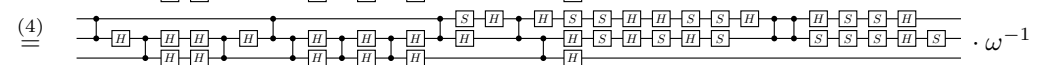
(2) 

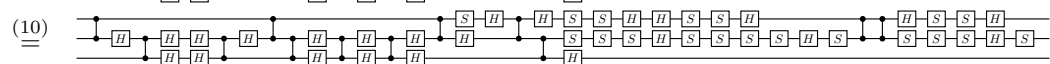
(3) 

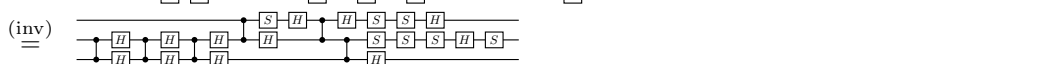
(2) 

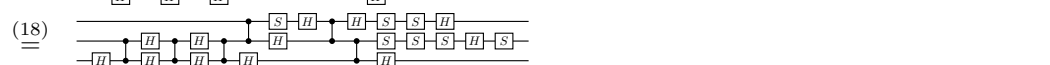
(3) 

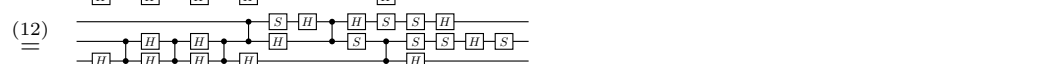
(2) 

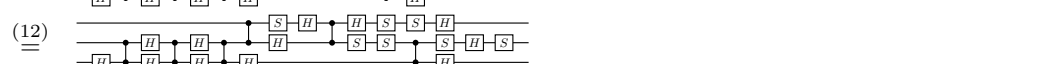
(4)   $\cdot \omega^{-1}$

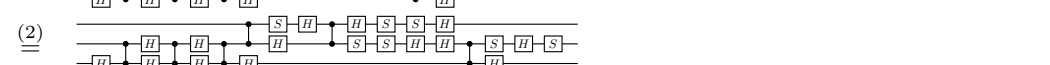
(10) 

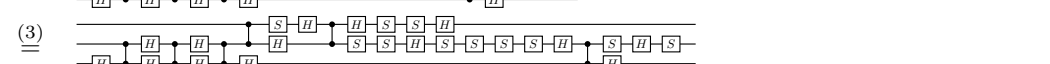
(inv) 

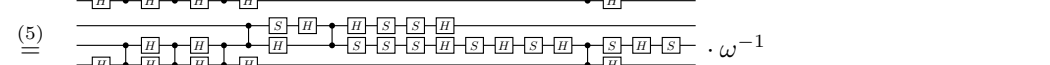
(18) 

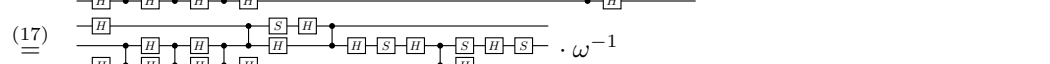
(12) 

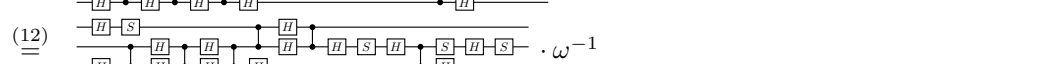
(12) 

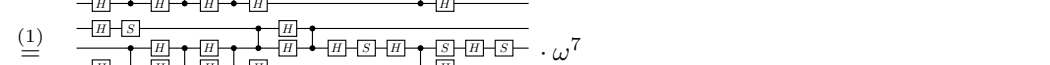
(2) 


(3) 

(5)   $\cdot \omega^{-1}$

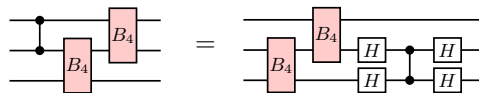
(17)   $\cdot \omega^{-1}$

(12)   $\cdot \omega^{-1}$

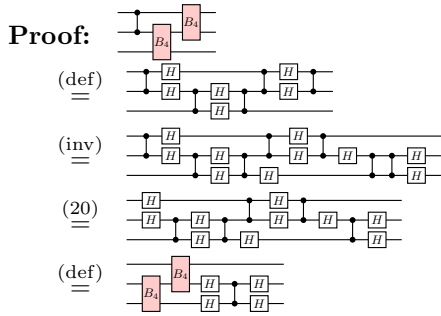
(1)   $\cdot \omega^7$

(def)   $\cdot \omega^7$

### 3.65 Equation

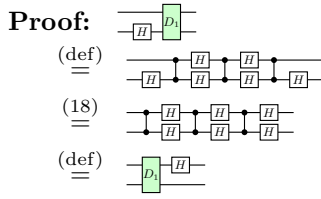


The equation shows two configurations of boxes  $B_4$  on a grid of nodes and connections. The left side shows two boxes  $B_4$  stacked vertically. The right side shows two boxes  $B_4$  stacked vertically, with two horizontal lines passing through them, each containing a box labeled  $H$ .



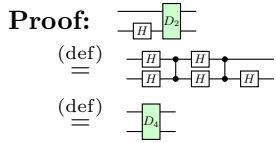
### 3.66 Equation

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} D_1 \\ | \\ H \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} D_1 \\ | \\ H \end{array}$$



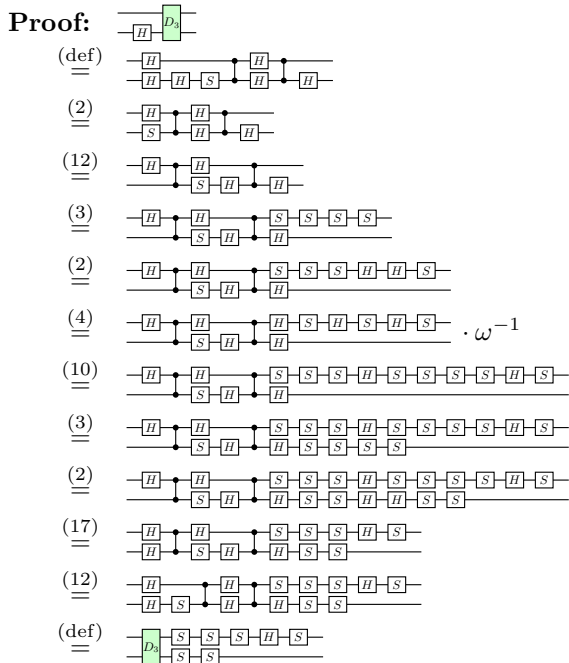
### 3.67 Equation

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} D_2 \\ | \\ H \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} D_4$$



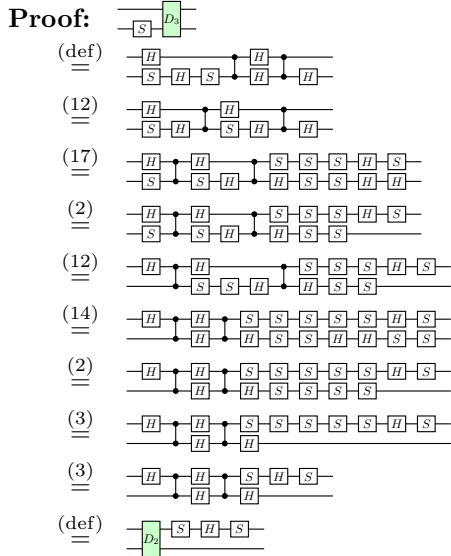
### 3.68 Equation

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} D_3 \\ | \\ H \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} D_3 \\ | \\ S \\ | \\ S \\ | \\ S \\ | \\ S \\ | \\ H \\ | \\ S \end{array}$$

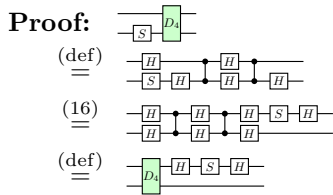
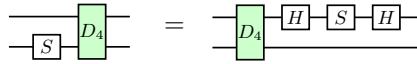




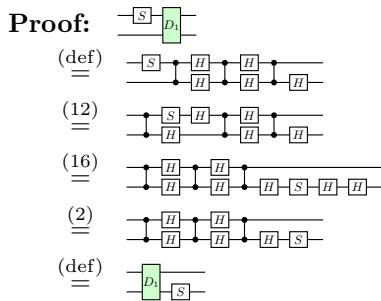
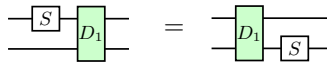




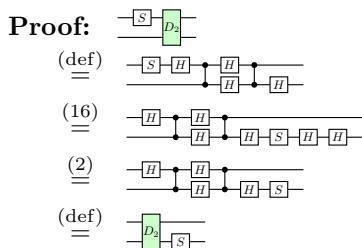
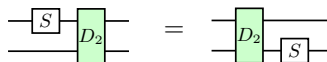
### 3.73 Equation



### 3.74 Equation

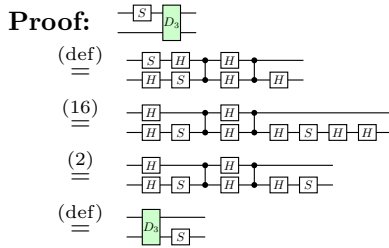


### 3.75 Equation



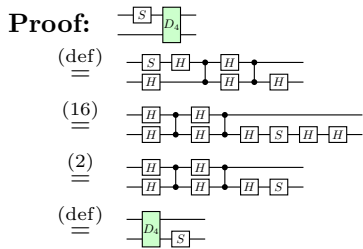
### 3.76 Equation

$$\begin{array}{c} \boxed{S} \\ \hline \end{array} \begin{array}{c} \boxed{D_3} \\ \hline \end{array} = \begin{array}{c} \boxed{D_3} \\ \hline \end{array} \begin{array}{c} \boxed{S} \\ \hline \end{array}$$



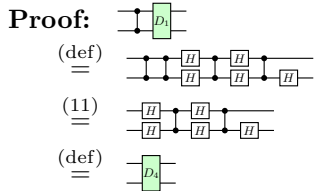
### 3.77 Equation

$$\begin{array}{c} \boxed{S} \\ \hline \end{array} \begin{array}{c} \boxed{D_4} \\ \hline \end{array} = \begin{array}{c} \boxed{D_4} \\ \hline \end{array} \begin{array}{c} \boxed{S} \\ \hline \end{array}$$



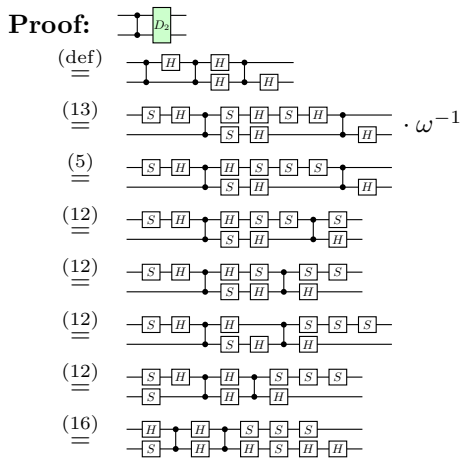
### 3.78 Equation

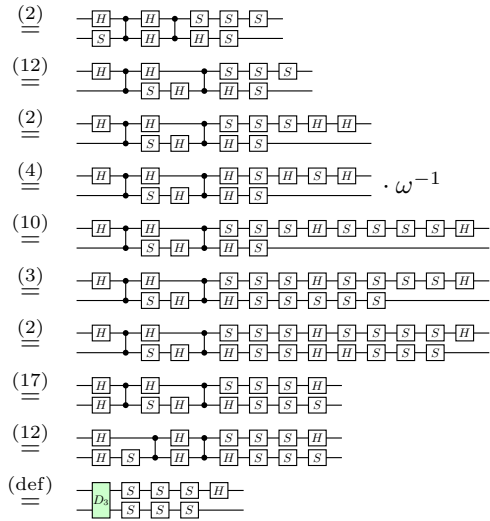
$$\begin{array}{c} \bullet \\ \bullet \\ \hline \end{array} \begin{array}{c} \boxed{D_1} \\ \hline \end{array} = \begin{array}{c} \boxed{D_4} \\ \hline \end{array}$$



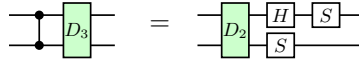
### 3.79 Equation

$$\begin{array}{c} \bullet \\ \bullet \\ \hline \end{array} \begin{array}{c} \boxed{D_2} \\ \hline \end{array} = \begin{array}{c} \boxed{D_3} \\ \hline \end{array} \begin{array}{c} \boxed{S} \\ \hline \end{array} \begin{array}{c} \boxed{S} \\ \hline \end{array} \begin{array}{c} \boxed{S} \\ \hline \end{array} \begin{array}{c} \boxed{H} \\ \hline \end{array}$$

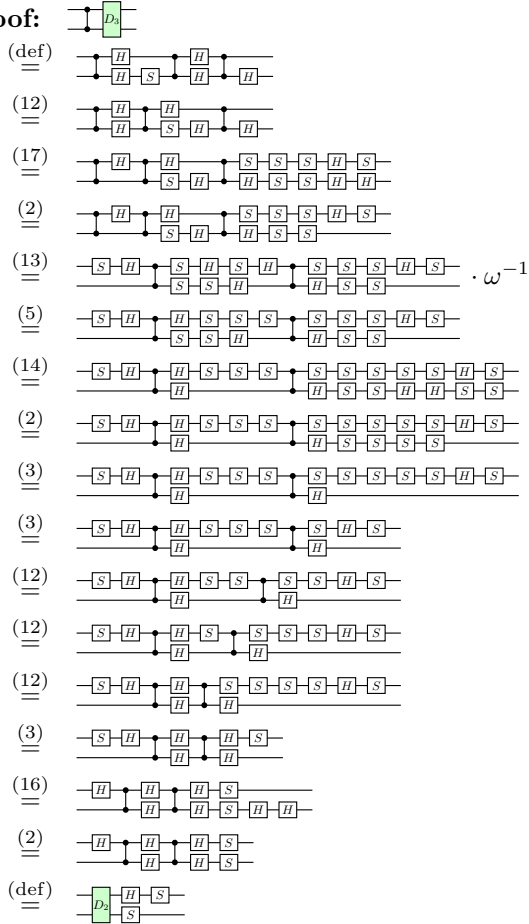




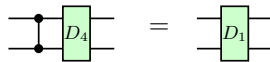
### 3.80 Equation

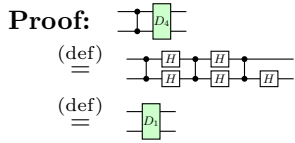


**Proof:**



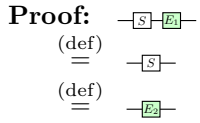
### 3.81 Equation





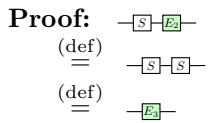
### 3.82 Equation

$$\boxed{S} \boxed{E_1} = \boxed{E_2}$$



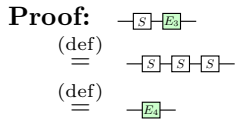
### 3.83 Equation

$$\boxed{S} \boxed{E_2} = \boxed{E_3}$$



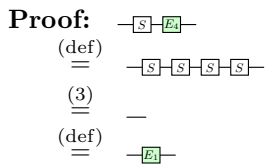
### 3.84 Equation

$$\boxed{S} \boxed{E_3} = \boxed{E_4}$$

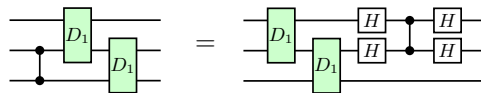


### 3.85 Equation

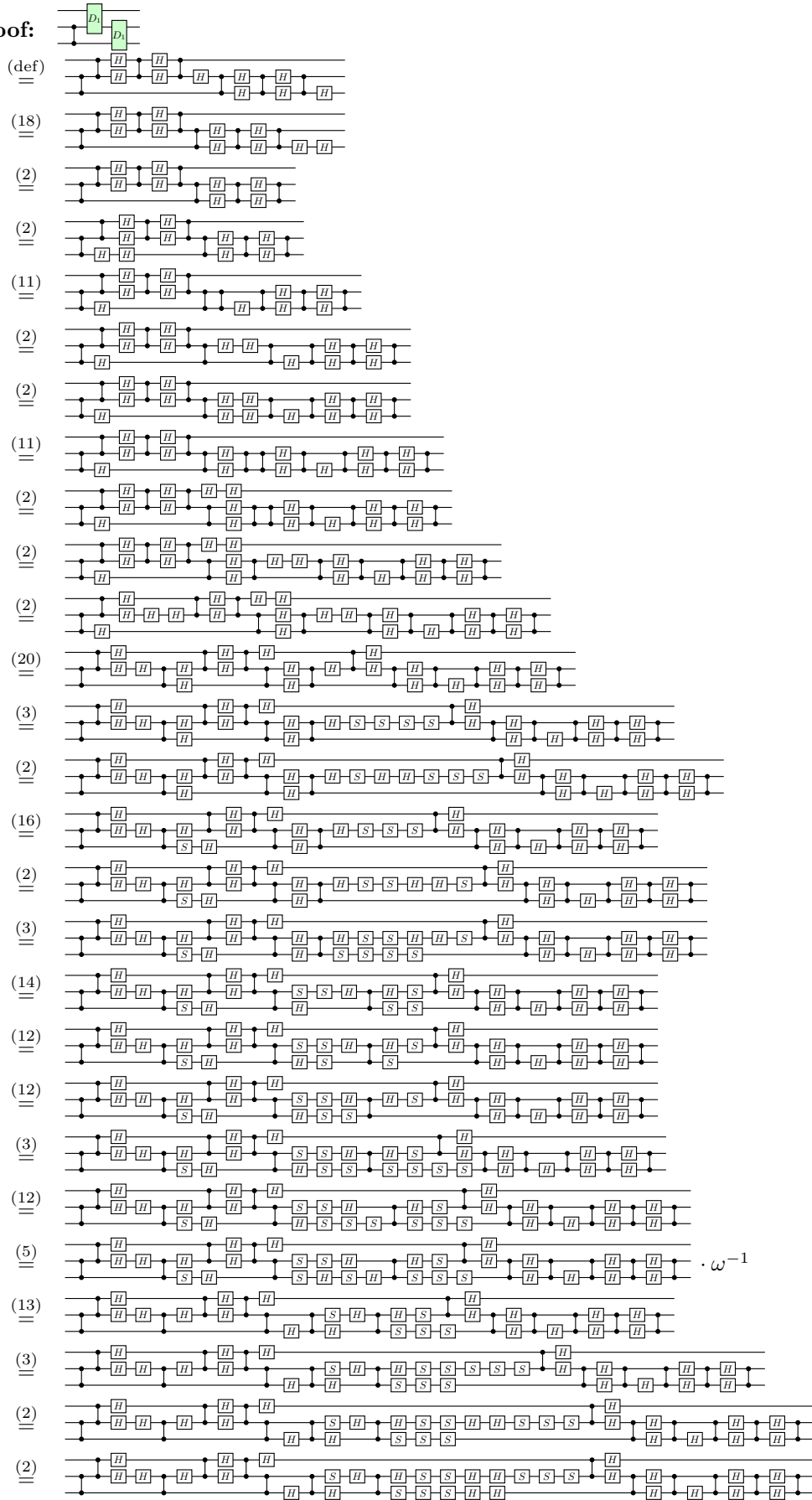
$$\boxed{S} \boxed{E_4} = \boxed{E_1}$$

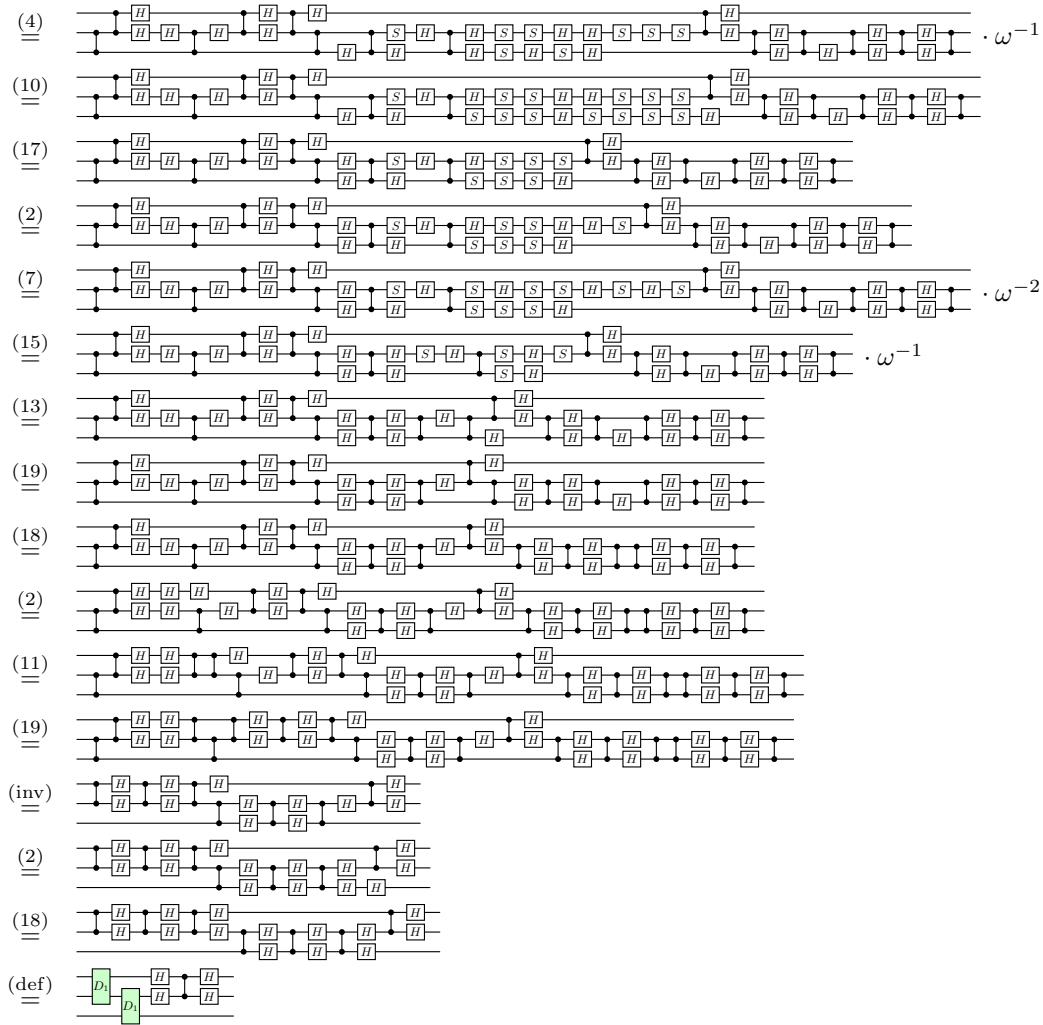


### 3.86 Equation

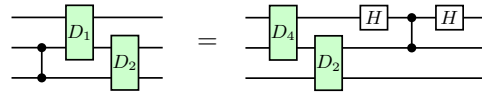


**Proof:**

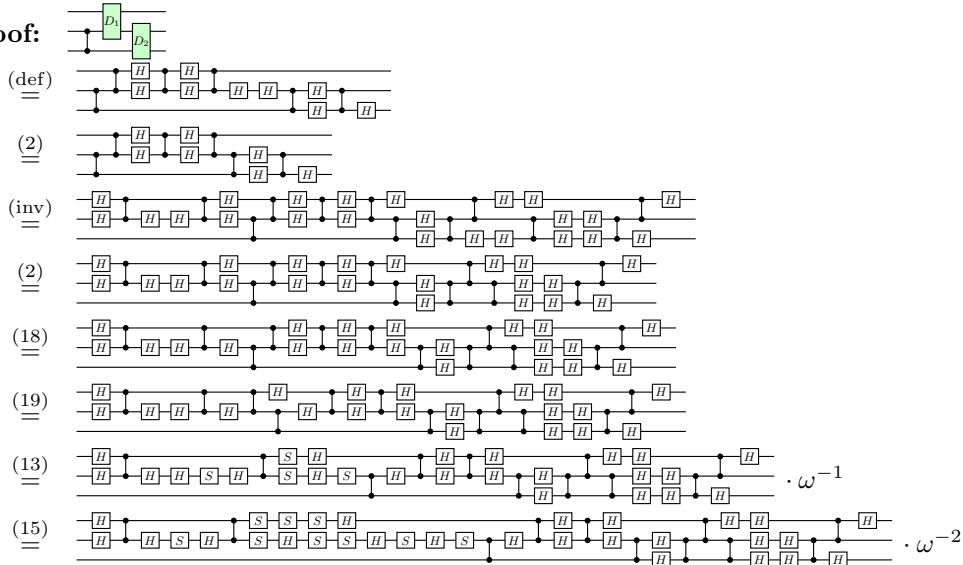


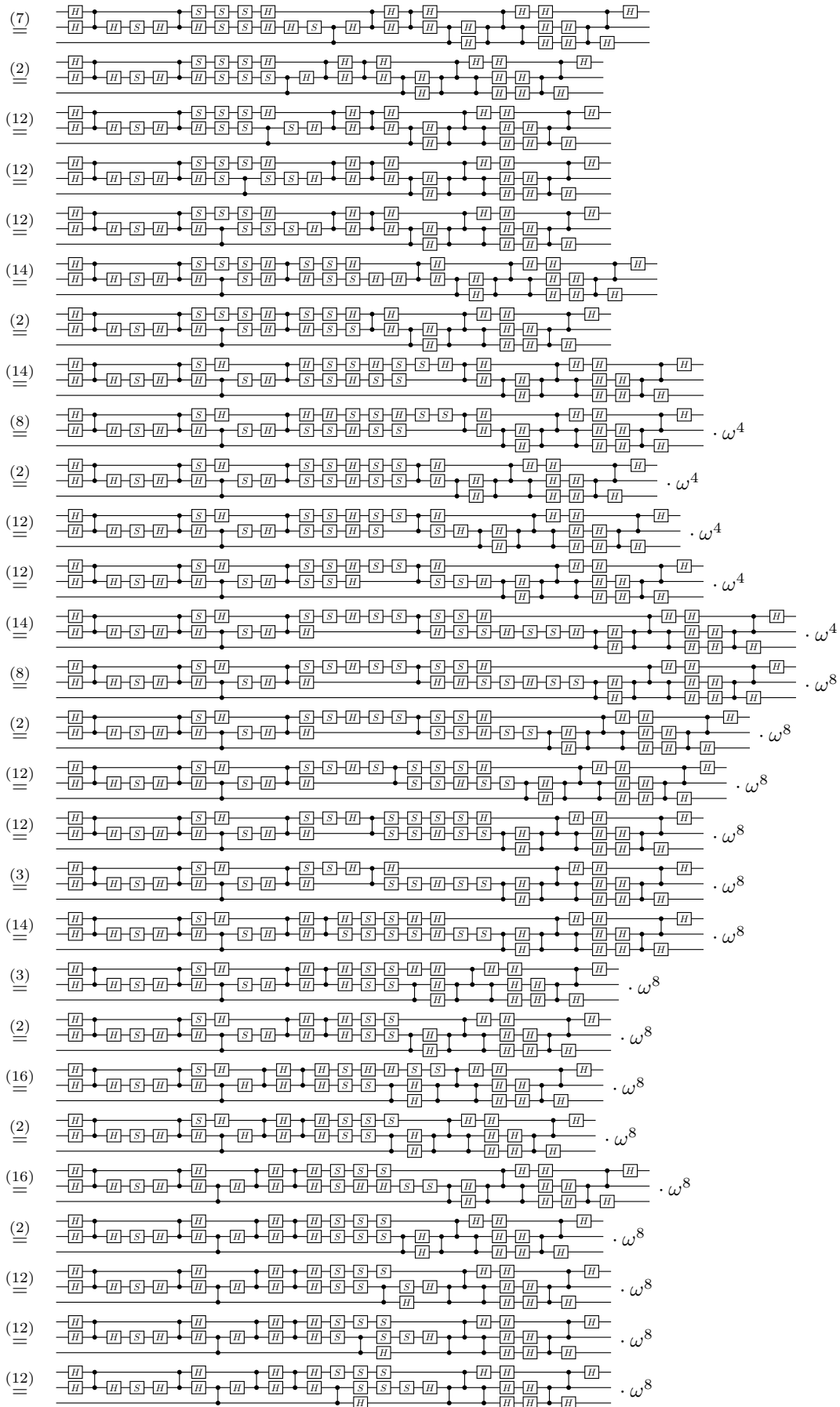


### 3.87 Equation

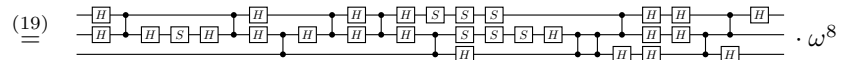


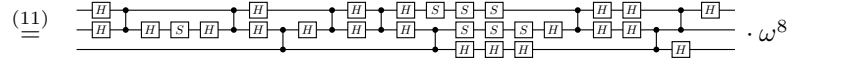
**Proof:**

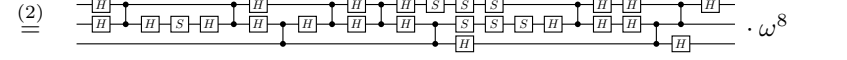


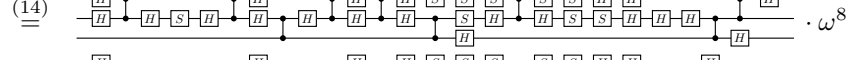


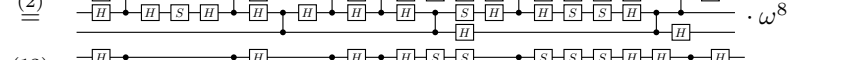


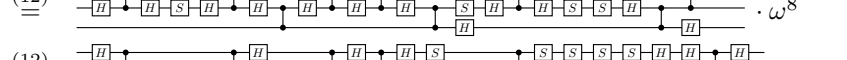
(19)  · ω<sup>8</sup>

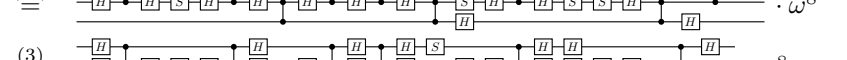
(11)  · ω<sup>8</sup>


(2)  · ω<sup>8</sup>

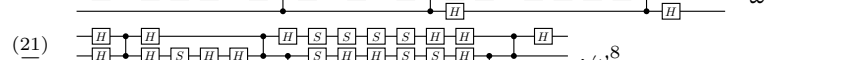
(14)  · ω<sup>8</sup>

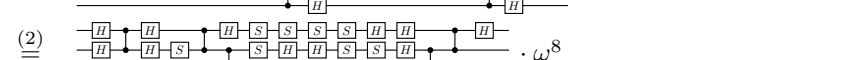
(2)  · ω<sup>8</sup>


(12)  · ω<sup>8</sup>

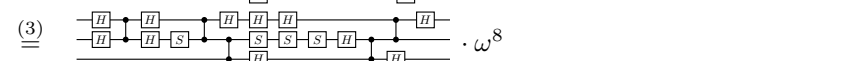
(12)  · ω<sup>8</sup>

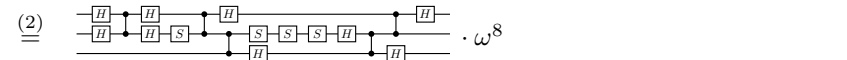
(3)  · ω<sup>8</sup>

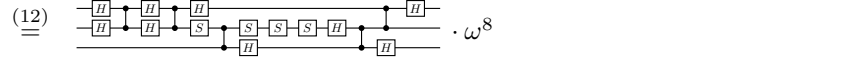
(12)  · ω<sup>8</sup>

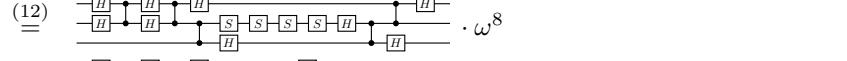
(21)  · ω<sup>8</sup>


(2)  · ω<sup>8</sup>

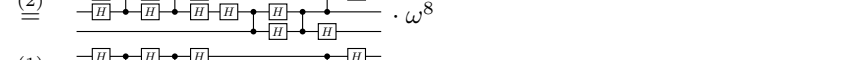
(2)  · ω<sup>8</sup>


(3)  · ω<sup>8</sup>


(2)  · ω<sup>8</sup>


(12)  · ω<sup>8</sup>

(12)  · ω<sup>8</sup>

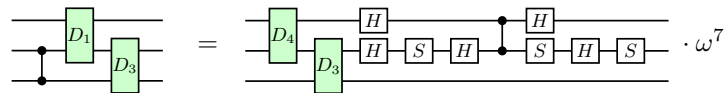
(3)  · ω<sup>8</sup>

(2)  · ω<sup>8</sup>


(1)  · ω<sup>8</sup>

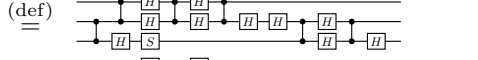
(def)  · ω<sup>8</sup>

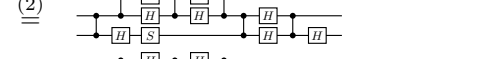
### 3.88 Equation

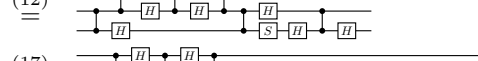
 · ω<sup>7</sup>

**Proof:**

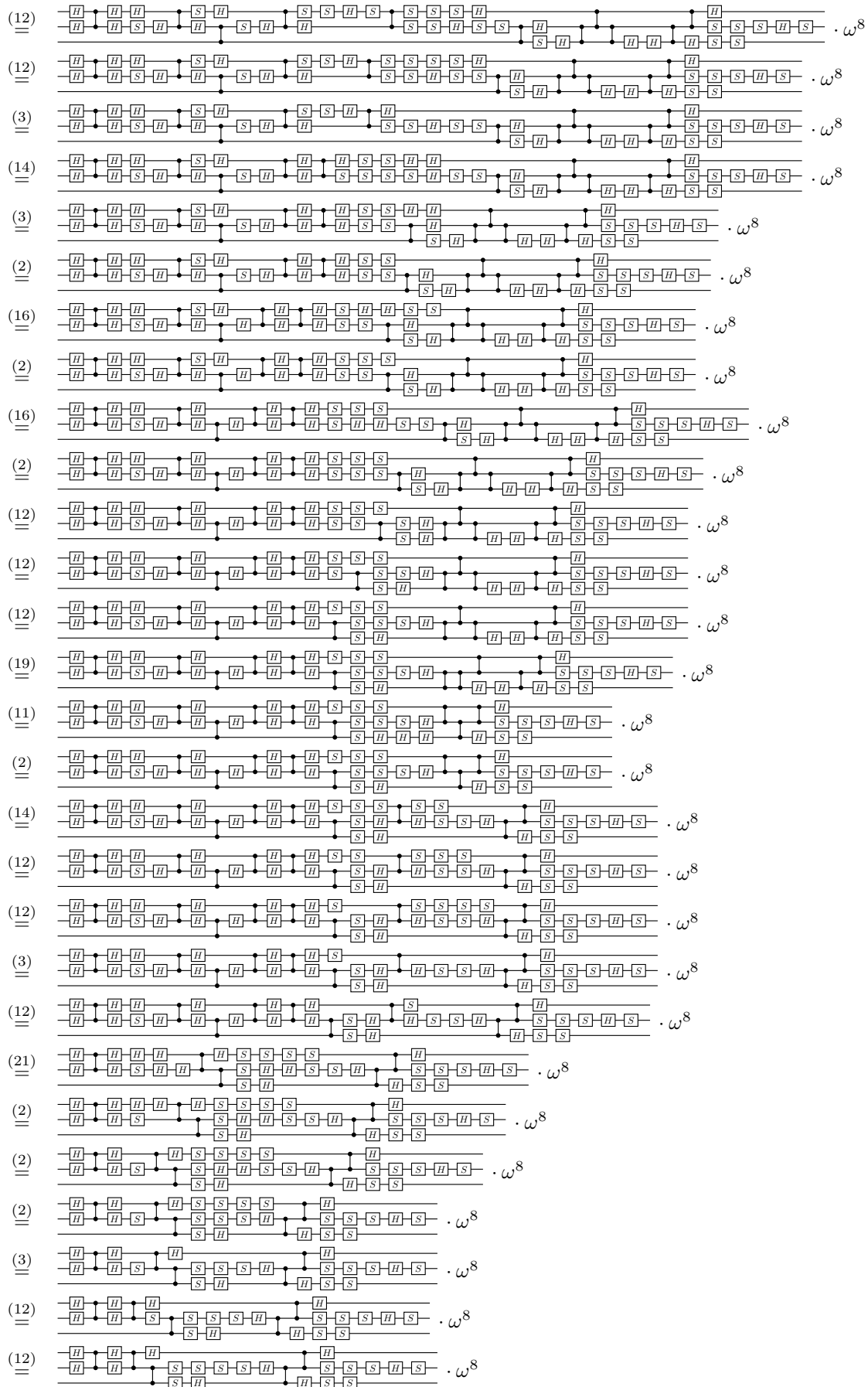
(def)  · ω<sup>8</sup>

(2)  · ω<sup>8</sup>

(12)  · ω<sup>8</sup>

(17)  · ω<sup>8</sup>





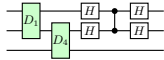
$$\begin{aligned}
& \underline{(3)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^8 \text{ phase]} \\
& \underline{(12)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^8 \text{ phase]} \\
& \underline{(12)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^8 \text{ phase]} \\
& \underline{(2)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^8 \text{ phase]} \\
& \underline{(3)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^8 \text{ phase]} \\
& \underline{(5)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase]} \\
& \underline{(2)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase]} \\
& \underline{(17)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase]} \\
& \underline{(12)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase]} \\
& \underline{(2)} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase]} \\
& \underline{(\text{def})} \quad \text{[Diagram with 4 qubits, H and S gates, and a } \omega^7 \text{ phase, with } D_4 \text{ gates highlighted]}
\end{aligned}$$

### 3.89 Equation

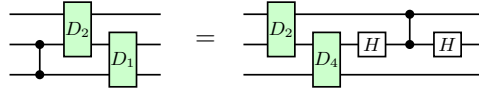
$$\text{[Diagram: } D_1 \text{ then } D_4 \text{]} = \text{[Diagram: } D_1 \text{ then } D_4 \text{ then } H \text{ then } H \text{]}$$

**Proof:**

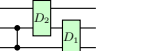
$$\begin{aligned}
& \underline{(\text{def})} \quad \text{[Diagram with } D_1 \text{ and } D_4 \text{ gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(20)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(11)} \quad \text{[Diagram with H gates]} \\
& \underline{(19)} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]} \\
& \underline{(\text{inv})} \quad \text{[Diagram with H gates]} \\
& \underline{(2)} \quad \text{[Diagram with H gates]}
\end{aligned}$$

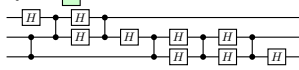
(def) 

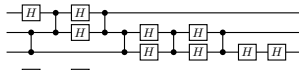
### 3.90 Equation

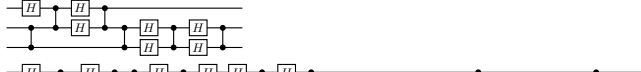


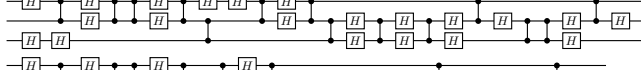
**Proof:**

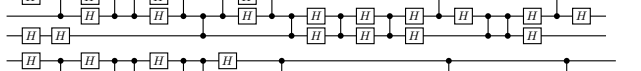
(def) 


(18) 

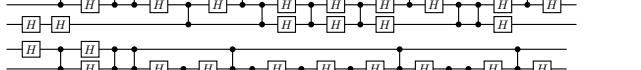
(2) 

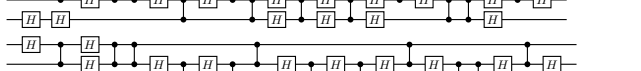
(inv) 

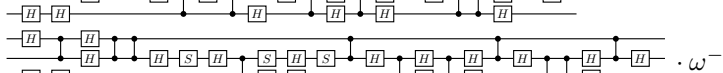
(2) 

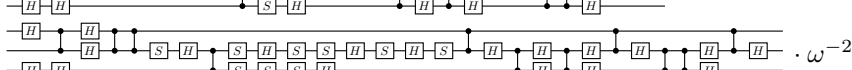
(19) 

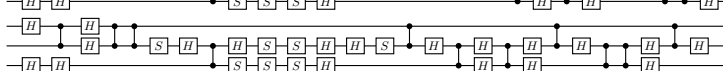
(11) 

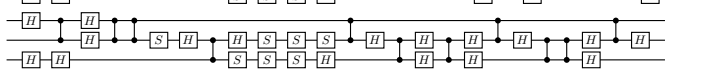
(2) 

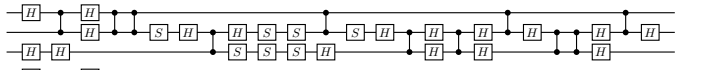
(19) 

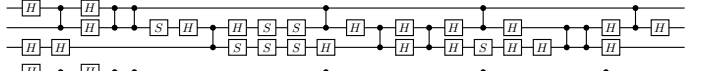
(13)   $\cdot \omega^{-1}$

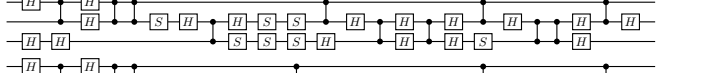
(15)   $\cdot \omega^{-2}$

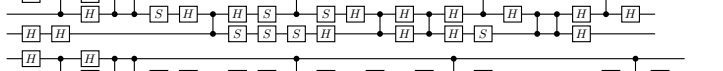
(7) 

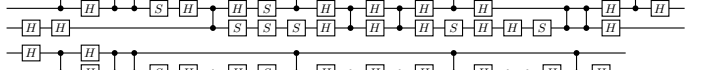
(2) 

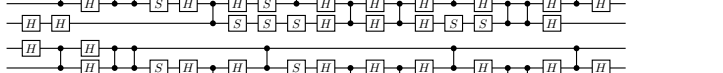
(12) 

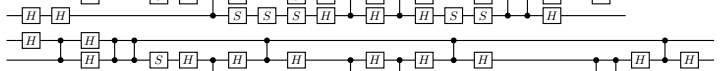
(16) 


(2) 

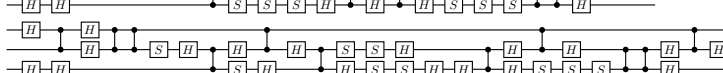
(12) 


(16) 

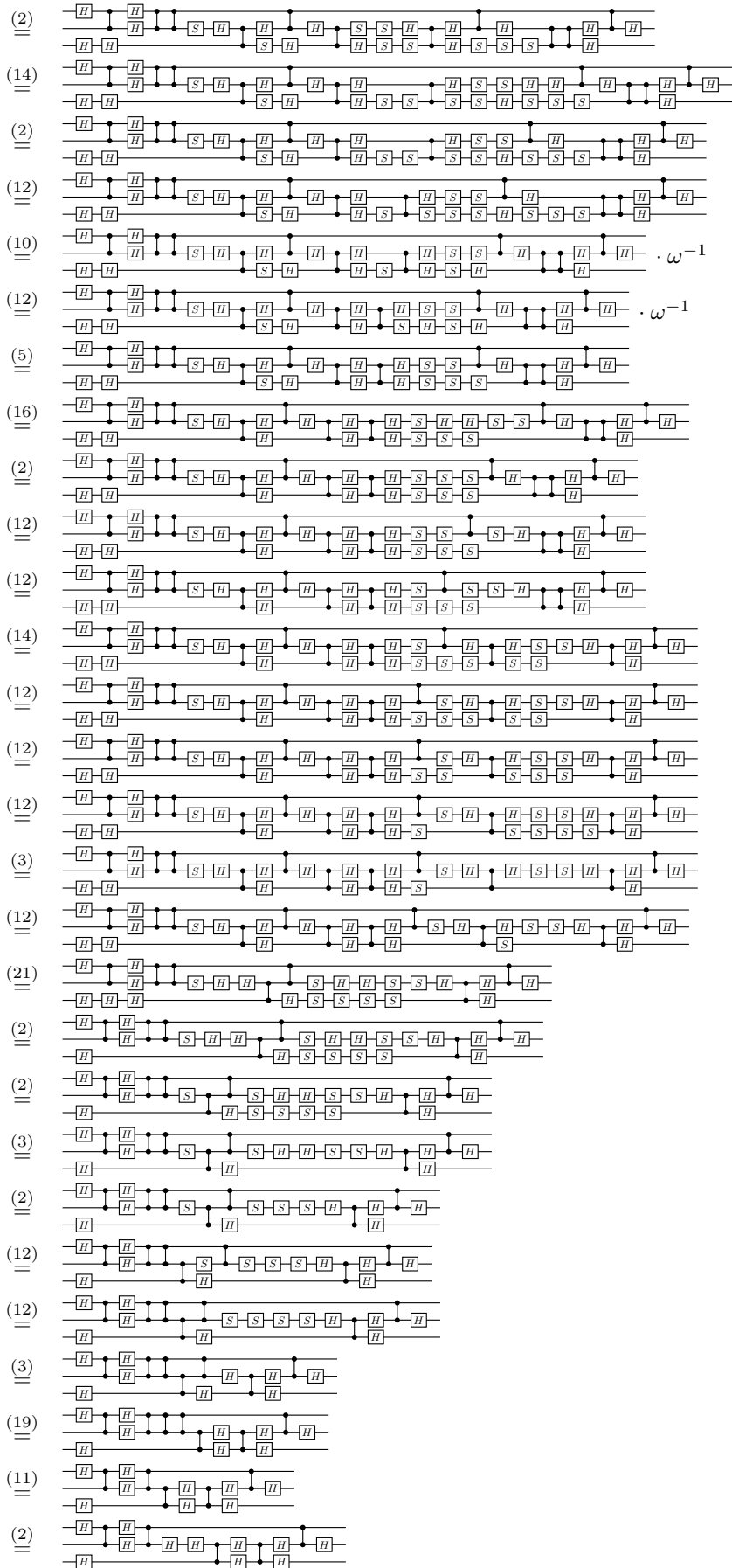
(2) 

(12) 

(16) 

(2) 

(14) 



(def)

### 3.91 Equation

**Proof:**

(def)

(2)

(11)

(19)

(3)

(12)

(3)

(12)

(12)

(3)

(2)

(3)

(2)

(14)

(7)  $\cdot \omega^{-2}$

(13)  $\cdot \omega^{-1}$

(12)  $\cdot \omega^{-1}$

(2)  $\cdot \omega^{-1}$

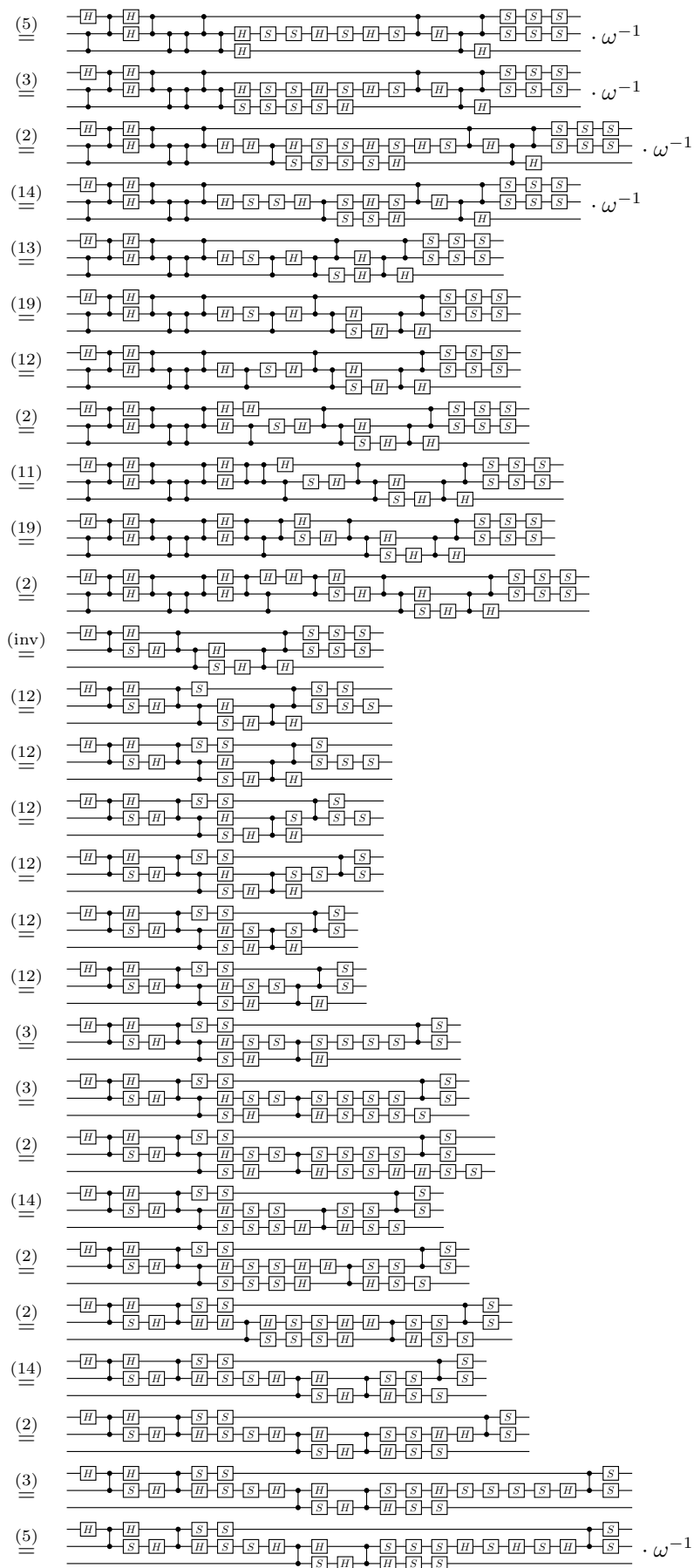
(15)

(11)

(19)

(10)  $\cdot \omega$

(4)





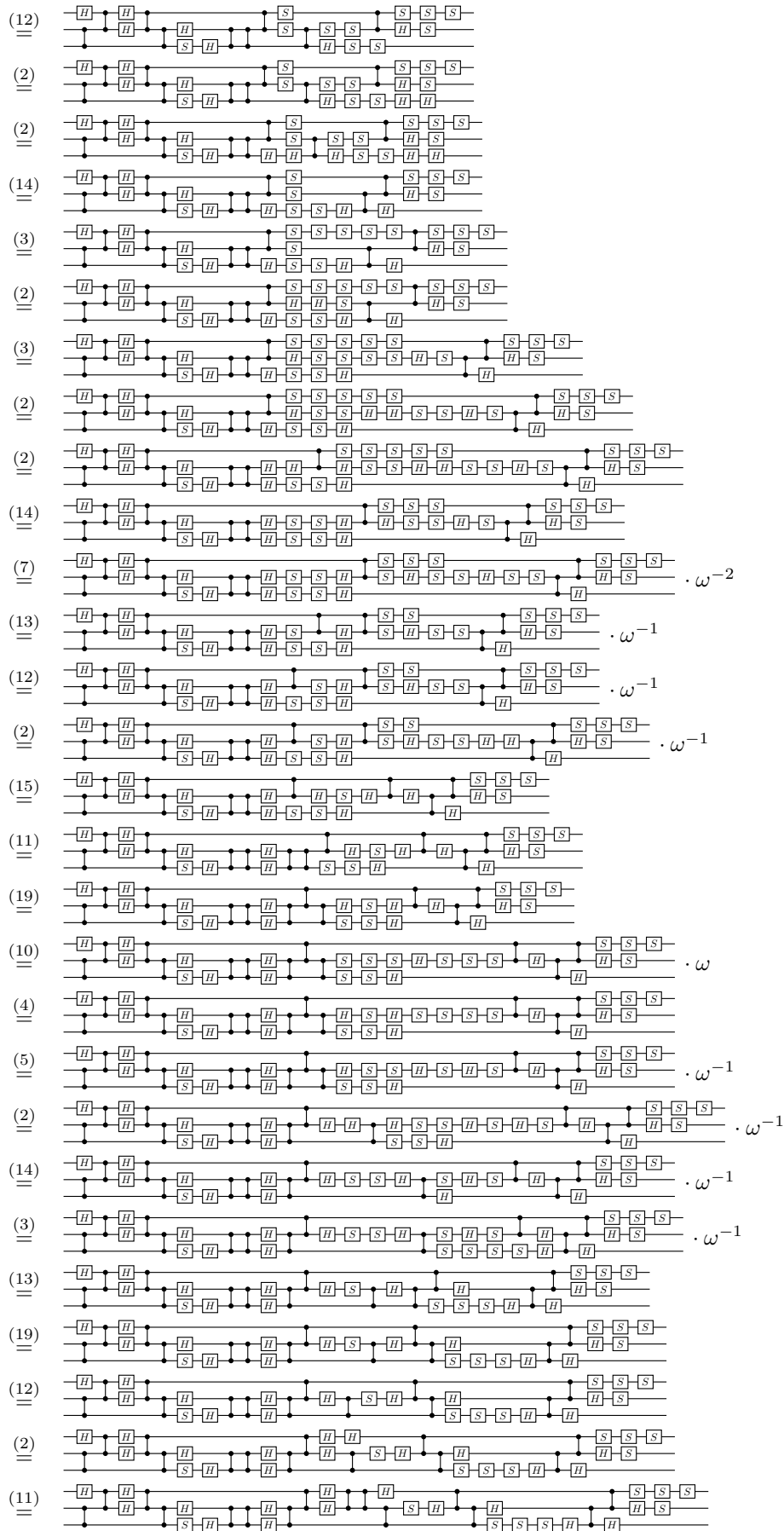
$$\begin{aligned}
& \underline{(2)} \quad \text{Diagram 1} \cdot \omega^{-1} \\
& \underline{(17)} \quad \text{Diagram 2} \cdot \omega^{-1} \\
& \underline{(12)} \quad \text{Diagram 3} \cdot \omega^{-1} \\
& \underline{(2)} \quad \text{Diagram 4} \cdot \omega^{-1} \\
& \underline{(3)} \quad \text{Diagram 5} \cdot \omega^{-1} \\
& \underline{(5)} \quad \text{Diagram 6} \cdot \omega^{-2} \\
& \underline{(17)} \quad \text{Diagram 7} \cdot \omega^{-2} \\
& \underline{(12)} \quad \text{Diagram 8} \cdot \omega^{-2} \\
& \underline{(2)} \quad \text{Diagram 9} \cdot \omega^{-2} \\
& \underline{(1)} \quad \text{Diagram 10} \cdot \omega^6 \\
& \underline{\text{(def)}} \quad \text{Diagram 11} \cdot \omega^6
\end{aligned}$$

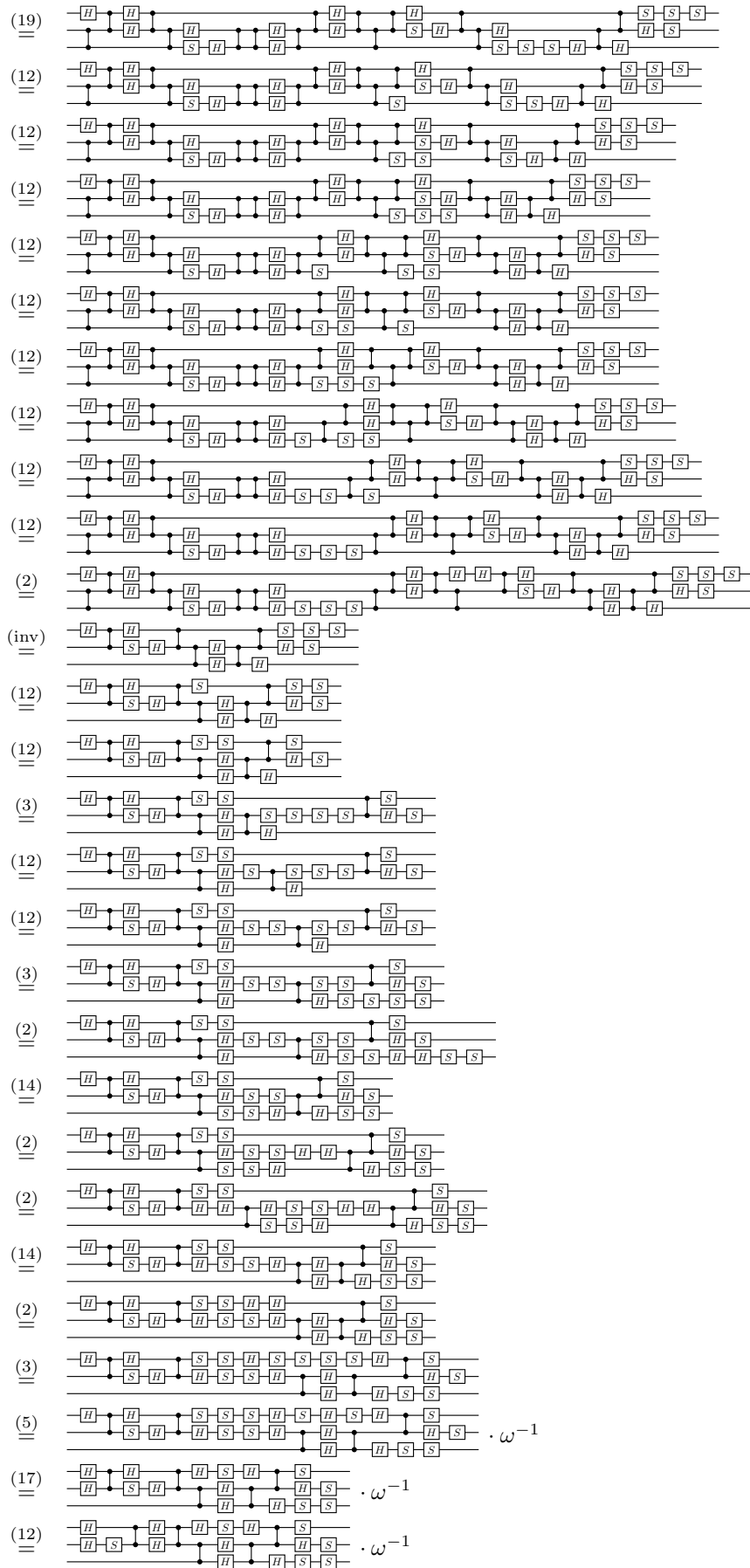
### 3.92 Equation

$$\text{Diagram A} = \text{Diagram B} \cdot \omega^7$$

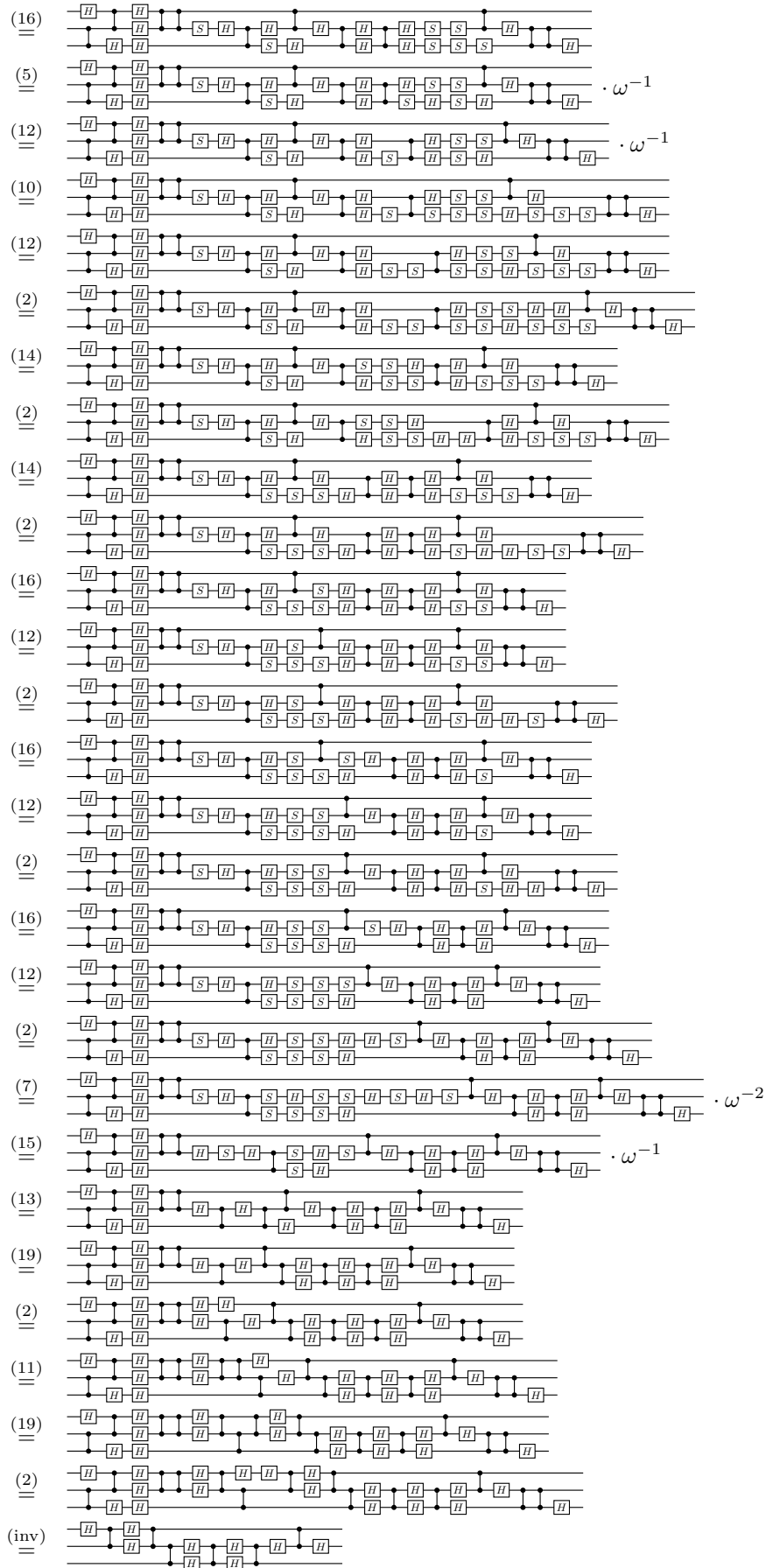
**Proof:**

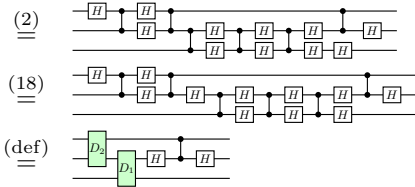
$$\begin{aligned}
& \underline{\text{(def)}} \quad \text{Diagram 1} \\
& \underline{(2)} \quad \text{Diagram 2} \\
& \underline{(12)} \quad \text{Diagram 3} \\
& \underline{(17)} \quad \text{Diagram 4} \\
& \underline{(2)} \quad \text{Diagram 5} \\
& \underline{(11)} \quad \text{Diagram 6} \\
& \underline{(11)} \quad \text{Diagram 7} \\
& \underline{(19)} \quad \text{Diagram 8} \\
& \underline{(3)} \quad \text{Diagram 9} \\
& \underline{(12)} \quad \text{Diagram 10} \\
& \underline{(12)} \quad \text{Diagram 11} \\
& \underline{(12)} \quad \text{Diagram 12} \\
& \underline{(12)} \quad \text{Diagram 13}
\end{aligned}$$







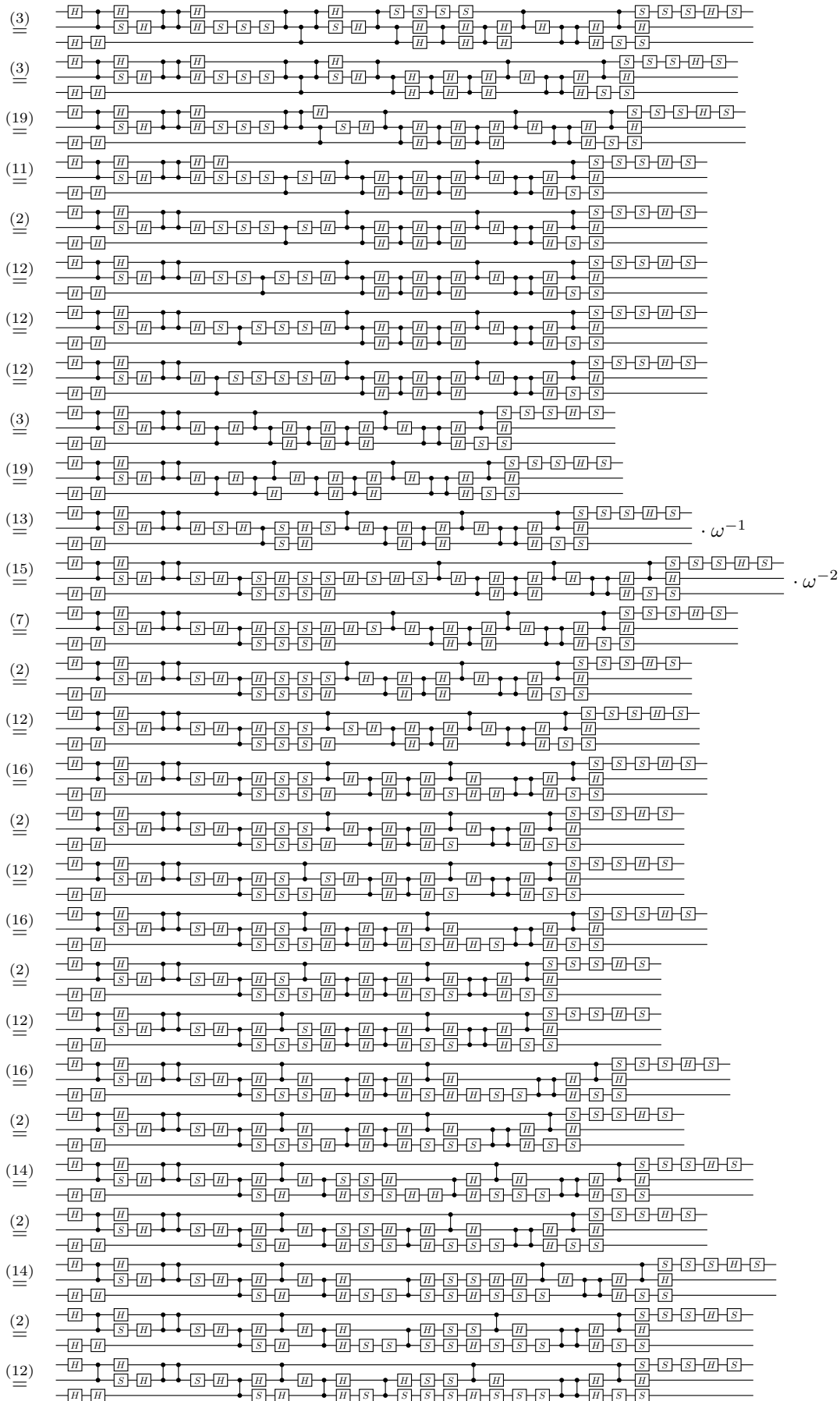


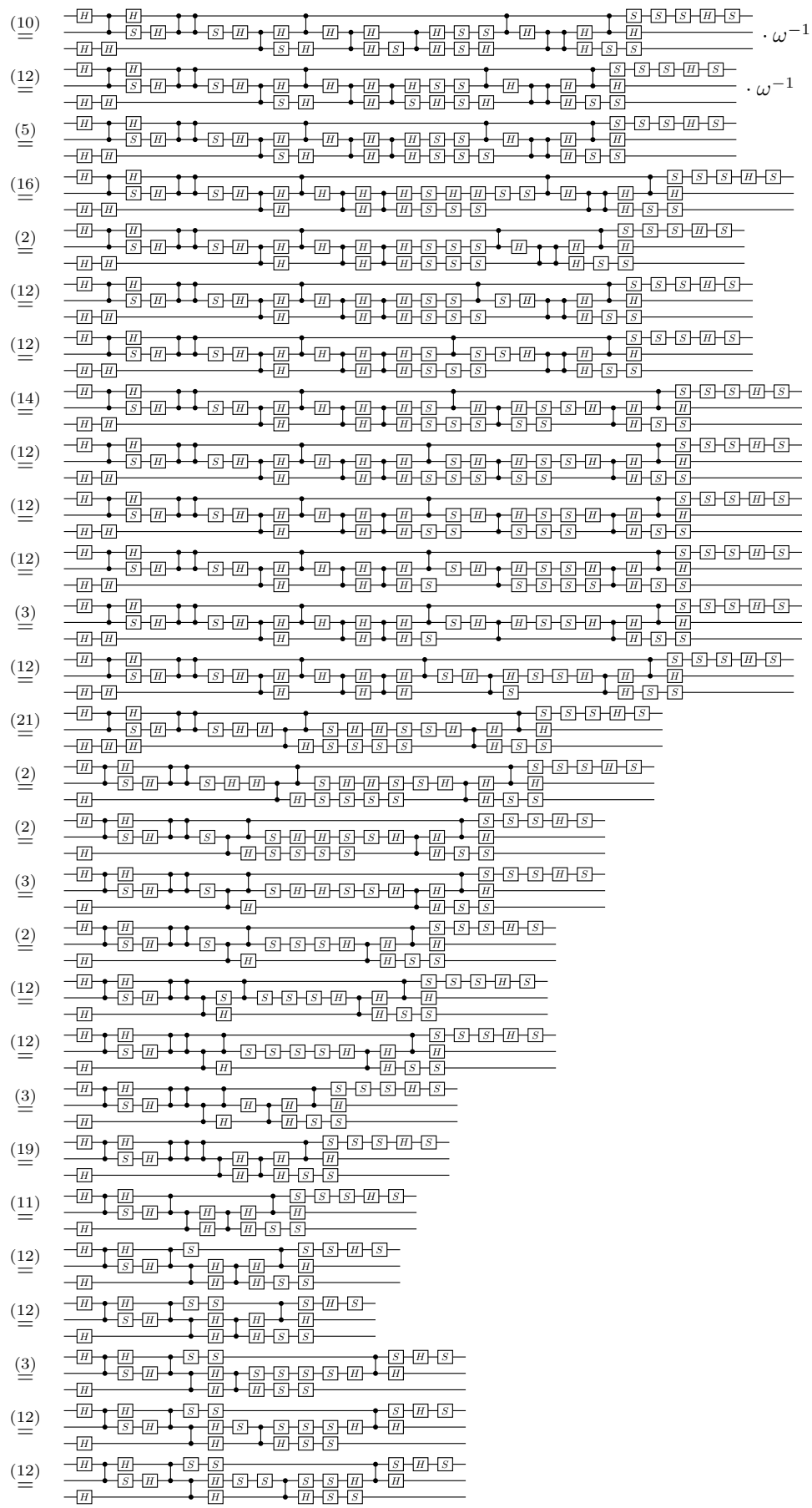


### 3.94 Equation

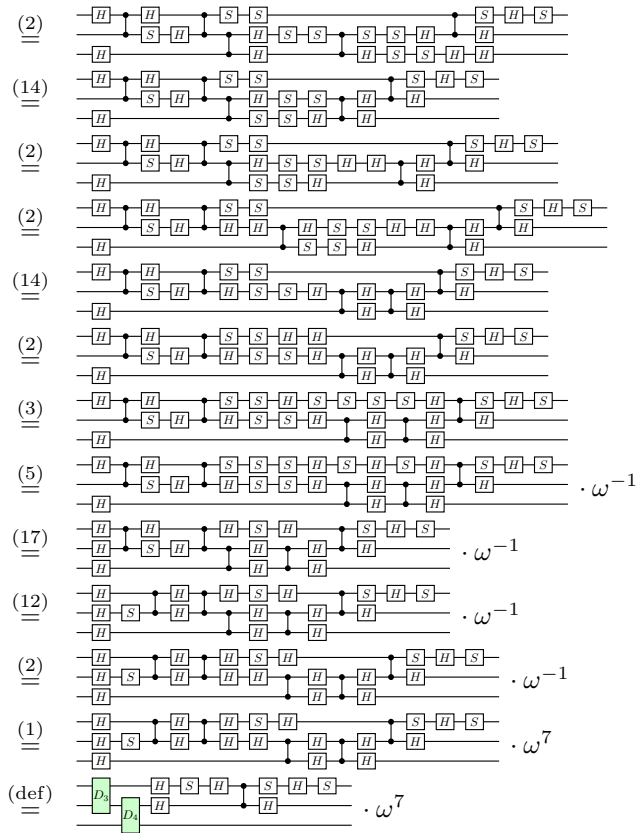
**Proof:**





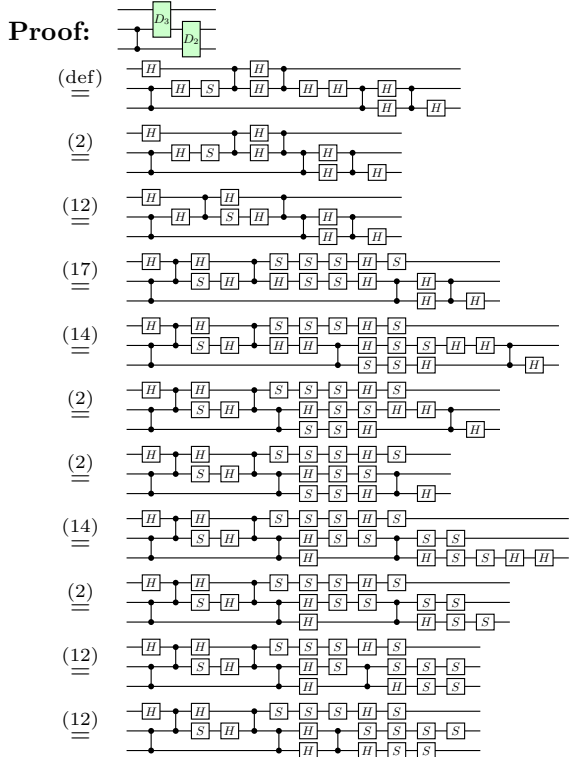


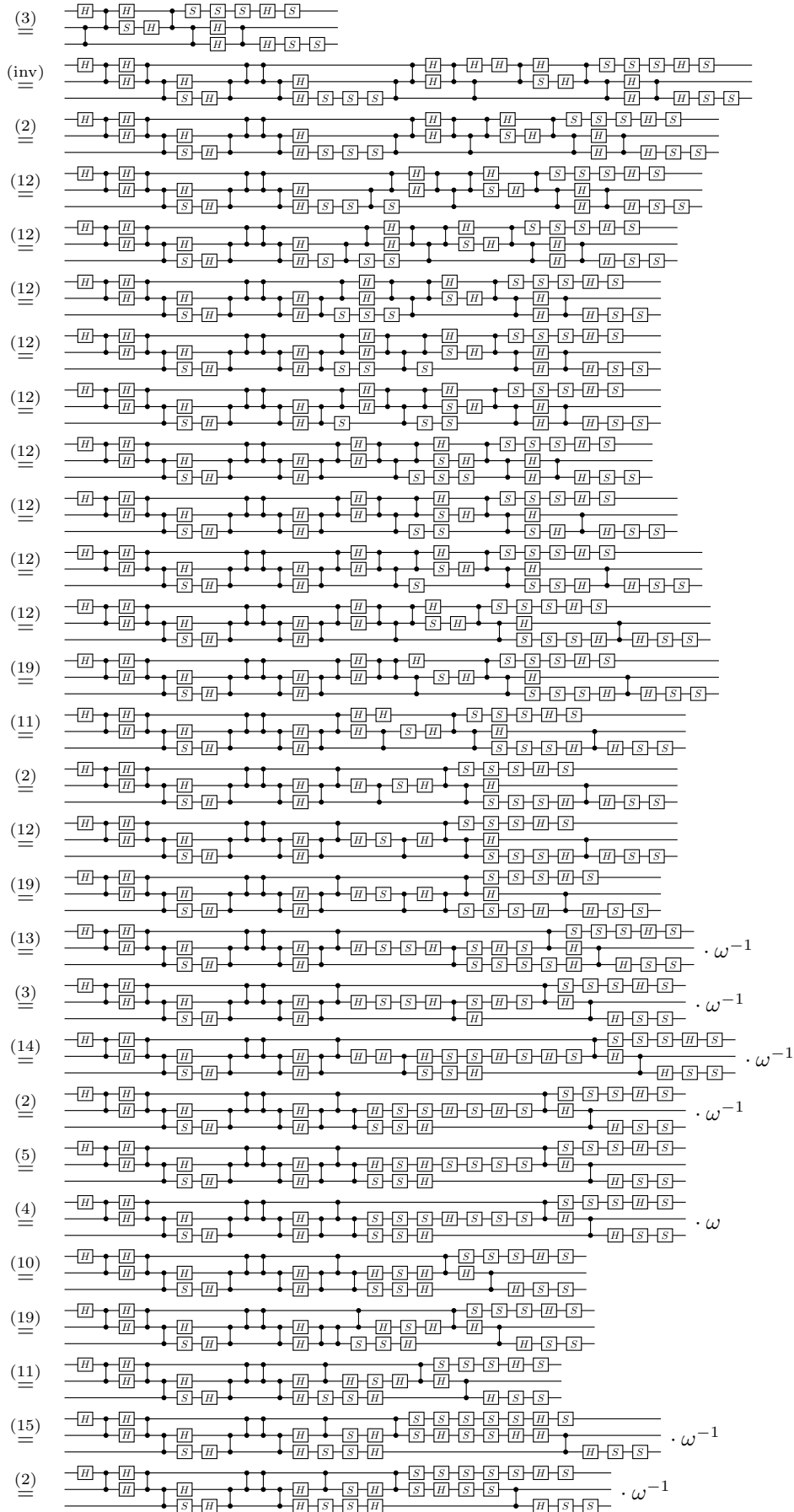


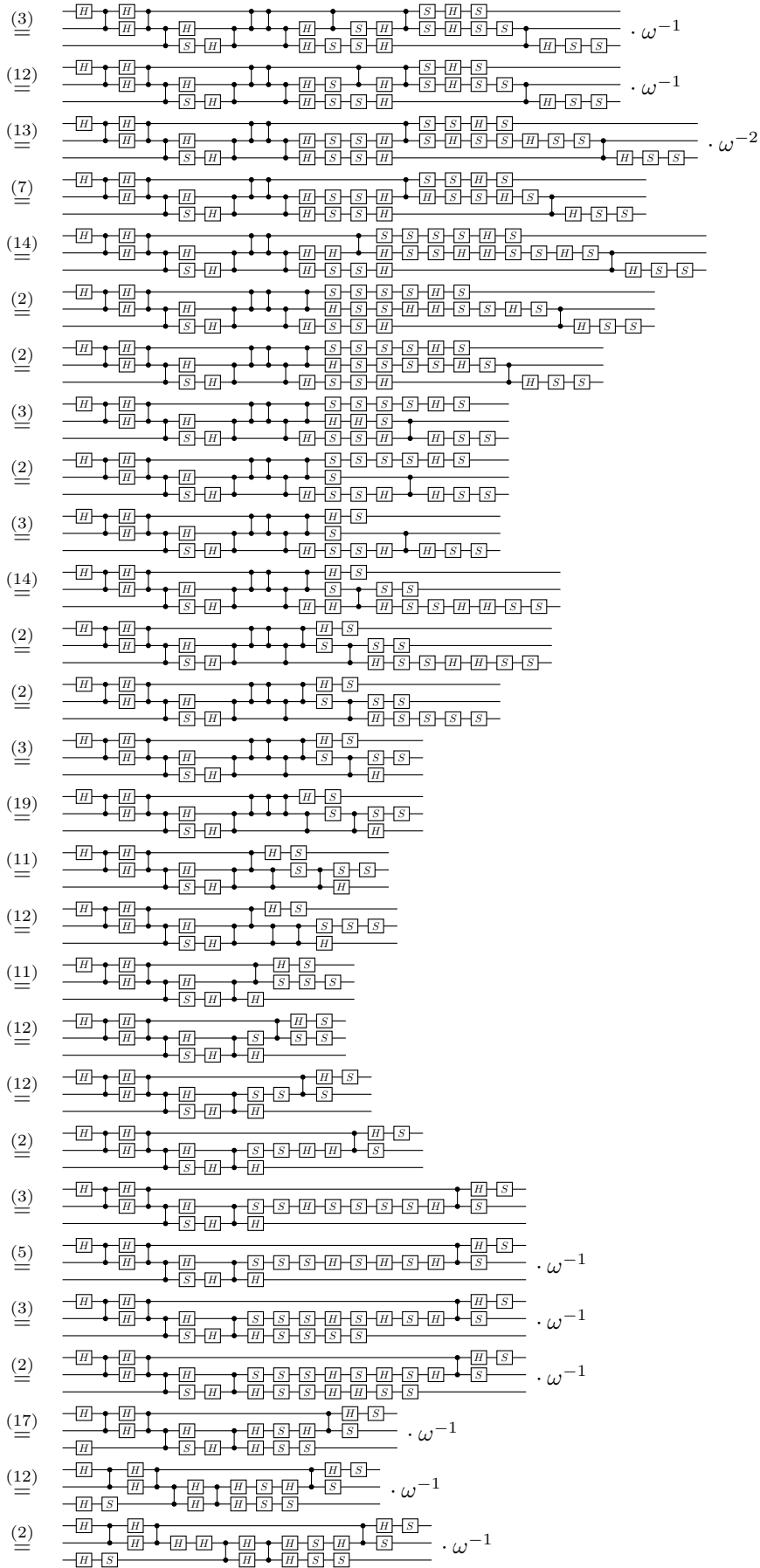


### 3.95 Equation

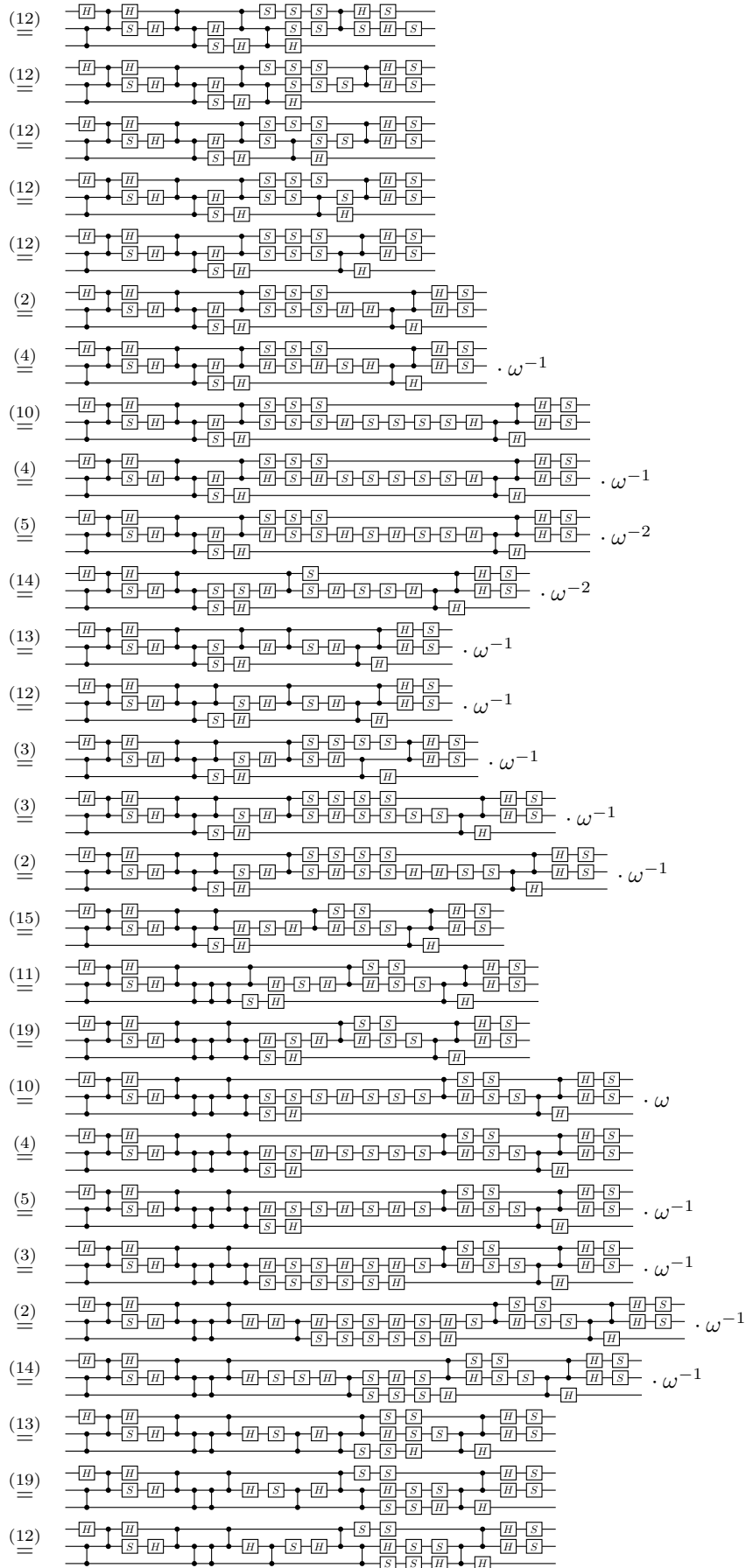
$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{D_3} \\ \boxed{D_2} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{D_2} \\ \boxed{D_3} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{S} \end{array} \begin{array}{c} \boxed{S} \\ \boxed{S} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{S} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{S} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{S} \end{array} \begin{array}{c} \boxed{H} \\ \boxed{S} \end{array} \cdot \omega^7$$

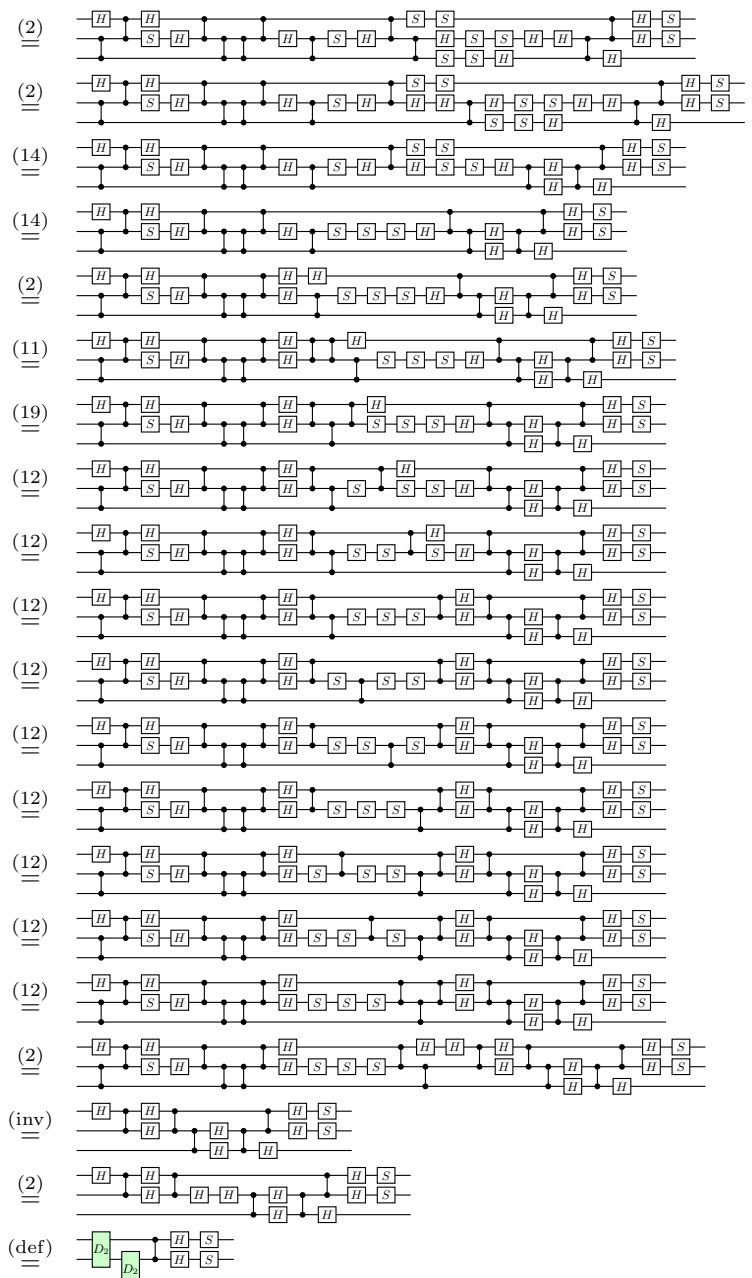




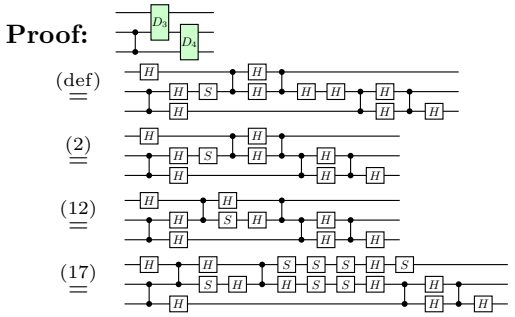


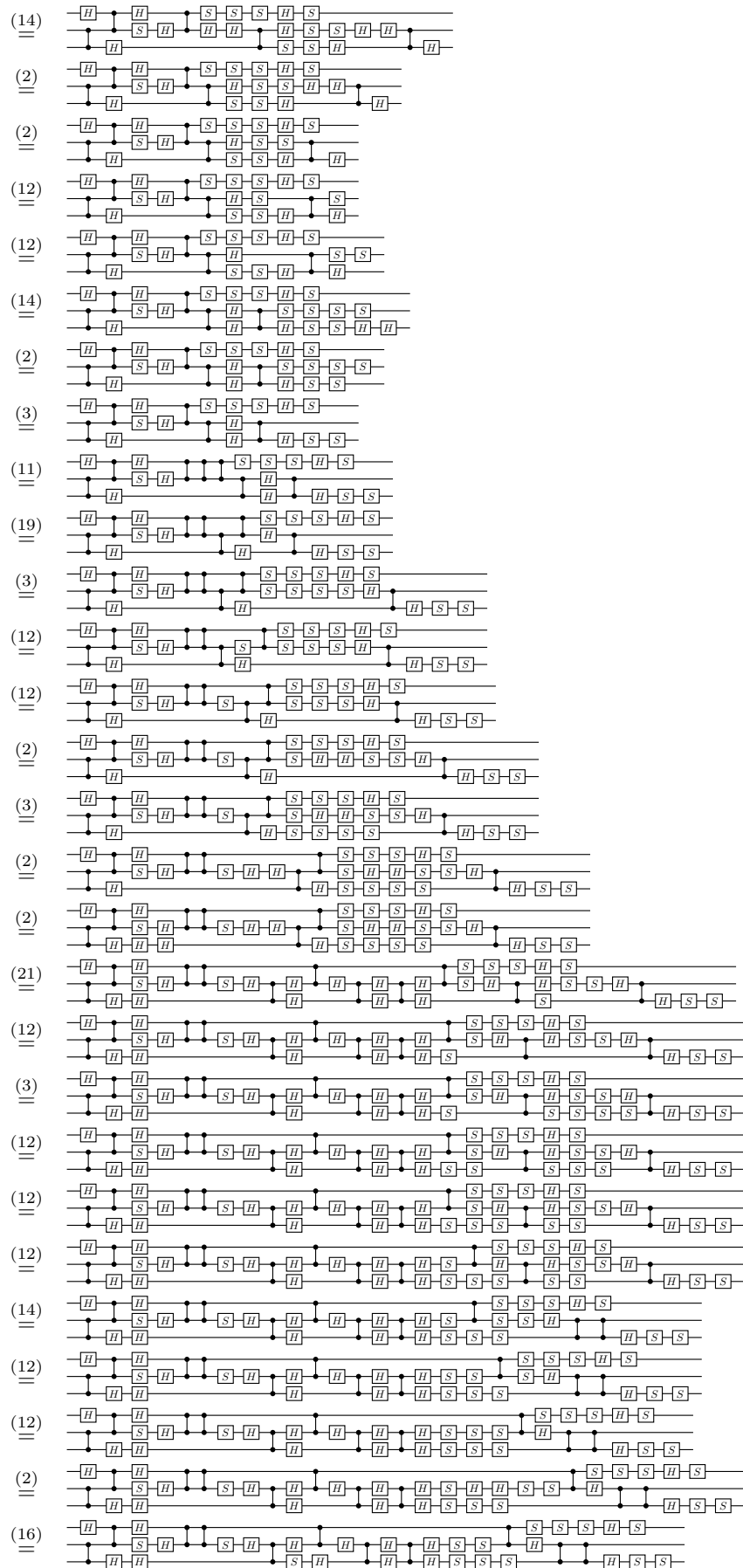


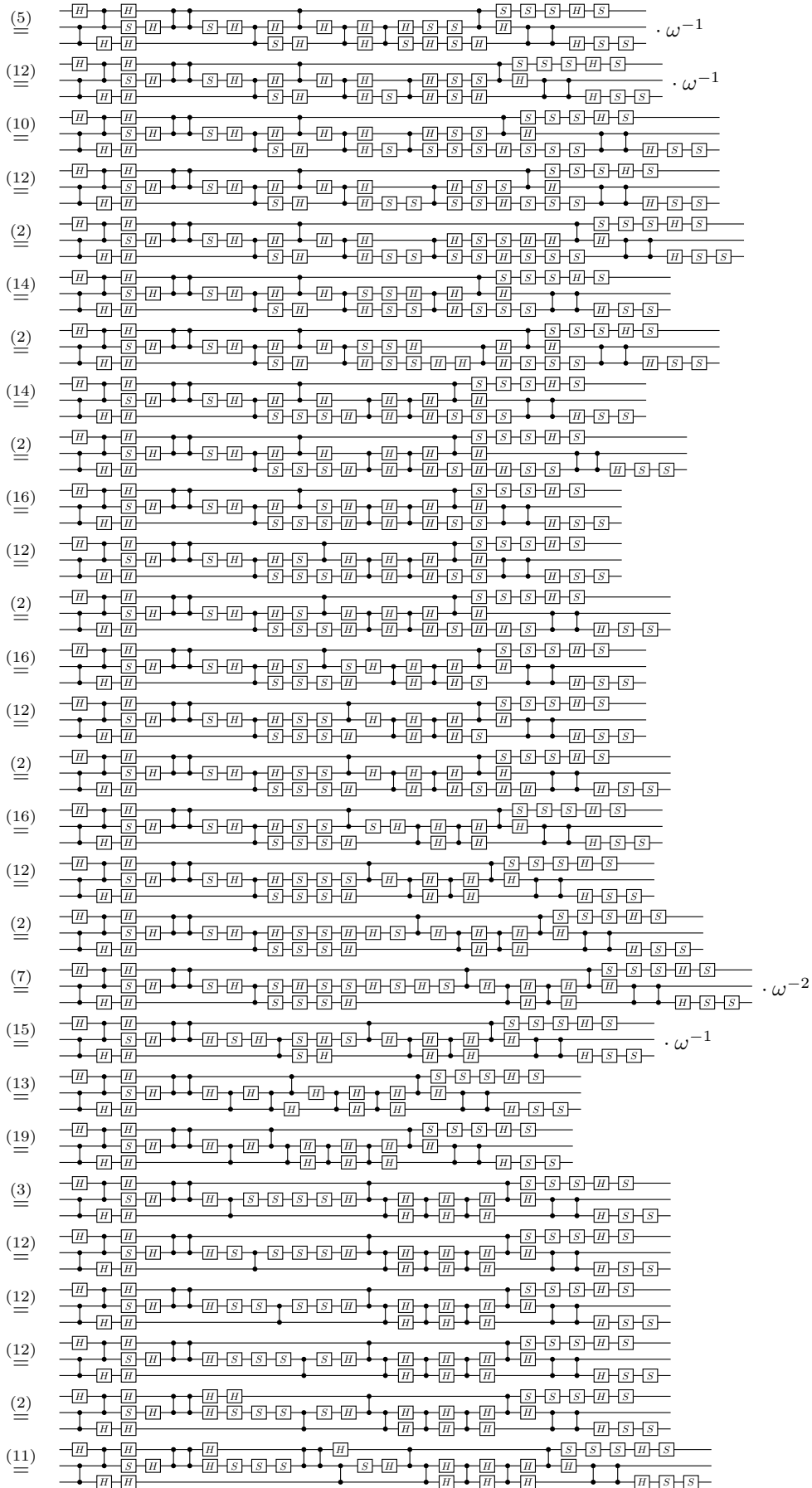




**3.97 Equation**



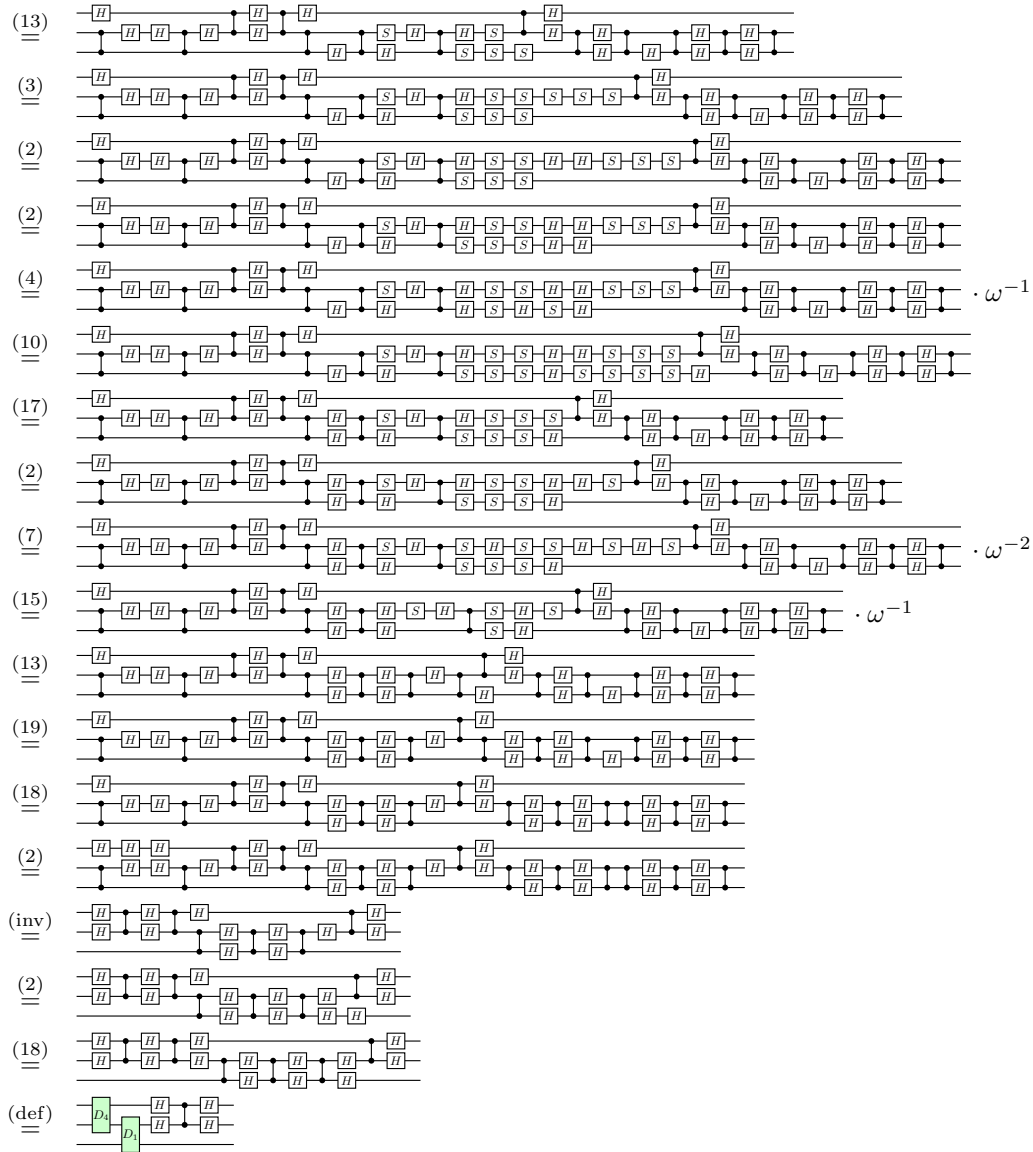








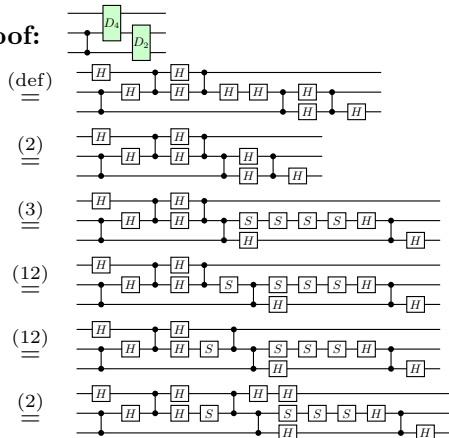


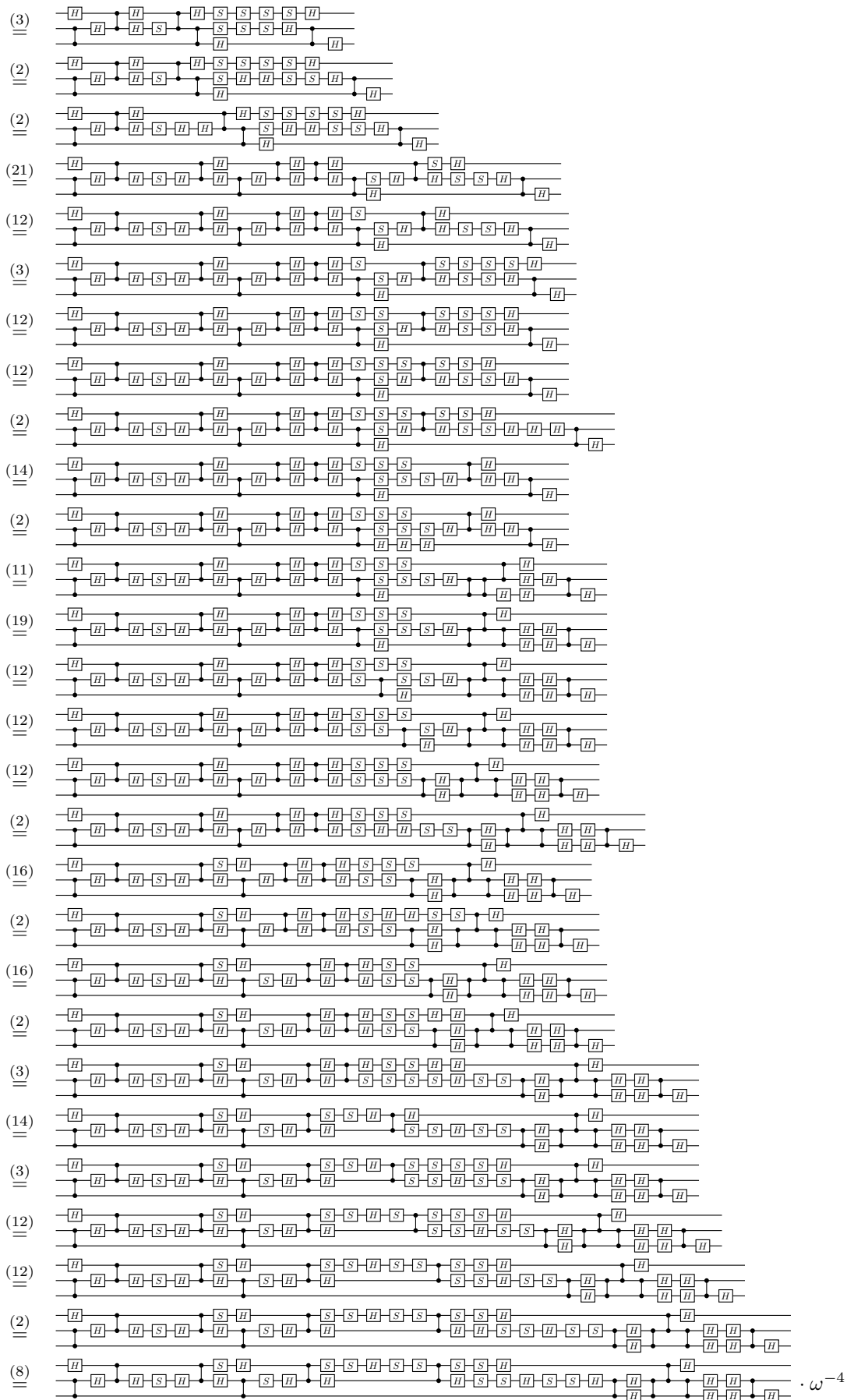


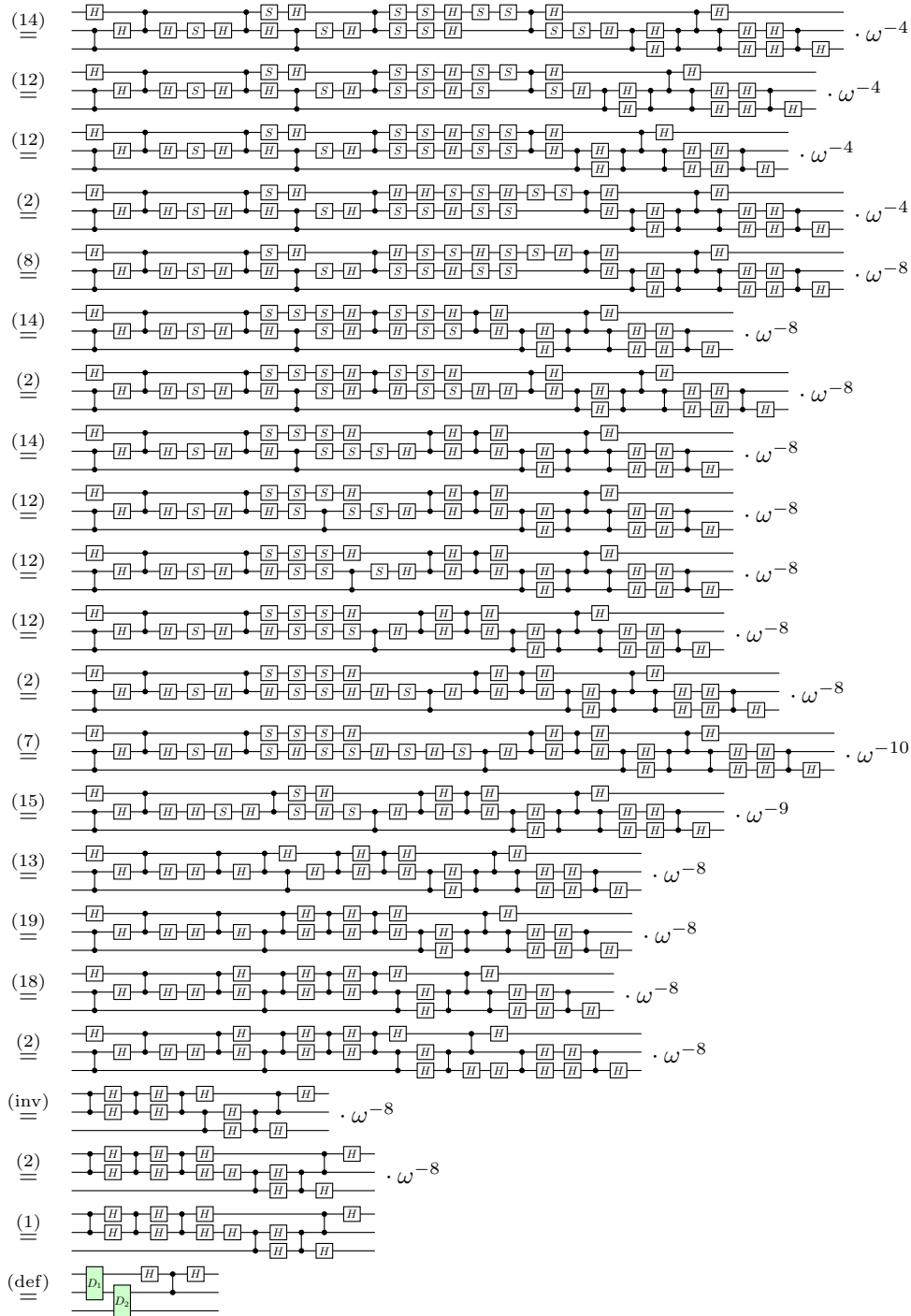
### 3.99 Equation

$$\text{Diagram 1} = \text{Diagram 2}$$

**Proof:**

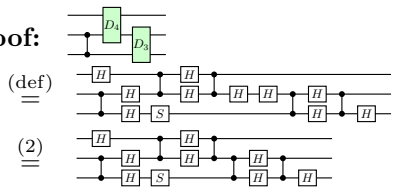


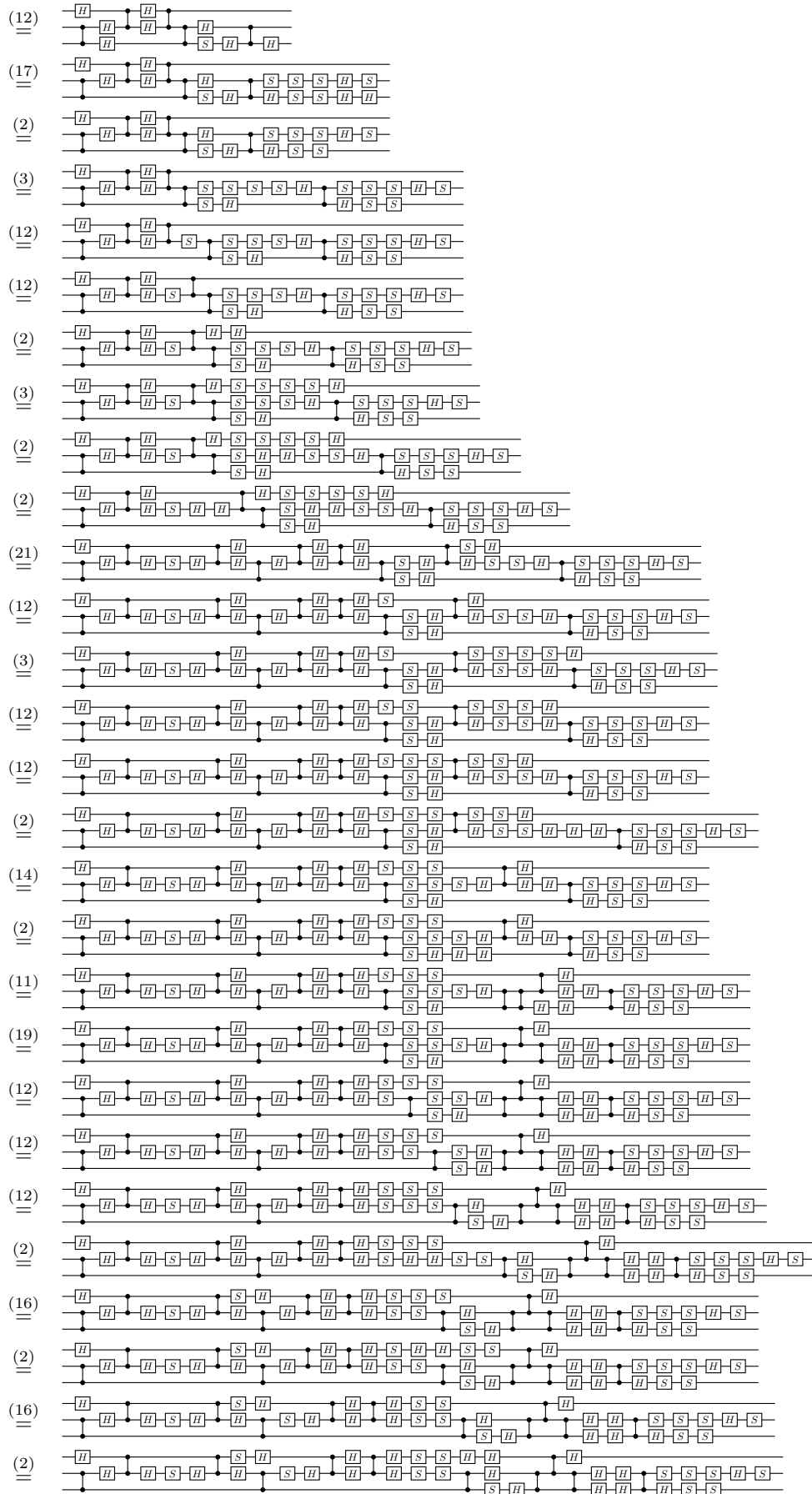


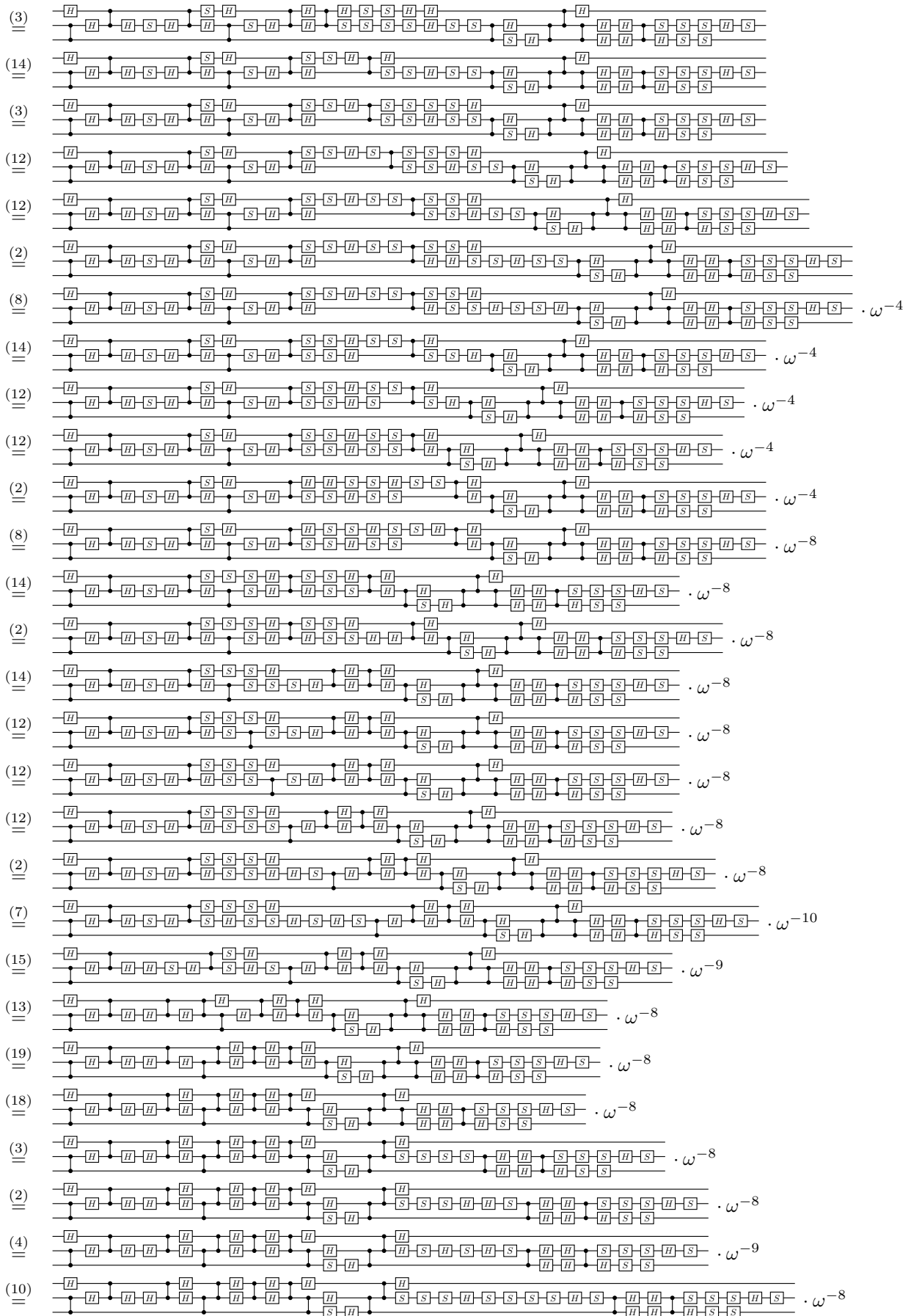


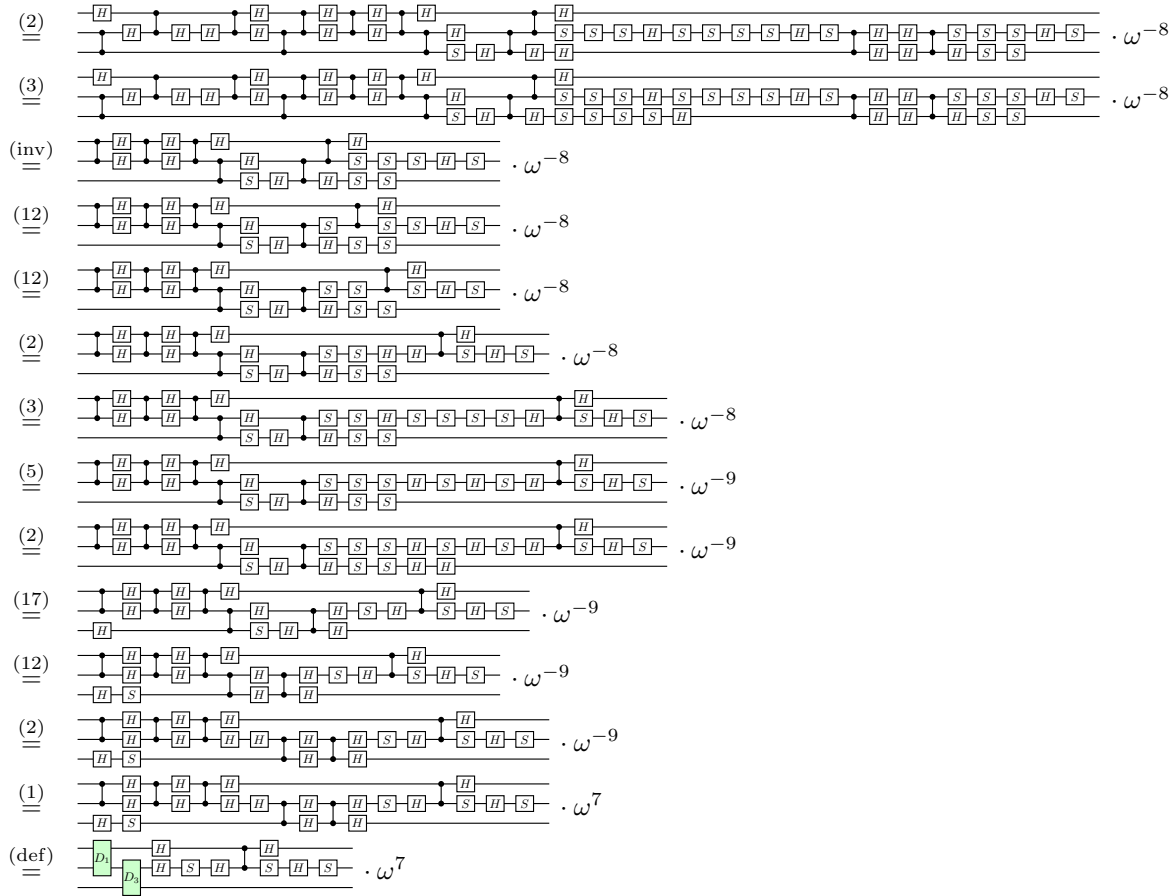
**3.100 Equation**

**Proof:**

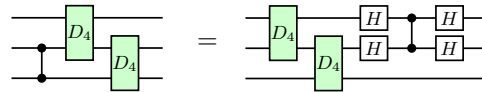








### 3.101 Equation



**Proof:**

