SQUARE PRINCIPLES WITH TAIL-END AGREEMENT

WILLIAM CHEN AND ITAY NEEMAN

ABSTRACT. This paper investigates the principles $\square^{\mathrm{ta}}_{\lambda,\delta}$, weakenings of $\square_\lambda$ which allow δ many clubs at each level but require them to agree on a tail-end. First, we prove that $\square^{\mathrm{ta}}_{\lambda,<\omega}$ implies $\square_\lambda$. Then, by forcing from a model with a measurable cardinal, we show that $\square_{\lambda,2}$ does not imply $\square^{\mathrm{ta}}_{\lambda,\delta}$ for regular λ, and $\square^{\mathrm{ta}}_{\delta^+,\delta}$ does not imply $\square_{\delta^+,<\delta}$. With a supercompact cardinal the former result can be extended to singular λ, and the latter can be improved to show that $\square^{\mathrm{ta}}_{\lambda,\delta}$ does not imply $\square_{\lambda,<\delta}$ for $\delta < \lambda$.

1. INTRODUCTION

Recently, Neeman [5] introduced the principles $\square^{\mathrm{ta}}_{\lambda,\delta}$ and $\square^{\mathrm{ta}}_{\lambda,<\delta}$, versions of Schimmerling's principles $\square_{\lambda,\delta}$ and $\square_{\lambda,<\delta}$ (see [6]) that require the clubs at each level of the sequence to agree on a tail-end. More precisely, for cardinals δ and λ, define a $\square^{\mathrm{ta}}_{\lambda,\delta}$ *sequence* to be a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda^+)\rangle$ such that for every $\alpha \in \mathrm{Lim}(\lambda^+)$,

(1) $\mathcal{C}_\alpha$ is a set of clubs of α, $1 \le |\mathcal{C}_\alpha| \le \delta$,
(2) for every $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) < \lambda$ if $\mathrm{cf}(\alpha) < \lambda$, and for every $\beta \in \mathrm{Lim}(C)$, $C \cap \beta \in \mathcal{C}_\beta$,
(3) for every $C, D \in \mathcal{C}_\alpha$ there exists $\beta < \alpha$ such that $C \setminus \beta = D \setminus \beta$.

The principle $\square^{\mathrm{ta}}_{\lambda,\delta}$ asserts the existence of a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence. We also define $\square^{\mathrm{ta}}_{\lambda,<\delta}$ asserting the existence of a sequence as above, except with $1 \le |\mathcal{C}_\alpha| < \delta$. $\square_{\lambda,\delta}$ and $\square_{\lambda,<\delta}$ are defined in the same way, but without the tail-end agreement condition (3).

Neeman observed that $\square^{\mathrm{ta}}_{\omega_1,\omega}$ is strong enough to carry out a construction of Shelah–Stanley [7] of a ω_2-Aronszajn tree which is not special (the construction originally used the principle $\square_{\omega_1}$). This is useful since $\square^{\mathrm{ta}}_{\omega_1,\omega}$ follows from certain higher analogues of the proper forcing axiom, but these analogues do not imply $\square_{\omega_1}$. $\square^{\mathrm{ta}}$ is strong enough to give some other consequences of $\square$. For example, it is not difficult to see that for any δ, $\square^{\mathrm{ta}}_{\lambda,\delta}$ implies that there is a nonreflecting stationary subset of λ^+, even though the weak square $\square_{\lambda,\lambda}$ does not.

Starting from a model with a Mahlo cardinal, Jensen [2] showed that $\square_{\lambda,\delta}$ does not imply $\square_{\lambda,\delta'}$, where $\delta' < \delta \le \lambda$ and λ is regular, and Cummings–Foreman–Magidor [1] extended the result to singular λ using a supercompact. Krueger and Schimmerling [3] showed $\square_{\lambda,\delta}$ does not imply $\square_{\lambda,<\delta}$ for $\delta \le \lambda$, and also achieved separation results involving partial square principles.

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It is natural to ask where the square principles with tail-end agreement fit into this picture. It is easy to see that $\square^{\text{ta}}_{\lambda,\delta}$ implies $\square_{\lambda,\delta}$, and $\square_\lambda$ implies $\square^{\text{ta}}_{\lambda,\delta}$. In Section 2, we will show that $\square^{\text{ta}}_{\lambda,<\omega}$ is actually equivalent to $\square_\lambda$. Section 3 proves $\square^{\text{ta}}_{\lambda,\delta}$ is not implied by $\square_{\lambda,2}$, and Section 4 proves that $\square^{\text{ta}}_{\lambda,\delta}$ does not imply $\square_{\lambda,<\delta}$ for $\delta < \lambda$. In particular, the principle $\square^{\text{ta}}_{\omega_1,\omega}$ considered in [5] is distinct from any of the square principles introduced in [6].

2. $\square^{\text{ta}}_{\lambda,<\omega}$ implies $\square_\lambda$

Suppose λ is an uncountable cardinal.

Theorem 1. *$\square^{\text{ta}}_{\lambda,<\omega}$ implies $\square_\lambda$.*

Proof. Let $\vec{\mathcal{C}}$ be a $\square^{\text{ta}}_{\lambda,<\omega}$ sequence.

For each $\alpha \in \text{Lim}(\lambda^+)$, set $\text{type}(\alpha) = |\mathcal{C}_\alpha|$. Define $g : \text{Lim}(\lambda^+) \to \lambda^+$ by $g(\alpha) =$ least β such that $\{C \setminus \beta : C \in \mathcal{C}_\alpha\}$ is a singleton (so $g(\alpha) = 0$ if $\text{type}(\alpha) = 1$). Let $D_\alpha = C \setminus \beta$ for some (any) $C \in \mathcal{C}_\alpha$. Call $\alpha \in \lambda^+$ *good* if $g \restriction \text{Lim}(D_\alpha)$ is bounded below α, and let $G \subseteq \lambda^+$ be the set of good points. Call $\alpha \in \lambda^+$ *bad* if there is $k < \omega$ with $g \restriction \{\beta \in \text{Lim}(D_\alpha) : \text{type}(\beta) = k\}$ unbounded below α, and let $B \subseteq \lambda^+$ be the set of bad points. Finally, call $\alpha \in \lambda^+$ *ugly* if it is neither good nor bad, i.e., $g \restriction \text{Lim}(D_\alpha)$ is unbounded in α but $g \restriction \{\beta \in \text{Lim}(D_\alpha) : \text{type}(\beta) = k\}$ is bounded below α for all $k < \omega$. Let $U \subseteq \lambda^+$ be the set of ugly points.

The first claim says that there are no ugly points of uncountable cofinality, allowing us to focus on good and bad points.

Claim 2.1. *If α is ugly, then $\text{cf}(\alpha) = \omega$.*

Proof. For each $k < \omega$, let $\alpha_k = \sup\{g(\beta) : \beta \in \text{Lim}(D_\alpha) \text{ and } \text{type}(\beta) = k\}$. Then since α is ugly, $\alpha_k < \alpha$ for every $k < \omega$ and $\sup\{\alpha_k : k < \omega\} = \alpha$. □

The next claim will be used frequently in the arguments that follow.

Claim 2.2. *Suppose $\alpha, \beta \in \text{Lim}(\lambda^+)$ and $\beta \in \text{Lim}(D_\alpha)$. Then $\text{type}(\alpha) \leq \text{type}(\beta)$ and $g(\alpha) \leq g(\beta)$. If furthermore $\text{type}(\alpha) = \text{type}(\beta)$, then $g(\alpha) = g(\beta)$.*

Proof. Since $\beta \in \text{Lim}(D_\alpha)$, $|\{C \cap \beta : C \in \mathcal{C}_\alpha\}| = \text{type}(\alpha)$. All of the clubs in this set must appear in $\mathcal{C}_\beta$, so $\text{type}(\alpha) \leq \text{type}(\beta)$ and $g(\alpha) \leq g(\beta)$. In case $\text{type}(\alpha) = \text{type}(\beta)$, $\mathcal{C}_\beta = \{C \cap \beta : C \in \mathcal{C}_\alpha\}$. □

Now we begin the analysis of good and bad points.

Claim 2.3. *If α is good, then all elements of $\text{Lim}(D_\alpha)$ above a bound for $g \restriction \text{Lim}(D_\alpha)$ are good. Furthermore, g is eventually constant on $\text{Lim}(D_\alpha)$.*

Proof. By coherence, if $\beta \in \text{Lim}(D_\alpha)$ is above a bound for $g \restriction \text{Lim}(D_\alpha)$, then all elements of $\text{Lim}(D_\beta)$ above that bound are also in $\text{Lim}(D_\alpha)$, proving the first part of the claim. By Claim 2.2 and coherence, if $\beta < \gamma$ both belong to $\text{Lim}(D_\alpha)$, and $\beta > g(\gamma)$, then $g(\beta) \geq g(\gamma)$. It follows that g is non-increasing on $\text{Lim}(D_\alpha)$ above a bound for $g \restriction \text{Lim}(D_\alpha)$, therefore it must be eventually constant. □

If α is bad, define $k_\alpha < \omega$ to be the least k such that $\{g(\beta) : \beta \in \text{Lim}(D_\alpha)$ and $\text{type}(\beta) = k\}$ is unbounded in α. Note $k_\alpha >$ *type*(α) by Claim 2.2. Define an increasing continuous sequence $\langle \alpha_\xi \rangle \subset \text{Lim}(D_\alpha)$ inductively. Set α_0 to be the least $\gamma \in \text{Lim}(D_\alpha)$ with $\text{type}(\gamma) = k_\alpha$, and $\alpha_{\xi+1} =$ the least $\gamma \in \text{Lim}(D_\alpha)$ with $g(\gamma) > \alpha_\xi$

and $\text{type}(\gamma) = k_\alpha$. Note that $g(\alpha_0) \geq g(\alpha)$ by Claim 2.2. Let E_α be the range of this sequence.

The following claim can be thought of as a version of Claim 2.3 for bad points.

Claim 2.4. *Suppose α is bad. Then E_α is closed unbounded in α, and every point of $\text{Lim}(E_\alpha)$ is bad and has type less than k_α. Furthermore, g is eventually constant on points of $\text{Lim}(D_\alpha)$ of type $< k_\alpha$.*

Proof. That E_α is closed unbounded in α follows immediately from the choice of k_α and the construction of E_α. For the rest of the claim, observe that for every limit $\rho < \text{ot}(E_\alpha)$, the sequence $\langle g(\alpha_\xi) : \xi < \rho \rangle$ is unbounded in α_ρ, and by coherence, a tail of $\langle \alpha_\xi : \xi < \rho \rangle$ is contained in $\text{Lim}(D_{\alpha_\rho})$. Therefore α_ρ is bad. Furthermore, the $\alpha_{\xi+1}$ are each by definition of type k_α. By Claim 2.2, $\text{type}(\alpha_\rho) \leq k_\alpha$, and $\text{type}(\alpha_\rho) \neq k_\alpha$ otherwise $\langle g(\alpha_\xi) : \xi < \rho \rangle$ could not be unbounded in α_ρ.

The second part of the claim is proved similarly as Claim 2.3, working above a bound for g restricted to points of $\text{Lim}(D_\alpha)$ of type $< k_\alpha$ (which can be taken below α by minimality of k_α). □

If α is bad, then every $\beta \in \text{Lim}(E_\alpha)$ is bad, so k_β is defined. The next claim shows that above a certain bound, $k_\beta = k_\alpha$, and gives a weak coherence property between E_α and E_β which will be useful later in our construction.

Claim 2.5. *Suppose α is bad, and let $\alpha' < \alpha$ be such that g is constant on points of $\text{Lim}(D_\alpha)$ of type $< k_\alpha$ which are greater than α'. Then for all $\beta \in \text{Lim}(E_\alpha) \setminus \alpha'$ we have $k_\beta = k_\alpha$ and $E_\alpha \cap (g(\beta), \beta) = E_\beta$.*

Proof. Let $\beta \in \text{Lim}(E_\alpha) \setminus \alpha'$, say $\beta = \alpha_\rho$ for a limit ordinal ρ. By Claim 2.4, β is bad. Moreover, $k_\beta \geq k_\alpha$ since D_β and $D_\alpha \cap \beta$ are equal on a tail-end below β, and so $\{g(\gamma) : \gamma \in \text{Lim}(D_\beta) \text{ and } \text{type}(\gamma) < k_\alpha\}$ must be bounded below β by the assumptions on α'. The reverse inequality $k_\beta \leq k_\alpha$ is witnessed by $\{\alpha_{\xi+1} : \xi < \rho\}$, which are all of type k_α by the construction.

By Claim 2.2, any $\gamma \in \text{Lim}(D_\beta)$ has $g(\gamma) \geq g(\beta)$. By coherence, $D_\beta = D_\alpha \cap [g(\beta), \beta)$. It follows that β_0 is the least γ in E_α above $g(\beta)$ (where β_0 is the least member of E_β). Now E_α and E_β are defined in the same way above β_0 by the coherence of $\vec{\mathcal{C}}$. □

Extend the definition of E_α to all of $\text{Lim}(\lambda^+)$ by setting $E_\alpha = \text{Lim}(D_\alpha)$ if α is good and $\text{ot}(D_\alpha)$ is a limit of limit ordinals, and E_α to be any sequence of order-type ω cofinal in α if α is ugly or $\text{ot}(D_\alpha) = \rho + \omega$ for some ordinal ρ.

We will define a function $h : \text{Lim}(\lambda^+) \to \lambda^+$. If α is good and $\text{ot}(D_\alpha)$ is a limit of limits, set $h(\alpha)$ to be the least $\gamma \in \text{Lim}(D_\alpha)$ such that g is constant on $\text{Lim}(D_\alpha) \setminus \gamma$. If α is bad, set $h(\alpha)$ to be the least $\gamma \in \text{Lim}(D_\alpha)$ such that g is constant on those points of $(\text{Lim}(D_\alpha) \setminus \gamma)$ with type $< k_\alpha$. Otherwise, set $h(\alpha) = g(\alpha)$.

Finally, define $F_\alpha = E_\alpha \setminus h(\alpha)$ for each $\alpha \in \text{Lim}(\lambda^+)$. We check that $\langle F_\alpha \rangle$ is a $\square_\lambda$ sequence.

Claim 2.6. *For any $\alpha \in \text{Lim}(\lambda^+)$, $g(\alpha) \leq h(\alpha) < \alpha$, and for any $\beta \in \text{Lim}(E_\alpha \setminus h(\alpha))$, we have $h(\beta) = h(\alpha)$.*

Proof. The value of $h(\alpha)$ is either a point of D_α or just $g(\alpha)$, so $g(\alpha) \leq h(\alpha)$. The inequality $h(\alpha) < \alpha$ follows from Claim 2.3 or Claim 2.4, depending on the case.

Now we prove the second part of the claim. Suppose α is good and $\text{ot}(D_\alpha)$ is a limit of limits. Then by definition of $h(\alpha)$ and the fact that $\beta \in \text{Lim}(E_\alpha) =$

$\mathrm{Lim}(\mathrm{Lim}(D_\alpha))$, β is also good and $\mathrm{ot}(D_\beta)$ is a limit of limits. Above $h(\alpha)$, g is constant on D_β with the eventual constant value of g on D_α. This value is also equal to $g(\beta)$, and by Claim 2.2, $g(\alpha) \leq g(\beta)$ so

$$D_\alpha \cap (g(\beta), \beta) = D_\beta. \tag{2.1}$$

The ordinal $h(\alpha)$ is defined to be an element of $\mathrm{Lim}(D_\alpha)$ with $g(h(\alpha)) = g(\beta)$, so in particular $h(\alpha) > g(\beta)$. Together with (2.1), this implies that $h(\beta)$ is computed using the same values as $h(\alpha)$, since $g(\alpha) \leq g(\beta) < h(\alpha) < \beta$. We conclude that $h(\beta) = h(\alpha)$.

The case where α is bad is similar: by Claim 2.5, β is bad with $k_\beta = k_\alpha$. Above $h(\alpha)$, g is constant on points of D_β of type $< k_\alpha$ with the eventual constant value of g on points of type $< k_\alpha$ in D_α. This value is also equal to $g(\beta)$ since $\mathrm{type}(\beta) < k_\alpha$ by Claim 2.4. By Claim 2.2, $D_\alpha \cap (g(\beta), \beta) = D_\beta$, so $h(\beta)$ is computed using the same values as $h(\alpha)$, and $h(\beta) = h(\alpha)$.

The claim is vacuously true for the remaining cases. □

Suppose $\alpha \in \mathrm{Lim}(\lambda^+)$ and $\beta \in \mathrm{Lim}(F_\alpha)$. If α is good, then β is also good and using the fact that $g(\alpha), g(\beta) \leq h(\beta)$ we have

$$F_\beta = \mathrm{Lim}(D_\beta) \setminus h(\beta) = (\mathrm{Lim}(D_\alpha) \cap \beta) \setminus h(\alpha) = F_\alpha \cap \beta.$$

Similarly, if α is bad then β is bad and we have

$$F_\beta = E_\beta \setminus h(\beta) = (E_\alpha \cap \beta) \setminus h(\alpha) = F_\alpha \cap \beta.$$

Here we used Claim 2.5 for the middle equality. □

3. $\square_{\lambda,2}$ does not imply $\square^{\mathrm{ta}}_{\lambda,\delta}$

Now we turn to separating $\square^{\mathrm{ta}}_{\lambda,\delta}$ from the hierarchy of principles $\square_{\lambda,\delta'}$ for various δ'. The methods we use, and the general structure of the proof, are similar to these used by [1] and [4] to separate square principles, and trace back to work of Jensen [2]. In this section we prove:

Theorem 2. *Suppose λ is an uncountable regular cardinal. If there is a measurable cardinal $\kappa > \lambda$, then there is a forcing extension preserving cardinals $\leq \lambda$ and $\geq \kappa$ in which $\square_{\lambda,2}$ holds and $\square^{\mathrm{ta}}_{\lambda,\delta}$ fails for all δ.*

Proof. Let $\mathbb{P}$ the Levy collapse $\mathrm{Col}(\lambda, <\kappa)$. For this section, let $\mathbb{Q}$ be the poset in $V^{\mathbb{P}}$ forcing a $\square_{\lambda,2}$-sequence using initial segments. More precisely, $\mathbb{Q}$ is the poset of all functions q ordered by end-extension such that in $V^{\mathbb{P}}$,

1. $\mathrm{dom}(q) = \mathrm{Lim}(\lambda^+) \cap (\alpha + 1)$ for some limit ordinal $\alpha < \lambda^+$.
2. For all $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a set of closed unbounded subsets of β of order type $\leq \lambda$, and $1 \leq |q(\beta)| \leq 2$.
3. For all $\beta \in \mathrm{dom}(q)$, if $C \in q(\beta)$ and $\gamma \in \mathrm{Lim}(C)$, then $C \cap \gamma \in q(\gamma)$.

In $V^{\mathbb{P}*\mathbb{Q}}$ let $\vec{\mathcal{C}}$ be the $\square_{\lambda,2}$ sequence added by $\mathbb{Q}$. Define $\mathbb{R}$ in $V^{\mathbb{P}*\mathbb{Q}}$ to be the poset of closed, bounded subsets $c \subseteq \kappa$ with the property that $c \cap \beta \in \mathcal{C}_\beta$ for any $\beta \in \mathrm{Lim}(c)$, ordered by end-extension. $\mathbb{R}$ adds a *thread* of $\vec{\mathcal{C}}$, i.e., a closed unbounded set $S \subseteq (\lambda^+)^{V^{\mathbb{P}*\mathbb{Q}}} = \kappa$ such that $S \cap \beta \in \mathcal{C}_\beta$ for all $\beta \in \mathrm{Lim}(S)$.

Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. We collect some useful facts about the various posets and their interactions with the embedding; proofs can be found in [4].

Fact 3.1. Let G, H, I be generics for $\mathbb{P}, \mathbb{Q}, \mathbb{R}$, respectively.

- In $V[G]$, $\mathbb{Q}$ is κ-distributive, and the set of *flat* conditions $\{(q, \check{r}) \in \mathbb{Q} * \mathbb{R} : r \in V[G] \text{ and } \max(\mathrm{dom}(q)) = \max(r)\}$ is dense and λ-closed. The condition $(q, \check{r})$ will be denoted as (q, r) for simplicity.
- $j(\mathbb{P}) = \mathrm{Col}(\lambda, < j(\kappa))$ and there is a complete embedding of $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ into $j(\mathbb{P})$ with λ-closed quotient forcing,
- letting J be generic for $j(\mathbb{P})/\mathbb{P} * \mathbb{Q} * \mathbb{R}$, there is a K generic for $j(\mathbb{Q})$ so that j can be extended to an elementary embedding $j : V[G * H] \to M[G * H * I * J * K]$ in the extension by $j(\mathbb{P} * \mathbb{Q})$.

In particular, all of the models we consider have the same $< \lambda$-sequences of ordinals.

We will show that $V[G * H]$ is a model satisfying the conclusion of the theorem. Clearly $\square_{\lambda,2}$ holds in $V[G * H]$, so assume towards a contradiction that $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha : \alpha < \kappa \rangle$ is a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence in $V[G * H]$ for some δ. Let $T \in j(\vec{\mathcal{D}})_\kappa$, so T threads $\vec{\mathcal{D}}$ in $V[G * H * I * J * K]$. Since $j(\mathbb{Q})$ is $j(\kappa)$-distributive in $M[G * H * I * J]$, T must be a member of $M[G * H * I * J]$, and hence also $V[G * H * I * J]$.

Lemma 3.2. *Suppose $V \subseteq W$ are models of set theory, λ is an uncountable cardinal in V, and $V \models$ "$\vec{\mathcal{D}}$ is a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence" for some δ. Then forcing with a countably closed poset $\mathbb{S}$ over W cannot add a new thread to $\vec{\mathcal{D}}$ (i.e., a thread not already in W).*

Proof. Assume towards a contradiction that $\dot{E}$ is an $\mathbb{S}$-name for a thread through $\vec{\mathcal{D}}$ which is forced to not be in W. Under this assumption, $(\lambda^+)^V$ has uncountable cofinality in W.

Claim 3.3. *For any $\alpha < (\lambda^+)^V$, and $s_0, s_1 \in \mathbb{S}$, there are $\beta > \alpha$ and $s_0' \leq s_0, s_1' \leq s_1$ deciding "$\beta \in \dot{E}$" differently.*

Suppose that s_0, s_1, and α witness that this fails. Let $J_0 \times J_1$ be generic for $\mathbb{S} \times \mathbb{S}$ over W such that $(s_0, s_1) \in J_0 \times J_1$. Then $\dot{E}[J_0]$ and $\dot{E}[J_1]$ have the same tail-end above α, and since their proper initial segments belong to W it follows that both belong to each of $W[J_0]$ and $W[J_1]$, and hence also W. This proves the claim.

Using the claim, we will recursively construct $s^i_j \in \mathbb{S}$ and ordinals $\alpha^i_j, \beta_j < \lambda^+$ for $i \in \{0, 1\}$ and $j < \omega$ satisfying the following properties:

- $s^i_{j+1} \leq s^i_j$ and $\alpha^0_j < \alpha^1_j < \beta_j < \alpha^0_{j+1}$,
- s^0_j and s^1_j decide $\beta_j \in \dot{E}$ differently,
- $s^i_{j+1} \Vdash \alpha^i_{j+1} \in \dot{E}$.

By countable closure of $\mathbb{S}$, let s^0 be a lower bound for $\langle s^0_j : j < \omega \rangle$, s^1 be a lower bound for $\langle s^1_j : j < \omega \rangle$, and $\beta^* = \sup\{\beta_j : j < \omega\}$. Note that $\beta^* < (\lambda^+)^V$, since $(\lambda^+)^V$ has uncountable cofinality in W. The values for $\dot{E}$ forced by s^0 and s^1 both have β^* as a limit point, but for each $j < \omega$ they disagree on whether $\beta_j \in \dot{E}$. Since $\{\beta_j : j < \omega\}$ is cofinal in β^* and $\dot{E}$ is forced to be a thread, this contradicts the tail-end agreement condition for $\vec{\mathcal{D}}$. □

By Lemma 3.2, T must be a member of $V[G * H * I]$. For the remainder of the proof, work in $V[G]$ and let $\dot{T}$ be a $\mathbb{Q} * \mathbb{R}$-name for T. Note that a $\square_{\lambda,\delta}$ sequence in $V[G * H]$ cannot be threaded in $V[G * H]$ since all initial segments of the thread are initial segments of some C_α and thus have order-type $< \lambda$. Hence $T \notin V[G*H]$ and we get the following claim:

Claim 3.4. *For any $q \in \mathbb{Q}$, $r_0, r_1 \in \mathbb{R}$, $\alpha < \lambda^+$, there are $\beta > \alpha, q' \leq q, r_0' \leq r_0, r_1' \leq r_1$ such that (q', r_0') and (q', r_1') decide "$\beta \in \dot{T}$" differently.*

Proof. Suppose that q, r_0, r_1, and α witness that the claim fails. Modifying H if necessary, we may assume $q \in H$. Working over $V[G*H]$, the argument proceeds as in the proof of Claim 3.3. □

Let $(q, r) \in \mathbb{Q} * \mathbb{R}$ force that $\dot{T}$ threads $\vec{\mathcal{D}}$. Using Claim 3.4 and the fact that $\dot{T}$ is forced to be unbounded in λ^+, recursively construct flat conditions $(q_j, r_j^i) \in \mathbb{Q} * \mathbb{R}$ and ordinals $\alpha_j^i, \beta_j < \lambda^+$ for $i \in \{0, 1\}$ and $j < \omega$ satisfying the following properties:

- $(q_j, r_j^i) \leq (q, r)$,
- $(q_{j+1}, r_{j+1}^i) \leq (q_j, r_j^i)$, and $\alpha_j^0 < \alpha_j^1 < \beta_j < \alpha_{j+1}^0$,
- (q_j, r_j^0) and (q_j, r_j^1) decide $\beta_j \in \dot{T}$ differently,
- $(q_{j+1}, r_{j+1}^i) \Vdash \alpha_{j+1}^i \in \dot{T}$.

Now let $\gamma^* = \sup\{\max \mathrm{dom}(q_j) : j < \omega\}$ and $\alpha^* = \sup\{\beta_j : j < \omega\}$. Define

$$\hat{r}^i = \bigcup\{r_j^i : j < \omega\} \cup \{\gamma^*\} \text{ for } i \in \{0, 1\},$$

$$\hat{q} = \bigcup\{q_j^i : j < \omega\} \cup \{(\gamma^*, \{\hat{r}^0 \cap \gamma^*, \hat{r}^1 \cap \gamma^*\})\}.$$

By the flatness we have maintained during the construction, we have for each $i \in \{0, 1\}$ that $\gamma^* = \sup\{\max \mathrm{dom}(r_j^i) : j < \omega\}$, so each $(\hat{q}, \hat{r}^i)$ is a condition in $\mathbb{Q} * \mathbb{R}$.

We can find $q^* \leq \hat{q}$ which decides the value of $\mathcal{D}_{\alpha^*}$, since no new subsets of $V[G]$ of size $< \lambda$ are added by $\mathbb{Q}$. For each $i \in \{0, 1\}$, $(q^*, \hat{r}^i) \Vdash$ "α^* is a limit point of $\dot{T}$" so $(q^*, \hat{r}^i) \Vdash \dot{T} \cap \alpha^* \in \mathcal{D}_{\alpha^*}$. But the values for $\dot{T}$ forced by $(q^*, \hat{r}^0)$ and $(q^*, \hat{r}^1)$ disagree on whether $\beta_j \in \dot{T}$, for each $j < \omega$. Since $\{\beta_j : j < \omega\}$ is cofinal in α^*, this contradicts the tail-end agreement condition for $\vec{\mathcal{D}}$. □

Starting with a supercompact cardinal instead of a measurable, we can get a version of Theorem 2 that applies to singular λ. This adapts the argument of Theorem 2 using ideas from Section 7 of [1].

Theorem 3. *Suppose λ is an infinite cardinal, μ is an uncountable regular cardinal $< \lambda$, and κ is a supercompact cardinal with $\mu < \kappa \leq \lambda$. Then there is a forcing extension preserving cardinals in $[0, \mu^+] \cup [\kappa, \lambda^+]$ in which $\square_{\lambda,2}$ holds and $\square_{\lambda,\delta}^{\mathrm{ta}}$ fails for all δ.*

Proof. We provide a rough sketch of the proof. Let $\mathbb{P} = \mathrm{Col}(\mu, < \kappa)$. Let $\mathbb{Q}$ be the poset defined in $V^{\mathbb{P}}$ forcing a $\square_{\lambda,2}$ sequence using initial segments, and let $\vec{\mathcal{C}}$ be the $\square_{\lambda,2}$ sequence added by $\mathbb{Q}$. Let $\mathbb{R}$ be the poset adding a thread through $\vec{\mathcal{C}}$ by closed initial segments of order-type $< \mu$.

If G is generic for $\mathbb{P}$ and H is generic for $\mathbb{Q}$, we claim that $V[G*H]$ is a model satisfying the conclusion of the theorem. Suppose for a contradiction that $\vec{\mathcal{D}}$ is a $\square_{\lambda,\delta}^{\mathrm{ta}}$ sequence in $V[G*H]$ for some δ. With $j : V \to M$ a 2^λ-supercompactness embedding, it can be shown that there is some forcing extension of $V[G*H]$ by $j(\mathbb{P} * \mathbb{Q})/(G * H)$ in which j can be extended to $V[G*H]$.

If $\vec{\mathcal{D}}$ is a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence, then define $\vec{\mathcal{D}}^{-\xi}$ by $\mathcal{D}^{-\xi}_\alpha = \mathcal{D}_\alpha$ if $\alpha \leq \xi$, and $\mathcal{D}^{-\xi}_\alpha = \{C \setminus \xi : C \in \mathcal{D}_\alpha\}$ if $\alpha > \xi$. It is straightforward to check that $\vec{\mathcal{D}}^{-\xi}$ is still a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence.

Claim 3.5. *If $\gamma = \sup j``\lambda^+$ and $A \in j(\vec{\mathcal{D}})_\gamma$, then there exists $\xi < \lambda^+$ such that $T = \{\alpha \in \lambda^+ \setminus (\xi+1) : j(\alpha) \in \mathrm{Lim}(A)\}$ generates a thread through $\vec{\mathcal{D}}^{-\xi}$.*

Since j is continuous at points of countable cofinality, $j``\lambda^+$ is an $\omega$-club subset of $\gamma$ and hence $\mathrm{Lim}(A) \cap j``\lambda^+$ is stationary in γ. The set $S = \{\alpha \in \lambda^+ : j(\alpha) \in \mathrm{Lim}(A)\}$ is unbounded in λ^+ since it is the pointwise j-preimage of $\mathrm{Lim}(A) \cap j``\lambda^+$. If $\alpha \in S$ then $A \cap j(\alpha) \in j(\vec{\mathcal{D}})_{j(\alpha)} = j(\mathcal{D}_\alpha)$. Let $\zeta(\alpha)$ be least so that there is $D_\alpha \in \mathcal{D}_\alpha$ such that $A \cap j(\alpha) \setminus \zeta(\alpha) = j(D_\alpha) \setminus \zeta(\alpha)$; by tail-end agreement for $j(\mathcal{D}_\alpha)$, $\zeta(\alpha) < j(\alpha)$. By Fodor's lemma, there is a stationary $B \subseteq \mathrm{Lim}(A) \cap j``\lambda^+$ and $\zeta_0 < \gamma$ so that $\zeta(\alpha) < \zeta_0$ if $j(\alpha) \in B$. Let $\xi < \lambda^+$ be such that $j(\xi) > \zeta_0$.

If $\beta < \alpha$ are in $T = S \setminus (\xi+1)$, then $j(D_\alpha) \cap j(\beta) \setminus j(\xi) = A \cap j(\beta) \setminus j(\xi) = j(D_\beta) \setminus j(\xi)$. By elementarity, $\beta \in \mathrm{Lim}(D_\alpha)$ and $D_\beta \setminus \xi = D_\alpha \cap \beta \setminus \xi$, so $\bigcup_{\alpha \in T} D_\alpha \setminus \xi$ threads $\vec{\mathcal{D}}^{-\xi}$. This proves the claim.

By the claim, replacing $\vec{\mathcal{D}}$ with $\vec{\mathcal{D}}^{-\xi}$, we may assume that $\{\alpha \in \lambda^+ \setminus (\xi+1) : j(\alpha) \in \mathrm{Lim}(A)\}$ generates a thread through $\vec{\mathcal{D}}$. The poset $\mathbb{R}$ collapses λ^+ to μ and can be absorbed into $j(\mathbb{P}*\mathbb{Q})/(G*H)$. As in the proof of the previous theorem it can be shown that the thread through $\vec{\mathcal{D}}$ in the extension of $V[G*H]$ by $j(\mathbb{P}*\mathbb{Q})/(G*H)$ must be added by $\mathbb{R}$, and that this leads to a contradiction. □

Considering large δ, all of the principles $\square^{\mathrm{ta}}_{\lambda,\delta}$ with $\delta \geq \lambda^+$ are equivalent. This can be easily seen by taking a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence $\vec{\mathcal{C}}$, and for each $\alpha \in \mathrm{Lim}(\lambda)$ fixing a particular $C_\alpha \in \mathcal{C}_\alpha$. Then define a $\square^{\mathrm{ta}}_{\lambda,\lambda^+}$ sequence $\vec{\mathcal{D}}$ by $\mathcal{D}_\beta = \{C_\alpha \cap \beta : \beta \in \mathrm{Lim}(C_\alpha)\}$. If $\lambda^{<\lambda} = \lambda$, then $|\mathcal{D}_\alpha| \leq \lambda$ for $\alpha < \lambda^+$ of cofinality $< \lambda$ (and $|\mathcal{D}_\alpha| = 1$ for α of cofinality λ), so $\square^{\mathrm{ta}}_{\lambda,\lambda^+}$ and $\square^{\mathrm{ta}}_{\lambda,\lambda}$ are also equivalent in this case.

This argument repeated with clubs not having to agree on a tail-end shows that $\square_{\lambda,\lambda^+}$ is just outright true; however, Theorem 2 shows that with a measurable cardinal, even $\square_{\lambda,2}$ does not imply $\square^{\mathrm{ta}}_{\lambda,\lambda^+}$.

4. $\square^{\mathrm{ta}}_{\lambda,\delta}$ does not imply $\square_{\lambda,<\delta}$

We will now show that $\square^{\mathrm{ta}}_{\lambda,\delta}$ does not imply $\square_{\lambda,<\delta}$ for certain $\delta < \lambda$. Using a measurable cardinal, we will show:

Theorem 4. *If δ is an infinite cardinal and there is a measurable cardinal $\kappa > \delta$, then there is a forcing extension preserving cardinals $\leq \delta^+$ and cardinals $\geq \kappa$ in which $\square^{\mathrm{ta}}_{\delta^+,\delta}$ holds and $\square_{\delta^+,<\delta}$ fails.*

Strengthening the large cardinal hypothesis to a supercompact cardinal, we can obtain:

Theorem 5. *Suppose $\delta < \lambda$ are infinite cardinals and there is a supercompact cardinal κ with $\delta < \kappa \leq \lambda$. Then there is a forcing extension preserving cardinals in $[0,\delta^+] \cup [\kappa,\lambda^+]$ in which $\square^{\mathrm{ta}}_{\lambda,\delta}$ holds and $\square_{\lambda,<\delta}$ fails.*

Theorem 5 does not apply when $\delta = \lambda$. If λ is regular and not inaccessible, then Theorem 4 can be extended to this case.

Theorem 6. *Suppose λ is an uncountable regular cardinal, λ is not strongly inaccessible, and there is a measurable cardinal $\kappa > \lambda$. Then there is a forcing extension preserving cardinals $\leq \lambda$ and cardinals $\geq \kappa$ in which $\square^{\mathrm{ta}}_{\lambda,\lambda}$ holds and $\square_{\lambda,<\lambda}$ fails.*

Proof of Theorem 4. Let $\lambda = \delta^+$. We will force to add a $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence with a certain extra property, and show that in the extension $\square_{\lambda,<\delta}$ fails. Let $\mathbb{P} = \mathrm{Col}(\lambda, <\kappa)$ be the Levy collapse as in the last section, and $\mathbb{Q}$ be the poset defined in $V^{\mathbb{P}}$ of all functions q ordered by end-extension such that

(i) $\mathrm{dom}(q) = \mathrm{Lim}(\lambda^+) \cap (\alpha + 1)$ for some limit ordinal $\alpha < \lambda^+$.
(ii) For all $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a set of closed unbounded subsets of β of order type $\leq \lambda$, and $1 \leq |q(\beta)| \leq \delta$.
(iii) If $C \in q(\beta)$ and $\gamma \in \mathrm{Lim}(C)$, then $C \cap \gamma \in q(\gamma)$.
(iv) For every $C, D \in q(\beta)$ there exists $\beta < \alpha$ such that $C \setminus \beta = D \setminus \beta$.
(v) If $\mathrm{cf}(\beta) \leq \delta$, then for every $C \in q(\beta)$, $\gamma \in \mathrm{Lim}(C)$, and $D \in q(\gamma)$,
$$D \cup (C \setminus \gamma) \in q(\beta).$$

In $V^{\mathbb{P}*\mathbb{Q}}$ let $\vec{\mathcal{C}}$ be the $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence added by $\mathbb{Q}$. Define $\mathbb{R}$ to be the poset which adds a thread through $\vec{\mathcal{C}}$, i.e., the poset of closed bounded subsets $c \subseteq \kappa$ with the property that $c \cap \beta \in \mathcal{C}_\beta$ for any $\beta \in \mathrm{Lim}(c)$, ordered by end-extension.

Claim 4.1. *Suppose q satisfies (i)–(iv) in the definition of $\mathbb{Q}$ with dom(q) = $\mathrm{Lim}(\lambda^+) \cap \alpha + 1$ for some $\alpha < \lambda^+$ which is a limit of limit ordinals, $\mathrm{cf}(\alpha) \leq \delta$. Suppose further that for any limit ordinal $\beta < \alpha$, $q \restriction (\beta + 1) \in \mathbb{Q}$.*

Define q^* as the function on dom(q) with $q^* \restriction \max(\mathit{dom}(q)) = q$ and
$$q^*(\alpha) = q(\alpha) \cup \{D \cup (C \setminus \beta) : C \in q(\alpha), \beta \in \mathrm{Lim}(C) \text{ and } D \in q(\beta)\}.$$
Then $q^* \in \mathbb{Q}$.

Proof. There are at most δ many $C \in q(\alpha)$ and δ many β in each such C, so $|q^*(\alpha)| \leq \delta$. (It is important here that $\lambda = \delta^+$, for otherwise there could be more than δ many elements of C.) The only nontrivial requirements to check in the definition of $\mathbb{Q}$ are (iii) and (v) at α.

To show (iii) at α, suppose $E \in q^*(\alpha)$ and $\gamma \in \mathrm{Lim}(E)$. We check that $E \cap \gamma \in q(\gamma)$. The less immediate case has $E = D \cup (C \setminus \beta)$ for some $C \in q(\alpha)$, $\beta \in \mathrm{Lim}(C)$, and $D \in q(\beta)$. If $\gamma \leq \beta$, then $E \cap \gamma = D \cap \gamma \in q(\gamma)$. If $\gamma > \beta$, then $E \cap \gamma = D \cup ((C \cap \gamma) \setminus \beta)$. By (iii) applied at γ, $C \cap \gamma \in q(\gamma)$, so by (v) applied at γ, $E \cap \gamma = D \cup ((C \cap \gamma) \setminus \beta) \in q(\gamma)$.

To show (v), suppose that $E \in q^*(\alpha)$, $\gamma \in \mathrm{Lim}(E)$, and $F \in q(\gamma)$. We check that $F \cup (E \setminus \gamma) \in q^*(\alpha)$. Again, the less immediate case has $E = D \cup (C \setminus \beta)$ for some $C \in q(\alpha)$, $\beta \in \mathrm{Lim}(C)$, and $D \in q(\beta)$. If $\gamma \geq \beta$, then $F \cup (E \setminus \gamma) = F \cup (C \setminus \gamma) \in q^*(\alpha)$. If $\gamma < \beta$, then $\gamma \in \mathrm{Lim}(D)$, so by (v) applied at level β, we have $F' := F \cup (D \setminus \gamma) \in q(\beta)$. Therefore, $F \cup (E \setminus \gamma) = F' \cup (C \setminus \gamma) \in q^*(\beta)$. □

In the situation of the claim, we call q^* the *completion* of q.

We have a version of Fact 3.1 for the new $\mathbb{Q}$ and $\mathbb{R}$. We will prove $\mathbb{Q}$ is κ-distributive by showing that it is $\lambda+1$-strategically closed (similarly to [1]). Recall that in our situation, κ has been collapsed to be λ^+.

Lemma 4.2. *The poset $\mathbb{Q}$ is $\lambda + 1$-strategically closed, therefore κ-distributive.*

Proof. Players I and II play elements of $\mathbb{Q}$, with II playing at even stages, i.e., limit stages and even successor stages. We describe a winning strategy for player II. Let q_ξ be the condition played at stage ξ and β_ξ be $\max \mathrm{dom}(q_\xi)$. At stage $\eta+2$, II plays $q_{\eta+2} \le q_{\eta+1}$ with $\beta_{\eta+2} = \beta_{\eta+1} + \omega$ and $q_{\eta+2}(\beta_{\eta+2}) = \{\{\beta_{\eta+1} + n : 1 \le n < \omega\}\}$.

If ξ is limit, define $A_\xi = \{\beta_\eta : \eta < \xi \text{ and } \eta \text{ even}\}$. II plays $q_\xi = \bigcup_{\eta<\xi} q_\eta \cup \{(\beta_\xi, \{A_\xi\})\}$, with $\beta_\xi = \sup_{\eta<\xi} \beta_\eta$. This is closed and unbounded in ξ by our construction so far. Furthermore, the construction ensures that for every $\gamma \in \mathrm{Lim}(A_\xi)$, $q_\xi(\gamma)$ is the singleton $\{A_\xi \cap \gamma\}$, so that coherence holds and condition (v) in the definition of $\mathbb{Q}$ is satisfied trivially at β_ξ. □

The other parts of Fact 3.1 carry over to this situation as well.

Fact 4.3. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, and G, H, I be generics for $\mathbb{P}, \mathbb{Q}, \mathbb{R}$, respectively.

- Working in $V^{\mathbb{P}}$, the set of flat conditions
$$\{(q, \check{r}) \in \mathbb{Q} * \mathbb{R} : r \in V[G] \text{ and } \max(\mathrm{dom}(q)) = \max(r)\}$$
is dense and λ-closed.
- There is a complete embedding of $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ into $j(\mathbb{P})$ with λ-closed quotient forcing.
- Letting J be generic for $j(\mathbb{P})/\mathbb{P} * \mathbb{Q} * \mathbb{R}$, there is a K generic for $j(\mathbb{Q})$ so that j can be extended to an elementary embedding $j : V[G * H] \to M[G * H * I * J * K]$ in the extension by $j(\mathbb{P} * \mathbb{Q})$.

Proof. We just prove the set of flat conditions is λ-closed, as this requires us to take a completion. Suppose $\langle (q_\xi, r_\xi) : \xi < \eta \rangle$ is a decreasing sequence of flat conditions of $\mathbb{Q} * \mathbb{R}$, where $\eta < \lambda$. Letting $\alpha = \sup\{\max \mathrm{dom}(q_\xi) : \xi < \eta\}$, $r = \bigcup_\xi r_\xi \cup \{\alpha\}$, and q be the completion of $\bigcup_\xi q_\xi \cup \{(\alpha, r \cap \alpha)\}$, we see that (q, r) is a flat condition strengthening all the conditions from the sequence. The other parts of the claim are also proved just like the analogous facts in [4], taking completions where necessary. □

As before, we will show that $V[G * H]$ is a model satisfying the conclusion of the theorem: $\square^{\mathrm{ta}}_{\lambda,\delta}$ holds in $V[G * H]$, so assume towards a contradiction that $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha : \alpha < \kappa \rangle$ is a $\square_{\lambda,<\delta}$ sequence in $V[G * H]$. Let $T \in j(\vec{\mathcal{D}})_\kappa$, so T threads $\vec{\mathcal{D}}$ in $V[G * H * I * J * K]$.

The version of Lemma 3.2 we need here is essentially the same as Lemma 4.5 in [4], whose proof easily adapts to our statement.

Lemma 4.4. *Suppose $V \subseteq W$ are models of set theory, λ is an uncountable regular cardinal in W, and $\vec{\mathcal{D}}$ is a $\square_{\lambda,<\lambda}$ sequence in V. Then forcing with a λ-closed poset over W cannot add a new thread to $\vec{\mathcal{D}}$.*

By $j(\kappa)$-distributivity of $j(\mathbb{Q})$ and Lemma 4.4, T must be a member of $V[G * H * I]$. Work in $V[G]$ and let $\dot{T}$ be a $\mathbb{Q} * \mathbb{R}$-name for T. Since $\vec{\mathcal{D}}$ is a $\square_{\lambda,<\delta}$ sequence in $V[G * H]$, it follows that $T \notin V[G * H]$, and therefore:

Claim 4.5. *For any $q \in \mathbb{Q}$, $r \in \mathbb{R}$, there are $\alpha < \lambda^+, q' \le q, r'_0, r'_1 \le r$ such that (q', r'_0) and (q', r'_1) decide "$\alpha \in \dot{T}$" differently.*

Fix $f : \delta \to \delta$ such that $f(k) \le k$ for each $k < \delta$, and for each $j < \delta$ there are unboundedly many $k < \delta$ with $f(k) = j$. We will recursively construct $\langle q_j : j \le \delta \rangle$, $\langle r^i_j : i < j \le \delta \rangle$, and $\langle \alpha_j : j < \delta \rangle$ such that for all $j \le \delta$:

(1) (q_1, r_1^0) forces that $\dot{T}$ is a thread of $\vec{\mathcal{C}}$.
(2) For all $i < j$, $(q_j, r_j^i) \in \mathbb{Q} * \mathbb{R}$ is flat and the order-type of r_j^i is $\rho + 1$ for some limit ordinal ρ. We will use the notation β_j for $\max(\mathrm{dom}(q_j))$.
(3) $\langle \alpha_k : k < \delta \rangle$ is a strictly increasing sequence of ordinals less than λ^+, and for each i the sequence $\langle (q_k, r_k^i) : i < k < \delta \rangle$ is decreasing in the $\mathbb{Q} * \mathbb{R}$ ordering,
(4) $(q_{j+1}, r_{j+1}^{f(j)}) \Vdash \alpha_j \in \dot{T}$.
(5) If $i, i' < j$ are distinct, then (q_j, r_j^i) and $(q_j, r_j^{i'})$ force distinct values for $\dot{T}$ below α_j.
(6) If $i, i' < j$, then $r_{j+1}^i \setminus \beta_j = r_{j+1}^{i'} \setminus \beta_j$.
(7) If j is limit, then $\beta_j = \sup\{\beta_k : k < j\}$, $r_j^i = \bigcup_{i<k<j} r_k^i \cup \{\beta_j\}$ for each $i < j$, and q_j is the completion of $\bigcup_{k<j} q_k \cup \{(\beta_j, \{r_j^i \cap \beta_j : i < j\})\}$.

Assume that we are at stage $j+1$ of the construction, so that (q_j, r_j^i) and α_i have been defined for all $i < j$. Using Claim 4.5, find $q'_{j+1} \leq q_j$, $r_{j+1,0}, r_{j+1,1} \leq r_j^0$, and $\gamma < \lambda^+$ such that $(q'_{j+1}, r_{j+1,0})$ and $(q'_{j+1}, r_{j+1,1})$ decide "$\gamma \in \dot{T}$" differently. By extending further, we can take $(q'_{j+1}, r_{j+1,0})$ and $(q'_{j+1}, r_{j+1,1})$ to satisfy (2) above. Let $\beta'_{j+1} = \max \mathrm{dom}(q'_{j+1})$.

We construct so that (4) holds. Since $r_{j+1,0} \in q'_{j+1}(\beta'_{j+1})$, and $\beta_j \in \mathrm{Lim}(r_{j+1,0})$ by (2), we can extend $(q_j, r_j^{f(j)})$ to $(q'_{j+1}, r_j^{f(j)} \cup (r_{j+1,0} \setminus \beta_j))$ using (v) of the definition of $\mathbb{Q}$. Extend this to a condition which forces $\alpha_j \in \dot{T}$ for some $\alpha_j < \lambda^+$ with $\alpha_j > \gamma$, $\alpha_j > \alpha_i$ for every $i < j$. Extend further to $(q_{j+1}, r_{j+1}^{f(j)})$ satisfying (2).

Set $r_{j+1}^0 = r_{j+1,0} \cup (r_{j+1}^{f(j)} \setminus \beta'_{j+1})$ and $r_{j+1}^j = r_{j+1,1} \cup (r_{j+1}^{f(j)} \setminus \beta'_{j+1})$. For $0 < i < j$, set $r_{j+1}^i = r_j^i \cup (r_{j+1}^{f(j)} \setminus \beta_j)$. By condition (v) from the definition of $\mathbb{Q}$, it follows that $(q_{j+1}, r_{j+1}^i) \in \mathbb{Q} * \mathbb{R}$ for all $i < j+1$.

Now suppose $j \leq \delta$ is limit. The construction is completely determined by (7). For any $i' < i < j$ we have $r_j^i \setminus \beta_{i+1} = r_j^{i'} \setminus \beta_{i+1}$, otherwise there is some $i < k < j$ where they disagree in $[\beta_k, \beta_{k+1})$, contradicting (6). Therefore all of the r_j^i agree on a tail-end and so q_j defined by (7) is really a member of $\mathbb{Q}$. It is straightforward to check inductively throughout that (1)–(7) above hold, so we have finished the construction.

Let $\alpha^* = \sup\{\alpha_j : j < \delta\}$. Find $q^* \leq q_\delta$ which decides the value of $\mathcal{D}_{\alpha^*}$. For all $i < \delta$,

$$(q^*, r_\delta^i) \Vdash \alpha^* \text{ is a limit point of } \dot{T}$$

since $\{\alpha_j : f(j) = i\}$ is unbounded in α^* and (q^*, r_δ^i) forces such α_j into $\dot{T}$. This means $(q^*, r_\delta^i) \Vdash \dot{T} \cap \alpha^* \in \mathcal{D}_{\alpha^*}$. If $i \neq j$, then (q^*, r_δ^i) and (q^*, r_δ^j) force different values for $\dot{T} \cap \alpha^*$ by (5). This gives δ many distinct elements of $\mathcal{D}_{\alpha^*}$, a contradiction, concluding the proof of Theorem 4. □

This proof can be modified slightly to give Theorem 6.

Proof of Theorem 6. Let μ be the least cardinal such that $2^\mu \geq \lambda$. Since λ is not strongly inaccessible, $\mu < \lambda$. Run the main construction in the proof of Theorem 4 for $\mu + 1$ many steps, but with i ranging over 2^j rather than j at stage j. This involves modifying the successor step to extend *each* r_j^i, not just r_j^0, in two incompatible ways. At each limit stage $j < \mu$, there are fewer than λ many r_j^i, so the construction can be continued. At stage μ, take a subset of the r^i_μ of size λ to form q_μ. Then the argument is completed as in the proof of Theorem 4. $\square$

The proof of Theorem 4 does not generalize immediately to the situation of Theorem 5, since closure of the set of flat conditions of $\mathbb{Q} * \mathbb{R}$ requires taking completions at limit levels, and therefore $\mathbb{Q}*\mathbb{R}$ (and hence also the quotient forcing $j(\mathbb{P})/\mathbb{P} * \mathbb{Q} * \mathbb{R}$) is only δ^+-closed. In the case where $\delta^+ < \lambda$, this is insufficient to show that T was not added by $j(\mathbb{P})/\mathbb{P} * \mathbb{Q} * \mathbb{R}$. To overcome this, we will use a technique similar to the argument in Section 7 of [1] separating different $\square_{\lambda,\delta}$ for singular λ.

Proof of Theorem 5. Let $\mathbb{P} = \mathrm{Col}(\delta^+, <\kappa)$. Let $\mathbb{Q}$ be the poset defined in $V^{\mathbb{P}}$ as in the proof of Theorem 4, and let $\vec{\mathcal{C}}$ be the $\square^{\mathrm{ta}}_{\lambda,\delta}$ sequence added by $\mathbb{Q}$. Let $\mathbb{R}$ be the poset adding a thread through $\vec{\mathcal{C}}$ by closed initial segments of order-type $< \delta^+$. It can be shown that the generic thread added by $\mathbb{R}$ has order-type δ^+.

Again, we will build elements of $\mathbb{Q}$ by taking completions. The statement of Claim 4.1 holds in the new situation, but we must be more careful in the proof to avoid taking too many elements of $q^*(\alpha)$.

Claim 4.6. *Suppose q satisfies (i)–(iv) in the definition of $\mathbb{Q}$ with dom(q) = $\mathrm{Lim}(\lambda^+) \cap \alpha + 1$ for some $\alpha < \lambda^+$ which is a limit of limit ordinals, $\mathrm{cf}(\alpha) \leq \delta$. Suppose further that for any limit ordinal $\beta < \alpha$, $q\restriction(\beta + 1) \in \mathbb{Q}$.*

Define q^* as the function on dom(q) with $q^* \restriction \max(dom(q)) = q$ and

$$q^*(\alpha) = q(\alpha) \cup \{D \cup (C \setminus \beta) : C \in q(\alpha), \beta \in \mathrm{Lim}(C) \text{ and } D \in q(\beta)\}.$$

Then $q^* \in \mathbb{Q}$.

Proof. Fix a particular $C_0 \in q(\alpha)$. Assume that $\mathrm{Lim}(C_0)$ is unbounded in α (the other case is similar, and easier). Let X be a subset of $\mathrm{Lim}(C_0)$ cofinal in α of order-type $\mathrm{cf}(\alpha)$. Define $\tilde{q}^*(\alpha) = q(\alpha) \cup \{D \cup (C \setminus \beta) : C \in q(\alpha), \beta \in \mathrm{Lim}(C) \cap X \text{ and } D \in q(\beta)\}$. This has at most δ many elements.

We claim $q^*(\alpha) \subseteq \tilde{q}^*(\alpha)$. Suppose $C \in q(\alpha), \beta \in \mathrm{Lim}(C)$ and $D \in q(\beta)$. Then there is some $\gamma > \beta$ in $\mathrm{Lim}(C) \cap X$ since X is unbounded in α and C and C_0 agree on a tail-end. By condition (v) of the definition of $\mathbb{Q}$ and since $C \cap \gamma \in q(\gamma)$, $D' = D \cup ((C \cap \gamma) \setminus \beta) \in q(\gamma)$. Now $D \cup (C \setminus \beta) = D' \cup (C \setminus \gamma) \in \tilde{q}^*(\alpha)$. $\square$

We get the basic facts about $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ as before. In our new situation, let $j : V \to M$ be a 2^λ-supercompactness embedding.

Fact 4.7. Let G, H, I be generics for $\mathbb{P}, \mathbb{Q}, \mathbb{R}$, respectively.

- In $V^{\mathbb{P}}$, the poset $\mathbb{Q}$ is $\lambda + 1$-strategically closed, therefore κ-distributive.
- The set of flat conditions $\{(q, \check{r}) \in \mathbb{Q} * \mathbb{R} : r \in V[G] \text{ and } \max(\mathrm{dom}(q)) = \max(r)\}$ is dense and δ^+-closed.
- There is a complete embedding of $\mathbb{P}*\mathbb{Q}*\mathbb{R}$ into $j(\mathbb{P})$ with δ^+-closed quotient forcing.
- Letting J be generic for $j(\mathbb{P})/\mathbb{P} * \mathbb{Q} * \mathbb{R}$, there is a K generic for $j(\mathbb{Q})$ so that j can be extended to an elementary embedding $j : V[G * H] \to M[G * H * I * J * K]$ in the extension by $j(\mathbb{P} * \mathbb{Q})$.

The first item is a parallel of Lemma 4.2. The second, which uses completions in an essential way and is therefore limited to δ^+-closure, is a parallel of the first item of Fact 4.3. The remaining items are similar to facts found in [1], and the proofs there can be adapted to our situation in a straightforward way.

Assume towards a contradiction that $\vec{\mathcal{D}}$ is a $\square_{\lambda,<\delta}$ sequence in $V[G*H]$. Let $\gamma = \sup j``\lambda^+$ and fix some $A \in j(\vec{\mathcal{D}})_\gamma$. Since $j(\mathbb{Q})$ is $j(\kappa)$-distributive, $A \in V[G*H*I*J]$.

In this situation, we have an analogue of Claim 3.5 which gives a thread through $\vec{\mathcal{D}}$ in $V[G*H*I*J]$. The fact that $\delta < \kappa$ allows us to avoid the use of tail-end agreement for $\vec{\mathcal{D}}$ needed in the proof of Claim 3.5.

Claim 4.8. *If $\gamma = \sup j``\lambda^+$ and $A \in j(\vec{\mathcal{D}})_\gamma$, then $S = \{\alpha \in \lambda^+ : j(\alpha) \in \mathrm{Lim}(A)\}$ generates a thread T through $\vec{\mathcal{D}}$.*

Since j is continuous at points of countable cofinality, $j``\lambda^+$ is an $\omega$-club subset of $\gamma$ and hence $\mathrm{Lim}(A) \cap j``\lambda^+$ is unbounded in γ. Therefore, its pointwise j-preimage $S = \{\alpha < \lambda^+ : j(\alpha) \in \mathrm{Lim}(A)\}$ is unbounded in λ^+. If $\alpha \in S$ then $A \cap j(\alpha) \in j(\vec{\mathcal{D}})_{j(\alpha)} = j(\mathcal{D}_\alpha)$. Since $\delta < \kappa$, $j(\mathcal{D}_\alpha) = j``\mathcal{D}_\alpha$, so there is $D_\alpha \in \mathcal{D}_\alpha$ such that $A \cap j(\alpha) = j(D_\alpha)$. If $\beta < \alpha$ are in S, then $j(D_\alpha) \cap j(\beta) = A \cap j(\beta) = j(D_\beta)$. By elementarity, $\beta \in \mathrm{Lim}(D_\alpha)$ and $D_\beta = D_\alpha \cap \beta$, so $T = \bigcup_{\alpha \in S} D_\alpha$ threads $\vec{\mathcal{D}}$. This proves the claim.

We require a version of Lemma 4.4 which assumes less closure, and also applies to singular cardinals. The following is implicit in [1]:

Lemma 4.9. *Let $\delta < \lambda$ be infinite cardinals. Suppose $V \subseteq W$ are models of set theory with the same cardinals $\leq \delta^+$, $W \models \mathrm{cf}((\lambda^+)^V) \geq \delta^+$, and $\vec{\mathcal{D}}$ is a $\square_{\lambda,<\delta}$ sequence in V. Then forcing with a δ^+-closed poset over W cannot add a new thread to $\vec{\mathcal{D}}$.*

Since the thread added by $\mathbb{R}$ has order-type δ^+, $V[G*H*I] \models \mathrm{cf}((\lambda^+)^V) = \delta^+$. By Lemma 4.9, $T \in V[G*H*I]$. The rest of the proof proceeds in exactly the same way as the proof of Theorem 4. $\square$

References

[1] Cummings, J., Foreman, M., & Magidor, M.: Squares, scales and stationary reflection. *Journal of Mathematical Logic*, 1(01), 35–98 (2001).

[2] Jensen, R. Some remarks on $\square$ below zero-pistol. Circulated notes.

[3] Krueger, J. & Schimmerling, E.: Separating weak partial square principles. *Annals of Pure and Applied Logic*, 165(2), 609–619 (2014).

[4] Magidor, M., & Lambie-Hanson, C.: On the strengths and weaknesses of weak squares. *Appalachian Set Theory: 2006-2012*, 406, 301 (2012).

[5] Neeman, I. Two applications of finite side conditions at ω_2.

[6] Schimmerling, E.: Combinatorial principles in the core model for one Woodin cardinal. *Annals of Pure and Applied Logic*, 74(2), 153–201 (1995).

[7] Shelah, S., & Stanley, L.: Weakly compact cardinals and nonspecial Aronszajn trees. *Proceedings of the American Mathematical Society*, 104(3), 887–897 (1988).

Department of Mathematics, UCLA
E-mail address: chenwb@math.ucla.edu

Department of Mathematics, UCLA
E-mail address: ineeman@math.ucla.edu