HIERARCHIES OF FORCING AXIOMS II

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Abstract. A Σ^2_1 **truth** for λ is a pair $\langle Q, \psi\rangle$ so that $Q \subseteq H_\lambda$, ψ is a first order formula with one free variable, and there exists $B \subseteq H_{\lambda^+}$ such that $(H_{\lambda^+}; \in, B) \models \psi[Q]$. A cardinal λ is Σ^2_1 **indescribable** just in case that for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there exists $\bar\lambda < \lambda$ so that $\bar\lambda$ is a cardinal and $\langle Q \cap H_{\bar\lambda}, \psi\rangle$ is a Σ^2_1 truth for $\bar\lambda$. More generally, an interval of cardinals $[\kappa, \lambda]$ with $\kappa \leq \lambda$ is Σ^2_1 **indescribable** if for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there exists $\bar\kappa \leq \bar\lambda < \kappa$, $\bar Q \subseteq H_{\bar\lambda}$, and $\pi \colon H_{\bar\lambda} \to H_\lambda$ so that $\bar\lambda$ is a cardinal, $\langle \bar Q, \psi\rangle$ is a Σ^2_1 truth for $\bar\lambda$, and π is elementary from $(H_{\bar\lambda}; \in, \bar\kappa, \bar Q)$ into $(H_\lambda; \in, \kappa, Q)$ with $\pi \restriction \bar\kappa = \mathrm{id}$.

We prove that the restriction of the proper forcing axiom to $\mathfrak{c}$-linked posets requires a Σ^2_1 indescribable cardinal in L, and that the restriction of the proper forcing axiom to $\mathfrak{c}^+$-linked posets, in a proper forcing extension of a fine structural model, requires a Σ^2_1 indescribable 1-gap $[\kappa, \kappa^+]$. These results show that the respective forward directions obtained in Hierarchies of Forcing Axioms I by Neeman and Schimmerling are optimal.

It is a well-known conjecture that the large cardinal consistency strength of PFA is a supercompact cardinal. This paper is the second in a pair of papers connecting a hierarchy of forcing axioms leading to PFA with a hierarchy of large cardinal axioms leading to supercompact.

Recall that a forcing notion $\mathbb{P}$ is **λ-linked** if it can be written as a union of sets P_ξ, $\xi < \lambda$, so that for each ξ, the conditions in P_ξ are pairwise compatible. PFA(λ-linked) is the restriction of PFA to λ-linked posets. The forcing axioms form a hierarchy, with PFA of course equivalent to the statement that PFA(λ-linked) holds for all λ. The following theorem deals with consistency strength at the low end of this hierarchy.

Theorem A. *The consistency strength of* PFA($\mathfrak{c}$*-linked*) *is precisely a* Σ^2_1 *indescribable cardinal. More specifically:*

1. *If* κ *is* Σ^2_1 *indescribable in a model* M *satisfying the* GCH *then there is a forcing extension of* M*, by a proper poset, in which* $\mathfrak{c} = \omega_2 = \kappa$ *and* PFA($\mathfrak{c}$*-linked*) *holds.*
2. *If* PFA($\mathfrak{c}$*-linked*) *holds then* $(\omega_2)^{\mathrm{V}}$ *is* Σ^2_1 *indescribable in* L.

The statement that $\mathfrak{c} = \omega_2$ in part (1) is redundant, as PFA($\mathfrak{c}$-linked) implies $\mathfrak{c} = \omega_2$. This was proved by Todorčević (see Bekkali [2]) and Veličković [15].

Part (1) is joint with Schimmerling: Neeman–Schimmerling [7] proves its semi-proper analogue, producing a semiproper forcing extension of M in which SPFA($\mathfrak{c}$-linked) holds, and the proof of (1) is identical except for the routine change of replacing semi-proper by proper throughout. It follows from part (2)

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0556223

and the semiproper analogue of part (1) proved in [7] that $\mathsf{PFA}(\mathfrak{c}\text{-linked})$ and $\mathsf{SPFA}(\mathfrak{c}\text{-linked})$ are equiconsistent, both having the consistency strength of a Σ^2_1 indescribable cardinal.

Part (2), which extracts strength from $\mathsf{PFA}(\mathfrak{c}\text{-linked})$, is Theorem 1.10 in this paper.

Todorčević [13] proved that PFA implies the failure of $\square_\lambda$ for all uncountable cardinals λ. By results of Dodd–Jensen and Welch [3, 16], this was known at the time to imply the existence of a model with a measurable cardinal. Derivations of strength from PFA have since used the failure of square principles under PFA as an intermediary, deriving the large cardinal strength from the failure of square. For examples of this we refer the reader to Schimmerling [9], Steel [12], and finally Schimmerling [8] which shows that already the failure of $\square_{\omega_2}$ and related square principles at ω_2 has substantial large cardinal strength. In a different direction, Goldstern–Shelah [4] measured the consistency strength of BPFA, a version of PFA restricting the antichains used to size ω_1, to be a Σ_1 reflecting cardinal. Miyamoto [6] defined a hierarchy of proper forcing axioms with BPFA at the bottom, and found the exact strength of the second axiom in the hierarchy, dealing with antichains of size ω_2. Here too the consistency strength can be derived using Todorčević's antisquare poset, see [14].

Let $\kappa = \omega_2^{\mathrm{V}}$. To prove the Σ^2_1 indescribability of κ in L we must reflect a first order statement about a subset B of $\tau = \kappa^+$, namely the witness to the Σ^2_1 truth, to a lower cardinal $\bar\tau = \bar\kappa^+$ of L. (All successors here are computed in L.) The main difficulty is in making sure that $\bar\tau$ reaches $\bar\kappa^+$. It is precisely for this that the proof of Miyamoto's result in Todorčević [14] relies on an antisquare poset. This route is not available to us here, as the relevant antisquare posets need not be $\mathfrak{c}$-linked. Instead we rely on representations of constructible levels L_α for $\alpha \in (\kappa, \tau)$ as direct limits of systems of canonical embeddings between levels of L below κ. These representations are related to the existence in L of a combinatorial object known as a morass, although an actual morass is not needed for the argument. We define a $\mathfrak{c}$-linked poset generating a system of representations which reach τ, and a witness that the object being reached is the successor of κ. A pseudo-generic for the poset allows us to reflect the first order statement from κ and τ to $\bar\kappa$ and $\bar\tau$, while ensuring that $\bar\tau$ is the successor of $\bar\kappa$.

Let us move now to higher levels of the forcing and large cardinal hierarchies. We begin by generalizing Σ^2_1 indescribability to gaps of cardinals.

By a Σ^2_1 **truth** for λ we mean a pair $\langle Q, \psi\rangle$ so that $Q \subseteq H_\lambda$, ψ is a first order formula with one free variable, and there exists $B \subseteq H_{\lambda^+}$ such that $(H_{\lambda^+}; \in, B) \models \psi[Q]$. A cardinal λ is then Σ^2_1 **indescribable** just in case that for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there exists $\bar\lambda < \lambda$ so that $\bar\lambda$ is a cardinal and $\langle Q \cap H_{\bar\lambda}, \psi\rangle$ is a Σ^2_1 truth for $\bar\lambda$. The following definition generalizes this.

Definition. A gap of cardinals $[\kappa, \lambda]$ with $\kappa \le \lambda$ is Σ^2_1 **indescribable** if for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there exists $\bar\kappa \le \bar\lambda < \kappa$, $\bar Q \subseteq H_{\bar\lambda}$, and $\pi\colon H_{\bar\lambda} \to H_\lambda$, such that:

1. $\bar\lambda$ is a cardinal and $\langle \bar Q, \psi\rangle$ is a Σ^2_1 truth for $\bar\lambda$.
2. π is elementary from $(H_{\bar\lambda}; \in, \bar\kappa, \bar Q)$ into $(H_\lambda; \in, \kappa, Q)$ with $\pi \restriction \bar\kappa = \mathrm{id}$.

We also say that κ is (λ, Σ^2_1)-subcompact in this case.

At the lowest end, $[\kappa, \kappa]$ is Σ^2_1 indescribable just in case that κ is Σ^2_1 indescribable. Σ^2_1 indescribability for a 1-gap $[\kappa, \kappa^+]$ is already substantially stronger, enough to imply the existence of superstrong extenders and many subcompact cardinals. At the upper end, $[\kappa, \lambda]$ is Σ^2_1 indescribable for all $\lambda \geq \kappa$ just in case that κ is supercompact.

Theorem A ties the lower end of the hierarchy of forcing axioms $\mathsf{PFA}(\lambda\text{-linked})$ to the lower end of the Σ^2_1 indescribability hierarchy. The next theorem moves one step up in both hierarchies:

Theorem B. *The large cardinal necessary to obtain* $\mathsf{PFA}(\mathfrak{c}^+\text{-}linked)$ *by proper forcing over a fine structural model is precisely a* Σ^2_1 *indescribable 1-gap. More specifically:*

1. *Suppose* $[\kappa, \kappa^+]$ *is* Σ^2_1 *indescribable in a model* M *satisfying* GCH*. Then there is a forcing extension of* M*, by a proper poset, in which* $\mathfrak{c} = \omega_2 = \kappa$ *and* $\mathsf{PFA}(\mathfrak{c}^+\text{-}linked)$ *holds.*
2. *Suppose* V *is a proper forcing extension of a fine structural model* M*, and* $\mathsf{PFA}(\mathfrak{c}^+\text{-}linked)$ *holds in* V*. Then* $[\kappa, \kappa^+]$ *is* Σ^2_1 *indescribable in* M *where* $\kappa = (\omega_2)^{\mathrm{V}}$*.*

Part (1) again is due to Neeman–Schimmerling [7]. Part (2) is Theorem 2.12 in this paper. It shows that the large cardinal assumption used in part (1) is optimal. This large cardinal assumption involves superstrong extenders. The fine structure of models with such extenders has been developed, and was applied for proofs of square in Schimmerling-Zeman [10, 11] and Zeman [17]. But core model theory is still very far below this level. By core model theory we mean the construction of a maximal fine structural model inside a given universe V. In proving part (2) we bypass the lack of core model theory by putting some assumptions on V that tie it to a fine structural model. Precisely we assume that V is a proper forcing extension of a fine structural model M. This assumption lets us work with M as if it were a maximal fine structural model in V. We expect that the assumption could be easily dropped when core model theory reaches the level of Σ^2_1 indescribable 1-gaps, resulting in an actual equiconsistency.

In light of the results above it is natural to tie the hierarchy of forcing axioms $\mathsf{PFA}(\lambda\text{-linked})$ to the hierarchy of Σ^2_1 indescribability, and conjecture that for $\lambda \geq \omega_2$, the large cardinal strength of $\mathfrak{c} = \omega_2 \wedge \mathsf{PFA}(\lambda\text{-linked})$ is a Σ^2_1 indescribable gap $[\omega_2, \lambda]$. The forward direction is known, proved in [7]. The reverse direction, beyond Theorems 1.10 and 2.12, awaits the development of fine structure theory beyond superstrongs, and core model theory beyond Woodin cardinals.

Remark. The development of the results in the paper owes a great deal to conversations between Ernest Schimmerling and the author.

§1. A Σ^2_1 indescribable cardinal. Throughout this section, cardinal successors are computed in L, not in V. A **point** is a limit ordinal β so that the following conditions hold, where α is uniquely determined from β by the first condition:

1. $\mathrm{L}_\beta \models$"α is the largest cardinal."

2. β is a cardinal in $\mathrm{L}_{\beta+1}$.
3. $\beta < \alpha^+$ (equivalently, β is not a cardinal in L).

REMARK 1.1. The demand that β remains a cardinal in $\mathrm{L}_{\beta+1}$ is not needed for any arguments in this section, but a parallel demand is useful in Section 2.

We refer to α as the **level** of the point β, and denote it $\alpha(\beta)$. Define $\gamma(\beta)$ to be least so that $\mathrm{L}_{\gamma(\beta)+1}$ has a surjection of α onto β. Such a level exists since $\beta < \alpha^+$. Since the surjection is coded by a subset of α, and since it does not belong to L_ν for any $\nu < \gamma(\beta)+1$, we have:

CLAIM 1.2. *Every element of* $\mathrm{L}_{\gamma(\beta)+1}$ *is definable in* $\mathrm{L}_{\gamma(\beta)+1}$ *from parameters in* $\alpha(\beta)$. $\dashv$

Let $\bar\alpha < \alpha$. Let X be the Skolem hull of $\bar\alpha$ in $\mathrm{L}_{\gamma(\beta)+1}$. $\bar\alpha$ is **stable** in β just in case that $\alpha \in X$ and $X \cap \alpha = \bar\alpha$. Let M be the transitive collapse of X, let $j\colon M \to \mathrm{L}_{\gamma(\beta)+1}$ be the anticollapse embedding, and let $\bar\beta$ be such that $j(\bar\beta) = \beta$. (Note also that $j(\bar\alpha) = \alpha$.) We call $\bar\beta$ the **projection** of β to level $\bar\alpha$, denoted $\mathrm{proj}_{\bar\alpha}(\beta)$. Using the elementarity of j it is easy to check that $\bar\beta$ is a point on level $\bar\alpha$, and that M is precisely $\mathrm{L}_{\gamma(\bar\beta)+1}$. We refer to $j\colon \mathrm{L}_{\gamma(\bar\beta)+1} \to \mathrm{L}_{\gamma(\beta)+1}$ as the **antiprojection embedding**, denoted $j_{\bar\beta,\beta}$. By Claim 1.2 the embedding is uniquely determined by $\bar\beta$ and β.

The following claims are easy to verify using the uniqueness and elementarity of the embeddings involved:

CLAIM 1.3. *Let* $\bar\alpha < \alpha < \alpha^*$. *Let* $\bar\beta$, β, *and* β^* *be points on levels* $\bar\alpha$, α, *and* α^*. *Suppose that* $\beta = \mathrm{proj}_\alpha(\beta^*)$ *and* $\bar\beta = \mathrm{proj}_{\bar\alpha}(\beta)$. *Then* $\bar\alpha$ *is stable in* β^*, $\mathrm{proj}_{\bar\alpha}(\beta^*) = \bar\beta$, *and* $j_{\bar\beta,\beta^*} = j_{\beta,\beta^*} \circ j_{\bar\beta,\beta}$. $\dashv$

CLAIM 1.4. *Let* $\bar\alpha < \alpha < \alpha^*$. *Let* $\bar\beta$ *be a point on level* $\bar\alpha$, β *a point on level* α, *and* β^* *a point on level* α^*. *Suppose that* $\beta = \mathrm{proj}_\alpha(\beta^*)$ *and* $\bar\beta = \mathrm{proj}_{\bar\alpha}(\beta^*)$. *Then* $\bar\beta = \mathrm{proj}_{\bar\alpha}(\beta)$. $\dashv$

CLAIM 1.5. *Let* $\beta < \beta^*$ *be points on the same level* α. *Let* $\bar\alpha < \alpha$ *be stable in* β^*, *let* $\bar\beta^* = \mathrm{proj}_{\bar\alpha}(\beta^*)$, *and let* j^* *denote* $j_{\bar\beta^*,\beta^*}$. *Suppose that* β *belongs to* $\mathrm{range}(j^*)$ *(in other words, it is definable in* $\mathrm{L}_{\gamma(\beta^*)+1}$ *from parameters in* $\bar\alpha$*). Then:*

1. $\bar\alpha$ *is stable in* β.

Let $\bar\beta = \mathrm{proj}_{\bar\alpha}(\beta)$.

2. $\bar\beta = (j^*)^{-1}(\beta)$. *(In particular* $\mathrm{proj}_{\bar\alpha}(\beta) < \mathrm{proj}_{\bar\alpha}(\beta^*)$.*)*
3. $j_{\bar\beta,\beta} = j^* \restriction \mathrm{L}_{\gamma(\bar\beta)+1}$. $\dashv$

A **thread** in τ is a sequence of points $T = \langle \beta_\alpha \mid \alpha \in C\rangle$ so that:

1. C is club in τ, and for each $\alpha \in C$, β_α is a point on level α.
2. Let $\alpha \in C$ and let $\bar\alpha < \alpha$. Then $\bar\alpha \in C$ iff $\bar\alpha$ is stable in β_α.
3. Let $\bar\alpha < \alpha$ both belong to C. Then $\beta_{\bar\alpha} = \mathrm{proj}_{\bar\alpha}(\beta_\alpha)$.

We refer to C as the domain of T, denoted $\mathrm{dom}(T)$, and to τ as the height of T, denoted $\mathrm{ht}(T)$. By Claim 1.3, the system of models and embeddings $\langle \mathrm{L}_{\gamma(\beta_\alpha)+1}, j_{\beta_\alpha,\beta_{\alpha'}} \mid \alpha, \alpha' \in C \wedge \alpha < \alpha'\rangle$ commutes. We use $\mathrm{dlm}(T)$ to denote the direct limit of this system, and refer to it as the **direct limit of the models**

of T. Let $\pi_{\alpha,\infty}\colon \mathrm{L}_{\gamma(\beta_\alpha)+1} \to \mathrm{dlm}(T)$ be the direct limit embeddings. If $\mathrm{dlm}(T)$ has the form $\mathrm{L}_{\gamma(\beta_\infty)+1}$ where $\beta_\infty = \pi_{\alpha,\infty}(\beta_\alpha)$ (for some/all $\alpha \in C$) then we say that β_∞ is the **limit** of T, denoted $\lim(T)$. Notice in this case that the direct limit embeddings $\pi_{\alpha,\infty}$ must be equal to the antiprojection embeddings $j_{\beta_\alpha,\beta_\infty}$ by Claim 1.2.

Remark 1.6. Typically we only consider threads with height of uncountable cofinality. The direct limit of the models of T is then wellfounded. By the elementarity of the direct limit embeddings $\mathrm{dlm}(T)$ must be a level of L, and in fact the first level which has a surjection of α onto β_∞. It follows that, for every thread T so that $\mathrm{ht}(T)$ has uncountable cofinality, the limit of T exists.

Claim 1.7. *Let β be a point on level τ, with τ a regular cardinal in* L. *Then there is thread $T \in \mathrm{L}$ of height τ with* $\lim(T) = \beta$.

Proof. The set D of $\alpha < \tau$ which are stable in β belongs to L. The set is closed, and using the regularity of τ a simple Lowenheim–Skolem argument inside L shows that D is unbounded in τ. The sequence $\langle \mathrm{proj}_\alpha(\beta) \mid \alpha \in D\rangle$ is therefore a thread. It is easy to check that the direct limit of this thread is $\mathrm{L}_{\gamma(\beta)+1}$ (and that the direct limit embeddings are $j_{\beta_\alpha,\beta}$). ⊣

Let β be a point on level α. If the set of levels stable in β is unbounded in α, then the sequence $\langle \mathrm{proj}_{\bar\alpha}(\beta) \mid \bar\alpha \text{ stable in } \beta\rangle$ is a thread with limit β, and is in fact the unique thread with limit β. We refer to it as the thread **leading** to β.

Let κ be a regular cardinal of L and let $\langle Q, \psi\rangle$ be a Σ^2_1 truth for κ. A point β on level $\alpha \le \kappa$ is said to **capture** $\langle Q, \psi\rangle$ just in case that:

1. $Q \cap \mathrm{L}_\alpha$ belongs to L_β.
2. There is $\eta < \gamma(\beta)$ and $B \subset \mathrm{L}_\beta$ in $\mathrm{L}_{\eta+1}$ so that $(\mathrm{L}_\beta; \in, B) \models \psi[Q \cap \mathrm{L}_\alpha]$.

The **witness** of β, denoted $\eta(\beta)$ is the least η witnessing condition (2). Notice then that there is a subset of L_β in $\mathrm{L}_{\eta(\beta)+1} - \mathrm{L}_{\eta(\beta)}$, and therefore:

Claim 1.8. *Every element of $\mathrm{L}_{\eta(\beta)+1}$ is definable in $\mathrm{L}_{\eta(\beta)+1}$ from parameters in L_β.* ⊣

The definitions in the next two paragraphs are made with reference to a fixed Σ^2_1 truth $\langle Q, \psi\rangle$, and apply to points which capture $\langle Q, \psi\rangle$.

Two points $\beta < \beta^*$ of the same level α are **compatible** just in case that there is an elementary embedding from $\mathrm{L}_{\eta(\beta)+1}$ into $\mathrm{L}_{\eta(\beta^*)+1}$ with critical point β. By Claim 1.8 the embedding is uniquely determined by β and β^*. We use φ_{β,β^*} to denote the embedding, and refer to it as a **horizontal embedding**, to emphasize that β and β^* are on the same level. If β, β^*, and β^{**} are compatible then using Claim 1.8 it is clear that $\varphi_{\beta,\beta^{**}} = \varphi_{\beta^*,\beta^{**}} \circ \varphi_{\beta,\beta^*}$.

For a set X of compatible points on the same level α, we use $\mathrm{hlim}(X)$ to denote the direct limit of the system $\langle \mathrm{L}_{\eta(\beta)+1}, \varphi_{\beta,\beta'} \mid \beta, \beta' \in X \wedge \beta < \beta'\rangle$. We refer to $\mathrm{hlim}(X)$ as a **horizontal direct limit**. If the direct limit is wellfounded then it must be a level of L, and by elementarity of the direct limit embeddings it must be the first level satisfying $(\exists B \subset \mathrm{L}_{\beta^*})(\mathrm{L}_{\beta^*}; \in, B) \models \psi[Q \cap \mathrm{L}_\alpha]$, where $\beta^* = \sup(X)$.

CLAIM 1.9. *Work in the settings of Claim 1.5. Suppose further that β and β^* capture $\langle Q, \psi \rangle$, and that $\bar{\alpha}$ is large enough that $Q \cap \mathrm{L}_\alpha$ is definable in $\mathrm{L}_{\gamma(\beta^*)+1}$ from parameters in $\bar{\alpha}$. Then:*

1. *$\bar{\beta}$ and $\bar{\beta}^*$ capture $\langle Q, \psi \rangle$.*
2. *$\bar{\beta}$ and $\bar{\beta}^*$ are compatible iff β and β^* are compatible.*
3. *(Assuming β and β^* are compatible.) φ_{β,β^*} is equal to $j^*(\varphi_{\bar{\beta},\bar{\beta}^*})$.*
4. *(Again assuming β and β^* are compatible.) $j_{\bar{\beta}^*,\beta^*} \circ \varphi_{\bar{\beta},\bar{\beta}^*} = \varphi_{\beta,\beta^*} \circ j_{\bar{\beta},\beta}$.*

PROOF. Note that $\eta(\beta)$ and $\eta(\beta^*)$ are both smaller than $\gamma(\beta^*)$. The fact that β and β^* capture $\langle Q, \psi \rangle$ can therefore be seen inside $\mathrm{L}_{\gamma(\beta^*)+1}$. Similarly the question of the compatibility of β and β^* can be answered inside $\mathrm{L}_{\gamma(\beta^*)+1}$, and φ_{β,β^*} can be identified inside $\mathrm{L}_{\gamma(\beta^*)+1}$. The first three conditions of the claim follow directly from these facts and the elementarity of j^*. Condition (4) follows from condition (3) here and condition (3) in Claim 1.5. $\dashv$

THEOREM 1.10. *Suppose that* PFA *holds for $\mathfrak{c}$-linked posets. Then ω_2^{V} is Σ^2_1 indescribable in L.*

PROOF. Let κ denote ω_2^{V}. κ is regular in L. Suppose that $\langle Q, \psi \rangle$ is a Σ^2_1 truth for κ in L. We aim to find $\bar{\kappa} < \kappa$ so that $\langle Q \cap \mathrm{L}_{\bar{\kappa}}, \psi \rangle$ is a Σ^2_1 truth for $\bar{\kappa}$ in L. Our plan is to force to add a set K of points below κ, so that the Σ^2_1 statement about Q can be expressed as a Π^1_1 statement about K. We then use the forcing axiom to reflect this statement, finding a system $\bar{K}$ of points below $\bar{\kappa} < \kappa$ satisfying the same Π^1_1 statement.

We first express the Σ^2_1 truth $\langle Q, \psi \rangle$ as a statement about a club E of points on level κ. We shall then force to add the set K so that the limits of threads through K are precisely the points in E.

CLAIM 1.11. *There is a club $E \subset \kappa^+$ so that every $\beta \in E$ is a point on level κ and captures $\langle Q, \psi \rangle$, so that every two points in E are compatible, and so that* $\mathrm{hlim}(E)$ *is wellfounded.*

PROOF. By assumption $\langle Q, \psi \rangle$ is a Σ^2_1 truth for κ in L. Let $\eta^* \geq \kappa^+$ be least so that a B witnessing this exists in L_{η^*+1}. For each $\beta \in (\kappa, \kappa^+)$ let X_β be the Skolem hull of $\beta \cup \{Q, \kappa^+, \eta^*\}$ in $\mathrm{L}_{\kappa^{++}}$. Let M_β be the transitive collapse of X_β, let $\pi_\beta \colon M_\beta \to X_\beta$ be the anticollapse embedding, and let $\eta_\beta = \pi_\beta^{-1}(\eta^*)$. Let $E \subset \kappa^+$ be a club so that for each $\beta \in E$, $X_\beta \cap \kappa^+ = \beta$. Notice then that $\pi_\beta(\beta) = \kappa^+$. It is easy to check that each β in E is a point that captures $\langle Q, \psi \rangle$ (further, $\eta(\beta) = \eta_\beta$), that any two points $\beta < \beta^*$ in E are compatible (further, φ_{β,β^*} is precisely $\pi_{\beta^*}^{-1} \circ \pi_\beta$), and that $\mathrm{hlim}(E) = \mathrm{L}_{\eta^*+1}$. $\dashv$

Define a poset $\mathbb{A}$ in V as follows. A condition is a countable set p of points so that:

(a) All the points in p capture $\langle Q, \psi \rangle$, and $\{\beta \in p \mid \alpha(\beta) = \kappa\} \subset E$.
(b) For every $\alpha < \kappa$, all the points in $\{\beta \in p \mid \alpha(\beta) = \alpha\}$ are compatible, and (assuming there are points in p on level α) $\mathrm{hlim}(\beta \in p \mid \alpha(\beta) = \alpha)$ is wellfounded.
(c) The set of $\{\alpha < \kappa \mid p$ has points on level $\alpha\}$ is closed (with a largest element).

We refer to $\{\beta \in p \mid \alpha(\beta) < \kappa\}$ as the **stem** of p, and to $\{\beta \in p \mid \alpha(\beta) = \kappa\}$ as the **commitment** of p. These sets are denoted $\mathrm{stem}(p)$ and $\mathrm{cmit}(p)$ respectively. We use $\mathrm{levels}(p)$ to denote the set of $\alpha < \kappa$ to that p has points on level α. The ordering of $\mathbb{A}$ is defined by setting $q \leq p$ just in case that:

(d) $p \subset q$.
(e) If $\alpha \in \mathrm{levels}(p)$ then p and q have the same points on level α. If $\alpha \in \mathrm{levels}(q) - \mathrm{levels}(p)$ then $\alpha \geq \sup(\mathrm{levels}(p))$.
(f) If $\alpha \in \mathrm{levels}(q) - \mathrm{levels}(p)$ then α is stable in every $\beta \in \mathrm{cmit}(p)$ and $\{\mathrm{proj}_\alpha(\beta) \mid \beta \in \mathrm{cmit}(p)\} \subset q$.
(g) If $\alpha \in \mathrm{levels}(q) - \mathrm{levels}(p)$ then α is large enough that: (1) for every $\beta < \beta'$ both in $\mathrm{cmit}(p)$, β is definable in $\mathrm{L}_{\gamma(\beta')+1}$ from parameters in α; and (2) for every $\beta \in \mathrm{cmit}(p)$, Q is definable in $\mathrm{L}_{\gamma(\beta)+1}$ from parameters in α.
(h) If $\alpha \in \mathrm{levels}(q) - \mathrm{levels}(p)$, $\beta, \beta' \in \mathrm{cmit}(p)$, and there are no elements of E between β and β', then there are no points in q between $\mathrm{proj}_\alpha(\beta)$ and $\mathrm{proj}_\alpha(\beta')$. Similarly if there are no elements of E below β, then there are no points in q on level α below $\mathrm{proj}_\alpha(\beta)$.

Claim 1.12. *Let p_n ($n < \omega$) be a sequence of conditions so that $p_{n+1} < p_n$ for each n. Then there is a condition q so that $(\forall n) q < p_n$.*

Proof. Let $p_\infty = \bigcup_{n<\omega} p_n$ and let $\alpha_\infty = \sup(\mathrm{levels}(p_\infty))$. By condition (f), α_∞ is a limit of ordinals stable in β, and therefore itself stable in β, for each $\beta \in \mathrm{cmit}(p_\infty)$. We may therefore set $a = \{\mathrm{proj}_{\alpha_\infty}(\beta) \mid \beta \in \mathrm{cmit}(p_\infty)\}$. By condition (g), α_∞ is large enough that for every $\beta < \beta'$ both in $\mathrm{cmit}(p_\infty)$, Q and β are definable in $\mathrm{L}_{\gamma(\beta')+1}$ from parameters in α_∞. We may therefore apply Claim 1.9 and conclude that all the points in a capture $\langle Q, \psi\rangle$, that they are all compatible, and that $\mathrm{hlim}(a)$ embeds into $\mathrm{hlim}(\mathrm{cmit}(p_\infty))$, which in turn embeds into $\mathrm{hlim}(E)$, and is therefore wellfounded. $q = p_\infty \cup a$ is therefore a condition in $\mathbb{A}$. A use of condition (2) in Claim 1.5 shows that if $\beta < \beta'$ both belong to $\mathrm{cmit}(p_\infty)$ and there are no elements of $\mathrm{cmit}(p_\infty)$ between them, then there are no elements of a between $\mathrm{proj}_{\alpha_\infty}(\beta)$ and $\mathrm{proj}_{\alpha_\infty}(\beta')$. Since $\mathrm{cmit}(p) \subset E$ this is enough to establish the first part of condition (h) for $\alpha = \alpha_\infty$ in verifying that $q < p_n$ for each n. The second part of condition (h) is similar, and the other conditions are easier. ⊣

Remark 1.13. $\mathbb{A}$ is countably closed, hence proper. Any two conditions in $\mathbb{A}$ with the same stem are compatible (their union is stronger than both). Since there are only $\kappa^\omega = (\omega_2^{\mathrm{V}})^\omega = \mathfrak{c}$ possible stems, $\mathbb{A}$ is $\mathfrak{c}$-linked.

$\mathbb{A}$ is proper and $\mathfrak{c}$-linked, but we are not yet done defining the poset to which we intend to apply $\mathsf{PFA}(\mathfrak{c}\text{-linked})$. We shall apply the axiom to a poset $\mathbb{A} * \dot{\mathbb{B}}$ where $\dot{\mathbb{B}}$ names a c.c.c. poset of size κ in $\mathrm{V}^{\mathbb{A}}$.

Claim 1.14. *Let p be a condition in $\mathbb{A}$. Let $\xi < \kappa$. Then there is $q \leq p$ so that q has points on levels above ξ.*

Proof. For $\beta < \beta'$ both in $\mathrm{cmit}(p)$ let $\nu_{\beta,\beta'} < \kappa$ be large enough that β is definable in $\mathrm{L}_{\gamma(\beta')+1}$ from parameters in $\nu_{\beta,\beta'}$. For $\beta \in \mathrm{cmit}(p)$ let ν_β be large enough that Q is definable in $\mathrm{L}_{\gamma(\beta)+1}$ from parameters in ν_β. For each $\beta \in \mathrm{cmit}(p)$ let T_β be the thread leading to β. The domain of T_β is club in κ, $\mathrm{cmit}(p)$

is countable, and $\kappa = \omega_2^{\mathrm{V}}$ has uncountable cofinality. So $\bigcap_{\beta \in \mathrm{cmit}(p)} \mathrm{dom}(T_\beta)$ is unbounded in κ. Let α belong to this intersection, with $\alpha > \xi$, $\alpha > \nu_\beta$, and $\alpha > \nu_{\beta,\beta'}$, for all $\beta, \beta' \in \mathrm{cmit}(p)$. Let $a = \{\mathrm{proj}_\alpha(\beta) \mid \beta \in \mathrm{cmit}(p)\}$. Using Claim 1.9 it is easy to check that all points in a are compatible, and that $\mathrm{hlim}(a)$ embeds into $\mathrm{hlim}(E)$ and is therefore wellfounded. So $q = p \cup a$ is a condition. It is easy to check that $q \leq p$ (condition (h) again uses Claim 1.5). $\dashv$

Let G be $\mathbb{A}$ generic over V. Let $K = \bigcup_{p \in G} \mathrm{stem}(p)$, and let $\dot{K}$ name K. By the last claim, K has points on unboundedly many levels below κ. For any $\alpha < \kappa$, the restriction of K to points of levels below α belongs to V, due to condition (e) in the definition of $\mathbb{A}$, and is countable in V. $\mathrm{levels}(K)$ is therefore a club of order type ω_1 in κ. In particular $\kappa = \omega_2^{\mathrm{V}}$ is collapsed to ω_1 in $\mathrm{V}[G]$.

A thread T of height κ is a thread **through** K if unboundedly many points of T belong to K.

Claim 1.15. *Let T be a thread of height κ and let $\beta = \lim(T)$. Then T is a thread through K iff $\beta \in E$.*

Proof. Suppose $\beta \in E$. By the genericity of G there is some $p \in G$ with $\beta \in p$. For every $\alpha > \sup(p)$ in $\mathrm{levels}(K)$, $\mathrm{proj}_\alpha(\beta)$ belongs to K. Since $\mathrm{proj}_\alpha(\beta)$ is a point in T it follows that T is a thread through K. Conversely suppose that $\beta \notin E$. Suppose initially that $\beta \not< \min(E)$, and let $\beta_1 < \beta$ be the largest element of E below β (recall that E is closed). Let $\beta_2 > \beta$ be the first element of E above β (recall that E is unbounded). By the genericity of G there is some $p \in G$ with $\beta_1, \beta_2 \in p$. Let $\nu < \kappa$ be large enough that β is definable in $\mathrm{L}_{\gamma(\beta_2)+1}$ from parameters in ν, and β_1 is definable in $\mathrm{L}_{\gamma(\beta)+1}$ from parameters in ν. Then for every $\alpha > \max\{\sup(p), \nu\}$ in $\mathrm{levels}(K)$, $\mathrm{proj}_\alpha(\beta_1) < \mathrm{proj}_\alpha(\beta) < \mathrm{proj}_\alpha(\beta_2)$, and using the first part of condition (h) in the definition of $\mathbb{A}$ it follows that $\mathrm{proj}_\alpha(\beta) \notin K$. Thus T has no points in K on levels above $\max\{\sup(p), \nu\}$. The case that $\beta < \min(E)$ is similar, using the second part of condition (h). $\dashv$

The proof of the last claim shows that T is a thread through K iff all sufficiently large points in T on levels in $\mathrm{levels}(K)$ belong to K. Let R_1 be the tree of attempts to contradict this. More precisely, a node in R_1 is a point β with $\alpha(\beta) \in \mathrm{levels}(K)$ and so that: (1) for unboundedly many $\bar{\alpha} < \alpha(\beta)$, $\mathrm{proj}_{\bar{\alpha}}(\beta)$ belongs to K; and (2) for unboundedly many $\bar{\alpha} < \alpha(\beta)$, $\bar{\alpha} \in \mathrm{levels}(K)$ yet $\mathrm{proj}_{\bar{\alpha}}(\beta) \notin K$ (possibly because $\bar{\alpha}$ is not stable in β and the projection is not defined). R_1 is ordered through projection: $\beta <_{R_1} \beta'$ iff $\beta = \mathrm{proj}_{\alpha(\beta)}(\beta')$. This order gives rise to a tree by Claim 1.4. Since a branch of length ω_1 through R_1 contradicts the fact that a thread T has unboundedly many points in K iff a tail-end of its points on levels in $\mathrm{levels}(K)$ are in K, we have:

Claim 1.16. *In $\mathrm{V}[G]$, there are no branches of length ω_1 through R_1.* $\dashv$

Let R_2 be the tree of attempts to create a thread with only boundedly many points of K to its right. More precisely, a node in R_2 is a pair $\langle \xi, \delta \rangle$ so that δ is a point, $\alpha(\delta) \in \mathrm{levels}(K)$, $\xi < \alpha(\delta)$, and for every $\bar{\alpha}$ which is stable in δ and greater than ξ, there are no points $\bar{\beta}$ of K on level $\bar{\alpha}$ with $\bar{\beta} > \mathrm{proj}_{\bar{\alpha}}(\delta)$. R_2 is ordered through projection on the second coordinate and equality on the first: $\langle \xi, \delta \rangle <_{R_2} \langle \xi', \delta' \rangle$ iff $\xi = \xi'$ and $\delta = \mathrm{proj}_{\alpha(\delta)}(\delta')$.

Claim 1.17. *In* $\mathrm{V}[G]$*, there are no branches of length* ω_1 *through* R_2*.*

Proof. Suppose for contradiction that $\langle\langle\xi,\delta_i\rangle \mid i<\omega_1\rangle$ is a branch through R_2. Since $\alpha(\delta_i)\in \mathrm{levels}(K)$ for each i, and since $\mathrm{levels}(K)$ is a set of order type ω_1 cofinal in κ, $\sup\{\alpha(\delta_i)\mid i<\omega_1\}$ is equal to κ. The sequence therefore generates a thread of height κ. Let δ^* be the limit of this thread. Let β^* be an element of E greater than δ^*. (This is possible since E is unbounded in κ^+.) Let $\bar{\alpha}<\kappa$ be such that $\bar{\alpha}>\xi$, δ^* is definable in $\mathrm{L}_{\gamma(\beta^*)+1}$ from parameters in $\bar{\alpha}$, $\bar{\alpha}$ is stable in β^*, and $\bar{\beta}=\mathrm{proj}_{\bar{\alpha}}(\beta^*)\in K$. (The last requirement is possible by Claim 1.15 since $\beta^*\in E$.) By Claim 1.5, $\bar{\alpha}$ is stable in δ^* and $\mathrm{proj}_{\bar{\alpha}}(\delta^*)<\bar{\beta}$. But for any i large enough that δ_i is on a level above $\bar{\alpha}$, this contradicts the fact that $\langle\xi,\delta_i\rangle$ is a node in R_2. ⊣

For a point δ define $\beta(\delta)$ to be the smallest $\beta>\delta$ in K on the same level as δ if there is one, and leave $\beta(\delta)$ undefined otherwise. The previous claim shows that for any thread $T=\langle\delta_\alpha\mid\alpha\in\mathrm{dom}(T)\rangle$ of height κ, $\beta(\delta_\alpha)$ is defined for unboundedly many $\alpha\in\mathrm{dom}(T)$.

Let R_3 be the following tree: A node in R_3 is a point δ so that $\alpha(\delta)\in\mathrm{levels}(K)$ and for every $\nu<\alpha(\delta)$ there are $\bar{\alpha}\neq\bar{\alpha}'$ between ν and $\alpha(\delta)$ so that $\beta(\mathrm{proj}_{\bar{\alpha}}(\delta))$ and $\beta(\mathrm{proj}_{\bar{\alpha}'}(\delta))$ are both defined, but neither is a projection of the other. R_3 is ordered through projection: $\delta<_{R_3}\delta'$ iff $\delta=\mathrm{proj}_{\alpha(\delta)}(\delta')$.

Claim 1.18. *In* $\mathrm{V}[G]$*, there are no branches of length* ω_1 *through* R_3*.*

Proof. Suppose for contradiction that $\langle\delta_i\mid i<\omega_1\rangle$ is a branch through R_3. The sequence then generates a thread of height κ. Let δ^* be the limit of this thread. Let $\beta_1^*\leq\delta^*$ be the largest element of E below δ^*, and let $\beta_2^*>\delta^*$ be the first element of E above β^*. An argument similar to that in the proof of Claim 1.15, using condition (h) in the definition of $\mathbb{A}$, shows that for all sufficiently large $\bar{\alpha}\in\mathrm{levels}(K)$: $\mathrm{proj}_{\bar{\alpha}}(\beta_1^*)\leq\mathrm{proj}_{\bar{\alpha}}(\delta^*)<\mathrm{proj}_{\bar{\alpha}}(\beta_2^*)$; $\mathrm{proj}_{\bar{\alpha}}(\beta_1^*)$ and $\mathrm{proj}_{\bar{\alpha}}(\beta_2^*)$ belong to K; and there are no elements of K between them. This implies that for all sufficiently large $\bar{\alpha}<\kappa$, if $\beta(\mathrm{proj}_{\bar{\alpha}}(\delta^*))$ is defined then it is equal to $\mathrm{proj}_{\bar{\alpha}}(\beta_2^*)$. It follows that there is $\nu<\kappa$ so that for all $\bar{\alpha}\neq\bar{\alpha}'$ between ν and κ, if $\beta(\mathrm{proj}_{\bar{\alpha}}(\delta^*))$ and $\beta(\mathrm{proj}_{\bar{\alpha}'}(\delta^*))$ are both defined then (they are equal to $\mathrm{proj}_{\bar{\alpha}}(\beta_2^*)$ and $\mathrm{proj}_{\bar{\alpha}'}(\beta_2^*)$ and hence) one is a projection of the other. But for any i large enough that δ_i is on a level above ν, this contradicts the fact that δ_i is a node in R_3. ⊣

Definition 1.19. A model M is said to satisfy the **singular square principle** if there is a map $\alpha\mapsto C_\alpha$, for $\alpha\in S=\{\text{singular cardinals of } M\}$, definable over M, so that:

(i) C_α is closed in α, contained in $S\cap\alpha$, and has order type strictly smaller than α.
(ii) If $\mathrm{cof}^M(\alpha)>\omega$ then C_α is unbounded in α. (Hence, in light of condition (i), C_α witnesses the singularity of α.)
(iii) If $\beta\in C_\alpha$ then $C_\beta=C_\alpha\cap\beta$.

By Jensen, L satisfies the singular square principle. Let S be the class of singular cardinals of L and let $\alpha\mapsto C_\alpha$, for $\alpha\in S$, witness the principle. Let R_0 be the tree of attempts to thread the sets C_α for $\alpha<\sup(\mathrm{levels}(K))=\kappa$.

Precisely, a node in R_0 is a pair $\langle \xi, \alpha \rangle$ so that α is a singular cardinal of L above the ξth element of levels(K). $\langle \xi, \alpha \rangle <_{R_0} \langle \xi', \alpha' \rangle$ iff $\xi < \xi'$ and $\alpha \in C_{\alpha'}$.

Claim 1.20. *In* V$[G]$*, there are no branches of length* ω_1 *through* R_0*.*

Proof. Suppose $\langle \langle \xi_i, \alpha_i \rangle \mid i < \omega_1 \rangle \in \mathrm{V}[G]$ is a branch through R_0. Let $D = \{\nu < \kappa \mid (\exists i) \nu \in C_{\alpha_i}\rangle$. Then $D \in \mathrm{V}[G]$ is club in κ. Let $\dot{D}$ name D, and suppose without loss of generality that it is outright forced in $\mathbb{A}$ that for any $\nu < \nu'$ both in $\dot{D}$, ν, ν' are singular in L and $\nu \in C_{\nu'}$. Using the fact that $\mathbb{A}$ is countably closed, it is easy to check that there is a club $\hat{D} \in \mathrm{V}$, so that for any $\nu < \nu'$ both in D and of cofinality ω in V, there is a condition in $\mathbb{A}$ forcing both ν and ν' into $\dot{D}$. It follows that for all $\nu < \nu'$ both in $\hat{D}$ and of cofinality ω, ν, ν' are singular in L and $\nu \in C_{\nu'}$. Using the fact that κ is regular in V fix $\alpha \in D$ of cofinality ω and so that $D \cap \alpha$ has order type α. Then α is singular in L and $D \cap \alpha \subset C_\alpha$. So C_α has order type α, contradicting condition (i) above. ⊣

The trees R_i for $i = 0, 1, 2, 3$ are defined with reference to K. When we wish to emphasize this dependence we write $R_i(K)$.

Let $\mathbb{B}$ be the poset for specializing the trees R_i, $i = 0, \dots, 3$. (See Jech [5, Equation (16.6)] or Baumgartner–Malitz–Reinhardt [1] for the definition. We are using the poset for specializing the disjoint union of the trees R_i.) Since the trees do not have branches of length ω_1 in V$[G]$, $\mathbb{B}$ is c.c.c. in V$[G]$.

Let $\dot{\mathbb{B}}$ name $\mathbb{B}$ and let $\mathbb{P}$ be the restriction of $\mathbb{A} * \dot{\mathbb{B}}$ to the set P of conditions $\langle p, \dot{f} \rangle$ in $\mathbb{A} * \dot{\mathbb{B}}$ so that p forces a value for $\dot{f}$. The restriction limits the number of conditions, so that the fact that $\mathbb{A}$ is $\mathfrak{c}$-linked and the fact that $\mathbb{B}$ has size $(\omega_2)^{\mathrm{V}}$ together imply that $\mathbb{P}$ is $\mathfrak{c}$-linked. Since $\mathbb{A}$ is countably closed and $\mathbb{B}$ is c.c.c., $\mathbb{A} * \dot{\mathbb{B}}$ is proper and hence so is $\mathbb{P}$. Apply $\mathsf{PFA}(\mathfrak{c}\text{-linked})$ to $\mathbb{P}$. Using an appropriate choice of dense sets we get a filter $\bar{G} * \bar{H} \subset \mathbb{P}$ so that:

1. $\bar{K} = \dot{K}[\bar{G}] = \bigcup_{p \in \bar{G}} \mathrm{stem}(p)$ is a set of points and each of the points in $\bar{K}$ captures $\langle Q, \psi \rangle$.
2. levels$(\bar{K})$ is a club of order type ω_1.
3. For every $\alpha \in$ levels$(\bar{K})$, all the points in $\bar{K}$ on level α are compatible, and the horizontal limit of these points is wellfounded.
4. Each of the trees $\bar{R}_i = R_i(\bar{K})$, $i = 0, 1, 2, 3$, is special, and therefore has no branches of length ω_1.

Standard arguments show that κ is a limit of cardinals of L. (If κ were the cardinal successor of some τ in L then the $\square_\tau$ sequence in L could be used to define a forcing that contradicts $\mathsf{PFA}(\mathfrak{c}\text{-linked})$. The argument is due to Todorčević.) We may therefore, through the choice of dense sets in $\mathbb{P}$, also make sure that:

5. $\bar{\kappa} = \sup(\text{levels}(\bar{K}))$ is a limit of cardinals of L.

By condition (2), $\bar{\kappa}$ has cofinality ω_1. Threads through $\bar{K}$ therefore have wellfounded direct limit models, and so every such thread has a limit. Let $\bar{E}$ be the set $\{\lim(T) \mid T \text{ a thread through } \bar{K}\}$. All points in $\bar{K}$ capture $\langle Q, \psi \rangle$, and by the elementarity of projection embeddings it follows that $\lim(T)$ captures $\langle Q, \psi \rangle$ whenever T is a thread through $\bar{K}$. Thus all points in $\bar{E}$ capture $\langle Q, \psi \rangle$. We intend to show that all these points are compatible, that they have a wellfounded horizonal limit, and that $\sup(\bar{E}) = \bar{\kappa}^+$. The horizonal limit of $\bar{E}$ then gives rise to a witness that $\langle Q \cap \mathrm{L}_{\bar{\kappa}}, \psi \rangle$ is a Σ^2_1 truth about $\bar{\kappa}$.

Claim 1.21. *Let β, β^* belong to $\bar{E}$. Then β and β^* are compatible.*

Proof. β is a limit of a thread through $\bar{K}$, and so there are unboundedly many $\alpha < \bar{\kappa}$ so that $\mathrm{proj}_\alpha(\beta) \in \bar{K}$. Using the fact that $\bar{R}_1$ has no branches of length ω_1 it follows that in fact $\mathrm{proj}_\alpha(\beta) \in \bar{K}$ for all sufficiently large $\alpha \in \mathrm{levels}(\bar{K})$. A similar argument applies to β^*. Thus there is $\nu < \bar{\kappa}$ so that for all $\alpha > \nu$ in $\mathrm{levels}(\bar{K})$, $\mathrm{proj}_\alpha(\beta)$ and $\mathrm{proj}_\alpha(\beta^*)$ are defined (meaning that α is stable in both), and both belong to $\bar{K}$. Using condition (3) above it follows that $\mathrm{proj}_\alpha(\beta)$ and $\mathrm{proj}_\alpha(\beta^*)$ are compatible.

Let $\alpha > \nu$ in $\mathrm{levels}(\bar{K})$ be large enough that β and $Q \cap \mathrm{L}_{\bar{\kappa}}$ are definable in $\mathrm{L}_{\gamma(\beta^*)+1}$ from parameters in α. By the last paragraph, α is stable in β^*, and $\mathrm{proj}_\alpha(\beta)$ and $\mathrm{proj}_\alpha(\beta^*)$ are compatible. Using Claim 1.9 it follows that β and β^* are compatible. ⊣

Claim 1.22. $\mathrm{hlim}(\bar{E})$ *is wellfounded.*

Proof. Suppose not. Let $\beta_i \in E$ and $\xi_i \leq \eta(\beta_i)$ be such that $\varphi_{\beta_i,\beta_{i+1}}(\xi_i) > \xi_{i+1}$ for each $i < \omega$. Let $\nu_i < \bar{\kappa}$ be large enough that β_i, $Q \cap \mathrm{L}_{\bar{\kappa}}$, and ξ_i are definable in $\mathrm{L}_{\gamma(\beta_{i+1})+1}$ from parameters in ν_i. Let $\nu'_i < \bar{\kappa}$ be large enough that $\mathrm{proj}_\alpha(\beta_i)$ is defined and belongs to $\bar{K}$ for every $\alpha > \nu'_i$ in $\mathrm{levels}(\bar{K})$. (We are using here the fact that R_1 has no branches of length ω_1, as in the previous claim.)

Let $\alpha \in \mathrm{levels}(\bar{K})$ be greater than $\sup\{\nu_i, \nu'_i \mid i < \omega\}$. Let $\bar{\beta}_i = \mathrm{proj}_\alpha(\beta_i)$ and let $\bar{\xi}_i = j^{-1}_{\bar{\beta}_i,\beta_i}(\xi_i) = j^{-1}_{\bar{\beta}_{i+1},\beta_{i+1}}(\xi_i)$. By Claim 1.9, $\varphi_{\bar{\beta}_i,\bar{\beta}_{i+1}}(\bar{\xi}_i) > \bar{\xi}_{i+1}$. But then since $\bar{\beta}_i \in \bar{K}$, the horizonal limit of the points of $\bar{K}$ on level α is illfounded. This contradicts condition (3) above. ⊣

Claim 1.23. $\bar{\kappa}$ *is regular in* L.

Proof. Suppose not. Since $\bar{\kappa}$ is a limit of cardinals in L it must then be a singular cardinal in L. So $C_{\bar{\kappa}}$ is defined, and is club in $\bar{\kappa}$. For each $\xi < \omega_1$ let α_ξ be an element of $C_{\bar{\kappa}}$ greater than the ξth element of $\mathrm{levels}(\bar{K})$, and greater than $\sup\{\alpha_\zeta \mid \zeta < \xi\}$. Notice that $C_{\alpha_\xi} = C_{\bar{\kappa}} \cap \alpha_\xi$ and therefore $\alpha_\zeta \in C_{\alpha_\xi}$ for $\zeta < \xi < \omega_1$. So $\langle \langle \xi, \alpha_\xi \rangle \mid \xi < \omega_1 \rangle$ is a branch of length ω_1 through $\bar{R}_0$, contradicting the fact that the tree is special. ⊣

Having established that $\bar{\kappa}$ is regular in L we may apply Claim 1.7 and conclude that for every point δ on level $\bar{\kappa}$, there is a thread of height $\bar{\kappa}$ with limit δ. We will use this in the following claim.

Claim 1.24. $\bar{E}$ *is unbounded in* $\bar{\kappa}^+$.

Proof. Fix a point $\delta \in (\bar{\kappa}, \bar{\kappa}^+)$. We produce $\beta \in \bar{E}$ with $\beta > \delta$.

Let C be the set of $\alpha < \bar{\kappa}$ which are stable in δ. By Claim 1.7, C is club in $\bar{\kappa}$ and $\langle \mathrm{proj}_\alpha(\delta) \mid \alpha \in C \rangle$ is a thread with limit δ.

Let D be the set of $\alpha \in \mathrm{levels}(\bar{K})$ so that there is a point β in $\bar{K}$ on level α with $\beta > \mathrm{proj}_\alpha(\delta)$. Let β_α be the least such.

Since R_2 has no branches of length ω_1, D is unbounded in $\bar{\kappa}$. Since R_3 has no branches of length ω_1, there is $\nu < \kappa$ so that for all $\alpha, \alpha' \in D$ between ν and $\bar{\kappa}$, one of β_α, $\beta_{\alpha'}$ is a projection of the other. It follows that $\{\beta_\alpha \mid \alpha \in D \wedge \alpha > \nu\}$ generate a thread. This is a thread through $\bar{K}$, and using Claim 1.5 it is easy to see that the limit of this thread is greater than δ. ⊣

Let $M = \mathrm{hlim}(\bar{E})$. The limit makes sense by Claim 1.21. M is wellfounded by Claim 1.22, and is therefore a level of L. (If $\mathrm{cof}(\bar{\kappa}^+) > \omega$ then the wellfoundedness of M is immediate, but $\mathrm{cof}(\bar{\kappa}^+) = \omega$ is possible, for example if $0^\sharp$ exists.) Each of the points in $\bar{E}$ captures $\langle Q, \psi \rangle$, and so using the elementarity of the horizonal limit embeddings it follows that M satisfies "there exists $B \subset \mathrm{L}_{\beta^*}$ so that $(\mathrm{L}_{\beta^*}; \in, B) \models \psi[Q \cap \mathrm{L}_{\bar{\kappa}}]$," where β^* stands for $\sup(\bar{E})$, which by Claim 1.24 is equal to $\bar{\kappa}^+$. Thus in L there exists $B \subset \mathrm{L}_{\bar{\kappa}^+}$ so that $(\mathrm{L}_{\bar{\kappa}^+}; \in, B) \models \psi[Q \cap \mathrm{L}_{\bar{\kappa}}]$. This completes the proof of Theorem 1.10. $\dashv$

§2. A Σ^2_1 indescribable 1-gap. Throughout this section we work with a class model $M = \mathcal{J}[\vec{E}]$ where $\vec{E}$ is a coherent sequence of short extenders in the style of Zeman [18]. $M \| \beta$ below denotes the structure $(\mathcal{J}_\beta[\vec{E} \restriction \beta]; E_\beta)$. We only need a few properties of the inner model $\mathcal{J}[\vec{E}]$, summarized in the following list:

- (Acceptability) If there is a subset of κ in $(M \| \gamma + 1) - (M \| \gamma)$ then there is a surjection of κ onto $M \| \gamma$ in $M \| \gamma + 1$.
- (Condensation) Suppose that κ is the largest cardinal in $M \| \gamma + 1$, X is an elementary substructure of $M \| \gamma + 1$, and $X \cap \kappa = \bar{\kappa} \leq \kappa$. Let P be the transitive collapse of X. Then either there is $\bar{\gamma}$ so that $P = M \| \bar{\gamma} + 1$, or (if $\bar{\kappa}$ indexes an extender in M) there is $\bar{\gamma}$ so that $P = \mathrm{Ult}(M, E_{\bar{\kappa}}) \| \bar{\gamma} + 1$.
- Acceptability and condensation hold not only for M, but also for ultrapowers of M.
- M satisfies the singular square principle (see Definition 1.19).
- $\square_\kappa$ holds in M for all κ which are not subcompact (and hence certainly for all singular κ).

The first three conditions are part of the standard theory of fine structural inner models. The last condition is due to Schimmerling-Zeman [10, 11]. The condition before last is due to Zeman [17].

The second possibility in the condensation statement forces us to work not only with initial segments of M, but also with initial segments of ultrapowers of M. The following lemma helps separate the two cases.

Lemma 2.1. *Let β be an ordinal. Then at most one of the following two conditions holds. Moreover, if condition (2) holds then there is exactly one ordinal α witnessing it. (The same is true, trivially, with condition (1).)*

1. *There is $\alpha < \beta$ so that $M \| \beta \models$"α is the largest cardinal," β remains a cardinal in $M \| \beta + 1$, yet β is not a cardinal in $\mathcal{J}[\vec{E}]$.*
2. *There is $\alpha < \beta$ so that α indexes an extender in $\vec{E}$, $\mathrm{Ult}(M, E_\alpha) \| \beta \models$"$\alpha$ is the largest cardinal," β remains a cardinal in $\mathrm{Ult}(M, E_\alpha) \| \beta + 1$, yet β is not a cardinal in $\mathrm{Ult}(M, E_\alpha)$.*

Proof. Suppose condition (2) holds and is witnessed by α. We prove that condition (1) fails, and that there is no $\alpha' > \alpha$ witnessing condition (2).

Note that E_α belongs to $M \| \beta + 1$, and since $\beta < (\alpha^+)^{\mathrm{Ult}(M, E_\alpha)}$, from E_α one can define, inside $M \| \beta + 1$, a surjection of $(\mathrm{crit}(E_\alpha)^{++})^{M \| \alpha} \times \mathrm{spt}(E_\alpha)$ onto β. So β is not a cardinal in $M \| \beta + 1$, in contradiction to condition (1).

Suppose now $\alpha' > \alpha$, and α' is also a witness for condition (2). From $M \| \alpha$ and E_α one can define a surjection of α onto β, and so certainly onto α'. It follows that α' is not a cardinal in $\mathcal{J}_{\alpha'+1}[\vec{E}\restriction\alpha']$, and therefore cannot index an extender in $\vec{E}$. $\dashv$

A **point** in this section is an ordinal β for which one of the conditions in Lemma 2.1 holds. If the first conditions holds for β then we refer to β as a **type one** point. If the second condition holds then β is a **type two** point. By the lemma, these two cases do not overlap. Again by the lemma, the ordinal α witnessing the condition is determined uniquely by β. We refer to this ordinal as $\alpha(\beta)$.

Remark 2.2. If α is a cardinal of M then it does not index an extender on $\vec{E}$, and it follows that all points on level α are of type one. If α indexes an extender on $\vec{E}$ then it is not a cardinal in $M\|\beta$ for any $\beta > \alpha$ and it follows that all points on level α are of type two.

For β of type one let $\gamma(\beta)$ be least so that $M\|\gamma(\beta)+1$ has a surjection of α onto β, and let M_β and $M_{\gamma(\beta)+1}$ denote $M\|\beta$ and $M\|\gamma(\beta)+1$. For β of type two define $\gamma(\beta)$ similarly but using $\mathrm{Ult}(M, E_\alpha)$ instead of M, and let M_β and $M_{\gamma(\beta)+1}$ denote $\mathrm{Ult}(M, E_\alpha)\|\beta$ and $\mathrm{Ult}(M, E_\alpha)\|\gamma(\beta)+1$.

Now define the notions stable, projection, antiprojection embedding, thread, direct limit of the models of a thread, and limit of a thread as in Section 1, but replacing $\mathrm{L}_{\gamma(\beta)+1}$ by $M_{\gamma(\beta)+1}$ throughout. Define the capturing of a Σ^2_1 truth as in Section 1, but replacing L by M if β is of type one, and by $\mathrm{Ult}(M, E_{\alpha(\beta)})$ if β is of type two. Define $\eta(\beta)$, compatibility, horizontal embeddings, and horizontal direct limits similarly. Let $M_{\eta(\beta)+1}$ denote $M\|\eta(\beta)+1$ if β is of type one, and $\mathrm{Ult}(M, E_{\alpha(\beta)})\|\eta(\beta)+1$ if β is of type two.

Claims 1.2 through 1.9 hold in the new settings, as their proofs depend only on acceptability and condensation. Let us just note that, because of the extra ultrapower clause in the condensation condition, the projection of a point of type one may very well be a point of type two, and this is the reason we require the two types. (It is also true that the projection of a point of type two may be a point of type one.) For the most part there is no need to distinguish between the types, as the same claims hold for both, albeit with different meanings for M_β, $M_{\eta(\beta)+1}$, and $M_{\gamma(\beta)+1}$.

Remark 1.6 need *not* hold in the new settings. The direct limit model $\mathrm{dlm}(T)$, even if it is wellfounded and indeed iterable, need not be a level of M as its extender sequence need not in general agree with that of M. There are a few ways to get around this problem. One is to assume that M satisfies some "core model like" maximality principles in V. Another, which we take in this paper, is to assume that V is a forcing extension of M by a proper poset.

Lemma 2.3. *Assume that* V *is a forcing extension of M by a proper poset. Let κ be a regular uncountable cardinal of M. Let T be a thread of height κ in* V. *Then* $\lim(T)$ *exists.*

Proof. Let $\mathbb{P} \in M$ and G be such that $\mathrm{V} = M[G]$, with $\mathbb{P}$ proper in M and G generic for $\mathbb{P}$ over M. Let $\dot{T}$ be a name for a thread of height κ. Let $T = \dot{T}[G]$. Let $Q = \mathrm{dlm}(T)$. If Q is a level of M then its first order properties imply that it has the form $M_{\gamma(\beta)+1}$ for a point β on level κ, and therefore $\lim(T)$ exists. Suppose then for contradiction that Q is *not* a level of M.

Let δ be largest so that $\vec{E}^M \restriction \delta = \vec{E}^Q \restriction \delta$ and so that δ is a limit of points on level κ. ($\vec{E}^M$ and $\vec{E}^Q$ agree to κ, and in fact they agree at least to the successor of κ in $\mathrm{L}(\vec{E}^M \restriction \kappa) = \mathrm{L}(\vec{E}^Q \restriction \kappa)$. The successor of κ in $\mathrm{L}(\vec{E}^M \restriction \kappa)$ is a limit of points on level κ, so δ is well defined and greater than or equal to this successor.) Without loss of generality we may assume that Q has only finitely many points on level κ above δ. (By a point of Q we mean an ordinal β satisfying one of the conditions in Lemma 2.1, but with M replaced by Q.) For if Q had infinitely many points on level κ above δ, then one of them, η say, would be such that $\vec{E}^Q \restriction \gamma(\eta)+1 \neq \vec{E}^M \restriction \gamma(\eta)+1$, meaning that already $Q \| \gamma(\eta)+1$ is not a level of M. We could then replace T by a thread T' so that $\mathrm{dlm}(T') = Q \| \gamma(\eta) + 1$.

Let $\langle \beta_\alpha \mid \alpha \in \mathrm{dom}(T) \rangle$ be the thread T, and let $j_\alpha \colon M_{\gamma(\beta_\alpha)+1} \to Q$ be the embeddings of the direct limit along the thread. Let $\beta = j_\alpha(\beta_\alpha)$ for some/any $\alpha \in \mathrm{dom}(T)$. β is then the largest point of Q on level κ. By the reasoning of the previous paragraph, there is $k < \omega$ so that β is the kth point of Q on level κ above δ. For each $\alpha \in \mathrm{dom}(T)$ let $\delta_\alpha = j_\alpha^{-1}(\delta)$. Then:

Claim 2.4. *β_α is the kth point (in the sense of M) on level α above δ_α.*

Proof. Since j_α is elementary, $\beta_\alpha = j_\alpha^{-1}(\beta)$ is the kth point of $M_{\gamma(\beta_\alpha)+1}$ above $\delta_\alpha = j_\alpha^{-1}(\delta)$. $M_{\gamma(\beta_\alpha)+1}$ is an initial segment, either of M or of $\mathrm{Ult}(M, E_\alpha)$. Using this it is easy to verify that being a point on level α is absolute between $M_{\gamma(\beta_\alpha)+1}$ and M. $\dashv$

The claim shows that the thread T can be recovered from the sequence $\langle \delta_\alpha \mid \alpha \in \mathrm{dom}(T) \rangle$. Our first step is to show that this sequence and the thread T both belong to M.

For each $\alpha < \kappa$ let Z_α be the Skolem hull of α in Q. Call α stable in Q just in case that $Z_\alpha \cap \kappa = \alpha$. Notice that $\alpha \in \mathrm{dom}(T)$ iff α is stable in Q, $M_{\gamma(\beta_\alpha)+1}$ is precisely the transitive collapse of Z_α in this case, and j_α is precisely the anticollapse embedding.

Claim 2.5. *Let α be stable in Q. Let $\xi < \nu < \delta$, with ν a point on level κ and a member of Z_α. Then $\xi \in Z_\alpha$ iff ξ is definable in $M_{\gamma(\nu)+1}$ from parameters in α.*

Proof. The right-to-left direction is immediate from the definitions as $M_{\gamma(\nu)+1}$ itself is definable in Q from ν. For the left-to-right direction: Suppose $\xi \in Z_\alpha$. Every ordinal below ν, and in fact every element of $M_{\gamma(\nu)+1}$, is definable in $M_{\gamma(\nu)+1}$ from parameters in κ. As $\nu \in Z_\alpha$ and Z_α is an elementary substructure of Q, it follows that every ordinal below ν in Z_α, and in particular the ordinal ξ, is definable in $M_{\gamma(\nu)+1}$ from parameters in $\kappa \cap Z_\alpha$, namely in α. $\dashv$

Claim 2.6. *Suppose δ has countable cofinality (in $\mathrm{V} = M[G]$ and therefore also in M). Then T belongs to M.*

Proof. Let $U \in M$ be a countable set of points on level κ, cofinal in δ. Fix $\tau < \kappa$ large enough that every point in U is definable in Q from parameters in τ. (This is possible since U is countable and κ has uncountable cofinality.)

Set $Y_\alpha = \{\xi < \delta \mid$ there is a point ν on level κ so that $\nu > \xi$, $\nu \in U$, and ξ is definable in $M_{\gamma(\nu)+1}$ from parameters in $\alpha\}$. By Claim 2.5, $Y_\alpha = Z_\alpha \cap \delta$ for every $\alpha > \tau$ which is stable in Q. Hence δ_α is precisely equal to the order type of Y_α.

But notice that Y_α is defined in M with no reference to the generic G. Thus there is a function $\alpha \mapsto \delta^*_\alpha$, inside M, so that $\delta_\alpha = \delta^*_\alpha$ for every $\alpha > \tau$ which is stable in Q. Using Claim 2.4 it follows that there is a function $\alpha \mapsto \beta^*_\alpha$, again inside M, so that $\beta_\alpha = \beta^*_\alpha$ for all sufficiently large $\alpha \in \mathrm{dom}(T)$. Suppose for simplicity that $\beta_\alpha = \beta^*_\alpha$ for all $\alpha \in \mathrm{dom}(T)$.

From the fact that every point in T is a projection of every greater point in T, and that T is closed under projections, it follows that:

$$\begin{aligned}\bar\beta \text{ is a point in } T &\iff \{\alpha < \kappa \mid \bar\beta \text{ is a projection of } \beta^*_\alpha\} \supset \mathrm{dom}(T) - \bar\beta\\ &\iff \{\alpha < \kappa \mid \bar\beta \text{ is a projection of } \beta^*_\alpha\} \cap (\mathrm{dom}(T) - \bar\beta) \neq \emptyset.\end{aligned}$$

From this, the fact that $\mathrm{dom}(T)$ is club, and the fact that $\mathbb{P}$ preserves stationary subsets of $\{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$ (a consequence of properness), it follows now that:

$$\begin{aligned}&\bar\beta \text{ is a point in } T \iff\\ &\quad \mathrm{V} \models \text{“}\{\alpha < \kappa \mid \bar\beta \text{ is a projection of } \beta^*_\alpha\} \text{ contains an } \omega\text{-club in } \kappa\text{”} \iff\\ &\qquad M \models \text{“}\{\alpha < \kappa \mid \bar\beta \text{ is a projection of } \beta^*_\alpha\} \text{ contains an } \omega\text{-club in } \kappa.\text{”}\end{aligned}$$

The final clause in the equivalence makes implicit use of the map $\alpha \mapsto \beta^*_\alpha$, and most importantly the presence of this map in M. It follows from the equivalence that T belongs to M. ⊣

Claim 2.7. *Suppose there is $\tau < \kappa$ so that the set $\{\xi < \delta \mid \xi$ is definable in Q from parameters in $\tau\}$ is cofinal in δ. Then T belongs to M.*

Proof. If δ has countable cofinality then $T \in M$ by the previous claim. So suppose that δ has uncountable cofinality. For $i < \omega$ let $Z_{\tau,i}$ consist of elements of Q which are Σ_i definable in Q from parameters in τ. Fix i large enough that $Z_{\tau,i} \cap \delta$ is cofinal in δ. (This is possible since $\bigcup_{i<\omega} Z_{\tau,i}$ is cofinal in δ, and δ has uncountable cofinality.) Let C be the set of limit points of $Z_{\tau,i}$ below δ. Then C is club in δ, and, because $Z_{\tau,i}$ itself is definable in Q from the parameters τ and β, $C \subset Z_\alpha$ for every $\alpha > \tau$.

For each $\xi < \delta$ and $\alpha < \kappa$ set $S_\alpha(\xi) = \{\eta \mid \xi < \eta < \delta$ and ξ is definable in $M_{\gamma(\nu)+1}$ from parameters in α, where ν is the first point on level κ above $\eta\}$. Notice that the map $\xi, \alpha \mapsto S_\alpha(\xi)$ is defined inside M, that is with no reference to G.

Suppose $\alpha > \tau$ is stable in Q. Then by Claim 2.5 and the fact that $Z_\alpha \supset C$,

$$\begin{aligned}\xi \text{ belongs to } Z_\alpha &\iff S_\alpha(\xi) \supset C - \xi\\ &\iff S_\alpha(\xi) \cap (C - \xi) \neq \emptyset.\end{aligned}$$

Using the fact that the map $\xi, \alpha \mapsto S_\alpha(\xi)$ belongs to M, and the fact that $\mathbb{P}$ preserves stationary subsets of $\{\eta < \delta \mid \mathrm{cof}(\eta) = \omega\}$, it therefore follows that:

$$\xi \text{ belongs to } Z_\alpha \iff M \models \text{“}S_\alpha(\xi) \text{ contains an } \omega\text{-club in } \delta.\text{”}$$

Working in M let $Y_\alpha = \{\xi < \delta \mid S_\alpha(\xi)$ contains an ω-club in $\delta\}$. The map $\alpha \mapsto Y_\alpha$ belongs to M, and we just saw that $Y_\alpha = Z_\alpha \cap \delta$ for every $\alpha > \tau$ which is stable in Q. We can now define maps $\alpha \mapsto \delta^*_\alpha$ and $\alpha \mapsto \beta^*_\alpha$ as in the proof of the previous claim, and follow the argument there to establish that T belongs to M. $\dashv$

Claim 2.8. *Suppose that δ has uncountable cofinality and that there is no $\tau < \kappa$ so that the set $\{\xi < \delta \mid \xi$ is definable in Q from parameters in $\tau\}$ is cofinal in δ. Then T belongs to M.*

Proof. Let θ be a regular cardinal much larger than κ. Say that a countable $X \subset \theta$ **extends to an elementary substructure** if there is $H \subset \mathrm{V}_\theta$ so that $\kappa, T, \delta, \beta, Q \in H$, $H \cap \theta = X$, and H is elementary in V_θ. Let $C = \{X \in [\theta]^\omega \mid X$ extends to an elementary substructure$\}$. C is club in $[\theta]^\omega$.

For $X \in C$, note that:

(i) $\alpha = \sup(X \cap \kappa)$ is stable in Q.
(ii) $X \cap \delta \subset Z_\alpha$.
(iii) $\sup(Z_\alpha \cap \delta) \leq \sup(X \cap \delta)$ (and hence by item (ii) the two are equal).

Item (i) holds since, by elementarity, α is a limit of levels of points in T. Item (ii) holds since every $\xi < \delta$ is definable in Q from parameters in κ, hence by elementarity every $\xi < \delta$ in X is definable in Q from parameters in $X \cap \kappa$ and so certainly from parameters in α. Item (iii) uses the assumption in Claim 2.8. For every $\tau \in X \cap \kappa$, $\sup\{\xi < \delta \mid \xi$ is definable in Q from parameters in $\tau\}$ belongs to X and is, by the claim assumption, smaller than δ. This supremum is therefore smaller than $\sup(X \cap \delta)$, and item (iii) follows.

Working in M set, for each $X \in [\theta]^\omega$, $\alpha_X = \sup(X \cap \kappa)$ and $Y_X = \{\xi < \delta \mid (\exists \nu > \xi)\ \nu$ is a point on level κ in $X \cap \delta$ and ξ is definable in $M_{\gamma(\nu)+1}$ from parameters in $\alpha_X\}$. Let δ_X be the ordertype of Y_X, and let β_X be the kth point on level α_X above δ_X. (k here is the number used in Claim 2.4.) Note that the map $X \mapsto \beta_X$ belongs to M.

By Claim 2.5 and items (i)–(iii), $Y_X = Z_{\alpha_X} \cap \delta$ whenever $X \in C$, and therefore $\delta_X = \delta_{\alpha_X}$ and $\beta_X = \beta_{\alpha_X}$. From this, the fact that C is club in $[\theta]^\omega$, that every point in T is a projection of every greater point in T, that T is closed under projections, and that $\mathbb{P}$ preserves stationary subsets of $[\theta]^\omega$, it follows that $\bar\beta$ is a point in T iff $M \models$"$\{X \in [\theta]^\omega \mid \bar\beta$ is a projection of $\beta_X\}$ contains a club in $[\theta]^\omega$." So T belongs to M. $\dashv$

The last three claims together establish that the thread T belongs to M. We now complete the proof of Lemma 2.3 by showing that $Q = \mathrm{dlm}(T)$ is a level of M. Fix some regular cardinal θ of M much larger than κ. Working inside M let X be an elementary substructure of $M\|\theta$, with $T, \kappa \in X$, $\mathrm{card}(X) < \kappa$, and $X \cap \kappa$ an ordinal. This is possible since κ is a regular cardinal in M. Let $\bar M$ be the transitive collapse of X and let $\pi\colon \bar M \to M$ be the anticollapse embedding. By condensation, $\bar M$ is an initial segment, either of $\mathrm{Ult}(M, E_\alpha)$ or of M, depending on whether α indexes an extender in M. Let $\alpha = X \cap \kappa$. It is easy to check that $X \cap Q = Z_\alpha$, and therefore $\pi^{-1}(Q)$ is precisely equal to $M_{\gamma(\beta_\alpha)+1}$ (where β_α is the point of T on level α). It follow from this, the meaning of $M_{\gamma(\beta_\alpha)+1}$, and the fact that $\pi^{-1}(Q) \in \bar{M}$, that $\pi^{-1}(Q)$ is a level of $\bar{M}$. By the elementarity of π then Q is a level of M. ⊣ (Lemma 2.3)

Claim 2.9. *Assume that* V *is a forcing extension of M by a proper poset. Let κ be a successor cardinal of M. Let $\langle Q, \psi\rangle$ be a Σ^2_1 truth in M. Let E be a set of points in M which capture $\langle Q, \psi\rangle$, all on level κ, and pairwise compatible. Suppose that E is cofinal in $(\kappa^+)^M$. Then the horizonal direct limit* hlim(E) *belongs to M.*

Proof. Let $\mu < (\kappa^+)^M$ be large enough that $Q \cap M\| \kappa$ belongs to $M\| \mu$. We work only with $\beta > \mu$, even when this is not stated explicitly.

Let $N = \mathrm{hlim}(E)$. For each $\beta < (\kappa^+)^M$ let Z_β be the Skolem hull of β in N. Let $C = \{\beta \mid Z_\beta \cap (\kappa^+)^M = \beta\}$. C is club in $(\kappa^+)^M$. Note that if $\beta \in C$ then κ is the largest cardinal in $M\|\beta$, and since κ is a successor cardinal of M it follows that β does not index an extender. Thus, by condensation, the transitive collapse of Z_β is a level of M. It is easy using this to verify that β is a point on level κ and that $M_{\eta(\beta)+1}$ is precisely the transitive collapse of Z_β. So $\mathrm{hlim}(C) = N = \mathrm{hlim}(E)$ and it is enough to prove that $C \in M$. Note that:

(i) If $\beta < \beta^*$ are both in C then there is an elementary embedding from $M_{\eta(\beta)+1}$ into $M_{\eta(\beta^*)+1}$ with critical point β.
(ii) If $\beta^* \in C$ and there is an elementary embedding from $M_{\eta(\beta)+1}$ into $M_{\eta(\beta^*)+1}$ with critical point β, then $\beta \in C$.

For $\beta < (\kappa^+)^M$ set $S_\beta = \{\beta^* < (\kappa^+)^M \mid$ there is an elementary embedding from $M_{\eta(\beta)+1}$ into $M_{\eta(\beta^*)+1}$ with critical point $\beta\}$. The map $\beta \mapsto S_\beta$ belongs to M.

Since the poset leading from M to V is proper, $(\kappa^+)^M$ has uncountable cofinality in V. From this, conditions (i) and (ii), and the fact that proper forcing extensions preserve stationary sets of $\{\beta < (\kappa^+)^M \mid \mathrm{cof}(\beta) = \omega\}$, it follows that $\beta \in C$ iff $M \models$"S_β contains an ω-club." So C belongs to M. ⊣

Definition 2.10. Let λ be a regular cardinal of M, of cofinality ω_1 in V. Let $u = \langle \mu_\xi \mid \xi < \omega_1\rangle$ be an increasing sequence of points on level λ. Let $a = \langle \lambda_\xi \mid \xi < \omega_1\rangle$ be increasing and cofinal in λ. Define $S = S(a, u)$ to be the tree of attempts to create a thread of height λ that dominates u. Precisely, a node in S is a pair $\langle \xi, \beta\rangle$ so that $\xi < \omega_1$, β is a point on a level $\alpha(\beta)$ above λ_ξ and below λ, $\alpha(\beta)$ is stable in each of the points μ_ζ for $\zeta < \xi$, and $\beta > \mathrm{proj}_{\alpha(\beta)}(\mu_\zeta)$ for each $\zeta < \xi$. S is ordered through the natural order on the first coordinate and projection on the second: $\langle \xi, \beta\rangle <_S \langle \xi', \beta'\rangle$ iff $\xi < \xi'$ and β is a projection of β'.

Claim 2.11. *(Assuming* V *is an extension of M by a proper poset.) S has a branch of length ω_1 iff* $\sup\{\mu_\xi \mid \xi < \omega_1\} < (\lambda^+)^M$.

Proof. Suppose first that $\sup\{\mu_\xi \mid \xi < \omega_1\} < (\lambda^+)^M$. Let β be a point on level λ, greater than $\sup\{\mu_\xi \mid \xi < \omega_1\}$. Let T be the thread leading to β. For each $\xi < \omega_1$ let $\alpha_\xi < \lambda$ be stable in β, larger than λ_ξ, and large enough that each of μ_ζ, $\zeta < \xi$, is definable in $M_{\gamma(\beta)+1}$ from parameters in α_ξ. Let $\beta_\xi = \mathrm{proj}_{\alpha_\xi}(\beta)$. Then $\langle\langle \xi, \beta_\xi\rangle \mid \xi < \omega_1\rangle$ is a branch through S, immediately by the definitions and by the properties in Claim 1.5.

Conversely, Suppose that C is cofinal in ω_1 and $\langle \xi, \beta_\xi \rangle \mid \xi \in C \rangle$ is a branch through S. Let T be the thread generated by this branch. Precisely, T consists of all points which are projections of points in $\{\beta_\xi \mid \xi \in C\}$. Then T is a thread of height λ, and by Lemma 2.3 the limit of T exists. Let $\beta = \lim(T)$. β is collapsed to λ by a function in $\mathrm{dlm}(T)$, and since $\mathrm{dlm}(T)$ is a level of M, $\beta < (\lambda^+)^M$. We claim that $\sup\{\mu_\xi \mid \xi < \omega_1\} \leq \beta$. Suppose not, and fix ζ so that $\beta < \mu_\zeta$. Let $\tau < \lambda$ be large enough that β is definable in $M_{\gamma(\mu_\zeta)+1}$ from parameters in τ. Let $\xi \in C$ be large enough that $\alpha(\beta_\xi) > \tau$ and $\xi > \zeta$. Using Claim 1.5, $\mathrm{proj}_{\alpha(\beta_\xi)}(\beta) < \mathrm{proj}_{\alpha(\beta_\xi)}(\mu_\zeta)$. But as $\mathrm{proj}_{\alpha(\beta_\xi)}(\beta) = \beta_\xi$, this contradicts the fact that $\langle \xi, \beta_\xi \rangle$ is a node in S. $\dashv$

Theorem 2.12. *Let M be a fine structural inner model. Suppose that there is a proper forcing extension of M that satisfies* $\mathsf{PFA}(\mathfrak{c}^+$*-linked). Let τ denote ω_2 of the extension. Then $[\tau, \tau^+]$ is Σ^2_1 indescribable in M.*

Proof. Suppose for definitiveness that the proper forcing extension of M that satisfies $\mathsf{PFA}(\mathfrak{c}^+$-linked) is V. Throughout this proof, cardinal successors are computed in M, except that $\mathfrak{c}^+$ is computed in V. Similarly H_λ always denotes $(H_\lambda)^M$.

Claim 2.13. *τ is a limit cardinal of M.*

Proof. Suppose not, and let λ be such that $\tau = \lambda^+$. Note that $\lambda > \omega$ for otherwise $(\omega_2)^{\mathrm{V}}$ would be equal to $(\omega_1)^M$.

By an argument of Todorčević, $\square(\omega_2)$ fails under $\mathsf{PFA}(\mathfrak{c}^+$-linked). If λ is singular in M then $\square_\lambda$ holds in M, see the properties of M listed at the start of the section. But then $\square((\lambda^+)^M)$ holds in V, in contradiction to $\mathsf{PFA}(\mathfrak{c}^+$-linked) as $(\lambda^+)^M = \tau = (\omega_2)^{\mathrm{V}}$. We may therefore assume that λ is regular in M. (A similar argument shows that λ is in fact subcompact in M, but we only need its regularity.) As $\omega < \lambda < (\omega_2)^{\mathrm{V}}$, λ has cofinality ω_1 in V. We shall use $\mathsf{PFA}(\mathfrak{c}^+$-linked) and Claim 2.11 to derive a contradiction.

Let $a = \langle \lambda_\xi \mid \xi < \omega_1 \rangle$ be increasing and cofinal in λ. Let $\mathbb{A}$ be the poset collapsing ω_2 to ω_1. Let G be $\mathbb{A}$–generic over V. In $\mathrm{V}[G]$ let $u = \langle \mu_\xi \mid \xi < \omega_1 \rangle$ be an increasing sequence of points on level λ, cofinal in $(\omega_2)^{\mathrm{V}} = (\lambda^+)^M$. By Claim 2.11 the tree $S(a, u)$ has no branches of length ω_1. Let $\mathbb{B} \in \mathrm{V}[G]$ be the poset to specialize this tree, and let $\dot{\mathbb{B}}$ be the canonical name for $\mathbb{B}$.

Let $\mathbb{P}$ be the restriction of $\mathbb{A} * \dot{\mathbb{B}}$ to conditions $\langle p, \dot{f} \rangle \in \mathbb{A} * \dot{\mathbb{B}}$ so that p forces a value to $\dot{f}$. $\mathbb{P}$ is proper, and has size $\mathfrak{c}$. Applying $\mathsf{PFA}(\mathfrak{c}^+$-linked) to this poset we obtain pseudo generics giving rise to:

1. An increasing sequence of points $\langle \eta_\xi \mid \xi < \omega_1 \rangle$ on level λ.
2. A function f specializing the tree $S(a, \vec{\eta})$.

The sequence $\langle \eta_\xi \mid \xi < \omega_1 \rangle$ given by the pseudo generic is of course not cofinal in $(\lambda^+)^M = (\omega_2)^{\mathrm{V}}$, since it exists in V. On the other hand from Claim 2.11 and the fact that $S(a, \vec{\eta})$ is special it follows that $\sup\{\eta_\xi \mid \xi < \omega_1\}$ is equal to $(\lambda^+)^M$, contradiction. $\dashv$

Let $\kappa = \tau^+$. Suppose that $\langle Q, \psi \rangle$ is a Σ^2_1 truth for κ in M. We intend to find $\bar{\tau} < \tau$, $\bar{Q} \subset H_{\bar{\kappa}}$ where $\bar{\kappa} = \bar{\tau}^+$, and an elementary $\pi \colon (H_{\bar{\kappa}}; \bar{Q}) \to (H_\kappa; Q)$ inside M, so that $\mathrm{crit}(\pi) = \bar{\tau}$, $\pi(\bar{\tau}) = \tau$, and $\langle \bar{Q}, \psi \rangle$ is a Σ^2_1 truth for $\bar{\kappa}$ in M. This will complete the proof of Theorem 2.12.

In part we follow the proof of Theorem 1.10, with points reinterpreted subject to the definition at the start of this section. The initial step, Claim 1.11, adapts trivially to show that there is a club $E \subset \kappa^+$, consisting of points on level κ, such that every two points in E are compatible. (There is no need here to note separately that $\text{hlim}(E)$ is wellfounded, as this follows from Claim 2.9. The underlying cause is the properness of the poset leading from M to V, which implies that κ^+ has uncountable cofinality in V.)

Let F be $\text{col}(\omega_1, \tau)$–generic over V. Working in V$[F]$, where κ is ω_2, define $\mathbb{A}$ following conditions (a)–(h) in the proof of Theorem 1.10, replacing $\mathrm{L}_{\gamma(\beta)+1}$ by $M_{\gamma(\beta)+1}$ throughout (and removing the requirement of wellfoundedness in condition (b)). Let G be $\mathbb{A}$ generic over V$[F]$. Let $K = \bigcup_{p \in G} \text{stem}(p)$.

As in the proof of Theorem 1.10, $\text{levels}(K)$ is a club of order type ω_1 in κ. $\kappa = \tau^+$ is therefore collapsed to ω_1 in V$[F][G]$. Fix a function $f \in$ V$[F][G]$, from ω_1 onto $M \| \kappa$. Let $\dot{f} \in$ V name f.

Let $R_1 = R_1(K)$, $R_2 = R_2(K)$, and $R_3 = R_3(K)$ be defined as in the proof of Theorem 1.10. Claims 1.12 through 1.18 all hold in the current context, and the trees therefore do not have branches of length ω_1 in V$[F][G]$.

Let $u = \langle \mu_\xi \mid \xi < \omega_1 \rangle$ enumerate $\text{levels}(K)$ in increasing order. The sequence is then cofinal in $\kappa = \tau^+$. Let $a = \langle \lambda_\xi \mid \xi < \omega_1 \rangle$ be a normal sequence cofinal in τ. Let $\dot{u}$ and $\dot{a}$ name the sequences u and a.

Let R_0 be defined as in the proof of Theorem 1.10, but using τ and a rather than κ and $\text{levels}(K)$. Precisely, S is the class of singular cardinals of M, $\langle C_\alpha \mid \alpha \in S \rangle$ is the sequence given by the singular square principle for M, and $R_0 = R_0(a)$ consists of pairs $\langle \xi, \alpha \rangle$ so that α is a singular cardinal of M above λ_ξ, ordered through the relation $\langle \xi, \alpha \rangle <_{R_0} \langle \xi', \alpha' \rangle$ iff $\xi < \xi'$ and $\alpha \in C_{\alpha'}$. An argument similar to that of Claim 1.20, using the fact that τ is regular in M, shows that there are no branches of length ω_1 through R_0 in V$[F][G]$.

Finally, let $R_4 = R_4(a, u)$ be the tree $S(a, u)$ of Definition 2.10. The sequence u is cofinal in $\kappa = \tau^+$, and so by claim 2.11, there are no branches of length ω_1 through R_4 in V$[F][G]$.

Let $\mathbb{B}$ be the poset for specializing the trees R_i, $i = 0, \ldots, 4$. $\mathbb{B}$ is c.c.c. in V$[F][G]$ since the trees do not have branches of length ω_1. Let $\dot{\mathbb{B}} \in$ V name $\mathbb{B}$. Let $\mathbb{P}$ be the restriction of the poset $\text{col}(\omega_1, \tau) * \dot{\mathbb{A}} * \dot{\mathbb{B}}$ to the set P of conditions $\langle p, \dot{q}, \dot{h} \rangle$ so that p forces a value for $\dot{q}$ and $\langle p, \dot{q} \rangle$ forces a value for $\dot{h}$. $\mathbb{P}$ is proper, since $\text{col}(\omega_1, \tau) * \dot{\mathbb{A}}$ is countably closed, and $\mathbb{B}$ is c.c.c. in V$[F][G]$. $\text{col}(\omega_1, \tau)$ has size $\mathfrak{c}$, $\mathbb{B}$ has size κ, and $\mathbb{A}$ is $\mathfrak{c}^+$-linked (since any two conditions with the same stem in $\mathbb{A}$ are compatible). It follows that $\mathbb{P}$ is $\mathfrak{c}^+$-linked. We apply $\mathsf{PFA}(\mathfrak{c}^+$-linked) to $\mathbb{P}$.

Through a suitable choice of dense sets we obtain a pseudo generic $\bar{F} * \bar{G} * \bar{H}$ so that:

1. $(\text{range}(\bar{f}); Q \cap \text{range}(\bar{f}))$ is elementary in $(M \| \kappa; Q)$, where $\bar{f} = \dot{f}[\bar{F} * \bar{G}]$. (Recall that $\dot{f}$ names $f = \dot{f}[F * G]$, a surjection of ω_1 onto $M \| \kappa$.)
2. For each $\xi < \omega_1$, if $\bar{f}(\xi) < \omega_2^{\mathrm{V}} = \tau$ then $\bar{f}(\xi) \subset \text{range}(\bar{f})$.
3. $\bar{u} = \dot{u}[\bar{F} * \bar{G}]$ is an increasing sequence of ordinals in $\text{range}(\bar{f})$ and $\sup(\bar{u}) = \sup(\text{range}(\bar{f}) \cap \kappa)$. Similarly $\bar{a} = \dot{a}[\bar{F} * \bar{G}]$ is an increasing sequence of ordinals in $\text{range}(\bar{f})$ and $\sup(\bar{a}) = \sup(\text{range}(\bar{f}) \cap \tau)$.

4. $\bar{K} = \dot{K}[\bar{F} * \bar{G}]$ is a set of points all the points in $\bar{K}$ belong to range$(\bar{f})$, and each of the points in $\bar{K}$ captures $\langle Q, \psi\rangle$.
5. levels$(\bar{K})$ is a club of order type ω_1, enumerated by the sequence $\bar{u}$.
6. For every $\alpha \in$ levels$(\bar{K})$, all the points in $\bar{K}$ on level α are compatible.
7. The trees $R_i(\bar{K}) \cap \mathrm{range}(\bar{f})$, $i = 1, 2, 3$, $R_0(\bar{a}) \cap \mathrm{range}(\bar{f})$, and $R_4(\bar{a}, \bar{u}) \cap \mathrm{range}(\bar{f})$ are special, and therefore have no branches of length ω_1.
8. $\bar{\tau} = \sup(\bar{a})$ is a limit of cardinals of M.

(For condition (8) notice that τ is a limit of cardinals of M, by Claim 2.13.)

Let N denote $M\|\kappa$ and let $\bar{N}$ be the transitive collapse of range$(\bar{f})$. Let $\pi\colon \bar{N} \to N$ be the anticollapse embedding. By conditions (2) and (3), $\bar{\tau} = \tau \cap \mathrm{range}(\bar{f})$ and therefore $\mathrm{crit}(\pi) = \bar{\tau}$ and $\pi(\bar{\tau}) = \tau$. Let $\bar{Q} = \pi^{-1}{}''Q$. By condition (1), π is elementary from $(\bar{N}; \bar{Q})$ into $(N; Q)$.

Since $\bar{\tau}$ a cardinal of M (in fact a limit of cardinals of M), it does not index an extender in $\vec{E}$. It follows by condensation that $\bar{N}$ is a level of M. Letting $\bar{\kappa} = \bar{N} \cap \mathrm{ON}$ we have $\bar{N} = M\|\bar{\kappa}$.

Claim 2.14. *$\bar{\tau}$ is a regular cardinal of M.*

Proof. Similar to the proof of Claim 1.23, using the fact that $R_0(\bar{a}) \cap \mathrm{range}(\bar{f}) = R_0(\bar{a})$ is special. $R_0(\bar{a}) \cap \mathrm{range}(\bar{f})$ is equal to $R_0(\bar{a})$ because of condition (2). ⊣

Claim 2.15. *$\bar{\kappa}$ is the successor of $\bar{\tau}$ in M.*

Proof. Let $\bar{R}_4 = \pi^{-1}{}''(R_4(\bar{a}, \bar{u}) \cap \mathrm{range}(\bar{f}))$. Notice that this is precisely the tree $S(\bar{a}, \pi^{-1}{}''\bar{u})$, and that $\sup(\pi^{-1}{}''\bar{u}) = \bar{\kappa}$. By condition (7), $\bar{R}_4$ has no branches of length ω_1. From this, the fact that $\bar{\tau} = \sup(\bar{a})$ is regular in M, and Claim 2.11, it follows that $\sup(\pi^{-1}{}''u) = (\bar{\tau}^{+})^M$. ⊣

Claim 2.16. *$\langle \bar{Q}, \psi\rangle$ is a Σ^2_1 truth in M.*

Proof. For each $i = 1, 2, 3$ let $\bar{R}_i = \pi^{-1}{}''R_i(\bar{K}) \cap \mathrm{range}(\bar{f})$. The trees are special by condition (7), and an argument similar to that in the proof of Theorem 1.10 shows from this that $\langle \bar{Q}, \psi\rangle$ is a Σ^2_1 truth in M. Let us just comment that Claims 1.21, 1.23, and 1.24 hold in the current context, in some cases using Lemma 2.3, and Claim 1.22 is replaced by Claim 2.9. ⊣

Claim 2.17. *π belongs to M.*

Proof. We show that the range of π can be identified in M.

For each ν between τ and κ let $\beta(\nu)$ be the first point on level κ above ν, and let $\gamma(\nu)$ denote $\gamma(\beta(\nu))$. Suppose for a moment that $\nu \in \mathrm{range}(\pi)$. Then $\beta(\nu), \gamma(\nu) \in \mathrm{range}(\pi)$ since the range of π is elementary in $M\|\kappa$. Since every ordinal below ν is definable in $M_{\gamma(\nu)+1}$ from parameters in τ, it follows again by elementarity that if an ordinal below ν belongs to range(π) then it is definable in $M_{\gamma(\nu)+1}$ from parameters in $\bar{\tau}$. The converse is true trivially.

Let $\delta = \sup(\mathrm{range}(\pi))$. Then δ has cofinality ω_1, and, since V is an extension of M by a proper subset, stationary subsets of $\{\alpha < \delta \mid \mathrm{cof}^M(\alpha) = \omega\}$ in M remain stationary in V. For each ξ let $S(\xi)$ be the set of $\nu < \delta$ so that $\nu > \xi$ and ξ is definable in $M_{\gamma(\nu)+1}$ from parameters in $\bar{\tau}$. The map $\xi \mapsto S(\xi)$ belongs to M. The argument of the previous paragraph, the fact that range(π) contains an

ω-club in δ (that is the club levels($\bar{K}$)), and the fact that stationarity is preserved from M to V, combine to imply that $\xi \in \text{range}(\pi)$ iff $M \models$"$S(\xi)$ contains an ω-club in δ." So range(π) can be identified in M and hence $\pi \in M$. $\dashv$

Since $\bar{\kappa} = \bar{\tau}^+$, the elements of $M \| \bar{\kappa}$ are precisely the elements of $H_{\bar{\tau}^+}$ (all in the sense of M). We thus have $\bar{\tau} < \tau$, $\bar{Q} \subset H_{\bar{\tau}^+}$ in M so that $\langle \bar{Q}, \psi \rangle$ is a Σ^2_1 truth in M, and an elementary embedding $\pi \colon (H_{\bar{\tau}^+}; \bar{Q}) \to (H_{\tau^+}; Q)$, also inside M, with crit(π) $= \bar{\tau}$ and $\pi(\bar{\tau}) = \tau$. This completes the proof of Theorem 2.12. $\dashv$

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