GAMES OF LENGTH ω_1

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Abstract. We prove determinacy for open length ω_1 games. Going further we introduce, and prove determinacy for, a stronger class of games of length ω_1, with payoff conditions involving the entire run, the club filter on ω_1, and a sequence of ω_1 disjoint stationary subsets of ω_1. The determinacy proofs use an iterable model with a class of indiscernible Woodin cardinals, and we show that the games precisely capture the theory of the minimal model for this assumption.

The purpose of this paper is to bring determinacy to the level of games of length ω_1.

For a set $A \subset \omega^{<\omega_1}$ define $G_{\text{open}-\omega_1}(A)$ to be the following game: Players I and II alternate playing natural numbers as in Diagram 1 to create $r \in \omega^{\omega_1}$. Player I wins if there exists some $\alpha < \omega_1$ so that $r \restriction \alpha$ belongs to A, and otherwise II wins. Such games are called **open** length ω_1 games, as victory by player I, if achieved, is secured at a strict initial segment of the run. By **definable** open length ω_1 games we mean games $G_{\text{open}-\omega_1}(A)$ with A which is Π^1_1 in the codes. (We could relax to projective in the codes, or to lightface definable over $\mathrm{L}(\mathbb{R})$, instead of Π^1_1. This would not affect the strength of the resulting class of games, since any number of extra real quantifiers in the payoff can be absorbed by moves in $G_{\text{open}-\omega_1}$.) These games trace back to Steel [7, 5] who proved various results assuming their determinacy, including propagation of scales and existence of definable winning strategies.

I	$r(0)$		$r(2)$		$\dots$ $r(2\xi)$		$\dots\dots$	
II		$r(1)$		$\dots$		$r(2\xi+1)$		$\dots\dots$

Diagram 1. Games of length ω_1.

We prove in this paper that these and even stronger games are determined, assuming the existence of an iterable model with a class of indiscernible Woodin cardinals. This precise large cardinal assumption had been expected, and indeed it was already known through work of Steel to be optimal, in the sense that no weaker large cardinal assumption proves the determinacy of definable open games of length ω_1. Steel showed this by noting that the minimal iterable model with a class of indiscernible Woodin cardinals does not satisfy definable open length ω_1 determinacy. It had also been known from work of Steel and

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Woodin that, assuming determinacy, the Σ_1 theory of this minimal model is recursively equivalent to the universal $\Game_{\text{open}-\omega_1}\Pi^1_1$ real, just as the Σ_1 theory of L is recursively equivalent to the universal $\Game_\omega \Pi^1_1$ real.

There are some intrinsic difficulties in trying to take determinacy beyond open games of length ω_1. The definability hierarchy of descriptive set theory, generally referenced in determinacy results to exclude non-determined games, applies to sets of reals. Countable sequences of reals can be coded by reals and so the hierarchy can be translated to apply to subsets of $\omega^{<\omega_1}$, namely to payoff sets in open games of length ω_1. But to go beyond open one has to consider subsets of ω^{ω_1}, and for this a mechanism for specifying the payoff is needed which goes beyond the hierarchy of definability of descriptive set theory.

We introduce such a mechanism in Section 1. The class of games we define subsumes and surpasses the class of open games of length ω_1, but its determinacy still follows from the existence of an iterable model with a class of indiscernible Woodin cardinals. We prove the determinacy in Section 3. Let N denote the minimal iterable model with a class of indiscernible Woodin cardinals. In Sections 2 and 4 we prove that the universal real for the pointclass resulting from our length ω_1 games is recursively equivalent to the theory of indiscernible Woodin cardinals for N, just as the universal $\Game_\omega(<\omega^2 - \Pi^1_1)$ real is recursively equivalent to the theory of Silver indiscernibles for L. This precise equivalence shows that our games, despite several eccentricities mentioned below, provide the correct notion of determinacy at the level of a class of indiscernible Woodin cardinals.

The main part of the paper of course is the determinacy proof in Section 3. The section assumes familiarity with parts of Neeman [4], specifically some rough knowledge of the definitions of δ–sequences, δ–names, and pullbacks in Chapter 4 of [4], fleeting knowledge of the end results of Chapter 5, and deeper familiarity with the game $\widehat{G}_{\text{branch}}$ of §6A, the end results about this game in §6G, and the structure of the construction in §§7B–7D. Section 2 assumes some knowledge of inner model theory.

Section 1 is a continuation of the introduction, and makes no special assumptions. The games we introduce there make reference to the club filter on ω_1: to secure victory a player must make sure a certain condition holds on a club. This is the first instance in the study of determinacy of payoff conditions which involve quantification over the club filter, but in retrospect such involvement is very natural, fitting nicely with the comparison games of inner model theory. The reference to the club filter has a couple of consequences which are unusual in the study of determinacy: the games have runs which are lost by both players; and their payoff conditions are defined relative to sequences of ω_1 disjoint stationary subsets of ω_1. Disjoint stationary subsets of ω_1 cannot be obtained in any canonical fashion, and in particular the payoff conditions in the games are not, strictly speaking, definable. But fortunately the most crucial question on each game, namely which player has a winning strategy, is independent of the particular sequence of stationary sets used. We prove this in Section 4.

§1. Games. Fix a number $k < \omega$. Given a set $C \subset \omega_1$ we use (as is standard) $[C]^k$ to denote the set of increasing k-tuples from C. We use $[C]^{<k}$ to denote the set of increasing tuples from C of length less than k.

Let $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ be a collection of mutually disjoint stationary subsets of ω_1, with a stationary set S_a associated to each tuple $a \in [\omega_1]^{<k}$.

Definition 1.1. $[\vec{S}]$ denotes the set:

$$\{\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\omega_1]^k \mid (\forall i < k)\, \alpha_i \in S_{\langle \alpha_0, \ldots, \alpha_{i-1} \rangle}\}.$$

Let $\mathcal{L}^+$ be the language of set theory with an added unary relation symbol $\dot{r}$. Let $\varphi(x_0, \ldots, x_{k-1})$ be a formula in $\mathcal{L}^+$. We write $(M; r) \models \varphi[a_0, \ldots, a_{k-1}]$ to mean that $\varphi[a_0, \ldots, a_{k-1}]$ holds in M with $\dot{r}$ interpreted by r. r has to be a subset of M for this to make sense, but it need not be an element of M.

Definition 1.2. $G_{\omega_1,k}(\vec{S}, \varphi)$ denotes the following game: Players I and II alternate playing ω_1 natural numbers in the manner of Diagram 1, producing together a sequence $r \in \omega^{\omega_1}$. If there is a club $C \subset \omega_1$ so that $(\mathrm{L}_{\omega_1}[r]; r) \models \varphi[\alpha_0, \ldots, \alpha_{k-1}]$ for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$ then player I wins the run r. If there is a club $C \subset \omega_1$ so that $(\mathrm{L}_{\omega_1}[r]; r) \models \neg\varphi[\alpha_0, \ldots, \alpha_{k-1}]$ for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$ then player II wins r. If neither condition holds then both players lose.

By $\mathrm{L}_{\omega_1}[r]$ we mean the sets which are constructible *relative* to r at a countable level. Formally r is a set of pairs in $\omega_1 \times \omega$. r is therefore a subset of $\mathrm{L}_{\omega_1}[r]$, and the use of $(\mathrm{L}_{\omega_1}[r]; r)$ makes sense.

Remark 1.3. Note that the two winning conditions in Definition 1.2 cannot *both* hold. (This uses the fact that each of the sets S_a is stationary in ω_1, and the demand in Definition 1.2 that C must be club in ω_1.) Thus at most one player wins each run of $G_{\omega_1,k}(\vec{S}, \varphi)$. For $k > 0$ it may well be that neither condition in Definition 1.2 holds. So there may well be runs of $G_{\omega_1,k}(\vec{S}, \varphi)$ which are won by neither player. We say that $G_{\omega_1,k}(\vec{S}, \varphi)$ is **determined** if one of the players has a winning strategy. This is the stronger of two candidates for the notion of determinacy. We require not just the existence of a strategy that avoids losing, but the existence of a strategy that actually wins.

Definable open length ω_1 games (described in the introduction) can be simulated by the games of Definition 1.2 with $k = 0$ and φ in Σ_1. The converse is also true: in the case of $k = 0$ and $\varphi \in \Sigma_1$ the game $G_{\omega_1,k}(\vec{S}, \varphi)$ is a definable open length ω_1 game. Determinacy for the games $G_{\omega_1,0}(\vec{S}, \varphi)$ with $\varphi \in \Sigma_1$ is therefore precisely equivalent to determinacy for definable open length ω_1 games.

Let us next consider Definition 1.2 in the case that $k = 1$. One can let $\vec{S}$ be given simply by $S_\emptyset = \omega_1$ in this case. A run r of $G_{\omega_1,k}(\vec{S}, \varphi)$ is then won by player I just in case that $\varphi[\alpha]$ holds (in $(\mathrm{L}_{\omega_1}[r]; r)$) on a club of $\alpha < \omega_1$; and by player II just in case that $\varphi[\alpha]$ fails on a club.

It is tempting to try and phrase similar payoff conditions also in the case $k > 1$, and thereby remove the use of $\vec{S}$ altogether. For example, for $k = 2$, one can try to say that I wins r iff there exists a club $C \subset \omega_1$ so that $\varphi[\alpha_0, \alpha_1]$ holds for all $\langle \alpha_0, \alpha_1 \rangle \in [C]^2$; and II wins iff there exists a club $C \subset \omega_1$ so that $\varphi[\alpha_0, \alpha_1]$ fails for all $\langle \alpha_0, \alpha_1 \rangle \in [C]^2$. But a definition of this kind leads to games which are trivially non-determined, for example the game (pointed out to the author by Greg Hjorth and based on a comment of Menachem Magidor) corresponding to the formula $\varphi(\alpha_0, \alpha_1) =$"$\dot{r}(\alpha_0 + 1) = \dot{r}(\alpha_1)$."

Another attempt to get rid of $\vec{S}$ involves placing a club quantifier only on one side of the payoff condition: say that I wins r iff there exists a club $C \subset \omega_1$ so that $(\mathrm{L}_{\omega_1}[r]; r) \models \varphi[\alpha_0, \ldots, \alpha_{k-1}]$ for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [C]^k$; and II wins *otherwise*. Let $G_k^{\mathrm{club}}(\varphi)$ denote the resulting game. It is consistent that the games $G_k^{\mathrm{club}}(\varphi)$ are all determined. In fact, assuming a sharp for a Woodin limit of Woodin cardinals, it is consistent that all zero-sum games of length ω_1 on natural numbers with ordinal definable payoff are determined, see Neeman [4, 7F.14, 7F.15]. The result is due to Woodin, who derived the consistency from long game determinacy proved in [4]. The large cardinal used for the consistency proof is perhaps misleading as an indication of the strength of the games, as the game quantifier corresponding to G_2^{club} is strong enough to define the set of $\omega_1 + 1$ iterable countable Mitchell-Steel premice, and such premice may have superstrong cardinals.

It is also consistent with large cardinals that the games $G_k^{\mathrm{club}}(\varphi)$ are *not* all determined. The argument for this is due to Larson [1]. Still the question remains whether large cardinals prove the determinacy of the games $G_k^{\mathrm{club}}(\varphi)$ under some combinatorial principle. (Of course the principle would have to fail in Larson's model.) Precisely, it is open whether there is a small forcing notion $\mathbb{P}$, and a large cardinal axiom which proves all games $G_k^{\mathrm{club}}(\varphi)$ determined in $\mathrm{V}^{\mathbb{P}}$. $\mathbb{P} = \mathrm{Col}(\omega_1, \mathbb{R})$ is a natural candidate proposed by Woodin. A proof of $G_k^{\mathrm{club}}(\varphi)$ determinacy in $\mathrm{V}^{\mathrm{Col}(\omega_1,\mathbb{R})}$ from large cardinals would likely have interesting consequences on Σ_2^2 absoluteness, see Woodin [8].

Returning now to the games of Definition 1.2 let us define the length ω_1 game quantifier $\Game_{\omega_1,k}$ in the natural way:

$$\Game_{\omega_1,k}(\vec{S}, \varphi) = \begin{cases} \text{True} & \text{if player I has a winning strategy in } G_{\omega_1,k}(\vec{S}, \varphi)\text{; and} \\ \text{False} & \text{otherwise.} \end{cases}$$

As defined above, $\Game_{\omega_1,k}$ takes two arguments corresponding to the two components of the payoff conditions in Definition 1.2: the argument φ corresponding to the definable part of the payoff, and the argument $\vec{S}$ corresponding to the non-definable part. Assuming the existence of an iterable model with indiscernible Woodin cardinals we shall see later that $\Game_{\omega_1,k}(\vec{S}, \varphi)$ depends only on φ. Specifically we shall see (in Corollary 4.2) that player I wins $G_{\omega_1,k}(\vec{S}, \varphi)$ if and only if she wins $G_{\omega_1,k}(\vec{S}^*, \varphi)$ whenever $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ and $\vec{S}^* = \langle S_a^* \mid a \in [\omega_1]^{<k} \rangle$ are two collections of mutually disjoint stationary sets. This is important since $\vec{S}$ is outside the realm of descriptive set theory and cannot be picked in any canonical way. The fact that the value of $\Game_{\omega_1,k}(\vec{S}, \varphi)$ does not depend on $\vec{S}$ allows us to remove $\vec{S}$ from the argument of the game quantifier, and obtain the following descriptive set theoretic operation:

$$\Game_{\omega_1,k}(\varphi) = \begin{cases} \text{True} & \text{if player I has a winning strategy in } G_{\omega_1,k}(\vec{S}, \varphi) \\ & \text{for some/all collections } \vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle \text{ of} \\ & \text{mutually disjoint stationary subsets of } \omega_1\text{; and} \\ \text{False} & \text{otherwise.} \end{cases}$$

Given a formula φ in $\mathcal{L}^+$ we use $\Game_{\omega_1}(\varphi)$ to denote $\Game_{\omega_1,k}(\varphi)$ where k is the number of free variables in φ. Given a set Φ of formulae in $\mathcal{L}^+$ we use $\Game_{\omega_1}\Phi$ to denote the set $\{\varphi \in \Phi \mid \Game_{\omega_1}(\varphi) = \text{True}\}$. (This set may be viewed as a real through some coding of formulae by natural numbers.) We refer to $\Game_{\omega_1}$ and $\Game_{\omega_1,k}$ as **length ω_1 game quantifiers**.

Let F consist of all formulae in $\mathcal{L}^+$, and let F_{0,Σ_1} consist of all Σ_1 formulae in $\mathcal{L}^+$ with no free variables. We shall see finally that $\Game_{\omega_1}F_{0,\Sigma_1}$ is recursively equivalent to the Σ_1 theory of the minimal iterable model with a class of indiscernible Woodin cardinals, and that $\Game_{\omega_1}F$ is recursively equivalent to the theory of the indiscernible Woodin cardinals in this model.

§2. Indiscernible Woodin cardinals. By 0^W (read "zero Woodin") we mean the minimal sound mouse M which has a top extender predicate E_M so that $\text{crit}(E_M)$ is Woodin in M.

A **mouse** here is a countable premouse which is $\omega_1 + 1$ iterable. **Premice** are models constructed from coherent sequences of extenders. There are two canonized meanings for "coherent," using Mitchell–Steel indexing [3], or Jensen indexing as in Zeman [9]. The proofs in this paper work under both methods. Soundness and minimality for mice involve fine structure, and we refer the reader to [3] and [9] for more on this.

Were it not for the demand in the definition of 0^W that $\text{crit}(E_M)$ is Woodin in M, we would simply be defining $0^\sharp$. With this final demand we are defining a parallel of $0^\sharp$, involving indiscernible *Woodin* cardinals. The existence of 0^W is not yet known, the main impediment being our inability to prove *iterability* at the level of indiscernible Woodin cardinals. Some sufficient form of iterability is widely expected to be true. But for the time being it can only be assumed. Assuming, say, $\omega_1 + 1$ iterability for countable elementary substructures of rank initial segments of V, the existence of 0^W follows from the existence of measurable Woodin cardinals in V.

We work throughout this paper under the assumption that 0^W exists.

Let $M = 0^W$ and let E_M be the top extender of M. Let $\kappa_M = \text{crit}(E_M)$ and let μ_M be the measure on κ_M induced by E_M, that is the measure given by $\mu_M(X) = 1$ iff $\kappa_M \in j(X)$ where j is the ultrapower embedding by E_M.

Remark 2.1. The minimality of M implies that E_M is an extender with a *single* generator. This means that E_M is generated by μ_M. In fact under Mitchell–Steel indexing it means that the two are literally equal.

Let $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \leq \xi \in \text{On}\rangle$ be the iteration determined by setting $M_0 = M$; letting $M_{\xi+1} = \text{Ult}(M_\xi, E_\xi)$ where $E_\xi = j_{0,\xi}(E_M)$; letting $j_{\xi,\xi+1}\colon M_\xi \to M_{\xi+1}$ be the ultrapower embedding by E_ξ; defining the remaining embeddings by compositions; and taking direct limits at limit stages.

Let D be the direct limit of the (class) system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \leq \xi \in \text{On}\rangle$. Let $N = D\|\text{On}$. D is simply the result of starting with $M = 0^W$ and iterating its top extender through the ordinals. N is obtained by cutting D to height On.

Let $\kappa_\xi = \text{crit}(E_\xi) = j_{0,\xi}(\kappa_M)$. Each κ_ξ is then Woodin in N—this is because κ_M is Woodin in M—and $\{\kappa_\xi \mid \xi \in \text{On}\}$ is a club of indiscernibles for N. We view N and $\{\kappa_\xi \mid \xi \in \mathrm{On}\}$ as parallels to the context of Woodin cardinals of L and its club of Silver indiscernibles.

By the **theory of k Woodin indiscernibles**, here denoted T_k, we mean the theory of $\kappa_0, \ldots, \kappa_{k-1}$ in N. Since $\{\kappa_\xi \mid \xi \in \mathrm{On}\}$ is a club of indiscernibles, T_k is equal to the theory of $\kappa_{\xi_0}, \ldots, \kappa_{\xi_{k-1}}$ in N for any increasing tuple $\langle \xi_0, \ldots, \xi_{k-1}\rangle$. Notice that a formula ψ with k free variables belongs to T_k just in case that the set $\{\langle \alpha_0, \ldots, \alpha_{k-1}\rangle \in [\kappa_M]^k \mid (M \| \kappa_M) \models \psi[\alpha_0, \ldots, \alpha_{k-1}]\}$ has $(\mu_M)^k$ measure 1 where $(\mu_M)^k$ is the kth power of μ_M.

Claim 2.2. *The Σ_1 theory of 0^W is r.e. in $\oplus_{k<\omega} T_k$, and $\oplus_{k<\omega} T_k$ is recursive in the Σ_1 theory of 0^W.*

Proof sketch. The paragraph above the claim shows how to obtain T_k from the Σ_1 theory of 0^W. To get the Σ_1 theory of 0^W from $\oplus_{k<\omega} T_k$, search for fragments of its last measure obtained by shifting indiscernibles. The argument uses the fact that every element of 0^W is definable over N from finitely many indiscernibles. The search is the reason we only get the Σ_1 theory to be r.e. in $\oplus_{k<\omega} T_k$. ⊣

The mouse 0^W Σ_1-projects to ω, and its first standard parameter is $\emptyset$. The mouse is therefore canonically coded by its Σ_1 theory. In light of this and the last claim, we refer to $\oplus_{k<\omega} T_k$ as the **real coding** 0^W, and when talking of 0^W as a real, for example in Corollary 2.14 below, we mean the real $\oplus_{k<\omega} T_k$. $\oplus_{k<\omega} T_k$ codes 0^W in much the same way that the theory of Silver indiscernibles for L codes the minimal sound mouse with a non-trivial top extender.

Let Φ_k denote the set of formulae, in the language $\mathcal{L}^+$ of Section 1, which have at most k free variables. We plan to show that T_k is recursive in $\Game_{\omega_1} \Phi_k$. To do this we must reduce the question of membership in T_k to the question of winning length ω_1 games of the kind defined in Section 1, with payoff formula in Φ_k.

Following the notation of Section 1 fix a collection $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k}\rangle$ of mutually disjoint stationary subsets of ω_1.

Fix a formula $\psi(x_0, \ldots, x_{k-1})$. We define below a length ω_1 game G^ψ, of the format of Definition 1.2 with payoff in Φ_k. We shall show later that player I has a winning strategy in G^ψ precisely when ψ belongs to T_k. The association $\psi \mapsto G^\psi$ will then allow us to reduce T_k to $\Game_{\omega_1} \Phi_k$.

Description of G^ψ. Players I and II use the first ω moves to play reals coding sound countable pre-mice P and Q respectively. P and Q must Σ_1-project to ω and must moreover satisfy the following conditions:

1. P must have a top extender E_P with $\mathrm{crit}(E_P)$ a Woodin cardinal in P. There must be no earlier extender in P with a critical point which is Woodin in the level at which the extender is added.
2. Similarly Q must have a top extender E_Q with $\mathrm{crit}(E_Q)$ Woodin in Q, and there must be no earlier extender in Q with a critical point which is Woodin in the level at which the extender is added.
3. Let $\kappa_P = \mathrm{crit}(E_P)$ and let μ_P be the measure on κ_P induced by E_P. The set of $\langle \alpha_0, \ldots, \alpha_{k-1}\rangle \in [\kappa_P]^k$ so that $(P \| \kappa_P) \models \psi[\alpha_0, \ldots, \alpha_{k-1}]$ must have $(\mu_P)^k$ measure 1, where $(\mu_P)^k$ denotes the kth power of μ_P.

4. Let $\kappa_Q = \text{crit}(E_Q)$ and let μ_Q be the measure on κ_Q induced by E_Q. The set of $\langle\alpha_0,\ldots,\alpha_{k-1}\rangle \in [\kappa_Q]^k$ so that $(Q\|\,\kappa_Q) \models \psi[\alpha_0,\ldots,\alpha_{k-1}]$ must *not* have $(\mu_Q)^k$ measure 1.

Conditions (1) and (3) are placed on player I. Conditions (2) and (4) are placed on player II.

For the rest of the game the players compare P and Q. More precisely they construct maximal non-overlapping iteration trees $\mathcal{U}$ and $\mathcal{V}$ on P and Q respectively, subject to the condition that for each ξ, $F^{\mathcal{U}}_\xi$ and $F^{\mathcal{V}}_\xi$ are given by the least disagreement between P_ξ and Q_ξ. (See [3] or [9] for the definition of fine structural iteration trees, and examples of comparisons. P_ξ and $F^{\mathcal{U}}_\xi$ here stand for the models and extenders of $\mathcal{U}$. Similarly Q_ξ and $F^{\mathcal{V}}_\xi$ stand for the models and extenders of $\mathcal{V}$. Notice that conditions (1)–(4) imply that no iterate of P can agree with an iterate of Q. So P_ξ and Q_ξ must indeed disagree.) This condition determines the trees, modulo a choice of branches $[0,\gamma]_{\mathcal{U}}$ and $[0,\gamma]_{\mathcal{V}}$ for limit γ. We ask player I to pick the branches $[0,\gamma]_{\mathcal{U}}$ used on the P side, and ask player II to pick the branches $[0,\gamma]_{\mathcal{V}}$ used on the Q side.

If ever a stage $\xi < \omega_1$ is reached so that either P_ξ or Q_ξ is illfounded, then the game ends. If P_ξ is illfounded then II wins. Otherwise (Q_ξ is illfounded and) I wins.

Suppose now that both players maintain wellfoundedness for ω_1 stages, producing iteration trees $\mathcal{U}$ and $\mathcal{V}$ of length ω_1. Let N_{end} be the lined-up part of the comparison. More precisely this is $\bigcup_{\xi<\omega_1} P_\xi\|\,\text{lh}(F^{\mathcal{U}}_\xi) = \bigcup_{\xi<\omega_1} Q_\xi\|\,\text{lh}(F^{\mathcal{V}}_\xi)$.

If there exists a club $C \subset \omega_1$ so that $N_{\text{end}} \models \psi[\alpha_0,\ldots,\alpha_{k-1}]$ for all tuples $\langle\alpha_0,\ldots,\alpha_{k-1}\rangle \in [\vec{S}] \cap [C]^k$ then player I wins. If there exists a club $C \subset \omega_1$ so that $N_{\text{end}} \models \neg\psi[\alpha_0,\ldots,\alpha_{k-1}]$ for all tuples $\langle\alpha_0,\ldots,\alpha_{k-1}\rangle \in [\vec{S}] \cap [C]^k$ then player II wins. Note that at most one of these conditions holds. If both conditions fail then both players lose. $\dashv$ (Description of G^ψ.)

LEMMA 2.3. *Suppose that $\psi \in T_k$. Then player* I *has a winning strategy in G^ψ.*

PROOF. Let Γ be an $\omega_1 + 1$ iteration strategy for 0^W. We describe how to play for I in G^ψ, and win.

Start by playing (a real coding) $P = 0^W$. 0^W trivially satisfies condition (1) in the definition of G^ψ. The assumption that $\psi \in T_k$ implies that it also satisfies condition (3).

Let the opponent play Q. Now play through the comparison of P and Q by letting the opponent pick branches for $\mathcal{V}$ on the Q side, and using Γ to pick branches for $\mathcal{U}$ on the P side.

Since Γ is an iteration strategy for $P = 0^W$, this method guarantees that all the models of $\mathcal{U}$ are wellfounded. If an illfoundedness is reached it can only be on the Q side, and must therefore result in victory for I, as required.

Suppose then that no illfoundedness is reached, so that the game ends with two iteration trees $\mathcal{U}$ and $\mathcal{V}$ on P and Q respectively, of length ω_1.

Γ is an $\omega_1 + 1$ iteration strategy for $P = 0^W$. We may therefore apply it to the tree $\mathcal{U}$. Let b be the cofinal branch through $\mathcal{U}$ given by Γ. Let P_{ω_1} be the direct limit of the models of $\mathcal{U}$ along b, and let $j_{\zeta,\omega_1}\colon P_\zeta \to P_{\omega_1}$, for sufficiently large $\zeta \in b$, be the direct limit embeddings.

A standard argument produces some $\xi_0 \in b$, some $\tau \in P_{\xi_0}$, and some $C \subset \omega_1$ so that:

(i) There are no truncations on b above ξ_0;
(ii) C is club in ω_1 and contained in $b - \xi_0$; and
(iii) $j_{\xi_0,\alpha}(\tau) = \alpha$ for every $\alpha \in C$.

It follows from condition (iii) that $\mathrm{crit}(j_{\alpha,\omega_1}) \leq \alpha$ for each $\alpha \in C$. By thinning C if needed we may in fact make sure that:

(iv) $\mathrm{crit}(j_{\alpha,\omega_1}) = \alpha$ for every $\alpha \in C$.

From conditions (iii) and (iv) it follows easily that $j_{\alpha,\omega_1}(\alpha) = \omega_1$ for each $\alpha \in C$, so $j_{\xi_0,\omega_1}(\tau) = \omega_1$.

Claim 2.4. *There is no cofinal branch through $\mathcal{V}$.*

Proof. $\mathcal{U}$ and $\mathcal{V}$ are length ω_1 trees generated through a comparison of two countable pre-mice. Cofinal branches through *both* trees would allow completing the standard comparison argument to derive a contradiction. (The standard argument then uses the contradiction to concludes that the comparison must have terminated before reaching ω_1.) So at least one of the trees has no cofinal branch. Since $\mathcal{U}$ has a cofinal branch, namely b, it must be that $\mathcal{V}$ does not. ⊣

Claim 2.5. *ω_1 is Woodin in P_{ω_1}.*

Proof. Suppose for contradiction that ω_1 is not Woodin in P_{ω_1}. Let $\eta \in P_{\omega_1}$ be least so that $\eta \geq \omega_1$ and ω_1 fails to be Woodin in $P_{\omega_1} \| \eta + 1$. Let $P^* = P_{\omega_1} \| \eta + 1$.

Let θ be some regular cardinal greater than ω_1. Let H be a countable Skolem hull of V_θ, with all relevant objects, including $\mathcal{U}$, $\mathcal{V}$, C, ξ_0, and τ, thrown into H. Let $\bar{H}$ be the transitive collapse of H and let $\pi\colon \bar{H} \to H$ be the anticollapse embedding.

Let $\alpha = \bar{H} \cap \omega_1$. It's easy to check that α belongs to C, and hence to b. It's also easy to check that $\pi^{-1}(\omega_1) = \alpha$, $\pi^{-1}(C) = C \cap \alpha$, $\pi^{-1}(b) = b \cap \alpha$, and $\pi^{-1}(P_{\omega_1}) = P_\alpha$. In other words, α, $C \cap \alpha$, $b \cap \alpha$, and P_α belong to $\bar{H}$, and are sent by π to ω_1, C, b, and P_{ω_1} respectively.

Let $\bar{\eta} = \pi^{-1}(\eta)$. Using the definition of η, the elementarity of π, and the fact that $\pi(P_\alpha) = P_{\omega_1}$ it's easy to see that $\bar{\eta}$ is least so that $\bar{\eta} \geq \alpha$ and α fails to be Woodin in $P_\alpha \| \bar{\eta} + 1$. Note that in particular $P_\alpha \| \bar{\eta}$ projects to α.

The fact that ξ_0 was thrown into H implies that $\alpha > \xi_0$. It follows that there are no truncations on b above α. From this, the fact that $P_\alpha \| \bar{\eta}$ projects to α, and the fact that $\mathrm{crit}(j_{\alpha,\omega_1}) = \alpha$, it follows that the least disagreement between P_α and Q_α must be above $\bar{\eta}$. So $P_\alpha \| \bar{\eta} + 1$ is an initial segment of Q_α.

Let $\bar{P}^* = \pi^{-1}(P^*)$. Notice that $\bar{P}^*$ is then equal to $P_\alpha \| \bar{\eta} + 1$. The arguments of the preceding paragraphs show that:

(v) $\bar{P}^* \models$"α is not a Woodin cardinal"; and
(vi) $\bar{P}^*$ is an initial segment of Q_α.

Let $\bar{\mathcal{V}} = \pi^{-1}(\mathcal{V})$. Note that $\bar{\mathcal{V}}$ is then equal to $\mathcal{V} \restriction \alpha$. Let g be $\mathrm{Col}(\omega, \alpha)$–generic over $\bar{H}$. Working inside $\bar{H}[g]$ let R be the tree of attempts to create a cofinal branch c through $\bar{\mathcal{V}}$ with the property that $\bar{P}^*$ is an initial segment of the direct limit along c. Such a tree can be defined using the fact that $\bar{P}^*$ and $\bar{\mathcal{V}}$ are countable in $\bar{H}[g]$.

Notice that in V there exists a branch through R: the branch $[0,\alpha]_{\mathcal{V}}$ leads to the direct limit Q_α, and $\bar{P}^*$ is an initial segment of this direct limit by condition (vi). Using absoluteness it follows that a branch through R must exist also in $\bar{H}[g]$. In other words $\bar{H}[g] \models$"there exists a branch c, cofinal in $\bar{\mathcal{V}}$, and such that $\bar{P}^*$ is an initial segment of the direct limit along c." There can only be *one* such branch c, since otherwise α would be Woodin with respect to all functions in $\bar{P}^*$ and this would contradict condition (v). The uniqueness of c in $\bar{H}[g]$ implies that c must in fact exist already in $\bar{H}$. Thus we conclude that:

(vii) $\bar{H} \models$"there exists a cofinal branch through $\bar{\mathcal{V}}$."

But now using the elementarity of π it follows that (in H) there exists a cofinal branch through $\pi(\bar{\mathcal{V}}) = \mathcal{V}$. This contradicts Claim 2.4. ⊣

Remark 2.6. Claim 2.5 is part of an argument due to John Steel, showing that length ω_1 iterability (as opposed to the stronger length $\omega_1 + 1$ iterability) suffices for identifying mice below 0^W. Steel's result is rephrased in this paper as Theorem 2.15 below.

For each $\alpha \in C$ let $\xi(\alpha) \geq \alpha$ be such that the successor of α in the branch b is $\xi(\alpha)+1$. Notice then that $\xi(\alpha)+1$ belongs to b, that $P_{\xi(\alpha)+1} = \mathrm{Ult}(P_\alpha, F^{\mathcal{U}}_{\xi(\alpha)})$, and that $j_{\alpha,\xi(\alpha)+1}$ is the ultrapower embedding of P_α by $F^{\mathcal{U}}_{\xi(\alpha)}$.

$F^{\mathcal{U}}_{\xi(\alpha)}$ is the first extender used for the embedding j_{α,ω_1}. The embedding has critical point α by condition (iv). So $F^{\mathcal{U}}_{\xi(\alpha)}$ must have critical point α.

Let E_P denote the top extender predicate of $P = 0^W$. Let $\kappa_P = \mathrm{crit}(E_P)$ and let μ_P be the measure on κ_P induced by E_P. For each $\alpha \in C$ let $\mu_\alpha = j_{0,\alpha}(\mu_P)$.

Claim 2.7. *$F^{\mathcal{U}}_{\xi(\alpha)}$ is equal to $j_{0,\xi(\alpha)}(E_P)$.*

Proof. From Claim 2.5 it follows that α is Woodin in P_α. This in turn implies that α is Woodin in $P_{\xi(\alpha)}$. We know that $\mathrm{crit}(F^{\mathcal{U}}_{\xi(\alpha)})$ is equal to α. $F^{\mathcal{U}}_{\xi(\alpha)}$ is thus an extender on the sequence of $P_{\xi(\alpha)}$, with Woodin critical point. In $P = 0^W$ only the top extender, namely E_P, has this property. The claim follows. ⊣

Claim 2.8. *α belongs to the branch of $\mathcal{U}$ leading to $\xi(\alpha)$, and (if $\xi(\alpha) \neq \alpha$ then) $\mathrm{crit}(j_{\alpha,\xi(\alpha)}) > \alpha$.*

Proof. By Claim 2.7, $\mathrm{crit}(F^{\mathcal{U}}_{\xi(\alpha)})$ is equal to $j_{0,\xi(\alpha)}(\kappa_P)$. Since $\mathrm{crit}(F^{\mathcal{U}}_{\xi(\alpha)}) = \alpha$ it follows that α belongs to the range of $j_{0,\xi(\alpha)}$. This implies that no extenders which overlap α are used on the branch of $\mathcal{U}$ leading to $\xi(\alpha)$. (An extender F is said to **overlap** α if $\alpha \in [\mathrm{crit}(F), \mathrm{lh}(F))$. Notice that in this case, at least for extenders below superstrong, α cannot belong to the range of the ultrapower embedding by F.)

The fact that $\mathrm{crit}(j_{\alpha,\omega_1}) = \alpha$ implies that extenders applied to models before P_α in $\mathcal{U}$ must have critical point below α, and extenders $F^{\mathcal{U}}_\zeta$ for $\zeta \geq \alpha$ must have length above α. The current claim follows easily from these observations and the conclusion of the last paragraph. ⊣

Claim 2.9. $j_{0,\omega_1}(\kappa_P) = \omega_1$.

Proof. By Claim 2.7, $j_{0,\xi(\alpha)}(\kappa_P) = \alpha$ for each $\alpha \in C$. Using Claim 2.8 it follows that $j_{0,\alpha}(\kappa_P) = \alpha$. Composing this with the fact that $j_{\alpha,\omega_1}(\alpha) = \omega_1$ we get $j_{0,\omega_1}(\kappa_P) = \omega_1$. ⊣

Claim 2.10. *Let $\alpha \in C$ and let $X \in P_\alpha$ be a subset of $j_{0,\alpha}(\kappa_P)$. Suppose that X has $j_{0,\alpha}(\mu_P)$ measure 1. Then $\alpha \in j_{\alpha,\omega_1}(X)$.*

Proof. Using Claim 2.8 and the fact that X has $j_{0,\alpha}(\mu_P)$ measure 1 we see that $j_{\alpha,\xi(\alpha)}(X)$ has $j_{0,\xi(\alpha)}(\mu_P)$ measure 1. From this and Claim 2.7 it follows that α belongs to the image of X under the ultrapower embedding by $F^{\mathcal{U}}_{\xi(\alpha)}$, and so α belongs to $j_{\alpha,\xi(\alpha)+1}(X)$. Now $\xi(\alpha)+1$ belongs to the branch b, and $\text{crit}(j_{\xi(\alpha)+1,\omega_1}) > \alpha$ since $\mathcal{U}$ is non-overlapping. So α belongs to $(j_{\xi(\alpha)+1,\omega_1} \circ j_{\alpha,\xi(\alpha)+1})(X)$, namely to $j_{\alpha,\omega_1}(X)$. ⊣

Corollary 2.11. *Let $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$ be a tuple in $[C]^k$. Then $(P_{\omega_1} \| \omega_1) \models \psi[\alpha_0, \dots, \alpha_{k-1}]$.*

Proof. The initial assumption of Lemma 2.3 is such that ψ belongs to T_k. Since $P = 0^W$ this implies that the set $\{\langle \beta_0, \dots, \beta_{k-1} \rangle \in ([\kappa]_P)^k \mid (P \| \kappa_P) \models \psi[\beta_0, \dots, \beta_{k-1}]\}$ has $(\mu_P)^k$ measure 1. The corollary follows from this using Claim 2.9 and most importantly Claim 2.10. ⊣

Corollary 2.11 establishes that player I wins the run of G^ψ that we constructed above. (In fact the corollary establishes more than the payoff condition. It establishes that $\psi[\alpha_0, \dots, \alpha_{k-1}]$ holds in $N_{\text{end}} = P_{\omega_1} \| \omega_1$ not only for all $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$, but outright for all $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [C]^k$.) The construction can therefore be formalized to give a winning strategy for player I in G^ψ. ⊣ (Lemma 2.3.)

An argument similar to that of Lemma 2.3 proves the following, dual lemma:

Lemma 2.12. *Suppose $\psi \notin T_k$. Then* II *has a winning strategy in G^ψ.* ⊣

Equipped with the definition of G^ψ and the lemmas above we can begin to characterize 0^W in terms of the game quantifier of Section 1:

Theorem 2.13. *T_k is recursive in $\Game_{\omega_1} \Phi_k$.*

Proof. The game G^ψ above clearly has the format of Definition 1.2, with payoff given by a formula in Φ_k. In fact it's clear that we can fix a recursive map $\psi(x_0, \dots, x_{k-1}) \mapsto \varphi^\psi \in \Phi_k$ so that for each formula $\psi(x_0, \dots, x_{k-1})$ the game G^ψ is precisely equal to the game $G_{\omega_1,k}(\vec{S}, \varphi^\psi)$ of Definition 1.2.

By Lemmas 2.3 and 2.12, $\psi \in T_k$ iff player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi^\psi)$. T_k is therefore equal to $\{\psi(x_0, \dots, x_{k-1}) \mid \Game_{\omega_1,k}(\varphi^\psi) = \text{True}\}$, and this set is recursive in $\Game_{\omega_1} \Phi_k$. ⊣

Corollary 2.14. *0^W is recursive in $\oplus_{k<\omega}(\Game_{\omega_1} \Phi_k)$.* ⊣

We shall see later that the reverse direction, that $\oplus_{k<\omega}(\Game_{\omega_1} \Phi_k)$ is recursive in 0^W, is also true. Thus the situation we obtain here for 0^W and games of length ω_1 precisely parallels the situation in the case of $0^\sharp$ and length ω games with $<\omega^2 - \Pi^1_1$ payoff. More precisely it parallels the fact that $0^\sharp$ is the recursive join of complete $\Game_\omega(\omega \cdot k - \Pi^1_1)$ reals, shown in Martin [2]. The analogy can be strengthened further. Recall that N, defined earlier in the section, is the model obtained by iterating the top extender of 0^W through the ordinals, and cutting the direct limit to height On. N is a parallel of L (obtained by iterating $0^\sharp$ through the ordinals and cutting the direct limit to height On) to the context of indiscernible Woodin cardinals. The following result parallels the fact that the Σ_1 theory of L is a Σ^1_2 real, or in other words a $\Game_\omega \Pi^1_1$ real.

THEOREM 2.15 (Steel). *Let T_{Σ_1} be the Σ_1 theory of N. Let Φ_{Σ_1} be the set of Σ_1 sentences in the language $\mathcal{L}^+$ of Section 1. Then T_{Σ_1} is recursive in $\Game_{\omega_1} \Phi_{\Sigma_1}$.*

Notice that both T_{Σ_1} and Φ_{Σ_1} only involve sentences, that is formulae with 0 free variables. We noted in Section 1 that the games $G_{\omega_1,0}(\vec{S}, \varphi)$ for $\varphi \in \Phi_{\Sigma_1}$ are simply the definable *open* length ω_1 games. Theorem 2.15 thus connects the open length ω_1 game quantifier to the Σ_1 theory of the minimal class model with indiscernible Woodin cardinals.

PROOF OF THEOREM 2.15. For a Σ_1 sentence ψ, the game G^ψ defined earlier in the section can be revised to have the format of a game $G_{\omega_1,0}(\varphi)$ with $\varphi \in \Phi_{\Sigma_1}$. The revision involves joining the payoff conditions of the cases of illfoundedness and wellfoundedness. More precisely, revise G^ψ to say that player I wins a run consisting of P, Q, $\mathcal{U}$, and $\mathcal{V}$ just in case that there exists some $\eta < \omega_1$ so that either:

- $(N_{\text{end}} \| \eta) \models \psi$; or else
- There is $\xi \leq \eta$ so that P_ξ and all previous models on $\mathcal{U}$ are wellfounded but Q_ξ is illfounded.

If such an $\eta < \omega_1$ does not exist then II wins.

It is clear that the revised G^ψ has the format of Definition 1.2 with payoff given by a Σ_1 sentence. More precisely there is a recursive map $\psi \mapsto \varphi^\psi \in \Phi_{\Sigma_1}$ so that for every Σ_1 sentence ψ, the revised game G^ψ is precisely equal to $G_{\omega_1,0}(\vec{S}, \varphi^\psi)$.

Lemmas 2.3 and 2.12 easily adapt to the revised game, showing that $\psi \in T_{\Sigma_1}$ iff player I has a winning strategy in $G_{\omega_1,0}(\vec{S}, \varphi^\psi)$. The theorem follows. ⊣

We shall see later that the reverse direction to Theorem 2.15, that $\Game_{\omega_1} \Phi_{\Sigma_1}$ is recursive in T_{Σ_1}, is also true. Again this parallels the situation at the level of $0^\sharp$ and length ω games, since $\Game_\omega \Pi^1_1$ statements are Σ_1 over L by Martin [2].

THEOREM 2.16 (Steel). *0^W is not a model of determinacy for definable open length ω_1 games, that is for games $G_{\omega_1,0}(\vec{S}, \varphi)$ with $\varphi \in \Phi_{\Sigma_1}$.*

PROOF. 0^W and N have the same reals and the same sets of reals. We may therefore prove the theorem for N instead of 0^W. Let $<_c$ denote the order of constructibility on reals in N. $x <_c y$ is Σ_1 over N. Relativizing the proof of Theorem 2.15 to reals, and running it inside N, we see that "$<_c$ belongs to the pointclass $\Game_{\text{open}-\omega_1} \Pi^1_1$" holds in N. (When running the proof of Theorem 2.15 inside N we use the fact that N knows how to iterate its countable initial segments, due to Steel [6].) From this it follows by standard arguments that determinacy for length ω games with $\Game_{\text{open}-\omega_1} \Pi^1_1$ payoff fails in N. Hence certainly $G_{\text{open}-\omega_1} \Pi^1_1$ determinacy fails in N, and equivalently there are games $G_{\omega_1,0}(\vec{S}, \varphi)$ with $\varphi \in \Phi_{\Sigma_1}$ which are not determined in N. ⊣

§3. Determinacy. We work in this section under the assumption that there exists a pair $\langle M, \mu_M\rangle$ satisfying:

(A1) M is a countable model of ZFC^*;[1]
(A2) μ_M is an external measure over M, and $\mathrm{Ult}(M, \mu_M)$ agrees with M up to its first strongly inaccessible cardinal above $\mathrm{crit}(\mu_M)$;
(A3) $\mathrm{crit}(\mu_M)$ is Woodin in M; and
(A4) $\langle M, \mu_M\rangle$ is $\omega_1 + 1$–iterable.

By an iteration tree on $\langle M, \mu_M\rangle$ we mean the natural modification of the standard definition, to allow the use of μ_M and its images, in addition to the use of internal extenders. Iterability in condition (A4) is meant with respect to this liberalized notion.

The existence of a pair $\langle M, \mu_M\rangle$ satisfying conditions (A1)–(A4) follows from the existence of 0^W: if $N = 0^W$ and μ^W is the top extender of 0^W then $\langle \mathrm{Ult}(N, \mu^W), \mu^W\rangle$ satisfies these conditions.

Essentially conditions (A1)–(A4) spell out the properties of 0^W (or more precisely the ultrapower of 0^W by its top extender) which we shall need in the construction below. Notice that none of these properties involves the fine structure of 0^W. So a pair $\langle M, \mu_M\rangle$ satisfying conditions (A1)–(A4) can be obtained from any countable iterable model with, e.g., a measurable Woodin cardinal.

Fix $k < \omega$. Fix a collection $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k}\rangle$ of mutually disjoint subsets of ω_1. (There is no need to assume that these sets are stationary.) Fix a formula φ in the language $\mathcal{L}^+$ of Section 1, with k free variables. We work to prove that $G_{\omega_1,k}(\vec{S}, \varphi)$ is determined.

Remark 3.1. To avoid some inconveniences in the proof let us assume that $k \geq 1$. We shall derive the case $k = 0$ from the case $k = 1$ later on.

Remark 3.2. Without loss of generality we may assume that $\bigcup_{a\in[\omega_1]^{<k}} S_a$ is equal to ω_1. This can always be arranged by increasing $S_\emptyset$, and an increase of this kind only serves to make the game $G_{\omega_1,k}(\vec{S}, \varphi)$ more demanding for both players.

We work throughout with the terminology of Neeman [4]. More specifically we need the definitions in Chapter 4 of [4], the end results in Chapters 5 and 6, and some of the definitions and intermediary claims in Chapter 7. We briefly and very informally introduce key points of the definitions and results below, as they become relevant, and give more specific references.

An **annotated position** t, defined in [4, §4A], is a sequence consisting of reals, and auxiliary objects which come up during the determinacy proofs. We use $\vec{z}(t)$ to denote the **real part** of t, defined precisely in [4, 4A.21]. $\vec{z}(t)$ is a sequence of reals. We use $r(t)$ to denote the concatenation of the reals in $\vec{z}(t)$. More precisely, if $\vec{z} = \langle z_\xi \mid \xi < \mathrm{lh}(\vec{z})\rangle$ say, then $r(t)$ is the sequence r defined by $r(\omega \cdot \xi + n) = z_\xi(n)$ for $\xi < \mathrm{lh}(\vec{z})$ and $n < \omega$. $r(t)$ is then a sequence of natural numbers of length $\omega \cdot \mathrm{lh}(\vec{z}(t))$, literally the concatenation of $\vec{z}$. We refer to $r(t)$ as the **concatenated** real part of t.

[1] See Neeman [4, Appendix A].

Let $\theta = \text{crit}(\mu_M)$. θ is a Woodin cardinal of M by condition (A3). Using this and the fact that θ is the critical point of a measure over M it follows that θ is also a limit of Woodin cardinals of M. So θ is a Woodin limit of Woodin cardinals in M.

Let $\mathbb{W}_\theta$ be the poset defined in [4, §4B]. This is a version of Woodin's extender algebra on identities in $M\|\theta$, restricted to the use of extenders which overlap Woodin cardinals, and designed specifically so that the generic object is an annotated position of length θ (rather than merely its real part). A θ–**sequence**, defined precisely in [4, 4D.1], is an annotated position of length θ which is generic for $\mathbb{W}_\theta$, meaning that it satisfies all the extender axioms of the algebra.

Let α_k denote θ. For the sake of Definitions 3.3 and 3.4 fix some tuple $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\theta]^k$, with each α_i a Woodin cardinal of M.

Definition 3.3. For expository simplicity fix some G which is $\mathbb{W}_\theta$–generic over M. Define $\dot{Y}_k(\alpha_0, \dots, \alpha_{k-1}) \in M$ to be the canonical $\mathbb{W}_\theta$–name for the set of θ–sequences $t \in M[G]$ so that $(\mathrm{L}_\theta[r(t)]; r(t)) \models \varphi[\alpha_0, \dots, \alpha_{k-1}]$.

$r(t)$ in Definition 3.3 is the concatenated real part of t mentioned above. In the case of a θ–sequence t, where θ is a Woodin limit of Woodin cardinals, $r(t)$ has length precisely θ.

Definition 3.3 sets our goal in the game $G_{\omega_1,k}(\vec{S}, \varphi)$. We want to play the game so that the set of tuples $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$ for which we enter interpretations of $\dot{Y}_k(\alpha_0, \dots, \alpha_{k-1})$ is large enough that it contains $[C]^k \cap [\vec{S}]$ for a club C.

Definition 3.4. For each $i < k$ define $\dot{Y}_i(\alpha_0, \dots, \alpha_{k-1})$ to be the (α_i, α_{i+1})–pullback of $\dot{Y}_{i+1}(\alpha_0, \dots, \alpha_{k-1})$ as computed in M. The definition is made by induction, working *downward* from $i = k-1$ to $i = 0$.

The precise definition of the pullback operation is given in [4, §§4C,4D]. $\dot{Y}_i(\alpha_0, \dots, \alpha_{k-1})$ is a name for a set of α_i–sequences. Roughly speaking the pullback operation is defined in such a way that $\dot{Y}_i$ names the set of sequences from which player I can play to enter an interpretation of a shift of $\dot{Y}_{i+1}$. The precise meaning of shift here is given by the definitions of the games $\widehat{G}_{\text{branch}}$ in [4, §6A]. These games set the rules for the construction of an iteration map by which $\dot{Y}_{i+1}$ is shifted.

Remember that we are working with a fixed formula φ in $\mathcal{L}^+$, and aiming to prove that $G_{\omega_1,k}(\vec{S}, \varphi)$ is determined. The definitions above are made with reference to φ; the formula comes in through Definition 3.3. We make the dependence more explicit in the following definition:

Definition 3.5. Define $U(\varphi)$ to be the set of $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\theta]^k$ so that $M \models \varphi_{\text{ini}}[\alpha_0, \dot{Y}_0(\alpha_0, \dots, \alpha_{k-1})]$.

The formula φ_{ini} is defined precisely in [4, Definition 5G.2]. Roughly speaking, if $\varphi_{\text{ini}}[\alpha_0, \dot{Y}_0(\alpha_0, \dots, \alpha_{k-1})]$ holds in M then player I has a strategy to enter an interpretation of a shift of $\dot{Y}_0(\alpha_0, \dots, \alpha_{k-1})$.

Let us now combine Definitions 3.3, 3.4, and 3.5: If $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in U(\varphi)$ then player I has a strategy to reach an annotated position t_0^* which enters an interpretation of a shift of $\dot{Y}_0(\alpha_0, \dots, \alpha_{k-1})$. From t_0^* player I then has a strategy to reach an annotated position t_1^* which enters a shift of $\dot{Y}_1(\alpha_0, \dots, \alpha_{k-1})$.

Continuing this way, player I then has a strategy from t_1^* to enter a shift of $\dot{Y}_2$, etc., until eventually reaching t_k^* which belongs to a shift of $\dot{Y}_k(\alpha_0, \ldots, \alpha_{k-1})$. Now Definition 3.3 is such that membership in a shift of $\dot{Y}_k(\alpha_0, \ldots, \alpha_{k-1})$ secures the instance of the payoff formula φ corresponding to the appropriate shift of $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$.

Thus, assuming that $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$ belongs to $U(\varphi)$, we intuitively expect player I to be able to play to secure an instance of the payoff formula φ, corresponding to a shift of $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$.

If many tuples $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$ belong to $U(\varphi)$ then we intuitively expect player I to be able to secure many instances of the payoff formula φ. If $U(\varphi)$ is so large that it has $(\mu_{M})^k$ measure 1 then we may even expect player I to secure enough instances of φ so as to win $G_{\omega_1,k}(\vec{S}, \varphi)$. This intuitive expectation is realized by the following lemma:

Lemma 3.6. *Suppose that $U(\varphi)$ has $(\mu_{M})^k$ measure 1, where $(\mu_{M})^k$ is the kth power of μ_{M}. Then player* I *has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$.*

We prove the lemma below. The proof relies heavily on the precise meaning of "entering a shift." The shifts are created according to the rules of the games $\widehat{G}_{\text{branch}}$, which the reader may find in [4, §6A]. Theorem 6G.1 of [4] formalizes the fact that from an annotated position t which belongs to a pullback of $\dot{Y}$, player I can win to enter a shift of $\dot{Y}$. Precisely, the theorem produces a strategy for player I in an instance of $\widehat{G}_{\text{branch}}$. In proving Lemma 3.6 we combine these strategies to form a strategy for player I in $G_{\omega_1,k}(\vec{S}, \varphi)$. The reader can survive without knowledge of how the strategies in $\widehat{G}_{\text{branch}}$ are produced in [4, Chapter 6]. But it is important to know the underlying game, described in [4, 6A], and it is helpful to know how the strategies can be used, for example in Chapter 7 of [4].

Remark 3.7. Notice that the function $\varphi \mapsto U(\varphi)$ is definable over M using θ as a parameter. The definition of the function is simply the combination of Definitions 3.3 through 3.5, the definition from θ of $\mathbb{W}_\theta$ in [4, §4B], and the definition of the pullback operation in [4, §§4C,4D]. All can be phrased over M, and only θ is needed as a parameter. For future reference fix a formula χ witnessing the definability of $\varphi \mapsto U(\varphi)$. More precisely fix a formula χ so that (for all M) $M \models \chi[\theta, \varphi, X]$ iff $X = U(\varphi)$ where $U(\varphi)$ is given by Definitions 3.3 through 3.5 on M and θ.

We prove Lemma 3.6 below. But first let us find an equivalent formulation to the statement that $U(\varphi)$ has $(\mu_{M})^k$ measure 1.

For each tuple $a \in [\theta]^{\leq k}$ we define below a θ–name $\dot{Y}(a) \in M$. We work by induction on the length of a, downward from length k to length 0. The definition results in a map $a \mapsto \dot{Y}(a)$ ($a \in [\theta]^{\leq k}$), which we denote $\dot{Y}$. We make sure as we proceed that the map belongs to M.

In the case of a tuple $a = \langle \alpha_0, \ldots, \alpha_{k-1} \rangle$ of length k set $\dot{Y}(a)$ equal to the name $\dot{Y}_k(\alpha_0, \ldots, \alpha_{k-1})$ of Definition 3.3. This defines the map $\dot{Y} \restriction [\theta]^k$. $\dot{Y} \restriction [\theta]^k$ belongs to M since Definition 3.3 is made inside M.

Let now $l < k$, and suppose that the map $\dot{Y} \restriction [\theta]^{l+1}$ is known and belongs to M. Let $M^* = \text{Ult}(M, \mu_{M})$ and let $j \colon M \to M^*$ be the ultrapower embedding.

For each $a \in [\theta]^l$ set $\dot{Y}(a)$ equal to the $(\theta, j(\theta))$–pullback of $j(\dot{Y} \restriction [\theta]^{l+1}])(a^\frown\langle\theta\rangle)$, as computed in M^*. This defines $\dot{Y} \restriction [\theta]^l$ inside M^*, and each $\dot{Y}(a)$ for $a \in [\theta]^l$ is a θ–name in M^*. Since M and M^* agree to θ, $(\mathbb{W}_\theta)^{M^*}$ is equal to $(\mathbb{W}_\theta)^M$. It follows that each $\dot{Y}(a)$ is also a θ–name in M. Notice that θ–names in M^* are essentially elements of $M^* \| \theta + 2$. It follows that the entire map $\dot{Y} \restriction [\theta]^l$ can be coded by an element of $M^* \| \theta + 2$. The agreement in condition (A2) above is such that $M^* \| \theta + 2$ is contained in M. So $\dot{Y} \restriction [\theta]^l$ belongs to M.

The two paragraphs above complete the definition of the map $a \mapsto \dot{Y}(a)$ for $a \in [\theta]^{\leq k}$, and show that the map belongs to M. We record some properties of the definition, crucial for future use, in Claim 3.8 below. Then in Claim 3.9 we connect the definition of $\dot{Y}$ to the statement "$U(\varphi)$ has $(\mu_M)^k$ measure 1" of Lemma 3.6.

CLAIM 3.8. *Let $a \in [\theta]^{\leq k}$. Let t belong to an interpretation of $\dot{Y}(a)$.*

1. *If $\text{lh}(a) = k$ then t belongs to an interpretation of $\dot{Y}_k(a)$ where $\dot{Y}_k(a)$ is the name of Definition 3.3.*
2. *If $\text{lh}(a) < k$ then t belongs to an interpretation of the $(\theta, j(\theta))$–pullback of $j(\dot{Y})(a^\frown\langle\theta\rangle)$, where $j\colon M \to \text{Ult}(M, \mu_M)$ is the ultrapower embedding of M by μ_M, and the pullback is computed inside* $\text{Ult}(M, \mu_M)$. ⊣

CLAIM 3.9. *Suppose that $U(\varphi)$ has $(\mu_M)^k$ measure 1. Then $M \models \varphi_{\text{ini}}[\theta, \dot{Y}(\emptyset)]$.*

PROOF. Let $\langle M_n, j_{n,m} \mid n \leq m \leq k\rangle$ be the finite iteration defined by setting $M_0 = M$; setting $M_{n+1} = \text{Ult}(M_n, j_{0,n}(\mu_M))$ and letting $j_{n,n+1}\colon M_n \to M_{n+1}$ be the ultrapower embedding by $j_{0,n}(\mu_M)$; and defining the remaining embeddings $j_{n,m}$ by composition. Let $\theta_n = j_{0,n}(\theta)$ for each $n \leq k$.

The agreement in condition (A2) above implies that for each $n \leq k$, M_n and M_k agree to the first strongly inaccessible cardinal of M_n above θ_n. This is more than enough to make sure that pullbacks of θ_n–names are absolute between M_n and M_k. The absoluteness follows from the local nature of the pullback operation; the pullback of a δ–name in a model N only involves objects at ranks approximately δ. (More precisely it involves elements up to the least pair of local indiscernibles, see [4, Definition 1A.15], of N above δ.)

Notice that M_k is simply equal to the ultrapower of M by $(\mu_M)^k$. A set $X \subset [\theta]^k$ in M has $(\mu_M)^k$ measure 1 iff $\langle\theta_0, \ldots, \theta_{k-1}\rangle$ belongs to $j_{0,k}(X)$.

Suppose now that $U(\varphi)$ has $(\mu_M)^k$ measure 1. It follows that $\langle\theta_0, \ldots, \theta_{k-1}\rangle$ belongs to $j_{0,k}(U(\varphi))$, and by Definition 3.5 this means that M_k satisfies the formula $\varphi_{\text{ini}}[\theta, j_{0,k}(\dot{Y}_0)(\theta_0, \ldots, \theta_{k-1})]$. It's easy to check, directly from Definition 3.4, the definition of the map $\dot{Y}$, and the absoluteness of pullbacks noted above, that $j_{0,k}(\dot{Y}_0)(\theta_0, \ldots, \theta_{k-1})$ is precisely equal to $\dot{Y}(\emptyset)$. So M_k satisfies $\varphi_{\text{ini}}[\theta, \dot{Y}(\emptyset)]$. $\varphi_{\text{ini}}[\theta, \ldots]$ involves pullbacks of δ–names for $\delta \leq \theta$. Using the absoluteness of pullbacks noted above it follows that φ_{ini} is absolute between M and M_k. So M satisfies $\varphi_{\text{ini}}[\theta, \dot{Y}(\emptyset)]$, as required. ⊣

REMARK 3.10. We say that a $\mathbb{W}_\theta$–name $\dot{B}$ is **symmetric** if for any two generics G_1 and G_2 for $\mathbb{W}_\theta$ over M, and for any x which belongs to $M[G_1] \cap M[G_2]$, $x \in \dot{B}[G_1] \iff x \in \dot{B}[G_2]$.

Notice that the condition of symmetry holds for $\dot{Y}(a)$. If $\mathrm{lh}(a) = k$ this follows from the fact that Definition 3.3 decides the matter of the membership of t in $\dot{Y}(a)[G]$ with no reference to G or even to $M[G]$, but just with reference to t. If $\mathrm{lh}(a) < k$ then the symmetry of $\dot{Y}(a)$ follows from the symmetry of pullbacks to Woodin limit of Woodin cardinals. These pullbacks are ultimately given by an application of case 2 in [4, §4D(2)]. The conditions there refer to a formula φ_{suc}, defined in [4, §4C(2)]. φ_{suc} is absolute between generic extensions of M by [4, Claim 4C.10], and using this it's easy to see that case 2 in [4, §4D(2)] defines a symmetric name.

Proof of Lemma 3.6. Let us now begin the proof of Lemma 3.6. Fix an iteration strategy Γ for $\langle M, \mu_M \rangle$. Fix an imaginary opponent willing to play for II in $G_{\omega_1,k}(\vec{S}, \varphi)$. We describe how to play against the opponent, and win.

The description takes the form of a construction. We work to construct:

(A) A function $\alpha \mapsto a_\alpha$ ($\alpha < \omega_1$) with $a_\alpha \in [\alpha]^{\leq k}$;
(B) A regular tot $\mathfrak{U}$ on $\langle M, \mu_M \rangle$, of length $\omega_1 + 0.2$, consistent with Γ; and
(C) A $\mathfrak{U}$–sequence $\langle \vec{w}, \vec{y} \rangle = \langle w_\xi, y_\xi \mid \xi \in K^{\mathfrak{U}} \rangle$.

The sequence in condition (C) consists of a play r in $G_{\omega_1,k}(\vec{S}, \varphi)$, obtained from $\vec{y}$ in the manner of Definition 3.11 below, and auxiliary moves, in $\vec{w}$. The "tot," or **tree of trees**, in condition (B) is an iteration tree making individual real moves in the play generic over collapses of Woodin cardinals which are not limits of Woodin cardinals, and making the play itself, plus the auxiliary information, generic for the extender algebra at Woodin limits of Woodin cardinals. (See [4, §7B] for the definitions relevant to these conditions. By a tot on $\langle M, \mu_M \rangle$ we mean the natural modification of the definition of [4, §7B], to allow the use of μ_M and its images, in addition to the use of internal extenders.) The sequence in condition (A) delineates our progress toward winning $G_{\omega_1,k}(\vec{S}, \varphi)$. We intend to make sure that, if a_α is a tuple of length k, then $(L_{\omega_1}[r]; r) \models \varphi[\hat{a}_\alpha]$, attaining an instance of our goal in the game. ($a \mapsto \hat{a}$ is a shift which is the identity on a club.) We then intend to prove that there is a club C so that the set $\{a_\alpha \mid \alpha < \omega_1\}$ contains $[C]^k \cap [\vec{S}]$, thereby securing our victory in $G_{\omega_1,k}(\vec{S}, \varphi)$. We will use $\vec{S}$ during the construction to guide the sequence $\alpha \mapsto a_\alpha$ and the tree structure of $\mathfrak{U}$, so that at the end the club C can be obtained very directly from a branch of length ω_1 through $\mathfrak{U}$.

Definition 3.11. Let $\vec{z} = \langle y_\zeta \mid \zeta < \omega_1$ and ζ is either zero or a successor ordinal$\rangle$. Define $r \in \omega^{\omega_1}$ by $r(\omega \cdot \xi + n) = y_{-1+\xi+1}(n)$ for $\xi < \omega_1$ and $n < \omega$. Both $\vec{z}$ and r are defined with reference to the construction, specifically with reference to the objects of condition (C).

$\vec{z}$ is the part of the $\mathfrak{U}$–sequence $\langle \vec{w}, \vec{y} \rangle$ which involves real numbers. r is simply the concatenation of the reals in $\vec{z}$. Notice that r, being an element of ω^{ω_1}, is a run of $G_{\omega_1,k}(\vec{S}, \varphi)$. We let the imaginary opponent contribute the odd half of r during the construction. All the other elements involved with conditions (A)–(C), including the even half of r, we construct ourselves. We shall verify at the end that r is won by player I in $G_{\omega_1,k}(\vec{S}, \varphi)$.

When working with $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$ we regularly use the notation of [4, §7B]. $\mathfrak{U}$ itself consists of a tree order U on $\omega_1 + 1$; models M_ξ for $\xi \leq \omega_1$ and Q_ξ for

$\xi < \omega_1$; embeddings $j_{\zeta,\xi} \colon M_\zeta \to M_\xi$ for $\zeta\ U\ \xi$ commuting in the natural way; length ω iteration trees $\mathcal{T}_\xi$ on M_ξ for $\xi < \omega_1$; infinite branches b_ξ through these trees; and objects E_ξ for $\xi < \omega_1$ which may either be extenders of Q_ξ or be equal to "undefined." The precise relationship between these objects is explained in conditions (S), (U), and (L) of [4, §7B].

The objects in $\mathfrak{U}$ give rise to ordinals $\delta_{\xi+1}$ and λ_ξ defined in [4, §7B] and characterized specifically in Claims 7B.5 through 7B.13 of [4]. $\mathfrak{U}$ and the sequence $\langle \vec{w}, \vec{y} \rangle$ of (C) above together give rise to annotated positions t_η ($\eta \le \omega_1$) in the manner of [4, Definition 7B.14]. They also give rise to *strands*, in the manner of [4, Definition 7B.17]. We need actually a generalization of this last notion, which we define next.

For the purpose of Definition 3.12 below fix some $\eta \le \omega_1$. By $[0,\eta]_{\mathfrak{U}}$ we mean the branch of $\mathfrak{U}$ leading to η, with η itself included. More precisely this is the set $\{\zeta \mid (\zeta\ U\ \eta) \vee (\zeta = \eta)\}$.

Again for the purpose of Definition 3.12 let $\beta + 1$ be the order type of $[0,\eta]_{\mathfrak{U}}$ and let $f \colon \beta + 1 \to [0,\eta]_{\mathfrak{U}}$ be an order preserving isomorphism. For each $\xi < \beta$, $f(\xi+1)$ is a successor ordinal. Still for the purpose of Definition 3.12 let E^*_ξ denote $E_{f(\xi+1)-1}$.

The notation above follows that leading to Definition 7B.17 of [4], which defines the **strand** (of $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$) **leading to** η to be the sequence

$$P_\eta = \langle \mathcal{T}_{f(\xi)}, b_{f(\xi)}, E^*_\xi, t_{f(\xi+1)} \mid \xi < \beta \rangle.$$

Intuitively this is the part of $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$ which corresponds to the branch of $\mathfrak{U}$ leading to η. It is observed following Definition 7B.17 of [4] that P_η has the format of a position of length β in the game $\widehat{G}_{\text{branch}}$ of [4, §6A], and this connection is key to the later constructions in [4, §§7C,7D].

Definition 3.12. Let ν belong to $[0,\eta]_{\mathfrak{U}}$. Let $\alpha = f^{-1}(\nu)$. By the **strand** (of $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$) **leading from** ν **to** η we mean the sequence

$$P_{\nu,\eta} = \langle \mathcal{T}_{f(\xi)}, b_{f(\xi)}, E^*_\xi, t_{f(\xi+1)} \mid \xi \in [\alpha, \beta) \rangle.$$

Definition 3.12 generalizes Definition 7B.17 of [4]. The strand leading to η in the sense of [4, 7B.17] is the same as the strand leading from 0 to η in the sense of Definition 3.12.

Notice that this more general definition retains the connection to $\widehat{G}_{\text{branch}}$. The strand leading from ν to η has the format of a position in an instance of $\widehat{G}_{\text{branch}}$, more specifically the instance appearing in condition (4) below.

We need one more notational ingredient before we can begin to be more specific on the construction of the objects in conditions (A)–(C) above.

For each $\alpha < \omega_1$ let $\hat{\alpha}$ denote $\text{rdm}(t_\alpha)$. (This definition is made with reference to $\mathfrak{U}$, or at least $\mathfrak{U} \restriction \alpha + 0.2$, which is needed to give rise to t_α.) For each tuple $a = \langle \alpha_0, \ldots, \alpha_{l-1} \rangle \in [\omega_1]^{\le k}$ let $\hat{a}$ denote the tuple $\langle \hat{\alpha}_0, \ldots, \hat{\alpha}_{l-1} \rangle$.

Remark 3.13. Note that $\alpha = \text{rdm}(t_\alpha)$ on a club, and therefore $\hat{\alpha} = \alpha$ on a club. These equalities follow from the fact that $\langle \text{rdm}(t_\alpha) \mid \alpha < \omega_1 \rangle$ is a continuous sequence of countable ordinals, monotone increasing, and strictly increasing at successors in the sense that $\text{rdm}(t_\alpha) < \text{rdm}(t_\beta)$ for $\alpha < \beta$ with α a successor.

For each $a \in [\omega_1]^{\leq k}$ of length greater than 0 let $\nu(a) = \max(a) + 1$. Let $\nu(\emptyset) = 0$.

Let A denote the set of $a \in [\omega_1]^{\leq k}$ so that player I has a winning strategy in the game $\widehat{G}_{\text{branch}}(M_{\nu(a)}, t_{\nu(a)}, j_{0,\nu(a)}(\theta))(j_{0,\nu(a)}(\dot{Y})(\hat{a}))$. We refer the reader to [4, §6A] for the definition of $\widehat{G}_{\text{branch}}$. Here we use the instance of $\widehat{G}_{\text{branch}}$ which corresponds to starting from the model $M_{\nu(a)}$ of $\mathfrak{U}$ and from the annotated position $t_{\nu(a)}$, and aiming to enter a shift of $j_{0,\nu(a)}(\dot{Y})(\hat{a})$. The target $j_{0,\nu(a)}(\dot{Y})(\hat{a})$ is the name associated to $\hat{a}$ by the shift to $M_{\nu(a)}$ of the map $\dot{Y}$ defined in connection with Claims 3.8 and 3.9 above.

A is defined with reference to $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$ which we have yet to construct. But regardless of the construction we have $M_0 = M$ (this is because $\mathfrak{U}$ is a tot on $\langle M, \mu_M \rangle$) and $t_0 = \emptyset$, or in other words t_0 equal to the empty annotated position of length 0. The question of membership of $a = \emptyset$ in A can therefore be considered already now, regardless of the construction. The following claim shows that $\emptyset \in A$. The claim makes a crucial use of the initial assumption in Lemma 3.6.

Claim 3.14. *The tuple $a = \emptyset$ belongs to A.*

Proof. The initial assumption in Lemma 3.6 states that $U(\varphi)$ has $(\mu_M)^k$ measure 1. By Claim 3.9 it follows that M satisfies $\varphi_{\text{ini}}[\theta, \dot{Y}(\emptyset)]$. By Corollary 6G.2 of [4] then player I has a winning strategy in $\widehat{G}_{\text{branch}}(M, \emptyset, \theta)(\dot{Y}(\emptyset))$. Since $M_0 = M$, $j_{0,0} = id$, and $t_0 = \emptyset$, it follows that $\emptyset \in A$. ⊣

For each a which belongs to A fix some winning strategy $\widehat{\Sigma}_{\text{branch}}(a)$ for player I in $\widehat{G}_{\text{branch}}(M_{\nu(a)}, t_{\nu(a)}, j_{0,\nu(a)}(\theta))(j_{0,\nu(a)}(\dot{Y})(\hat{a}))$. Given a non-terminal position P in $\widehat{G}_{\text{branch}}(M_{\nu(a)}, t_{\nu(a)}, j_{0,\nu(a)}(\theta))(j_{0,\nu(a)}(\dot{Y})(\hat{a}))$ let $\widehat{\Sigma}_{\text{branch}}(a)[P]$ be the restriction of $\widehat{\Sigma}_{\text{branch}}(a)$ to the mega-round which precisely follows P, that is to mega-round $\text{lh}(P)$ following the position P.

We intend to make sure that the following conditions hold for each $\alpha < \omega_1$ (except for *external limit* α, which we shall define and discuss later):

1. The tuple a_α belongs to A;
2. All the ordinals in $\hat{a}_\alpha$ are smaller than $j_{0,\nu(a_\alpha)}(\theta)$;
3. $\nu(a_\alpha)$ belongs to $[0, \alpha]_{\mathfrak{U}}$; and
4. The strand $P_{\nu(a_\alpha),\alpha}$ is a legal position in the game
$$\widehat{G}_{\text{branch}}(M_\nu, t_\nu, j_{0,\nu}(\theta))(j_{0,\nu}(\dot{Y})(\hat{a}_\alpha))$$
(where ν abbreviates $\nu(a_\alpha)$), non-terminal in this game, and played according to $\widehat{\Sigma}_{\text{branch}}(a_\alpha)$.

Condition (4) is the most important one. The other conditions are simply needed to make sense of condition (4). Condition (1) is needed to make $\widehat{\Sigma}_{\text{branch}}(a_\alpha)$ meaningful, condition (2) is needed to make $j_{0,\nu}(\dot{Y})(\hat{a}_\alpha)$ meaningful, and condition (3) is needed to make $P_{\nu,\eta}$ meaningful.

We begin the construction by setting $a_0 = \emptyset$ (as we must since a_0 has to belong to $[0]^{\leq k}$). Condition (1) then holds for $\alpha = 0$ by Claim 3.14. Condition (2) holds trivially since $\hat{a}_0 = \emptyset$. Condition (3) holds trivially since $\nu(a_0) = 0$. Condition (4) also holds trivially, since $P_{0,0}$ is the empty position.

We then construct in stages $\eta < \omega_1$, starting with $\eta = 0$.

At the start of a successor (or zero) stage η we have $\mathfrak{U}\restriction\eta + 0.2$; the sequences $\vec{w}\restriction\eta$ and $\vec{y}\restriction\eta$; and the association $\alpha \mapsto a_\alpha$ for $\alpha \leq \eta$. Notice that this is enough to determine $\alpha \mapsto \hat{\alpha}$ for $\alpha \leq \eta$, enough to determine membership in A for $a \in [\eta]^{\leq k}$, and enough to determine strands leading to η. Thus at the start of a successor (or zero) stage η we have enough information to determine the truth value of conditions (1)–(4) for $\alpha \leq \eta$. Inductively we know that these conditions hold true.

Successor and zero stages. Conditions (1), (3), and (4) for $\alpha = \eta$ tell us that $a = a_\eta$ belongs to A so that $\widehat{\Sigma}_{\text{branch}}(a)$ is defined; that $\nu = \nu(a_\eta)$ belongs to $[0, \eta]_{\mathfrak{U}}$ so that $P_{\nu,\eta}$ is defined; and that $P_{\nu,\eta}$ is legal and non-terminal in $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, j_{0,\nu}(\theta))(j_{0,\nu}(\dot{Y})(\hat{a}))$, and played according to $\widehat{\Sigma}_{\text{branch}}(a)$, so that $\widehat{\Sigma}_{\text{branch}}(a)[P_{\nu,\eta}]$ is defined.

$\widehat{\Sigma}_{\text{branch}}(a)[P_{\nu,\eta}]$, the iteration strategy Γ, and the imaginary opponent, combine to produce w_η, y_η, $\mathcal{T}_\eta$, and b_η according to rules (S1)–(S4) of $\widehat{G}_{\text{branch}}$ in [4, §6A]. $\mathcal{T}_\eta$ and b_η determine $\mathfrak{U}\restriction\eta + 1$, with a final model Q_η equal to the direct limit of the models of $\mathcal{T}_\eta$ along b_η. Working over Q_η let $t^\dagger_\eta = t_\eta$—$, w_\eta, y_\eta$. This is the annotated position over Q_η obtained by extending t_η of [4, Definition 7B.14] with the moves w_η and y_η produced above.

Case 1. If $t^\dagger_\eta$ is obstruction free over Q_η. In this case let $\mathfrak{U}\restriction\eta + 1.2$ be the extension of $\mathfrak{U}\restriction\eta{+}1$ determined by the assignment $E_\eta =$"undefined." Let $a_{\eta+1} = a$ (namely to a_η). An argument similar to that of [4, Lemma 7C.7] shows that $P_{\nu,\eta+1}$ is then legal in $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, j_{0,\nu}(\theta))(j_{0,\nu}(\dot{Y})(\hat{a}))$, non-terminal, and played according to $\widehat{\Sigma}_{\text{branch}}(a)$. This secures condition (4) for $\eta + 1$. Conditions (1) and (2) for $\eta{+}1$ follow trivially from the same condition for η, since $a_{\eta+1} = a_\eta$ through the assignment above. Condition (3) for $\eta + 1$ also follows trivially since the extension of $\mathfrak{U}\restriction\eta$ made above is such that $\eta\ U\ \eta + 1$. $\dashv$ (Case 1.)

Case 2. If $t^\dagger_\eta$ is obstructed over Q_η. An argument similar to that of [4, Claim 7C.6] shows that $t^\dagger_\eta$ is I–acceptably obstructed over Q_η. (The key point is that annotated positions which are obstructed but not I–acceptably obstructed cause a loss for player I in $\widehat{G}_{\text{branch}}$, and therefore cannot occur in plays according to $\widehat{\Sigma}_{\text{branch}}(a)$ which is winning for I.)

Let $\langle E, \vec{\sigma}\rangle$ then be a I–acceptable obstruction for $t^\dagger_\eta$ over Q_η. crit(E) is a limit of Woodin cardinals in Q_η but not itself Woodin. By [4, Claim 7B.6] there exists some $\gamma \leq \eta$ so that γ is a standard limit in $\mathfrak{U}$ and crit(E) is equal to λ_γ. (λ_γ is one of the objects defined in [4, §7B]. Our reasoning above is similar to that in [4, §7C(2)].)

Let $\mathfrak{U}\restriction\eta + 1.2$ be the extension of $\mathfrak{U}\restriction\eta + 1$ determined by:

(a) $E_\eta = E$; and
(b) The U–predecessor of $\eta + 1$ is γ.

These assignments are similar to the ones made in [4, §7C(2)].

Let $a^* = a_\gamma$ and let $\nu^* = \nu(a_\gamma)$. Make the assignment:

(c) $a_{\eta+1} = a^*$ (equal to a_γ that is).

Conditions (1)–(3) for $\eta+1$ then follow from the same conditions for γ. An argument similar to that of [4, Lemma 7C.13] shows that $P_{\nu^*,\eta+1}$ is legal in $\widehat{G}_{\text{branch}}(M_{\nu^*}, t_{\nu^*}, j_{0,\nu^*}(\theta))(j_{0,\nu^*}(\dot{Y})(\hat{a}^*))$, non-terminal, and played according to $\widehat{\Sigma}_{\text{branch}}(a^*)$, thereby securing condition (4) for $\eta+1$. Note that $P_{\eta+1}$ in the current case extends P_γ, rather than P_η, and this is why we pass to $a^* = a_\gamma$ and $\nu^* = \nu(a_\gamma)$ above. The fact that $P_{\eta+1}$ extends P_γ is connected to the leap taken in the proof of [4, Lemma 7C.13]. ⊣ (Case 2.)

The two cases above complete the construction in stage η in the case that η is a successor or zero, and put us in a position to pass to stage $\eta+1$. ⊣ (Successor and zero stages.)

At the start of a limit stage η we have $\mathfrak{U}\restriction\eta$, the sequences $\vec{w}\restriction\eta$ and $\vec{y}\restriction\eta$, and the association $\alpha \mapsto a_\alpha$ for $\alpha < \eta$. We know inductively that conditions (1)–(4) hold true for all $\alpha < \eta$.

Let c_η be the cofinal branch through $\mathfrak{U}\restriction\eta$ picked by the iteration strategy Γ. Let $\mathfrak{U}\restriction\eta+0.2$ be the extension of $\mathfrak{U}\restriction\eta$ determined by this branch, in other words determined by setting $[0,\eta)_{\mathfrak{U}} = c_\eta$.

If $\eta = \omega_1$ then this assignment for $\mathfrak{U}\restriction\eta+0.2$ completes the construction of the items of conditions (A)–(C). We pass to the verification of victory by player I, starting with Claim 3.25 below.

Suppose then that $\eta < \omega_1$. We must continue with the construction of $\mathfrak{U}\restriction\eta+1.2$, w_η, y_η, and $a_{\eta+1}$, which are needed at the start of stage $\eta+1$. We divide the construction of these objects into three cases. The first is similar to the case of successor and zero stages above, and results in a $P_{\eta+1}$ which either extends P_η or extends P_γ for some limit $\gamma \le \eta$. The other two cases, which we handle later, are of a different nature.

Internal limit. If $\alpha \mapsto a_\alpha$ is constant on a tail-end of c_η, and λ_η (the relative domain of t_η) is not equal to $j_{0,\eta}(\theta)$.

Let $\zeta < \eta$ be large enough that $\alpha \mapsto a_\alpha$ is constant for $\alpha \in [\zeta,\eta)_{\mathfrak{U}}$. Set $a_\eta = a_\zeta$. Conditions (1)–(4) for η then follow from the fact that the same conditions hold for all $\alpha \in [\zeta,\eta)_{\mathfrak{U}}$, and the facts that η is countable and M_η (being a model on a tot consistent with the iteration strategy Γ) is wellfounded. The last two facts are needed to see that $P_{\nu(a_\eta),\eta}$, which is equal to $\bigcup_{\alpha\in[\zeta,\eta)_{\mathfrak{U}}} P_{\nu(a_\alpha),\alpha}$, is not terminal through one of the snags (I3) and (I4) in [4, §6A].

If η is a phantom limit in $\mathfrak{U}$ then set $\mathcal{T}_\eta$ equal to the trivial length ω iteration tree consisting entirely of padding, set b_η to be the unique branch through this tree, and set $E_\eta =$"undefined." These assignments determine $\mathfrak{U}\restriction\eta+1.2$ in such a way that $M_{\eta+1} = M_\eta$ and $j_{\eta,\eta+1} = id$. Notice that there is no need to define w_η and y_η in this case, since phantom limits are excluded from $K^{\mathfrak{U}}$ which is the domain of $\mathfrak{U}$–sequences. (See [4, §7B] for the relevant definitions.) Set $a_{\eta+1} = a_\eta$. Conditions (1)–(3) for $\eta+1$ then follow directly from the same conditions for η. The same is true of condition (4), since $P_{\nu(a_{\eta+1}),\eta+1}$ here extends $P_{\nu(a_\eta),\eta}$ with just a *trivial* mega-round subject to the rules of the phantom limit case in [4, §6A]. Let us just note that the fact that $P_{\nu(a_{\eta+1}),\eta+1}$ is non-terminal, which is needed for condition (4), uses the internal limit case assumption that λ_η is not equal to $j_{0,\eta}(\theta)$. Without this assumption $P_{\nu(a_{\eta+1}),\eta+1}$ would be terminal through the condition (P2) in [4, §6A].

Suppose next that η is not a phantom limit. In other words suppose that η is a standard limit in $\mathfrak{U}$. So far we constructed $\mathfrak{U}\restriction\eta+0.2$ and a_η, and secured conditions (1)–(4) for η. We proceed now to construct $\mathfrak{U}\restriction\eta+1.2$, w_η, y_η, and $a_{\eta+1}$, working along the lines of the successor and zero stages described above.

Let a denote a_η and let ν denote $\nu(a_\eta)$. $\widehat{\Sigma}_{\text{branch}}(a)[P_{\nu,\eta}]$ and the iteration strategy Γ combine to produce w_η, $\mathcal{T}_\eta$, b_η, and y_η according to rules (L1)–(L4) of $\widehat{G}_{\text{branch}}$ in [4, §6A]. $\mathcal{T}_\eta$ and b_η determine $\mathfrak{U}\restriction\eta+1$, with a final model Q_η. Working over Q_η let $t^\dagger_\eta = t_\eta$—$, w_\eta, y_\eta$.

If $t^\dagger_\eta$ is obstruction free over Q_η then let $\mathfrak{U}\restriction\eta+1.2$ be the extension of $\mathfrak{U}\restriction\eta+1$ determined by the assignment $E_\eta =$"undefined." Let $a_{\eta+1} = a$ (namely equal to a_η). With these assignments conditions (1)–(3) for $\eta+1$ follow from the same conditions for η, secured above. Moreover an argument similar to that of [4, Lemma 7C.7] shows that $P_{\nu,\eta+1}$ is legal in $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, j_{0,\nu}(\theta))(j_{0,\nu}(\dot{Y})(\hat{a}))$, non-terminal, and played according to $\widehat{\Sigma}_{\text{branch}}(a)$. This secures condition (4) for $\eta+1$. Notice how the work here is similar to that in case 1 of the successor and zero stages above. Indeed the parallel constructions in [4] were combined into one case; [4, Lemma 7C.7] applies to both successors and standard limits.

If $t^\dagger_\eta$ is obstructed over Q_η then by an argument similar to that of [4, Claim 7C.6] it must be I–acceptably obstructed. Let $\langle E, \vec{\sigma}\rangle$ be a I–acceptable obstruction for $t^\dagger_\eta$ over Q_η in this case. Let $\gamma \leq \eta$ be such that crit(E) is equal to λ_γ. Let a^* denote a_γ and let ν^* denote $\nu(a_\gamma)$. Let $\mathfrak{U}\restriction\eta+1.2$ be the extension of $\mathfrak{U}\restriction\eta+1$ determined by the assignments:

(a) $E_\eta = E$; and
(b) The U–predecessor of $\eta+1$ is γ.

Let $a_{\eta+1} = a_\gamma$. Conditions (1)–(3) for $\eta+1$ then follow from the same conditions for γ, and an argument similar to that of [4, Lemma 7C.13] shows that $P_{\nu^*,\eta+1}$ is legal in $\widehat{G}_{\text{branch}}(M_{\nu^*}, t_{\nu^*}, j_{0,\nu^*}(\theta))(j_{0,\nu^*}(\dot{Y})(\hat{a}^*))$, non-terminal, and played according to $\widehat{\Sigma}_{\text{branch}}(a^*)$, thereby securing condition (4) for $\eta+1$. Notice how the work here is similar to that in case 2 of the successor and zero stages above. Again the parallel constructions in [4] were in fact combined into one case; [4, Lemma 7C.13] applies to both successors and standard limits.

The descriptions above divide into three subcases: phantom limit; standard limit with $t^\dagger_\eta$ obstruction free over Q_η; and standard limit with $t^\dagger_\eta$ obstructed over Q_η. In each of the subcases we constructed to the point of obtaining $\mathfrak{U}\restriction\eta+1.2$, $\vec{w}\restriction\eta+1$, $\vec{y}\restriction\eta+1$, a_η, and $a_\eta+1$, and secured conditions (1)–(4) for η and $\eta+1$. This puts us in the position necessary to pass to stage $\eta+1$. ⊣ (Internal limit.)

In both the construction for the successor and zero stages and the construction for internal limits we obtained the following condition for $\alpha = \eta+1$:

5. (If α is a successor.) Let ζ be the U–predecessor of α. Then crit$(j_{\zeta,\alpha})$ is greater than or equal to the relative domain of t_ζ, with equality possible only if ζ is a limit.

If $t^\dagger_\eta$ is obstruction free over Q_η, the U–predecessor of $\eta+1$ is η and $j_{\eta,\eta+1}$ is the direct limit embedding along the branch b_η of $\mathcal{T}_\eta$. Condition (5) for $\alpha = \eta+1$ follows from the restrictions in the rules of $\widehat{G}_{\text{branch}}$, specifically rules (S3) and (L2) in [4, §6A], which force $\mathcal{T}_\eta$ to only use critical points strictly above rdm(t_η).

If $t^\dagger_\eta$ is obstructed over Q_η, the U–predecessor of $\eta+1$ is γ where γ is such that $\mathrm{crit}(E_\eta)=\lambda_\gamma$. γ is a limit in this case and $j_{\gamma,\eta+1}$ is the ultrapower embedding of M_γ by E_η, so $\mathrm{crit}(j_{\gamma,\eta+1})=\mathrm{crit}(E_\eta)=\lambda_\gamma$. Condition (5) for $\alpha=\eta+1$ follows from this since λ_γ is equal to the relative domain of t_γ by [4, Claim 7B.15]. Finally, if η is a phantom limit then $j_{\eta,\eta+1}$ is the identity and condition (5) for $\alpha=\eta+1$ is taken to hold vacuously.

We intend to maintain condition (5) for $\eta+1$ also in the case that η falls under the additional limit cases described below. Before proceeding with these limit cases let us establish some necessary claims. The claims assume conditions (1)–(5) for $\alpha<\eta$, except that Claim 3.15 assumes condition (5) also for η, and Claim 3.16 assumes that a_η is known and that conditions (3) and (5) hold true for η. In the case of condition (5) these extra assumptions are *vacuous* if η is a limit.

Claim 3.15. *(Assuming that condition (5) holds true for η.) Let $\zeta\ U\ \eta\leq\omega_1$. Then* $\mathrm{crit}(j_{\zeta,\eta})\geq\mathrm{rdm}(t_\zeta)$ *with equality possible only if ζ is a limit.*

Proof. Immediate using condition (5) on successor ordinals α so that $\alpha\in(\zeta,\eta]_U$. ⊣

Claim 3.16. *(Assuming that a_η is known, and conditions (3) and (5) hold true for η.) $\hat{a}_\eta$ is not moved by $j_{\nu(a_\eta),\eta}$.*

Proof. If $a_\eta=\emptyset$ then $\hat{a}_\eta=\emptyset$ and the claim holds trivially. So suppose $a_\eta\neq\emptyset$. $\nu(a_\eta)$ in this case is $\max(a_\eta)+1$ by definition. In particular it is a successor, and so $\mathrm{crit}(j_{\nu(a_\eta),\eta})>\mathrm{rdm}(t_{\nu(a_\eta)})$ strictly by Claim 3.15.

Now $\hat{a}_\eta$ consists of the ordinals $\mathrm{rdm}(t_\alpha)$ for $\alpha\in a_\eta$. All these ordinals are smaller than or equal to $\mathrm{rdm}(t_{\max(a_\eta)})$, which in turn is smaller than or equal to $\mathrm{rdm}(t_{\max(a_\eta)+1})=\mathrm{rdm}(t_{\nu(a_\eta)})$, which as we saw above is strictly below the critical point of $j_{\nu(a_\eta),\eta}$. So the ordinals in $\hat{a}_\eta$ are not moved by $j_{\nu(a_\eta),\eta}$, and it follows that $\hat{a}_\eta$ too is not moved. ⊣

Claim 3.17. *Let ζ be such that $\zeta+1<\eta$. Then $t_{\zeta+1}$ is $M_{\zeta+1}$–clear.*

Proof. $P_{\nu(a_{\zeta+1}),\zeta+1}$ is legal in an instance of $\widehat{G}_{\mathrm{branch}}$ by condition (4). It's easy to see directly from Definition 3.12 that the outcome of $P_{\zeta+1}$ is equal to $\langle M_{\zeta+1}, j_{\nu(a_{\zeta+1}),\zeta+1}, t_{\zeta+1}\rangle$. Using [4, Remark 6A.4] it follows that $t_{\zeta+1}$ is $M_{\zeta+1}$–clear. ⊣

Claim 3.18. *Let $\eta\leq\omega_1$ be a limit. Then t_η is M_η–clear.*

Proof. By Claim 3.17, $t_{\zeta+1}$ is $M_{\zeta+1}$–clear for each ζ such that $\zeta+1<\eta$ and in particular for each ζ so that $\zeta+1\in[0,\eta)_U$. Since $\mathrm{crit}(j_{\zeta+1,\eta})\geq\mathrm{rdm}(t_{\zeta+1})$ by Claim 3.15, $t_{\zeta+1}$ is also M_η–clear. Since $t_\eta=\bigcup_{\zeta+1\in[0,\eta)_U}t_{\zeta+1}$ for limit η it follows that t_η is M_η–clear. ⊣

Claim 3.19. *Let $\zeta\ U\ \eta<\omega_1$. Suppose that all limit stages in $[\zeta,\eta]_U$ fall under the case of the internal limit above. Then $a_\eta=a_\zeta$.*

Proof. The claim assumes that for each $\bar\eta+1\in[\zeta,\eta]_U$, $a_{\bar\eta+1}$ is defined according to either the successor and zero construction or the terminal limit construction. Either way $a_{\bar\eta+1}$ is then equal to a_ξ where ξ is the U–predecessor of $\bar{\eta}+1$. Similarly for limit $\bar{\eta} \in [\zeta, \eta]_{\mathfrak{U}}$ the claim assumes that $a_{\bar{\eta}}$ is defined according to the terminal limit construction. $a_{\bar{\eta}}$ is then equal to a_ξ for all sufficiently large $\xi \; U \; \bar{\eta}$. Now by induction on $\bar{\eta}$ it follows that $a_{\bar{\eta}} = a_\zeta$ for all $\bar{\eta} \in [\zeta, \eta]_{\mathfrak{U}}$, and in particular $a_\eta = a_\zeta$. ⊣

CLAIM 3.20. *Let $\zeta \; U \; \eta \leq \omega_1$, and let a equal a_ζ. Suppose that η is a limit and $\lambda_\eta = j_{0,\eta}(\theta)$. Suppose that t_ζ belongs to an interpretation of $j_{0,\zeta}(\dot{Y})(\hat{a})$ over M_ζ. Then t_η belongs to an interpretation of $j_{0,\eta}(\dot{Y})(\hat{a})$ over M_η.*

PROOF. Roughly speaking this is simply a matter of rephrasing the fact that t_ζ belongs to an interpretation of $j_{0,\zeta}(\dot{Y})(\hat{a})$ as a statement over M_ζ, and then using the elementarity of $j_{\zeta,\eta}$ to transfer this statement to M_η.

Let $\dot{C}$ denote $j_{0,\zeta}(\dot{Y})(\hat{a})$. $\dot{C}$ is a name for a set of $j_{0,\zeta}(\theta)$–sequences. Since t_ζ belongs to an interpretation of $\dot{C}$ it follows that t_ζ is a $j_{0,\zeta}(\theta)$–sequence over M_ζ. In other words t_ζ is an M_ζ–clear annotated position of relative domain $j_{0,\zeta}(\theta)$. (See [4, §4B] for the relevant definitions.) Let G be the filter associated to t_ζ by [4, Definition 4B.23] carried over M_ζ. (This is the filter consisting of identities in the extender algebra which are satisfied by t_ζ.) G is then $j_{0,\zeta}(\mathbb{W}_\theta)$–generic over M_ζ by [4, Corollary 4B.30], and t_ζ belongs to $M_\zeta[G]$. From the symmetry of $\dot{C}$ given by Remark 3.10, and the fact that t_ζ belongs to an interpretation of $\dot{C}$, it follows that $t_\zeta \in \dot{C}[G]$.

Let $\dot{t}$ be the name of [4, Definition 4B.39] carried over M_ζ, so that $\dot{t}[G]$ is simply equal to t_ζ. Rephrasing the conclusion of the last paragraph we see that $\dot{t}[G] \in \dot{C}[G]$. Let $[\sigma] \in G$ be a condition forcing this.

By Claim 3.15, $j_{\zeta,\eta}$ has critical point at least the relative domain of t_ζ, which as we observed above is equal to $j_{0,\zeta}(\theta)$. Conditions in $j_{0,\zeta}(\mathbb{W}_\theta)$ are elements of $M_\zeta \| j_{0,\zeta}(\theta)$. So $j_{\zeta,\eta}$ does not move conditions in $j_{0,\zeta}(\mathbb{W}_\theta)$, and in particular it does not move $[\sigma]$.

Let H be the filter associated to t_η by [4, Definition 4B.23] carried over M_η. t_η has relative domain λ_η which is equal to $j_{0,\eta}(\theta)$ by assumption. Moreover t_η is M_η–clear by Claim 3.18. Using [4, Corollary 4B.30] it follows that H is $j_{0,\eta}(\mathbb{W}_\theta)$–generic over M_η.

Let $\dot{s}$ denote the name of [4, Definition 4B.39] carried over M_η, so that $\dot{s}[H]$ is simply equal to t_η.

Let $\dot{D}$ denote $j_{0,\eta}(\dot{Y})(\hat{a})$. Note then that $\dot{D} = j_{\zeta,\eta}(\dot{C})$; $\hat{a}$ is not moved by $j_{\zeta,\eta}$, since the map has critical point at least $\text{rdm}(t_\zeta)$ by Claim 3.15, $\text{rdm}(t_\zeta) = j_{0,\zeta}(\theta)$ as we observed above, and the ordinals in $\hat{a}$ are all below $j_{0,\nu(a_\zeta)}(\theta)$ and therefore certainly below $j_{0,\zeta}(\theta)$, by condition (2).

Applying $j_{\zeta,\eta}$ to the fact that $[\sigma] \Vdash_{j_{0,\zeta}(\mathbb{W}_\theta)}$ "$\dot{t} \in \dot{C}$," and using the fact that $[\sigma]$ is not moved by $j_{\zeta,\eta}$, we see that $[\sigma] \Vdash_{j_{0,\eta}(\mathbb{W}_\theta)}$ "$\dot{s} \in \dot{D}$."

The fact that $[\sigma] \in G$ implies that $t_\zeta \models \sigma$. (See [4, Definition 4B.23].) Since t_η extends t_ζ it follows that $t_\eta \models \sigma$. From this in turn it follows that $[\sigma] \in H$. Since $[\sigma]$ forces "$\dot{s} \in \dot{D}$" we conclude that $\dot{s}[H] \in \dot{D}[H]$, in other words t_η belongs to an interpretation of $\dot{D} = j_{0,\eta}(\dot{Y})(\hat{a})$. ⊣

We return now to the construction in the case of limit η. We have $\mathfrak{U} \restriction \eta + 0.2$, extending $\mathfrak{U} \restriction \eta$ in a manner consistent with the iteration strategy Γ. We know that conditions (1)–(5) hold true for $\alpha < \eta$. c_η denotes the branch $[0,\eta)_{\mathfrak{U}}$ picked by Γ for the extension to $\mathfrak{U}\restriction \eta + 0.2$.

So far we handled the case that $\alpha \mapsto a_\alpha$ is constant on a tail-end of c_η, and λ_η is not equal to $j_{0,\eta}(\theta)$. We handle the remaining cases next.

Terminal limit. If $\alpha \mapsto a_\alpha$ is constant on a tail-end of c_η, and λ_η *is* equal to $j_{0,\eta}(\theta)$.

Let $\zeta < \eta$ be large enough that $\alpha \mapsto a_\alpha$ is constant for $\alpha \in [\zeta,\eta)_{\mathfrak{U}}$. Set $a_\eta = a_\zeta$. As in the case of the internal limit this secures conditions (1)–(4) for η. Condition (5) for η is vacuous since η is a limit.

Let a denote a_η and let ν denote $\nu(a)$. Let θ_ν denote $j_{0,\nu}(\theta)$ and let $\dot{Y}_\nu$ denote $j_{0,\nu}(\dot{Y})$. By condition (4), $P_{\nu,\eta}$ is legal in $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, \theta_\nu)(\dot{Y}_\nu(\hat{a}))$, and played according to $\widehat{\Sigma}_{\text{branch}}(a)$. The outcome of $P_{\nu,\eta}$ is equal to $\langle M_\eta, j_{\nu,\eta}, t_\eta\rangle$.

$\text{rdm}(t_\eta)$ is equal to λ_η which by the case assumption is equal to $j_{0,\eta}(\theta)$. In particular it follows that $\text{rdm}(t_\eta)$ is Woodin in M_η, so mega-round $\beta = \text{lh}(P_{\nu,\eta})$ of $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, \theta_\nu)(\dot{Y}_\nu(\hat{a}))$ following $P_{\nu,\eta}$ is played according to the rules of the phantom limit case in [4, §6A].

Let P^+ be the one mega-round extension of $P_{\nu,\eta}$ generated by the trivial moves of the phantom limit case. The settings in the phantom limit case are such that the outcome of P^+ is simply equal to the outcome of $P_{\nu,\eta}$, namely to $\langle M_\eta, j_{\nu,\eta}, t_\eta\rangle$. (No moves are actually made in phantom limit cases, see [4, §6A].)

We saw above that $\text{rdm}(t_\eta) = j_{0,\eta}(\theta)$. In other words $\text{rdm}(t_\eta) = j_{\nu,\eta}(\theta_\nu)$. It follows from this that P^+ is *terminal* in $\widehat{G}_{\text{branch}}(M_\nu, t_\nu, \theta_\nu)(\dot{Y}_\nu(\hat{a}))$ through the payoff condition (P2) in [4, §6A]. P^+, being consistent with $\widehat{\Sigma}_{\text{branch}}(a)$, must be won by player I. Looking at the payoff condition (P2) in [4, §6A] and folding into it the fact that the outcome of P^+ is $\langle M_\eta, j_{\nu,\eta}, t_\eta\rangle$, we see that t_η belongs to an interpretation of $j_{\nu,\eta}(\dot{Y}_\nu(\hat{a}))$. By Claim 3.16 $\hat{a}$ is not moved by $j_{\nu,\eta}$. So t_η belongs to an interpretation of $j_{\nu,\eta}(\dot{Y}_\nu)(\hat{a} = \hat{a}_\eta)$. In other words:

(∗) t_η belongs to an interpretation of $j_{0,\eta}(\dot{Y})(\hat{a}_\eta)$.

Remark 3.21. Condition (∗) is in some sense the crux of the construction. It shows that we construct in a way that enters instances of shifts of $\dot{Y}$ determined by the assignment $a \mapsto a_\alpha$. The choices for this assignment (at least the crucial ones) will be made in the external limit case below. Combining these choices with condition (∗) we will then show that the construction leads to a run of $G_{\omega_1,k}(\vec{S},\varphi)$ which is won by player I.

Let $\bar{\eta} \leq \eta$ be the least element of $[0,\eta]_{\mathfrak{U}}$ which falls under the conditions of the terminal limit case. ($\bar{\eta}$ may be equal to η, but it may also be smaller.)

By condition (∗) for $\bar{\eta}$, $t_{\bar{\eta}}$ belongs to an interpretation of $j_{0,\bar{\eta}}(\dot{Y})(\hat{a}_{\bar{\eta}})$. Using the choice of $\bar{\eta}$ and instances of Claim 3.19 at and below $\bar{\eta}$, it is easy to check that $a_{\bar{\eta}}$ is equal to a_0, which was set equal to $\emptyset$. So $t_{\bar{\eta}}$ belongs to an interpretation of $j_{0,\bar{\eta}}(\dot{Y})(\emptyset)$. Using now Claim 3.20 with $\zeta = \bar{\eta}$ it follows that:

(i) t_η belongs to an interpretation of $j_{0,\eta}(\dot{Y})(\emptyset)$.

Remember that our goal is to bring the construction to the point necessary for passing to stage $\eta + 1$. We have to define $\mathfrak{U}\restriction \eta + 1.2$, assign a value to $a_{\eta+1}$, and verify that conditions (1)–(5) hold true for $\eta+1$ with the assignments made.

(There is no need to define w_η and y_η since η is a phantom limit in $\mathfrak{U}$, and therefore excluded from the domain of $\mathfrak{U}$–sequences.)

Let μ_η denote $j_{0,\eta}(\mu_{M})$ and let θ_η denote $j_{0,\eta}(\theta)$. μ_η is an external measure over M_η, with critical point equal to θ_η.

Set $\mathcal{T}_\eta$ equal to the trivial length ω iteration tree which consists entirely of padding, and set b_η equal to the unique branch through this tree. Set E_η equal to μ_η, and set the U–predecessor of $\eta+1$ equal to η. These assignments determine $\mathfrak{U}\restriction \eta+1.2$, and do so in such a way that $M_{\eta+1} = \mathrm{Ult}(M_\eta, \mu_\eta)$ and $j_{\eta,\eta+1}$ is the ultrapower embedding. $t_{\eta+1}$ is equal to t_η since η is a phantom limit in $\mathfrak{U}$.

Set $a_{\eta+1} = \langle \eta \rangle$. $\hat{a}_{\eta+1}$ is then equal to $\langle \mathrm{rdm}(t_\eta) \rangle$. We noted above that $\mathrm{rdm}(t_\eta) = j_{0,\eta}(\theta)$. So $\hat{a}_{\eta+1} = \langle \theta_\eta \rangle$.

Remark 3.22. In making the assignment $a_{\eta+1} = \langle \eta \rangle$ we use the assumption for convenience made in Remark 3.1. We need $a_{\eta+1} \in [\eta+1]^{\leq k}$, and for this we need $k \geq 1$.

Let $\theta_{\eta+1}$ denote $j_{0,\eta+1}(\theta)$. Working over M_η and applying the second part of Claim 3.8 to condition (i) above we see that t_η belongs to an interpretation of the $(\theta_\eta, \theta_{\eta+1})$–pullback of $j_{0,\eta+1}(\dot{Y})(\emptyset^\frown\langle\theta_\eta\rangle)$, where the pullback is computed in $M_{\eta+1}$. In other words $t_{\eta+1} = t_\eta$ belongs to an interpretation of the $(\theta_\eta, \theta_{\eta+1})$–pullback of $j_{0,\eta+1}(\dot{Y})(\hat{a}_{\eta+1})$, computed over $M_{\eta+1}$.

Using [4, Theorem 6G.1] it follows that player I has a winning strategy in $\widehat{G}_{\mathrm{branch}}(M_{\eta+1}, t_{\eta+1}, \theta_{\eta+1})(j_{0,\eta+1}(\dot{Y})(\hat{a}_{\eta+1}))$. This secures condition (1) for $\eta+1$. Conditions (2)–(5) for $\eta+1$ can be verified directly from the assignments made above. $\dashv$ (Terminal limit.)

We have still one limit case left to handle, the external limit case below. But first let us establish the following claims:

Claim 3.23. *Let $\eta \leq \omega_1$ be a limit. Suppose that $\alpha \mapsto a_\alpha$ is not constant on any tail-end of $[0,\eta)_{\mathfrak{U}}$. Let ζ belong to $[0,\eta)_{\mathfrak{U}}$. Then there exists some $\bar{\eta} \in [\zeta,\eta)_{\mathfrak{U}}$ so that $\bar{\eta}$ is a terminal limit.*

Proof. Otherwise an induction using instances of Claim 3.19 shows that every limit $\bar{\eta} \in [\zeta,\eta)_{\mathfrak{U}}$ is internal, and that $a_{\bar{\eta}} = a_\zeta$ for every $\bar{\eta} \in [\zeta,\eta)_{\mathfrak{U}}$. But this contradicts the assumption that $\alpha \mapsto a_\alpha$ is not constant on any tail-end of $[0,\eta)_{\mathfrak{U}}$. $\dashv$

Claim 3.24. *Let $\eta \leq \omega_1$ be a limit. Suppose that $\alpha \mapsto a_\alpha$ is not constant on any tail-end of $[0,\eta)_{\mathfrak{U}}$. Then $\lambda_\eta = j_{0,\eta}(\theta)$.*

Proof. Let $I \subset [0,\eta)_{\mathfrak{U}}$ be the set of $\bar{\eta} \in [0,\eta)_{\mathfrak{U}}$ which are terminal limits. By Claim 3.23, I is cofinal in η. The terminal limit case assumptions are such that $\lambda_{\bar{\eta}} = j_{0,\bar{\eta}}(\theta)$ for each $\bar{\eta} \in I$. $\mathrm{crit}(j_{\bar{\eta},\eta})$ is greater than or equal to $\lambda_{\bar{\eta}}$ by Claim 3.15. Combining all these facts it follows that $j_{0,\eta}(\theta)$ is equal to $\sup_{\bar{\eta}\in I} \lambda_{\bar{\eta}}$. This supremum is equal to $\lambda_{\sup(I)=\eta}$ since the sequence $\langle \lambda_\alpha = \mathrm{rdm}(t_\alpha) \mid \alpha \leq \omega_1 \rangle$ is continuous. $\dashv$

External limit. If $\alpha \mapsto a_\alpha$ is *not* constant on any tail-end of c_η.

Begin by setting $a_\eta = \emptyset$. There is not much significance to this assignment, since conditions (1)–(4) are not needed for external limit η, and condition (5) is vacuous. We must proceed to construct $\mathfrak{U}\restriction \eta + 1.2$, and set value to $a_{\eta+1}$, in a way that secures conditions (1)–(5) for $\eta + 1$.

Let $a \in [\omega_1]^{<k}$ be the unique tuple so that $\eta \in S_a$. A tuple of this kind must exist because of the assumption made in Remark 3.2. There can only be one tuple of this kind since the sets in $\vec{S}$ are mutually disjoint.

We divide the construction for external limits into two cases, depending on whether a appears as an a_ζ for some $\zeta \in c_\eta = [0, \eta)_{\mathfrak{U}}$, or not.

Case 1. Suppose first that there exists some $\zeta \in [0, \eta)_{\mathfrak{U}}$ so that a is equal to a_ζ. Let $\bar{\eta}$ be the first terminal limit in $[\zeta, \eta)_{\mathfrak{U}}$. (A terminal limit of this kind must exist by Claim 3.23.) By Claim 3.19 on $\bar{\eta}$, $a_{\bar{\eta}}$ is equal to a_ζ, which is equal to a. By condition $(*)$ of the terminal limit case for $\bar{\eta}$ we have then:

(ii) $t_{\bar{\eta}}$ belongs to an interpretation of $j_{0,\bar{\eta}}(\dot{Y})(\hat{a})$.

By Claim 3.24, $\lambda_\eta = j_{0,\eta}(\theta)$. Applying Claim 3.20 (with the current $\bar{\eta}$ standing for ζ of that claim) it follows from this and from condition (ii) that:

(iii) t_η belongs to an interpretation of $j_{0,\eta}(\dot{Y})(\hat{a})$.

From now on continue along the lines of the construction in the terminal limit case, from condition (i) onward, only setting $a_{\eta+1}$ equal to $a ^\frown \langle \eta \rangle$ instead of $\emptyset ^\frown \langle \eta \rangle$, and using condition (iii) instead of condition (i). Except for these two changes the constructions are the same, and we therefore omit further details. ⊣ (Case 1.)

Case 2. Suppose next that there is *no* $\zeta \in [0, \eta)_{\mathfrak{U}}$ so that a is equal to a_ζ. Let $\bar{\eta}$ be the first terminal limit in $[0, \eta)_{\mathfrak{U}}$. (A terminal limit must exist in $[0, \eta)_{\mathfrak{U}}$ by Claim 3.23.) By Claim 3.19 on $\bar{\eta}$, $a_{\bar{\eta}}$ is equal to a_0, which is equal to $\emptyset$. By condition $(*)$ of the terminal limit case for $\bar{\eta}$:

(iv) $t_{\bar{\eta}}$ belongs to an interpretation of $j_{0,\bar{\eta}}(\dot{Y})(\hat{\emptyset})$.

Using Claim 3.20 (with the current $\bar{\eta}$ standing for ζ of that claim) it follows that:

(v) t_η belongs to an interpretation of $j_{0,\eta}(\dot{Y})(\emptyset)$.

This is the same as condition (i) in the construction for terminal limits. Continue by precisely following the construction there. ⊣ (Case 2.)

For future reference let us record the following fact, which simply expresses the settings in case 1:

(vi) Let $a \in [\omega_1]^{<k}$ be such that $\eta \in S_a$. If there exists some $\zeta \in [0, \eta)_{\mathfrak{U}}$ so that $a_\zeta = a$ then $a_{\eta+1}$ is equal to $a ^\frown \langle \eta \rangle$.

Of the two cases above, case 1 is the more important. It aims to fit $a_{\eta+1}$ with an element of $[\vec{S}]$. We shall see below that a fit is obtained sufficiently often to make sure that the set $\{a_\alpha \mid \alpha < \omega_1\}$ generated through the construction contains $[\vec{S}] \cap C$ for some C which is club in ω_1. ⊣ (External limit.)

The case of successor and zero stages above, and the three limit cases (internal, terminal, and external), complete the construction of the items in conditions (A)–(C) listed at the start of the proof of Lemma 3.6. Among other objects we constructed a run $r \in \omega^{\omega_1}$, given by Definition 3.11. The odd half of r was created by the imaginary opponent. (This was done through the use of the imaginary opponent in the successor and zero stages.) The even half was created by the mechanisms of the construction.

The construction can thus be formalized into a strategy for I in the length ω_1 game of Diagram 1. It remains to verify that this strategy is winning for I in $G_{\omega_1,k}(\vec{S},\varphi)$. In other words it remains to verify that the run r obtained through the construction via Definition 3.11 is won by player I.

We show this through a series of claims. We work to produce a club $C \subset \omega_1$ so that: every tuple $a \in [\vec{S}] \cap [C]^k$ belongs to $\{a_\zeta \mid \zeta \in [0,\omega_1)_{\mathfrak{U}}\}$; and $(L_{\omega_1}[r];r) \models \varphi[\alpha_0,\dots,\alpha_{k-1}]$ for every tuple $\langle\alpha_0,\dots,\alpha_{k-1}\rangle$ which belongs to $\{a_\zeta \mid \zeta \in [0,\omega_1)_{\mathfrak{U}}\} \cap [C]^k$. The first property is obtained in Corollary 3.31. Its proof ultimately relies on condition (vi) in the external limit construction. The second property is (in essence) obtained in Claim 3.32. Its proof ultimately relies on condition $(*)$ in the terminal limit construction and on the nature of the names given by $\dot{Y}$, specifically on the connection to φ in Definition 3.3. The two properties combined imply that $(L_{\omega_1}[r];r) \models \varphi[\alpha_0,\dots,\alpha_{k-1}]$ for every tuple $\langle\alpha_0,\dots,\alpha_{k-1}\rangle \in [\vec{S}] \cap [C]^k$, showing that r is won by player I in $G_{\omega_1,k}(\vec{S},\varphi)$.

CLAIM 3.25. *$\alpha \mapsto a_\alpha$ is* not *constant on any tail-end of $[0,\omega_1)_{\mathfrak{U}}$.*

PROOF. Suppose for contradiction that $\zeta \in [0,\omega_1)_{\mathfrak{U}}$ is such that $\alpha \mapsto a_\alpha$ is constant on $[\zeta,\omega_1)_{\mathfrak{U}}$. Let a denote a_ζ, let ν denote $\nu(a)$, let θ_ν denote $j_{0,\nu}(\theta)$, and let $\dot{Y}_\nu$ denote $j_{0,\nu}(\dot{Y})$. By condition (4), $P_{\nu,\alpha}$ is legal and non-terminal in $\widehat{G}_{\text{branch}}(M_\nu,t_\nu,\theta_\nu)(\dot{Y}_\nu(a))$, and moreover played according to $\widehat{\Sigma}_{\text{branch}}(a)$, for each $\alpha \in [\zeta,\omega_1)_{\mathfrak{U}}$. Since P_{ν,ω_1} is equal to $\bigcup_{\alpha\in[\zeta,\omega_1)_{\mathfrak{U}}} P_{\nu,\alpha}$, it follows that P_{ν,ω_1} is legal in $\widehat{G}_{\text{branch}}(M_\nu,t_\nu,\theta_\nu)(\dot{Y}_\nu(a))$, and moreover played according to $\widehat{\Sigma}_{\text{branch}}(a)$. But P_{ν,ω_1} has length ω_1 and is therefore lost by player I through the snag (I4) in [4, §6A]. This is a contradiction since $\widehat{\Sigma}_{\text{branch}}(a)$ is a winning strategy for I. ⊣

CLAIM 3.26. *There is a set $C_1 \subset [0,\omega_1)_{\mathfrak{U}}$ so that C_1 is club in ω_1 and every $\eta \in C_1$ is an external limit.*

PROOF. Let $C_1 \subset [0,\omega_1)_{\mathfrak{U}}$ be a club so that for every $\eta \in C_1$ the function $\alpha \mapsto a_\alpha$ is not constant on any tail-end of $[0,\eta)_{\mathfrak{U}}$. The existence of such a club follows directly from Claim 3.25 and the fact that $[0,\omega_1)_{\mathfrak{U}}$ is club in ω_1. Every $\eta \in C_1$ is an external limit directly by definition. ⊣

CLAIM 3.27. *Let $\eta+1 < \omega_1$. Let ζ be the U-predecessor of $\eta+1$. Then at least one of the following possibilities holds:*

1. *$\zeta = \eta$; or*
2. *λ_ζ is a limit of Woodin cardinals in M_ζ, but not itself Woodin.*

PROOF. In most cases of the construction we set the U–predecessor of $\eta+1$ equal to η. The only exceptions were in the obstructed cases of the construction for successor and zero stages, and similarly in the obstructed cases of the construction for internal limits. In those cases the U–predecessor of $\eta+1$ was set equal to an ordinal γ determined by the critical point appearing in an obstruction for $t^\dagger_\eta$. It was observed there (see specifically case 2 in the construction for successor and zero stages) that λ_γ is a limit of Woodin cardinals in Q_η but not itself Woodin, and this yields condition (2) of the current claim. ⊣

CLAIM 3.28. *Let $\zeta \in [0,\omega_1)_{\mathfrak{U}}$ be an external limit. Then the successor of ζ in $[0,\omega_1)_{\mathfrak{U}}$ is $\zeta+1$.*

Proof. Let $\eta+1$ be the successor of ζ in $[0,\omega_1)_{\mathfrak{U}}$. Then the U–predecessor of $\eta+1$ is ζ. Now apply Claim 3.27. The second condition of the claim cannot hold, since λ_ζ for an external limit ζ is equal to $j_{0,\zeta}(\theta)$ by Claim 3.24, and $j_{0,\zeta}(\theta)$ is a Woodin limit of Woodin cardinals in M_ζ. Thus the second condition of the claim must hold. In other words $\eta=\zeta$ and $\eta+1$—which by its very choice is the successor of ζ in $[0,\omega_1)_{\mathfrak{U}}$—is equal to $\zeta+1$. ⊣

Claim 3.29. *Let η belong to C_1. Then $\eta+1$ belongs to $[0,\omega_1)_{\mathfrak{U}}$.*

Proof. Membership in C_1 implies that η is an external limit and belongs to $[0,\omega_1)_{\mathfrak{U}}$. By the previous claim then, the successor of η in $[0,\omega_1)_{\mathfrak{U}}$ is $\eta+1$. In particular $\eta+1$ belongs to $[0,\omega_1)_{\mathfrak{U}}$. ⊣

Let T denote the set:

$$\{\langle\alpha_0,\dots,\alpha_{l-1}\rangle\in[\omega_1]^{\leq k}\mid(\forall i<l)\,\alpha_i\in S_{\langle\alpha_0,\dots,\alpha_{i-1}\rangle}\}.$$

Notice that $T\cap[\omega_1]^k$ is precisely equal to $\vec{S}$. In fact the definition of T is identical to that of $[\vec{S}]$ in Definition 1.1, except that here we consider not just tuples of length k, but tuples of any length $\leq k$.

Claim 3.30. *Let a belong to $T\cap[C_1]^{\leq k}$. Let ν denote $\nu(a)$. (Recall that this is 0 if $a=\emptyset$, and $\max(a)+1$ otherwise.) Then:*

1. *ν belongs to $[0,\omega_1)_{\mathfrak{U}}$; and*
2. *$a=a_\nu$.*

Proof. For $a=\emptyset$ the claim holds trivially since a_0 was set equal to $\emptyset$ at the start of the construction.

Fix $l<k$ and suppose inductively that the claim holds for all a of length l. We prove that it holds for all a^* of length $l+1$.

Fix $a^*\in T\cap[C_1]^{\leq k}$ of length $l+1$. Let $\eta=\max(a^*)$. Let $a=a^*{\restriction}l$. Then $a^*=a^\frown\langle\eta\rangle$, and $\eta\in S_a$ for otherwise a^* would not belong to T. We have $a\in T\cap[C_1]^{\leq k}$ and $\eta\in C_1$. In particular η is an external limit.

Let ν denote $\nu(a)$. By the inductive assumption the claim holds for a, and since $a\in T\cap[C_1]^{\leq k}$ it follows that ν belongs to $[0,\omega_1)_{\mathfrak{U}}$ and $a=a_\nu$. In particular there exists some $\zeta\in[0,\eta)_{\mathfrak{U}}$ so that $a=a_\zeta$. (Take $\zeta=\nu$.) Using now condition (vi) in the external limit case it follows that $a_{\eta+1}=a^\frown\langle\eta\rangle$. In other words $a_{\eta+1}$ is equal to a^*.

We have $\eta+1\in[0,\omega_1)_{\mathfrak{U}}$ by Claim 3.29 since $\eta\in C_1$. We showed in the previous paragraph that $a^*=a_{\eta+1}$. These two conclusions prove the current claim for a^* since $\nu^*=\max(a^*)+1$ is simply $\eta+1$. ⊣

Corollary 3.31. *Let a belong to $[\vec{S}]\cap[C_1]^k$. Then there exists $\zeta\in[0,\omega_1)_{\mathfrak{U}}$ so that $a=a_\zeta$.*

Proof. Immediate from the last claim, since $[\vec{S}]\cap[C_1]^k\subset T\cap[C_1]^{\leq k}$. ⊣

Claim 3.32. *Let $\zeta\in[0,\omega_1)_{\mathfrak{U}}$, and suppose that a_ζ is a tuple of length k, equal to $\langle\alpha_0,\dots,\alpha_{k-1}\rangle$ say. Then $(L_{\omega_1}[r];r)\models\varphi[\hat{\alpha}_0,\dots,\hat{\alpha}_{k-1}]$. ($r$ here is the run of $G_{\omega_1,k}(\vec{S},\varphi)$ created by the construction through Definition 3.11.)*

PROOF. Let $\bar{\eta}$ be the first terminal limit in $[\zeta, \omega_1)_{\mathfrak{U}}$. Such a terminal limit must exist, since otherwise an induction using Claim 3.19 shows that $a_\eta = a_\zeta$ for all $\eta \in [\zeta, \omega_1)_{\mathfrak{U}}$, contradicting Claim 3.25.

By condition $(*)$ of the terminal limit construction for $\bar{\eta}$, $t_{\bar{\eta}}$ belongs to an interpretation of $j_{0,\bar{\eta}}(\dot{Y})(\hat{a}_{\bar{\eta}})$. By Claims 3.25 and 3.24, $\lambda_{\omega_1} = j_{0,\omega_1}(\theta)$. Using Claim 3.20 with $\eta = \omega_1$ and the current $\bar{\eta}$ standing for ζ of that claim, it follows from the conclusions of the last two sentences that t_{ω_1} belongs to an interpretation of $j_{0,\omega_1}(\dot{Y})(\hat{a}_{\bar{\eta}})$. By Claim 3.19 on ζ and $\bar{\eta}$, $a_{\bar{\eta}}$ is equal to a_ζ. So t_{ω_1} belongs to an interpretation of $j_{0,\omega_1}(\dot{Y})(\hat{a}_\zeta)$.

The conclusion of the last paragraph holds for every $\zeta \in [0, \omega_1)_{\mathfrak{U}}$, regardless of the length of a_ζ. Ultimately it traces back to the construction for terminal limits, and most importantly to condition $(*)$ of the terminal limit case.

Here we assume that a_ζ is a tuple of length k. Membership in an interpretation of $j_{0,\omega_1}(\dot{Y})(\hat{a}_\zeta)$ is thus the same as membership in an interpretation of $j_{0,\omega_1}(\dot{Y}_k)(\hat{a}_\zeta)$ by condition (1) of Claim 3.8 (shifted to M_{ω_1}). So t_{ω_1} belongs to an interpretation of $j_{0,\omega_1}(\dot{Y}_k)(\hat{a}_\zeta)$. By Definition 3.3 this means that:

$$(\mathrm{L}_{j_{0,\omega_1}(\theta)}[r(t_{\omega_1})]; r(t_{\omega_1})) \models \varphi[\hat{\alpha}_0, \dots, \hat{\alpha}_{k-1}].$$

($\alpha_0, \dots, \alpha_{k-1}$ here are the ordinals forming the tuple a_ζ. Notice then that $\hat{a}_\zeta$ is equal to $\langle \hat{\alpha}_0, \dots, \hat{\alpha}_{k-1} \rangle$.)

$j_{0,\omega_1}(\theta)$ is equal to λ_{ω_1} by Claims 3.24 and 3.25, and $\lambda_{\omega_1} = \mathrm{rdm}(t_{\omega_1})$ is equal to ω_1. $r(t_{\omega_1})$ is the concatenation of the reals in $\vec{z}(t_{\omega_1})$, the real part of t_{ω_1}. $\vec{z}(t_{\omega_1})$ is equal to $\langle y_{-1+\xi+1} \mid \xi < \omega_1 \rangle$ by [4, Claim 7B.16]. The concatenation leading to $r(t_{\omega_1})$ is therefore precisely the same as the concatenation leading to r in Definition 3.11. Substituting $j_{0,\omega_1}(\theta) = \omega_1$ and $r(t_{\omega_1}) = r$ in the equation above we get:

$$(\mathrm{L}_{\omega_1}[r]; r) \models \varphi[\hat{\alpha}_0, \dots, \hat{\alpha}_{k-1}],$$

as required. ⊣

COROLLARY 3.33. *There is a club $C \subset \omega_1$ so that $(\mathrm{L}[r]; r) \models \varphi[\alpha_0, \dots, \alpha_{k-1}]$ for every tuple $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$.*

PROOF. Fix a club $C_2 \subset \omega_1$ so that $\hat{\alpha} = \alpha$ for every $\alpha \in C_2$. This is possible by Remark 3.13. Let $C = C_1 \cap C_2$.

Fix $a = \langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$. By Corollary 3.31 there exists $\zeta \in [0, \omega_1)_{\mathfrak{U}}$ so that $a = a_\zeta$. By Claim 3.32 then $(L_{\omega_1}[r]; r) \models \varphi[\hat{\alpha}_0, \dots, \hat{\alpha}_{k-1}]$. Now $\hat{\alpha}_i = \alpha_i$ for each $i = 0, \dots, k-1$ since each of the ordinals α_i belongs to C, and therefore to C_2. So $(L_{\omega_1}[r]; r) \models \varphi[\alpha_0, \dots, \alpha_{k-1}]$. ⊣

Corollary 3.33 shows that r is won by player I in $G_{\omega_1,k}(\vec{S}, \varphi)$. This completes the proof of Lemma 3.6. ⊣ (Lemma 3.6.)

Lemma 3.6 provides a criterion for the existence of a winning strategy for I in $G_{\omega_1,k}(\vec{S}, \varphi)$. We work now to mirror the lemma, and obtain a similar criterion for the existence of a winning strategy for II. Later on we shall see that at least one of criterions must hold.

For the sake of Definitions 3.34 and 3.35 fix some tuple $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\theta]^k$. Let α_k denote θ.

Definition 3.34. For expository simplicity fix some G which is $\mathbb{W}_\theta$–generic over M. Define $\dot{Z}_k(\alpha_0, \dots, \alpha_{k-1}) \in M$ to be the canonical $\mathbb{W}_\theta$–name for the set of θ–sequences $t \in M[G]$ so that $(\mathrm{L}_\theta[r(t)]; r(t)) \models \neg\varphi[\alpha_0, \dots, \alpha_{k-1}]$.

Definition 3.34 mirrors Definition 3.3. Notice how here the reference to φ involves its *failure* in $(\mathrm{L}_\theta[r(t)]; r(t))$.

Definition 3.35. For each $i < k$ define $\dot{Z}_i(\alpha_0, \dots, \alpha_{k-1})$ to be the *mirrored* (α_i, α_{i+1})–pullback of $\dot{Z}_{i+1}(\alpha_0, \dots, \alpha_{k-1})$ as computed in M. The definition is made by induction, working downward from $i = k-1$ to $i = 0$. We refer the reader to [4, §4E] for the definition of the mirrored pullback operation.

Definition 3.36. Define $V(\varphi)$ to be the set of $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in [\theta]^k$ so that $M \models \psi_{\mathrm{ini}}[\alpha_0, \dot{Z}_0(\alpha_0, \dots, \alpha_{k-1})]$.

ψ_{ini} here is the formula of [4, Definition 5G.2]. Notice how Definitions 3.34 through 3.36 precisely mirror Definitions 3.3 through 3.5. By mirroring precisely the argument of Lemma 3.6 we get:

Lemma 3.37. *Suppose that $V(\varphi)$ has $(\mu_M)^k$ measure 1. Then player* II *has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$.* ⊣

To establish the determinacy of $G_{\omega_1,k}(\vec{S}, \varphi)$ it is now enough to prove that at least one of $U(\varphi)$ and $V(\varphi)$ has $(\mu_M)^k$ measure 1.

Lemma 3.38. *It cannot be that both $U(\varphi)$ and $V(\varphi)$ have $(\mu_M)^k$ measure 0.*

Proof. Let $U^* = [\theta]^k - U(\varphi)$ and let $V^* = [\theta]^k - V(\varphi)$. Suppose for contradiction that both U^* and V^* have $(\mu_M)^k$ measure 1. It follows in particular that their intersection is non-empty. Fix then a tuple $\langle \alpha_0, \dots, \alpha_{k-1} \rangle \in U^* \cap V^*$.

Let $\dot{Y}_i$ denote $\dot{Y}_i(\alpha_0, \dots, \alpha_{k-1})$ and let $\dot{Z}_i$ denote $\dot{Z}_i(\alpha_0, \dots, \alpha_{k-1})$. The fact that $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$ belongs to neither $U(\varphi)$ nor $V(\varphi)$ means that:

- $M \not\models \varphi_{\mathrm{ini}}[\alpha_0, \dot{Y}_0]$ and $M \not\models \psi_{\mathrm{ini}}[\alpha_0, \dot{Z}_0]$.

By [4, Corollary 5G.3] it follows that there exists a supernice, saturated α_0–sequence t_0 over M which avoids $\dot{Y}_0$ and $\dot{Z}_0$. Inductive applications of [4, Theorem 5G.1] then produce supernice, saturated α_{i+1}–sequences t_{i+1} for $i < k$ so that each t_{i+1} extends t_i and avoids $\dot{Y}_{i+1}$ and $\dot{Z}_{i+1}$. This ultimately results in a θ–sequence t_k which avoids $\dot{Y}_k$ and $\dot{Z}_k$, meaning that there is some $\mathbb{W}_\theta$–generic G so that t_k belongs to $M[G]$, yet t_k does not belong to either $\dot{Y}_k[G]$ or $\dot{Z}_k[G]$. But this is a contradiction since $\dot{Y}_k$ and $\dot{Z}_k$ by definition name complementary sets of θ–sequences. ⊣

Theorem 3.39. *Suppose that 0^W exists. Let $k < \omega$. Let $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ be a sequence of mutually disjoint subsets of ω_1. Let $\varphi(x_0, \dots, x_{k-1})$ be a formula of $\mathcal{L}^+$. Then the game $G_{\omega_1,k}(\vec{S}, \varphi)$ is determined.*

Proof. Suppose first that $k \geq 1$. The theorem is then an immediate consequence of Lemmas 3.6, 3.37, and 3.38. If $U(\varphi)$ has $(\mu_M)^k$ measure 1 then player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$ by Lemma 3.6. If $V(\varphi)$ has $(\mu_M)^k$ measure 1 then player II has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$ by Lemma 3.37. At least one of these cases must hold by Lemma 3.38.

The restriction to $k \geq 1$ in the previous paragraph has to do with the assumption for convenience in Remark 3.1. Let us next derive determinacy in the case $k = 0$ from determinacy in the case $k = 1$. This is a simple matter of adding a dummy variable to φ, and making a slight adjustment to $\vec{S}$. Suppose $k = 0$. Let $S_\emptyset = \omega_1$, and let $\vec{S}^* = \langle S_\emptyset \rangle$. Let $\varphi^*(x_0) = \varphi$. (That is let φ^* be obtained by adding a dummy variable x_0 to φ.) The games $G_{\omega_1,1}(\vec{S}^*, \varphi^*)$ and $G_{\omega_1,0}(\vec{S}, \varphi)$ are then precisely the same, and the determinacy of the latter follows from the determinacy of the former, established in the previous paragraph. $\dashv$

REMARK 3.40. We do not need the actual fine structural mouse 0^W for Theorem 3.39, only its large cardinal strength. The theorem holds under the coarse assumption that there exists a pair $\langle M, \mu_M \rangle$ satisfying conditions (A1)–(A4) at the start of this section.

§4. Definability. We work now to reduce statements involving $\Game_{\omega_1}$ to statements about 0^W.

Let $M_0 = 0^W$ and let μ_0 be the top extender of 0^W. Let $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \leq \xi \in \mathrm{On} \rangle$ be the iteration determined by letting $M_{\xi+1} = \mathrm{Ult}(M_\xi, \mu_\xi)$ where $\mu_\xi = j_{0,\xi}(\mu_0)$; letting $j_{\xi,\xi+1} \colon M_\xi \to M_{\xi+1}$ be the ultrapower embedding by μ_ξ; defining the remaining embeddings by compositions; and taking direct limits at limit stages. Let $\kappa_\xi = \mathrm{crit}(\mu_\xi) = j_{0,\xi}(\kappa_0)$. Let N be the direct limit of the (class) system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \leq \xi \in \mathrm{On} \rangle$, cut to height On. This is the class model defined in Section 2, and the theory of k Woodin indiscernibles defined there is the theory of $\kappa_0, \ldots, \kappa_{k-1}$ in N.

Let $M = M_1 \| \kappa_1$. Fix $k < \omega$. Let χ be the formula of Remark 3.7. This is the formula which defines the function $\varphi \mapsto U(\varphi)$ of Definition 3.5. The following is a summary of the properties of χ and the map $\varphi \mapsto U(\varphi)$, taken from Section 3, needed for the definability results below.

- $U(\varphi)$ is a subset of $[\kappa_0]^k$ in M.
- If $U(\varphi)$ has $(\mu_0)^k$ measure 1, then player I has a winning strategy in the game $G_{\omega_1,k}(\vec{S}, \varphi)$.
- If $U(\varphi)$ does not have $(\mu_0)^k$ measure 1, then player II has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$.
- $M \models \chi[\kappa_0, \varphi, X]$ iff $X = U(\varphi)$.

Precise references for the proofs of these properties are given as they are used, in the proof of the next lemma. (The proof of the second property takes the bulk of Section 3.)

LEMMA 4.1. *Let $k < \omega$. Let $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ be a collection of mutually disjoint stationary subsets of ω_1. Let $\varphi(x_0, \ldots, x_{k-1})$ be a formula of $\mathcal{L}^+$. Then player* I *wins $G_{\omega_1,k}(\vec{S}, \varphi)$ iff $N \models$ "$(\forall X)\, (\chi(\kappa_k, \varphi, X) \to \langle \kappa_0, \ldots, \kappa_{k-1} \rangle \in X)$."*

PROOF. Let $\langle M, \mu_M \rangle$ be the pair $\langle M_1 \| \kappa_1, \mu_0 \rangle$, that is the pair consisting of the top extender of 0^W, and an initial segment of the *ultrapower* of 0^W by its top extender. Note that $\langle M, \mu_M \rangle$ then satisfies conditions (A1)–(A4) of Section 3. We work now with the results of that section, applied specifically to this particular pair.

Let $U(\varphi)$ be given by Definition 3.5 (applied on the pair $\langle M, \mu_M\rangle$ given by $\langle M_1 \| \kappa_1, \mu_0\rangle$).

By Remark 3.7, $U(\varphi)$ is the unique $X \in M_1 \| \kappa_1$ so that $(M_1 \| \kappa_1) \models \chi[\kappa_0, \varphi, X]$. Applying to this the elementary embedding $j_{0,k}$ we see that $j_{0,k}(U(\varphi))$ is the unique $X \in M_{k+1} \| \kappa_{k+1}$ so that $(M_{k+1} \| \kappa_{k+1}) \models \chi[\kappa_k, \varphi, X]$. Applying next the embedding $j_{k+1,\infty}$ it follows that:

(i) $j_{0,k}(U(\varphi))$ is the unique $X \in N$ so that $N \models \chi[\kappa_k, \varphi, X]$.

Suppose now that $N \models$"$(\forall X)\ (\chi(\kappa_k, \varphi, X) \rightarrow \langle \kappa_0, \dots, \kappa_{k-1}\rangle \in X)$." Using condition (i) it follows that $\langle \kappa_0, \dots, \kappa_{k-1}\rangle$ belongs to $j_{0,k}(U(\varphi))$. From this it follows that $U(\varphi)$ has $(\mu_0)^k$ measure 1. Using Lemma 3.6 it follows that player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$, as required.

Suppose next that $N \not\models$"$(\forall X)\ (\chi(\kappa_k, \varphi, X) \rightarrow \langle \kappa_0, \dots, \kappa_{k-1}\rangle \in X)$." Using condition (i) it follows that $\langle \kappa_0, \dots, \kappa_{k-1}\rangle$ does not belong to $j_{0,k}(U(\varphi))$, and from this it follows that $U(\varphi)$ has $(\mu_0)^k$ measure 0. By Lemma 3.38, $V(\varphi)$ has $(\mu_0)^k$ measure 1. By Lemma 3.37 then player II has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$. Since the sets S_a $(a \in [\omega_1]^{<k})$ are assumed here to all be stationary, it cannot be that both I and II win $G_{\omega_1,k}(\vec{S}, \varphi)$. So player I does *not* have a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$, as required. ⊣

Corollary 4.2. *(Assuming the existence of 0^W.) Let $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k}\rangle$ and $\vec{S}^* = \langle S^*_a \mid a \in [\omega_1]^{<k}\rangle$ each be a collection of mutually disjoint stationary subsets of ω_1. Then player* I *has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$ iff she has a winning strategy in $G_{\omega_1,k}(\vec{S}^*, \varphi)$.*

Proof. Simply note that player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$ iff $N \models$"$(\forall X)\ (\chi(\kappa_k, \varphi, X) \rightarrow \langle \kappa_0, \dots, \kappa_{k-1}\rangle \in X)$" iff player I has a winning strategy in $G_{\omega_1,k}(\vec{S}^*, \varphi)$. Each of the equivalences follows by an application of Lemma 4.1, the first with $\vec{S}$ and the second with $\vec{S}^*$. ⊣

Corollary 4.2 is needed to make sense of the game quantifier $⅁_{\omega_1,k}(\varphi)$ of Section 1. Having made sense of the game quantifier we can use Lemma 4.1 further, to complete the connection between $⅁_{\omega_1}$ and 0^W discussed in Section 2. Recall that Φ_k denotes the set of formulae, in the language $\mathcal{L}^+$ of Section 1, with at most k free variables. Recall that T_k denotes the theory of k indiscernible Woodin cardinals, that is the theory of $\kappa_0, \dots, \kappa_{k-1}$ in the model N defined above.

Theorem 4.3. *$⅁_{\omega_1}\Phi_k$ is recursive in T_{k+1}.*

Proof. Immediate from Lemma 4.1, since the question of whether or not $N \models$"$(\forall X)\ (\chi(\kappa_k, \varphi, X) \rightarrow \langle \kappa_0, \dots, \kappa_{k-1}\rangle \in X)$" is answered by T_{k+1}. ⊣

Corollary 4.4. *Each of 0^W and $\oplus_{k<\omega}⅁_{\omega_1}\Phi_k$ is recursive in the other.*

Proof. This is just the combination of Theorem 2.13, Theorem 4.3, and Claim 2.2. ⊣

We pass now to the specific case of $k = 0$ and φ in Σ_1. Fix a Σ_1 sentence φ in $\mathcal{L}^+$. (Let us emphasize the fact that we are working with the case $k = 0$, and φ has no free variables.) Let $\varphi^*(x_0)$ denote φ with a dummy variable x_0 added. Let $S_\emptyset = \omega_1$ and let $\vec{S}^* = \langle S_\emptyset\rangle$. Notice then that $G_{\omega_1,0}(\emptyset, \varphi)$ is precisely the same game as $G_{\omega_1,1}(\vec{S}^*, \varphi^*)$.

DEFINITION 4.5. Let M be a model of ZFC^*. Let $\alpha_0 < \alpha_1$ be Woodin cardinals of M, with α_1 a Woodin limit of Woodin cardinals in M. For expository simplicity let G be $\mathbb{W}_{\alpha_1}$–generic over M. Define $\dot{Y}_1^* \in M$ to be the canonical $\mathbb{W}_{\alpha_1}$–name for the set of α_1–sequences t so that $(\mathrm{L}_{\alpha_1}[r(\vec{t})]; r(\vec{t})) \models \varphi$ (equivalently, $(\mathrm{L}_{\alpha_1}[r(\vec{t})]; r(\vec{t})]) \models \varphi^*[\alpha_0]$). Define $\dot{Y}_0^*(\alpha_0, \alpha_1)$ to be the (α_0, α_1)–pullback of $\dot{Y}_1^*$ as computed in M.

Notice the similarities between Definition 4.5 and Definitions 3.3 and 3.4 in Section 3 in the case $k = 1$. Indeed, Definitions 3.3 and 3.4 for $k = 1$ are the instance of Definition 4.5 corresponding to $\alpha_1 = \theta$, namely to $\alpha_1 = \mathrm{crit}(\mu_M)$ in the notation of Section 3.

LEMMA 4.6. *Suppose there exist some iterable, countable model M and some $\alpha_0 < \alpha_1$ in M so that $M \models \varphi_{\mathrm{ini}}[\alpha_0, \dot{Y}_0(\alpha_0, \alpha_1)]$. Then player* I *wins $G_{\omega_1,0}(\emptyset, \varphi)$.*

PROOF. Fix M, α_0, and α_1 as in the claim. Fix an iteration strategy Γ for M. Working with an imaginary opponent who plays for II in $G_{\omega_1,0}(\emptyset, \varphi)$, construct a regular tot $\mathfrak{U}$ on M of length $\gamma + 0.2$ for some $\gamma < \omega_1$, and a $\mathfrak{U}$–sequence $\langle w_\xi, y_\xi \mid \xi \in K^{\mathfrak{U}} \rangle$, so that:

(i) $\mathfrak{U}$ is consistent with Γ (in particular all models of $\mathfrak{U}$ are wellfounded); and

(ii) The final annotated position t_η induced by $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$ belongs to an interpretation of $j_{0,\gamma}(\dot{Y}_1^*(\alpha_0, \alpha_1))$.

The construction is an application of the methods of [4, Chapters 6,7] and some elements in the proof of Lemma 3.6, using the fact that $M \models \varphi_{\mathrm{ini}}[\alpha_0, \dot{Y}_0(\alpha_0, \alpha_1)]$. We leave the exact details to the pleasure of the reader.

Let $\bar{r} = r(t_\gamma)$. This is the concatenated real part of the annotated position t_γ induced by $\mathfrak{U}$ and $\langle \vec{w}, \vec{y} \rangle$. Let $\beta = \mathrm{lh}(\bar{r})$. Notice that β is countable, since $\mathfrak{U}$ has countable length. From condition (ii) and the definition of $\dot{Y}_1^*$ it follows that:

(iii) $(\mathrm{L}_\beta[\bar{r}]; \bar{r}) \models \varphi$.

$\bar{r}$ was produced through a construction involving an imaginary opponent playing for II in $G_{\omega_1,0}(\emptyset, \varphi)$. The opponent contributed the odd half of $\bar{r}$ and the mechanism of the construction gave rise to the even half.

Continue now to construct and extend $\bar{r}$ to a full run $r \in \omega^{\omega_1}$ of $G_{\omega_1,0}(\emptyset, \varphi)$. Let the imaginary opponent play the odd half of the extension, that is the moves $r(\xi)$ for odd $\xi \geq \beta$. For the even half play always $r(\xi) = 0$.

Since φ is Σ_1, condition (iii) and the fact that r extends $\bar{r}$ imply that $(\mathrm{L}_{\omega_1}[r]; r)$ satisfies φ. So r is won by player I in $G_{\omega_1,0}(\emptyset, \varphi)$, as required. ⊣

REMARK 4.7. The end argument in the proof of Lemma 4.6 illustrates the fact that $G_{\omega_1,0}(\emptyset, \varphi)$ in the case of a Σ_1 sentence φ is an *open* game of length ω_1. Victory for player I is secured already at the initial stage $\bar{r}$, and the subsequent moves are irrelevant.

LEMMA 4.8. *Let $\langle M, \mu_M \rangle$ satisfy assumptions (A1)–(A4) in Section 3. Suppose there are* no *$\alpha_0 < \alpha_1$ in M so that $M \models \varphi_{\mathrm{ini}}[\alpha_0, \dot{Y}_0(\alpha_0, \alpha_1)]$. Then player* II *wins $G_{\omega_1,0}(\emptyset, \varphi)$.*

PROOF. Following the notation in Section 3 let $\theta = \mathrm{crit}(\mu_M)$. The assumption of the current lemma, taken with $\alpha_1 = \theta$, in particular implies that $\{\alpha_0 < \theta \mid$

$M \models \varphi_{\rm ini}[\alpha_0, \dot{Y}_0^*(\alpha_0, \alpha_1)]\}$ is empty. Notice that this is precisely the set $U(\varphi^*)$ of Definition 3.5, applied over M with $k = 1$ and with the formula φ^* obtained by adding a dummy variable to φ. The fact that this set is empty certainly implies that it has μ_M measure 0. Using Lemmas 3.38 and 3.37 it follows that player II has a winning strategy in $G_{\omega_1,1}(\vec{S}^*, \varphi^*)$. We noted above that this game is precisely the same as $G_{\omega_1,0}(\emptyset, \varphi)$. ⊣

Corollary 4.9. *(Assuming 0^W exists and letting N be the result of iterating the top extender of 0^W through the ordinals.) Player* I *wins $G_{\omega_1,0}(\emptyset, \varphi)$ iff there exists some initial segment M of N, and some $\alpha_0 < \alpha_1$ in M so that $M \models \varphi_{\rm ini}[\alpha_0, \dot{Y}_0(\alpha_0, \alpha_1)]$.*

Proof. Recall that $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \leq \xi \in {\rm On}\rangle$ is the transfinite iteration leading to N. M_0 is 0^W and μ_0 is the top extender of 0^W. $M_{\xi+1}$ and $j_{\xi,\xi+1}$ are always obtained through an ultrapower by $\mu_\xi = j_{0,\xi}(\mu_0)$, and N is the direct limit of the entire system. Recall that κ_ξ denotes the critical point of μ_ξ. We noted in the proof of Lemma 4.1 that the pair $\langle M_1 \| \kappa_1, \mu_0\rangle$ satisfies assumptions (A1)–(A4) of Section 1.

Suppose first that there are *no* M, α_0, and α_1 as in the corollary. Taking $M = M_1 \| \kappa_1$ it follows in particular that there are no $\alpha_0 < \alpha_1 \in M_1 \| \kappa_1$ so that $(M_1 \| \kappa_1) \models \varphi_{\rm ini}[\alpha_0, \dot{Y}_0(\alpha_0, \alpha_1)]$. Applying Lemma 4.8 with the pair $\langle M_1 \| \kappa_1, \mu_0\rangle$ it follows that player II has a winning strategy in $G_{\omega_1,0}(\emptyset, \varphi)$, and therefore player I does not.

Suppose next that there *are* M, α_0, and α_1 as in the corollary. Using the elementarity of $j_{1,\infty}$, which embeds $M_1 \| \kappa_1$ into N, it follows that there exist M, α_0, and α_1 as in the corollary with the additional property that M is an initial segment of $M_1 \| \kappa_1$. In particular then M is countable and iterable. Applying Lemma 4.6 to M, α_0, and α_1, it follows that player I wins $G_{\omega_1,0}(\emptyset, \varphi)$. ⊣

Recall that T_{Σ_1} denotes the Σ_1 theory of N (with no parameters) and Φ_{Σ_1} denotes the set of Σ_1 sentences in $\mathcal{L}^+$. In Section 2 we saw that T_{Σ_1} is recursive in $\Game_{\omega_1} \Phi_{\Sigma_1}$. We can now establish the reverse direction:

Theorem 4.10. *$\Game_{\omega_1} \Phi_{\Sigma_1}$ is recursive in T_{Σ_1}.*

Proof. Direct from Corollary 4.9 since the condition in the corollary is Σ_1 over N. ⊣

Remark 4.11. As part of his proof of Π^2_1 generic absoluteness under CH, Woodin shows that assuming the existence of 0^W, failures of $\Game_{{\rm closed}-\omega_1} \Pi^1_1$ statements in V are witnessed by existential statements on mice below indiscernible Woodin cardinals. More precisely, there is a recursive association $A \mapsto \psi_A$, from Π^1_1 sets to Σ_1 formulae, so that for every Π^1_1 set A, the closed player does not have a winning strategy in $\Game_{{\rm open}-\omega_1}(A)$ iff $N \models \psi_A$. From this and determinacy it follows that $\Game_{{\rm open}-\omega_1} \Pi^1_1$ is recursive in T_{Σ_1}. $\Game_{{\rm open}-\omega_1} \Pi^1_1$ is recursively equivalent to $\Game_{\omega_1} \Phi_{\Sigma_1}$, so Theorem 4.10 is a consequence of Woodin's argument plus determinacy for open length ω_1 games. (The following is a sketch of the relevant part of Woodin's argument. For a Woodin limit of Woodin cardinals δ, let $\mathbb{P}_\delta$ be the poset adding a sequence of reals $\langle x_\xi \mid \xi < \delta\rangle$ generic over the sequence of extenders algebras at Woodin cardinals below δ, and a strategy σ

generic for the extender algebra at δ, acting on countable length plays coded by reals in $\{x_\xi \mid \xi < \delta\}$. Let ψ_A state that there is some δ so that, in the extension by $\mathbb{P}_\delta$, σ is forced to not be winning for the closed player in $\Game_{\text{open}-\omega_1}(A)$, meaning that there is a play coded by some x_ξ which defeats σ. It is easy to see that if $N \models \psi_A$ then in V the closed player does not have a winning strategy in $\Game_{\text{open}-\omega_1}(A)$: assuming there is such a strategy, iterate to make it—or more precisely its restriction to the plays appearing in the extension—generic, and obtain a contradiction. Conversely, suppose $N \not\models \psi_A$, and let δ be an indiscernible Woodin cardinal of N. Work with $\mathbb{P}_\delta$. Fix a condition p forcing σ to be winning for the closed player against plays coded by reals in $\{x_\xi \mid \xi < \delta\}$. Now play for the closed player by following an interpretation of σ, below the condition p, in iterates making all initial segments of the opponent's play generic. The measure on δ allows continuing this process to ω_1, progressively fixing more of the interpretation of σ as the game and the iteration proceed.)

We finish this section with a note on the definability of the winning strategies constructed in Section 3. Given an iteration strategy Γ let Γ^c be the restriction of Γ to countable iteration trees. Let Γ^{ω_1} be the restriction of Γ to trees of length ω_1. The proof of Lemma 3.6 refers to an iteration strategy Γ for $\langle M, \mu_M \rangle$. The proof can be viewed as converting Γ^c into a strategy for I in $G_{\omega_1,k}(\vec{S}, \varphi)$, and then using Γ^{ω_1} to obtain the club that witnesses victory for I in the game. (The club C in Corollary 3.33 is essentially the branch $[0, \omega_1)_{\mathcal{U}}$ given by Γ^{ω_1}, only thinned to the set of external limits, and then thinned further to the set of fixed points of the map $\alpha \mapsto \hat{\alpha}$.) The first part of this observation can be formalized to Lemma 4.12 below.

By a **pseudo-strategy** for player I in a game of length ω_1 we mean a function Σ^* defined on pairs $\langle p, w \rangle$ so that p is a position in the game where it is I's turn to play, and w is a wellordering of order type $\mathrm{lh}(p)$ on a subset of ω. Informally we think of Σ^* as providing moves for I granted codes for countable ordinals. A run r of the game is **consistent** with the pseudo-strategy Σ^* if there is a sequence $\langle w_\xi \mid \xi < \omega_1 \rangle$ so that $r(\xi) = \Sigma^*(r{\restriction}\xi, w_\xi)$ for each ξ so that it is I's turn to move following $r{\restriction}\xi$. A **winning** pseudo-strategy for player I is a pseudo strategy Σ^* so that all runs consistent with Σ^* are won by I. The corresponding notions for II are defined similarly.

LEMMA 4.12. *Let $\langle M, \mu_M \rangle$ satisfy assumptions (A1)–(A4) in Section 3. Let $\widetilde{M}$ be a real coding M. Let Γ be an $\omega_1 + 1$ iteration strategy for $\langle M, \mu_M \rangle$. Let Γ^c be the restriction of Γ to countable trees. Let $\widetilde{\Gamma}^c$ be the set of reals which code iteration trees according to Γ^c.*

Let $k < \omega$. Let $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ be a sequence of mutually disjoint stationary subsets of ω_1, and let $\widetilde{S}$ be the set of reals which code tuples $\langle \xi, a \rangle$ so that $\xi \in S_a$.

Let $\varphi(x_0, \ldots, x_{k-1})$ be a formula of $\mathcal{L}^+$.

Then the player who wins $G_{\omega_1,k}(\vec{S}, \varphi)$ has a winning pseudo-strategy in the pointclass $\Delta^1_1(\widetilde{\Gamma}^c, \widetilde{S}, \widetilde{M})$.

PROOF. It is enough to check that the proof of Lemma 3.6 gives rise to a $\Delta^1_1(\widetilde{\Gamma}^c, \widetilde{S}, \widetilde{M})$ pseudo-strategy. The construction in that proof can be formalized and seen to be $\Delta^1_1(\widetilde{\Gamma}^c, \widetilde{S}, \widetilde{M})$, at least modulo the appeal to the strategies $\widehat{\Sigma}_{\text{branch}}(\dots)$. These strategies are obtained through applications of Theorem 6G.1 and Corollary 6G.2 of [4]. The proofs of these results in [4] are constructive, and lead to $\Delta^1_1(w)$ strategies in games $\widehat{G}_{\text{branch}}(P, t, \delta^*)(\dot{C}^*)$, provided that the parameter w codes an enumeration of $P \| \delta^* + 1$ of order type ω. In the instances which come up during the construction of a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi)$ in Section 3, the enumeration needed in each round α of $G_{\omega_1,k}(\vec{S}, \varphi)$ can be obtained from an enumeration of the initial model M, and an enumeration of α. It is because of the need for the enumeration of α that we only get a definable winning pseudo-strategy, and not outright a definable winning strategy. ⊣

REMARK 4.13. Steel [7] showed that open length ω_1 games won by the open player have definable winning strategies, and asked whether games won by the closed player have definable winning pseudo-strategies. Lemma 4.12 shows that the answer is yes, granted a definable iteration strategy Γ^c, since for $k = 0$ the set $\widetilde{S}$ is simply empty.

The lemma also gives definable winning pseudo-strategies in games $G_{\omega_1,k}(\vec{S}, \varphi)$ for $k = 1$, as one can take $\vec{S} = \langle S_\emptyset \rangle$ with $S_\emptyset = \omega_1$ in this case, so that $\widetilde{S}$ consists simply of all reals coding pairs $\langle \xi, \emptyset \rangle$, $\xi < \omega_1$. But for $k \geq 2$ the parameter $\widetilde{S}$ must code disjoint stationary subsets of ω_1, taking it outside the realm of definability.

§5. Relativizations. So far we worked only with *lightface* games of length ω_1, games where the payoff is determined by a formula φ with no parameters. Let us now consider ways to allow parameters into the definitions. We consider two ways: the first allows a real as parameter; and the second allows a set of reals (or more precisely a subset of $H(\omega_1)$). The results of Sections 2 through 4 can be relativized to admit parameters of these kinds. The relativization of the results is straightforward, and we therefore confine the discussion here to the definitions.

We begin by considering real parameters.

DEFINITION 5.1. Let $x \in \omega^\omega$ be a real number. Let k, $\vec{S}$, and φ be as in Definition 1.2. Define $G_{\omega_1,k}(\vec{S}, \varphi, x)$ to follow the rules and payoff in Definition 1.2, only replacing the reference to $(\mathrm{L}_{\omega_1}[r]; r)$ in the payoff condition with a reference to $(\mathrm{L}_{\omega_1}[x^\frown r]; x^\frown r)$.

The results of Section 3 easily relativize to x and yield the determinacy of $G_{\omega_1,k}(\vec{S}, \varphi, x)$ assuming the existence of a pair $\langle M, \mu_M \rangle$ satisfying conditions (A1)–(A4) of Section 3 with the additional demand that $x \in M$. The relativization also shows that the player who wins $G_{\omega_1,k}(\vec{S}, \varphi, x)$ for one sequence $\vec{S}$ wins the game for all such sequences. Using this invariance define $\Game_{\omega_1,k}(\varphi, x)$ to be "True" if player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, \varphi, x)$ for some/all sequences $\vec{S} = \langle S_a \mid a \in [\omega_1]^{<k} \rangle$ of mutually disjoint stationary subsets of ω_1, and "False" otherwise. Define $\Game_{\omega_1}(\varphi, x)$ to be $\Game_{\omega_1,k}(\varphi, x)$ where k is the

number of free variables in φ. Given a set $A \subset \Phi \times \mathbb{R}$ define $\Game_{\omega_1} A$ to be $\{\langle \varphi, x\rangle \in A \mid \Game_{\omega_1}(\varphi, x) = \text{True}\}$.

Next we consider allowing reference to a set of hereditarily countable objects, that is to a subset of $H(\omega_1)$.

Fix $A \subset H(\omega_1)$. By $\mathrm{L}^A_{\omega_1}[r]$ we mean the collection of sets constructible (at a countable stage) relative to both r and A. $(\mathrm{L}^A_{\omega_1}[r]; r)$ is the structure with universe $\mathrm{L}^A_{\omega_1}[r]$ and *two* predicates, $A \cap (\mathrm{L}^A_{\omega_1}[r])$ and r. The language describing this structure is $\mathcal{L}^{++}$, obtained by adding a relation symbol $\dot{A}$ and a function symbol $\dot{r}$ to the language of set theory. $\mathrm{L}^A_{\omega_1}[r]$ is reached by closing under sets definable in this language, namely using the definition $\mathrm{L}^A_{\alpha+1}[r] = \{x \subset \mathrm{L}^A_\alpha[r] \mid x$ is definable from parameters over $(\mathrm{L}^A_\alpha[r]; \in, r \cap \mathrm{L}^A_\alpha[r], A \cap \mathrm{L}^A_\alpha[r])\}$.

DEFINITION 5.2. Let A be a subset of $H(\omega_1)$. Let k and $\vec{S}$ be as in Definition 1.2. Let $\varphi(x_0, \ldots, x_{k-1})$ be a formula in $\mathcal{L}^{++}$. Define $G_{\omega_1,k}(\vec{S}, A, \varphi)$ to follow the rules and payoff in Definition 1.2, only replacing the reference to $(\mathrm{L}_{\omega_1}[r]; r)$ in the payoff condition with a reference to $(\mathrm{L}^A_{\omega_1}[r]; r)$.

Let M be a model of ZFC^*. Let θ be a Woodin limit of Woodin cardinals in M. Let $\dot{B}$ be a $\mathbb{W}_\theta$–name in M. $\dot{B}$ is said to **capture** A over M just in case that $\dot{B}[G] = A \cap M[G]$ for every $G \in \mathrm{V}$ which is $\mathbb{W}_\theta$–generic over M. This is an adaptation to our context of the notion of capturing due to Woodin, tracing back to his core model induction. The related definitions below similarly adapt notions due to Woodin.

Let Γ be an iteration strategy for M. $\dot{B}$ is said to **capture** A **over** (M, Γ) just in case that $j(\dot{B})$ captures A over M^* for every iteration $j\colon M \to M^*$ consistent with Γ and so that $j(\theta) \le \omega_1$.

If μ_M is an external measure over M and Γ is an iteration strategy for $\langle M, \mu_M\rangle$, then we say that $\dot{B}$ **captures** A **over** $(\langle M, \mu_M\rangle, \Gamma)$ just in case that the condition of the previous paragraph holds, only allowing now uses in j of images of μ_M, in addition to uses of internal extenders.

We say that A **can be captured** over (M, Γ) if there is a name $\dot{B}$ in M which captures A over (M, Γ), and similarly for $(\langle M, \mu_M\rangle, \Gamma)$.

The following result relativizes Theorem 3.39 and Corollary 4.2 to the games of Definition 5.2.

THEOREM 5.3. *Suppose that there exists a pair $\langle M, \mu_M\rangle$ and an $\omega_1{+}1$ iteration strategy Γ for $\langle M, \mu_M\rangle$ so that:*

1. *$\langle M, \mu_M\rangle$ satisfies conditions (A1)–(A3) in Section 3; and*
2. *A can be captured over $(\langle M, \mu_M\rangle, \Gamma)$.*

Then the games $G_{\omega_1,k}(\vec{S}, A, \varphi)$ are determined. Moreover the question of which player has a winning strategy in $G_{\omega_1,k}(\vec{S}, A, \varphi)$ is independent of $\vec{S}$. ⊣

The final clause in Theorem 5.3 allows defining a relativized game quantifier in the natural manner: $\Game_{\omega_1,k}(A, \varphi)$ is "True" iff player I has a winning strategy in $G_{\omega_1,k}(\vec{S}, A, \varphi)$ for some/all $\vec{S}$. As usual define then $\Game_{\omega_1}(A, \varphi)$ to stand for $\Game_{\omega_1,k}(A, \varphi)$ where k is the number of free variables in φ.

One can of course combine Definitions 5.1 and 5.2, to phrase the games $G_{\omega_1,k}(\vec{S}, A, \varphi, x)$ with reference to both a set $A \subset H(\omega_1)$ and a real x, and to phrase the corresponding game quantifier $\Game_{\omega_1}(A, \varphi, x)$.

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