# Forcing with side conditions

Itay Neeman

Department of Mathematics
University of California Los Angeles
Los Angeles, CA 90095

www.math.ucla.edu/~ineeman

Oberwolfach, January 2011

# Forcing axioms



Developed in late 1960s early 1970s, initially to crystalize center points for applications of iterated forcing.

**Martin's axiom** (MA, for $\omega_1$ antichains): for any c.c.c. poset $\mathbb{P}$ and any collection $F$ of $\omega_1$ maximal antichains of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every antichain in $F$.

Obtained through an iteration of enough c.c.c. posets. Can then be used axiomatically as a starting point for consistency proofs that would otherwise require an iteration of c.c.c. posets.

Key points in proving consistency of MA:

- **(a)** Finite support iteration of c.c.c. posets does not collapse $\omega_1$, and in fact the iteration poset is itself c.c.c.
- **(b)** Can "close off", that is reach a point where enough c.c.c. posets have been hit to ensure MA.

## Proper forcing

There are classes of posets other than c.c.c. which also preserve $\omega_1$.

### Definition

Let $\mathbb{P}$ be a poset. Let $\kappa$ be large enough that $\mathbb{P} \in H(\kappa)$. $p \in \mathbb{P}$ is a master condition for $M \prec H(\kappa)$ if

1. $p$ forces that every maximal antichain $A$ of $\mathbb{P}$ that belongs to $M$ is met by the generic filter inside $M$.

Equivalently any of:

2. $p$ forces that $\dot{G} \cap \check{M}$ is generic over $M$.
3. $p$ forces that $M[\dot{G}] \prec H(\theta)[\dot{G}]$ and $M[\dot{G}] \cap V = M$.

### Definition

$\mathbb{P}$ is proper if for all large enough $\kappa$ and all countable $M \prec H(\kappa)$, every condition in $M$ extends to a master condition for $M$.

Proper posets do not collapse $\omega_1$; immediate from (3).

# PFA

**Proper forcing axiom** (PFA): the parallel of MA for proper posets. Again used axiomatically as a starting point for consistency proofs.

Key points in consistency proof of PFA:

**(a)** Countable support iteration of proper posets does not collapse $\omega_1$, and is indeed proper.

**(b)** Can close off, assuming a supercompact cardinal.

For (b), fix a supercompact cardinal $\theta$. Iterate up to $\theta$ hitting proper posets given by a Laver function. At stage $\theta$, using properties of the Laver function and supercompactness, have covered enough posets to ensure PFA holds.

### Note

*In addition to (a) and (b), important also that iteration does not collapse $\theta$, but this is clear.*

## Higher analogs

In the case of MA, the forcing axiom has higher analogs, and in fact strengthenings.

For example it is consistent that for all c.c.c. posets, all maximal antichain in families of size $\omega_2$ can be simultaneously met by a filter.

Initial expectation was that similar analogs should exist for PFA.

Naive attempt: demand existence of master conditions also for models of size $\omega_1$.

Posets in the resulting class preserve $\omega_1$ and $\omega_2$ (certainly a necessary property for a higher analog).

But preservation under iteration fails.

## More on iteration

When iterating proper posets, countable support used to ensure properness of iteration at limits of countable cofinality. Basic idea for preservation, e.g. at stage $\omega$, for iteration $\langle \mathbb{P}_\xi \mid \xi \leq \theta \rangle$ of posets $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \theta \rangle$:

Let $D_n$ enumerate all dense sets of $\mathbb{P}_\omega$ that belong to $M$. Diagonalize to create a condition $p \in \mathbb{P}_\omega$ which "almost meets" each of them, meaning that below $p$, $D_n$ is reduced to a dense set in $\mathbb{P}_n$. This can be done extending only coordinates $\geq n$ when handling $D_n$, so that the construction converges to a condition $p$.

Properness of the individual posets iterated then allows extending $p$ to a master condition in $\mathbb{P}_\omega$.

Similar diagonalization used at all limits of cofinality $\omega$. Existence of master conditions at limits of greater cofinality is a consequence of their existence at limits of cofinality $\omega$, because of the use of countable support.

# Higher analogs and iteration



For diagonalization process at a limit $\alpha$ over dense sets in *M*, important that $\mathrm{cof}(\alpha) = |M|$. Can then create a condition of length $\alpha$ which almost meets each dense set in *M*.

Method breaks down if there are models of two different sizes. Problems with models of size $\omega_1$ at cofinality $\omega$, and then with models of size $\omega$ at cofinality $\omega_1$.

Seemingly a terminal barrier for higher analog of PFA.

Moreover, in contrast with MA, PFA actually implies that the continuum is $\omega_2$. This is further evidence against higher analogs, though strictly speaking only implies that analog is not a *strengthening* of PFA.

# Models as side conditions

Models are used as side conditions in several very nice applications of PFA.

For example, fix $\theta$ and consider the following posets $\mathbb{P}$.

Conditions are increasing finite sequences $M_0 \in M_1 \in \cdots \in M_n$ of countable $\Sigma_1$ elementary submodels of $H(\theta)$.

(Abusing notation slightly regard the condition as a set $s = \{M_0, \ldots, M_n\}$. No loss of information since the order of the sequence is determined from the models.)

Poset order is the natural one, reverse inclusion.

$\mathbb{P}$ is proper. For $\delta > \theta$ and $M^* \prec H(\delta)$, any condition $s$ with $M = M^* \cap H(\theta) \in s$ is a master condition for $M^*$. In fact a strong master condition: forces that the generic filter for $\mathbb{P}$ is also generic (over $V$) for $\mathbb{P} \cap M$.

# Models as side conditions (cont.)



This is among the simplest examples, and models are not needed.

Can cast the forcing in terms that use the ordinals $M_i \cap \omega_1$ instead of the models $M_i$.

Due to Baumgartner, adds a club in $\omega_1$ with finite conditions.

Other, much more sophisticated uses of models as side conditions. Models used to enforce properness.

## Clubs in $\omega_2$

Around 2003, higher analog found for adding clubs with finite conditions.

Friedman, Mitchell independently force to add a club subset of $\omega_2$, with finite conditions. Mitchell also adds club subsets to inaccessible $\theta$, turning $\theta$ to $\omega_2$.

Use countable models as side conditions to enforce properness (and in particular preservation of $\omega_1$).

Proofs are quite complicated. Sequence of models is no longer increasing, and there is a careful agreement condition between countable models on the sequence.

Can be simplified substantially by explicitly adding models of greater size.

We illustrate in the case of adding a club subset to an inaccessible $\theta$ while converting it to $\omega_2$. Similar definitions work for adding club subset of $\omega_2$.

## Adding club in $\theta$ with finite conditions



### Definition

A node is an a model $M$ of one of the following types:

1. $M \prec_1 H(\theta)$ is countable. (Countable type nodes.)
2. $M = H(\kappa) \prec_1 H(\theta)$ with $\kappa$ of cofinality at least $\omega_1$. (Rank type nodes.)

A side condition is an increasing sequence of nodes $M_0 \in M_1 \in \cdots \in M_n$ which is closed under intersections.

As before can regard the condition as a set $s = \{M_0, \ldots, M_n\}$ with no loss of information.

$\mathbb{P}_{\text{side}}$ is the poset of side conditions, ordered by reverse inclusion.

# Adding club in $\theta$ with finite conditions (cont.)



**Lemma**

*If $s$ is a side condition and $Q \in s$, then $s$ is a strong master condition for $Q$.*

**Sketch of proof.**

Define $\mathrm{res}_Q(s)$, the residue of $s$ in $Q$, to be $\{M \in s \mid M \in Q\}$.

Using closure of $s$ under intersections can show $\mathrm{res}_Q(s)$ is increasing. It is also closed under intersections by closure of $s$ and elementarity of $Q$. So $\mathrm{res}_Q(s)$ is a side condition.

Prove that any side condition $t \in Q$ which extends $\mathrm{res}_Q(s)$ is compatible with $s$. This is enough to establish lemma.

Proof of compatibility is straightforward if $Q$ if of rank type, a bit more involved if $Q$ is of countable type. □

## Adding club in $\theta$ with finite conditions (cont.)

Lemma holds, with same proof, if nodes are restricted to belong to a given class $\mathcal{C}$, so long as:

1. If $W \in M$ of rank and countable type respectively both belong to $\mathcal{C}$, then $M \cap W \in \mathcal{C}$.
2. In the situation of condition (1), $M \cap W \in W$.

(1) needed for closure of side condition under intersection to make sense. (2) clear when working in $H(\theta)$, but meaningful in parallel forcing to add club in $\omega_2$.

For Lemma 4 to be useful also need $\mathcal{C}$ to be stationary in both $\mathcal{P}_{<\omega_1}(H(\theta))$ and $\mathcal{P}_{<\theta}(H(\theta))$. Can then use forcing to add clubs through stationary sets.

$\mathbb{P}_{\text{side}}$ has same collapsing effect as $\text{col}(\omega_1, <\theta)$, but does not add branches through trees of height $\omega_1$ in $V$. Very useful for arguments on the tree property. Definition here simplifies Friedman/Mitchell proofs, gives more flexible higher cardinal analogs, helps reproving many tree property results (but these will be covered in another talk).

## Another proof of the consistency of PFA

$\mathbb{P}_{\text{side}}$ can be used to reprove the consistency of PFA, with finite support.

Let $\theta$ be supercompact, $f$ a Laver function, $\mathbb{P}_{\text{side}}$ the poset of side conditions with nodes elementary in $(H(\theta); f)$.

### Definition

Condition in $\mathbb{A}$ are pairs $\langle s, p\rangle$ where:

1. $s \in \mathbb{P}_{\text{side}}$.
2. $p$ is a function with $\mathrm{dom}(p) \subseteq \{\kappa \mid H(\kappa) \in s\}$.
3. $p(\kappa)$ is defined only if
   - **3.1** $f(\kappa)$ is a name in the poset $\mathbb{A} \cap H(\kappa)$. Call it $\dot{\mathbb{Q}}_\kappa$.
   - **3.2** $\langle s \cap H(\kappa), \emptyset\rangle$ forces in $\mathbb{A} \cap H(\kappa)$ that $\dot{\mathbb{Q}}_\kappa$ is proper.
4. When defined, $p(\kappa)$ is an $\mathbb{A} \cap H(\kappa)$-name, forced by $\langle s \cap H(\kappa), p \restriction \kappa\rangle$ to be (a) in $\dot{\mathbb{Q}}_\kappa$, (b) a master condition for each countable $M \in s$ with $\kappa \in M$.

$\langle s^*, p^*\rangle < \langle s, p\rangle$ iff $s^* \supseteq s$, and for each $\kappa \in \mathrm{dom}(p)$, $\langle s^* \cap H(\kappa), p^* \restriction \kappa\rangle$ forces that $p^*(\kappa)$ extends $p(\kappa)$.

# Another proof of the consistency of PFA, comments



Formally this is a definition of $\mathbb{A} \cap H(\kappa)$ by induction on $\kappa$.

Note that $\mathrm{dom}(p)$ is finite.

Use of side conditions allows proving that $\mathbb{A}$ is proper. (Proof is again by induction on $\kappa$.) In particular $\omega_1$ is preserved.

Must also show $\theta$ is preserved. Since $\mathbb{A}$ is not quite an iteration, this is not automatic. Use the fact that any $s$ with $H(\kappa) \in s$ is a strong master condition for $H(\kappa)$ in $\mathbb{P}_{\text{side}}$ to get preservation of $\theta$.

## Why?

Why bother with a finite support proof of the consistency of PFA?

Recall the question of higher analogs.

Impediment for higher analogs is the need, in preservation theorem for iteration, for exact match between size of support and size of models. (Can only have exact match for models of one size.) This need is eliminated in a finite support proof.

Get higher analog?

Not so fast....

In finite support proof of PFA, needed $\mathbb{P}_{\text{side}}$ to preserve two cardinals, $\omega_1$ and $\theta$.

For a higher analog, need a poset of side conditions which preserves *three* cardinals, $\omega_1$, $\omega_2$, and $\theta$.

## Side conditions preserving three cardinals

A pre-cursor exists in Mitchell's proof that $I(\omega_2)$ can be trivial. This proof involves preservation of three cardinals: $\omega_1$, a weakly compact cardinal $\kappa$ which is turned into $\omega_2$, and $\kappa^+$.

Need a diferent poset, to decouple the third cardinal from $\kappa$, so that the third preserved cardinal can be supercompact.

Can be done, but poset is quite complicated.

As expected involves nodes of three types, countable, $\omega_1$, and rank type.

But not all nodes are elementary.

The presence of non-elementary nodes causes substantial technical complications (including closure requirements beyond closure under intersections) that will be ignored for the rest of this talk.

# High analog of properness



Fix a class $\mathcal{C}$ of $M \prec_1 V$ of countable and $\omega_1$ sizes so that:

1. If $P \in M$ of sizes $\omega_1$ and countable respectively both belong to $\mathcal{C}$, then so does $M \cap P$.
2. In the situation of condition (1), $M \cap P \in P$.
3. Whenever $\mathrm{cof}(\kappa) \geq \omega_2$ and $H(\kappa) \prec_1 V$, $\mathcal{C} \cap H(\kappa)$ is stationary in both $\mathcal{P}_{<\omega_1}(H(\kappa))$ and $\mathcal{P}_{<\omega_2}(H(\kappa))$.

Existence of such a class has to be assumed. It is not always possible to arrange condition (2).

$\mathbb{P}_{\text{side}}$ below is poset of side conditions with nodes (of two sizes, countable and $\omega_1$) in $\mathcal{C}$.

## High analog of properness (cont.)



Fix a poset $\mathbb{Q}$.

The notion of a master condition $q \in \mathbb{Q}$ for $M \in \mathcal{C}$ is defined in the obvious way, both for countable $M$ and for $M$ of size $\omega_1$.

### Definition

Let $s \in \mathbb{P}_{\text{side}}$. Then $q \in \mathbb{Q}$ is a master condition for $s$ if $q$ is a master condition for each $M \in s$.

### Remark

*This definition neglects the presence of non-elementary nodes. A master condition for a non-elementary node M is a condition which is a master condition for a certain countable, $\in$-linear, set $S_M$ of elementary nodes of size $\omega_1$ that belong to M.*

Call $H(\kappa)$ appropriate for $\mathbb{Q}$ if $\mathbb{Q} \in H(\kappa) \prec_1 V$ and $\mathrm{cof}(\kappa) \geq \omega_2$. A side condition $s$ is appropriate for $\mathbb{Q}$ and $H(\kappa)$ if $\mathbb{Q} \in M \prec H(\kappa)$ for each $M \in s$.

# High analog of properness (cont.)



## Definition

$\mathbb{Q}$ is $\{\omega, \omega_1\}$-proper if there is $\kappa$ and a function $\mathrm{mc}$ so that $H(\kappa)$ is appropriate for $\mathbb{Q}$, and for every $s$ which is appropriate for $\mathbb{Q}$ and $H(\kappa)$:

1. Every $p \in \mathrm{mc}(s)$ is a master condition for $s$.
2. For every $M \in s$ and every $p \in \mathrm{mc}(\mathrm{res}_M(s))$ that belongs to $M$, there is $q \in \mathrm{mc}(s)$ extending $p$.
3. If $s = \emptyset$ then $\mathrm{mc}(s) = \mathbb{Q}$.

## Remark

*Restricted to s with only countable models, this is equivalent to properness. (2) in this case follows from "for all $p \in M \prec H(\kappa)$, there is a master condition q for M extending p." But this statement by itself is weaker than (2) in case of side conditions with models of two sizes.*

Again neglect the issue of non-elementary nodes.

# High analog of PFA



### Definition

The $\{\omega, \omega_1\}$-proper forcing axiom states that for every $\{\omega, \omega_1\}$-proper poset $\mathbb{Q}$, and every collection $F$ of $\omega_2$ maximal antichains of $\mathbb{Q}$, there is a filter on $\mathbb{Q}$ that meets every antichain in $F$.

### Theorem

*Assume $\theta$ is supercompact. Then the $\{\omega, \omega_1\}$-proper forcing axiom holds in a forcing extension of $V$.*

Fairly broad. Includes all c.c.c. posets, and posets to collapse cardinals to $\omega_2$ (but with finite conditions).

Can also add "anti-thread" through square seq. above $\omega_2$ (again with finite conditions). So axiom implies failure of $\square$ above $\omega_2$, and in particular has large cardinal strength.

A starting point for higher analogs to consequences of PFA. Further applications require more work.