Handbook of Set Theory

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Contents

I. Determinacy in $\mathrm{L}(\mathbb{R})$

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Given a set $C \subseteq \omega^\omega$ define $G_\omega(C)$, the length ω game with payoff set C, to be played as follows: Players I and II collaborate to produce an infinite sequence $x = \langle x(i) \mid i < \omega \rangle$ of natural numbers. They take turns as in Diagram 1, I picking $x(i)$ for even i and II picking $x(i)$ for odd i. If at the end the sequence x they produce belongs to C then player I wins; and otherwise player II wins. $G_\omega(C)$, or any other game for that matter, is *determined* if one of the two players has a *winning strategy*, namely a strategy for the game that wins against all possible plays by the opponent. The set C is said to be determined if the corresponding game $G_\omega(C)$ is determined. Determinacy is said to hold for a pointclass Γ if all sets of reals in Γ are determined. (Following standard abuse of notation we identify $\mathbb{R}$ with ω^ω.)

$$
\begin{array}{c|cccccc}
\mathrm{I} & x(0) & & x(2) & & \ldots\ldots & \\
\hline
\mathrm{II} & & x(1) & & x(3) & & \ldots\ldots
\end{array}
$$

Diagram 1: The game $G_\omega(C)$.

Perhaps surprisingly, determinacy has turned out to have a crucial and central role in the study of definable sets of reals. This role resulted from two lines of discoveries. On the one hand it was seen that determinacy for definable sets of reals, taken as an axiom, can be used to prove many desirable results about these sets, and indeed to obtain a rich and powerful structure theory. On the other hand it was seen that determinacy can be proved for definable sets of reals, from large cardinal axioms.

The earliest work on consequences of determinacy, by Banach, Mazur, and Ulam [23] at the famous Scottish Café in the 1930's, Oxtoby [35], Davis [3], and Mycielski–Swierczkowski [27], established that determinacy for a

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pointclass Γ implies that all sets of reals in Γ have the Baire property, have the perfect set property, and are Lebesgue measurable. Later on Blackwell [2] used the determinacy of open sets to prove Kuratowski's reduction theorem. (In modern terminology this theorem states that for any $\mathbf{\Pi}^1_1$ sets A, B, there are $A^* \subseteq A$ and $B^* \subseteq B$ so that $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \emptyset$.) Inspired by his methods, Martin [15] and Addison–Moschovakis [1] used determinacy for projective sets to prove reduction for each of the pointclasses $\mathbf{\Pi}^1_n$, $n > 1$ odd, and indeed prove for these pointclasses some of the structural properties that hold for $\mathbf{\Pi}^1_1$. Their results initiated a wider study of consequences of the *axiom of determinacy* (AD), that is the assertion that all sets of reals are determined, proposed initially by Mycielski–Steinhaus [26]. Over time this line of research, which the reader may find in Moschovakis [25], Jackson [7], and of course the Cabal volumes [12, 9, 10, 11], established determinacy axioms as natural assumptions in the study of definable sets of reals.

It should be emphasized that AD was not studied as an assumption about V. (It contradicts the axiom of choice.) Rather, it was studied as an assumption about more restrictive models, models which contain all the reals but have only definable sets of reals. A prime example was the model $\mathrm{L}(\mathbb{R})$, consisting of all sets which are constructible from $\{\mathbb{R}\} \cup \mathbb{R}$. It was known by work of Solovay [37] that this model need not satisfy the axiom of choice, and that in fact it is consistent that all sets of reals in this model are Lebesgue measurable. The extra assumption of AD allowed for a very careful analysis of $\mathrm{L}(\mathbb{R})$, in terms that combined descriptive set theory, fine structure, and infinitary combinatorics. It seemed plausible that if there were a model of AD, $\mathrm{L}(\mathbb{R})$ would be it.

Research into the consequences of determinacy was to some extent done on faith. The established hierarchy of strength in set theory involved large cardinals axioms, that is axioms asserting the existence of elementary embeddings from the universe of sets into transitive subclasses, not determinacy axioms. A great deal of work has been done in set theory on large cardinal axioms, Kanamori [8] is a good reference, and large cardinals have come to be regarded as the backbone of the universe of sets, providing a hierarchy of consistency strengths against which all other statements are measured. From $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ one could obtain objects in $\mathrm{L}(\mathbb{R})$ which are very strongly reminiscent of large cardinal axioms in V, suggesting a connection between the two. Perhaps the most well known of the early results in this direction is Solovay's proof that ω_1 is measurable under AD. Further justification for the use of $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ was provided by proofs of determinacy for simply definable sets: for open sets in Gale–Stewart [6], for Borel sets in Martin [18, 17], and for $\mathbf{\Pi}^1_1$ sets from a measurable cardinal in Martin [16], to name the most well known. Additional results, inspired by Solovay's proof that ω_1 is measurable under AD and Martin's proof of $\mathbf{\Pi}^1_1$ determinacy from a measurable, identified detailed and systematic correspondences of strength, relating models for many measurable cardinals to determinacy for pointclasses just above $\boldsymbol{\Pi}^1_1$. These levels are well below the pointclass of all sets in $\mathrm{L}(\mathbb{R})$, but still the accumulated evidence of the results suggested that there should be a proof of $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ from large cardinals, and conversely a construction of inner models with these large cardinals from $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$. In 1985 the faith in this connection was fully justified. A sequence of results of Foreman, Magidor, Martin, Shelah, Steel, and Woodin, see [5, 36, 21, 22, 43] for the papers involved and the introduction in [31] for an overview, brought the identification of a new class of large cardinals, known now as Woodin cardinals, new structures of iterated ultrapowers, known now as iteration trees, and new proofs of determinacy, including a proof of $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$. Additional results later on obtained Woodin cardinals from determinacy axioms, and indeed established a deep and intricate connection between the descriptive set theory of $\mathrm{L}(\mathbb{R})$ under AD, and inner models for Woodin cardinals.

In this chapter we prove $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ from Woodin cardinals. Our exposition is complete and self contained: the necessary large cardinals are introduced in Section 1, and every result about them which is needed in the course of proving $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ is included in the chapter, mostly in Sections 2 and 3. The climb to $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ is carried out progressively in the remaining sections. In Section 4 we introduce homogeneously Suslin sets and present a proof of determinacy for $\boldsymbol{\Pi}^1_1$ sets from a measurable cardinal. In Section 5 we move up and present a proof of projective determinacy from Woodin cardinals. The proof in essence converts the quantifiers over reals appearing in the definition of a projective set to quantifiers over iteration trees and branches through the trees, and these quantifiers in turn are tamed by the iterability results in Section 2. In Section 6 we improve on the results in Section 5 by reducing the large cardinal assumption needed for the determinacy of universally Baire sets. The section also lays the grounds for Section 7, where we show that models with Woodin cardinals can be iterated to absorb an arbitrary given real into a generic extension. Finally, in Section 8 we derive $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$.

There is much more to be said about proofs of determinacy that cannot be fitted within the scope of this chapter. Martin [18, 19] and Neeman [33] for example prove weaker forms of determinacy (from weaker assumptions) using completely different methods, which handle increments of payoff complexity corresponding to countable unions, rather than real quantifiers. Perhaps more importantly there are strengthenings of $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ in two directions, one involving stronger payoff sets, and the other involving longer games. In the former direction the reader should consult Steel [40], which contains a proof of Woodin's derived model theorem, a fundamental theorem connecting models of AD to symmetric extensions of models of choice with Woodin cardinals, and uses this theorem to establish AD in models substantially stronger than $\mathrm{L}(\mathbb{R})$. In the latter direction the reader should consult Neeman [31, 28], which contain proofs of determinacy for games of fixed countable lengths, variable countable lengths, and length ω_1.

Historical Remarks. With some exceptions, noted individually inside the various sections, the following remarks summarize credits for the material in the chapter. Extenders were introduced by Mitchell [24], then simplified to their present forms by Jensen. The related material on ultrapowers in Section 1 is by now folklore within set theory. Its history can be found in Kanamori [8]. The material on iteration trees in Section 1 is due to Martin–Steel [22] and so is all the material in Section 2. The material in Section 3 is due to Martin–Steel [21], and follows the exposition of Neeman [31]. The material in Section 4 is due to Martin. The material in Section 5 is due to Martin–Steel [21]. (The exposition here is specifically geared to easing the transition to the next section.) The material in Sections 6 and 7 is due to Neeman. $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ from infinitely many Woodin cardinals and a measurable cardinal above them is due to Woodin, proved using the methods of stationary tower forcing (see Larson [14]) and an appeal to the main theorem, Theorem 5.11, in Martin–Steel [21]. A proof using Woodin's genericity iterations [39, 4.3] and fine structure instead of stationary tower forcing is due to Steel, and the proof reached in this chapter (using a second form of genericity iterations and no fine structure) is due to Neeman.

1. Extenders and Iteration Trees

Throughout this chapter we shall deal with elementary embeddings of the universe into transitive classes. Here we develop tools for the study of such embeddings. Most basic among them is the ultrapower construction, which allows the creation of an embedding $\pi\colon V \to M$ from the restriction of such an embedding to a set. We begin by characterizing the restrictions.

1.1 Remark. By embedding we always mean elementary embedding, even when this is not said explicitly. As a matter of convention when we say a wellfounded model of set theory we mean a *transitive* model equipped with the standard membership relation $\in$. More generally we always take the wellfounded parts of our models to be transitive.

Let $(*, *)$ denote the Gödel pairing operation on ordinals. Given sets of ordinals A and B define $A \times B$ to be $\{(\alpha, \beta) \mid \alpha \in A \wedge \beta \in B\}$. Note that $A \times B$ is then a set of ordinals too. We refer to it as the *product* of A and B. In general define finite products of sets of ordinals as follows: For $n = 0$ set $\Pi_{i\leq n} A_i$ equal to A_0; for $n > 0$ set $\Pi_{i\leq n} A_i$ equal to $(\Pi_{i\leq n-1} A_i) \times A_n$. Define finite sequences of ordinals similarly by setting the empty sequence equal to 0, setting (α) equal to α, and setting $(\alpha_0, \dots, \alpha_n)$ equal to $((\alpha_0, \dots, \alpha_{n-1}), \alpha_n)$ for $n > 0$.

If A is a set of ordinal sequences of length n, and $\sigma\colon n \to n$ is a permutation of n, then define σA by setting $(\alpha_0, \ldots, \alpha_{n-1}) \in \sigma A \iff (\alpha_{\sigma^{-1}(0)}, \ldots, \alpha_{\sigma^{-1}(n-1)}) \in A$.

If A is a set of ordinal sequences of length $n+1$, then define $\mathrm{bp}(A)$ to be the set $\{(\alpha_0, \ldots, \alpha_{n-1}) \mid (\exists \xi \in \alpha_0)(\alpha_0, \ldots, \alpha_{n-1}, \xi) \in A\}$. $\mathrm{bp}(A)$ is the *bounded projection* of A.

By a *fiber* through a sequence of sets $\langle A_i \mid i < \omega \rangle$ we mean a sequence $\langle \alpha_i \mid i < \omega \rangle$ so that $(\alpha_0, \ldots, \alpha_{i-1}) \in A_i$ for every $i < \omega$.

1.2 Definition. A *(short) extender* is a function E that satisfies the following conditions:

1. The domain of E is equal to $\mathcal{P}(\kappa)$ for an ordinal κ closed under Gödel pairing.

2. E sends ordinals to ordinals and sets of ordinals to sets of ordinals.

3. $E(\alpha) = \alpha$ for $\alpha < \kappa$, and $E(\kappa) \neq \kappa$.

4. E respects products, intersections, set differences, membership, the predicates of equality and membership, permutations, and bounded projections. More precisely this means that for all $A, B \in \mathrm{dom}(E)$, all ordinals $\alpha \in \mathrm{dom}(E)$, and all permutations σ of the appropriate format:

   (a) $E(A \times B) = E(A) \times E(B)$, $E(A \cap B) = E(A) \cap E(B)$, and $E(A - B) = E(A) - E(B)$.

   (b) $\alpha \in A \Longrightarrow E(\alpha) \in E(A)$.

   (c) $E(\{(\alpha, \beta) \in A \times A \mid \alpha = \beta\})$ is equal to $\{(\alpha, \beta) \in E(A) \times E(A) \mid \alpha = \beta\}$, and similarly with $\alpha \in \beta$ replacing $\alpha = \beta$.

   (d) $E(\sigma A) = \sigma E(A)$.

   (e) $E(\mathrm{bp}(A)) = \mathrm{bp}(E(A))$.

5. E is *countably complete*. Precisely, this means that for any sequence $\langle A_i \mid i < \omega \rangle$ of sets which are each in the domain of E, if there exists a fiber through $\langle E(A_i) \mid i < \omega \rangle$ then there exists also a fiber through $\langle A_i \mid i < \omega \rangle$.

The first ordinal moved by E is called the *critical point* of E, denoted $\mathrm{crit}(E)$. By condition (3), this critical point is precisely equal to the ordinal κ of condition (1). The set $\bigcup_{A \in \mathrm{dom}(E)} E(A)$ is called the *support* of E, denoted $\mathrm{spt}(E)$. Using condition (2) it is easy to see that the support of E is an ordinal.

1.3 Remark. Condition (3) limits our definition to extenders with domains consisting of just the subsets of the extenders' critical points. It is this condition that makes our extenders "short." We shall see later that it has the effect of limiting the strength of embeddings generated by our (short) extenders to a level known as superstrong. This level is more than adequate for our needs. We shall therefore deal exclusively with short extenders in this chapter, and refer to them simply as extenders. For a more general definition see Neeman [32].

1.4 Definition. A (two-valued) *measure* over a set U is a function μ from $\mathcal{P}(U)$ into $\{0,1\}$ with the properties that $\mu(\emptyset) = 0$, $\mu(U) = 1$, and $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for any disjoint $X, Y \subseteq U$.

1.5 Remark. Given $a \in \mathrm{spt}(E)$ define $E_a \colon \mathcal{P}(\kappa) \to \{0,1\}$ to be the function given by $E_a(X) = 1$ if $a \in E(X)$, and 0 otherwise. E_a is then a measure over κ. It has been customary to define extenders by specifying properties of the sequence $\langle E_a \mid a \in \mathrm{spt}(E)\rangle$ equivalent to the properties of E specified in Definition 1.2. For a definition of extender through properties of $\langle E_a \mid a \in \mathrm{spt}(E)\rangle$ see Martin–Steel [21, §1A] (short extenders) and Kanamori [8, §26] (the general case).

By a *pre-extender* over a model Q we mean an object E that satisfies conditions (1)–(4) in Definition 1.2, with $\mathcal{P}(\kappa)$ in condition (1) replaced by $\mathcal{P}^Q(\kappa)$, but not necessarily condition (5). The point of this distinction is that condition (5) involves second order quantification over E, whereas conditions (1)–(4) involve only E, the powerset of κ, and bounded quantifiers over the transitive closure of E. By removing condition (5) we obtain a notion that is absolute in the sense given by Claim 1.7:

1.6 Definition. Two models Q and N *agree to* an ordinal ρ if (ρ is contained in the wellfounded part of both models, and) $\mathcal{P}^Q(\xi) = \mathcal{P}^N(\xi)$ for each $\xi < \rho$. Q and N *agree past* an ordinal κ if they agree to $\kappa + 1$.

1.7 Claim. *Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree past κ. Then E is a pre-extender over Q.*

Extenders are naturally induced by elementary embeddings. Let $\pi \colon V \to M$ be a non-trivial elementary embedding of V into some wellfounded class model M. Let κ be the critical point of π, namely the first ordinal moved by π. Let $\lambda \leq \pi(\kappa)$ be an ordinal closed under Gödel pairing. Define the λ-*restriction* of π to be the map E given by:

(R1) $\mathrm{dom}(E) = \mathcal{P}(\kappa)$.

(R2) $E(X) = \pi(X) \cap \lambda$ for each $X \in \mathrm{dom}(E)$.

It is then easy to check that E is an extender. The items in condition (4) of Definition 1.2 follow directly from the elementarity of π and, in the case of condition (4e), the absoluteness between M and V of formulae with only bounded quantifiers. Condition (5) follows from the elementarity of π and the *wellfoundedness* of M. If a fiber through $\langle E(A_i) \mid i < \omega\rangle$ exists in V then using the wellfoundedness of M such a fiber must also exist in M. Its existence can then be pulled back via π to yield a fiber through $\langle A_i \mid i < \omega\rangle$.

1.8 Remark. The λ-restriction makes sense also in the case of an embedding into an illfounded model M, so long as the wellfounded part of M contains λ. But countable completeness may fail in this case, and the λ-restriction need only be a pre-extender.

The description above shows how extenders are induced by elementary embeddings into wellfounded models. Extenders also give rise to such elementary embeddings, through the ultrapower construction, which we describe next.

Let ZFC^- consist of the standard axioms of ZFC excluding the powerset axiom. Fix a model Q of ZFC^- and a pre-extender E over Q. Let $\kappa = \mathrm{crit}(E)$. Let $\mathcal{F}$ be the class of functions $f \in Q$ so that $\mathrm{dom}(f) \subseteq \kappa$. Let $\mathcal{D} = \{\langle f, a\rangle \mid f \in \mathcal{F} \wedge a \in E(\mathrm{dom}(f))\}$.

For two functions $f, g \in \mathcal{F}$ set $Z^{=}_{f,g} = \{(\alpha, \beta) \mid f(\alpha) = g(\beta)\}$ and $Z^{\in}_{f,g} = \{(\alpha, \beta) \mid f(\alpha) \in g(\beta)\}$. Both $Z^{=}_{f,g}$ and $Z^{\in}_{f,g}$ are then subsets of κ in Q, and therefore elements of the domain of E.

Define a relation $\sim$ on $\mathcal{D}$ by setting $\langle f, a\rangle \sim \langle g, b\rangle$ iff $(a, b) \in E(Z^{=}_{f,g})$. One can check using condition (4) in Definition 1.2 that $\sim$ is an equivalence relation. Let $[f, a]$ denote the equivalence class of $\langle f, a\rangle$. Let $\mathcal{D}^*$ denote $\mathcal{D}/{\sim}$. Define a relation R on $\mathcal{D}^*$ by setting $[f, a]\ R\ [g, b]$ iff $(a, b) \in E(Z^{\in}_{f,g})$. Again using condition (4) in Definition 1.2 one can check that R is well defined.

The following property, known as Łoś's Theorem, can be proved from the various definitions, by induction on the complexity of φ:

1.9 Theorem (Łoś). *Let $[f_1, a_1], \ldots, [f_n, a_n]$ be elements of $\mathcal{D}^*$. Let $\varphi = \varphi(v_1, \ldots, v_n)$ be a formula. Let Z be the set*

$$\{(\alpha_1, \ldots, \alpha_n) \mid Q \models \varphi[f_1(\alpha_1), \ldots, f_n(\alpha_n)]\}.$$

Then $(\mathcal{D}^*, R) \models \varphi[[f_1, a_1], \ldots, [f_n, a_n]]$ iff $(a_1, \ldots, a_n)$ belongs to $E(Z)$.

For each set x let c_x be the function with domain $\{0\}$ and value $c_x(0) = x$. From Łoś's Theorem it follows that the map $x \mapsto [c_x, 0]$ is elementary, from Q into $(\mathcal{D}^*, R)$. (In particular then $(\mathcal{D}^*, R)$ satisfies ZFC^-.)

1.10 Definition. The *ultrapower* of Q by E, denoted $\mathrm{Ult}(Q, E)$, is the structure $(\mathcal{D}^*; R)$. The *ultrapower embedding* is the map $j\colon Q \to \mathrm{Ult}(Q, E)$ defined by $j(x) = [c_x, 0]$.

$$\begin{tikzcd} & M \\ V \arrow[ur, "\pi"] \arrow[r, "j"'] & N = \mathrm{Ult}(V,E) \arrow[u, "k"'] \end{tikzcd}$$

Diagram 2: The original map π and the ultrapower map j.

In general $\mathrm{Ult}(Q, E)$ need not be wellfounded. (If it is then we of course identify it with its transitive collapse, and identify R with $\in$.) But notice that wellfoundedness is a consequence of countable completeness: if $\langle [f_i, a_i] \mid i < \omega\rangle$ is an infinite descending sequence in R, then the sequence of sets $A_i = \{(\alpha_0, \ldots, \alpha_{i-1}) \mid f_0(\alpha_0) \ni f_1(\alpha_1) \ni \ldots f_{i-1}(\alpha_{i-1})\}$ violates countable completeness. Ultrapowers by extenders, as opposed to mere pre-extenders, are therefore wellfounded.

Let $\lambda = \mathrm{spt}(E)$. Using the various definitions one can prove the following two properties of the ultrapower. The first relates the ultrapower embedding back to the extender E, and the second describes a certain minimality of the ultrapower:

(U1) The λ-restriction of j is precisely equal to E.

(U2) Every element of $\mathrm{Ult}(Q, E)$ has the form $j(f)(a)$ for some function $f \in \mathcal{F}$ and some $a \in \lambda$.

These properties determine the ultrapower and the embedding completely.

The following lemma relates an embedding $\pi \colon V \to M$ to the ultrapower embedding by the extender over V derived from π. It shows that the ultrapower by the λ-restriction of π captures π up to λ.

1.11 Lemma. *Let $\pi \colon V \to M$ be an elementary embedding of V into a wellfounded model M, and let $\kappa = \mathrm{crit}(\pi)$. Let $\lambda \le \pi(\kappa)$ be an ordinal closed under Gödel pairing. Let E be the λ-restriction of π. Let $N = \mathrm{Ult}(V, E)$ and let $j \colon V \to N$ be the ultrapower embedding.*

Then there is an elementary embedding $k \colon N \to M$ with $\pi = k \circ j$ (see Diagram 2) and $\mathrm{crit}(k) \ge \lambda$. ⊣

1.12 Exercise. Let μ be a two-valued measure over a cardinal κ. Let $\mathcal{F}$ be the class of functions from κ into V. For $f, g \in \mathcal{F}$ set $f \sim g$ iff $\{\xi < \kappa \mid f(\xi) = g(\xi)\}$ has measure one. Show that $\sim$ is an equivalence relation. Let $\mathcal{F}^* = \mathcal{F}/\sim$. For $f \in \mathcal{F}$ let $[f]$ denote the equivalence class of f. Define a relation R on $\mathcal{F}^*$ by $[f]\ R\ [g]$ iff $\{\xi < \kappa \mid f(\xi) \in g(\xi)\}$ has measure one. Show that R is well defined.

Define $\mathrm{Ult}(V,\mu)$, the *ultrapower* of V by μ, to be the structure $(\mathcal{F}^*; R)$, and define the *ultrapower embedding* $j\colon V \to \mathrm{Ult}(V,\mu)$ by $j(x) = [c_x]$ where $c_x\colon \kappa \to V$ is the constant function which takes the value x.

Show that ultrapower embedding is elementary. Show that if μ is countably complete, meaning that $\mu(\bigcap_{n<\omega} X_n) = 1$ whenever $\langle X_n \mid n < \omega\rangle$ is a sequence of sets of measure one, then the ultrapower is wellfounded.

1.13 Exercise. The *seed* of a measure μ is the element $[id]$ of the ultrapower, where $id\colon \kappa \to V$ is the identity function. Let s be the seed of μ. Prove that every element of $\mathrm{Ult}(V,\mu)$ has the form $j(f)(s)$, where $j\colon V \to \mathrm{Ult}(V,\mu)$ is the ultrapower embedding.

1.14 Exercise. A (two-valued) measure μ over a set U is called *non-principal* just in case that $\mu(\{\xi\}) = 0$ for each singleton $\{\xi\}$. μ is *κ-complete* if $\mu(\bigcap_{\alpha<\tau} X_\alpha) = 1$ whenever $\tau < \kappa$ and $X_\alpha \subseteq U$ $(\alpha < \tau)$ are all sets of measure one. A cardinal κ is called *measurable* if there is a two-valued, non-principal, κ-complete measure over κ. Let κ be measurable, let μ witness this, and let $j\colon V \to \mathrm{Ult}(V,\mu)$ be the ultrapower embedding. Show that $\mathrm{crit}(j) = \kappa$.

1.15 Exercise. Let κ be measurable and let μ witness this. Let $M = \mathrm{Ult}(V,\mu)$. Prove that $\mathcal{P}(\kappa) \subseteq M$, and that $\mathcal{P}(\mathcal{P}(\kappa)) \not\subseteq M$.

Hint. To see that $\mathcal{P}(\kappa) \subseteq M$, note that $j(X) \cap \kappa = X$ for each $X \subseteq \kappa$ (where $j\colon V \to M$ is the ultrapower embedding).

To see that $\mathcal{P}(\mathcal{P}(\kappa)) \not\subseteq M$, prove that $\mu \notin M$: Suppose for contradiction that $\mu \in \mathrm{Ult}(V,\mu)$. Without loss of generality you may assume that κ is the smallest cardinal carrying a measure μ with $\mu \in \mathrm{Ult}(V,\mu)$. Derive a contradiction to the analogous minimality of $j(\kappa)$ in M by showing that $\mu \in \mathrm{Ult}(M,\mu)$. ⊣

1.16 Definition. An embedding $\pi\colon V \to M$ is *α-strong* just in case that $\mathcal{P}(\xi) \subseteq M$ for all $\xi < \alpha$. An extender E is *α-strong* just in case that $\mathcal{P}(\xi) \subseteq \mathrm{Ult}(V,E)$ for all $\xi < \alpha$. The *strength* of $\pi\colon V \to M$ is defined to be the largest α so that π is α-strong. The strength of an extender E is defined similarly, using the ultrapower, and is denoted $\mathrm{Strength}(E)$. (Notice that the strength of an embedding is always a cardinal.) An embedding π with critical point κ is *superstrong* if it is $\pi(\kappa)$-strong. A cardinal κ is *α-strong* if it is the critical point of an α-strong embedding, and *superstrong* if it is the critical point of a superstrong embedding.

Measurable cardinals lie at the low end of the hierarchy of strength: assuming the GCH, an ultrapower embedding by a measure on κ is κ^+-strong and no more. Superstrong embeddings lie much higher in the hierarchy. These embedding are the most we can hope to capture using (short) extenders:

1.17 Lemma. *Let E be a (short) extender with critical point κ. Let j be the ultrapower embedding by E. Then E is at most $j(\kappa)$-strong.*

Proof. Using the ultrapower construction and the elementarity of j, one can see that every element x of $j(\kappa^+)$ has the form $j(f)(a)$ for a function $f\colon \kappa \to \kappa^+$ and an $a \in \mathrm{dom}(j(f)) = j(\kappa)$. (The fact that f can be taken to have domain κ traces back to the fact that the domain of E consists precisely of the subsets of its critical point, in other words to the fact that E is a *short* extender.) It follows that $j(\kappa^+)$ has cardinality at most $\theta = (\kappa^+)^\kappa \cdot j(\kappa)$. If j is $j(\kappa)$-strong then $j(\kappa)$ is a strong limit cardinal in V, and a quick calculation shows that $\theta = j(\kappa)$. Thus $j(\kappa^+) = (j(\kappa)^+)^{\mathrm{Ult}(V,E)}$ has cardinality $j(\kappa)$ in V, and from this it follows that $\mathrm{Ult}(V, E)$ must be missing some subsets of $j(\kappa)$. So E is not $j(\kappa) + 1$-strong. ⊣

1.18 Lemma. *Let $\pi\colon V \to M$ with critical point κ. Suppose that π is α-strong where $\alpha \leq \pi(\kappa)$. Let $\lambda \leq \pi(\kappa)$ be an ordinal closed under Gödel pairing and such that $\lambda \geq (2^{<\alpha})^M$. Then the λ-restriction of π is an α-strong extender.*

Proof. Immediate from Lemma 1.11. ⊣

Lemma 1.18 shows that (short) extenders are adequate means for capturing the strength of embeddings at or below the level of superstrong. On the other hand Lemma 1.17 shows that (short) extenders cannot capture embeddings beyond superstrong. Such stronger embeddings can be captured using the general extenders mentioned in Remark 1.3, but for our purpose in this chapter the greater generality is not necessary.

1.19 Definition. We write $Q\|\alpha$ to denote V_α^Q. We say that Q and N *agree well beyond* κ if the first inaccessible above κ is the same in both Q and N, and, letting $\alpha > \kappa$ be this inaccessible, $Q\|\alpha = N\|\alpha$. Given further embeddings $i\colon Q \to Q^*$ and $j\colon N \to N^*$ we say that i and j *agree well beyond* κ if $i{\restriction}(Q\|\alpha \cup \{\alpha\}) = j{\restriction}(N\|\alpha \cup \{\alpha\})$.

We shall use the notion of Definition 1.19 as an all purpose security blanket, giving us (more than) enough room in several arguments below.

1.20 Claim. *Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree well beyond κ, so that (in particular) E is a pre-extender over Q. Let i be the ultrapower embedding of Q by E, and let j be the ultrapower embedding of N by E. Then i and j agree well beyond κ, and* $\mathrm{Ult}(Q, E)$ *and* $\mathrm{Ult}(N, E)$ *agree well beyond $i(\kappa) = j(\kappa)$.*

Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree well beyond κ, so that in particular E is a pre-extender over Q.

$$\begin{tikzcd}
Q^* \arrow[rr, bend left, "i^*"] & E^* \in N^* & \mathrm{Ult}(Q^*, E^*) \\
Q \arrow[u, "\pi"] \arrow[rr, bend right, "i"'] & E \in N \arrow[u, "\sigma"] & \mathrm{Ult}(Q, E) \arrow[u, "\tau"]
\end{tikzcd}$$

Diagram 3: Copying the ultrapower of Q by E to an ultrapower of Q^* by E^*.

Let $\pi\colon Q \to Q^*$ and $\sigma\colon N \to N^*$ be elementary. Let $E^* = \sigma(E)$. Suppose that π and σ agree well beyond κ. Hence in particular Q^* and N^* agree well beyond $\pi(\kappa) = \sigma(\kappa)$, and E^* is therefore a pre-extender over Q^*. The models and embeddings are presented in Diagram 3.

For an element $x = [f, a]$ of $\mathrm{Ult}(Q, E)$ define $\tau(x)$ to be the element $[\pi(f), \sigma(a)]$ of $\mathrm{Ult}(Q^*, E^*)$.

Then τ is a well defined (meaning invariant under the choice of representatives for $x \in \mathrm{Ult}(Q, E)$) elementary embedding from $\mathrm{Ult}(Q, E)$ into $\mathrm{Ult}(Q^*, E^*)$; $\tau \restriction \mathrm{spt}(E) = \sigma \restriction \mathrm{spt}(E)$; and τ makes Diagram 3, with i and i^* being the relevant ultrapower embeddings, commute.

The ultrapower of Q^* by E^* is called the *copy*, via the pair $\langle \pi, \sigma \rangle$, of the ultrapower of Q by E. τ is called the *copy embedding*. Note that the definition of τ involves both π and σ, and the agreement between these two embeddings is important for the proof that τ is well defined.

1.21 Remark. Recall that every element of $\mathrm{Ult}(Q, E)$ has the form $i(f)(a)$ for a function $f \in Q$ and an ordinal $a \in \mathrm{spt}(E)$. The copy embedding τ is characterized completely by the condition $\tau(i(f)(a)) = (i^* \circ \pi)(f)(\sigma(a))$ for all f and a.

Next we describe how to repeatedly form ultrapowers by extenders, to obtain a chain, or a tree, of models. For the record let us start by defining direct limits.

1.22 Definition. Let $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \alpha \rangle$ be a system of models M_ξ and elementary embeddings $j_{\zeta,\xi}\colon M_\zeta \to M_\xi$, commuting in the natural way. Let $\mathcal{D} = \{\langle \xi, x \rangle \mid \xi < \alpha, x \in M_\xi\}$.

Define an equivalence relation $\sim$ on $\mathcal{D}$ by setting $\langle \xi, x \rangle \sim \langle \xi', x' \rangle$ iff $j_{\xi,\nu}(x) = j_{\xi',\nu}(x)$ where $\nu = \max\{\xi, \xi'\}$. Let $\mathcal{D}^* = \mathcal{D}/\sim$.

Define a relation R on $\mathcal{D}^*$ by setting $[\xi, x]\ R\ [\xi', x']$ iff $j_{\xi,\nu}(x) \in j_{\xi',\nu}(x)$ where again $\nu = \max\{\xi, \xi'\}$. It is easy to check that R is well defined.

The structure $M^* = (\mathcal{D}; R)$ is called the *direct limit* of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \alpha \rangle$. The embeddings $j_{\xi,*}\colon M_\xi \to M^*$ determined by

$j_\xi(x) = [\xi, x]$ are called the *direct limit embeddings*. It is easy to check that these embeddings commute with the embeddings $j_{\zeta,\xi}$ in the natural way.

1.23 Remark. If $(\mathcal{D}^*; R)$ is wellfounded then we identify it with its transitive collapse, and identify R with $\in$.

We pass now to the matter of iterated ultrapowers.

1.24 Definition. A *tree order* is an order T on an ordinal α so that:

1. T is a suborder of $<\restriction(\alpha \times \alpha)$.
2. For each $\eta < \alpha$, the set $\{\xi \mid \xi \; T \; \eta\}$ is linearly ordered by T.
3. For each ξ so that $\xi + 1 < \alpha$, the ordinal $\xi + 1$ is a successor in T.
4. For each limit ordinal $\gamma < \alpha$, the set $\{\xi \mid \xi \; T \; \gamma\}$ is cofinal in γ.

1.25 Definition. An *iteration tree* $\mathcal{T}$ of length α on a model M consists of a tree order T on α and a sequence $\langle E_\xi \mid \xi + 1 < \alpha \rangle$, so that the following conditions hold with an additional sequence $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \; T \; \xi < \alpha \rangle$ which is determined completely by the conditions:

1. $M_0 = M$.
2. For each ξ so that $\xi + 1 < \alpha$, E_ξ is an extender of M_ξ, or $E_\xi =$"pad."
3. (a) If $E_\xi =$"pad" then $M_{\xi+1} = M_\xi$, the T-predecessor of $\xi + 1$ is ξ, and $j_{\xi,\xi+1}$ is the identity.
   (b) If $E_\xi \neq$"pad" then $M_{\xi+1} = \mathrm{Ult}(M_\zeta, E_\xi)$ and $j_{\zeta,\xi+1} \colon M_\zeta \to M_{\xi+1}$ is the ultrapower embedding, where ζ is the T-predecessor of $\xi+1$. It is implicit in this condition that M_ζ must agree with M_ξ past $\mathrm{crit}(E_\xi)$, so that E_ξ is a pre-extender over M_ζ by Claim 1.7.
4. For limit $\lambda < \alpha$, M_λ is the direct limit of the system $\langle M_\zeta, j_{\zeta,\xi} \mid \zeta \; T \; \xi \; T \; \lambda \rangle$, and $j_{\zeta,\lambda} \colon M_\zeta \to M_\lambda$ for $\zeta \; T \; \lambda$ are the direct limit embeddings.
5. The remaining embeddings $j_{\zeta,\xi}$ for $\zeta \; T \; \xi < \alpha$ are obtained through composition.

M_ξ and $j_{\zeta,\xi}$ for $\zeta \; T \; \xi < \alpha$ are the *models* and *embeddings of* $\mathcal{T}$. We view them as part of $\mathcal{T}$, though formally they are not.

1.26 Remark. The inclusion of pads in iteration tree is convenient for purposes of indexing in various constructions, and we shall use it later on. But for much of the discussion below we make the implicit assumption that the iteration tree considered has no pads. This assumption poses no loss of generality.

$$\begin{array}{ccc} M_{n+1} & & \\ \uparrow{\scriptstyle j_{k,n+1}} & & E_n \in M_n \\ M_k & & \end{array}$$

Diagram 4: Forming M_{n+1}

We shall only need iteration trees of length ω in this chapter. We shall construct these trees recursively. In stage n of the construction we shall have the models $M_0, \ldots, M_n$. During the stage we shall pick an extender E_n in M_n, and pick further some $k \leq n$ so that M_k and M_n agree past $\mathrm{crit}(E_n)$. We shall then set k to be the T-predecessor of $n+1$ and set $M_{n+1} = \mathrm{Ult}(M_k, E_n)$. This is illustrated in Diagram 4. After ω stages of a construction of this kind we obtain an iteration tree of length ω.

A *branch* through an iteration tree $\mathcal{T}$ is a set b which is linearly ordered by T. The branch is *cofinal* if $\sup(b) = \mathrm{lh}(\mathcal{T})$. By the *direct limit* along b, denoted $M_b^{\mathcal{T}}$ or simply M_b, we mean the direct limit of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \; T \; \xi \in b\rangle$. We use $j^{\mathcal{T}}_{\zeta,b}$, or simply $j_{\zeta,b}$, to denote the direct limit embeddings of this system.

The branch b is called *wellfounded* just in case that the model M_b is wellfounded.

2. Iterability

The existence of wellfounded cofinal branches through certain iteration trees is crucial to proofs of determinacy. This existence is part of the general topic of iterability. In this section we briefly describe the topic, point out its most important open problem, and sketch a proof of the specific iterability necessary for the determinacy results in this chapter.

Let M be a model of ZFC^-. In the *(full) iteration game* on M players "good" and "bad" collaborate to construct an iteration tree $\mathcal{T}$ of length $\omega_1^V + 1$ on M. "bad" plays all the extenders, and determines the T-predecessor of $\xi+1$ for each ξ. "good" plays the branches $\{\zeta \mid \zeta \; T \; \lambda\}$ for limit λ, thereby determining the T-predecessors of λ and the direct limit model M_λ. Note that "good" is also responsible for the final move, which determines $M_{\omega_1^V}$.

If ever a model along the tree is reached which is illfounded then "bad" wins. Otherwise "good" wins. M is *(fully) iterable* if "good" has a winning strategy in this game. An *iteration strategy* for M is a strategy for the good player in the iteration game on M. The *Strategic Branch Hypothesis* (SBH) asserts that every countable model which embeds into a rank initial segment

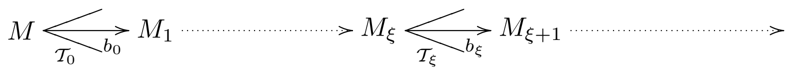

Diagram 5: A weak iteration of M.

of V is iterable.

As stated the hypothesis is more general than necessary. The iteration trees that come up in applications follow a specific format, and only the restriction of SBH to trees of such format is needed.

Call an iteration tree $\mathcal{T}$ on M *nice* if:

1. The extenders used in $\mathcal{T}$ have increasing strengths. More precisely, $\langle \mathrm{Strength}^{M_\xi}(E_\xi) \mid \xi + 1 < \mathrm{lh}(\mathcal{T})\rangle$ is strictly increasing.

2. For each ξ, $\mathrm{Strength}^{M_\xi}(E_\xi)$ is inaccessible in M_ξ.

3. For each ξ, $\mathrm{spt}(E_\xi) = \mathrm{Strength}^{M_\xi}(E_\xi)$.

2.1 Remark. Throughout this chapter, whenever a result claims the existence of an iteration tree, the iteration tree is nice. In the later sections we often neglect to mention this explicitly.

A model N is λ-*closed* if every subset of N of size λ in V belongs to N.

2.2 Exercise. Let $\mathcal{T}$ be a nice, finite iteration tree on V. Prove that each of the models in $\mathcal{T}$ is countably closed, and conclude from this that each of the models in $\mathcal{T}$ is wellfounded. Prove further that each of the models in $\mathcal{T}$ is $2^{\aleph_0}$-closed.

Hint. Prove the general fact that if $Q \models$"E is an extender with inaccessible support," N agrees with Q past the critical point of E, and both N and Q are countably (respectively $2^{\aleph_0}$) closed, then $\mathrm{Ult}(N, E)$ is countably (respectively $2^{\aleph_0}$) closed. Wellfoundedness follows from countable closure, since by elementarity each of the models in $\mathcal{T}$ satisfies internally that "there are no infinite descending sequences of ordinals." $\dashv$

Call M *iterable for nice trees* if "good" has a winning strategy in the iteration game on M when "bad" is restricted to extenders which give rise to nice trees. Let nSBH be the assertion that every countable model which embeds elementarily into a rank initial segment of V is iterable for nice trees. nSBH is a technical weakening of SBH, sufficient for all known applications. A proof of nSBH would constitute a substantial breakthrough in the study of large cardinals, particularly in inner model theory.

For the sake of the determinacy proofs in this chapter we need only a weak form of iterability, involving linear compositions of trees of length ω.

$$\begin{tikzcd} V\|\theta & \\ M \arrow[u, "\pi"] \arrow[r, "j_b"', "\mathcal{T}"] & M_b \arrow[ul, "\sigma"'] \end{tikzcd}$$

Diagram 6: Theorem 2.3.

This iterability was proved by Martin–Steel [22]. We now proceed to state the iterability precisely, and give its proof.

A *weak iteration* of M of length α consists of objects M_ξ, $\mathcal{T}_\xi$, b_ξ for $\xi < \alpha$ and embeddings $j_{\zeta,\xi}\colon M_\zeta \to M_\xi$ for $\zeta < \xi < \alpha$, so that:

1. $M_0 = M$.
2. For each $\xi < \alpha$, $\mathcal{T}_\xi$ is a nice iteration tree of length ω on M_ξ; b_ξ is a cofinal branch through $\mathcal{T}_\xi$; $M_{\xi+1}$ is the direct limit along b_ξ; and $j_{\xi,\xi+1}\colon M_\xi \to M_{\xi+1}$ is the direct limit embedding along b_ξ.
3. For limit $\lambda < \alpha$, M_λ is the direct limit of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \lambda\rangle$ and $j_{\zeta,\lambda}\colon M_\zeta \to M_\lambda$ are the direct limit embeddings.
4. The remaining embeddings $j_{\zeta,\xi}$ are obtained by composition.

A weak iteration is thus a linear composition of length ω iteration trees.

In the *weak iteration game* on M players “good” and “bad” collaborate to produce a weak iteration of M, of length ω_1^V. “Bad” plays the iteration trees $\mathcal{T}_\xi$ and “good” plays the branches b_ξ. (These moves determine the iteration completely.) If ever a model M_ξ, $\xi < \omega_1$, is reached which is illfounded, then “bad” wins. Otherwise “good” wins. M is *weakly iterable* if “good” has a winning strategy in the weak iteration game on M.

2.3 Theorem. *Let $\pi\colon M \to V\|\theta$ be elementary with M countable. Let $\mathcal{T}$ be a nice iteration tree of length ω on M. Then there is a cofinal branch b through $\mathcal{T}$, and an embedding $\sigma\colon M_b \to V\|\theta$, so that $\sigma \circ j_b = \pi$. (Note that b is then a wellfounded branch, since M_b embeds into $V\|\theta$.)*

2.4 Corollary. *Let $\pi\colon M \to V\|\theta$ be elementary with M countable. Then “good” has a winning strategy in the weak iteration game on M.*

Proof. Immediate through iterated applications of Theorem 2.3. The good player should simply keep choosing branches given by the theorem, successively embedding each $M_{\xi+1}$ into $V\|\theta$, and preserving commutativity which is needed for the limits.

The idea of proving iterability by embedding back into V, simple only in retrospect, was first used by Jensen in the context of linear iterations. $\dashv$

Theorem 2.3 and Corollary 2.4 provide the iterability necessary for the determinacy proofs in this chapter. In the remainder of this section we give the proof of the theorem.

2.5 Definition. Let $\mathcal{T}$ be a nice iteration tree of length ω on a model M, giving rise to models and embeddings $\langle M_m, j_{m,n} \mid m\ T\ n < \omega\rangle$. $\mathcal{T}$ is *continuously illfounded* if there exists a sequence of ordinals $\alpha_n \in M_n$ $(n < \omega)$ so that $j_{m,n}(\alpha_m) > \alpha_n$ whenever $m\ T\ n$.

Note that a continuously illfounded iteration tree has *no* wellfounded cofinal branches. Indeed, for any cofinal branch b, the sequence $j_{n,b}(\alpha_n)$ for $n \in b$ witnesses that M_b is illfounded. Continuously illfounded iteration trees, on countable models M which embed into rank initial segments of V, thus contradict Theorem 2.3 in a very strong way. We begin by showing that in fact any counterexample to Theorem 2.3 gives rise to a continuously illfounded iteration tree.

2.6 Lemma. *Let $\pi\colon M \to V\|\theta$ be elementary with M countable. Let $\mathcal{T}$ be a nice iteration tree of length ω on M, and suppose that the conclusion of Theorem 2.3* fails *for $\mathcal{T}$. Then there is a continuously illfounded nice iteration tree on V.*

Proof. Let E_n, M_n, and $j_{m,n}$ $(m\ T\ n < \omega)$ denote the extenders, models, and embeddings of $\mathcal{T}$. Working recursively define a length ω iteration tree $\mathcal{T}^*$ on V, and embeddings $\pi_n\colon M_n \to M_n^*$ through the conditions:

- $M_0^* = V$ and $\pi_0 = \pi$.
- $E_n^* = \pi_n(E_n)$.
- The T^*-predecessor of $n+1$ is the same as the T-predecessor of $n+1$.
- $M_{n+1}^* = \mathrm{Ult}(M_k^*, E_n^*)$ where k is the T-predecessor of $n+1$, and π_{n+1} is the copy embedding via the pair $\langle \pi_k, \pi_n\rangle$.

It is easy to check that this definition goes through, giving rise to a nice iteration tree $\mathcal{T}^*$ and the commuting diagram presented in Diagram 7. We will show that $\mathcal{T}^*$ is continuously illfounded.

2.7 Definition. The tree $\mathcal{T}^*$ defined through the conditions above is the *copy* of $\mathcal{T}$ via $\pi\colon M \to V$. It is denoted $\pi\mathcal{T}$.

From the fact that M is countable it follows that each M_n is countable. Let $\vec{e}^{\,n} = \langle e_l^n \mid l < \omega\rangle$ enumerate M_n. Given an embedding σ with domain M_n, we use $\sigma{\restriction}l$ to denote the restriction of σ to $\{e_0^n, \dots, e_{l-1}^n\}$, and we write $M_n{\restriction}l$ to denote $\{e_0^n, \dots, e_{l-1}^n\}$.

Working in V let R be the tree of attempts to create a cofinal branch b through $\mathcal{T}$ and a commuting system of embeddings realizing the models

$$\begin{tikzcd}
\mathcal{T}^* & V \arrow[r, dotted, no head] & M_k^* \arrow[rr, bend left, "j^*_{k,n_1}"] & M_n^* & M_{n+1}^* \arrow[r, dotted, no head] & {} \\
\mathcal{T} & M_0 \arrow[u, "\pi"'] \arrow[r, dotted, no head] & M_k \arrow[u, "\pi_k"'] \arrow[rr, bend right, "j_{k,n+1}"'] & M_n \arrow[u, "\pi_n"'] & M_{n+1} \arrow[u, "\pi_{n+1}"'] \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

Diagram 7: $\mathcal{T}$ and $\mathcal{T}^*$.

along b into V. More precisely, a node in R consists of a finite branch a through T, and of partial embeddings $\sigma_i\colon M_i \to V$, $i \in a$, satisfying the following conditions (where l is the length of a):

- For each i the domain of σ_i is precisely $M_i \restriction l$.
- (Commutativity) If $i \mathrel{T} i' \in a$, $x \in M_i \restriction l$, $x' \in M_{i'} \restriction l$, and $x' = j_{i,i'}(x)$, then $\sigma_{i'}(x') = \sigma_i(x)$.
- σ_0 is equal to $\pi \restriction l$.

The tree R consists of these nodes, ordered naturally by extension for each component.

An infinite branch through R gives rise to a an infinite branch $b = \{n_0, n_1, \ldots\}$ through T, and an embedding σ_∞ of the direct limit along b into V, with the commutativity $\sigma_\infty \circ j_b = \pi$. Thus, an infinite branch through R produces precisely the objects b and σ necessary for the conclusion of Theorem 2.3.

The assumption of the current lemma is that $\mathcal{T}$ witnesses the failure of Theorem 2.3. The tree R must therefore have *no* infinite branches. Let $\varphi\colon R \to \mathrm{On}$ be a rank function, that is a function assigning to each node in R an ordinal, in such a way that if a node s' extends a node s then $\varphi(s') < \varphi(s)$. The existence of such a function follows from the fact that R has no infinite branches.

For each finite branch $a = \langle 0 = n_0 \mathrel{T} n_1 \ldots \mathrel{T} n_{l-1} \rangle$ through T, let s_a consist of a itself and the embeddings $(\pi_{n_{l-1}} \circ j_{n_i, n_{l-1}}) \restriction l$ for each $i < l$. Using the commutativity of Diagram 7 it is easy to check that s_a is a node in $j^*_{0,n_{l-1}}(R)$.

For $k < \omega$ let s_k be the node s_a where a is the branch of T ending at k. s_k is then a node in $j^*_{0,k}(R)$. For $k \mathrel{T} k'$ it is easy to check, again using the commutativity of Diagram 7, that $s_{k'}$ extends $j^*_{k,k'}(s_k)$.

Let $\alpha_k = j^*_{0,k}(\varphi)(s_k)$. This is the rank of the node s_k of $j^*_{0,k}(R)$ given by the shift of the rank function φ to M^*_k. From the fact that $s_{k'}$ extends

$j^*_{k,k'}(s_k)$ for $k \; T \; k'$ it follows that $\alpha_{k'} < j_{k,k'}(\alpha_k)$. The ordinals $\langle \alpha_k \mid k < \omega \rangle$ therefore witness that $\mathcal{T}^*$ is continuously illfounded. ⊣

2.8 Lemma. *Let $\mathcal{U}$ be a nice length ω iteration tree on V. Then $\mathcal{U}$ is* not *continuously illfounded.*

Proof. Suppose for contradiction that $\mathcal{U}$ is a nice, length ω, continuously illfounded iteration tree on V, and let $\langle \beta_n \mid n < \omega \rangle$ witness this. Let η be large enough that $\mathcal{U}$ belongs to $V \| \eta$. By replacing each β_n with the β_nth regular cardinal of $M^{\mathcal{U}}_n$ above $j^{\mathcal{U}}_{0,n}(\eta)$ we may assume that β_n is regular in $M^{\mathcal{U}}_n$ for each n, and larger than $j^{\mathcal{U}}_{0,n}(\eta)$.

Let θ be large enough that both $\mathcal{U}$ and $\langle \beta_n \mid n < \omega \rangle$ belong to $V \| \theta$. Let H be a countable Skolem hull of $V \| \theta$ with $\mathcal{U}$ and $\langle \beta_n \mid n < \omega \rangle$ elements of H. Let M be the transitive collapse of H and let $\pi \colon M \to V \| \theta$ be the anticollapse embedding. Let $\mathcal{T} = \pi^{-1}(\mathcal{U})$ and let $\langle \alpha_n \mid n < \omega \rangle = \pi^{-1}(\langle \beta_n \mid n < \omega \rangle)$. Then $\mathcal{T}$ is a nice, length ω, continuously illfounded iteration tree on M; $\langle \alpha_n \mid n < \omega \rangle$ witnesses this; for each n, α_n is regular in $M_n = M^{\mathcal{T}}_n$; and, for each n, $E_n = E^{\mathcal{T}}_n$ belongs to $M_n \| \alpha_n$. (The last clause follows from the fact that β_n is greater than $j^{\mathcal{U}}_{0,n}(\eta)$, obtained in the previous paragraph, and the fact that η was chosen large enough that $E^{\mathcal{U}}_n \in V \| j^{\mathcal{U}}_{0,n}(\eta)$.)

Let M_n, E_n, and $j_{m,n}$ ($m \; T \; n < \omega$) be the models and embeddings of $\mathcal{T}$. Let ρ_n be the strength of E_n in M_n. The sequence $\langle \rho_n \mid n < \omega \rangle$ is increasing, and for each $n < n^*$, M_n and M_{n^*} agree to ρ_n.

Let $P_0 = V \| \beta_0$ and let $\sigma_0 = \pi \restriction (M \| \alpha_0)$. We work by recursion to produce models P_n and embeddings σ_n satisfying the following conditions:

1. σ_n is elementary from $M_n \| \alpha_n$ into P_n.
2. σ_n belongs to P_n and is countable in P_n.
3. For $\bar{n} < n$, $\sigma_{\bar{n}}$ and σ_n agree on $M_{\bar{n}} \| \rho_{\bar{n}}$.

We shall construct so that:

(i) For each n, $P_{n+1} \in P_n$.

At the end of the construction we shall thus have an infinite $\in$-decreasing sequence, a contradiction.

We already have conditions (1) and (2) for $n = 0$, and condition (3) is vacuous for $n = 0$. Suppose inductively that we have conditions (1)–(3) for n. We describe how to construct P_{n+1} and σ_{n+1}.

Let k be the T-predecessor of $n+1$, so that M_{n+1} is the ultrapower of M_k by E_n. We wish to copy this ultrapower to an ultrapower of P_k via the pair $\langle \sigma_k, \sigma_n \rangle$. We cannot quite manage this, since the domain of σ_k is $M_k \restriction \alpha_k$ rather than M_k. We adjust our wishes as follows: Let $\gamma = j_{k,n+1}(\alpha_k)$. $M_{n+1} \| \gamma$ is then the ultrapower of $M_k \| \alpha_k$ by E_n. Now let P^*_n be the copy of this ultrapower via the pair $\langle\sigma_k, \sigma_n\rangle$, and let $\sigma_n^*\colon M_{n+1}\|\gamma \to P_n^*$ be the copy embedding.

We would have liked to simply set $P_{n+1} = P_n^*$ and σ_{n+1} equal to the restriction of σ_n^* to $M_{n+1}\|\alpha_{n+1}$. There are two problems with this desire. First, P_n^* does not belong to P_n, so we lose condition (i), the crucial condition in our scheme for a contradiction. Second, σ_n^* does not belong to P_n^*, so we lose condition (2). We handle the second problem first.

2.9 Claim. *Let τ denote the restriction of σ_n^* to $M_{n+1}\|\rho_n$. Then τ belongs to P_n^*.*

Proof. Let φ_n denote $\sigma_n(\rho_n)$. Let F_n denote $\sigma_n(E_n)$.

P_n^* is the ultrapower of P_k by F_n. F_n is φ_n-strong in P_n. It follows that P_n^* and P_n agree to φ_n.

The definition of copy embeddings is such that σ_n^* and σ_n agree on the support of E_n. This support must contain ρ_n, since otherwise E_n could not be ρ_n-strong. σ_n^* and σ_n thus agree on ρ_n. By condition (2) and the inaccessibility of φ_n in P_n, $\sigma_n\restriction\rho_n$ belongs to $P_n\restriction\varphi_n$. Since P_n and P_n^* agree to φ_n, $\sigma_n\restriction\rho_n$ belongs to P_n^*. Now σ_n^* is the same as σ_n up to ρ_n, so $\sigma_n^*\restriction\rho_n$ belongs to P_n^*. From this, using the inaccessibility of ρ_n in M_{n+1}, one can argue that $\sigma_n^*\restriction(M_{n+1}\|\rho_n)$ belongs to P_n^*. $\dashv$

Let $\alpha_n^* = \sigma_n^*(\alpha_{n+1})$. Notice that the definition makes sense, as α_{n+1} is smaller than $\gamma = j_{k,n+1}(\alpha_k)$, and therefore belongs to the domain of σ_n^*.

2.10 Claim. *There is an elementary embedding $\sigma_n^{**}\colon M_{n+1}\|\alpha_{n+1} \to P_n^*\|\alpha_n^*$ so that:*

- *The restriction of σ_n^{**} to $M_{n+1}\|\rho_n$ is equal to τ.*
- *$\sigma_n^{**}(\rho_n) = \varphi_n$.*
- *σ_n^{**} belongs to P_n^* and is countable in P_n^*.*

Notice that σ_n^*, restricted to $M_{n+1}\|\alpha_{n+1}$, already satisfies the first two demands of the claim. Replacing it by an embedding σ_n^{**} that also satisfies the third demand solves our "second problem" mentioned above.

Proof of Claim 2.10. This is a simple matter of absoluteness. Using the fact that τ belongs to P_n^* we can put together, inside P_n^*, the tree of attempts to construct an embedding σ_n^{**} satisfying the demands of the claim. This tree of attempts has an infinite branch in V, given by the restriction of σ_n^* to $M_{n+1}\|\alpha_{n+1}$. By absoluteness then it has an infinite branch inside P_n^*. $\dashv$

Let $P_n^{**} = P_n^*\|\alpha_n^*$. Note that P_n^{**} is then a *strict* rank initial segment of P_n^*, ultimately because $\alpha_{n+1} < j_{k,n+1}(\alpha_n)$.

Taking $P_{n+1} = P_n^{**}$ and $\sigma_{n+1} = \sigma_n^{**}$ would satisfy conditions (1)–(3). But we need one more adjustment to obtain condition (i), the crucial condition in our scheme for a contradiction. This final adjustment hinges on the fact that P_n^{**} is a strict initial segment of P_n^*, and therefore an element of P_n^*. Let H be the Skolem hull of $P_n^{**}\|\varphi_n \cup \{\varphi_n, \sigma_n^{**}\}$ inside P_n^{**}. Let P_{n+1} be the transitive collapse of H, and let $j\colon P_{n+1} \to H$ be the anticollapse embedding. Let $\sigma_{n+1} = j^{-1} \circ \sigma_n^{**}$. It is easy to check that conditions (1)–(3) hold with these assignments.

Since P_n^{**} and σ_n^{**} belong to P_n^*, the Skolem hull H taken above has cardinality φ_n *inside* P_n^*. It follows that P_{n+1} can be coded by a subset of φ_n inside P_n^*. Now P_n^* is equal to $\mathrm{Ult}(P_k, F_n)$. Since P_k and P_n agree well beyond the critical point of F_n, the ultrapowers $\mathrm{Ult}(P_k, F_n)$ and $\mathrm{Ult}(P_n, F_n)$ agree well beyond the image of this critical point (Claim 1.20). This image in turn is at least φ_n, that is the strength of F_n, since F_n is a short extender. (See Lemma 1.17.) It follows that all subsets of φ_n in $P_n^* = \mathrm{Ult}(P_k, F_n)$ belong also to $\mathrm{Ult}(P_n, F_n)$. Now $\mathrm{Ult}(P_n, F_n)$ can be computed over P_n (as $F_n \in P_n$). So all subsets of φ_n in P_n^* belong to P_n. We noted at the start of this paragraph that P_{n+1} can be coded by such a subset. So P_{n+1} belongs to P_n, and we have condition (i), as required. $\dashv$

Lemmas 2.6 and 2.8 combine to prove Theorem 2.3.

2.11 Remark. The contradiction in Lemma 2.8 is obtained through the very last adjustment in the proof, replacing P_n^{**} by a Skolem hull which belongs to P_n. It is crucial for that final adjustment that P_n^{**} is a strict rank initial segment of P_n^*, and this is where the continuous illfoundedness of $\mathcal{T}$ is used. The ordinals witnessing the continuous illfoundedness provide the necessary drops in rank.

2.12 Lemma. *Let $\mathcal{T}$ be a nice iteration tree of length ω on V. Then $\mathcal{T}$ has a cofinal branch leading to a wellfounded direct limit.*

Proof. Suppose not. For each cofinal branch b through $\mathcal{T}$ fix a sequence $\langle \alpha_n^b \mid n \in b\rangle$ witnessing that the direct limit along b is illfounded, more precisely satisfying $j_{m,n}(\alpha_m^b) > \alpha_n^b$ for all $m < n$ both in b. Let θ be large enough that all the ordinals α_n^b are smaller than θ.

For each $n < \omega$ let B_n be the set of cofinal branches b through $\mathcal{T}$ with $n \in b$. Let F_n be the set of functions from B_n into θ. Let $\prec$ be the following relation: $\langle n, f\rangle \prec \langle m, g\rangle$ iff $f \in F_n$, $g \in F_m$, $m\ T\ n$, and $f(b) < g(b)$ for every $b \in F_n$. The relation $\prec$ is wellfounded: if $\langle n_i, f_i \mid i < \omega\rangle$ were an infinite descending chain in $\prec$, then $\langle f_i(b) \mid i < \omega\rangle$, where b is the cofinal branch through $\mathcal{T}$ generated by $\{n_i \mid i < \omega\}$, would be an infinite descending sequence of ordinals.

For each $n < \omega$ let φ_n be the function $b \mapsto \alpha_n^b$, defined on $b \in B_n$, that is on branches b so that $n \in b$. By Exercise 2.2, each of the models M_n of $\mathcal{T}$ is $2^{\aleph_0}$-closed, and it follows that for each $n < \omega$, φ_n belongs to M_n. Let $\prec_n$ denote the relation $j_{0,n}(\prec)$. Using the fact that $j_{m,n}(\alpha_m^b) > \alpha_n^b$ for all b

and all $m < n$ both in b, it is easy to check that $\langle n, \varphi_n \rangle \prec_n \langle m, j_{m,n}(\varphi_m) \rangle$ whenever $m \ T \ n$. Letting γ_n be the rank of φ_n in $\prec_n$ it follows that $\gamma_n < j_{m,n}(\gamma_m)$ whenever $m \ T \ n$. But then the sequence $\langle \gamma_n \mid n < \omega \rangle$ is a witness that $\mathcal{T}$ is continuously illfounded, contradicting Lemma 2.8. ⊣

3. Creating Iteration Trees

The creation of iteration trees with non-linear tree orders is not a simple matter. Recall that the model M_{n+1} in an iteration tree $\mathcal{T}$ is created by picking an extender $E_n \in M_n$, picking $k \leq n$ so that M_k and M_n agree past $\mathrm{crit}(E_n)$, and setting $M_{n+1} = \mathrm{Ult}(M_k, E_n)$. The agreement between M_k and M_n is necessary for the ultrapower to make sense. The agreement can be obtained trivially by taking $k = n$. But doing this repeatedly would generate a linear iteration, that is an iteration with the simple tree order $0 \ T \ 1 \ T \ 2 \cdots$. For the creation of iteration trees with more complicated orders we need a way of ensuring that M_n has extenders with critical points within the level of agreement between M_n and previous models in the tree.

This section introduces the large cardinals and machinery that will allow us to create iteration trees with as complicated a tree order as we wish. The results here are due to Martin–Steel [21]. The terminology follows Neeman [31, §1A(1)].

3.1 Definition. u is called a *(κ, n)-type*, where κ is a limit ordinal and $n < \omega$, if u is a set of formulae involving n free variables $v_0 \ldots v_{n-1}$, a constant $\widetilde{\delta}$, and additional constants $\widetilde{c}$ for each $c \in V \| \kappa \cup \{\kappa\}$.

A (κ, n)-type can be coded by a subset of $(V\|\kappa)^{<\omega}$. Since κ is assumed to be a limit ordinal, $(V\|\kappa)^{<\omega} \subseteq V\|\kappa$. We may therefore view (κ, n)-types as subsets of $V\|\kappa$.

We refer to κ as the *domain* of u, denoted $\mathrm{dom}(u)$. For $\tau \leq \kappa$ and $m \leq n$, we let

$$\begin{aligned}\mathrm{proj}^m_\tau(u) = \{\, \phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{m-1}) \mid {} & k \in \mathbb{N},\ c_0, \ldots, c_k \in V\|\tau \cup \{\tau\}, \\ & \phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \in u, \text{ and } \phi \text{ makes no men-} \\ & \text{tion of } v_m, \ldots, v_{n-1}\}.\end{aligned}$$

We use $\mathrm{proj}_\tau(u)$ to denote $\mathrm{proj}^n_\tau(u)$, and $\mathrm{proj}^m(u)$ to denote $\mathrm{proj}^m_\kappa(u)$.

3.2 Definition. We say that a (κ, n)-type u is *realized (relative to δ) by $x_0, \ldots, x_{n-1}$ in $V\|\eta$* just in case that:

- $x_0, \ldots, x_{n-1}$ and δ are elements of $V\|\eta$.
- For any $k < \omega$, any $c_0, \ldots, c_k \in V\|\kappa \cup \{\kappa\}$, and any formula $\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1})$, $\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \in u$ if and only if $V\|\eta \models \phi[\delta, c_0, \ldots, c_k, x_0, \ldots, x_{n-1}]$. (Implicitly we must have $\eta > \kappa$ and $\eta > \delta$.)

We call u the *κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta$* (relative to δ) if u is the unique (κ, n)-type which is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$. A (κ, n)-type u is *realizable* (relative to δ) if it is realized by some $x_0, \ldots, x_{n-1}$ in some $V\|\eta$.

We often neglect to mention the set δ involved in the realization. In applications δ is usually fixed, and clear from the context.

3.3 Note. If u is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$, then $\mathrm{proj}^m_\tau(u)$ is realized by $x_0, \ldots, x_{m-1}$ in $V\|\eta$.

3.4 Definition. If the formula "there exists a largest ordinal," and the formula "$\widetilde{\kappa}, \widetilde{\delta}, v_0, \ldots, v_{n-1} \in V\|\nu$, where ν is the largest ordinal" are both elements of the (κ, n)-type u we define

$$\begin{aligned} u^- = \{\, \phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \mid\ & k \in \mathbb{N},\ c_0, \ldots, c_k \in V\|\kappa \cup \{\kappa\}, \\ & \text{and the formula "} V\|\nu \models \phi[\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}] \\ & \text{where } \nu \text{ is the largest ordinal" is an element of } u\}. \end{aligned}$$

3.5 Note. If $\kappa, \delta, x_0, \ldots, x_{n-1} \in V\|\eta$ and u is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta + 1$ then u^- is defined and is realized by the same $x_0, \ldots, x_{n-1}$ in $V\|\eta$.

3.6 Definition. Let u be a (κ, n)-type, and let w be a (τ, m)-type. We say that w is a *subtype* of u (and write $w < u$) if:

- $\tau < \kappa$.
- $m \geq n$.
- The formula "there is an ordinal ν and $v_n, \ldots, v_{m-1} \in V\|\nu$ such that $\widetilde{w}$ is realized by some permutation of $v_0, \ldots, v_{m-1}$ in $V\|\nu$" is an element of the type u.

3.7 Note. Let u be the κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta$. Then w is a subtype of u iff there is $\tau < \kappa$, $\nu < \eta$, $m \geq n$, and sets $x_n, \ldots, x_{m-1}$ so that w is the τ-type of some permutation of $x_0, \ldots, x_{m-1}$ in $V\|\nu$.

3.8 Remark. Definition 3.6 makes no mention of realizability but only stipulates that one particular formula belongs to u. It is immediate then that the property $w < u$ is absolute for any two models of set theory which have w and u as elements.

3.9 Definition. We say that a (τ, m)-type w *exceeds* the (κ, n)-type u, if:

- $\tau > \kappa$.
- $m \geq n$.
- There exist ordinals ν, η, and sets $x_0, \ldots, x_{m-1} \in V\|\nu$ such that

  - u is realized by $x_0, \ldots, x_{n-1}$ in $V \| \eta$,
  - w is realized by some permutation of $x_0, \ldots, x_{m-1}$ in $V \| \nu$, and
  - $\nu + 1 < \eta$.

ν, η, and $x_0, \ldots, x_{m-1}$ are said to witness the fact that w exceeds u.

3.10 Remark. The definition here is slightly more liberal than the corresponding definition in Neeman [31], where it is required that w be realized by $x_0, \ldots, x_{m-1}$ in their original order, not by a permutation of $x_0, \ldots, x_{m-1}$. A similar comment applies to Definition 3.6.

3.11 Note. Let u be the κ-type of $x_0, \ldots, x_{n-1}$ in $V \| \eta$. Suppose there is $\tau > \kappa$, ν with $\nu + 1 < \eta$, $m \geq n$, and $x_n, \ldots, x_{m-1}$ so that w is the τ-type of a permutation of $x_0, \ldots, x_{m-1}$ in $V \| \nu$. Then w exceeds u. This should be compared with Note 3.7. There τ is smaller than κ, and here τ must be larger than κ.

3.12 Definition. Let $\kappa < \lambda$ be ordinals, E a λ-strong extender with $\mathrm{crit}(E) = \kappa$ and u a type with $\mathrm{dom}(u) = \kappa$. Let $i_E \colon V \to \mathrm{Ult}(V, E)$ be the ultrapower embedding. We define $\mathrm{Stretch}^E_\lambda(u)$ to be equal to $\mathrm{proj}_\lambda(i_E(u))$.

$i_E(u)$ in Definition 3.12 is a type in $\mathrm{Ult}(V, E)$ with domain $i_E(\kappa)$. $i_E(\kappa)$ is at least as large as λ by Lemma 1.17, since E is λ-strong. So the projection to λ in Definition 3.12 makes sense.

3.13 Definition. A (κ, n)-type u is called *elastic* just in case that u^- is defined and u contains the following formulae:

- "$\widetilde{\delta}$ is an inaccessible cardinal."
- "Let ν be the largest ordinal. Then for all $\lambda < \widetilde{\delta}$ there exists an extender $E \in V \| \widetilde{\delta}$ such that
  - $\mathrm{crit}(E) = \widetilde{\kappa}$, $\mathrm{spt}(E) = \mathrm{Strength}(E)$, $\mathrm{Strength}(E)$ is an inaccessible cardinal greater than λ, and
  - $\mathrm{Stretch}^E_\lambda(u^-)$ is realized (relative to $\widetilde{\delta}$) by $v_0, \ldots, v_{n-1}$ in $V \| \nu$."

Formally the last clause should begin with "$\mathrm{Stretch}^E_\lambda(w)$, where w is the type of $v_0, \ldots, v_{n-1}$ in $V \| \nu$," instead of $\mathrm{Stretch}^E_\lambda(u^-)$, as u^- is not a parameter in formulae in u.

3.14 Remark. The requirements on support and of inaccessible strength in Definition 3.13 are not part of the definition in Neeman [31]. They are added in this chapter so as to make sure, later on, that our iteration trees are nice.

3.15 Remark. Definition 3.13 makes no mention of realizability but only stipulates that certain formulae belong to u. It is immediate then that the property of being elastic is absolute between models of set theory.

Ordinarily if u is realized by $x_0, \ldots, x_{n-1}$ in $V \| \nu + 1$ then $\mathrm{Stretch}^E_\lambda(u^-)$ is realized by $i_E(x_0), \ldots, i_E(x_{n-1})$ in $\mathrm{Ult}(V, E) \| i_E(\nu)$, and relative to $i_E(\delta)$. The demand in Definition 3.13 that it must also be realized by $x_0, \ldots, x_{n-1}$ in $V \| \nu$, and relative to δ, places a requirement of certain strength on the extender E. The existence of realizable elastic types is dependent on the existence of enough extenders with such strength.

3.16 Definition. Let H be a set. Let E be an extender and let $\kappa = \mathrm{crit}(E)$. Let j be the ultrapower embedding by E. Let $\alpha \leq j(\kappa)$. E is said to be *α-strong with respect to H* if (a) it is α-strong; and (b) $j(H \cap \kappa)$ and H agree to α, i.e., $j(H \cap \kappa) \cap \alpha = H \cap \alpha$.

A cardinal κ is said to be *α-strong with respect to H* if it is the critical point of an extender which is α-strong with respect to H.

A cardinal κ is said to be *$<\alpha$-strong with respect to H* if it is β-strong with respect to H for each $\beta < \alpha$.

3.17 Lemma. *Let τ be the critical point of a superstrong extender. Let $H \subseteq \tau$. Then there is $\kappa < \tau$ which is $<\tau$-strong with respect to H.*

Proof. Let E be a superstrong extender with critical point τ, let $M = \mathrm{Ult}(V, E)$, and let $\pi\colon V \to M$ be the ultrapower embedding. Let $\tau^* = \pi(\tau)$. For each $\alpha < \tau^*$ let F_α be the λ-restriction of π, where $\lambda < \tau^*$ is the least ordinal satisfying the requirements in Lemma 1.18 relative to α. Notice that F_α is then an element of $V \| \tau^*$, and therefore, through of the agreement between V and M, an element of M. Notice further that, by Lemma 1.18, F_α is α-strong. Let j_α be the ultrapower embedding by F_α, and notice finally that $j_\alpha(H)$ and $\pi(H)$ agree up to λ, meaning that $j_\alpha(H) \cap \lambda = \pi(H) \cap \lambda$. Since $H = \pi(H) \cap \kappa$ it follows that F_α is α-strong with respect to $\pi(H)$.

The extenders F_α, $\alpha < \tau^*$, thus witness that τ is $<\tau^*$-strong in M with respect to $\pi(H)$. So M is a model of the statement "there is $\kappa < \tau^*$ which is $<\tau^*$-strong with respect to $\pi(H)$." Using the elementarity of π to pull this statement back to V it follows that there is $\kappa < \tau$ which is $<\tau$-strong with respect to H. ⊣

3.18 Definition. A cardinal δ is called a *Woodin cardinal* if for every $H \subseteq \delta$, there exists $\kappa < \delta$ which is $<\delta$-strong with respect to H.

Lemma 3.17 shows that the critical point of a superstrong extender is Woodin. The next exercise shows that there are Woodin cardinals below the critical point. In fact Woodin cardinals sit *well* below such critical points in the large cardinal hierarchy, and there are many large cardinal axioms strictly between the existence of Woodin cardinals and the existence of superstrong extenders.

3.19 Exercise. Let E be a superstrong extender. Show that there are Woodin cardinals below the critical point of E. In fact, show that the critical point of E is a limit of Woodin cardinals.

3.20 Exercise. Let δ be a Woodin cardinal. Show that δ is a limit of (strongly) inaccessible cardinals, and that it is (strongly) inaccessible itself.

3.21 Exercise. Let δ be a Woodin cardinal. Let $H \subseteq \delta$ and let κ be $<\delta$-strong with respect to H. Let $\alpha < \delta$ be given. Prove that there is an extender E with critical point κ so that E is α-strong with respect to H, and so that $\mathrm{spt}(E) = \mathrm{Strength}(E)$ and $\mathrm{Strength}(E)$ is an inaccessible cardinal greater than α.

Hint. Let $\lambda < \delta$ be the first inaccessible cardinal above α. Using the fact that κ is $<\delta$-strong with respect to H, get an extender F with critical point κ so that F is λ-strong with respect to H. In particular then F is α-strong with respect to H, and $\mathrm{Strength}(F) \geq \lambda$. Let π be the ultrapower embedding by F, and let E be the λ-restriction of π. Show that the strength of E is precisely λ, and that E is α-strong with respect to H. ⊣

3.22 Lemma. *Let δ be a Woodin cardinal. Let $\eta > \delta$, and let $x_0, \dots, x_{n-1}$ be elements of $V\|\eta$. Then there exist unboundedly many $\kappa < \delta$ such that the κ-type of $x_0, \dots, x_{n-1}$ in $V\|\eta + 1$ relative to δ is elastic.*

Proof. For each (strongly) inaccessible $\gamma < \delta$ let A_γ be the γ-type of $x_0, \dots, x_{n-1}$ in $V\|\eta$ relative to δ, viewed as a subset of γ. Let $H = \{(\xi, \gamma) \mid \xi \in A_\gamma\}$, where $(*, *)$ is the Gödel pairing.

Let κ be $<\delta$-strong with respect to H. Let u be the κ-type of $x_0, \dots, x_{n-1}$ in $V\|\eta + 1$.

It is easy to check that if λ^* is closed under Gödel pairing and E is λ^*-strong with respect to H, then for every $\lambda < \lambda^*$, $\mathrm{Stretch}^E_\lambda(u^-)$ is realized by $x_0, \dots, x_{n-1}$ in $V\|\eta$. Using Exercise 3.21 it follows that the formula in the second clause of Definition 3.13 holds for $x_0, \dots, x_{n-1}$ in $V\|\eta + 1$, and is therefore an element of u. By Exercise 3.20, δ is inaccessible, and so the formula in the first clause of Definition 3.13 belongs to u. This shows that u is elastic.

We have so far obtained one cardinal $\kappa < \delta$ so that the κ-type of $x_0, \dots, x_{n-1}$ in $V\|\eta$ is elastic. We leave it to the reader to show that there are unboundedly many. ⊣

We now know that Woodin cardinals provide the strength necessary for the existence of many elastic types. The usefulness of elastic types appears through the following lemma. The lemma essentially says that an elastic type u which is exceeded by a type w can be stretched to a supertype of w.

3.23 Lemma (One-step Lemma). *Assume that u is an elastic type, and that w exceeds u (with all realizations relative to δ). Let $\tau = \mathrm{dom}(w)$ and let $\kappa = \mathrm{dom}(u)$. Suppose that $\tau < \delta$. Then there exists an extender $E \in V\|\delta$ so that*

- $\mathrm{crit}(E) = \kappa$, $\mathrm{spt}(E) = \mathrm{Strength}(E)$*, the strength of E is an inaccessible cardinal greater than τ, and*
- $w < \mathrm{Stretch}^E_{\tau+\omega}(u)$.

Proof. Let $\nu, \eta, x_0, \ldots, x_{m-1}$ witness that w exceeds u. Since u^- exists, η is a successor ordinal. Say $\eta = \bar{\eta} + 1$. Pick $E \in V\|\delta$ so that $\mathrm{crit}(E) = \kappa$, E has inaccessible strength greater than τ, and $\mathrm{Stretch}^E_{\tau+\omega}(u^-)$ is realized by $x_0, \ldots, x_{n-1}$ in $V\|\bar{\eta}$ relative to δ. This is possible since u is elastic.

Then w is a subtype of $\mathrm{Stretch}^E_{\tau+\omega}(u^-)$, as it is realized by a permutation of $x_0, \ldots, x_{n-1}, x_n, \ldots, x_m$ in $V\|\nu$ and $\nu < \bar{\eta}$. Simple properties of realizable types now imply that w is a subtype of $\mathrm{Stretch}^E_{\tau+\omega}(u)$. $\dashv$

We now have the tools necessary for the creation of iteration trees. We work for the rest of the section under the assumption that δ is a Woodin cardinal.

3.24 Lemma. *Let $M_0 = V$. There is an iteration tree with the structure of models presented in the following diagram:*

$$\begin{tikzcd} M_0 \arrow[rr, bend left] & = \quad M_1 \arrow[rr, bend right] & M_2 & M_3 \end{tikzcd}$$

Proof. Let η be an ordinal greater than δ. Let $\kappa_0 < \delta$ be such that the κ_0-type of η in $V\|\eta+5$ is elastic. Let u_0 be this type.

Let $\kappa_1 > \kappa_0$ be such that the κ_1-type of η in $V\|\eta+3$ is elastic. Let u_1 be this type.

Notice that u_1 exceeds u_0. Using the one-step lemma pick a κ_1+1-strong extender $E_1 \in M_1\|\delta$ so that $\mathrm{crit}(E_1) = \kappa_0$, and u_1 is a subtype of $\mathrm{Stretch}^{E_1}_{\kappa_1+\omega}(u_0)$.

Set $M_2 = \mathrm{Ult}(M_0, E_1)$, and let $j_{0,2}\colon M_0 \to M_2$ be the ultrapower embedding. Then u_1 is a subtype of $j_{0,2}(u_0)$. By the elementarity of $j_{0,2}$, $j_{0,2}(u_0)$ is realized by $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta)+5$. It follows from this and from the fact that u_1 is a subtype of $j_{0,2}(u_0)$, that u_1 is also realized in M_2, specifically it must be realized by $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta)+3$. The level $j_{0,2}(\eta)+3$ is reached by observing that u_1 contains the formula "$v_0 + 2$ is the largest ordinal."

Working now in M_2, let $\kappa_2 > \kappa_1$ be such that the κ_2-type of $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta)+1$ is elastic. Let u_2 be this type. Notice that u_2 then exceeds u_1, inside M_2. This uses the realization of u_1 in M_2, reached in the previous paragraph. Applying the one-step lemma pick an extender $E_2 \in M_2$ which stretches u_1 to a supertype of u_2. E_2 has critical point κ_1, and κ_1 is within the level of agreement between M_2 and M_1. E_2 can therefore be applied to M_1. Set $M_3 = \mathrm{Ult}(M_1, E_2)$. $\dashv$

3.25 Exercise. Construct an iteration tree with the structure presented in the following diagram:

$$\begin{tikzcd} M_0 \arrow[rr, bend left] & = \; M_1 \arrow[rr, bend right] & M_2 \arrow[rr, bend left] & M_3 & M_4 \end{tikzcd}$$

3.26 Exercise. Construct a length ω iteration tree with the tree order presented in the following diagram:

$$\begin{tikzcd} M_0 \arrow[rr, bend left] & = \; M_1 \arrow[rr, bend right] & M_2 \arrow[rr, bend left] & M_3 \arrow[rr, bend right] & M_4 & M_5 \cdots\cdots \end{tikzcd}$$

Hint. The following definition is useful:

3.27 Definition. Let $\nu_{\mathrm{L}} < \nu_{\mathrm{H}}$ be ordinals greater than δ. We say that $\langle \nu_{\mathrm{L}}, \nu_{\mathrm{H}} \rangle$ is a pair of *local indiscernibles* relative to δ just in case that:

$$(V\|\nu_{\mathrm{L}} + \omega) \models \phi[\nu_{\mathrm{L}}, c_0, \ldots, c_{k-1}] \iff (V\|\nu_{\mathrm{H}} + \omega) \models \phi[\nu_{\mathrm{H}}, c_0, \ldots, c_{k-1}]$$

for any $k < \omega$, any formula ϕ with $k{+}1$ free variables, and any $c_0, \ldots, c_{k-1} \in V\|\delta + \omega$.

Given local indiscernibles $\nu_{\mathrm{L}} < \nu_{\mathrm{H}}$, note that a type u is realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 1$ iff it is realized by ν_{H} in $V\|\nu_{\mathrm{H}} + 1$. Notice further that if u is realized by ν_{H} in $V\|\nu_{\mathrm{H}} + 1$, then any type of larger domain, which is realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 3$, exceeds $\mathrm{proj}^0(u)$, because $\nu_{\mathrm{L}} + 3 < \nu_{\mathrm{H}} + 1$. (It should be pointed out that the use of the projection is necessary here, to pass to a type which does not involve ν_{H} as a parameter.) In sum then you have:

3.28 Claim. *Let u be κ-type realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. Let $\tau > \kappa$ and let w be a τ-type realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 3$. Then w exceeds* $\mathrm{proj}^0(u)$.

You have also the following claim, directly from the definitions:

3.29 Claim. *Let α be an ordinal greater than δ. Let u be a κ-type realized by α in $V\|\alpha + 3$. Let $\tau > \kappa$ and let w be a τ-type realized by α in $V\|\alpha + 1$. Then w exceeds u.*

Use the two claims alternately, to construct the iteration tree required for the exercise, types $u_n \in M_n$, and ordinals α_n for $n < \omega$ odd, with $\alpha_1 = \nu_{\mathrm{L}}$, so that:

1. For even $n < \omega$, u_n is realized by $j_{0,n}(\nu_{\mathrm{L}})$ in $M_n\|j_{0,n}(\nu_{\mathrm{L}}) + 3$.
2. For odd $n < \omega$, u_{n-1} is realized by α_n in $M_n\|\alpha_n{+}3$, and u_n is realized by α_n in $M_n\|\alpha_n + 1$.

The construction is similar to that of the previous exercise, except that the use of the projection introduces some changes. The ordinals α_n for $n > 1$ odd are chosen using the third clause of Definition 3.6, applied to the fact that u_{n-1} is a subtype of $j_{n-2,n}(\mathrm{proj}^0(u_{n-2}))$. If you get $\alpha_n < j_{n-2,n}(\alpha_{n-2})$ for $n > 1$ odd, then you are on the right track. $\dashv$

3.30 Exercise. Go back to the last exercise, and make sure that the tree you construct is nice.

4. Homogeneously Suslin Sets

By a *tree* on a set X we mean a set $T \subseteq X^{<\omega}$, closed under initial segments. We use $[T]$ to denote the set of infinite branches through T, that is the set $\{x \in X^\omega \mid (\forall n) x{\restriction}n \in T\}$. Given a tree T on $X \times Y$ we often think of T as a subset of $X^{<\omega} \times Y^{<\omega}$ rather than $(X \times Y)^{<\omega}$, and similarly we think of $[T]$ as a subset of $X^\omega \times Y^\omega$. For T a tree on $X \times Y$ we use $p[T]$ to denote the *projection* of $[T]$ to X^ω, namely the set $\{x \in X^\omega \mid (\exists y)\langle x, y\rangle \in [T]\}$. We use T_s (for $s \in X^{<\omega}$) to denote the set $\{t \in Y^{<\omega} \mid \langle s, t\rangle \in T\}$, and use T_x (for $x \in X^\omega$) to denote the tree $\bigcup_{n<\omega} T_{x{\restriction}n}$. x is an element of $p[T]$ iff $[T_x]$ is non-empty. We sometimes apply similar terminology in the case that T is a tree on a product of more than two sets, for example $p[T] = \{x \mid (\exists y)(\exists z)\langle x, y, z\rangle \in [T]\}$ in the case that T is a tree on $X \times Y \times Z$.

Recall that a set $A \subseteq X^\omega$ is $\mathbf{\Sigma}^1_1$ iff there is a tree R on $X \times \omega$ so that $A = p[R]$. A set is $\mathbf{\Pi}^1_n$ if its complement is $\mathbf{\Sigma}^1_n$; and a set $A \subseteq X^\omega$ is $\mathbf{\Sigma}^1_{n+1}$ (for $n \geq 1$) if there is a $\mathbf{\Pi}^1_n$ set $B \subseteq X^\omega \times \omega^\omega$ so that $x \in A \Leftrightarrow (\exists y)\langle x, y\rangle \in B$. A set is *projective* if it is $\mathbf{\Pi}^1_n$ for some $n < \omega$. The projective sets are thus obtained from closed sets using complementations and projections along the real line.

The sets of reals in the very first level $\mathrm{L}_1(\mathbb{R})$ are precisely the projective sets, and our climb to $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ begins at the low end of the projective hierarchy. We prove determinacy for $\mathbf{\Pi}^1_1$ sets assuming measurable cardinals. The proof, due to Martin [16], can with hindsight be divided into two parts: a proof, using a measurable cardinal κ, that all $\mathbf{\Pi}^1_1$ sets are κ-homogeneously Suslin (see below for the definition); and a proof that all homogeneously Suslin sets are determined.

Let γ be an ordinal and let $m < n < \omega$. For $Z \subseteq \gamma^m$ let $Z^* = \{f \in \gamma^n \mid f{\restriction}m \in Z\}$. A measure ν over γ^n is an *extension* of a measure μ over γ^m just in case that for every $Z \subseteq \gamma^m$, $\mu(Z) = 1 \rightarrow \nu(Z^*) = 1$.

A *tower of measures* over γ is a sequence $\langle \mu_n \mid n < \omega\rangle$ so that:

(i) μ_n is a measure over γ^n for each n.

(ii) μ_n is an extension of μ_m for all $m < n < \omega$.

The tower is *countably complete* just in case that:

(iii) If $\mu_n(Z_n) = 1$ for each n then there is a fiber through $\langle Z_n \mid n < \omega \rangle$, namely a sequence $\langle \alpha_i \mid i < \omega \rangle$ so that $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in Z_n$ for each n.

For sequences s and t we write $s \leq t$ to mean that s is an initial segment of t, and $s < t$ to mean that s is a proper initial segment of t.

4.1 Definition. A tree T on $X \times \gamma$ is *homogeneous* if there is a sequence of measures $\langle \mu_s \mid s \in X^{<\omega} \rangle$ so that:

1. For each $s \in X^{<\omega}$, μ_s is a measure over T_s (equivalently, over $\gamma^{\mathrm{lh}(s)}$ with $\mu_s(T_s) = 1$), and μ_s is $\mathrm{card}(X)^+$-complete.

2. If $s \leq t$ then μ_t is an extension of μ_s.

It follows from condition (2) that for every $x \in X^\omega$, the sequence $\langle \mu_{x\restriction n} \mid n < \omega \rangle$ is a tower.

3. If $x \in p[T]$ then the tower $\langle \mu_{x\restriction n} \mid n < \omega \rangle$ is countably complete.

T is *κ-homogeneous* if in addition each of the measures μ_s is κ-complete.

4.2 Exercise. Let T be a homogeneous tree on $X \times \gamma$. Prove that there is a system $\langle M_s, f_s, j_{s,t} \mid s \leq t \in X^{<\omega} \rangle$ of (wellfounded) models M_s, nodes f_s, and embeddings $j_{s,t}$ satisfying the following conditions:

1. $j_{s,t} \colon M_s \to M_t$ for each $s \leq t$, $\mathrm{crit}(j_{s,t})$ is larger than $\mathrm{card}(X)$, $M_\emptyset = V$, and the system $\langle M_s, j_{s,t} \mid s \leq t \in X^{<\omega} \rangle$ commutes in the natural way.

2. $f_s \in j_{\emptyset,s}(T_s)$ for each $s \in X^{<\omega}$, and the nodes $\langle f_s \mid s \in X^{<\omega} \rangle$ cohere in the natural way, meaning that $s < t \Rightarrow j_{s,t}(f_s) < f_t$.

3. If $x \in p[T]$ then the system $\langle M_s, j_{s,t} \mid s \leq t < x \rangle$ has a wellfounded direct limit.

Hint. Let $M_s = \mathrm{Ult}(V, \mu_s)$ and let $j_s \colon V \to M_s$ be the ultrapower embedding. Let f_s be the seed of the measure μ_s. Notice that f_s is an element of $j_s(T_s)$.

Recall that each element of M_s has the form $j_s(g)(f_s)$ for some function $g \colon \gamma^{\mathrm{lh}(s)} \to V$. For $s \leq t \in X^{<\omega}$ define an embedding $j_{s,t} \colon M_s \to M_t$ by letting $j_{s,t}$ send $j_s(g)(f_s)$ to $j_t(g)(f_t \restriction \mathrm{lh}(f_s))$.

Prove that the resulting system satisfies conditions (1)–(3). $\dashv$

4.3 Exercise. Let T be a tree on $X \times \gamma$ and suppose that there is a system $\langle M_s, f_s, j_{s,t} \mid s \leq t \in X^{<\omega} \rangle$ satisfying the conditions in Exercise 4.2. Prove that T is homogeneous.

Suppose in addition that the embeddings $j_{s,t}$ all have critical points at least κ. Show that T is κ-homogeneous.

Hint. Set $\mu_s(Z) = 1$ iff $f_s \in j_{\emptyset,s}(Z)$. Prove that the resulting system of measures $\langle \mu_s \mid s \in X^{<\omega} \rangle$ satisfies the conditions in Definition 4.1. $\dashv$

The existence of a system satisfying the conditions in Definition 4.1 is thus equivalent to the existence of a system satisfying the conditions in Exercise 4.2. We use the two systems alternately, and refer to both of them as *homogeneity systems* for the tree T.

4.4 Exercise. Show that the converse of condition (3) in Exercise 4.2 follows from conditions (1) and (2) in the exercise. Condition (3) can therefore be strengthened to an equivalence, and so can condition (3) in Definition 4.1.

Hint. Fix x. Let M_x be the direct limit of the system $\langle M_s, j_{s,t} \mid s \leq t < x \rangle$, and let $j_{s,x} \colon M_s \to M_x$ for $s < x$ be the direct limit embeddings. Let $f_x = \bigcup_{s<x} j_{s,x}(f_s)$, and notice that using condition (2), f_x is an infinite branch through $j_{\emptyset,x}(T_x)$. Use the wellfoundedness of M_x to find some infinite branch f through $j_{\emptyset,x}(T_x)$ with $f \in M_x$, and then using the elementarity of $j_{\emptyset,x}$ argue that $x \in p[T]$. $\dashv$

A set $A \subseteq X^\omega$ is *Suslin* if there is an ordinal γ and a tree T on $X \times \gamma$ so that $p[T] = A$. $A \subseteq X^\omega$ is *homogeneously Suslin* if in addition T can be taken to be homogeneous, and *κ-homogeneously Suslin* if T can be taken to be κ-homogeneous. These definitions are due independently to Kechris and Martin. In the context of the axiom of choice, which we employ throughout the chapter, every $A \subseteq X^\omega$ is Suslin. But of course not every set is homogeneously Suslin.

Let κ be a measurable cardinal. Fix a set $X \in V\|\kappa$ and a $\mathbf{\Pi}^1_1$ set $A \subseteq X^\omega$. We aim to show that A is κ-homogeneously Suslin.

4.5 Exercise. Let $R \subseteq \omega^{<\omega}$ be a tree. The *Brouwer–Kleene order* on R is the strict order $\prec$ defined by the condition: $s \prec t$ iff s extends t or $s(n) < t(n)$ where n is least so that $s(n) \neq t(n)$. Prove that $\prec$ is illfounded iff R has an infinite branch.

4.6 Exercise. Show that there is a map $s \mapsto \prec_s$, defined on $s \in X^{<\omega}$, so that:

- $\prec_s$ is a linear order on $\mathrm{lh}(s)$.
- If $s \leq t$ then $\prec_s \subseteq \prec_t$.
- $x \in A$ iff $\prec_x$ is wellfounded, where $\prec_x = \bigcup_{n<\omega} \prec_{x\restriction n}$.

The last condition is the most important one. The first two conditions are needed to make sense of $\prec_x$.

Hint to Exercise 4.6. Let $R \subseteq (X \times \omega)^{<\omega}$ be a tree so that $p[R]$ is precisely equal to the complement of A. Define the map $s \mapsto \prec_s$ in such a way that for each $x \in X^\omega$, $\prec_x$ is isomorphic to the Brouwer–Kleene order on R_x. $\dashv$

Let $T \subseteq X \times \kappa$ be the tree consisting of nodes $\langle s, f \rangle$ so that f has the form $\langle \alpha_0, \ldots, \alpha_{\mathrm{lh}(s)-1} \rangle$ with $\alpha_i < \kappa$ for each i, and $\alpha_i < \alpha_j$ iff $i \prec_s j$.

4.7 Exercise. Show that $p[T] = A$.

Since κ is measurable, there is an elementary embedding $j \colon V \to M$ with $\mathrm{crit}(j) = \kappa$. Let μ be the measure over κ defined by $\mu(Z) = 1$ iff $\kappa \in j(Z)$.

4.8 Exercise. Prove that μ is a κ-complete, non-principal measure on κ.

4.9 Exercise. A function $f \colon \kappa \to \kappa$ is *pressing down* if $f(\alpha) < \alpha$ for all $\alpha < \kappa$. A measure over κ is called *normal* if every pressing down function on κ is constant on a set of measure one. Prove that the measure μ defined above is normal.

4.10 Exercise. The *diagonal intersection* of the sets Z_α $(\alpha < \kappa)$ is defined to be the set $\{\xi \in \kappa \mid (\forall \alpha < \xi) \xi \in Z_\alpha\}$. Prove, for the measure μ defined above, that the diagonal intersection of sets of measure one has measure one.

For each $s \in X^{<\omega}$ and each $C \subseteq \kappa$ define C^s to be the set of tuples $\langle \alpha_0, \ldots, \alpha_{\mathrm{lh}(s)-1} \rangle$ with $\alpha_i \in C$ for each i, and $\alpha_i < \alpha_j$ iff $i \prec_s j$. Define a filter $\mathcal{F}_s$ over $\kappa^{\mathrm{lh}(s)}$ by setting $Z \in \mathcal{F}_s$ iff there exists a set $C \subseteq \kappa$ so that $Z \supseteq C^s$ and $\mu(C) = 1$.

4.11 Exercise. Prove that $\mathcal{F}_s$ is an ultrafilter over $\kappa^{\mathrm{lh}(s)}$, meaning that for every $Z \subseteq \kappa^{\mathrm{lh}(s)}$, either $Z \in \mathcal{F}_s$ or else $\kappa^s - Z \in \mathcal{F}_s$.

Hint. Work by induction on the length of s. The inductive step makes several uses of Exercises 4.9 and 4.10. $\dashv$

Define a two-valued measure μ_s on κ^s by setting $\mu_s(Z) = 1$ iff $Z \in \mathcal{F}_s$.

4.12 Exercise. Prove that μ_s is κ-complete.

4.13 Exercise. Let $s \leq t \in X^{<\omega}$. Prove that μ_t extends μ_s.

4.14 Exercise. Let $x \in X^\omega$, and suppose that x belongs to A, so that $\prec_x$ is wellfounded. Prove that the tower $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is countably complete.

Hint. Suppose that $\mu_{x \restriction n}(Z_n) = 1$ for each $n < \omega$. Fix C_n so that $\mu(C_n) = 1$ and $C_n^s \subseteq Z_n$. Let $C = \bigcap_{n<\omega} C_n$. Then $C^s \subseteq Z_n$ for each n, and $\mu(C) = 1$ by countable completeness. Since $x \in A$, $\prec_x$ is wellfounded. The order $\prec_x$ can therefore be embedded into the ordinals, and in fact into C since C is uncountable. Pick then a sequence $\langle \alpha_i \mid i < \omega \rangle$ of ordinals in C so that $i \prec_x j$ iff $\alpha_i < \alpha_j$. The sequence $\langle \alpha_i \mid i < \omega \rangle$ is a fiber through $\langle Z_n \mid n < \omega \rangle$. $\dashv$

4.15 Theorem. *Let κ be a measurable cardinal. Let X belong to $V\|\kappa$ and let $A \subseteq X^\omega$ be $\mathbf{\Pi}^1_1$. Then A is κ-homogeneously Suslin.*

Proof. Let $T \subseteq (X \times \kappa)^{<\omega}$ be the tree defined above and let μ_s be the measures defined above. Exercises 4.12 through 4.14 establish that $\langle \mu_s \mid s \in X^{<\omega}\rangle$ is a κ-homogeneity system for T. ⊣

Next we prove that homogeneously Suslin sets are determined. We work for the rest of the section with some set X and a homogeneously Suslin set $A \subseteq X^\omega$. Let T and $\langle \mu_s \mid s \in X^{<\omega}\rangle$ witness that A is homogeneously Suslin.

Define G^* to be the game played according to Diagram 8 and the following rules:

- $x_n \in X$ for each $n < \omega$.
- $\langle x_0, \alpha_0, \dots, x_{n-1}, \alpha_{n-1}\rangle \in T$ for each $n < \omega$.

The first rule is a requirement on player I if n is even, and on player II if n is odd. The second rule is a requirement on player I. A player who violates a rule loses. Infinite runs of G^* are won by player I.

I	x_0 α_0		α_1	x_2 α_2		α_3	$\dots$
II		x_1			x_3		$\dots$

Diagram 8: The game G^*.

4.16 Exercise. Prove that G^* is determined.

Hint. You are asked to prove the famous theorem of Gale–Stewart [6] that infinite games with closed payoff are determined. Let S be the set of positions in G^* from which player II has a winning strategy. If the initial position belongs to S, then player II has a winning strategy in G^*. Suppose that the initial position does not belong to S, and prove that there is a strategy for player I which stays on positions outside S, and that this strategy is winning. ⊣

4.17 Exercise. Suppose that player I has a winning strategy in G^*. Prove that player I has a winning strategy in $G_\omega(A)$.

Hint. Let σ^* be a winning strategy for I in G^*. Call a position $p = \langle x_0, \dots, x_{n-1}\rangle$ in $G_\omega(A)$ nice if it can be expanded to a position $p^* = \langle x_0, \alpha_0, \dots, x_{n-1}, \alpha_{n-1}\rangle$ in G^* so that p^* is according to σ^*. Note that if such an expansion exists, then it is unique. Define a strategy σ for I in $G_\omega(A)$ by setting $\sigma(p) = \sigma^*(p^*)$. Show that every infinite run according to σ belongs to $p[T]$, and is therefore won by player I in $G_\omega(A)$. ⊣

4.18 Lemma. *Suppose that player* II *has a winning strategy in* G^*. *Then player* II *has a winning strategy in* $G_\omega(A)$.

Proof. Let σ^* be a winning strategy for II in G^*.

Let $s = \langle x_0, \dots, x_{i-1} \rangle$ be a position of odd length in $G_\omega(A)$. For each $\varphi = \langle \alpha_0, \dots, \alpha_{i-1} \rangle \in T_s$, let $h_s(\varphi)$ be σ^*'s move following the position $\langle x_0, \alpha_0, \dots, x_{i-1}, \alpha_{i-1} \rangle$ in G^*. h_s is then a function from T_s into X. By the completeness of μ_s there is a specific move x_i so that:

(∗) $\{\varphi \mid h_s(\varphi) = x_i\}$ has μ_s-measure one.

Define $\sigma(s)$ to be equal to this x_i.

Suppose now that $x = \langle x_i \mid i < \omega \rangle$ is an infinite run of $G_\omega(A)$, played according to σ. We have to show that x is won by player II.

Using condition (∗) fix for each odd $n < \omega$ a set $Z_n \subseteq T_{x \restriction n}$ so that $h_{x \restriction n}(\varphi) = x_n$ for every $\varphi \in Z_n$ and $\mu_{x \restriction n}(Z_n) = 1$. For even $n < \omega$ let $Z_n = T_{x \restriction n}$.

Suppose for contradiction that $x \in A$. Then $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is countably complete and so there is a fiber $\langle \alpha_i \mid i < \omega \rangle$ for the sequence $\langle Z_n \mid n < \omega \rangle$. In other words there is a sequence $\langle \alpha_i \mid i < \omega \rangle$ in $[T_x]$ so that $h_{x \restriction n}(\langle \alpha_0, \dots, \alpha_{n-1} \rangle) = x_n$ for each odd $n < \omega$. But then $\langle x_i, \alpha_i \mid i < \omega \rangle$ is a run of G^* and is consistent with σ^*. This is a contradiction, since σ^* is a winning strategy for player II, and infinite runs of G^* are won by player I. ⊣

4.19 Corollary. *Let* $A \subseteq X^\omega$ *be homogeneously Suslin. Then* $G_\omega(A)$ *is determined.*

Proof. By Exercise 4.16, G^* is determined. By Exercise 4.17 and Lemma 4.18, the player who has a winning strategy in G^* has a winning strategy in $G_\omega(A)$. ⊣

Theorem 4.15 and Corollary 4.19 establish the determinacy of $\mathbf{\Pi}^1_1$ subsets of ω^ω, assuming the existence of a measurable cardinal. In the next section we deal with $\mathbf{\Pi}^1_2$ sets.

5. Projections and Complementations

Martin and Steel [21] use Woodin cardinals to propagate the property of being homogeneously Suslin under complementation and existential real quantification, proving in this manner that all projective sets are determined. In this section we present their results. We begin by proving that if δ is a Woodin cardinal, and $A \subseteq X^\omega \times \omega^\omega$ is δ^+-homogeneously Suslin, then the set $B = \{x \in X^\omega \mid (\forall y) \langle x, y \rangle \notin A\}$ is determined. We then go on to show that B is κ-homogeneously Suslin for all $\kappa < \delta$. Together with the results in

Section 4 this shows that all $\mathbf{\Pi}^1_{n+1}$ sets are determined, assuming n Woodin cardinals and a measurable cardinal above them.

Let δ be a Woodin cardinal. Let X be a set in $V\|\delta$, and let $A \subseteq X^\omega \times \omega^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+-homogeneously Suslin, and let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ (for some ordinal γ) and $\langle \mu_{s,t} \mid \langle s,t\rangle \in (X \times \omega)^{<\omega}\rangle$ witness this.

5.1 Remark. The objects in the homogeneity system are given for pairs $\langle s,t\rangle \in X^{<\omega} \times \omega^{<\omega}$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$. We sometimes write $\mu_{s,t}$ or $S_{s,t}$ also when s and t are of different length. We mean $\mu_{s\restriction n,t\restriction n}$ where $n = \min\{\mathrm{lh}(s), \mathrm{lh}(t)\}$, and similarly with $S_{s,t}$. We also write $\mu_{x,t}$ for $x \in X^\omega$ to mean $\mu_{x\restriction n,t}$ where $n = \mathrm{lh}(t)$, and similarly with $S_{x,t}$.

5.2 Exercise (Martin–Solovay [20]). Let t_i $(i < \omega)$ enumerate $\omega^{<\omega}$. The *Martin–Solovay tree* for the complement of $p[A]$, where $A \subseteq X^\omega \times \omega^\omega$ is the projection of a tree S with homogeneity system $\langle \mu_{s,t} \mid \langle s,t\rangle \in (X \times \omega)^{<\omega}\rangle$, is the tree of attempts to create $x \in X^\omega$ and a sequence $\langle \rho_i \mid i < \omega\rangle$ so that:

(i) ρ_i is a partial function from $S_{x\restriction \mathrm{lh}(t_i),t_i}$ into $|S|^+$, and the domain of ρ_i has $\mu_{x\restriction \mathrm{lh}(t_i),t_i}$-measure one.

(ii) If $t_i < t_j$ then $\rho_i(f\restriction \mathrm{lh}(t_i)) > \rho_j(f)$ for every $f \in \mathrm{dom}(\rho_j)$.

Prove that this tree projects to $X^\omega - p[A]$.

Definitions 5.3 and 5.6 below essentially code a subset of the Martin–Solovay tree for B by a relation on types. We will use this coding to prove that $G_\omega(B)$ is determined, and that in fact B is homogeneously Suslin. Martin–Steel [21] proved that the Martin–Solovay tree itself is homogeneous. We work with types, rather than the Martin–Solovay tree of functions, in preparation for Section 6.

The constructions below use the definitions of Section 3. By type here we always mean a type with domain less than δ and greater than $\mathrm{rank}(X)$. All realizations in V are relative to the fixed Woodin cardinal δ. The variable v_0 in each type will always be realized by S. (Realizations in iterates M of V are made relative to the appropriate image of δ, and with the first variable realized by the image of S.)

5.3 Definition. Let $\langle s,t\rangle \in X^{<\omega} \times \omega^{<\omega}$, with $\mathrm{lh}(s) = \mathrm{lh}(t) = k$ say. Let w be a $k+2$-type. Define $Z_{s,t}$ to be the set of $f \in S_{s,t}$ for which $(\exists \eta \in \mathrm{On})(\exists \alpha > \max\{\delta, \mathrm{rank}(S)\})$ so that w is realized by S, $\langle 0, f(0)\rangle, \ldots, \langle k-1, f(k-1)\rangle$, and α in $V\|\eta$. Define $\rho_{s,t}\colon Z_{s,t} \to \mathrm{On}$ by setting $\rho_{s,t}(f)$ equal to the least η witnessing the existential statement above.

5.4 Remark. Both $Z_{s,t}$ and $\rho_{s,t}$ depend on w. When we wish to emphasize the dependence we write $Z_{s,t}(w)$ and $\rho_{s,t}(w)$.

Definition 5.3 lets us view types as defining partial functions $\rho_{s,t}$ from $S_{s,t}$ into the ordinals. The domain of the partial function $\rho_{s,t}$ is $Z_{s,t}$. Connecting the definition to the homogeneity system, let us say that w is $\langle s,t\rangle$-*nice* if $Z_{s,t}$ has $\mu_{s,t}$-measure one.

5.5 Claim. *Let w be a $k+2$-type and suppose that w is $\langle s,t\rangle$-nice. Then w contains the formula "$\{v_1,\dots,v_k\}$ is a node in the tree $(v_0)_{\widetilde{s},\widetilde{t}}$."*

Note that both s and t belong to the domain of w, since they are elements of $X^{<\omega}$, and the domain of w is greater than $\operatorname{rank}(X)$ (see the comment following Remark 5.1). The reference to $\widetilde{s}$ and $\widetilde{t}$ in a formula which may potentially belong to w therefore makes sense. $(v_0)_{\widetilde{s},\widetilde{t}}$ in the formula stands for the tree of nodes g so that $\langle s,t,g\rangle$ belongs to the interpretation of v_0.

Proof of Claim 5.5. Let f be any element of $Z_{s,t}(w)$. ($Z_{s,t}$ has $\mu_{s,t}$-measure one, and so certainly it is not empty.) Then

1. w is realized by S, $\langle 0,f(0)\rangle$, …, $\langle k-1,f(k-1)\rangle$, α in $V\|\eta$ for some α and η.

2. $\langle s,t,f\rangle$ belongs to S, meaning that f, which is formally equal to the set $\{\langle 0,f(0)\rangle,\dots,\langle k-1,f(k-1)\rangle\}$, belongs to $S_{s,t}$.

It follows that the formula in the claim belongs to w. ⊣

5.6 Definition. Let s' and t' extend s and t (perhaps not strictly), with $\operatorname{lh}(s')=\operatorname{lh}(t')=k'$. Let w be $\langle s,t\rangle$-nice and let w' be $\langle s',t'\rangle$-nice. We write $w'\prec w$ to mean that the set $\{f'\in S_{s',t'}\mid \rho_{s',t'}(w')(f')<\rho_{s,t}(w)(f'\restriction k)\}$ has $\mu_{s',t'}$-measure one.

5.7 Claim. *The relation $\prec$ is transitive.* ⊣

5.8 Definition. Given a $k+2$-type w we use $\operatorname{dcp}(w)$ (pronounced "decap w") to denote $\operatorname{proj}^{k+1}(w)$. If w is realized by S, $\langle 0,f(0)\rangle$, …, $\langle k-1,f(k-1)\rangle$, and α, then $\operatorname{dcp}(w)$ is realized by S, $\langle 0,f(0)\rangle$, …, and $\langle k-1,f(k-1)\rangle$.

5.9 Claim. *Let w be $\langle s,t\rangle$-nice, and suppose that w contains the formula "$v_{k+1}+2$ exists" (where $k=\operatorname{lh}(s)=\operatorname{lh}(t)$, and w is a $k+2$-type). Let s' and t' extend s and t, with $\operatorname{lh}(s')=\operatorname{lh}(t')=k'$. Then there is a $k'+2$-type u so that:*

1. *u is $\langle s',t'\rangle$-nice.*

2. *u contains the formula "$v_{k'+1}$ is the largest ordinal."*

3. *$\operatorname{dcp}(u)$ is elastic.*

4. *u exceeds w.*

5. $u \prec w$.

Proof. Fix for a moment some $f' \in S_{s',t'}$, and suppose that $f' \restriction k \in Z_{s,t}(w)$. Let $\eta = \rho_{s,t}(f' \restriction k)$, so that w is realized by S, $\langle 0, f'(0) \rangle$, …, $\langle k-1, f'(k-1) \rangle$, and some $\alpha > \max\{\delta, \mathrm{rank}(S)\}$ in $V \| \eta$. Since w contains the formula "$v_{k+1} + 2$ exists," it must be that $\eta > \alpha + 2$.

Let $\tau < \delta$ be such that the τ-type of S, $\langle 0, f'(0) \rangle$, …, and $\langle k'-1, f'(k'-1) \rangle$ in $V \| \alpha + 1$ is elastic, and such that $\tau > \mathrm{dom}(w)$. Such a τ exists by Lemma 3.22. Let u be the τ-type of S, $\langle 0, f'(0) \rangle$, …, $\langle k'-1, f'(k'-1) \rangle$, and α in $V \| \alpha + 1$. Then u contains the formula "$v_{k'+1}$ is the largest ordinal," u exceeds w, and $\mathrm{dcp}(u)$ is elastic.

The type u defined above depends on the node $f' \in S_{s',t'}$ used. To emphasize the dependence let us from now on write $u(f')$ to denote this type. Let us similarly write $\alpha(f')$ and $\eta(f')$ to emphasize the dependence of α and η on f'.

The function $f' \mapsto u(f')$ maps $\{f' \in S_{s',t'} \mid f' \restriction k \in Z_{s,t}\}$ into $V \| \delta$. Using the fact that $Z_{s,t}$ has $\mu_{s,t}$-measure one it is easy to check that the domain of this function has $\mu_{s',t'}$-measure one. From this and the δ^+-completeness of the measures it follows that the function is fixed on a set of $\mu_{s',t'}$-measure one. Thus, there exists a particular type u, and a set $Z \subseteq S_{s',t'}$, so that $u(f') = u$ for each $f' \in Z$, and Z has $\mu_{s',t'}$-measure one.

Clearly $Z_{s',t'}(u) \supseteq Z$, and it follows from this that u is $\langle s', t' \rangle$-nice. It is also clear that $\rho_{s',t'}(u)(f') \leq \alpha(f') + 1 < \eta(f')$ for each $f' \in Z$, and it follows from this that $u \prec w$. ⊣

5.10 Claim. *Let u be $\langle s, t \rangle$-nice, where* $\mathrm{lh}(s) = \mathrm{lh}(t) = k$. *Let w be a $k+2$-type, containing the formula "$v_{k+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$." Suppose that w is a subtype of* $\mathrm{dcp}(u)$. *Then w is $\langle s, t \rangle$-nice, and $w \prec u$.*

Proof. Fix for a moment some $f \in Z_{s,t}(u)$. Let $\eta = \rho_{s,t}(u)(f)$, so that u is realized by S, $\langle 0, f(0) \rangle$, …, $\langle k-1, f(k-1) \rangle$, and some α in $V \| \eta$.

Since w is a subtype of $\mathrm{dcp}(u)$, there must be some β and some ν so that w is realized by S, $\langle 0, f(0) \rangle$, …, $\langle k-1, f(k-1) \rangle$, and β in $V \| \nu$, and so that $\nu < \eta$. Since w contains the formula "$v_{k+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$," β is greater than $\max\{\delta, \mathrm{rank}(S)\}$.

It follows from the argument of the previous paragraph that, for each $f \in Z_{s,t}(u)$, there exists ν and $\beta > \max\{\delta, \mathrm{rank}(S)\}$ so that w is realized by S, $\langle 0, f(0) \rangle$, …, $\langle k-1, f(k-1) \rangle$, and β in $V \| \nu$, and that the least ν witnessing this is smaller than $\rho_{s,t}(u)(f)$. In other words $f \in Z_{s,t}(w)$ and $\rho_{s,t}(w)(f) < \rho_{s,t}(u)(f)$, for each $f \in Z_{s,t}(u)$. Since $Z_{s,t}(u)$ has $\mu_{s,t}$-measure one this implies that w is $\langle s, t \rangle$-nice and that $w \prec u$. ⊣

5.11 Claim. *Let $x \in X^\omega$. Suppose that there are types $\langle w_t \mid t \in \omega^{<\omega} \rangle$ so that:*

1. Each w_t is $\langle x, t \rangle$-nice.

2. *For each* $t < t^* \in \omega^{<\omega}$, $w_{t^*} \prec w_t$.

Then $x \in B$.

Proof. We have to show that $(\forall y \in \omega^\omega)\langle x, y\rangle \notin A$. Fix $y \in \omega^\omega$. For each $n < \omega$ let μ_n denote $\mu_{x\restriction n, y\restriction n}$. Let ρ_n denote $\rho_{x,y\restriction n}(w_{y\restriction n})$. ρ_n is a partial function with domain a μ_n-measure one subset of $S_{x\restriction n, y\restriction n}$.

Set $Z_0 = S_{\emptyset,\emptyset}$ and for each $n < \omega$ set $Z_{n+1} = \{f \in S_{x\restriction n+1, y\restriction n+1} \mid \rho_{n+1}(f) < \rho_n(f\restriction n)\}$. By assumption $w_{y\restriction n+1} \prec w_{y\restriction n}$ so Z_{n+1} has μ_{n+1}-measure one.

Suppose for contradiction that $\langle x, y\rangle \in A$. The tower $\langle \mu_n \mid n < \omega\rangle$ is then countably complete by Definition 4.1, so the sequence $\langle Z_n \mid n < \omega\rangle$ has a fiber, $f = \langle \alpha_i \mid i < \omega\rangle$ say. Then $f\restriction n+1 \in Z_{n+1}$ for each $n < \omega$, meaning that $\rho_{n+1}(f\restriction n+1) < \rho_n(f\restriction n)$, so that $\langle \rho_n(f\restriction n) \mid n < \omega\rangle$ is an infinite descending sequence of ordinals, contradiction. ⊣

Let $\langle \nu_{\mathrm{L}}, \nu_{\mathrm{H}}\rangle$ be the lexicographically least pair of local indiscernible of V relative to $\max\{\delta, \mathrm{rank}(S)\}$, minimizing first over the second coordinate.

5.12 Claim. *For each* $\kappa < \delta$*, the* κ*-type of* S *and* ν_{L} *in* $V\|\nu_{\mathrm{L}} + 1$ *is* $\langle\emptyset,\emptyset\rangle$*-nice.* ⊣

For $t \in \omega^{<\omega}$ let $\mathrm{pred}(t)$ denote $t\restriction(\mathrm{lh}(t) - 1)$.

5.13 Definition. Define G^*, illustrated in Diagram 9, to be played according to the following rules:

1. $x_n \in X$.
2. $t_n \in \omega^{<\omega}$.
3. u_n is a $k_n + 2$-type, $\mathrm{dcp}(u_n)$ is elastic, and u_n contains the formula "$\{v_1, \ldots, v_{k_n}\}$ is a node in the tree $(v_0)_{\widetilde{s}_n, \widetilde{t}_n}$," where $k_n = \mathrm{lh}(t_n)$ and $s_n = x\restriction k_n$.
4. If $n > 0$ then $\mathrm{dom}(u_n) > \mathrm{dom}(u_{n-1})$. (And $\mathrm{dom}(u_0) > \mathrm{rank}(X)$, see the comment following Remark 5.1.)
5. If $t_n = \emptyset$ then u_n is realized by S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$.
6. If $t_n \neq \emptyset$ then $l_n < n$ is such that $t_{l_n} = \mathrm{pred}(t_n)$, and u_n exceeds w_{l_n}.
7. w_n too is a $k_n + 2$-type, w_n is a subtype of $\mathrm{dcp}(u_n)$, and w_n contains the formulae "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$" and "$v_{k_n+1} + 2$ exists and is the largest ordinal."

The first player to violate any of the rules loses. Infinite runs where all rules have been followed are won by player I.

$$\begin{array}{c|cccccccccc}
\mathrm{I} & & w_0 & x_0 & & w_1 & & & w_2 & x_2 & \dots \\
\hline
\mathrm{II} & l_0, t_0, u_0 & & & l_1, t_1, u_1 & & x_1 & l_2, t_2, u_2 & & & \dots
\end{array}$$

Diagram 9: The game G^*.

5.14 Lemma. *Suppose that player* I *has a winning strategy in* G^*. *Then player* I *has a winning strategy in* $G_\omega(B)$.

Proof. Let σ^* be a winning strategy for player I in G^*. Let $\langle t_n^* \mid n < \omega\rangle$ enumerate $\omega^{<\omega}$ in such a way that $(\forall t \in \omega^{<\omega})$ pred(t) is enumerated before t. In particular $t_0^* = \emptyset$. For $n > 0$ let $l_n^* < n$ be such that $\mathrm{pred}(t_n^*) = t^*_{l_n^*}$. Let $l_0^* = 0$.

Fix an opponent willing to play for II in $G_\omega(B)$. We describe how to play against the opponent, and win. Our description takes the form of a construction of a run of G^*. σ^* supplies moves for I. The opponent supplies the moves $x_1, x_3, x_5, \cdots$ for II. It is up to us to come up with the remaining moves, l_n, t_n, u_n for $n < \omega$. We make sure as we play that:

1. $t_n = t_n^*$ and $l_n = l_n^*$.
2. u_n contains the formula "v_{k_n+1} is the largest ordinal" where $k_n = \mathrm{lh}(t_n)$.
3. u_n is $\langle x, t_n\rangle$-nice.

(We write $\langle x, t_n\rangle$-nice, but notice that only $x{\restriction}\mathrm{lh}(t_n)$ is relevant to the condition.)

w_n, by the rules of G^*, is a $k_n + 2$-type, is a subtype of $\mathrm{dcp}(u_n)$, and contains the formula "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$." It follows by Claim 5.10 that:

(i) w_n is $\langle x, t_n\rangle$-nice.

(ii) $w_n \prec u_n$.

Let us now describe how to play l_n, t_n, and u_n. We begin with the case $n = 0$. Set $t_0 = \emptyset$ and $l_0 = 0$. Using Lemma 3.22 let $\kappa_0 < \delta$ be such that the κ_0-type of S in $V\|\nu_{\mathrm{L}} + 1$ is elastic. Set u_0 to be the κ_0-type of S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. These assignments determine the moves l_0, t_0, and u_0. It is easy to check that they satisfy the relevant rules of G^*, and conditions (1)–(3) above for $n = 0$.

Suppose next that rounds 0 through $n-1$ have been played, subject to the relevant rules and to conditions (1)–(3) above. Set $t_n = t_n^*$ and $l_n = l_n^*$. Note that by condition (i), w_{l_n} is $\langle x, t_{l_n}\rangle$-nice. Let $k_n = \mathrm{lh}(t_n)$. Using Claim 5.9, set u_n to be a $k_n + 2$-type so that: u_n is $\langle x, t_n\rangle$-nice; u_n contains the formula "v_{k_n+1} is the largest ordinal"; dcp(u_n) is elastic; u_n exceeds w_{l_n}; and $u_n \prec w_{l_n}$. These assignments determine the moves l_n, t_n, and u_n. It is again easy to check that they satisfy the relevant rules of G^*, and conditions (1)–(3) above. For the record let us note that we have also the following condition:

(iii) $u_n \prec w_{l_n}$.

The assignments made above, together with moves supplied by σ^* and by the opponent, determine an infinite run $\langle l_n, t_n, w_n, u_n, x_n \mid n < \omega\rangle$ of G^*. It remains to check that the real $x = \langle x_n \mid n < \omega\rangle$ constructed as part of this run is won by player I in $G_\omega(B)$.

By conditions (ii) and (iii), $w_n \prec w_{l_n}$ for each $n > 0$. It follows from this that $w_n \prec w_m$ whenever $t_n > t_m$. By Claim 5.11, $x \in B$. So x is won by player I in $G_\omega(B)$, as required. ⊣

5.15 Lemma. *Suppose that player* II *has a winning strategy in G^*. Then player* II *has a winning strategy in $G_\omega(B)$.*

Proof. Let σ^* be a winning strategy for player II in G^*. Fix an opponent willing to play for I in $G_\omega(B)$. We describe how to play against the opponent, and win. Again our description takes the form of a construction. But this time we do not construct a run of G^*. Rather we construct an iteration tree $\mathcal{T}$ with an even branch consisting of $\{0, 2, 4, \ldots\}$, and a run of $j_{\text{even}}(G^*)$, played according to $j_{\text{even}}(\sigma^*)$.

Precisely, we construct:

(A) l_n, t_n, u_n, w_n, and x_n for $n < \omega$.

(B) An iteration tree $\mathcal{T}$ giving rise to models M_k for $k < \omega$ and embeddings $j_{l,k}$ for $l \; T \; k < \omega$.

(C) Nodes $g_n \in j_{0,2n+1}(S)_{x,t_n}$ for $n < \omega$.

x in the last condition is the sequence $\langle x_n \mid n < \omega\rangle$, although of course only $x \restriction \text{lh}(t_n)$ is relevant to the condition.

We construct so that:

- $0 \; T \; 2 \; T \; 4 \cdots$.
- If $t_n \neq \emptyset$ then the T-predecessor of $2n+1$ is $2l_n + 1$.
- If $t_n = \emptyset$ then the T-predecessor of $2n+1$ is $2n$.

Note that these conditions determine the tree order T completely.

Let $p_0 = \emptyset$ and recursively define

$$p_{n+1} = j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n, x_n\rangle.$$

We construct so that p_n is a position in $j_{0,2n}(G^*)$, played according to $j_{0,2n}(\sigma^*)$. This amounts to maintaining the following conditions:

1. l_n, t_n, and u_n are the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n.

2. w_n is a legal move for player I following the position $j_{2n,2n+2}(p_n)^\frown \langle l_n, t_n, j_{2n,2n+2}(u_n)\rangle$.

3. If n is odd then x_n is the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$.

Notice that conditions (1) and (3) determine l_n, t_n, and u_n for each n, and x_n for odd n.

Let k_n denote $\mathrm{lh}(t_n)$. Condition (C) above already places some restriction on the nature of g_n. It must be a sequence of length k_n, and $\langle x{\restriction}k_n, t_n, g_n\rangle$ must belong to $j_{0,2n+1}(S)$. We maintain the following additional condition during the construction:

4. w_n is realized by $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle$, $\ldots$, $\langle k_n - 1, g_n(k_n-1)\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$.

Notice that from this it automatically follows that w_n is a k_n+2-type and that it contains the formulae "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$" and "$v_{k_n+1}+2$ exists and is the largest ordinal," as demanded by rule (7) of G^*.

Finally, we maintain the conditions:

5. w_n is elastic.

6. M_{2n+1} agrees with all later models of $\mathcal{T}$, that is all models M_i for $i > 2n+1$, past $\mathrm{dom}(w_n)$. w_n belongs to M_i for each $i > 2n+1$.

7. All the extenders used in $\mathcal{T}$ have critical points above $\mathrm{rank}(X)$. For each $m > n$, the critical point of $j_{2n+2,2m+2}$ is greater than the domain of w_n. In particular $j_{2n+2,2m+2}(w_n) = w_n$ for each $m \geq n$.

5.16 Remark. It follows from the last condition that p_n has the form $\langle l_i, t_i, j_{2i,2n}(u_i), w_i, x_i \mid i < n\rangle$.

Let us now describe the construction in round n, assuming inductively that we have already constructed the objects corresponding to rounds 0 through $n-1$, and that we maintained conditions (1)–(7) for these rounds.

Set l_n, t_n, and u_n to be the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n, in line with condition (1). The construction continues subject to one of the following cases:

Case 1, $t_n = \emptyset$. The rules of G^* are such that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $M_{2n}\|j_{0,2n}(\nu_{\mathrm{L}})+1$. From the local indiscernibility of ν_{L} and ν_{H} it follows that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{H}})$ in $M_{2n}\|j_{0,2n}(\nu_{\mathrm{H}})+1$. Working in M_{2n} using Lemma 3.22, let $\tau < j_{0,2n}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $j_{0,2n}(\nu_{\mathrm{L}})+3$

is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\mathrm{dcp}(u_n)$ in M_{2n}.

Set E_{2n} ="pad" so that $M_{2n+1} = M_{2n}$ and $j_{2n,2n+1}$ is the identity. Using the one-step lemma, Lemma 3.23, in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$. It is easy now to check that w_n satisfies the conditions of rule (7) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, t_n, u_n\rangle)$.

Finally, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n){}^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in $G_\omega(B)$ following $\langle x_0, \dots, x_{n-1}\rangle$ if n is even. This completes the round. $\dashv$ (Case 1)

Case 2, $t_n \neq \emptyset$. The rules of $j_{0,2n}(G^*)$ following the position p_n are such that u_n exceeds w_{l_n} in M_{2n}. (We are making an implicit use of Remark 5.16 here.) Let κ denote the domain of u_n. Using the one-step lemma in M_{2n} find an extender E_{2n} with critical point $\mathrm{dom}(w_{l_n})$, so that u_n is a subtype of $\mathrm{Stretch}^{E_{2n}}_{\kappa+\omega}(w_{l_n})$. Set $M_{2n+1} = \mathrm{Ult}(M_{2l_n+1}, E_{2n})$, and set $j_{2l_n+1,2n+1}$ to be the ultrapower embedding, so that u_n is a subtype of $j_{2l_n+1,2n+1}(w_{l_n})$.

5.17 Exercise. Complete the precise details of this construction, verifying that there is enough agreement between the various models to make sense of the ultrapower taken.

Let k denote $\mathrm{lh}(t_n)$. Note that $t_{l_n} = \mathrm{pred}(t_n)$, so $\mathrm{lh}(t_{l_n}) = k-1$. Let $\bar{k}$ denote $k-1$. Let $\bar{g}$ denote g_{l_n}, and let $\bar{g}' = j_{2l_n+1,2n+1}(\bar{g})$.

Now w_{l_n} is realized by $j_{0,2l_n+1}(S)$, $\langle 0, \bar{g}(0)\rangle$, $\dots$, $\langle \bar{k}-1, \bar{g}(\bar{k}-1)\rangle$ and $j_{0,2l_n+1}(\nu_{\mathrm{L}})$ in $M_{2l_n+1}\|j_{0,2l_n+1}(\nu_{\mathrm{L}})+3$. Using the elementarity of the embedding $j_{2l_n+1,2n+1}$ it follows that $j_{2l_n+1,2n+1}(w_{l_n})$ is realized by $j_{0,2n+1}(S)$, $\langle 0, \bar{g}'(0)\rangle$, $\dots$, $\langle \bar{k}-1, \bar{g}'(\bar{k}-1)\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$. Since u_n is a subtype of $j_{2ln+1,2n+1}(w_{l_n})$ it must be realized, by the same objects and one more object, at a lower rank. Combining this with the fact that u_n is a $k+2$-type which contains the formula in rule (3) of the definition of G^* (Definition 5.13), we see that there must exist some set z so that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, \bar{g}'(0)\rangle$, $\dots$, $\langle \bar{k}-1, \bar{g}'(\bar{k}-1)\rangle$, $\langle \bar{k}, z\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+1$, and that moreover the function $g = \bar{g}' \cup \{\langle \bar{k}, z\rangle\}$ is a node in $j_{0,2n+1}(S)_{x,t_n}$. Set g_n equal to this function g, securing the demands of condition (C) above. For the record let us note that:

(i) g_n extends $j_{2l_n+1,2n+1}(g_{l_n})$.

We now continue very much as we did in case 1. Using the local indiscernibility of ν_{L} and ν_{H}, we see that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle$, $\dots$, $\langle k-1, g_n(k-1)\rangle$, and $j_{0,2n+1}(\nu_{\mathrm{H}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{H}})+1$. Working in M_{2n+1} using Lemma 3.22, let $\tau < j_{0,2n+1}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle, \ldots, \langle k-1, g_n(k-1)\rangle$, $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$ is elastic. Let w_n be this type. w_n then exceeds $\mathrm{dcp}(u_n)$ in M_{2n+1}.

Using the one-step lemma in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$, with critical point equal to the domain of u_n, so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$, and this secures the main requirement on w_n posed by rule (7) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, t_n, u_n\rangle)$.

Finally, as in case 1, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n){}^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in $G_\omega(B)$ following $\langle x_0, \ldots, x_{n-1}\rangle$ if n is even. This completes the round. ⊣ (Case 2)

5.18 Exercise. Verify that the construction described above maintains conditions (1)–(7).

It remains now to check that every sequence $x = \langle x_n \mid n < \omega\rangle \in X^\omega$ that can be obtained by following the construction described above (with moves x_n for even n supplied by the opponent) is won by player II in $G_\omega(B)$.

Let x, $\mathcal{T}$, $\langle l_n, t_n, u_n, w_n \mid n < \omega\rangle$, and $\langle g_n \mid n < \omega\rangle$ be obtained through the construction above. We work through a series of claims to show that $x \notin B$.

5.19 Claim. *The even branch of $\mathcal{T}$ has an illfounded direct limit.*

Proof. Suppose for contradiction that M_{even} is wellfounded. Let R be the tree of attempts to construct an infinite run of G^*, played according to σ^*. Note that $j_{\mathrm{even}}(R)$ has an infinite branch, consisting of $\bigcup_{n<\omega} j_{2n,\mathrm{even}}(p_n)$. Since M_{even} is wellfounded, the existence of an infinite branch through $j_{\mathrm{even}}(R)$ reflects to M_{even}. Thus, $M_{\mathrm{even}} \models$"there is an infinite run of $j_{\mathrm{even}}(G^*)$, played according to $j_{\mathrm{even}}(\sigma^*)$." Using the elementarity of j_{even} it follows that $V \models$"there is an infinite run of G^* played according to σ^*." But this contradicts the fact that σ^* is a winning strategy for player II, the player who loses infinite runs. ⊣

Let θ be a regular cardinal, large enough that all the objects involved in the construction belong to $V\|\theta$. Let H be a countable elementary substructure of $V\|\theta$, with x, $\mathcal{T}$, $\langle l_n, t_n, u_n, w_n \mid n < \omega\rangle$ and $\langle g_n \mid n < \omega\rangle$ in H. Let P be the transitive collapse of H, and let $\pi\colon P \to H$ be the anti-collapse embedding. Let $\mathcal{U} = \pi^{-1}(\mathcal{T})$ and let $h_n = \pi^{-1}(g_n)$. Let P_i and $\bar{j}_{i,i'}$ denote the models and embeddings of $\mathcal{U}$. Let $\bar{S}$ denote $\pi^{-1}(S)$. Let $\bar{x}_i = \pi^{-1}(x_i)$ and let $\bar{x} = \langle \bar{x}_i \mid i < \omega\rangle$.

Using Theorem 2.3 find an infinite branch b through $\mathcal{U}$ so that there is an embedding $\sigma\colon \bar{P}_b \to V\|\theta$ with $\sigma \circ \bar{j}_b = \pi$.

5.20 Claim. *b is not the even branch.*

Proof. The fact that $\bar{P}_b$ embeds into $V\|\theta$ implies that it is wellfounded. $\bar{P}_{\text{even}}$ is not wellfounded, by Claim 5.19. ⊣

Let $m_0, m_1, \ldots$ be such that $2m_0+1, 2m_1+1, 2m_2+2, \ldots$ lists, in increasing order, all the odd models in b. The tree structure of $\mathcal{T}$, and hence of $\mathcal{U}$, is such that:

- $t_{m_0} = \emptyset$.
- $\text{pred}(t_{m_{i+1}}) = t_{m_i}$.

From the last condition and from condition (i) of the construction it follows that:

- $h_{m_{i+1}}$ extends $\bar{j}_{2m_i+1, 2m_{i+1}+1}(h_{m_i})$.

Letting $h^*_i = \bar{j}_{2m_i+1,b}(h_{m_i})$ it follows that:

- h^*_{i+1} extends h^*_i for each i.

Let $y = \bigcup_{i<\omega} t_{m_i}$ and let $h^* = \bigcup_{i<\omega} h^*_i$. Condition (C) of the construction implies that $\langle \bar{x}\restriction i, y\restriction i, h^*\restriction i\rangle$ is a node in $\bar{j}_b(\bar{S})$. Applying the embedding $\sigma\colon P_b \to V\|\theta$ to this statement, and using the fact that $\sigma \circ \bar{j}_b = \pi$, it follows that $\langle x\restriction i, y\restriction i, \sigma(h^*\restriction i)\rangle$ is a node in $\pi(\bar{S}) = S$. This is true for each i, and hence:

5.21 Claim. $\langle x, y\rangle \in p[S]$.

Proof. Let $h^{**} = \bigcup_{i<\omega} \sigma(h^*\restriction i)$. The argument of the previous paragraph shows that $\langle x, y, h^{**}\rangle$ is an infinite branch through S. ⊣

Recall that $A = p[S]$ and that $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. From the last claim it follows that $x \notin B$, and therefore x is won by player II in $G_\omega(B)$, as required. ⊣

5.22 Definition. Let M be a model of ZFC^-. Let X belong to M and let $S \in M$ be a tree on $X \times U$ for some set $U \in M$. Define $\text{gp}(S)$, the *generalized projection* of S, by setting $x \in \text{gp}(S)$ iff there exists a length ω iteration tree $\mathcal{T}$ on M, using only extenders with critical points above $\text{rank}(X)$, so that for every wellfounded cofinal branch b of $\mathcal{T}$, $x \in p[j^{\mathcal{T}}_b(S)]$. An iteration tree $\mathcal{T}$ witnessing that $x \in \text{gp}(S)$ is said to put x in a *shifted projection* of S. Notice that the tree must be such that $x \in p[j^{\mathcal{T}}_b(S)]$ for *all* wellfounded cofinal branches of $\mathcal{T}$.

5.23 Exercise. Let M be a model of ZFC and let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$ and let $S \in M$ be a tree on $X \times \omega \times \gamma$ for some ordinal γ. Let G^* be the game of Definition 5.13 but relativized to M. Suppose $M \models$"player II has a winning strategy in G^*." Prove that there is a strategy σ for player II in the game on X so that, in V, every infinite play according to σ belongs to $\mathrm{gp}(S)$.

Hint. Let $\sigma^* \in M$ be a winning strategy for player II in G^*. Imitate the construction in the proof of Lemma 5.15 to define a strategy σ for II in the game on X. Show that if $x \in X^\omega$ and $\mathcal{T}$ are produced by the construction in the proof of Lemma 5.15, then $\mathcal{T}$ witnesses that x belongs to a shifted projection of S: Claim 5.19 shows that the even branch of $\mathcal{T}$ is illfounded, and the argument following Claim 5.20 can be modified to produce, for each cofinal branch b other than the even branch, some y and f so that $\langle x, y, f\rangle \in [j_b(S)]$. ⊣

Lemmas 5.14 and 5.15 combine to show that $G_\omega(B)$ is determined: G^* is determined since it is a closed game, and by Lemmas 5.14 and 5.15 the player who has a winning strategy in G^* has a winning strategy in $G_\omega(B)$. We thus obtained the following theorem:

5.24 Theorem. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$ and let $A \subseteq (X \times \omega)^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+ homogeneously Suslin. Then B is determined.*

In the next section we weaken the assumption, from homogeneously Suslin to universally Baire. But first we continue toward a proof that B is homogeneously Suslin.

Let Γ be the map that assigns to each position $q^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ in the game G^* the move $\langle l_n, t_n, u_n\rangle$ described in the proof of Lemma 5.14. By this we mean the move that the construction there would produce for round n, assuming that the moves of the previous rounds were $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$. (The construction appears between conditions (ii) and (iii) in the proof of Lemma 5.14. Notice that this part does not depend on the strategy σ^*.) If the moves in $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ do not satisfy the inductive conditions in the proof of Lemma 5.14, then leave $\Gamma(q^*)$ undefined.

Given a sequence $q = \langle x_i, w_i \mid i < n\rangle$ define q^* to be the sequence $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ where for each $m < n$, $\langle l_m, t_m, u_m\rangle$ is equal to $\Gamma(q^*\restriction m)$. If for some $m < n$, $q^*\restriction m$ is not a legal position in G^* or $\Gamma(q^*\restriction m)$ is undefined, then leave q^* undefined.

Let $R \subseteq (X \times V\|\delta)^{<\omega}$ be the tree of sequences $q = \langle x_i, w_i \mid i < n\rangle$ so that q^* is defined.

5.25 Exercise. Suppose that $x \in p[R]$. Prove that $x \in B$.

Hint. Let $\langle w_i \mid i < \omega\rangle$ be such that $\langle x_i, w_i \mid i < n\rangle \in R$ for each $n < \omega$. Let q_n denote $\langle x_i, w_i \mid i < n\rangle$. Note that for each $n < \omega$, q_n^* is defined. Let $q^* = \bigcup_{n<\omega} q_n^*$. Check that q^* is an infinite run of G^*, satisfying all the conditions in the proof of Lemma 5.14. Use the final argument in that proof to conclude that $x \in B$. $\dashv$

Given $z \in X^\omega$ let $\langle l_n^z, t_n^z, u_n^z, w_n^z, x_n^z \mid n < \omega\rangle$, $\mathcal{T}^z$, and $\langle g_n^z \mid n < \omega\rangle$ be the objects obtained by constructing subject to the conditions in the proof of Lemma 5.15, with condition (1) replaced by the condition "$\langle l_n, t_n, u_n\rangle = j_{0,2n}(\Gamma)(p_n)$," and condition (3) replaced by the condition "$x_n = z_n$ for all n." These two replacements remove the use of the opponent and of σ^* in the construction. The use of σ^* is replaced by a use of Γ and of the odd half of z. The use of the opponent is replaced by a use of the even half of z.

Notice that the dependence of the construction on z is continuous, in the sense that knowledge of $z\restriction n$ suffices to determine the construction in rounds 0 through $n-1$. These rounds construct, among other things, $\mathcal{T}^z\restriction 2n{+}1$, and $\langle w_0, \ldots, w_{n-1}\rangle$. We have therefore maps $s \mapsto \mathcal{T}^s$, $s \mapsto \langle l_i^s, t_i^s, u_i^s, w_i^s, x_i^s \mid i < \mathrm{lh}(s)\rangle$, and $s \mapsto \langle g_i^s \mid i < \mathrm{lh}(s)\rangle$, defined on $s \in X^{<\omega}$, with the properties:

- $\mathcal{T}^s$ is an iteration tree of length $2\,\mathrm{lh}(s)+1$, leading to a final model indexed $2\,\mathrm{lh}(s)$.
- $\mathcal{T}^z = \bigcup_{n<\omega} \mathcal{T}^{z\restriction n}$.
- $l_i^z = l_i^s$ whenever z extends s and $i < \mathrm{lh}(s)$, and similarly with t_i^z, u_i^z, w_i^z, x_i^z, and g_i^z.

Let M_i^s, for $i \le 2\,\mathrm{lh}(s)$, be the models of the tree $\mathcal{T}^s$. Let $j_{i,i'}^s$ be the embeddings of the tree.

5.26 Exercise. Show that $\langle x_i^s, w_i^s \mid i < \mathrm{lh}(s)\rangle$ belongs to $j_{0,2\,\mathrm{lh}(s)}^s(R)$.

Hint. Let $q = \langle x_i^s, w_i^s \mid i < \mathrm{lh}(s)\rangle$. Let $p = \langle l_i^s, t_i^s, j_{2i,2\,\mathrm{lh}(s)}^s(u_i^s), w_i^s, x_i^s \mid i < \mathrm{lh}(s)\rangle$. Use the fact that $\langle l_i^s, t_i^s, u_i^s\rangle = j_{0,2i}(\Gamma)(p\restriction i)$ to show that q^* (in the sense of $M_{2\,\mathrm{lh}(s)}^s$) is equal to p. $\dashv$

Define M_s to be the last model of the tree $\mathcal{T}^s$, namely the model $M_{2\,\mathrm{lh}(s)}^s$. Define $j_{s,s^*}\colon M_s \to M_{s^*}$ to be the embedding $j_{2\,\mathrm{lh}(s),2\,\mathrm{lh}(s^*)}^{s^*}$. Define φ_s to be the sequence $\langle w_i^s \mid i < \mathrm{lh}(s)\rangle$.

5.27 Exercise. Prove that R is homogeneous by showing that the system $\langle M_s, \varphi_s, j_{s,s^*} \mid s < s^* \in X^{<\omega}\rangle$ satisfies the conditions in Exercise 4.2. Conclude that B is homogeneously Suslin.

Hint. Condition (2) of Exercise 4.2 follows from the previous exercise. For condition (3): The direct limit of $\langle M_s, j_{s,s^*} \mid s < s^* < x\rangle$ is simply the direct limit along the even branch of $\mathcal{T}^x$. You can use its illfoundedness as a replacement for Claim 5.19, and proceed from there as in the proof of Lemma 5.15, to show that $x \notin B$, and hence by Exercise 5.25, $x \notin p[R]$. To conclude that B is homogeneously Suslin you now only need the converse to Exercise 5.25. To prove it use the fact that illfoundedness of the direct limit of $\langle M_s, j_{s,s^*} \mid s < s^* < x\rangle$ implies not only $x \notin p[R]$, but $x \notin B$. ⊣

5.28 Exercise. Prove that the Martin–Solovay tree for B (see Exercise 5.2) is homogeneous.

Hint. Embed R into the Martin–Solovay tree for B, and use the embedding to transfer the homogeneity measures on R to the Martin–Solovay tree. ⊣

The exercises above establish that B is homogeneously Suslin. With a small additional adjustment we obtain the following:

5.29 Exercise. Let δ be a Woodin cardinal. Let X belong to $V\|\delta$ and let $A \subseteq (X \times \omega)^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+ homogeneously Suslin. Then B is κ-homogeneously Suslin for each $\kappa < \delta$.

Hint. Fix $\kappa < \delta$. Revise the construction in the proof of Lemma 5.14 to make sure that $\mathrm{dom}(u_0) > \kappa$. Show that if Γ is defined using this revised construction, then the embeddings j_{s,s^*} obtained above all have critical points above κ. ⊣

5.30 Corollary. *Suppose that there are n Woodin cardinals and a measurable cardinal above them. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_{n+1}$. Then A is homogeneously Suslin.*

Proof. Let $\delta_1 < \cdots < \delta_n$ be the Woodin cardinals, and let $\kappa > \delta_n$ be the measurable cardinal. Let $\delta_0 = \aleph_0$.

Let $A_k \subseteq (\omega^\omega)^k$ be such that A_{n+1} is $\mathbf{\Pi}^1_1$, $A_k = \{\langle x, y_1, \ldots, y_{k-1}\rangle \mid (\forall y_k)\langle x, y_1, \ldots, y_k\rangle \notin A_{k+1}\}$ for each $k \le n$, and $A_1 = A$.

By Theorem 4.15, A_{n+1} is $(\delta_n)^+$-homogeneously Suslin. Successive applications of Exercise 5.29, starting from $k = n$ and working down to $k = 1$, show that A_k is $(\delta_{k-1})^+$-homogeneously Suslin. Finally then $A = A_1$ is homogeneously Suslin. ⊣

5.31 Corollary. *Suppose that there are n Woodin cardinals and a measurable cardinal above them. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_{n+1}$. Then $G_\omega(A)$ is determined.*

6. Universally Baire Sets

Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let S be a tree on $X \times \omega \times \gamma$ for some ordinal γ, let $A = p[S] \subseteq X^\omega \times \omega^\omega$, and let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. In the previous section we showed that if S is δ^+-homogeneous then $G_\omega(B)$ is determined. Here we work without the assumption of homogeneity, and try to salvage as much determinacy as we can. We cannot hope for actual determinacy since every set is Suslin under the axiom of choice, but not every set is determined. The approximation for determinacy that we salvage is the following lemma. Recalling a standard notation, $\mathrm{Col}(\omega, \delta)$ is the poset that adjoins a map from ω onto δ using restrictions of the map to finite sets as conditions.

6.1 Lemma. *Let g be $\mathrm{Col}(\omega, \delta)$-generic over V. In $V[g]$ define B^* to be the set $\{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin p[S]\}$, where X^ω, the quantifier $(\forall y)$, and the projection $p[S]$ are all computed in $V[g]$. Then at least one of the following cases hold:*

1. *In V, player* II *has a winning strategy in the game $G_\omega(B)$.*
2. *In $V[g]$, player* I *has a winning strategy in $G_\omega(B^*)$.*

With a sufficiently absolute set B the lemma can be used to obtain actual determinacy, as we shall see later on.

Proof of Lemma 6.1. Let G^* be the game defined in the previous section, specifically in Definition 5.13. Notice that the game is defined with no reference to the homogeneity system of the previous section, and so we may use it in the current context. Notice further that Lemma 5.15 is proved without use of the homogeneity system. It too applies in the current context, showing that if player II has a winning strategy in G^* then player II has a winning strategy in $G_\omega(B)$. To complete the proof of Lemma 6.1 it thus suffices to show that if player I has a winning strategy in G^*, then condition (2) of Lemma 6.1 holds true.

Let σ^* be a winning strategy for player I in G^*. Let $\rho\colon \delta \to V\|\delta$ be a bijection. To be precise we emphasize that both σ^* and ρ are taken in V. Working now in $V[g]$, notice that $\rho \circ g$ is a bijection of ω and $V\|\delta$.

In Lemma 5.14 we used the homogeneity measures for S to ascribe auxiliary moves for player II in G^* while playing against σ^*. We cannot do the same here since T is not assumed to be homogeneous. Instead, we plan to ascribe to player II the $\rho \circ g$-first legal move in each round.

6.2 Claim. *Let $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ be a legal position in G^*. Then there is a move $\langle l_n, t_n, u_n\rangle$ which is legal for player* II *in G^* following p^*.*

Proof. Let $\zeta < \delta$ be large enough that all the moves made in p belong to $V\|\zeta$. Using Lemma 3.22 let $\kappa < \delta$ be such that the κ-type of S in $V\|\nu_{\mathrm{L}} + 1$ is elastic, and such that $\kappa > \zeta$. Set u to be the κ-type of S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. (ν_{L} here is taken from the lexicographically least pair of local indiscernibles relative to $\max\{\delta, \mathrm{rank}(S)\}$.) It is easy to check that the triple $\langle 0, \emptyset, u\rangle$ is legal for II in G^* following p^*. It falls under the case of rule (5) in Definition 5.13. $\dashv$

Call a number $e < \omega$ *valid* at a position $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ in G^* just in case that $(\rho \circ g)(e)$ is a legal move for player II in G^* following p^*. By this we mean that $(\rho \circ g)(e)$ is equal to a tuple $\langle l_n, t_n, u_n\rangle \in V\|\delta$ that satisfies the relevant rules in Definition 5.13. By the last claim there is always a number which is valid at p^*.

6.3 Definition. A position $\langle x_0, \ldots, x_{n-1}\rangle$ in $G_\omega(B^*)$ is *nice* if it can be expanded to a position $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle^\frown\langle l_n, t_n, u_n, w_n\rangle$ in G^* so that:

1. p^* is according to σ^*.
2. For each $m \leq n$, $\langle l_m, t_m, u_m\rangle$ is equal to $(\rho \circ g)(e)$ for the *least* number e which is valid at $p^*\restriction m$.

Notice that if p is nice then the expansion p^* is unique: condition (1) uniquely determines w_m for each $m \leq n$, and condition (2) uniquely determines l_m, t_m, and u_m for each $m \leq n$. Define a strategy σ for player I in $G_\omega(B^*)$ by setting $\sigma(p) = \sigma(p^*)$ in the case that p is a nice position of even length. (It is easy to check that all finite plays by σ lead to nice positions. So there is no need to define σ on positions which are not nice.)

The generic g comes in to the definition of σ through condition (2) in Definition 6.3. σ is thus not an element of V, but of $V[g]$. We now aim to show that, in $V[g]$, σ is winning for I in $G_\omega(B^*)$.

Let $x \in V[g]$ be an infinite run, played according to σ. Suppose for contradiction that $x \notin B$, and let $y \in V[g]$ and $f \in V[g]$ be such that $\langle x, y, f\rangle$ is an infinite branch through S.

For each $n < \omega$ let p^*_n be the unique expansion of $x\restriction n$ that satisfies the conditions of Definition 6.3. Let $p^* = \bigcup_{n<\omega} p^*_n$. Let l_i, t_i, u_i, and w_i be such that $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < \omega\rangle$. Let e_n be the least number valid at $p^*\restriction n$, so that $\langle l_i, t_i, u_i\rangle = (\rho \circ g)(e_i)$.

We work recursively to construct sequences $n_0 < n_1 < \ldots$ and $\alpha_0, \alpha_1, \ldots$ so that for each i:

1. $t_{n_i} = y\restriction i$.
2. u_{n_i} is realized by S, $\langle 0, f(0)\rangle$, $\ldots$, $\langle i-1, f(i-1)\rangle$, and α_i in $V\|\alpha_i + 1$.

Set to begin with $n_0 = 0$ and $\alpha_0 = \nu_{\mathrm{L}}$. The rules of G^* are such that $t_0 = \emptyset$ and u_0 is the type of S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. Conditions (1) and (2) for $i = 0$ therefore hold with these settings.

Suppose now that n_i and α_i have been defined and that conditions (1) and (2) hold for i. The rules of G^* are such that w_{n_i} is a subtype of $\mathrm{dcp}(u_{n_i})$, and must therefore be realized at a lower level. In fact, using the realization of u_{n_i} given by condition (2) above, the specific requirements in rule (7) in Definition 5.13 are such that there must exist some ordinal $\beta < \alpha_i$ so that

w_{n_i} is realized by S, $\langle 0, f(0)\rangle$, ..., $\langle i-1, f(i-1)\rangle$, and β in $V\|\beta+3$, and so that $\beta > \max\{\delta, \mathrm{rank}(S)\}$.

Let α_{i+1} be this ordinal β. For the record we note that:

(i) $\alpha_{i+1} < \alpha_i$.

(ii) w_{n_i} is realized by S, $\langle 0, f(0)\rangle$, ..., $\langle i-1, f(i-1)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+3$.

It remains to define n_{i+1} in such a way that conditions (1) and (2) hold for $i+1$.

6.4 Claim. *Let $E = \max\{e_0, \ldots, e_{n_i}\}$. There exists $e < \omega$ and $\kappa < \delta$ so that:*

(a) $(\rho \circ g)(e)$ has the form $\langle l, t, u\rangle$ with $l = n_i$, $t = y{\restriction}i+1$, and u equal to the κ-type of S, $\langle 0, f(0)\rangle$, ..., $\langle i, f(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+1$.

(b) $\mathrm{dcp}(u)$ is elastic.

(c) $e > E$.

(d) κ is large enough that $(\rho \circ g)(0)$, ..., $(\rho \circ g)(e-1)$ all belong to $V\|\kappa$.

Proof. Let $D \subseteq \mathrm{Col}(\omega, \delta)$ be the set of conditions q so that conditions (a)–(d) hold for some $e < \mathrm{dom}(q)$ and $\kappa < \delta$, with $(\rho \circ g)$ replaced by $(\rho \circ q)$ in conditions (a) and (d). Notice that D is defined in V: it only makes reference to $f{\restriction}i+1$ and $y{\restriction}i+1$. Using Lemma 3.22 it is easy to check that D is dense. Thus $g \cap D$ is non-empty and the claim follows. $\dashv$

Let e be given by the last claim. Let $\langle l, t, u\rangle = (\rho \circ g)(e)$, and let $\kappa = \mathrm{dom}(u)$.

6.5 Claim. *$\langle l, t, u\rangle$ is a legal move for player* II *in G^* following $p^*{\restriction}n$, for every n such that:*

1. $n > n_i$.

2. $\mathrm{dom}(u_{n-1}) < \kappa$.

Proof. This is easy to verify, using conditions (1), (ii), (a), and (b) above, and the fact that $\langle x{\restriction}i+1, y{\restriction}i+1, f{\restriction}i+1\rangle$ is a node in S. $\dashv$

6.6 Claim. *There exists $n < \omega$ so that $e_n = e$.*

Proof. Let n be least so that $e_n \geq e$. Since $e > E = \max\{e_0, \ldots, e_{n_i}\}$, certainly $n > n_i$. Note that $e_{n-1} < e$ and so from condition (d) it follows that $(\rho \circ g)(e_{n-1})$ belongs to $V\|\kappa$. In particular then u_{n-1} belongs to $V\|\kappa$, so certainly $\mathrm{dom}(u_{n-1}) < \kappa$. Applying Claim 6.5 it follows that $\langle l, t, u\rangle$ is legal for II in G^* following $p^*{\restriction}n$, and hence e is valid at $p^*{\restriction}n$. Since e_n is the least number which is valid at $p^*{\restriction}n$, it must be that $e_n \leq e$. We have $e_n \geq e$ by the initial choice of n. Thus $e_n = e$. $\dashv$

Set n_{i+1} equal to the number n given by the last claim. By condition (a) of Claim 6.4 then, $l_{n_{i+1}}$, $t_{n_{i+1}}$, and $u_{n_{i+1}}$ are such that $l_{n_{i+1}} = n_i$, $t_{n_{i+1}} = y{\restriction}i+1$, and $u_{n_{i+1}}$ is equal to the κ-type of S, $\langle 0, f(0)\rangle, \ldots, \langle i, f(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+1$. In particular conditions (1) and (2) hold for $i+1$.

Working by recursion we completed the construction of the sequences $\langle n_i \mid i < \omega\rangle$ and $\langle \alpha_i \mid i < \omega\rangle$. By condition (i) above the sequence $\langle \alpha_i \mid i < \omega\rangle$ is descending. The construction of this infinite descending sequence was based on the assumption that $\langle x, y, f\rangle$ is an infinite branch through S. (This assumption was used in the proof of Claim 6.5.) The assumption must therefore be false, and this shows that x, an arbitrary play according to σ^* in $V[g]$, must belong to B^*. This completes the proof of Lemma 6.1. $\dashv$

6.7 Corollary. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be $\mathrm{Col}(\omega, \delta)$-generic over V. Then at least one of the following holds:*

1. *$V \models$ "player II has a winning strategy in the game $G_\omega(\neg p[T])$."*

2. *$V[g] \models$ "player I has a winning strategy in the game $G_\omega(\neg p[T])$."*

($\neg p[T]$ here is the complement of the projection of T. Notice that $\neg p[T]$ need not be the same in $V[g]$ and in V.)

Proof. Immediate from Lemma 6.1 by introducing a vacuous coordinate, more precisely by using the tree $S = \{\langle s, t, f\rangle \in (X \times \omega \times \gamma)^{<\omega} \mid \langle s, f\rangle \in T\}$. $\dashv$

6.8 Exercise. It may seem that we are losing ground in passing from the lemma to the corollary, but in fact we are not. Prove that Lemma 6.1 is a consequence of Corollary 6.7.

Hint. Let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ be given. Let $\varphi\colon \omega \times \gamma \to \gamma'$ be a bijection of $\omega \times \gamma$ onto an ordinal γ'. Define a tree T on $X \times \gamma'$ in such a way that $\langle x, y, f\rangle \in [S]$ iff $\langle x, g\rangle \in [T]$ where $g(n) = \varphi(\langle y_n, f(n)\rangle)$. Use Corollary 6.7 with T. $\dashv$

6.9 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Let $T \in M$ be a tree on $X \times \gamma$ for some ordinal γ. Let g be $\mathrm{Col}(\omega, \delta)$-generic over M. Prove that at least one of the following holds:

1. There is a strategy σ for player II in the game on X so that, in V, every infinite play according to σ belongs to $\mathrm{gp}(T)$.

2. There is a strategy $\sigma \in M[g]$ for player I in the game on X so that, in $M[g]$, every infinite play according to σ avoids $p[T]$.

Hint. Relativize the proof of Corollary 6.7 to M, but replace the use of Lemma 5.15, which ultimately leads to the case of condition (1) in Corollary 6.7, with a use of Exercise 5.23. $\dashv$

6.10 Corollary. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be* $\mathrm{Col}(\omega, \delta)$*-generic over V. Then at least one of the following holds:*

1. $V \models$ *"player* I *has a winning strategy in the game $G_\omega(p[T])$."*

2. $V[g] \models$ *"player* II *has a winning strategy in the game $G_\omega(p[T])$."*

(Notice that $p[T]$ need not be the same in $V[g]$ and in V.)

Proof. Immediate from Corollary 6.7, using continuous substitution to reverse the roles of the players. Let us just point out that both here and in Corollary 6.7, the player who has a winning strategy in V is the player who wants to get into $p[T]$, and the player who has a winning strategy in $V[g]$ is the player who wants to avoid $p[T]$. $\dashv$

We can use various forms of absoluteness to obtain actual determinacy, either in V or in $V[g]$, from Corollary 6.10:

6.11 Lemma. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be* $\mathrm{Col}(\omega, \delta)$*-generic over V. Suppose that there is a tree S in V so that $V[g] \models$ "$p[S] = \neg p[T]$." Then $V[g] \models$ "$G_\omega(p[T])$ is determined."*

Proof. It is enough to show that if case 1 of Corollary 6.10 holds, then player I wins $G_\omega(p[T])$ also in $V[g]$.

Suppose then that player I wins $G_\omega(p[T])$ in V, and let σ witness this. Let R be the tree of attempts to construct a pair $\langle x, f\rangle$ so that $x \in X^\omega$ is a play according to σ, and $\langle x, f\rangle \in [S]$.

The tree R belongs to V. An infinite branch in V through R would produce an x which belongs to both $p[T]$ and $p[S]$. But then the same x, taken in $V[g]$, would exhibit a contradiction to the assumption of the lemma that $(p[S])^{V[g]}$ and $(p[T])^{V[g]}$ are complementary.

Thus R has no infinite branches in V. By absoluteness R has no infinite branches in $V[g]$ either. It follows that all plays according to σ in $V[g]$ belong to the complement of $(p[S])^{V[g]}$, which by assumption is $(p[T])^{V[g]}$. So σ witnesses that player I wins $G_\omega(p[T])$ in $V[g]$. $\dashv$

6.12 Corollary (Woodin). *Let δ be a Woodin cardinal and let g be a* $\mathrm{Col}(\omega, \delta)$*-generic filter over V. Then $V[g]$ is a model of Δ^1_2 (lightface) determinacy.*

Let X be hereditarily countable. A set $C \subseteq X^\omega$ is *λ-universally Baire* if all its continuous preimages, to topological spaces with regular open bases of cardinality $\leq \lambda$, have the property of Baire. C is ∞-universally Baire if it is λ-universally Baire for all cardinals λ. Feng–Magidor–Woodin [4] provides the following convenient characterization of universally Baire sets, and the basic results in Exercises 6.15 and 6.16:

6.13 Definition. A pair of trees T and T^* on $X \times \gamma$ and $X \times \gamma^*$ respectively is *exhaustive* for a poset $\mathbb{P}$ if the statement "$p[T] \cup p[T^*] = X^\omega$" is forced to hold in all generic extensions of V by $\mathbb{P}$.

6.14 Fact (Feng–Magidor–Woodin [4]). Let X be hereditarily countable, let $C \subseteq X^\omega$, and let λ be an infinite cardinal. C is λ-universally Baire iff there are trees T and T^* so that:

1. $p[T] = C$ and $p[T^*] = X^\omega - C$.
2. The pair $\langle T, T^* \rangle$ is exhaustive for all posets of size $\leq \lambda$.

6.15 Exercise. Suppose T and T^* are trees so that:

1. $p[T] \cap p[T^*]$ is empty.
2. $\langle T, T^* \rangle$ is exhaustive for $\mathrm{Col}(\omega, \lambda)$.

Prove that $p[T^*] = \mathbb{R} - p[T]$, and that $p[T]$ is λ-universally Baire.

Hint. Use condition (2) and simple absoluteness to argue that $p[T] \cup p[T^*] = \mathbb{R}$. This establishes that $p[T^*] = \mathbb{R} - p[T]$. Basic forcing arguments show that condition (2) here is equivalent to the corresponding condition in Fact 6.14. ⊣

6.16 Exercise (Feng–Magidor–Woodin [4]). A set $C \subseteq X^\omega$ is *weakly homogeneously Suslin* (respectively, *weakly λ-homogeneously Suslin*) if it is the projection to X^ω of a homogeneously Suslin (respectively, λ-homogeneously Suslin) subset of $X^\omega \times \omega^\omega$. Prove that if C is weakly λ^+-homogeneously Suslin then it is λ-universally Baire.

Hint. Let $A \subseteq X^\omega \times \omega^\omega$ be λ^+-homogeneously Suslin with $p[A] = C$. Let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ be a λ^+-homogeneous tree projecting to A, and let $\langle \mu_{s,t} \mid \langle s, t \rangle \in (X \times \omega)^{<\omega} \rangle$ be a λ^+-homogeneity system for S.

Let T be equal to S, viewed as a tree on $X \times (\omega \times \gamma)$, so that T projects to $p[A] = C$. Let T^* be the Martin–Solovay tree for the complement of $p[A]$, defined in Exercise 5.2. Prove that $\langle T, T^* \rangle$ is exhaustive for every poset $\mathbb{P}$ of size $\leq \lambda$. You will need the following claim, which follows from the completeness of the measures $\mu_{s,t}$: Let $\dot{\rho} \in V^{\mathbb{P}}$ be a function from $S_{s,t}$ into the ordinals. Then there is a $\mu_{s,t}$-measure one set Z so that $\dot{\rho} \restriction \check{Z}$ is forced to belong to V. ⊣

Using the characterization in Fact 6.14 we can prove, from a Woodin cardinal δ, that δ-universally Baire sets are determined. In light of Exercise 6.16 this is a strengthening of Theorem 5.24:

6.17 Theorem. *Suppose that C is δ-universally Baire, and that δ is a Woodin cardinal. Then $G_\omega(C)$ is determined.*

Proof. Let T and T^* witness that C is δ-universally Baire. Apply Corollary 6.10 with T and Corollary 6.7 with T^*.

If case 1 of Corollary 6.10 with T holds, then player I wins $G_\omega(C)$ in V. If case 1 of Corollary 6.7 with T^* holds, then player II wins $G_\omega(C)$ in V. Thus it suffices to show that it cannot be that case 2 holds in both applications.

Suppose for contradiction that case 2 holds in both applications. Then in $V[g]$ player II wins $G_\omega(p[T])$ and player I wins $G_\omega(\neg p[T^*])$. Pitting I's winning strategy against II's winning strategy we obtain a real $x \in V[g]$ which does not belong to $(p[T])^{V[g]}$ and does belong to $(\neg p[T^*])^{V[g]}$. In other words x belongs to neither $(p[T])^{V[g]}$ nor $(p[T^*])^{V[g]}$. But this contradicts the fact that $\langle T, T^*\rangle$ is exhaustive for $\mathrm{Col}(\omega, \delta)$. $\dashv$

Our plan for the future is to prove $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ by proving, from large cardinals, that the least non-determined set in $\mathrm{L}(\mathbb{R})$, if it exists, is universally Baire, and then appealing to Theorem 6.17 to conclude that in fact the set is determined.

7. Genericity Iterations

Given a tree S on $X \times U_1 \times U_2$, define $\mathrm{dp}(S)$, the *demanding projection* of S, by putting $x \in \mathrm{dp}(S)$ iff there exists $f_1 \colon \omega \to U_1$ and $f_2 \colon \omega \to U_2$ so that $\langle x, f_1, f_2\rangle \in [S]$ and so that f_1 is onto U_1. It is the final clause, that f_1 must be onto U_1, that makes the demanding projection more demanding than the standard projection $p[S]$.

Let M be a model of ZFC and let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$ and let $S \in M$ be a tree on $X \times U_1 \times U_2$ for some sets $U_1, U_2 \in M$. For convenience suppose that $U_1 \cap U_2 = \emptyset$. For further convenience suppose that U_1 and U_2 are the smallest (meaning $\subseteq$-minimal) sets so that S is a tree on $X \times U_1 \times U_2$. U_1 and U_2 are then definable from S.

Define $\mathrm{gdp}(S)$, the *generalized demanding projection* of S, by setting $x \in \mathrm{gdp}(S)$ iff there exists a length ω iteration tree $\mathcal{T}$ on M, using only extenders with critical points above $\mathrm{rank}(X)$, so that for every wellfounded cofinal branch b of $\mathcal{T}$, $x \in \mathrm{dp}(j_b^{\mathcal{T}}(S))$.

An iteration tree $\mathcal{T}$ witnessing that $x \in \mathrm{gdp}(S)$ is said to put x in a *shifted demanding projection* of S. Note that the tree must be such that $x \in \mathrm{dp}(j_b^{\mathcal{T}}(S))$ for *every* cofinal wellfounded branch of $\mathcal{T}$.

$$\begin{array}{c|cccccccccc}
\mathrm{I} & & w_0 & x_0 & & w_1 & & & w_2 & x_2 & \dots \\
\hline
\mathrm{II} & l_0, u_0 & & & l_1, u_1 & & x_1 & l_2, u_2 & & & \dots
\end{array}$$

Diagram 10: The game G^*.

The generalized projection here is similar to the one in Definition 5.22, only using the demanding projection instead of the standard projection. We work next to obtain some parallel to the result in Exercise 6.9, for the generalized demanding projection. We work with the objects M, X, δ, and S fixed. We assume throughout that U_1 and $\mathcal{P}^M(\delta)$ are countable in V, so that in V there are surjections onto U_1, and there are $\mathrm{Col}(\omega, \delta)$ filters which are generic over M.

7.1 Definition. Working inside M, define G^* to be played according to Diagram 10 and the following rules:

1. $x_n \in X$.
2. u_n is a $2k_n + 2$-type for some number k_n, $\mathrm{dcp}(u_n)$ is elastic, and, setting $s_n = x{\restriction}k_n$, u_n contains the formula "$\langle \widetilde{s}_n, a, b\rangle$ is a node in v_0 where $a = \{v_1, v_3, \dots, v_{2k_n - 1}\}$ and $b = \{v_2, v_4, \dots, v_{2k_n}\}$."
3. If $n > 0$ then $\mathrm{dom}(u_n) > \mathrm{dom}(u_{n-1})$. $\mathrm{dom}(u_0) > \mathrm{rank}(X)$.
4. If $k_n = 0$ then u_n is realized by S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$.
5. If $k_n \neq 0$ then $l_n < n$ is such that $k_{l_n} = k_n - 1$, and u_n exceeds w_{l_n}.
6. w_n is a $2k_n + 3$-type, w_n is a subtype of $\mathrm{dcp}(u_n)$, and w_n contains the formulae "$v_{2k_n+2} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$," "$v_{2k_n+2} + 2$ exists and is the largest ordinal," and "v_{2k_n+1} has the form $\langle k_n, z\rangle$ with $z \in A_1$, where A_1, A_2 are the smallest sets so that v_0 is a tree on $\widetilde{X} \times A_1 \times A_2$."

The first player to violate any of the rules loses. Infinite runs where all rules have been followed are won by player I.

7.2 Remark. The key difference between the definition here and that in Section 5 is the addition of variables to the types. The use of these variables is governed by rules (2) and (6). Rule (2) is such that the sets realizing $v_1, \dots, v_{2k}$ must form a node $\langle a, b\rangle$ of S_x. Rule (6) is such that v_{2k+1} must be realized by a pair $\langle k, z\rangle$ with $z \in U_1$.

A smaller difference is the elimination here of the moves t_n of Section 5. These moves correspond to the vacuous coordinate in the derivation of Corollary 6.7 from Lemma 6.1, and are not needed in a direct proof.

For $x \in X^\omega$ define $G(\neg S_x)$ to be the following game: players I and II alternate moves as in Diagram 11 to construct sequences $f_1 = \langle f_1(n) \mid n < \omega\rangle \in (U_1)^\omega$ and $f_2 = \langle f_2(n) \mid n < \omega\rangle \in (U_2)^\omega$. If at any point $\langle x\restriction n, f_1\restriction n, f_2\restriction n\rangle \notin S$ then player I wins. Otherwise player II wins.

$$\begin{array}{c|ccccc} \mathrm{I} & f_1(0) & & f_1(1) & & \cdots \\ \hline \mathrm{II} & & f_2(0) & & f_2(1) & & \cdots \end{array}$$

Diagram 11: The game $G(\neg S_x)$.

Define $\Game(\neg S)$ by setting $x \in \Game(\neg S)$ iff I has a winning strategy in $G(\neg S_x)$.

7.3 Exercise. Suppose $x \notin \mathrm{dp}(S)$. Prove that $x \in \Game(\neg S)$.

7.4 Lemma. *Let g be* $\mathrm{Col}(\omega, \delta)$*-generic over M, and let $B = \Game(\neg S)$ in the sense of $M[g]$. Suppose that $M \models$ "player* I *has a winning strategy in G^*." Then $M[g] \models$ "player* I *has a winning strategy in $G_\omega(B)$."*

Proof. We adapt the construction in the proof of Lemma 6.1.

Let $\sigma \in M$ be a winning strategy for player I in G^*. Let $\rho \in M$ be a bijection of δ onto $V\|\delta$. Call a number $e < \omega$ *valid* at a position $p^* = \langle l_i, u_i, w_i, x_i \mid i < n\rangle$ in G^* just in case that $(\rho \circ g)(e)$ is a legal move for player II in G^* following p^*. Adapting the proof of Claim 6.2, it is easy to see that player II always has a legal move in G^*, so that there is always a number which is valid at p^*.

7.5 Definition. Call a position $\langle x_0, \ldots, x_{n-1}\rangle$ in $G_\omega(B)$ *nice* if it can be expanded to a position $p^* = \langle l_i, u_i, w_i, x_i \mid i < n\rangle^\frown\langle l_n, u_n, w_n\rangle$ in G^* so that:

1. p^* is according to σ^*.
2. For each $m \leq n$, $\langle l_m, u_m\rangle$ is equal to $(\rho \circ g)(e)$ for the *least* number e which is valid at $p^*\restriction m$.

Notice that if p is nice then the expansion p^* is unique. Define a strategy σ for player I in $G_\omega(B)$ by setting $\sigma(p) = \sigma(p^*)$ in the case that p is a nice position of even length. (All finite plays by σ lead to nice positions, so there is no need to define σ on positions which are not nice.)

We now aim to show that, in $M[g]$, σ is winning for I in $G_\omega(B)$. Again we adapt the argument in the proof of Lemma 6.1.

Let $x \in M[g]$ be an infinite run of $G_\omega(B)$, played according to σ. Suppose for contradiction that $x \notin B$. This implies that there is a strategy $\tau \in M[g]$ which is winning for II in $G(\neg S_x)$. We intend to use τ and the nature of rule (6) in Definition 7.1 as replacements for the infinite branch through S_x used in the proof of Lemma 6.1.

For each $n < \omega$ let p^*_n be the unique expansion of $x\restriction n$ that satisfies the conditions of Definition 7.5. Let $p^* = \bigcup_{n<\omega} p^*_n$. Let l_i, u_i, and w_i be such that $p^* = \langle l_i, u_i, w_i, x_i \mid i < \omega\rangle$. Let e_i be the least number valid at $p^*\restriction n$, so that $\langle l_i, u_i\rangle = (\rho \circ g)(e_i)$.

We work recursively to construct $f_1 \in (U_1)^\omega$, $f_2 \in (U_2)^\omega$, and sequences $n_0 < n_1 < \dots$ and $\alpha_0, \alpha_1, \dots$ so that for each i:

1. $k_{n_i} = i$ (see Definition 7.1 for the definition of k_n).
2. $\langle x\restriction i, f_1\restriction i, f_2\restriction i\rangle \in S$.
3. $\langle f_1\restriction i, f_2\restriction i\rangle$, viewed as a position in $G(\neg S_x)$, is according to τ.
4. u_{n_i} is realized by S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i-1, f_1(i-1)\rangle$, $\langle i-1, f_2(i-1)\rangle$ and α_i in $V\|\alpha_i + 1$.

As in the proof of Lemma 6.1, we shall have $\alpha_{i+1} < \alpha_i$, leading to a contradiction.

Set to begin with $n_0 = 0$, $\alpha_0 = \nu_{\mathrm{L}}$, $f_1\restriction 0 = \emptyset$, and $f_2\restriction 0 = \emptyset$. It is easy to check that these assignments satisfy conditions (1)–(4). In the case of condition (4) note that condition (5) in Definition 7.1 implies that $k_0 = 0$, whence by condition (4) of the definition, u_0 is realized by S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$.

Suppose now that n_i, α_i, $f_1\restriction i$, and $f_2\restriction i$ have been defined and that conditions (1)–(4) hold for i. The rules of G^* are such that w_{n_i} is a subtype of u_{n_i}. Using the realization of u_{n_i} given by condition (4) and the conditions placed on w_{n_i} by rule (6) in Definition 7.1, it follows that there is $\beta < \alpha_i$ and $z \in U_1$ so that w_{n_i} is realized by S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i-1, f_1(i-1)\rangle$, $\langle i-1, f_2(i-1)\rangle$, $\langle i, z\rangle$, and β, in $V\|\beta + 3$, and so that $\beta > \max\{\delta, \mathrm{rank}(S)\}$.

Let $\alpha_{i+1} = \beta$ and let $f_1(i) = z$. Let $f_2(i)$ be τ's reply to the move $f_1(i) = z$ following the position $\langle f_1\restriction i, f_2\restriction i\rangle$ in $G(\neg S_x)$. Since τ is a winning strategy for II in $G(\neg S_x)$, $\langle x\restriction i+1, f_1\restriction i+1, f_2\restriction i+1\rangle$ is a node in S.

7.6 Remark. The use of rule (6) in Definition 7.1 to obtain $f_1(i)$, and the use of τ to obtain $f_2(i)$, together replace the use of the infinite branch through S in the proof of Lemma 6.1.

We have so far determined α_{i+1}, $f_1\restriction i+1$, and $f_2\restriction i+1$. It remains to determine n_{i+1}.

7.7 Claim. *Let $E = \max\{e_0, \dots, e_{n_i}\}$. There exists $e < \omega$ and $\kappa < \delta$ so that:*

(a) $(\rho\circ g)(e)$ *has the form* $\langle l, u\rangle$ *with* $l = n_i$*, and* u *equal to the* κ*-type of* S*,* $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i, f_1(i)\rangle$, $\langle i, f_2(i)\rangle$*, and* α_{i+1} *in* $V\|\alpha_{i+1} + 1$.

(b) $\mathrm{dcp}(u)$ *is elastic,* $e > E$*, and* κ *is large enough that* $(\rho \circ q)(0)$, $\dots$, $(\rho \circ q)(e-1)$ *all belong to* $V\|\kappa$.

Proof. Similar to the proof of Claim 6.4. ⊣

Let e be given by the last claim. Let $\langle l, u\rangle = (\rho \circ g)(e)$, and let $\kappa = \text{dom}(u)$. An argument similar to that in the proof of Claim 6.5, using the fact that $\langle x{\restriction}i+1, f_1{\restriction}i+1, f_2{\restriction}i+1\rangle$ is a node in S, shows that $\langle l, u\rangle$ is a legal move for player I following $p^*{\restriction}n$. An argument similar to that in the proof of Claim 6.6 produces $n < \omega$ so that $e_n = e$. Set n_{i+1} equal to this n. By condition (a) then, $l_{n_{i+1}} = n_i$ and $u_{n_{i+1}}$ is equal to the κ-type of S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, …, $\langle i, f_1(i)\rangle$, $\langle i, f_2(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+1$. It is easy now to check that conditions (1)–(4) hold for $i+1$.

The recursive construction above is such that $\alpha_{i+1} < \alpha_i$ for each $i < \omega$. This contradiction, similar to the one obtained in the proof of Lemma 6.1, completes the proof of Lemma 7.4. ⊣

7.8 Lemma. *Suppose that player* II *has a winning strategy in* G^**. Then there is a strategy* σ *for player* II *in the game on* X *so that, in* V*, every infinite play according to* σ *belongs to* gdp(S)*.*

Proof. We adapt the solution for Exercise 5.23 to the current setting.

Let $\sigma \in M$ be a winning strategy for player II in G^*. Fix an opponent, willing to play for I in the game on X. We describe how to play against the opponent, making sure that each infinite play according to our description ends up in gdp(S). As usual our description takes the form of a construction. Precisely, we construct:

(A) l_n, u_n, w_n, and x_n for $n < \omega$.

(B) An iteration tree $\mathcal{T}$ on M giving rise to models M_k for $k < \omega$ and embeddings $j_{l,k}$ for $l\ T\ k < \omega$.

(C) Nodes $\langle a_n, b_n\rangle \in j_{0,2n+1}(S)_x$ for $n < \omega$.

(D) $z_n \in j_{0,2n+1}(U_1)$ for $n < \omega$.

This list of objects is similar to the one in the proof of Lemma 5.15, and our construction too will be similar to the one in that proof.

As in Lemma 5.15 we construct so that: $0\ T\ 2\ T\ 4 \cdots$; if $k_n \neq 0$ then the T-predecessor of $2n+1$ is $2l_n+1$; and if $k_n = 0$ then the T-predecessor of $2n+1$ is $2n$. k_n here is such that u_n is a $2k_n+2$-type, see Definition 7.1.

Let $p_0 = \emptyset$ and recursively define

$$p_{n+1} = j_{2n,2n+2}(p_n)^\frown\langle l_n, j_{2n,2n+2}(u_n), w_n, x_n\rangle.$$

We construct so that p_n is a position in $j_{0,2n}(G^*)$, played according to $j_{0,2n}(\sigma^*)$. In addition we maintain the conditions:

1. w_n is realized by the objects $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, $\ldots$, $\langle k_n - 1, a_n(k_n - 1)\rangle$, $\langle k_n - 1, b_n(k_n - 1)\rangle$, $\langle k_n, z_n\rangle$, and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\mathrm{L}}) + 3$.

2. w_n is elastic.

3. M_{2n+1} agrees with all later models of $\mathcal{T}$, that is all models M_i for $i > 2n + 1$, past $\mathrm{dom}(w_n)$. w_n belongs to M_i for each $i > 2n + 1$.

4. All the extenders used in $\mathcal{T}$ have critical points above $\mathrm{rank}(X)$. For each $m > n$, the critical point of $j_{2n+2,2m+2}$ is greater than the domain of w_n. In particular $j_{2n+2,2m+2}(w_n) = w_n$ for each $m \geq n$.

Notice that from condition (1) and the fact that $z_n \in j_{0,2n+1}(U_1)$ it automatically follows that w_n is a $2k_n + 3$-type and that it contains the formulae required by rule (6) of G^*.

To begin round n of the construction set l_n, and u_n to be the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n. Let k_n be such that u_n is a $2k_n + 2$-type. The construction in round n continues subject to one of the following cases:

Case 1, $k_n = 0$. The rules of G^* are such that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $M_{2n} \| j_{0,2n}(\nu_{\mathrm{L}}) + 1$. From the local indiscernibility of ν_{L} and ν_{H} it follows that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{H}})$ in $M_{2n} \| j_{0,2n}(\nu_{\mathrm{H}}) + 1$. Pick a set $z_n \in j_{0,2n}(U_1)$. We shall say more on how this set should be picked, later on. Working in M_{2n} using Lemma 3.22, let $\tau < j_{0,2n}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n}(S)$, $\langle 0, z_n\rangle$, and $j_{0,2n}(\nu_{\mathrm{L}})$ in $j_{0,2n}(\nu_{\mathrm{L}}) + 3$ is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\mathrm{dcp}(u_n)$ in M_{2n}.

Set $E_{2n} =$“pad” so that $M_{2n+1} = M_{2n}$ and $j_{2n,2n+1}$ is the identity. Using the one-step lemma, Lemma 3.23, in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$. It is easy now to check that w_n satisfies the conditions of rule (6) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n {}^\frown \langle l_n, u_n\rangle)$.

Finally, set x_n to be the move played $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown \langle l_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in the game on X following $\langle x_0, \ldots, x_{n-1}\rangle$ if n is even. This completes the round. ⊣ (Case 1)

Case 2, $k_n \neq 0$. The rules of $j_{0,2n}(G^*)$ following the position p_n are such that u_n exceeds w_{l_n} in M_{2n}. Let κ denote the domain of u_n. Using the one-step lemma in M_{2n} find an extender E_{2n} with critical point $\mathrm{dom}(w_{l_n})$, so that u_n is a subtype of $\mathrm{Stretch}^{E_{2n}}_{\kappa+\omega}(w_{l_n})$. Set $M_{2n+1} = \mathrm{Ult}(M_{2l_n+1}, E_{2n})$, and set $j_{2l_n+1,2n+1}$ to be the ultrapower embedding, so that u_n is a subtype of $j_{2l_n+1,2n+1}(w_{l_n})$.

Let $k = k_n$ be such that u_n is a $2k_n + 2$-type. Let $\bar{k}$ denote $k - 1$. The rules of G^* are such that w_{l_n} is a $2\bar{k} + 3$-type. Let $\bar{a}$, $\bar{b}$, and $\bar{z}$ denote a_{l_n}, b_{l_n}, and z_{l_n}. Let $a = j_{2l_n+1,2n+1}(\bar{a})$ and similarly with b and z.

Our construction is such that w_{l_n} is realized by $j_{0,2l_n+1}(S)$, $\langle 0, \bar{a}(0)\rangle$, $\langle 0, \bar{b}(0)\rangle$, $\ldots$, $\langle \bar{k} - 1, \bar{a}(\bar{k} - 1)\rangle$, $\langle \bar{k} - 1, \bar{b}(\bar{k} - 1)\rangle$, $\langle \bar{k}, z\rangle$, and $j_{0,2l_n+1}(\nu_{\rm L})$ in $M_{2l_n+1}\|j_{0,2l_n+1}(\nu_{\rm L}) + 3$. Using the elementarity of $j_{2l_n+1,2n+1}$, the fact that u_n is a subtype of $j_{2l_n+1,2n+1}$, and the conditions placed on u_n by rule (2) of Definition 7.1, it follows that there must exist some set z' so that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a(0)\rangle$, $\langle 0, b(0)\rangle$, $\ldots$, $\langle \bar{k} - 1, a(\bar{k} - 1)\rangle$, $\langle \bar{k}-1, b(\bar{k}-1)\rangle$, $\langle \bar{k}, z\rangle$, $\langle \bar{k}, z'\rangle$, and $j_{0,2n+1}(\nu_{\rm L})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\rm L})+1$, and that moreover $\langle a^\frown\langle z\rangle, b^\frown\langle z'\rangle\rangle$ is a node in $j_{0,2n+1}(S)_x$. Set $a_n = a^\frown\langle z\rangle$ and set $b_n = b^\frown\langle z'\rangle$. Then $\langle a_n, b_n\rangle$ is a node in $j_{0,2n+1}(S)$, and u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, $\ldots$, $\langle k - 1, a_n(k - 1)\rangle$, $\langle k - 1, b_n(k - 1)\rangle$, and $j_{0,2n+1}(\nu_{\rm L})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\rm L}) + 1$. For the record we note that:

(i) a_n extends $j_{2l_n+1,2n+1}(a_{l_n})$, and similarly with b_n.

(ii) $j_{2l_n+1,2n+1}(z_{l_n})$ belongs to the range of a_n.

From here we continue as in case 1.

By the local indiscernibility of $\nu_{\rm L}$ and $\nu_{\rm H}$, u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, $\ldots$, $\langle k-1, a_n(k-1)\rangle$, $\langle k-1, b_n(k-1)\rangle$, and $j_{0,2n+1}(\nu_{\rm H})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\rm H}) + 1$

Pick some set $z_n \in j_{0,2n+1}(U_1)$. We shall say more on how this set should be picked, later on. Working in M_{2n+1} using Lemma 3.22, let $\tau < j_{0,2n+1}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, $\ldots$, $\langle k - 1, a_n(k - 1)\rangle$, $\langle k - 1, b_n(k - 1)\rangle$, $\langle k, z_n\rangle$, and $j_{0,2n+1}(\nu_{\rm L})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\rm L}) + 3$ is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\mathrm{dcp}(u_n)$ in M_{2n+1}.

Using the one-step lemma, Lemma 3.23, in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. As in case 1, w_n satisfies the conditions of rule (6) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, u_n\rangle)$.

Finally, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in the game on X following $\langle x_0, \ldots, x_{n-1}\rangle$ if n is even. This completes the round. $\dashv$ (Case 2)

The description above completes the construction, except that we have yet to specify how the sets z_n are picked. Note that the structure of the iteration tree $\mathcal{T}$ is such that cofinal branches other than the even branch have the form $0, 2, \ldots, 2m_0, 2m_0 + 1, 2m_1 + 1, \ldots$ for some increasing sequence

$\{m_i\}$. The sets z_n should be picked during the construction in such a way that:

(iii) For every cofinal branch b other than the even branch, for every odd node $2m+1 \in b$, and for every set $y \in j_{0,2m+1}(U_1)$, there exists a node $2m^*+1 \in b$, with $m^* > m$, so that z_{m^*} is equal to $j_{2m+1,2m^*+1}(y)$.

Securing this through some condition on the way z_n is chosen is a simple matter of book-keeping, using the fact that U_1 is countable in V. Let us just note that this book-keeping cannot in general be phrased inside M, since U_1 is only assumed to be countable in V. Thus the strategy σ which our construction describes need not be an element of M.

With the construction complete, it remains to check that every sequence $x = \langle x_n \mid n < \omega\rangle \in X^\omega$ that can be obtained by following the construction, with moves x_n for even n supplied by the opponent, belongs to $\mathrm{gdp}(S)$.

Let x, $\mathcal{T}$, $\langle l_n, u_n, w_n \mid n < \omega\rangle$, $\langle a_n \mid n < \omega\rangle$, $\langle b_n \mid n < \omega\rangle$ and $\langle z_n \mid n < \omega\rangle$ be obtained through the construction above. We work through a series of claims to show that x belongs to $\mathrm{gdp}(S)$.

7.9 Claim. *The even branch of $\mathcal{T}$ has an illfounded direct limit.*

Proof. Identical to the proof of Claim 5.19. ⊣

7.10 Claim. *Let b be a branch of $\mathcal{T}$ other than the even branch. Let $\{m_i\}$ be such that $b = \{0, 2, \ldots, 2m_0, 2m_0{+}1, \ldots, 2m_i{+}1, \ldots\}$. Let $a_i^* = j_{2m_i+1,b}(a_i)$ and let $b_i^* = j_{2m_i+1,b}(b_i)$. Let $a^* = \bigcup_{i<\omega} a_i^*$ and let $b^* = \bigcup_{i<\omega} b_i^*$. Then:*

1. *$\langle x, a^*, b^*\rangle \in [j_{0,b}(S)]$.*
2. *a^* is onto $j_{0,b}(U_1)$.*

Proof. Note first that by condition (i), $\bigcup_{i<\omega} a_i^*$ and $\bigcup_{i<\omega} b_i^*$ are both increasing unions giving rise to infinite sequences. By condition (C), below Lemma 7.8, $\langle x{\restriction}i, a_i^*, b_i^*\rangle$ is a node in $j_{0,b}(S)$ for each i. Thus $\langle x, a^*, b^*\rangle$ is an infinite branch through $j_{0,b}(S)$.

By conditions (ii), $j_{2m_i+1,b}(z_{m_i})$ belongs to the range of a^* for each i. From this and condition (iii) it follows that a^* is onto $j_{0,b}(U_1)$. ⊣

Claims 7.9 and 7.10 together combine to show that $x \in \mathrm{dp}(j_{0,b}(S))$ for every wellfounded cofinal branch b of $\mathcal{T}$. $\mathcal{T}$ therefore witnesses that $x \in \mathrm{gdp}(S)$. ⊣

7.11 Corollary. *Let M be a model of* ZFC*. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$.*

Let $S \in M$ be a tree. Suppose that both S and $\mathcal{P}^M(\delta)$ are countable in V. Let g be $\mathrm{Col}(\omega,\delta)$-generic over M.

Then at least one of the following conditions holds:

1. *There is a strategy σ for player* II *in the game on X so that, in V, every infinite run according to σ belongs to* $\mathrm{gdp}(S)$.

2. *There is a strategy $\sigma \in M[g]$ for player* I *in the game on X so that, in $M[g]$, every infinite run according to σ belongs to $\Game(\neg S)$.*

Proof. Immediate from Lemma 7.4, Lemma 7.8, and the fact that the game G^* is closed and therefore determined in M. $\dashv$

Sometimes we want to restrict players on X to some specific subtree of $X^{<\omega}$. The the next exercise is useful in such circumstances.

7.12 Exercise. Work in the setting of Corollary 7.11, and in addition to the objects there let $R \in M$ be a tree on X with no terminal nodes. Show that at least one of the conditions in the corollary holds, with "game on X" replaced by "game on R" in both conditions.

Hint. Define $\pi \colon X^{<\omega} \to R$ so that $\mathrm{lh}(\pi(s)) = \mathrm{lh}(s)$ for each $s \in X^{<\omega}$, $s < t \Rightarrow \pi(s) < \pi(t)$ for all $s, t \in X^{<\omega}$, and so that π is onto R. Let $\hat{S} = \{\langle s, u_1, u_2\rangle \mid \langle \pi(s), u_1, u_2\rangle \in S\}$. Use Corollary 7.11 on $\hat{S}$. $\dashv$

One can use Corollary 7.11 to directly obtain determinacy results. Here instead we use the corollary to obtain a genericity result, and then use the genericity result in conjunction with Theorem 6.17 to obtain determinacy.

7.13 Definition. Let $\mathbb{P} \in M$ be a poset. An iteration tree $\mathcal{T}$ on M is said to *absorb* x to an extension by an image of $\mathbb{P}$ just in case that for every wellfounded cofinal branch b through $\mathcal{T}$, there is a generic extension $M_b^{\mathcal{T}}[g]$ of $M_b^{\mathcal{T}}$ by the poset $j_{0,b}^{\mathcal{T}}(\mathbb{P})$, so that $x \in M_b^{\mathcal{T}}[g]$.

7.14 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.

Let U_1 be the set of dense sets in $\mathrm{Col}(\omega, \delta)$. Let A be the set of canonical names in M for functions from ω into X. Let U_2 be the union of A with the set of conditions in $\mathrm{Col}(\omega, \delta)$. Working in M let $S \subseteq (X \times U_1 \times U_2)^{<\omega}$ be the tree of attempts to construct sequences $x = \langle x_0, x_1, \dots\rangle \in X^\omega$, $\langle D_0, D_1, \dots\rangle \in (U_1)^\omega$, and $\langle \dot{x}, p_1, p_2, \dots\rangle \in (U_2)^\omega$ so that:

1. $\dot{x} \in A$ and $p_n \in \mathrm{Col}(\omega, \delta)$ for each n.

2. $p_{n+1} < p_n$ and $p_{n+1} \in D_n$ for each n.

3. $p_n \Vdash$"$\dot{x}(\check{n}) = \check{x}_n$" for each n.

Prove that $x \in \mathrm{dp}(S)$ iff there is a g which is $\mathrm{Col}(\omega, \delta)$-generic over M with $x \in M[g]$.

7.15 Exercise. Continuing to work with the tree of the previous exercise, prove that $x \in \Game(\neg S)$ iff there is *no* g which is $\mathrm{Col}(\omega, \delta)$-generic over M with $x \in M[g]$.

7.16 Theorem. *Let M be a model of* ZFC. *Let δ be a Woodin cardinal of M. Let X belong to $M \| \delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.*

Then for every $x \in X^\omega$ there is a length ω iteration tree $\mathcal{T}$ on M which absorbs x into an extension by an image of $\mathrm{Col}(\omega, \delta)$.

Note that in particular any real number in V can be absorbed into a generic extension of an iterate of M.

Proof of Theorem 7.16. Let g be $\mathrm{Col}(\omega, \delta)$-generic over M, and apply Corollary 7.11 to the tree S of Exercise 7.14. Notice that condition (2) of the corollary cannot hold: the strategy σ in that condition belongs to $M[g]$, and certainly then there are plays $x \in X^\omega$ which are according to σ, and which belong to $M[g]$. But from Exercise 7.15 and the fact that x belongs to $M[g]$ it follows that $x \notin \Game(\neg S)$, while from condition (2) of the corollary and the fact that x is according to σ it follows that $x \in \Game(\neg S)$.

Thus condition (1) of the corollary must hold, and this immediately implies that for every sequence $\langle x_0, x_2, \dots \rangle \in X^\omega$, there is a sequence $\langle x_1, x_3, \dots \rangle \in X^\omega$ and a length ω iteration tree $\mathcal{T}$ on M, so that the combined sequence $x = \langle x_0, x_1, \dots \rangle$ belongs to $\mathrm{dp}(j_{0,b})(S)$ for every cofinal wellfounded branch b of $\mathcal{T}$. By Exercise 7.14 then, x belongs to a generic extension of $M_b^{\mathcal{T}}$ by $j_{0,b}(\mathrm{Col}(\omega, \delta))$. So $\mathcal{T}$ absorbs x, and therefore certainly $\langle x_0, x_2, \dots \rangle$, into an extension by an image of $\mathrm{Col}(\omega, \delta)$. $\dashv$

Theorem 7.16 was proved in Neeman [29, 30]. It is the second of two genericity results. The first is due to Woodin [42]. Woodin's theorem uses a forcing notion which has the δ chain condition, and it does not require any assumption on the size of δ or its power set in V. These properties often make it more useful than Theorem 7.16, see for example Neeman–Zapletal [34]. On the other hand Woodin's theorem requires full iterability for trees of lengths up to ω_1, and in our setting this is a disadvantage.

7.17 Definition. Let M be a model of ZFC, let δ be a cardinal of M, let $X \in M \| \delta$, and let $\dot{A} \in M$ be a $\mathrm{Col}(\omega, \delta)$-name for a subset of X^ω.

$x \in X^\omega$ belongs to the *generalized interpretation* of $\dot{A}$ if there exists a length ω iteration tree $\mathcal{T}$ on M using only extenders with critical points above $\mathrm{rank}(X)$, and a map $h\colon \omega \to \mathrm{On}^{<\omega}$, so that for every wellfounded cofinal branch b of $\mathcal{T}$:

1. $h_b = \bigcup_{n \in b} h(n)$ is $\mathrm{Col}(\omega, j_b^{\mathcal{T}}(\delta))$-generic over $M_b^{\mathcal{T}}$.

2. x belongs to $j_b^{\mathcal{T}}(\dot{A})[h_b]$.

7.18 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V. Let g be $\mathrm{Col}(\omega,\delta)$-generic over M.

Let $\dot{A} \in M$ be a $\mathrm{Col}(\omega,\delta)$-name for a subset of X^ω. Prove that at least one of the following conditions holds:

1. In V, player I has a winning strategy in $G_\omega(A^*)$, where A^* is the generalized interpretation of $\dot{A}$.
2. In $M[g]$, player II has a winning strategy in $G_\omega(\dot{A}[g])$.

Hint. First note that by changing the roles of the players (and modifying the name $\dot{A}$ accordingly) the exercise can be reduced to proving that at least one of the following conditions holds:

1. There is a strategy σ for player II so that, in V, every play according to σ belongs to the generalized interpretation of $\dot{A}$.
2. There is a strategy $\sigma \in M[g]$ for player I so that, in $M[g]$, every play according to σ belongs to the complement of $\dot{A}[g]$.

This in turn can be derived from Corollary 7.11, with a tree S similar to the one defined in Exercise 7.14, but replacing the set A used in that exercise with the name $\dot{A}$. $\dashv$

Exercise 7.18 appeared in Neeman [29]. When applied with an iterable model M and a name $\dot{A}$ for a set defined by an absolute condition, the exercise leads to determinacy, and Neeman [29] uses it to prove projective determinacy and indeed $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$.

Tracing through the construction leading to the exercise, the reader can check that in condition (1), the tree $\mathcal{T}$ and the function h witnessing that x belongs to the generalized interpretation of $\dot{A}$ depend on x continuously. This element of continuity is expressed more explicitly in Lemma 1.7 of Neeman [29]. It is crucial for proofs of determinacy of long games, but we shall not get into this here. The interested reader may find more in Neeman [31].

7.19 Exercise (Windßus [41], see [13, Lemma 4.5, Theorem 5.2]). Let $\pi\colon P \to V\|\theta$ be elementary, with P countable. Let $\bar{\kappa} \in P$. Let A be the set of sequences $\langle u_i \mid i < \omega\rangle \in P^\omega$ so that:

(i) u_i is a (nice) finite iteration tree on P. If $i < j$ then u_j extends u_i, so that $\mathcal{U} = \bigcup_{i<\omega} u_i$ is a (nice) iteration tree of length ω. The trees use only extenders with critical points above $\bar{\kappa}$.

(ii) Let $n_i + 1 = \mathrm{lh}(u_i)$. Then $b = \{n_i \mid i < \omega\}$ is a branch through U.

(iii) The direct limit of the models of $\pi\mathcal{U}$ along b is wellfounded. (Recall that $\pi\mathcal{U}$ is the copy of $\mathcal{U}$ via π, see Definition 2.7. It is an iteration tree on V.)

Prove that A is $\pi(\bar{\kappa})$-homogeneously Suslin.

Proof. The proof builds on that of Lemma 2.12. Let B be the set of sequences $\langle u_i \mid i < \omega\rangle$ satisfying conditions (i) and (ii), but such that the direct limit of $\pi\mathcal{U}$ along b is illfounded. For each $x = \langle u_i \mid i < \omega\rangle$ in B fix a sequence $\langle \alpha_i^x \mid i < \omega\rangle$ witnessing the illfoundedness, more precisely a sequence so that:

1. for all $i < \omega$, $j^{\pi u_{i+1}}_{n_i, n_{i+1}}(\alpha_i^x) > \alpha_{i+1}^x$.

Let θ be larger than all the ordinals α_i^x.

For $s = \langle u_0, \dots, u_{i-1}\rangle$ let B_s be the set of $x \in B$ which extend s. Let T be the tree of attempts to construct sequences $x = \langle u_i \mid i < \omega\rangle$ and $\langle \sigma_i \mid i < \omega\rangle$ so that:

2. x satisfies conditions (i) and (ii).
3. $\sigma_i \colon B_{\langle u_0,\dots,u_i\rangle} \to \theta$.
4. For all i and all $y \in B_{\langle u_0,\dots,u_{i+1}\rangle}$, $\sigma_i(y) > \sigma_{i+1}(y)$.

Prove that $x \in B \implies x \notin p[T]$, and hence $p[T] \subseteq A$. You will prove that $A \subseteq p[T]$ later on.

Let $M_\emptyset = V$. For $s = \langle u_0, \dots, u_i\rangle$ let M_s be the final model $M^{\pi u_i}_{n_i}$ of the copied tree πu_i. Let φ_s be the function $x \mapsto \alpha_i^x$, defined for $x \in B_s$, where α_i^x are the ordinals witnessing condition (1) above. The models of πu_i are $2^{\aleph_0}$-closed by Exercise 2.2, and hence $\varphi_s \in M_s$.

For $t = \langle u_0, \dots, u_{i^*}\rangle$ extending $s = \langle u_0, \dots, u_i\rangle$ let $j_{s,t} \colon M_s \to M_t$ be the embedding $j^{\pi u_{i^*}}_{n_i, n_{i^*}}$. Let $j_{\emptyset,t} \colon V \to M_t$ be the embedding $j^{\pi u_{i^*}}_{0,i^*}$. Notice that all these embeddings have critical points above $\pi(\bar{\kappa})$.

Show using condition (1) that $f_s = \langle j_{s\restriction 1,s}(\varphi_{s\restriction 1}), j_{s\restriction 2,s}(\varphi_{s\restriction 2}), \dots, \varphi_s\rangle$ is a node in $j_{\emptyset,s}(T_s)$, and use the models M_s, embeddings $j_{s,t}$, and nodes f_s to assemble a homogeneity system for T along the conditions of Exercise 4.2. Finally use the converse of condition (3) of Exercise 4.2, given by Exercise 4.4, to show that $A \subseteq p[T]$. ⊣

7.20 Exercise (Woodin, see [14, Theorem 3.3.8]). Let δ be Woodin in V and let $A \subseteq \omega^\omega$ be δ-universally Baire. Prove that A is weakly κ-homogeneously Suslin for each $\kappa < \delta$.

Hint. Fix κ. Let $\langle T, T^*\rangle$ witness that A is δ-universally Baire. Let θ be large enough that δ, T, and T^* belong to $V\|\theta$. Let $\pi \colon P \to V\|\theta$ be elementary, with P countable and κ, δ, T, and T^* in the range of π. Let $\bar{\kappa}$ be such that $\pi(\bar{\kappa}) = \kappa$, and similarly with $\bar{\delta}$, $\bar{T}$, and $\bar{T}^*$.

Let B be the set of tuples $\langle x, \mathcal{U}, b, n, \dot{x}, g\rangle$ so that: $x \in \omega^\omega$; $\mathcal{U}$ is a (nice) length ω iteration tree on P using only extenders with critical points above $\bar{\kappa}$; b is a cofinal branch through U, leading to a wellfounded direct limit in the copy tree $\pi\mathcal{U}$ on V; $n \in b$; $\dot{x} \in P_n$ is a name in $\mathrm{Col}(\omega, j_{0,n}(\bar{\delta}))$, forced by the empty condition to be a real belonging to $p[j_{0,n}(\bar{T})]$; g is $\mathrm{Col}(\omega, j_b(\bar{\delta}))$-generic over P_b; and $j_{n,b}(\dot{x})[g] = x$.

Show using Exercise 7.19 that B is κ-homogeneously Suslin. Then show using Theorem 7.16 and Lemma 2.12 that $x \in A$ iff $(\exists\mathcal{U})(\exists b)(\exists n)(\exists\dot{x})(\exists g)$ $\langle x, \mathcal{U}, b, n, \dot{x}, g\rangle \in B$. The quantifiers all involve elements of P and P^ω, which are isomorphic to ω and ω^ω. Use this to present A as the projection of a κ-homogeneously Suslin subset of $\omega^\omega \times \omega^\omega$. $\dashv$

7.21 Remark. If κ is a limit of Woodin cardinals, then for any $A \subseteq \omega^\omega$, Exercises 5.29, 6.16, and 7.20 together imply that A is $<\kappa$-universally Baire iff A is $<\kappa$-homogeneously Suslin iff A is weakly $<\kappa$-homogeneously Suslin.

7.22 Exercise. Let $j \colon M \to N$ be elementary. Let h be $\mathrm{Col}(\omega, \kappa)$-generic over M. Suppose that $\mathrm{crit}(j) > \kappa$. Prove that j can be extended to an embedding $j^* \colon M[h] \to N[h]$.

Hint. Define j^* by setting $j^*(\dot{a}[h]) = (j(\dot{a}))[h]$. Show that j^* is well defined and elementary. $\dashv$

7.23 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.

Let $\kappa < \delta$. Let h be $\mathrm{Col}(\omega, \kappa)$-generic over M.

Let $x \in X^\omega$. Then there is a length ω iteration tree $\mathcal{T}$ on M so that:

1. All the extenders used in $\mathcal{T}$ have critical points above κ. (In particular then the embeddings along branches of $\mathcal{T}$ extend to act on $M[h]$.)

2. For every cofinal wellfounded branch b of $\mathcal{T}$, there is g which is $\mathrm{Col}(\omega, j_b(\delta))$-generic over $M_b[h]$, and so that x belongs to $M_b[h][g]$.

Note that in particular any real in V can be absorbed into a generic extension of $M_b[h]$ for an iterate M_b of M.

Hint to Exercise 7.23. Let $\widehat{X} = M\|\kappa + \omega$. Let $R \subseteq \widehat{X}^{<\omega}$ be the tree of attempts to construct a sequence $\langle\langle x_0, q_0\rangle, E_0, \langle x_1, q_1\rangle, E_1, \ldots\rangle$ so that:

1. $x_n \in X$ for each n, and q_n is a condition in $\mathrm{Col}(\omega, \kappa)$.

2. E_n is a dense subset of $\mathrm{Col}(\omega, \kappa)$ for each n.

3. $q_{n+1} < q_n$ and $q_{n+1} \in E_n$ for each n.

For clarity let us point out that in games on R, player I plays the objects $\langle x_n, q_n\rangle$, and player II plays the objects E_n.

Working in M let U_1 be the set of $\mathrm{Col}(\omega,\kappa)$-names for dense subsets of $\mathrm{Col}(\omega,\delta)$, let A be the set of canonical $\mathrm{Col}(\omega,\kappa)\times\mathrm{Col}(\omega,\delta)$-names for functions from ω into X, and let U_2 be the union of A with the set of conditions in $\mathrm{Col}(\omega,\delta)$.

Let $S \subseteq (\widehat{X}\times U_1\times U_2)^{<\omega}$ be the tree of attempts to construct a sequence $\langle\langle x_0,q_0\rangle, E_0, \langle x_1,q_1\rangle, E_1,\dots\rangle \in [R]$, a sequence $\langle \dot{D}_0,\dot{D}_1,\dots\rangle\in (U_1)^\omega$, and a sequence $\langle \dot{x}, p_1, p_2,\dots\rangle \in (U_2)^\omega$ so that:

1. $\dot{x}\in A$ and $p_n\in\mathrm{Col}(\omega,\delta)$ for each n.
2. For each n and each $i\le n$, $p_{n+1}<p_n$ and $q_{n+1}\not\Vdash^{\mathrm{Col}(\omega,\kappa)}$ "$\check{p}_{i+1}\notin\dot{D}_i$."
3. For each n and each $i\le n$, $\langle q_n,p_n\rangle\not\Vdash^{\mathrm{Col}(\omega,\kappa)\times\mathrm{Col}(\omega,\delta)}$ "$\dot{x}(\check{i})\neq\check{x}_i$."

Apply Exercise 7.12 to $\widehat{X}$, R, and S as defined above. Argue first that case (2) cannot hold. (For this you will need the following forcing claim: Let g be $\mathrm{Col}(\omega,\delta)$-generic over M. Let h^* belong to $M[g]$ and suppose that h^* is $\mathrm{Col}(\omega,\kappa)$-generic over M. Then there exists a g^* which is $\mathrm{Col}(\omega,\delta)$-generic over $M[h^*]$ and so that $M[h^*][g^*]=M[g]$.) Then use case (1) of Exercise 7.12 to reach the conclusion of the current exercise. ⊣

7.24 Remark. Let $\kappa_1<\kappa_2<\cdots<\kappa_i=\kappa$. $\mathrm{Col}(\omega,\kappa)$ is then isomorphic to $\mathrm{Col}(\omega,\kappa_1)\times\cdots\times\mathrm{Col}(\omega,\kappa_i)$. Exercise 7.23 can therefore be rephrased to replace h by a generic $h_1\times\cdots\times h_i$ for $\mathrm{Col}(\omega,\kappa_1)\times\cdots\times\mathrm{Col}(\omega,\kappa_i)$. This sets the stage for an iterated use of the exercise, assuming an increasing sequence of Woodin cardinals. We shall make such a use in the next section.

8. Determinacy in $\mathrm{L}(\mathbb{R})$

Let M be a model of ZFC and let $\delta_0<\delta_1<\cdots$ be ω Woodin cardinals in M. Let $\delta_\infty=\sup_{n<\omega}\delta_n$. Suppose that $\mathcal{P}^M(\delta_\infty)$ is countable in V.

Let $\mathbb{P}$ be the finite support product $\mathrm{Col}(\omega,\delta_0)\times\mathrm{Col}(\omega,\delta_1)\times\cdots$.

Given a filter $G=\langle g_i\mid i<\omega\rangle$ which is $\mathbb{P}$-generic over M define $R^*[G]$ to be $\bigcup_{n<\omega}\mathbb{R}^{M[G\restriction n]}$. We refer to $R^*[G]$ as the reals in the *symmetric collapse* of M induced by G. We refer to $\mathrm{L}_{M\cap\mathrm{On}}(R^*[G])$ as the *derived model* of M induced by G. (This is $\mathrm{L}(R^*[G])$ if M is a class model.)

8.1 Remark. Let $v_1,\dots,v_k\in M[G\restriction n]$. Let $\mathbb{P}_{\mathrm{L}}=\mathrm{Col}(\omega,\delta_0)\times\cdots\times\mathrm{Col}(\omega,\delta_{n-1})$, so that $G\restriction n$ is $\mathbb{P}_{\mathrm{L}}$-generic over M, and let $\mathbb{P}_{\mathrm{H}}=\mathrm{Col}(\omega,\delta_n)\times\cdots$. Because of the symmetry of $\mathbb{P}_{\mathrm{H}}$, any statement $\varphi[v_1,\dots,v_k]$ which holds in $M(R^*[G])$ must be forced to hold in $M(R^*[G])$ by the *empty condition* in $\mathbb{P}_{\mathrm{H}}$ over $M[G\restriction n]$.

8.2 Exercise. Let R^* denote the reals of the symmetric collapse of M induced by G, and let W denote the derived model of M induced by G. Prove that $\mathbb{R}^W = R^*$.

Hint. The inclusion $\mathbb{R}^W \supseteq R^*$ is clear. For the reverse inclusion: let $b \in \mathbb{R}^W$. b is definable in W from some parameters in $R^* \cup (\mathrm{On} \cap M)$. Thus there is some $n < \omega$ so that the parameters defining b belong to $M[G{\restriction}n]$. Use this and the symmetry given by Remark 8.1 to argue that b belongs to $M[G{\restriction}n]$, and therefore $b \in R^*$. ⊣

Exercise 8.2 makes no use of the assumption that δ_∞ is a limit of Woodin cardinals in M. But without this assumption the derived model need not even satisfy the axiom of dependent choice for reals, and in such circumstances the conclusion of the exercise is less meaningful than it appears.

8.3 Definition. By a $\Sigma_1(\mathbb{R})$ *statement over* $\mathrm{L}(\mathbb{R})$, $\Sigma_1(\mathbb{R})$ for short, we mean a statement of the form $(\exists Q \supseteq \mathbb{R})Q \models \psi[x_1, \dots, x_n]$, where $x_1, \dots, x_n \in \mathbb{R}$.

We say that $\mathrm{L}_\alpha(R)$ is an *initial segment* of $\mathrm{L}_\beta(R)$ if: (1) $\alpha \leq \beta$; and (2) $\mathbb{R}^{\mathrm{L}_\alpha(R)} = \mathbb{R}^{\mathrm{L}_\beta(R)} = R$.

8.4 Claim. *Suppose that* $\mathrm{L}_\alpha(R)$ *is an initial segment of* $\mathrm{L}_\beta(R)$*. Then any* $\Sigma_1(R)$ *statement true in* $\mathrm{L}_\alpha(R)$ *is also true in* $\mathrm{L}_\beta(R)$*.* ⊣

The failure of $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ is $\Sigma_1(\mathbb{R})$, and so is the failure of dependent choice for reals in $\mathrm{L}(\mathbb{R})$.

8.5 Lemma. *Let* $\varphi[x_1, \dots, x_k]$ *be* $\Sigma_1(\mathbb{R})$ *over* $\mathrm{L}(\mathbb{R})$*. Suppose that* $x_1, \dots, x_k$ *belong to the symmetric collapse of* M *induced by* G*. Suppose that* M *is countable and embeds into a rank initial segment of* V*. Then if* $\varphi[x_1, \dots, x_k]$ *holds in the derived model of* M *induced by* G*, it must hold also in (the true)* $\mathrm{L}(\mathbb{R})$*.*

Proof. Let Σ be the weak iteration strategy for M given by Corollary 2.4. Let θ be a cardinal large enough that M, Σ, G, and $\mathbb{R}$ all belong to $V\|\theta$, and so that $V\|\theta$ satisfies enough of ZFC for the argument below. Let X be a countable elementary substructure of $V\|\theta$ containing these objects. Let P be the transitive collapse of X and let $\tau \colon P \to V\|\theta$ be the anti-collapse embedding. Notice that M, being countable, is not moved by the collapse. So $\tau(M) = M$. Notice further that $\tau^{-1}(\Sigma)$ is simply equal to $\Sigma \cap P$. This is because the iteration trees which come up in weak iteration games on M are countable, and not moved by τ.

Let $\langle a_i \mid n \leq i < \omega\rangle$ be an enumeration of the reals of P, which is $\mathrm{Col}(\omega, \mathbb{R}^P)$-generic over P. Let $M_0 = M_1 = \cdots = M_n = M$ and let $j_{i,i'}$ for $i \leq i' \leq n$ be the identity. For $i < n$ let $h_i = g_i$. Below let h^i denote $h_0 \times h_1 \times \cdots \times h_{i-1}$. Using repeated applications of Exercise 7.23 and Remark 7.24 construct $\mathcal{T}_i$, b_i, M_i, and h_i for $i \geq n$, and a commuting system of embeddings $j_{i,i'} \colon M_i \to M_{i'}$ for $i \leq i' < \omega$ so that:

1. $\mathcal{T}_i$ is a length ω iteration tree on M_i, using only extenders with critical points above $j_{0,i}(\delta_{i-1})$.
2. b_i is the cofinal branch through $\mathcal{T}_i$ given by Σ (equivalently by $\bar{\Sigma}$).
3. M_{i+1} is the direct limit of the models of $\mathcal{T}_i$ along b_i. $j_{i,i+1}\colon M_i \to M_{i+1}$ is the direct limit embedding.
4. h_i is $\mathrm{Col}(\omega, j_{0,i+1}(\delta_i))$-generic over $M_{i+1}[h^i]$.
5. a_i belongs to $M_{i+1}[h^i \times h_i]$.

The key point in the construction is the last condition, condition (5). It is obtained through an application of Exercise 7.23, inside P, on the model $M_i[h^i]$, to absorb the real a_i into a generic extension of an iterate. $\mathcal{T}_i$ is the iteration tree given by the exercise.

The construction is dependent on the sequence $\langle a_i \mid n \leq i < \omega\rangle$ which does not belong to P. Thus the sequence $\langle M_i, \mathcal{T}_i, b_i, h_i \mid i < \omega\rangle$ does not belong to P. But notice that every stage of the construction is done inside P. Each of the individual objects in the sequence is therefore an element of P (and countable in P, since M is countable in P). Using this and some book-keeping it is easy to arrange that:

(i) For every $i < \omega$, and every $D \in M_i$ which is dense in $j_{0,i}(\mathbb{P})$, there exists some $i^* > i$ so that the filter $h_0 \times \cdots \times h_{i^*-1}$ meets $j_{i,i^*}(D)$.

The book-keeping requires an enumeration of $\bigcup_{i<\omega} M_i$. Notice that there are such enumerations in $P[a_i \mid n \leq i < \omega]$ since each M_i is countable in P, and therefore coded by a real.

Let M_∞ be the direct limit of the system $\langle M_i, j_{i,i'} \mid i \leq i' < \omega\rangle$, and let $j_{i,\infty}$ be the direct limit maps. M_∞ is wellfounded since it is obtained in a play of the weak iteration game according to Σ.

From condition (1) it follows that $\mathrm{crit}(j_{i^*,\infty}) \geq j_{0,i^*}(\delta_{i-1})$ for every $i^* < \omega$. Conditions in h^{i^*} are therefore not moved by $j_{i^*,\infty}$. From this and condition (i) it follows that $H = \langle h_i \mid i < \omega\rangle$ is $j_{0,\infty}(\mathbb{P})$-generic over M_∞.

8.6 Claim. *$\varphi[x_1, \ldots, x_k]$ holds in the derived model of M_∞ induced by H.*

Proof. We know that $\varphi[x_1, \ldots, x_k]$ holds in the derived model of M induced by G. By Remark 8.1 this statement, let us denote it $(*)$, is forced, over $M[g_0 \times \cdots \times g_{n-1}] = M[h_0 \times \cdots \times h_{n-1}]$ by the empty condition in $\mathbb{P}_{\mathrm{H}}$. $j_{0,\infty}$ has critical point above δ_{n-1} and therefore extends to an elementary embedding of $M[h_0 \times \cdots \times h_{n-1}]$ into $M_\infty[h_0 \times \cdots \times h_{n-1}]$. $x_1, \ldots, x_k$, being reals, are not moved by the embedding. From this and elementarity if follows that the statement $(*)$ is forced to hold also over $M_\infty[h_0 \times \cdots \times h_{n-1}]$. It follows that $\varphi[x_1, \ldots, x_k]$ holds in the derived model of M_∞ induced by H. $\dashv$

8.7 Claim. $R^*(H) = \mathbb{R}^P$.

Proof. From the restriction on the critical points in condition (1) it follows that $\mathbb{R} \cap M_\infty[H{\restriction}i] = \mathbb{R} \cap M_i[H{\restriction}i]$. Since M_i and $H{\restriction}i$ belong to P it follows that $\mathbb{R} \cap M_\infty[H{\restriction}i] \subseteq P$, and hence $R^*(H) \subseteq \mathbb{R}^P$.

Conversely, every real in P belongs to $\{a_i \mid n \leq i < \omega\}$, and is, by construction, an element of $M_{i+1}[h^i][h_i] = M_{i+1}[H{\restriction}i+1]$ for some i. Using the restriction on the critical points in condition (1), $\mathbb{R} \cap M_{i+1}[H{\restriction}i+1] = \mathbb{R} \cap M_\infty[H{\restriction}i+1]$. So $\mathbb{R}^P \subseteq R^*(H)$. $\dashv$

8.8 Claim. $\varphi[x_1, \dots, x_k]$ *holds in* $(\mathrm{L}(\mathbb{R}))^P$.

Proof. Notice that the ordinals of M_∞ are contained in the ordinals of P. (This is because M_∞ belongs to $P[a_i \mid n \leq i < \omega]$.) From this and the last claim it follows that the derived model of M_∞ induced by H is an initial segment of the model $(\mathrm{L}(\mathbb{R}))^P$. By Claim 8.6, $\varphi[x_1, \dots, x_k]$ holds in the former model. From this and the fact that φ is $\Sigma_1(\mathbb{R})$ it follows that $\varphi[x_1, \dots, x_k]$ holds also in the latter. $\dashv$

We showed so far that $\varphi[x_1, \dots, x_k]$ holds in $(\mathrm{L}(\mathbb{R}))^P$, where P is the transitive collapse of a Skolem hull of a rank initial segment of V. Using the elementarity of the anti-collapse embedding it follows that $\varphi[x_1, \dots, x_k]$ holds in $(\mathrm{L}(\mathbb{R}))^{V\|\theta}$, and since $\varphi[x_1, \dots, x_k]$ is $\Sigma_1(\mathbb{R})$ this implies that it holds in $(\mathrm{L}(\mathbb{R}))^V$. $\dashv$

8.9 Lemma. *Suppose that* $\langle \eta_i \mid i < \omega \rangle$ *is an increasing sequence of Woodin cardinals of* V. *Let* $\mathbb{Q}$ *be the finite support product* $\mathrm{Col}(\omega, \eta_1) \times \mathrm{Col}(\omega, \eta_2) \times \cdots$. *Let* $H = \langle h_i \mid i < \omega \rangle$ *be* $\mathbb{Q}$*-generic over* V.

Then the derived model of V *induced by* H *satisfies the axiom of dependent choice for reals (and hence the full axiom of dependent choice).*

Proof. Suppose not. Let θ be a cardinal large enough that $\mathbb{Q} \in V\|\theta$ and so that $V\|\theta$ satisfies the fragment of ZFC that must be assumed in a model M for Lemma 8.5 to hold for the model. Let $\pi\colon M \to V\|\theta$ be elementary, with M countable and $\mathbb{Q} \in \mathrm{range}(\pi)$. By elementarity, dependent choice for reals fails in the derived models of M. The failure of dependent choice for reals is $\Sigma_1(\mathbb{R})$. Thus by Lemma 8.5 dependent choice for reals must fail also in the true L($\mathbb{R}$). But this is a contradiction. Dependent choice for reals in the true L($\mathbb{R}$) follows from the axiom of choice in V and the fact that countable sequences of reals can be coded by reals. $\dashv$

8.10 Theorem. *Suppose that* $\langle \eta_i \mid i < \omega \rangle$ *is an increasing sequence of Woodin cardinals of* V. *Let* $\mathbb{Q}$ *be the finite support product* $\mathrm{Col}(\omega, \eta_1) \times \mathrm{Col}(\omega, \eta_2) \times \cdots$. *Let* $H = \langle h_i \mid i < \omega \rangle$ *be* $\mathbb{Q}$*-generic over* V.

Then the derived model of V *induced by* H *satisfies* AD.

Proof. Let R^* denote $R^*[H]$, and suppose for contradiction that there is a set $A \in \mathrm{L}(R^*)$ so that $A \subseteq R^*$ and $G_\omega(A)$ is not determined in $\mathrm{L}(R^*)$.

Since every set in $\mathrm{L}(R^*)$ is definable from real and ordinal parameters in a level of $\mathrm{L}(R^*)$, there must be a parameter $a \in R^*$, a formula φ, and ordinals γ, ζ so that

$$x \in A \iff \mathrm{L}_\gamma(R^*) \models \varphi[x, a, \zeta].$$

Without loss of generality we may assume that $a \in \mathbb{R}^V$. Otherwise we may simply replace V by $V[h_0 \times \cdots \times h_i]$ for i large enough that $a \in \mathbb{R}^{V[h_0 \times \cdots \times h_i]}$.

Again without loss of generality we may assume that $\langle \gamma, \zeta \rangle$ is the lexicographically least pair of ordinals for which the set $\{x \mid \mathrm{L}_\gamma(R^*) \models \varphi[x, a, \zeta]\}$ is not determined. By the symmetry of the collapse, this minimality of $\langle \gamma, \zeta \rangle$ is forced by the empty condition in $\mathbb{Q}$ over V.

8.11 Remark. We refer to A as the *least* non-determined set definable from a and ordinal parameters in $\mathrm{L}(R^*)$.

Let θ be a cardinal larger than $\sup_{i<\omega} \eta_i$, larger than γ, and so that $V \| \theta$ satisfies the fragment of ZFC that must be assumed in a model M for Lemma 8.5 to hold for the model. Let $\dot{R}^* \in V$ be the canonical name for $R^*[H]$.

8.12 Definition. Working in V let $T_{\mathrm{in}} \subseteq \omega \times V \| \theta$ be the tree of attempts to construct a real x, and a sequence $\langle \langle e_i, f_i \rangle \mid i < \omega \rangle \in (V \| \theta)^\omega$ so that:

1. $\{e_i \mid i < \omega\}$ is an elementary substructure of $V \| \theta$.

Let M be the transitive collapse of $\{e_i \mid i < \omega\}$, and let $\pi \colon M \to V \| \theta$ be the anticollapse embedding.

2. $e_0 = a$, e_1 is equal to $\langle \eta_i \mid i < \omega \rangle$, $e_2 = \mathbb{Q}$, $e_3 = \dot{R}^*$, $e_4 = \gamma$, $e_5 = \zeta$, and e_6 is a name for a real in the symmetric collapse of V by $\mathbb{Q}$.
3. It is forced by the empty condition in $\mathbb{Q}$ that $\mathrm{L}_{\check{\gamma}}(\dot{R}^*) \models \varphi[e_6, \check{a}, \check{\zeta}]$.

Let $\dot{x}$ denote $\pi^{-1}(e_6)$. Let $\mathbb{P}$ denote $\pi^{-1}(\mathbb{Q})$.

4. The set $G = \{\pi^{-1}(e_{f_i}) \mid i < \omega\}$ forms a $\mathbb{P}$-generic filter over M.
5. $\dot{x}[G]$ is equal to x.

Let $T_{\mathrm{out}} \in V$ be defined similarly, only changing "$\models$" in condition (3) to "$\not\models$."

8.13 Remark. We emphasize that both T_{in} and T_{out} are defined in V, that is with no reference to H.

8.14 Remark. Let $x \in p[T_{\text{in}}]$ and let $\langle\langle e_i, f_i\rangle \mid i < \omega\rangle$ witness this. Let M, π, and G be as in Definition 8.12. Note in this case that the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinal parameters, and x belongs to the least such set." This follows from the minimality of $\langle\gamma, \zeta\rangle$, the elementarity of π, condition (3) of Definition 8.12, and condition (5) of the definition.

Similarly, if $x \in p[T_{\text{out}}]$, then the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinal parameters, and x belongs to the *complement* of the least such set."

8.15 Claim. *The pair $\langle T_{\text{in}}, T_{\text{out}}\rangle$ is exhaustive for* $\mathrm{Col}(\omega, \eta_0)$.

Proof. Let x be a real in $V[h_0]$. Recall that $A = \{x \mid \mathrm{L}_\gamma(R^*) \models \varphi[x, a, \zeta]\}$. If $x \in A$ then a Skolem hull argument in $V[H]$ easily shows that $x \in (p[T_{\text{in}}])^{V[H]}$, and from this by absoluteness it follows that $x \in (p[T_{\text{in}}])^{V[h_0]}$. If $x \notin A$ then a similar argument shows that $x \in (p[T_{\text{out}}])^{V[h_0]}$. ⊣

8.16 Claim. *Let x be a real in V. Suppose that $x \in p[T_{\text{in}}]$. Then, in* $\mathrm{L}(\mathbb{R})$, *there is a non-determined set definable from a and ordinal parameters, and x belongs to the least such set.*

Proof. Let $\langle\langle e_i, f_i\rangle \mid i < \omega\rangle$ witness that $x \in p[T_{\text{in}}]$. Let M, π, $\dot{x}$, and G be as in Definition 8.12. By Remark 8.14, the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinals parameters, and x belongs to the least such set." This statement is $\Sigma_1(\mathbb{R})$. By Lemma 8.5 the statement must hold of x and a in the true $\mathrm{L}(\mathbb{R})$. ⊣

8.17 Claim. *Let x be a real in V. Suppose that $x \in p[T_{\text{out}}]$. Then, in* $\mathrm{L}(\mathbb{R})$, *there is a non-determined set definable from a and ordinal parameters, and x belongs to the* complement *of the least such set.*

Proof. Similar to the proof of the previous claim. ⊣

8.18 Claim. $V \models$ *"$p[T_{\text{in}}] \cap p[T_{\text{out}}] = \emptyset$."*

Proof. This follows immediately from the last two claims: x cannot belong to both the least non-determined set and its complement. ⊣

From Claims 8.15 and 8.18, and Exercise 6.15, it follows that, in V, $p[T_{\text{out}}]$ is precisely equal to the complement of $p[T_{\text{in}}]$. In particular this means that, in the true $\mathrm{L}(\mathbb{R})$, there is a non-determined set definable from a and ordinal parameter, for otherwise both $p[T_{\text{in}}]$ and $p[T_{\text{out}}]$ would be empty by Claims 8.16 and 8.17. $p[T_{\text{in}}]$ is equal to the least such set.

Again from Exercise 6.15, $p[T_{\text{in}}]$ is η_0-universally Baire. By Theorem 6.17, $G_\omega(p[T_{\text{in}}])$ must be determined. But this is a contradiction since $p[T_{\text{in}}]$ is the least *non*-determined set. The contradiction completes the proof of Theorem 8.10 ⊣

8.19 Definition. Let $A \subseteq \mathbb{R}$ in V be $<\eta$-universally Baire, that is κ-universally Baire for each $\kappa < \eta$. Let H be $\mathrm{Col}(\omega, {<}\eta)$-generic over V and let $\mathbb{R}^* = R^*[H] = \bigcup_{\alpha<\eta} \mathbb{R}^{V[H\restriction\alpha]}$. The set A has a *canonical extension* to a set $A^* \subseteq \mathbb{R}^*$, defined as follows: $x \in \mathbb{R}^{V[H\restriction\alpha]}$ belongs to A^* iff $x \in p[T]$ for some, and equivalently any, pair $\langle T, T^*\rangle \in V$ witnessing that A is α-universally Baire. (The equivalence is easy to prove using the conditions in Fact 6.14, and makes the canonical extension useful.)

8.20 Exercise. Let η be a limit of Woodin cardinals, and let H be a $\mathrm{Col}(\omega, {<}\eta)$-generic filter over V. Let $A \subseteq \mathbb{R}$ in V be $<\eta$-universally Baire (equivalently, by Remark 7.21, $<\eta$-homogeneously Suslin, or weakly $<\eta$-homogeneously Suslin). Let A^* be the canonical extension of A to a subset of $\mathbb{R}^* = R^*[H]$. Prove that $\mathrm{L}(\mathbb{R}^*, A^*)$ satisfies AD.

Exercise 8.20 is a first step towards Woodin's derived model theorem, which the reader can find in Steel [38]. Assuming enough large cardinals, it can be shown that there are universally Baire sets which do not belong to $\mathrm{L}(\mathbb{R})$, and in that case Exercise 8.20 is a proper strengthening of Theorem 8.10, taking determinacy to sets outside $\mathrm{L}(\mathbb{R}^*)$.

Hint to Exercise 8.20. Adapt the proof of Theorem 8.10, replacing $\mathrm{L}(\mathbb{R})$ by $\mathrm{L}(\mathbb{R}, A)$ and, for countable N and $\sigma\colon N \to V\|\theta$, replacing derived models of N by models of the form $\mathrm{L}_{N\cap\mathrm{On}}(\bar{\mathbb{R}}^*, \bar{A}^*)$ where $\bar{\mathbb{R}}^*$ is the set of reals of the derived model and $\bar{A}^*$ is the canonical extension of $\bar{A} = \sigma^{-1}(A)$ to a subset of $\bar{\mathbb{R}}^*$. (Notice that all the countable models which come up during the proof of Theorem 8.10 embed into rank initial segments of V, either directly by construction or because they are obtained through uses of Theorem 2.3.) You will need the following observation, which is easily verified, to connect $\mathrm{L}(\bar{\mathbb{R}}^*, \bar{A}^*)$ with $\mathrm{L}(\mathbb{R}, A)$: Let $\sigma\colon N \to V\|\theta$ be elementary, with N countable and $\sigma(\bar{A}) = A$, $\sigma(\bar{\eta}) = \eta$. Let $\bar{H} \in V$ be $\mathrm{Col}(\omega, \bar{\eta})$-generic over N. Let $\bar{\mathbb{R}}^* = \bigcup_{\alpha<\bar{\eta}} \mathbb{R}^{N[\bar{H}\restriction\alpha]}$ and let $\bar{A}^*$ be the canonical extension of $\bar{A}$ to a subset of $\bar{\mathbb{R}}^*$, as defined inside $N[\bar{H}]$. Then for every $x \in \bar{\mathbb{R}}^*$, $x \in \bar{A}^* \iff x \in A$. ⊣

8.21 Theorem. *Suppose that there is a model M of* ZFC *so that:*

- *M has ω Woodin cardinals, and a measurable cardinal above them.*
- *M is countable in V.*
- *M is weakly iterable.*

Then the true $\mathrm{L}(\mathbb{R})$ *satisfies* AD*.*

Proof. Let Σ be a weak iteration strategy for M. Let θ be a cardinal large enough that $\Sigma \in V\|\theta$, and so that $V\|\theta$ satisfies enough of ZFC for the argument below. Let X be a countable elementary substructure of $V\|\theta$

with $M, \Sigma \in X$. Let P be the transitive collapse of X and let $\tau\colon P \to V\|\theta$ be the anti-collapse embedding. We intend to show that $(\mathrm{L}(\mathbb{R}))^P$ satisfies AD, and then use the elementarity of τ.

Let $\langle \delta_i \mid i < \omega \rangle \in M$ be an increasing sequence of Woodin cardinals of M, and let ρ be a measurable cardinal of M above these Woodin cardinals. Let $\mathbb{P}$ denote the finite support product $\mathrm{Col}(\omega, \delta_0) \times \mathrm{Col}(\omega, \delta_1) \times \cdots$.

Using iterated applications of Exercise 7.23 construct a weak iteration $\langle M_i, j_{i,i'} \mid i \leq i' \leq \omega \rangle$ of $M_0 = M$, and a filter H, so that: the iteration is according to Σ, H is $j_{0,\omega}(\mathbb{P})$-generic over M_ω, and $R^*[H]$ is precisely equal to $\mathbb{R} \cap P$. The construction is similar to the main construction in the proof of Lemma 8.5.

By Theorem 8.10, the derived model of M_ω induced by H satisfies AD. This model is an initial segment of $(\mathrm{L}(\mathbb{R}))^P$: it has the reals that P has, but it does not have all the ordinals P has. We now add ordinals by passing from M_ω to an iterate of M_ω obtained through ultrapowers by a measure on ρ and its images.

Let μ witness that ρ is measurable in M. Extend the iteration $\langle M_i, j_{i,i'} \mid i \leq i' \leq \omega \rangle$ of M to a weak iteration of length ω_1 by setting $M_{\xi+1} = \mathrm{Ult}(M_\xi, j_{0,\xi}(\mu))$ for each $\xi \geq \omega$ and setting $j_{\xi,\xi+1}$ to be the ultrapower embedding. This completely determines the iteration.

Let η_α denote the ordinal height of M_α, that is $\mathrm{On} \cap M_\alpha$.

8.22 Exercise. Show that $\eta_\alpha \geq \alpha$.

Hint. The map $\xi \mapsto j_{0,\xi}(\rho)$ embeds $\alpha - \omega$ into the ordinals of M_α. ⊣

Note that, for $\alpha \geq \omega$, $j_{\omega,\alpha}$ has critical point $j_{0,\omega}(\rho)$, and this is larger than $j_{0,\omega}(\sup_{i<\omega} \delta_i)$. It follows that H is generic also over M_α, and that the reals of the symmetric collapse induced by H over M_α are the same as the reals of the symmetric collapse induced by H over M_ω, which in turn are the same as the reals of P. Thus, for each $\alpha \geq \omega$:

(i) The derived model of M_α induced by H is equal to $\mathrm{L}_{\eta_\alpha}(\mathbb{R}^P)$.

From this and Theorem 8.10 it follows that:

(ii) $\mathrm{L}_{\eta_\alpha}(\mathbb{R}^P)$ satisfies AD.

Using (i) and Exercise 8.2:

(iii) $\mathbb{R}^{\mathrm{L}_{\eta_\alpha}(\mathbb{R}^P)}$ is equal to $\mathbb{R}^P$.

Using Exercise 8.22 fix some $\alpha < \omega_1$ so that $\eta_\alpha > \mathrm{On} \cap P$. By condition (iii) then, $(\mathrm{L}(\mathbb{R}))^P$ is an initial segment of $\mathrm{L}_{\eta_\alpha}(\mathbb{R}^P)$. From this and condition (ii) it follows that $(\mathrm{L}(\mathbb{R}))^P$ satisfies AD. Using the elementarity of τ it follows that $(\mathrm{L}(\mathbb{R}))^{V\|\theta} = \mathrm{L}_\theta(\mathbb{R})$ satisfies AD. Since θ could be chosen arbitrarily large, it follows finally that $\mathrm{L}(\mathbb{R})$ satisfies AD. ⊣

8.23 Remark. Readers familiar with sharps can verify, by adapting the proof given above, that the assumption in Theorem 8.21 can be weakened, from demanding that M has ω Woodin cardinals and a measurable cardinal above them, to demanding that M is a sharp for ω Woodin cardinals.

8.24 Theorem. *Suppose that in V there are ω Woodin cardinals and a measurable cardinal above them. Then* $\mathrm{L}(\mathbb{R})$ *satisfies* AD.

Proof. Let θ be a cardinal large enough that $V\|\theta \models$"there are ω Woodin cardinals and a measurable cardinal above them," and so that $V\|\theta$ satisfies the fragment of ZFC necessary in a model M for Theorem 8.21 to hold for the model. Let X be a countable elementary substructure of $V\|\theta$ and let M be the transitive collapse of X. Then $M \models$"there are ω Woodin cardinals and a measurable cardinal above them," M is countable in V, and, by Corollary 2.4, M is weakly iterable. Applying Theorem 8.21 it follows that $\mathrm{L}(\mathbb{R})$ satisfies AD. $\dashv$

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