CODING ALONG TREES AND GENERIC ABSOLUTENESS

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1. Introduction

There is a rich tradition in set theory of investigating the extent to which forcing can change the truth value of assertions about ordinals and reals, especially in the presence of large cardinals. It is a now-classical theorem of Woodin that, in the presence of suitable large cardinals, the theory of $L(\mathbb{R})$ cannot be changed by set forcing (see Larson [12]), and, conversely, if the theory of $L(\mathbb{R})$ cannot be changed by set forcing, then AD holds in $L(\mathbb{R})$ (see Steel [23]). Adding parameters to the theory changes the situation quite drastically, though: for example, any forcing collapsing ω_1 changes the theory of $L(\mathbb{R})$ with ordinal parameters. Neeman and Zapletal [19] showed that, assuming roughly a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ with real and ordinal parameters cannot be changed by any proper forcing. We call the conclusion of their theorem $L(\mathbb{R})$*-absoluteness for proper posets*. Likewise, if $\mathcal{C}$ is a class of posets, then $L(\mathbb{R})$*-absoluteness for all posets in* $\mathcal{C}$ is the assertion that the theory of $L(\mathbb{R})$ with ordinal and real parameters cannot be changed by any forcing in $\mathcal{C}$.

Schindler [22] identified the consistency strength of $L(\mathbb{R})$-absoluteness for proper posets to be exactly what he called a *remarkable* cardinal (see Definition 3.2). The definition is a natural weakening of Magidor's characterization of supercompactness [14], in which the embedding is required to exist only in some generic extension of V. This large-cardinal assumption sits far below the level of Woodin cardinals and $\mathsf{AD}^{L(\mathbb{R})}$: while a remarkable cardinal must be weakly compact, if $0^\sharp$ exists then every Silver indiscernible is remarkable in L.

Theorem 1.1 (Schindler)**.**

(a) Assume $V = L$. If κ is a remarkable cardinal, then $L(\mathbb{R})$-absoluteness for proper posets holds in the extension by the Levy collapse to make $\kappa = \aleph_1$.

(b) Conversely, if $L(\mathbb{R})$-absoluteness for proper posets holds, then $\aleph_1$ is remarkable in L.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1363364.

Schindler's lower-bound argument uses the reshaping and almost-disjoint coding methods of Jensen (see Jensen & Solovay [10]). The reshaping poset to which he applies $L(\mathbb{R})$-absoluteness is not σ-closed $*$ ccc or indeed proper in any strong sense. (For more on the properness, distributivity, and stationary-preservation of the reshaping poset, see [21].) Lower bounds for the Proper Forcing Axiom are typically obtained using anti-square posets, which take the form σ-closed $*$ ccc. (For a particularly relevant example, see [3].) One might expect by analogy with the forcing-axiom case that *proper* in Theorem 1.1(b) can be replaced by *σ-closed $*$ ccc*. Our main theorem confirms that expectation.

Theorem 1.2. If $L(\mathbb{R})$-absoluteness holds for σ-closed $*$ ccc posets, then $\aleph_1$ is remarkable in L.

Whereas Schindler's proof uses almost-disjoint coding, our proof expands on the coding methods introduced by Harrington–Shelah, who showed that $L(\mathbb{R})$-absoluteness for ccc posets is equiconsistent with the existence of a weakly compact cardinal. (A version of this theorem appears in Harrington–Shelah [8, Theorem C]. See [7] for the upper-bound portion of the argument, due to Kunen.) Assuming that $\aleph_1^V$ is not weakly compact in L, they code an uncountable sequence of reals into a single real by specializing an Aronszajn tree. Since $\aleph_1$ can be weakly compact in L without being remarkable in L, we must expand these coding techniques to trees that may have large levels. The ccc part of our poset will be a modified specializing poset, following Harrington–Shelah, but first we must add the tree to be specialized. The trees we use bear some resemblance to those of [16], and in Section 5 we adapt the methods of that article to obtain a finer lower bound on the consistency strength of $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets.

In Sections 6 and 7 we generalize a reflection principle that arises in our analysis of trees on ω_1 and make some initial observations, leaving many questions open for future work.

2. The coding argument

In this section we show how to use our generic-absoluteness assumption to establish a principle about trees on ω_1. In the next section, we will use that principle in a σ-closed extension to deduce that $\aleph_1^V$ is remarkable in L.

The first difficulty in adapting the coding methods of [8] is that our trees will not belong to L, so they too will have to be coded. A more substantive difficulty is that, although our trees will have size $\aleph_1$, they will typically have uncountable levels. With a view toward defining a suitable countable analogue of the α^{th} level of a tree, we present our trees as increasing sequences of countable subtrees.

Definition 2.1. A *tree presentation* is a sequence $\vec{T} = \langle T_\alpha : \alpha \leq \omega_1 \rangle$ of trees satisfying the following conditions.

(i) $\vec{T}$ is *concrete*: Each T_α is a tree on a subset of ω_1. Moreover, the height of T_α is a limit ordinal, and T_α is countable iff α is countable.
(ii) $\vec{T}$ is *increasing*: if $\alpha < \beta$ then $T_\alpha \subseteq T_\beta$. Moreover, if $\alpha < \beta$, then T_β is an end-extension of T_α: that is, if $s \leq_{T_\beta} t$ and $t \in T_\alpha$, then $s \in T_\alpha$ also.
(iii) $\vec{T}$ is *continuous*: if $\alpha \leq \omega_1$ is a limit ordinal, then $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$.

Remarks.

(1) Any two presentations of the same tree agree on a club. That is, if $\langle T_\alpha : \alpha \leq \omega_1 \rangle$ and $\langle U_\alpha : \alpha \leq \omega_1 \rangle$ are two tree presentations with $T_{\omega_1} = U_{\omega_1}$, then $T_\alpha = U_\alpha$ for a club of $\alpha < \omega_1$.
(2) If T_{ω_1} is an Aronszajn tree, then one naturally obtains a tree presentation of T_{ω_1} by taking T_α to be the set of nodes on level $< \alpha$.

We will typically use $\leq$ or $\leq_{\vec{T}}$ to refer to the order on any tree in a tree presentation.

Definition 2.2. Let $\vec{T} = \langle T_\alpha : \alpha \leq \omega_1 \rangle$ be a tree presentation. The α^{th} *boundary of* $\vec{T}$ is the set of suprema in $T_{\omega_1} \setminus T_\alpha$ of branches through T_α. That is, a node $t \in T_{\omega_1}$ lies on the α^{th} boundary of $\vec{T}$ iff $t \notin T_\alpha$ but

$$\{s \in T_{\omega_1} : s < t\} \subseteq T_\alpha.$$

Definition 2.3. Let X be a subset of ω_1 and $\vec{T}$ be a tree presentation. We say that X is *codable along* $\vec{T}$ if T_{ω_1} has no uncountable branches, but the set

$$\{\alpha < \omega_1 : T_\alpha \text{ has a cofinal branch in } L[X \cap \alpha]\}$$

is club in ω_1.

The *Tree Reflection Principle at* $\aleph_1$, abbreviated $\mathrm{TRP}(\aleph_1)$, is the assertion

$$(\forall X \subseteq \omega_1)(\forall \text{tree presentations } \vec{T})\ X \text{ is not codable along } \vec{T}.$$

Here we are primarily interested in presentations of trees that have uncountable levels, but Definition 2.3 is relevant even for $\aleph_1$-Aronszajn trees. Even if T_α has many cofinal branches (as will be the case if T_{ω_1} has only countable levels, for example), it may not have any cofinal branches in $L[X \cap \alpha]$. In fact, $\mathrm{TRP}(\aleph_1)$ is consistent (see Theorem 2.4), so it does not contradict the existence of $\aleph_1$-Aronszajn trees.

Our main theorem, Theorem 1.2, factors conveniently through the Tree Reflection Principle.

Theorem 2.4. $L(\mathbb{R})$-absoluteness for ccc posets implies $\mathrm{TRP}(\aleph_1)$.

Since σ-closed posets do not add reals, $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets holds if and only if $L(\mathbb{R})$-absoluteness for ccc posets holds in every extension by a σ-closed poset. With this observation in hand, we obtain the following corollary.

Corollary 2.5. $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets implies that $\mathrm{TRP}(\aleph_1)$ holds in every σ-closed forcing extension.

Combining the forward direction of Schindler's Theorem 1.1 with Corollary 2.5 gives the following consistency result.

Theorem 2.6. Assume $V = L$. If κ is a remarkable cardinal, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the following holds: $\mathrm{TRP}(\aleph_1)$ holds in every extension by a σ-closed forcing.

Theorem 2.6 can be proved directly, without going through Theorem 2.4.

The following theorem will be proved as Theorem 3.1, completing the proof of Theorem 1.2.

Theorem. If $\mathrm{TRP}(\aleph_1)$ holds in every σ-closed forcing extension, then $\kappa = \aleph_1^V$ is remarkable in L.

Now we turn toward the proof of Theorem 2.4.

2.1. **Coding along trees.** The following lemma gets to the heart of our coding argument.

Lemma 2.7. Suppose that X is codable along $\vec{T}$. Then, in a ccc generic extension, there is a real g such that $L[g]$ has uncountably many reals.

We may assume that $\aleph_1^{L[X\cap\alpha]}$ is countable for every $\alpha < \omega_1$; otherwise we could take the trivial poset and let g be any real coding $X \cap \alpha$.

Let $C \subseteq \omega_1$ be a club witnessing that X is codable along $\vec{T}$. In a series of claims we modify $\vec{T}$ until it is suitable for coding.

Claim 2.8. There is a tree presentation $\vec{T}' = \langle T'_\alpha : \alpha \leq \omega_1\rangle$ such that T'_{ω_1} has no cofinal branches and for every $\alpha \in C$, the subtree T'_α has infinitely many cofinal branches in $L[X \cap \alpha]$. Moreover, for any finite subset F of T_{ω_1}, there are infinitely many cofinal branches of T'_α in $L[X \cap \alpha]$ each disjoint from F.

Proof of Claim 2.8. Let T'_α be the image under a definable bijection $\omega_1 \times \omega \to \omega_1$ of $T_\alpha \times \omega$ ordered as the disjoint union of infinitely many copies of T_α. ∎

By replacing $\vec{T}$ with $\vec{T}'$ if necessary, we may assume that $\vec{T}$ satisfies the conclusion of Claim 2.8.

Claim 2.9. There is a tree presentation $\vec{T}' = \langle T'_\alpha : \alpha \leq \omega_1\rangle$ satisfying the conclusion of Claim 2.8 such that T'_{ω_1} has no cofinal branches and if $b \in L[X \cap \alpha]$ is a cofinal branch through T_α, then b has a supremum in $T_{\alpha+1}$.

Proof of Claim 2.9. Notice first that T_α has countably many cofinal branches in $L[X \cap \alpha]$, by our assumption that $\aleph_1^{L[X\cap\alpha]}$ is countable. Recursively define

- $T'_0 = T_0$,
- $T'_\gamma = \bigcup_{\beta<\gamma} T'_\gamma$ for γ limit, and
- $T'_{\alpha+1} = T'_\alpha \cup S_\alpha$,

where the order on T'_α is extended to make S_α exactly the set of suprema of cofinal branches through T_α that belong to $L[X \cap \alpha]$. Notice that T_α and T'_α have the same branches, so in fact $T'_{\alpha+1}$ has suprema of every branch in $L[X \cap \alpha]$ through T'_α.

To satisfy concreteness, the S_α should be chosen to be sets of countable ordinals, and in the end a definable bijection $\omega_1 \times 2 \to \omega_1$ should be used to ensure that the T'_α are trees on ω_1. We leave it to the reader to fill in the details. ∎

The point of Claim 2.9 is to guarantee that the specializing function will not diverge along a countable branch of the tree. Again, by replacing $\vec{T}$ with $\vec{T}'$ if necessary, we may assume that $\vec{T}$ satisfies the conclusion of Claim 2.9.

Now we can mimic the coding argument in [8], using the presentation $\langle T_\alpha : \alpha \leq \omega_1\rangle$ in place of the sequence of initial segments of the tree.

We review the definition of the modified specializing poset defined in [8], tweaked for our purposes.

Definition 2.10. Let $\langle d_\alpha : \alpha < \omega_1\rangle$ be a sequence of reals, construed here as subsets of ω. Conditions in the poset $\mathbb{P}(d_\alpha : \alpha < \omega_1)$ are finite partial specializing functions $T_{\omega_1} \rightharpoonup \mathbb{Q}$ that code $\vec{d}$ along $\vec{T}$. Precisely, a condition is a finite partial function $p\colon T_{\omega_1} \rightharpoonup \mathbb{Q}$ with the following properties:

(a) $s < t$ in T_{ω_1} implies $p(s) < p(t)$ in $\mathbb{Q}$, and
(b) if t is on the α^{th} boundary of $\vec{T}$ and belongs to the domain of p, then either $p(t) \in d_\alpha$ or $p(t)$ is not an integer.

Baumgartner's original argument for the ccc of the specializing poset (see [9, Lemma 16.18], for example) can be repeated to show:

Claim 2.11. If T_{ω_1} has no uncountable branches, then $\mathbb{P}(d_\alpha : \alpha < \omega_1)$ has the ccc.

Suppose that G is $\mathbb{P}(d_\alpha : \alpha < \omega_1)$-generic and consider the generic specializing function $f = \bigcup G\colon T_{\omega_1} \to \mathbb{Q}$.

Remarks.

(1) As in [8], a crucial observation for us is that f is continuous: if t is a node on a limit level of T_{ω_1} then $f(t) = \sup\{f(s) : s < t\}$.
(2) The other crucial observation is that $f''B_\alpha \cap \mathbb{Z} = d_\alpha$ for $\alpha \in C$, where B_α is the intersection of the α^{th} boundary of $\vec{T}$ with $L[X \cap \alpha]$. The $\subseteq$

inclusion follows directly from condition (b) in the definition of the poset; the other inclusion follows from a genericity argument, using the properties of the presentation obtained in Claims 2.8 and 2.9.

Proof of Lemma 2.7. It will be convenient and harmless to assume that T_0 has infinite height. For $\alpha < \omega_1$ we will write α^* for $\min(C \setminus (\alpha+1))$.

The ccc poset will be a length-ω finite-support iteration of posets of the form $\mathbb{P}(d_\alpha : \alpha < \omega_1)$. Since each iterate is ccc, the full iteration is ccc.[1]

First, let $\langle d^0_\alpha : \alpha < \omega_1\rangle$ be any sequence of ω_1 distinct reals such that, for all $\alpha \in C$, d^0_α codes α^*, T_{α^*}, and $X \cap \alpha^*$. Suppose inductively that f_n is the generic specializing function added by $\mathbb{P}(d^n_\alpha : \alpha < \omega_1)$. Let d^{n+1}_α be a real coding $f_n \restriction T_{\alpha^*}$.

In the extension by the full iteration, let g be any real coding (the countable sequence of reals) $\langle d^n_\beta : n < \omega,\ \beta < \min(C)\rangle$.

We will verify that X, C, $\langle d^n_\alpha : \alpha < \omega_1\rangle$, and $\vec{T}$ all belong to $L[g]$. This suffices, since it implies in particular that $\langle d^0_\alpha : \alpha < \omega_1\rangle \in L[g]$. We prove by induction on $\alpha \in C$ that d^n_α belongs to $L[g]$. Moreover, the proof gives a uniform definition of d^n_α in $L[g]$ from n and α; this uniformity in turn is used for the limit case of the proof.

The base case of $\alpha = \min(C)$ is immediate from the choice of g.

Suppose first that the induction hypothesis holds for $\alpha \in C$; we will show that it holds for α^*. From d^0_α we can decode α^*, T_{α^*}, and $X \cap \alpha^*$. From d^{n+1}_α we can decode $f_n \restriction T_{\alpha^*}$, which gives us $d^n_{\alpha^*}$ by Remark (2).

The interesting case is when $\alpha \in \lim(C)$. As in Remark (2), let B_α be the intersection of the α^{th} boundary of $\vec{T}$ with $L[X \cap \alpha]$. We claim that B_α belongs to $L[g]$. By the inductive assumption and the *uniformity* of the proof, the sequence $\langle d^n_\beta : n < \omega,\ \beta \in C \cap \alpha\rangle$ belongs to $L[g]$. From this sequence one can decode $X \cap \alpha$, T_α, and $f \restriction T_\alpha$. From T_α and $X \cap \alpha$ one can decode B_α.

Moreover, if $s \in B_\alpha$, then we can (in $L[g]$) compute $f_n(s)$: the continuity of f implies that $f_n(s) = \sup\{f_n(t) : t \le s,\ t \in T_\alpha\}$. Now $d^n_\alpha = f_n'' B_\alpha \cap \mathbb{Z}$ belongs to $L[g]$, and the induction is complete. ∎

We need one more fact, which is well known.

Theorem 2.12 (Woodin)**.** Suppose that $L(\mathbb{R})$-absoluteness holds for ccc posets. Then $\aleph_1$ is inaccessible to the reals, meaning that $\aleph_1^{L[x]} < \aleph_1$ for every real x.

Theorem 2.12 follows from a lemma of Woodin [24]: If X is an uncountable sequence of reals in V and c is Cohen-generic over V, then in $V[c]$ there is no random real over $L(X, c)$. In fact, full $L(\mathbb{R})$-absoluteness for ccc posets

[1] Note that the tree T_{ω_1} is special after the first forcing, so none of the later posets can add uncountable branches to it.

is not needed for Theorem 2.12; the theorem can be proved from the absoluteness of Σ^1_4 sentences to Cohen and Random extensions. (See Bagaria [1, Theorem 2.1.1.3].) Likewise, Theorem 2.4 requires only the absoluteness of Σ^1_4 sentences to ccc extensions, since the existence of a real x for which $\aleph_1^{L[x]}$ is uncountable can be expressed as a Σ^1_4 sentence.

Proof of Theorem 2.4. The theorem follows immediately from Lemma 2.7, the definition of "codability along $\vec{T}$," and Theorem 2.12. ∎

3. The reflection argument

We will complete our proof of Theorem 1.2 by proving the following theorem.

Theorem 3.1. Suppose that $\mathrm{TRP}(\aleph_1)$ holds in every extension by a σ-closed poset. Then $\kappa = \aleph_1^V$ is a remarkable cardinal in L.

Definition 3.2 (Schindler)**.** Let κ be a cardinal and $\lambda \geq \kappa$ another cardinal. We say that κ is *λ-remarkable* if there is a cardinal $\overline{\lambda} < \kappa$ and in the extension $V^{\mathrm{Coll}(\omega,<\kappa)}$ an elementary embedding $j\colon H(\overline{\lambda}) \to H(\lambda)$ such that $j(\overline{\kappa}) = \kappa$, where $\overline{\kappa}$ is the critical point of j.

If κ is λ-remarkable for every $\lambda > \kappa$, then we just say that κ is *remarkable*.

Remarks.

(1) If we replaced $V^{\mathrm{Coll}(\omega,<\kappa)}$ with V in the definition, we would get supercompactness, by a theorem of Magidor [13]. (See [11, Theorem 22.10].)
(2) The embedding j in the definition is (in $V^{\mathrm{Coll}(\omega,<\kappa)}$) a countable object; its existence is therefore absolute between any two extensions of V where $\kappa = \aleph_1$.
(3) By standard arguments, if κ is λ^+-remarkable, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the set of $X \in [H(\lambda)]^\omega$ for which the anticollapse of X witnesses the λ-remarkability of κ is stationary in $[H(\lambda)]^\omega$. (See [20].)

See Schindler [22, Lemma 1.6] for a characterization of remarkability that does not refer to forcing.

In consistency strength, remarkable cardinals sit strictly between ineffable and ω-Erdős cardinals. (This is proved in [22].) For a more detailed analysis of the consistency strength of remarkable cardinals, see [6, §4], but for our purposes the following will suffice.

Fact (Schindler [22])**.**

(a) Every remarkable cardinal is weakly compact.
(b) If $0^\sharp$ exists, then every Silver indiscernible is remarkable in L.
(c) Remarkability is downward-absolute to L.

Since their discovery by Schindler, remarkable cardinals have been analyzed from many perspectives. For instance, Cheng and Gitman [4] define a version of the Laver Preparation for remarkable cardinals and use it to obtain an indestructibly remarkable cardinal, and their ideas are extended in [3] to obtain a weak version of the Proper Forcing Axiom from a remarkable cardinal. It is also shown there that this weak version of PFA implies that $\aleph_2^V$ is remarkable in L. In Section 7, we will show that their forcing axiom implies a strong form of the Tree Property at $\aleph_2$.

3.1. **Remarkability from branchless trees.** Let $\kappa = \aleph_1^V$. Let $\lambda \geq \kappa$ be an ordinal and $f\colon \kappa \to \lambda$ a bijection. (We will later assume that λ is a cardinal of L and force to add f.) The rest of the definitions in this section are made relative to f and λ.

In order to reflect properties of $\langle \kappa, \lambda \rangle$ to a pair $\langle \overline{\kappa}, \overline{\lambda} \rangle$ of countable ordinals, we need to arrange that $\overline{\kappa}$ is $\aleph_1$ in a suitable inner model. For this we use the following tree, a version of the tree of attempts to express κ as an ordinal of countable cofinality, modified to have height ω_1.

Definition 3.3. Let S, a tree of height κ, be defined as follows. Nodes are pairs $\langle \alpha, s \rangle$, where $\alpha < \kappa$ is an ordinal and s is a strictly increasing finite sequence of ordinals, each less than κ. We also require that $\alpha \in [s_{n-1}, s_n)$, where $s = \langle s_0, s_1, \ldots, s_{n-1}, s_n \rangle$. Nodes are ordered by the ordinal order in the first coordinate and by extension of sequences in the second.

If we define S_β as in Definition 3.3 but after replacing "$< \kappa$" with "$< \beta$," it is easy to see that we get a tree presentation $\langle S_\beta : \beta \leq \omega_1 \rangle$ in the sense of Definition 2.1.

Let $\beta \leq \omega_1$. If b were an uncountable branch through S_β, then the first coordinates of nodes on b would have to be unbounded in β. Conversely, any cofinal ω-sequence in β defines a cofinal branch through S_β. That is, we have:

Lemma 3.4. *S_β has a cofinal branch iff β has countable cofinality.*

Our second tree will capture the fact that λ is a cardinal of L.

Definition 3.5. If δ is an ordinal, then we write $\gamma(\delta)$ for the least ordinal γ (if any exist) such that δ is not a cardinal in $L_{\gamma+1}$.

Remarks. Notice that if $\gamma(\delta)$ is defined, then

(1) $\gamma(\delta) < (\delta^+)^L$, and
(2) every element of $L_{\gamma(\delta)+1}$ is definable in $L_{\gamma(\delta)+1}$ from parameters in δ.

The following definition will apply to the rest of this section. We will define *point* differently in Section 5.

Definition 3.6. Let $\alpha < \kappa$. Define ε_α to be the ordertype of $f''\alpha$.

A *point* is a countable ordinal α such that $\gamma(\varepsilon_\alpha)$ is defined. Points are ordered as follows. If α and α^* are points, then we say $\alpha <_T \alpha^*$ iff:

(i) $\alpha < \alpha^*$ (as ordinals).

(ii) Let $j\colon \varepsilon_\alpha \to f''\alpha$ and $j^*\colon \varepsilon_{\alpha^*} \to f''\alpha^*$ be the anticollapse embeddings. Let $\pi\colon \varepsilon_\alpha \to \varepsilon_{\alpha^*}$ be the composite $(j^*)^{-1} \circ j$. Then π extends to an elementary embedding $\widehat{\pi}\colon L_{\gamma(\varepsilon_\alpha)+1} \to L_{\gamma(\varepsilon_{\alpha^*})+1}$ with $\widehat{\pi}(\varepsilon_\alpha) = \varepsilon_{\alpha^*}$.

Notice that the extended embedding in (ii) is determined uniquely from π, by Remark (2) above.

The definition of *point* depends on both the ordinal λ and the function f. When we wish to emphasize this dependence, we will call a point a $\langle\lambda, f\rangle$-*point*.

Let $T = T(\lambda, f)$ be the tree of increasing sequences of $\langle\lambda, f\rangle$-points.

Lemma 3.7. If there is an uncountable branch through $T(\lambda, f)$, then λ is not a cardinal of L.

Proof. Let $\langle\alpha_\xi : \xi < \omega_1\rangle$ be a branch through T of length ω_1 and for convenience put $\varepsilon(\xi) = \varepsilon_{\alpha_\xi}$. For $\xi < \zeta < \omega_1$ let $\pi_{\xi,\zeta}\colon \varepsilon(\xi) \to \varepsilon(\zeta)$ and $\widehat{\pi}_{\xi,\zeta}\colon L_{\gamma(\varepsilon(\xi))+1} \to L_{\gamma(\varepsilon(\zeta))+1}$ be the maps determined by (ii) above. It is easy to see that the $\pi_{\xi,\zeta}$ commute, so the maps $\widehat{\pi}_{\xi,\zeta}$ also commute. Let M_∞ be the direct limit of the system $\langle L_{\gamma(\varepsilon(\xi))+1}, \widehat{\pi}_{\xi,\zeta} : \xi < \zeta < \omega_1\rangle$, which is wellfounded since it is taken along a sequence of length ω_1. Let $\widehat{\pi}_{\xi,\infty}\colon L_{\gamma(\varepsilon(\xi))+1} \to M_\infty$ be the direct-limit maps. Because the maps $\widehat{\pi}_{\xi,\zeta}$ extend the maps $\pi_{\xi,\zeta}$ determined by the anticollapses of $f''\alpha$, because $\sup_\xi \alpha_\xi = \kappa = \omega_1$, because $\widehat{\pi}_{\xi,\zeta}(\epsilon(\xi)) = \epsilon(\zeta)$, and because $f''\kappa = \lambda$, the model M_∞ is an end-extension of $L_{\lambda+1}$, and $\widehat{\pi}_{\xi,\infty}(\varepsilon(\xi)) = \lambda$ for every ξ. But then, by the elementarity of these maps, λ is not a cardinal in M_∞, so λ is not a cardinal in L. ∎

Lemma 3.8. If λ is not a cardinal of L, then there is in $L[f]$ a branch of length ω_1 through $T(\lambda, f)$.

Proof. Let $\gamma^* \geq \lambda$ be least so that λ is not a cardinal in L_{γ^*+1}. For each $\alpha < \omega_1$, let H_α be the hull in L_{γ^*+1} of $f''\alpha$. There is a club C of α for which $H_\alpha \cap \lambda = f''\alpha$.

For every $\alpha \in C$, let M_α be the transitive collapse of H_α, let $j_\alpha\colon M_\alpha \to H_\alpha$ be the anticollapse embedding, so that $j_\alpha(\varepsilon_\alpha) = \lambda$ and $M_\alpha = L_{\gamma(\varepsilon_\alpha)+1}$ by elementarity. Every $\alpha \in C$ is a point, and, for $\alpha < \alpha'$ each in C, the map $(j_{\alpha'})^{-1} \circ j_\alpha$ witnesses that $\alpha <_T \alpha'$. So C is a branch through T.

It is clear that this construction can be carried out in any inner model with f as an element, in particular in $L[f]$. ∎

The tree $T = T(\lambda, f)$ has a natural tree presentation, namely $\langle T \cap \alpha : \alpha \leq \omega_1\rangle$.

Proof of Theorem 3.1. Assume the Tree Reflection Principle at $\aleph_1$ in all countably closed forcing extensions and let $\lambda \geq \kappa$ be a cardinal of L. We must show that κ is λ-remarkable in L. That is, we must find $\overline{\kappa} < \overline{\lambda} < \kappa$ and (possibly in a generic extension of V) an embedding $j\colon L_{\overline{\lambda}} \to L_\lambda$ such that $\overline{\kappa}$ is the critical point of j and $j(\overline{\kappa}) = \kappa$. The definition requires the domain of j to be $H(\overline{\lambda})^L$, so we must also arrange that $\overline{\lambda}$ is a cardinal of L; this is the main difficulty.

By first forcing with the (countably closed) poset $\mathrm{Coll}(\aleph_1, \lambda)$ if necessary, we can assume that $|\lambda| = \aleph_1$ in V. (By assumption, $\mathrm{TRP}(\aleph_1)$ holds in this extension.) Fix a bijection $f\colon \kappa \to \lambda$. The embedding we find to witness remarkability will extend the anticollapse of a set $f''\overline{\kappa}$.

The following is routine.

Claim 3.9. There are a set $X \subseteq \kappa$ and a club of $\overline{\kappa} < \kappa$ satisfying the following. Let H be the hull in L_λ of $f''\overline{\kappa}$, and let $\rho\colon H \to L_{\overline{\lambda}}$ be the collapsing map.

(a) $f''\overline{\kappa} \cap \kappa = \overline{\kappa}$.
(b) $H \cap \mathrm{On} = f''\overline{\kappa}$.
(c) The map $\rho \circ f\colon \overline{\kappa} \to \overline{\lambda}$ belongs to $L[X \cap \overline{\kappa}]$.
(d) For every $\alpha < \overline{\kappa}$, $L[X \cap \overline{\kappa}] \models |\alpha| \leq \aleph_0$. ∎

Fix $X \subseteq \kappa$ as given by Claim 3.9. We can use Lemma 3.4 to see that S_κ has no uncountable branches in V. Since λ is a cardinal of L, we can use Lemma 3.7 to see that $T(\lambda, f)$ has no uncountable branches in V. We can therefore apply $\mathrm{TRP}(\aleph_1)$ to the natural tree presentation of $S \cup T(\lambda, f)$ to obtain a stationary set of $\overline{\kappa} < \kappa$ such that $S_{\overline{\kappa}}$ and $T(\lambda, f) \cap \overline{\kappa}$ each have no cofinal branches in $L[X \cap \overline{\kappa}]$. Fix such a $\overline{\kappa}$ that also satisfies the conclusions of Claim 3.9. Lemma 3.4 applied in $L[X \cap \overline{\kappa}]$ to $S_{\overline{\kappa}}$ implies that $\overline{\kappa}$ has uncountable cofinality in $L[X \cap \overline{\kappa}]$; combining this fact with item (d) of Claim 3.9, we see that $\overline{\kappa} = \aleph_1^{L[X \cap \overline{\kappa}]}$.

Let H be the hull in L_λ of $f''\overline{\kappa}$, and notice that H is countable. By Condensation, H collapses to an initial segment $L_{\overline{\lambda}}$ of L; let $\rho\colon H \to L_{\overline{\lambda}}$ be the collapsing map. We aim to show that ρ^{-1} witnesses the λ-remarkability of κ. By clauses (a) and (b) of Claim 3.9, ρ^{-1} has critical point $\overline{\kappa}$, and the image of its critical point is κ.

Let $\overline{f}\colon \overline{\kappa} \to \overline{\lambda}$ be the map $\rho \circ f$. Clause (c) of Claim 3.9 ensures that $\overline{f}$ belongs to $L[X \cap \overline{\kappa}]$.

Claim 3.10. The tree $T(\lambda, f) \cap \overline{\kappa}$ is exactly the tree $T(\overline{\lambda}, \overline{f})$ as computed in the model $L[X \cap \overline{\kappa}]$.

Proof of Claim 3.10. The key observation is that ρ restricts to an order isomorphism $f''\alpha \to \overline{f}''\alpha$ for every $\alpha < \overline{\kappa}$. This and the fact that $\overline{\kappa} = \aleph_1^{L[X \cap \overline{\kappa}]}$ imply that $\alpha < \overline{\kappa}$ is a $\langle \lambda, f \rangle$-point in V iff it is a $\langle \overline{\lambda}, \overline{f} \rangle$-point in $L[X \cap \overline{\kappa}]$.

To see that the tree orderings agree for $\langle\lambda, f\rangle$-points and for $\langle\overline{\lambda}, \overline{f}\rangle$-points, notice that the maps π in item (ii) of Definition 3.6 are the same for $\langle\lambda, f\rangle$-points and $\langle\overline{\lambda}, \overline{f}\rangle$-points, since the order-isomorphisms given by ρ commute with the maps j and j^* and their inverses. ■

The point of Claim 3.10 is that

$$L[X \cap \overline{\kappa}] \vDash T(\overline{\lambda}, \overline{f}) \text{ has no cofinal branches,}$$

so we can apply Lemma 3.8 in $L[X \cap \overline{\kappa}]$ to $T(\overline{\lambda}, \overline{f})$, concluding that $\overline{\lambda}$ is a cardinal of L. (Here we have used the fact that $L[X \cap \overline{\kappa}] \supseteq L[\overline{f}]$, which follows from item (c) of Claim 3.9.) This concludes the proof that $\rho^{-1}: L_{\overline{\lambda}} \to L_\lambda$ witnesses the λ-remarkability of κ in L. ■

4. Generic absoluteness from remarkability

In this section we review the upper-bound portion of Schindler's theorem (Theorem 1.1(a)), that $L(\mathbb{R})$-absoluteness for proper posets holds in the extension by the Levy collapse to turn a remarkable cardinal into $\aleph_1$. The point of this diversion is to emphasize the level-by-level upper bound that his argument gives. We will need to strengthen our definition of λ*-remarkability* so that it better resembles the definition of λ*-subcompactness*.

Definition 4.1. Let $\kappa \leq \lambda$ be cardinals. We say that κ is *strongly* λ*-remarkable* if for every $X \subseteq \lambda$ there is in $V^{\mathrm{Coll}(\omega,<\kappa)}$ an embedding $j: \langle H(\overline{\lambda}), \overline{X}, \overline{\kappa}\rangle \to \langle H(\lambda), X, \kappa\rangle$ such that $\overline{\kappa} < \kappa$ and $j \restriction \overline{\kappa}$ is the identity.

A standard argument shows that if κ is strongly λ-remarkable, then in $V^{\mathrm{Coll}(\omega,<\kappa)}$ the set of $M \in [H(\lambda)]^\omega$ whose anticollapses witness the strong λ-remarkability of κ is stationary in $[H(\lambda)]^\omega$.

A cardinal is remarkable if and only if it is strongly λ-remarkable for all λ, since

$$\lambda^+\text{-remarkable} \quad \Rightarrow \quad \text{strongly } \lambda\text{-remarkable} \quad \Rightarrow \quad \lambda\text{-remarkable.}$$

Recall that a poset $\mathbb{P}$ is μ*-linked* if there is a function $f: \mathbb{P} \to \mu$ such that $f(p) = f(q)$ only if p and q are compatible conditions.

We show that Schindler's argument gives the following refined version of Theorem 1.1(a).

Theorem 4.2. Assume $V = L$. If κ is strongly μ^+-remarkable, then in the extension $V^{\mathrm{Coll}(\omega,<\kappa)}$, $L(\mathbb{R})$-absoluteness holds for posets that are proper and μ-linked.

By well-known arguments (see [22, Lemma 2.2] and also [2, Lemma 1.2]), it suffices to show the following:

Theorem. Assume $V = L$. Suppose that κ is strongly μ^+-remarkable, G is $\mathrm{Coll}(\omega, {<}\kappa)$-generic over V, and H is generic over $V[G]$ for some proper, μ-linked poset $\mathbb{P} \in V[G]$. Suppose further that x is a real in $V[G][H]$. Then in V there are a poset Q_x of size $< \kappa$ and a Q_x-generic F over V such that $x \in V[F]$.

Assume $V = L$. We first reduce the problem to posets of size μ^+.

Claim 4.3. If $\mathbb{P}$ is a μ-linked poset and τ is a $\mathbb{P}$-name for a real, then $\mathbb{P}$ has a μ-linked complete subposet $\overline{\mathbb{P}}$ of size μ^+ such that τ is a $\overline{\mathbb{P}}$-name.

Consequently, the empty condition forces $\tau[\dot{G} \cap \overline{\mathbb{P}}] = \tau[\dot{G}]$, where $\dot{G}$ is the canonical name for the generic.

Proof. Since μ-linked posets have the μ^+-cc, we can assume that the name τ has size μ. Let $f\colon \mathbb{P} \to \mu$ witness the linkedness of $\mathbb{P}$. Choose θ sufficiently large, and let $X \prec H(\theta)$ be an elementary submodel of size μ^+ such that $X^\mu \subseteq X$ and $\tau \in X$. Then $\mathbb{P} \cap X$ is as desired. Notice that $f \restriction \mathbb{P} \cap X$ witnesses that $\mathbb{P} \cap X$ is μ-linked, and every maximal antichain of $\mathbb{P} \cap X$ is maximal in $\mathbb{P}$, by the elementarity of X and the fact that X is closed under μ-sequences. ∎

In $V[G]$, let $\mathbb{P}$ be a μ-linked, proper poset, and let τ be a $\mathbb{P}$-name for the real x. In light of Claim 4.3, we can assume that $\mathbb{P}$ has size μ^+. By replacing $\mathbb{P}$ with an isomorphic copy, we can further assume that $\mathbb{P} \subseteq \mu^+$. Let $f\colon \mu^+ \to \mu$ witness that $\mathbb{P}$ is μ-linked. Both $\mathbb{P}$ and f are coded by subsets of μ^+, so we can apply the strong μ^+-remarkability of κ to find (in $V[G]$) an embedding $j\colon L_{\overline{\mu}^+} \to L_{\mu^+}$ such that

- j has critical point $\overline{\kappa}$,
- $j(\overline{\kappa}) = \kappa$,
- there are $\overline{\mathbb{P}}$, $\overline{\tau}$, and $\overline{f}$ such that j is an elementary map $\langle L_{\overline{\mu}^+}, \overline{\mathbb{P}}, \overline{\tau}, \overline{f}\rangle \to \langle L_{\mu^+}, \mathbb{P}, \tau, f\rangle$,
- and $\overline{\mu}$ is a cardinal of L.

As noted earlier, images of embeddings of this type form a stationary subset of $[L_{\mu^+}]^\omega$. We can therefore extend j to an embedding $L_{\overline{\theta}} \to M \prec L_\theta$, which we also call j, for which M is countable and $\mathbb{P}, \tau, f \in M$. (We do not assume that $\overline{\theta}$ is a cardinal of L.) Take θ large enough to witness the properness of $\mathbb{P}$ in $V[G]$.

Claim 4.4. $M[G] \cap \mathrm{On} = M \cap \mathrm{On}$.

Proof of Claim. This just uses the κ-cc of the collapse. Suppose that $\sigma \in M$ is a $\mathrm{Coll}(\omega, {<}\kappa)$-name for an ordinal. Since $\mathrm{Coll}(\omega, {<}\kappa)$ has the κ-cc, we can assume that σ has size $< \kappa$. By elementarity of M the size of σ belongs to $M \cap \kappa = \overline{\kappa}$. Since $\overline{\kappa} \subseteq M$ it follows that $\sigma \subseteq M$. It follows by standard forcing arguments that $\sigma[G] \in M$. ∎

Claim 4.5. $M[G] \prec L_\theta[G]$.

Proof. This is a standard forcing argument, using no special properties of the poset. ∎

So we have an extended anticollapse embedding, which we also call j, from a model of ordinal height $\overline{\theta}$ onto its image $M[G]$. By a condensation argument, the domain of j must be exactly $L_{\overline{\theta}}[\overline{G}]$, where $\overline{G} = j^{-1}{}''G = G \cap \overline{\kappa}$. (This follows from the fact that $M \cap \kappa = \overline{\kappa}$.)

Claim 4.6. The set of countable $N \prec L_\theta[G]$ such that $N = M[G]$ for some $M = N \cap V \prec L_\theta$ is club in $[L_\theta[G]]^\omega$.

Proof. This is another straightforward application of the collapse's chain condition. The set is clearly closed. Consider some countable set $X \subseteq L_\theta[G]$. There is a countable set X^* of $\mathrm{Coll}(\omega, {<}\kappa)$-names, each of size $< \kappa$ (i.e., countable in $V[G]$), for the members of X. Choose a countable model $M \prec L_\theta$ such that $X^* \subseteq M$. Since each name $\sigma \in X^*$ is countable, it is a subset of M, so $M[G] \supseteq X$. ∎

In light of Claim 4.6 and the properness of $\mathbb{P}$, we can choose our model M so that $\mathbb{P}$ has master conditions for $M[G]$; that is, we can choose $M \prec L_\theta$ so that H is $\mathbb{P}$-generic over $M[G]$.

We have an embedding $j\colon L_{\overline{\theta}}[\overline{G}] \to M[G] \prec L_\theta[G]$ with critical point $\overline{\kappa}$. Recall that $\overline{\mathbb{P}} = j^{-1}(\mathbb{P})$ and $\overline{\tau}$ is the $\overline{\mathbb{P}}$-name $j^{-1}(\tau)$. Let $\overline{H} = j^{-1}{}''H$. The elementarity of j implies that $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\theta}}[\overline{G}]$.

Claim 4.7. $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L[\overline{G}]$.

Proof. The strength of the remarkability assumption in L comes from the ability to choose the height of the embedding's domain to be a *cardinal* of L, and it is here that we make use of that. By the $\overline{\kappa}$-cc of $\mathrm{Coll}(\omega, {<}\overline{\kappa})$, the L-cardinal $\overline{\mu}^+$ remains a cardinal in $L[\overline{G}]$, the extension by $\mathrm{Coll}(\omega, {<}\overline{\kappa})$. Every antichain of $\overline{\mathbb{P}}$ has size $\leq \overline{\mu}$, since $\overline{f}$ witnesses the $\overline{\mu}$-linkedness of $\overline{\mathbb{P}}$. It follows that every antichain of $\overline{\mathbb{P}}$ in $L[\overline{G}]$ belongs to $L_{\overline{\mu}^+}[\overline{G}]$. And now, since $\overline{H}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\theta}}[\overline{G}] \supseteq L_{\overline{\mu}^+}[\overline{G}]$, it must also be generic over $L[\overline{G}]$. ∎

Now we're ready to argue that the real x belongs to $L[\overline{G}][\overline{H}]$, a generic extension of L by the small poset $Q_x := \mathrm{Coll}(\omega, {<}\overline{\kappa}) * \dot{\overline{\mathbb{P}}}$. This follows, in $L[\overline{G}]$, from the fact that $\overline{\tau}$ is a $\overline{\mathbb{P}}$-name for x:

$$\begin{aligned} n \in x &\text{ iff } (\exists p \in H) p \Vdash_{\mathbb{P}} \check{n} \in \tau \\ &\text{ iff } (\exists p \in H \cap M[G]) p \Vdash_{\mathbb{P}} \check{n} \in \tau \\ &\text{ iff } (\exists \overline{p} \in \overline{H} \cap L_{\overline{\theta}}[\overline{G}]) \overline{p} \Vdash_{\overline{\mathbb{P}}} \check{n} \in \overline{\tau} \\ &\text{ iff } \check{n} \in \overline{\tau}[\overline{H}]. \end{aligned}$$

The second "iff" uses the fact that H is generic over $M[G]$, the third "iff" uses the elementarity of j, and the fourth uses the Forcing Theorem, for which we needed to know that $\overline{H}$ was $\overline{\mathbb{P}}$-generic over $L[\overline{G}]$. ∎

Remark. Although Neeman & Zapletal obtained $L(\mathbb{R})$-absoluteness for reasonable[2] posets, Schindler's proof here seems to use properness in an essential way. Indeed, Schindler has shown [21] that $L(\mathbb{R})$-absoluteness for reasonable posets is strong enough to give an inner model with a strong cardinal, much stronger in consistency strength than the existence of a remarkable cardinal.

5. A better lower bound

In the previous section, we showed that the consistency of a strongly λ^+-remarkable cardinal is enough to imply the consistency of $L(\mathbb{R})$-absoluteness for λ-linked proper posets. The arguments of Sections 2 and 3 give a naive level-by-level lower bound, too: $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets that are λ-linked implies the λ-remarkability of $\aleph_1^V$ in L. While we do have an equiconsistency between full remarkability and $L(\mathbb{R})$-absoluteness for all σ-closed $*$ ccc posets, we do not have a level-by-level equiconsistency. In this section, we improve the naive lower bound to get closer to a level-by-level equiconsistency. To do this, we adapt the methods of Neeman [16].

Definition 5.1 (See [16])**.** Let λ be a cardinal. A Σ^2_1 *truth for* λ is a pair $\langle Q, \psi\rangle$ such that $Q \subseteq \lambda$, ψ is a first-order formula with one free variable, and there is a set $B \subseteq H(\lambda^+)$ (called the *witness* for $\langle Q, \psi\rangle$) such that $\langle H(\lambda^+), \in, B\rangle \models \psi[Q]$.

An interval $[\kappa, \lambda]$ of cardinals is called Σ^2_1-*indescribable* if for every Σ^2_1 truth $\langle Q, \psi\rangle$ for λ, there are cardinals $\overline{\kappa} \le \overline{\lambda} < \kappa$, $\overline{Q} \subseteq \overline{\lambda}$, and an embedding $j\colon H(\overline{\lambda}) \to H(\lambda)$ such that $\langle \overline{Q}, \psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$, and j is elementary from $\langle H(\overline{\lambda}), \in, \overline{\kappa}, \overline{Q}\rangle$ to $\langle H(\lambda), \in, \kappa, Q\rangle$ with $j \restriction \overline{\kappa} = \mathrm{id}$.

We define what it means for the interval $[\kappa, \lambda]$ to be Σ^2_1-*remarkable* by weakening the definition of "Σ^2_1-indescribable" to require only that the embedding j exist in $V^{\mathrm{Coll}(\omega,<\kappa)}$.

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[2]A poset $\mathbb{P}$ is *reasonable* if for all uncountable cardinals κ, $([\kappa]^\omega)^V$ is stationary in $([\kappa]^\omega)^{V^{\mathbb{P}}}$. See [5].

Notice that the interval $[\kappa,\lambda]$ is Σ^2_1-indescribable for every $\lambda \geq \kappa$ if and only if κ is supercompact. (See [18].) A similar argument shows that, likewise, the interval $[\kappa,\lambda]$ is Σ^2_1-remarkable for every $\lambda \geq \kappa$ if and only if κ is remarkable.

Theorem 5.2. Let $\kappa = \aleph_1^V$, and let $\lambda \geq \kappa$ be a cardinal of L. Assume $L(\mathbb{R})$-absoluteness for σ-closed $*$ ccc posets that are also $\mathfrak{c} \cdot |\lambda|$-linked. Then, in L, the interval $[\kappa,\lambda]$ is Σ^2_1-remarkable.

The reader may, of course, replace $\mathfrak{c} \cdot |\lambda|$ with $|\lambda|$ at the cost of assuming CH, but there are models where $L(\mathbb{R})$-absoluteness for proper posets holds but $2^{\aleph_0} \geq \aleph_2$. For example, force over L with $\mathrm{Coll}(\omega, {<}\kappa)$, where κ is remarkable, and then force over the extension to add $\aleph_2$ Cohen reals.

To prove Theorem 5.2 we must repeat the argument of Section 3, finding an embedding $j\colon L_{\bar{\lambda}} \to L_\lambda$ that witnesses the λ-remarkability of κ and also reflects a prescribed Σ^2_1 truth $\langle Q,\psi\rangle$ for λ.

We adapt the methods of [16], especially those of the first section of that paper, though we will reflect a gap $[\kappa,\lambda]$ of cardinals instead of a single cardinal κ (which in that paper is ω_2^V). Whereas the embeddings of [16] extend inclusion maps $\bar{\varepsilon} \to \varepsilon$, ours will extend reflections of a bijection $\kappa \to \lambda$.

Our strategy will initially resemble our strategy in Section 3, though we will have to work harder to reflect a Σ^2_1 truth. We first collapse λ to have size κ, fixing a bijection $f\colon \kappa \to \lambda$ in the extension. It will be convenient to assume that $f(0) = \kappa$, so that in what follows we needn't worry about taking α large enough that $\kappa \in f''\alpha$. We will ultimately capture the Σ^2_1 truth by applying the Tree Reflection Principle to many more trees than we considered in Section 3.

If we could collapse λ^{+L}, then we could establish the Σ^2_1-remarkability of $[\kappa,\lambda^{+L}]$, which implies our desired conclusion. In particular, we needn't consider the case when λ^{+L} has countable cofinality in V.

5.1. **Basic definitions.**

Definitions in this subsection are made relative to a function f whose domain is $\kappa = \aleph_1$ and whose range is an ordinal λ. We caution the reader that our reflection argument will later require us to relativize these definitions to a different function $\bar{f}$ whose domain, while countable, is $\aleph_1$ in an inner model of V. The dependence on f will not always be made explicit, both to maintain readability and because later we will show that many definitions do not change after relativizing.

Definition 5.3. For $\alpha \leq \kappa$, let π_α denote the collapsing map of $f''\alpha$, and let ε_α denote range(π_α). Finally, we write $f_{\alpha,\alpha'}\colon \varepsilon_\alpha \to \varepsilon_{\alpha'}$ for the function $\pi_{\alpha'} \circ \pi_\alpha^{-1}$.

Fix a club $C \subseteq \kappa$ on which $\alpha \mapsto \varepsilon_\alpha$ is one-to-one.

An *f-point* is a limit ordinal β such that there is $\alpha \in C \cup \{\kappa\}$ satisfying the following conditions.

- $L_\beta \models \varepsilon_\alpha$ is the largest cardinal, and
- β is not a cardinal of L.

The α in the definition is unique (by choice of C), is called the *level* of β, and is denoted $\alpha(\beta)$. We will write $\varepsilon(\beta)$ for $\varepsilon_{\alpha(\beta)}$. (The reader is encouraged to think of α as reflecting κ and of ε_α as reflecting λ.)

Notice that $\alpha(\beta)$ depends on the club C, but only on $C \cap \beta$.

NB. Unlike in [16], the level of an f-point β is not the largest cardinal in L_β.

We will continue to use Definition 3.5: write $\gamma(\beta)$ for the least γ (if any exist) such that β is not a cardinal in $L_{\gamma+1}$.

Let β be a point on level α. Let $\overline{\alpha} < \alpha$ and let H be the hull in $L_{\gamma(\beta)+1}$ of $\operatorname{range}(f_{\overline{\alpha},\alpha})$ ($= \pi_\alpha''(f''\overline{\alpha})$). We say that $\overline{\alpha}$ is *stable* in the f-point β if $\overline{\alpha} \subseteq f''\overline{\alpha}$ and $H \cap \varepsilon_\alpha = \operatorname{range}(f_{\overline{\alpha},\alpha})$. In that case, the anticollapse of H is an embedding $M \to L_{\gamma(\beta)+1}$ that extends $f_{\overline{\alpha},\alpha}$ and has critical point $\overline{\alpha}$. Its domain M must be a level of L; in fact, it must be $L_{\gamma(\overline{\beta})+1}$ for some $\overline{\beta}$, which we denote $\operatorname{proj}_{\overline{\alpha}}(\beta)$, and $\overline{\beta}$ must be an f-point on level $\overline{\alpha}$. The antiprojection embedding, which we call $j_{\overline{\beta},\beta}\colon L_{\gamma(\overline{\beta})+1} \to L_{\gamma(\beta)+1}$, is unique because $\overline{\beta}$ projects below $\varepsilon_{\overline{\alpha}}$.

Claim 5.4. Suppose that β is an f-point and that $\overline{\alpha} < \alpha$ is an ordinal. There is at most one f-point $\overline{\beta}$ on level $\overline{\alpha}$ such that there is an elementary embedding $j\colon L_{\gamma(\overline{\beta})+1} \to L_{\gamma(\beta)+1}$ with critical point $\overline{\alpha}$.

The proofs of the following two claims are straightforward and analogous to the proofs of Claims 1.3–1.4 of [16].

Claim 5.5 (Commutativity of projections)**.** Suppose that $\overline{\alpha} < \alpha < \alpha^*$ and that $\overline{\beta}$, β, and β^* are all f-points, respectively on levels $\overline{\alpha}$, α, and α^*. Suppose further that $\beta = \operatorname{proj}_\alpha(\beta^*)$ and $\overline{\beta} = \operatorname{proj}_{\overline{\alpha}}(\beta)$. (We mean this to imply that $\overline{\alpha}$ is stable in β and α is stable in β^*.) Then $\overline{\alpha}$ is stable in β^*, $\operatorname{proj}_{\overline{\alpha}}(\beta^*) = \overline{\beta}$, and $j_{\overline{\beta},\beta^*} = j_{\beta,\beta^*} \circ j_{\overline{\beta},\beta}$.

Claim 5.6 (The projection ordering is "treelike")**.** Suppose that $\overline{\alpha} < \alpha < \alpha^*$ and that $\overline{\beta}$, β, and β^* are all f-points, respectively on levels $\overline{\alpha}$, α, and α^*. Suppose that $\beta = \operatorname{proj}_\alpha(\beta^*)$ and $\overline{\beta} = \operatorname{proj}_{\overline{\alpha}}(\beta^*)$. Then $\overline{\beta} = \operatorname{proj}_{\overline{\alpha}}(\beta)$.

The following claim is analogous to Claim 1.5 of [16] and is proved similarly. Notice that item (2) follows from Claim 5.4.

Claim 5.7 (Points on the same level I)**.** Suppose that β and β^* are points on the same level α, with $\beta < \beta^*$. Let $\overline{\alpha} < \alpha$ be stable in β^*, let $\overline{\beta}^* = \operatorname{proj}_{\overline{\alpha}}(\beta^*)$, and let j^* denote $j_{\overline{\beta}^*,\beta^*}$. Suppose that β is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\operatorname{range}(f_{\overline{\alpha},\alpha})$ (that is, β belongs to the range of j^*). Then:

(1) $\overline{\alpha}$ is stable in β.

Let $\overline{\beta} = \operatorname{proj}_{\overline{\alpha}}(\beta)$ and let $j = j_{\overline{\beta},\beta}$.

(2) $\overline{\beta} = (j^*)^{-1}(\beta)$. (In particular, $\mathrm{proj}_{\overline{\alpha}}(\beta) < \mathrm{proj}_{\overline{\alpha}}(\beta^*)$.)
(3) $j = j^* \restriction L_{\gamma(\overline{\beta})+1}$.

A *thread* of f-points is a sequence $T = \langle \beta_\alpha : \alpha \in D\rangle$ such that:

(i) D is club in ω_1, and, for each $\alpha \in D$, β_α is an f-point.
(ii) Let $\alpha \in D$ and let $\overline{\alpha} < \alpha$. Then $\overline{\alpha} \in D$ iff $\overline{\alpha}$ is stable in β_α.
(iii) Let $\overline{\alpha} < \alpha$ both belong to D. Then $\beta_{\overline{\alpha}} = \mathrm{proj}_{\overline{\alpha}}(\beta_\alpha)$.

By Claim 5.5, the system

$$\langle L_{\gamma(\beta_\alpha)+1}, j_{\overline{\beta},\beta} : \overline{\alpha}, \alpha \in C, \overline{\alpha} < \alpha\rangle$$

commutes. We write $\mathrm{dlm}(T)$ to denote the direct limit of this system. Since this direct limit is taken along a sequence of uncountable cofinality, the direct limit is well-founded and therefore a level of L; in fact, it must equal $L_{\gamma(\beta^*)+1}$ for some β^*, and the direct-limit embeddings must be the antiprojection embeddings j_{β_α,β^*}. We call β^* the *limit* of T and write $\beta^* = \lim(T)$.

Claim 5.8. Let β be an f-point on level ω_1. Then there is a thread $T \in L[f, C]$ with $\lim(T) = \beta$.

Contrast with Claim 1.7 of [16], which draws the stronger conclusion $T \in L$.

Proof. The hulls in $L_{\gamma(\beta)+1}$ of $f''\alpha$ for α stable in β form a thread whose limit is β. ∎

5.2. **Capturing the Σ^2_1 statement.** In [16], a Σ^2_1 statement $\langle Q, \psi\rangle$ is captured by taking direct limits of levels of L that capture $\langle Q \cap \alpha, \psi\rangle$ for various α; here, we will need to approximate Q using f, rather than by simply using initial segments of Q.

The definitions in this section are made relative to a function f as in the previous section and also to a subset Q of the range of f. Let ψ be a formula with one free variable. For $\alpha \leq \kappa$, put $Q_\alpha := \pi_\alpha''(Q \cap f''\alpha)$.

We say that an f-point β *captures* $\langle Q, \psi\rangle$ if the following hold:

(i) $Q_\alpha \in L_\beta$.
(ii) There is $\eta < \gamma(\beta)$ and there is $B \subseteq L_\beta$ in $L_{\eta+1}$ such that $\langle L_\beta; \in, B\rangle \models \psi[Q_\alpha]$.

The *witness* of β, denoted $\eta(\beta)$, is the least η witnessing condition (ii). There is a subset of L_β in $L_{\eta(\beta)+1} \setminus L_{\eta(\beta)}$, and so the following holds.

Claim 5.9. Every element of $L_{\eta(\beta)+1}$ is definable in $L_{\eta(\beta)+1}$ from parameters in L_β.

If $\beta < \beta^*$ are f-points on the same level α that each capture $\langle Q, \psi\rangle$, then we say β and β^* are *compatible* if there is an elementary embedding $L_{\eta(\beta)+1} \to L_{\eta(\beta^*)+1}$ with critical point β. If β and β^* are compatible f-points on level α, then the embedding witnessing this is unique by Claim 5.9 and it is denoted φ_{β,β^*}.

Claim 5.10. $\varphi_{\beta^*,\beta,^{**}} \circ \varphi_{\beta,\beta^*} = \varphi_{\beta,\beta^{**}}$.

Let Y be a set of compatible f-points on the same level α. The direct limit of the system $\langle L_{\eta(\beta)+1}, \varphi_{\beta,\beta'} : \beta, \beta' \in Y \wedge \beta < \beta' \rangle$ is denoted $\mathrm{hlim}(Y)$ and is called a *horizontal direct limit* to emphasize that f-points in Y are on the same level. If $\mathrm{hlim}(Y)$ is wellfounded, then it is a level of L and by elementarity of the direct-limit embeddings it must be the first level of L satisfying

$$(\exists B \subseteq L_{\beta^*}) \langle L_{\beta^*}, \in, B \rangle \vDash \psi[Q_\alpha].$$

Notice that $\beta^* = \sup(Y)$, because $\mathrm{crit}(\varphi_{\beta,\beta'}) = \beta$ and $\varphi_{\beta,\beta'}(\beta) = \beta'$.

There is no reason to expect $Q \cap f''\alpha$ to belong to L, since f need not. Luckily, we can reflect membership of Q in L to membership of Q_α in L_β for many points β, as the next claim shows.

Claim 5.11. Suppose that $\overline{\beta}$ and β are f-points and that $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$. Let $j = j_{\overline{\beta},\beta}$. Then $(j^{-1})''Q_\alpha = Q_{\overline{\alpha}}$. In particular, if $Q_\alpha \in L_\beta$, then $j^{-1}(Q_\alpha) = Q_{\overline{\alpha}}$.

Proof. This follows immediately from the definition of Q_α and the fact that j extends $f_{\overline{\alpha},\alpha} = \pi_\alpha \circ \pi_{\overline{\alpha}}^{-1}$. ∎

Claim 5.12 (Points on the same level II)**.** Suppose that β and β^* are points on the same level α, with $\beta < \beta^*$. Let $\overline{\alpha} < \alpha$ be stable in β^*, let $\overline{\beta}^* = \mathrm{proj}_{\overline{\alpha}}(\beta^*)$, and let j^* denote $j_{\overline{\beta}^*,\beta^*}$. Suppose that β is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\mathrm{range}(f_{\overline{\alpha},\alpha})$ (that is, β belongs to the range of j^*).

Recall from Claim 5.7 that $\overline{\alpha}$ is stable in β. Let $\overline{\beta} = \mathrm{proj}_{\overline{\alpha}}(\beta)$ and let $j = j_{\overline{\beta},\beta}$.

Suppose also that β and β^* capture $\langle Q, \psi \rangle$ and that $\overline{\alpha}$ is large enough that Q_α is definable in $L_{\gamma(\beta^*)+1}$ from parameters in $\mathrm{range}(f_{\overline{\alpha},\alpha})$.

Then:

(1) $\overline{\beta}$ and $\overline{\beta}^*$ capture $\langle Q, \psi \rangle$.
(2) $\overline{\beta}$ and $\overline{\beta}^*$ are compatible iff β and β^* are compatible.

Assume that β and β^* are compatible. Let $\varphi = \varphi_{\beta,\beta^*}$ and $\overline{\varphi} = \varphi_{\overline{\beta},\overline{\beta}^*}$.

(3) $\varphi = j^*(\overline{\varphi})$.
(4) $j^* \circ \overline{\varphi} = \varphi \circ j$.

Proof. The proof resembles the proof of Claim 1.9 in [16], except that our Claim 5.11 is needed for item (1). ∎

5.3. **The forcing and the trees.** Now we suppose that $f : \kappa \to \lambda$ is a bijection and that $\langle Q, \psi \rangle$ is a Σ^2_1 truth for λ in L.

Following [16], we express the Σ^2_1 truth as a statement about a club E of points on level κ and then force to add a set K so that limits of threads through

K are exactly the points in E, effectively turning the Σ^2_1 statement into a Π^1_1 statement.

Claim 5.13. There is a club $E \subseteq \lambda^{+L}$ such that:

(a) every $\beta \in E$ is an f-point on level κ,
(b) β captures $\langle Q, \psi \rangle$,
(c) for any two $\beta, \beta^* \in E$, the points β and β^* are compatible, and
(d) $\mathrm{hlim}(E)$ is wellfounded.

Proof. Follow the proof of Claim 1.11 of [16], replacing κ with λ throughout. ∎

Definition 5.14. As in [16], we add a system $K \subseteq \kappa$ by countable conditions as follows. (The only clause in this definition that differs materially from its analogue in [16, Definition after Claim 1.11] is clause (g).) A condition in $\mathbb{A}$ is a countable set $p \subseteq \kappa \cup (\lambda, \lambda^{+L})$ of f-points satisfying the following conditions.

(a) Every point in p captures $\langle Q, \psi \rangle$, and if $\beta \in p \setminus \kappa$ then $\beta \in E$.
(b) For every $\alpha < \kappa$ all the points in p on level α are compatible, and their horizontal direct limit is wellfounded.
(c) The set $\{\alpha < \kappa : p \text{ has points on level } \alpha\}$ is closed, with a largest element.

The set of f-points in p on levels $< \kappa$ is called the *stem* of p, denoted $\mathrm{stem}(p)$, and the set of f-points in p on level κ is called the *commitment* of p, denoted $\mathrm{cmit}(p)$. Let $\mathrm{levels}(p)$ denote the set of $\alpha < \kappa$ so that p has f-points on level α. The ordering on $\mathbb{A}$ is defined by setting $q \leq p$ iff the following conditions are satisfied.

(d) $p \subseteq q$.
(e) If $\alpha \in \mathrm{levels}(p)$ then p and q have the same f-points on level α. If $\alpha \in \mathrm{levels}(q) \setminus \mathrm{levels}(p)$ then $\alpha \geq \sup(\mathrm{levels}(p))$.
(f) If $\alpha \in \mathrm{levels}(q) \setminus \mathrm{levels}(p)$ then α is stable in β for every $\beta \in \mathrm{cmit}(p)$, and
$$\left\{\mathrm{proj}_\alpha(\beta) : \beta \in \mathrm{cmit}(p)\right\} \subseteq q.$$
(g) If $\alpha \in \mathrm{levels}(q) \setminus \mathrm{levels}(p)$ then α must be large enough that both of the following hold.
  (i) For every pair $\beta < \beta'$ such that β and β' both belong to $\mathrm{cmit}(p)$, β is definable in $L_{\gamma(\beta')+1}$ from parameters in $f''\alpha$; and
  (ii) for every f-point $\beta \in \mathrm{cmit}(p)$, Q is definable in $L_{\gamma(\beta)+1}$ from parameters in $f''\alpha$.
(h) If $\alpha \in \mathrm{levels}(q) \setminus \mathrm{levels}(p)$, β, $\beta' \in \mathrm{cmit}(p)$, and there are no elements of E between β and β', then there are no f-points in q between $\mathrm{proj}_\alpha(\beta)$ and $\mathrm{proj}_\alpha(\beta')$. Similarly, if there are no elements of E below β, then there are no f-points in q on level α below $\mathrm{proj}_\alpha(\beta)$.

The proofs of the following two claims follow the proofs of their analogues Claims 1.12 and 1.14 of [16], with the use and proof of clause (g) modified in an obvious way.

Claim 5.15. Let $\langle p_n : n < \omega \rangle$ be a decreasing sequence of conditions in $\mathbb{A}$. Then there is a condition q such that $q \le p_n$ for every n.

Claim 5.16. Let p be a condition in $\mathbb{A}$ and let $\xi < \kappa$. There is $q \le p$ such that q has f-points on levels above ξ.

For Claim 5.16, notice that the proof of Claim 1.14 in [16], even though $\kappa = \aleph_2^V$ there, needs only that κ has uncountable cofinality in V.

Notice that $\mathbb{A}$ is κ-linked, since any two conditions with the same stem are compatible.

Let G be $\mathbb{A}$-generic over V, and let $K = \bigcup_{p \in G} \mathrm{stem}(p)$. The remaining definitions in this section depend on K, f, and C, and will later be relativized to $K \cap \overline{\kappa}$, $\overline{f}$, and $C \cap \overline{\kappa}$. A thread T (of height ω_1) is a thread *through* K if unboundedly many f-points of T belong to K.

We will reflect the Σ^2_1 statement by reflecting the branchlessness of several trees, which we define now.

Definition 5.17. Let R_1 be the tree of attempts to build a thread through K with unboundedly many f-points on levels in levels(K) that do not belong to K. More precisely, a node in R_1 is an f-point β with $\alpha(\beta) \in \mathrm{levels}(K)$ so that

(1) for unboundedly many $\overline{\alpha} < \alpha$, $\mathrm{proj}_{\overline{\alpha}}(\beta) \in K$, and
(2) for unboundedly many $\overline{\alpha} < \alpha$ in levels(K), $\mathrm{proj}_{\overline{\alpha}}(\beta) \notin K$ (possibly because $\overline{\alpha}$ is not stable in β and so $\mathrm{proj}_{\overline{\alpha}}(\beta)$ is not defined).

The ordering on R_1 is defined by projection: $\beta <_{R_1} \beta'$ iff $\beta = \mathrm{proj}_{\overline{\alpha}}(\beta')$. This is a tree ordering by Claim 5.6. For $\alpha < \kappa$ write $R_1 \restriction \alpha$ for the tree R_1 restricted to nodes $< \alpha$. This gives a tree presentation $\langle R_1 \restriction \alpha : \alpha \le \kappa \rangle$ in the sense of Definition 2.1.

The following claim is proved exactly like Claim 1.15 of [16], except that (as usual) "definable from parameters in ν" should be replaced by "definable from parameters in $f''\nu$."

Claim 5.18. Let T be a thread of height κ and let $\beta = \lim(T)$. The following are equivalent.

(1) T is a thread through K.
(2) $\beta \in E$.
(3) All sufficiently large points in T on levels in levels(K) belong to K.

It follows that in $V[G]$, R_1 has no cofinal branches.

Definition 5.19. Let R_2 be the tree of attempts to build a thread with only boundedly many points of K to its right. That is, a node in R_2 is a pair $\langle \xi, \delta \rangle$

so that δ is an f-point, $\alpha(\delta) \in \text{levels}(K)$, $\xi < \alpha(\delta)$, and for every $\overline{\alpha}$ that is stable in δ and greater than ξ, there are no points $\overline{\beta}$ of K on level $\overline{\alpha}$ with $\overline{\beta} > \text{proj}_{\overline{\alpha}}(\delta)$. The ordering on R_2 is given by equality on the first coordinate and projection on the second: $\langle \xi, \delta \rangle <_{R_2} \langle \xi', \delta' \rangle$ iff $\xi = \xi'$ and $\delta = \text{proj}_{\alpha(\delta)}(\delta')$. The fact that this gives a tree ordering again follows from Claim 5.6.

For $\alpha < \kappa$ we write $R_2 \restriction \alpha$ to mean the tree R_2 restricted to nodes $\langle \xi, \delta \rangle \in \alpha \times \alpha$. This gives a tree presentation $\langle R_2 \restriction \alpha : \alpha \leq \kappa \rangle$ in the sense of Definition 2.1.

To prove the following claim, copy the proof of Claim 1.17 in [16], replacing "definable from parameters in $\overline{\alpha}$" by "definable from parameters in $f''\overline{\alpha}$."

Claim 5.20. In $V[G]$, R_2 has no cofinal branches.

Definition 5.21. For an f-point δ define $\beta(\delta)$ to be the smallest $\beta > \delta$ in K on the same level as δ if there is one, and leave $\beta(\delta)$ undefined otherwise. If T is a thread of height ω_1, then this function $\delta \mapsto \beta(\delta)$ is defined on unboundedly many points of T. (See the proof of Claim 1.17 in [16].)

A node in R_3 is an f-point δ such that $\alpha(\delta) \in \text{levels}(K)$ and for every $\nu < \alpha(\delta)$ there are $\overline{\alpha} \neq \overline{\alpha}'$ between ν and $\alpha(\delta)$ such that $\beta(\text{proj}_{\overline{\alpha}}(\delta))$ and $\beta(\text{proj}_{\overline{\alpha}'}(\delta))$ are each defined, but neither is a projection of the other. The ordering on R_3 is given by projection: $\delta <_{R_3} \delta'$ iff $\delta = \text{proj}_{\alpha(\delta)}(\delta')$.

For $\alpha < \kappa$ we use $R_3 \restriction \alpha$ to denote the restriction of R_3 to nodes $\delta < \alpha$. This gives a tree presentation $\langle R_3 \restriction \alpha : \alpha \leq \kappa \rangle$ in the sense of Definition 2.1.

The following claim is analogous to Claim 1.18 of [16] and is proved similarly.

Claim 5.22. In $V[G]$, R_3 has no uncountable branches.

5.4. **Relativizing definitions.** Anticipating our need to reflect $f\colon \kappa \to \lambda$ to a countable function $\overline{f}\colon \overline{\kappa} \to \overline{\lambda}$ and work in an inner model in which $\overline{\kappa} = \aleph_1$, we collect some claims relating the definitions of the preceding sections relative to f to those same definitions relative to $\overline{f}$. For now we need only assume that $\overline{\kappa}$ is (in V) a countable ordinal and that $\overline{f} = \pi_{\overline{\kappa}} \circ (f \restriction \overline{\kappa})$. Write $\overline{\lambda}$ for $\varepsilon_{\overline{\kappa}}$, the range of $\overline{f}$.

Where necessary, we will use superscripts to distinguish between a definition relative to f and its analogue relative to $\overline{f}$, so for instance we write π^f_α for the collapsing map $f''\alpha \to \varepsilon_\alpha$ and $\pi^{\overline{f}}_\alpha$ for the collapsing map $\overline{f}''\alpha \to \varepsilon_\alpha$.

The definition of $\alpha(\beta)$ depends on the club C, but only on $C \cap \beta$, so it will be harmless to ignore this dependence.

Claim 5.23. Let $\beta < \overline{\kappa}$.

(a) β is an f-point iff it is an $\overline{f}$-point, and in that case $\alpha^f(\beta) = \alpha^{\overline{f}}(\beta)$ and $\varepsilon^f(\beta) = \varepsilon^{\overline{f}}(\beta)$.

(b) $\pi^f_\alpha = \pi^{\overline{f}}_\alpha \circ (\pi^f_{\overline{\kappa}} \restriction f''\alpha)$.

Let $\alpha = \alpha(\beta)$ and let $\overline{\alpha} < \alpha$.

(c) $f_{\overline{\alpha},\alpha} = \overline{f}_{\overline{\alpha},\alpha}$.

(d) $\overline{\alpha}$ is f-stable in β iff it is $\overline{f}$-stable in β, and the definitions of the projection map and antiprojection map do not depend on whether f- or $\overline{f}$-points are considered.

Proof. (a) It is enough to check that $\varepsilon^f(\beta) = \varepsilon^{\overline{f}}(\beta)$, and this follows from the definition of $\overline{f}$ and the fact that $\pi^f_{\overline{\kappa}}$ is an order-isomorphism.

(b) is very important but follows immediately from the uniqueness of the Mostowski collapse.

(c) Follows from the definition of $f_{\overline{\alpha},\alpha}$ and item (b):

$$\begin{aligned} f_{\overline{\alpha},\alpha} &= \pi^f_\alpha \circ (\pi^f_{\overline{\alpha}})^{-1} = \pi^{\overline{f}}_\alpha \circ (\pi^f_{\overline{\kappa}} \restriction f''\alpha) \circ ((\pi^f_{\overline{\kappa}})^{-1} \restriction \overline{f}''\overline{\alpha}) \circ (\pi^{\overline{f}}_{\overline{\alpha}})^{-1} \\ &= \pi^{\overline{f}}_\alpha \circ (\pi^{\overline{f}}_{\overline{\alpha}})^{-1} = \overline{f}_{\overline{\alpha},\alpha}. \end{aligned}$$

(d) Assuming that β is either f- or $\overline{f}$-stable, we get $\overline{f}''\overline{\alpha} = f''\overline{\alpha}$, since $\pi^f_{\overline{\kappa}}$ is the identity on $\overline{\alpha}$. The rest follows from item (c). ∎

Claim 5.24. Let $\overline{Q} = Q_{\overline{\kappa}} = \pi^f_{\overline{\kappa}}{}''(Q \cap f''\overline{\kappa})$. The definition of Q_α can be reinterpreted using $\overline{Q}$ instead of Q; that is, by $\overline{Q}_\alpha$ we mean $\pi^{\overline{f}}_{\overline{\alpha}}{}''(Q_{\overline{\kappa}} \cap \overline{f}''\alpha)$.

(a) $Q_\alpha = \overline{Q}_\alpha$.

(b) An f-point β f-captures $\langle Q, \psi\rangle$ iff it $\overline{f}$-captures $\langle \overline{Q}, \psi\rangle$, and in that case the definitions of $\eta(\beta)$, compatibility, the horizontal embeddings $\varphi_{\beta,\beta'}$, and horizontal direct limit do not depend on whether f-points capturing $\langle Q, \psi\rangle$ or $\overline{f}$-points capturing $\langle \overline{Q}, \psi\rangle$ are considered.

Proof. (a) Since $\pi^f_{\overline{\kappa}}$ is a bijection, $\overline{Q} \cap \overline{f}''\alpha = \pi^f_{\overline{\kappa}}{}''(Q \cap f''\alpha)$. Now apply Claim 5.23(b).

(b) All of this follows immediately from (a). ∎

The definitions of the trees R_1, R_2, and R_3 depend on C, f, and K. The dependence on C and K can be safely ignored, since we are reflecting each of C and K to one of its initial segments. (Contrast this with the situation in [16], where K is not reflected to an initial segment of itself.) So we write $R_i(f)$ to emphasize the dependence on f.

The next claim follows easily from Claim 5.23.

Claim 5.25. Suppose that M is an inner model in which $\overline{\kappa} = \aleph_1$. For each $i = 1, 2, 3$, $R_i(f) \restriction \overline{\kappa} = R^M_i(\overline{f})$.

5.5. **Proof of the theorem.** After forcing with $\mathrm{Coll}(\kappa,\lambda)$, a σ-closed, $\mathfrak{c}\cdot|\lambda|$-linked poset to collapse λ to have size $\aleph_1$, fix a bijection $f\colon\kappa\to\lambda$. Assume for convenience that $f(0)=\kappa$. Suppose that $\langle Q,\psi\rangle$ is a Σ^2_1 truth for λ in L. Fix E as given by Claim 5.13 and force with the σ-closed poset $\mathbb{A}$ to add the system K of f-points. Let G be the $\mathrm{Coll}(\kappa,\lambda)*\dot{\mathbb{A}}$-generic over V. $\mathrm{Coll}(\kappa,\lambda)$ has size $\mathfrak{c}\cdot|\lambda|$, and $\mathbb{A}$ is κ-linked.

$L(\mathbb{R})$-absoluteness for ccc posets of size κ will hold in $V[G]$, and that is enough to apply Theorem 2.4, since the posets used in the proof of that theorem all have size $\leq\kappa$.

We will need the trees S and T from Section 3 to ensure that $\overline{\kappa}$ and $\overline{\lambda}$ are cardinals in $L[\overline{X}]$.

Choose $X\subseteq\kappa$ so that, for a club of $\overline{\kappa}$, $X\cap\overline{\kappa}$ codes $C\cap\overline{\kappa}$, $\pi^f_{\overline{\kappa}}\circ f\restriction\overline{\kappa}$, and $K\cap\overline{\kappa}$. The trees R_1, R_2, R_3, S, and T are trees of size $\aleph_1$ with natural tree presentations, and they have no uncountable branches in $V[G]$. We can therefore apply Theorem 2.4 to find $\overline{\kappa}<\kappa$ satisfying the following:

(1) $\overline{\kappa}$ is a regular cardinal of L.
(2) $\overline{X}:=X\cap\overline{\kappa}$ codes $\overline{C}:=C\cap\overline{\kappa}$, $\overline{f}:=\pi^f_{\overline{\kappa}}\circ f\restriction\overline{\kappa}$, and $\overline{K}:=K\cap\overline{\kappa}$.
(3) $f''\overline{\kappa}\cap\kappa=\overline{\kappa}$, and $\mathrm{Hull}_{L_\lambda}(f''\overline{\kappa})\cap\lambda=f''\overline{\kappa}$.
(4) $R_1\restriction\overline{\kappa}$, $R_2\restriction\overline{\kappa}$, $R_3\restriction\overline{\kappa}$, $S\restriction\overline{\kappa}$, and $T\restriction\overline{\kappa}$ all have no cofinal branches in $L[\overline{X}]$.

Remark. We do not require the tree R_0 of [16] for item (1), since we can use the Harrington–Shelah theorem that $L(\mathbb{R})$-absoluteness for ccc posets implies the weak compactness of κ in L.

We will show that the anticollapse $j\colon L_{\overline{\lambda}}\to L_\lambda$ of $\mathrm{Hull}_{L_\lambda}(f''\overline{\kappa})$ witnesses the Σ^2_1-remarkability of $[\kappa,\lambda]$. Argue as in the proof of Theorem 1.2 to see that $\overline{\kappa}=\aleph_1^{L[\overline{X}]}$ and that $\overline{\lambda}$ is a cardinal of L, using the fact that $S\restriction\overline{\kappa}$ and $T\restriction\overline{\kappa}$ have no cofinal branches in $L[\overline{X}]$. It remains to show that $\langle j^{-1}{}''Q,\psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L.

First, notice that

$$j^{-1}{}''Q=\pi^f_{\overline{\kappa}}{}''Q=\pi^f_{\overline{\kappa}}{}''(Q\cap f''\kappa)=\overline{Q}.$$

So we need to show that $\langle\overline{Q},\psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L.

Work in $L[\overline{X}]$. Let $\overline{E}$ be the set of limits of threads through $\overline{K}$. $\overline{K}$ is a set of f-points below $\overline{\kappa}$, so by Claim 5.24 every point in $\overline{K}$ is an $\overline{f}$-point capturing $\langle\overline{Q},\psi\rangle$. So $\overline{E}$ is a set of points on level $\overline{\kappa}$, each capturing $\langle\overline{Q},\psi\rangle$.

Claim 5.26. $\overline{E}$ consists of compatible points.

Proof. Repeat the proof of Claim 1.21 in [16], except that there is no need for us to appeal to condition (3) on page 10 there, since our $\overline{K}$ is just an initial segment of K. (As usual, "definable from parameters in α" should be changed to "definable from parameters in $f''\alpha$" and $Q \cap L_{\overline{\kappa}}$ should be replaced by $Q_{\overline{\kappa}}$.) ∎

Claim 5.27. $\mathrm{hlim}(\overline{E})$ is wellfounded.

Proof. The proof of Claim 1.22 of [16] can be repeated, changing definability from parameters in ν to definability from parameters in $f''\nu$ and changing $Q \cap L_{\overline{\kappa}}$ to $Q_{\overline{\kappa}}$ throughout. ∎

Claim 5.28. $\overline{E}$ is unbounded in $\overline{\lambda}^{+L}$.

Proof. The proof follows the proof of Claim 1.24 in [16], but we give it here in full, since it is the heart of the argument. Fix an $\overline{f}$-point $\delta \in (\overline{\lambda}, \overline{\lambda}^{+L})$. We must find a $\beta \in \overline{E}$ with $\beta > \delta$. Let B be the set of $\alpha < \overline{\kappa}$ that are stable in δ. Apply Claim 5.8 (in $L[\overline{X}]$) to conclude that B is club in $\overline{\kappa}$ and $\langle \mathrm{proj}_\alpha(\delta) : \alpha \in B\rangle$ is a thread with limit δ. (This differs slightly from the proof in [16]: there, the thread belongs to L, whereas here it only belongs to $L[\overline{X}]$.)

Let D be the set of $\alpha \in \mathrm{levels}(\overline{K})$ such that there is a $\overline{f}$-point β in $\overline{K}$ on level α with $\beta > \mathrm{proj}_\alpha(\delta)$. Let β_α be the least such β.

Since $R_2 \restriction \overline{\kappa}$ has no cofinal branches in $L[\overline{X}]$, D is unbounded in $\overline{\kappa}$. Since $R_3 \restriction \overline{\kappa}$ has no cofinal branches in $L[\overline{X}]$, there is $\nu < \overline{\kappa}$ such that for all $\alpha, \alpha' \in D \cap (\nu, \overline{\kappa})$, one of β_α and $\beta_{\alpha'}$ is a projection of the other. It follows that $\{\beta_\alpha : \alpha \in D \wedge \alpha > \nu\}$ generates a thread. This is a thread through $\overline{K}$, which has limit greater than δ by Claim 5.7. ∎

Let $M = \mathrm{hlim}(\overline{E})$. By Claim 5.27, M is wellfounded and is therefore a level of L. (Since $\overline{\lambda}^{+L}$ is countable in V, this is nontrivial.) Each of the points in $\overline{E}$ captures $\langle \overline{Q}, \psi\rangle$, so by the elementarity of the horizontal embeddings $\varphi_{\beta,\beta'}$ it follows that M satisfies, "There exists $B \subseteq L_{\beta^*}$ such that $\langle L_{\beta^*}, B\rangle \models \psi[\overline{Q}]$." Here $\beta^* = \sup(\overline{E})$, which by Claim 5.28 equals $\overline{\lambda}^{+L}$. Since M is an initial segment of L, we have shown that $\langle \overline{Q}, \psi\rangle$ is a Σ^2_1 truth for $\overline{\lambda}$ in L, completing the proof of the theorem. ∎

6. The Tree Reflection Principle at larger cardinals

We conclude by discussing two natural generalizations of the Tree Reflection Principle. The first simply generalizes TRP to cardinals larger than $\aleph_1$.

Definition 6.1. For a cardinal κ, the *Tree Reflection Principle* at κ, abbreviated $\mathrm{TRP}(\kappa)$, is the following assertion:

For all $X \subseteq \kappa$ and all trees T on κ, either T has a cofinal branch or

$$\{\alpha < \kappa : T \restriction \alpha \text{ has no cofinal branch in } L[X \cap \alpha]\}$$

is stationary in κ.

(For clarity we emphasize that here $T \upharpoonright \alpha$ means the restriction of T to the subset α, not the truncation of T to the first α levels.)

Proposition 6.2. Suppose that $\lambda \geq \kappa^+$. If κ is λ-subcompact and G is $\mathrm{Coll}(\kappa, \lambda)$-generic over V, then $\mathrm{TRP}(\kappa)$ holds in $V[G]$.

The argument resembles the proof of Theorem 4.2, so we provide only a sketch here.

Proof. Let $\dot{X}$ and $\dot{T}$ be $\mathrm{Coll}(\kappa, \lambda)$-names for a set $X \subseteq \kappa$ and a tree $T \subseteq \kappa$. The collapse has the λ^+-cc, so we can assume that the names $\dot{X}$ and $\dot{T}$ are subsets of λ. Let p be a condition that forces "$\dot{T}$ has no cofinal branches." We use the subcompactness assumption to find a map

$$e \colon \langle H(\overline{\lambda}), \overline{\dot{X}}, \overline{\dot{T}} \rangle \to M \prec \langle H(\lambda), X, T \rangle$$

with critical point $\overline{\kappa}$ and $e(\overline{\kappa}) = \kappa$. Using the $<\kappa$-closure of $\mathrm{Coll}(\kappa, \lambda)$, we can extend p to a master condition p^* for M, and we can further insist that p^* forces both

$$(\dot{T} \cap M) \upharpoonright \overline{\kappa} = \dot{T} \upharpoonright \overline{\kappa} \qquad \text{and} \qquad (\dot{X} \cap M) \upharpoonright \overline{\kappa} = \dot{X} \upharpoonright \overline{\kappa}.$$

Suppose that $G \subseteq \mathrm{Coll}(\kappa, \lambda)$ is generic over V with $p^* \in G$, so that $G \cap M$ is generic over M. Then $\overline{G} := e^{-1\prime\prime}(G \cap M)$ is a $\mathrm{Coll}(\overline{\kappa}, \overline{\lambda})$-generic filter over $H(\overline{\lambda})$ containing the condition $e^{-1}(p)$, which forces that $\overline{T} := \overline{\dot{T}}[\overline{G}] = T \cap \overline{\kappa}$ has no cofinal branches in $H(\overline{\lambda})[\overline{G}]$.

Put $\overline{X} = \overline{\dot{X}}[\overline{G}] = X \cap \overline{\kappa}$. Since $\overline{\lambda} \geq \overline{\kappa}^+ \geq \overline{\kappa}^{+L[\overline{X}]}$, every subset of $\overline{\kappa}$ in $L[\overline{X}]$ belongs to $L_{\overline{\lambda}}[\overline{X}]$. Since $L_{\overline{\lambda}}[\overline{X}] \subseteq H(\overline{\lambda})[\overline{G}]$, we have shown that $\overline{T}$ has no cofinal branches in $L[\overline{X}]$. ∎

7. The Strong Tree Reflection Principle

It seems natural to drop the coding requirement in $\mathrm{TRP}(\kappa)$ in favor of a purely combinatorial assertion about trees.

Definition 7.1. The *Strong Tree Reflection Property* at κ, denoted $\mathrm{STRP}(\kappa)$, is the assertion:

For all trees T on κ, either T has a cofinal branch or

$$\{\alpha < \kappa : T \upharpoonright \alpha \text{ has no cofinal branch}\}$$

is stationary in κ.

Unlike the $\mathrm{TRP}(\kappa)$, the $\mathrm{STRP}(\kappa)$ implies the Tree Property at κ: if every level of a tree T has size $< \kappa$, then the initial segments of T form a club in $[T]^{<\kappa}$. And the usual Π^1_1-reflection argument to show that a weakly compact cardinal has the Tree Property shows that it also has the Strong Tree Reflection Property.

Proposition 7.2. If κ is weakly compact, then $\mathrm{STRP}(\kappa)$ holds.

If $V = L$, then $\mathrm{STRP}(\kappa)$ and $\mathrm{TRP}(\kappa)$ are equivalent. Since the cardinals in L with the usual Tree Property are exactly the weakly compact cardinals, it follows that in L the STRP, TRP, and usual Tree Property all coincide and hold exactly at the weakly compact cardinals of L.

7.1. **A counterexample to STRP.** Suppose that κ is a cardinal for which $\kappa^{<\kappa} = \kappa$. Under this assumption we can adapt the tree of Definition 3.3 to provide a counterexample to $\mathrm{STRP}(\kappa^+)$.

For β an ordinal let S_β be the tree with nodes $\langle \alpha, s\rangle$, where $s \in \beta^{<\kappa}$ is a strictly increasing sequence of double-successor length — say $s\colon \eta + 2 \to \beta$ for concreteness — and α is an ordinal in the interval $[s(\eta), s(\eta+1))$. Nodes are ordered by the ordinal order in the first coordinate and by extension of sequences in the second.

Notice that $S_{\kappa^+} = \bigcup_{\beta<\kappa^+} S_\beta$. Notice also that S_β has a cofinal branch in a model $M \supseteq \kappa^+$ if and only if $\mathrm{cf}^M(\beta) \le \kappa$. So S_{κ^+} has no cofinal branches, yet S_β has a cofinal branch for every $\beta < \kappa^+$, and the cardinal-arithmetic assumption is not needed to see this. But we need to know that S_{κ^+} can be coded as a subset of κ^+ in such a way that a club of its initial segments take the form S_β, and for this we need to know that S_β has cardinality κ. The assumption $\kappa^{<\kappa} = \kappa$ is exactly what we need.

So S_{κ^+}, a tree of size and height κ^+, has no cofinal branches, yet a club of its subtrees have cofinal branches. It is therefore a counterexample to $\mathrm{STRP}(\kappa^+)$. Of course, for $\mathrm{STRP}(\kappa^+)$, a κ^+-Aronszajn tree would also suffice, but the previous example may be relevant to the following question.

Question 7.3. Does the Tree Property at κ^+ imply $\mathrm{STRP}(\kappa^+)$?

7.2. **A weak form of PFA.** Bagaria–Gitman–Schindler [3] isolate a weak form of the Proper Forcing Axiom that they show to be equiconsistent with a remarkable cardinal. Here we use their forcing axiom to prove that $\mathrm{STRP}(\aleph_2)$ is consistent. We do not know whether STRP can hold at other successor cardinals.

Definition 7.4. The *weak Proper Forcing Axiom* (wPFA) is the following assertion. If $\mathbf{M} = \langle M; \in, \langle R_\alpha : \alpha < \omega_1\rangle\rangle$ is a transitive model with $\aleph_1$-many predicates, $\varphi(x)$ is a Σ_1 formula, and $\varphi[\mathbf{M}]$ is forced to hold by some proper poset, then there is (in V) a transitive model $\overline{\mathbf{M}} = \langle \overline{M}; \in, \langle \overline{R}_\alpha : \alpha < \omega_1\rangle\rangle$ such that $\varphi[\overline{\mathbf{M}}]$ holds and there is in a set-forcing extension of V an elementary embedding $j\colon \overline{\mathbf{M}} \to \mathbf{M}$.

Proposition 7.5. wPFA implies $\mathrm{STRP}(\aleph_2)$.

Proof. Assume wPFA, let T be a tree on ω_2, and suppose that T has no cofinal branches.

Consider the poset $\mathbb{P}$ of finite $\in$-increasing sequences of countable $M \preceq H(\omega_2)$, ordered by reverse inclusion. This poset is strongly proper for countable structures: any condition s with $M \in s$ is a strong master condition for M. It follows (see e.g. [17, Lemma 3.7], which follows Mitchell's [15, Lemma 2.10]) that $\mathbb{P}$ does not add branches of length ω_1 to trees in V. Notice that $\mathbb{P}$ also collapses ω_2 to ω_1.

Let $\mathbf{M}$ be the structure $H(\aleph_3)$ together with the tree ordering $<_T$ and a relation coding each countable ordinal. Take φ to be the formula asserting the existence of a club $C \subseteq \omega_2$ and a specializing function $f\colon T \restriction C \to \omega$, a Σ_1 formula. (By $T \restriction C$ we mean the restriction of T to nodes on levels in the club C.) The proper forcing $\mathbb{P} * \dot{\mathbb{S}}$, where $\mathbb{S}$ is the usual (ccc) poset to specialize $T \restriction C$, forces $\varphi[\mathbf{M}]$. Apply wPFA to find $\overline{\mathbf{M}}$, $\overline{T}$, $\overline{C}$, and $\overline{f}$ such that $\overline{C}$ is club in $\overline{\kappa} := \omega_2^{\overline{\mathbf{M}}}$, and $\overline{f}\colon \overline{T} \restriction \overline{C} \to \omega$ is a specializing function.

Since $\overline{M} \supseteq \omega_1$, the critical point of the virtual elementary embedding $j\colon \overline{\mathbf{M}} \to \mathbf{M}$ must be at least $\overline{\kappa} = \omega_2^{\overline{\mathbf{M}}}$, which in particular implies that $\overline{T} = T \cap \overline{\kappa}$. So $T \restriction \overline{C}$ is special, and therefore $T \cap \overline{\kappa}$ has no cofinal branches.

We have shown how to find a single $\overline{\kappa}$ for which $T \cap \overline{\kappa}$ has no cofinal branches; to find stationarily many, one simply adds a club in ω_2 as a predicate of $\mathbf{M}$ to ensure that $\overline{\kappa}$ belongs to that club. ∎

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