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SYNCHRONIZATION OF TIME-DELAYED SYSTEMS WITH DISCONTINUOUS COUPLING

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This paper concerns the synchronization of time-delayed systems with periodic on-off coupling. Based on the stability theory and the comparison theorem of time-delayed differential equations, sufficient conditions for complete synchronization of systems with constant delay and time-varying delay are established. Compared with the results based on the Krasovskii–Lyapunov method, the sufficient conditions established in this paper are less restrictive. The theoretical results show that two time-delayed systems can achieve complete synchronization when the average coupling strength is sufficiently large. Numeric evidence shows that the synchronization speed depends on the coupling strength, on-off rate and time delay.

Keywords: time-delayed system, complete synchronization, discontinuous coupling

Classification: 34F05, 34H10

1. INTRODUCTION

Since the pioneer work of Pecora and Carroll [18], chaos synchronization in coupled systems has been extensively studied in many areas, such as biological systems, information processing, secure communications, economical systems, etc [3]. There are many different kinds of synchronization, such as complete synchronization [24], generalized synchronization [15], anti-synchronization [17], projective synchronization [7], etc.

As a typical dynamics behaviour of coupled nonlinear systems, synchronization is ubiquitous in nature, technology, and society. Sun *et al.* [27] showed that the velocity synchronization of multi-agent systems with mismatched parameters is achieved when the sampled period is chosen appropriately. Synchronization of duplex networks was investigated in Ref. [11]. Tan et al. illustrated that the common decision of a group can well reflect the concerns of all group members with evolutionary dynamics [30]. In Ref. [31], a microscopic deterministic formulation is developed for analyzing and controlling the evolutionary game dynamics on complex networks. To verify the presence of generalized synchronization of complex networks, a rigorous theoretical basis for the applicability of auxiliary system approach is established in Ref. [37]. Due to different applications of synchronization, a wide variety of approaches and controllers have been proposed, including adaptive control [4, 12, 13, 23, 34, 35], finite-time control [1, 29, 32], sliding mode control [5, 16, 20, 21, 22, 33] and so on.

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Most previous works focused on the synchronization of coupled systems with continuous coupling [2, 19, 36]. In this case, the synchronization condition can be derived by using the Lyapunov function or functional method. However, in real-word systems, the dynamical interactions between two communities or networks may be switched off sometimes. For example, the dynamics of biological populations are affected by climate, such as temperature and seasonality, the interactions between them may be activated and depressed periodically. This phenomenon can be described in on-off coupling to some extent. Therefore, the study of synchronization between two on-off coupled chaotic systems is very important to the perspective of control theory and practical applications. In Refs. [25, 28], the synchronization of coupled systems with on-off coupling was investigated.

The effect of time delay, which arise from a realistic consideration of finite communication times and processing speeds, is a key issue that has received increasing attention from many fields of science and engineering [6, 9, 14, 26]. It has been discovered that time delays have great influence on the behavior of complex dynamical systems. In spite of these investigations, a systematic study of the synchronization of time-delayed systems with on-off coupling has been lacking. From recent works [6, 9, 14, 26], the Krasovskii–Lyapunov theory is useful for discussing the synchronization of coupled time-delayed systems with continuous coupling. According to the Krasovskii–Lyapunov theory, coupled time-delayed systems can realize synchronization if the coupling strength is sufficiently large for all time. However, this condition is not valid for the general case where the coupling is switched off in some time intervals. Therefore, it is necessary and important to propose the synchronization conditions for time-delayed systems with on-off coupling.

Motivated by the above analysis, in this paper, for the first time we investigate the synchronization of time-delayed systems with discontinuous coupling. Based on the stability theory of differential equations and the comparison theorem, sufficient conditions for systems with both constant delay and time-varying delay are presented. Different from the Krasovskii–Lyapunov method, we use the comparison theorem of differential equations to obtain the sufficient conditions for time-delayed systems with periodic on-off coupling. Compared with the results based on the Krasovskii–Lyapunov method, the sufficient conditions established in this paper are less restrictive. In particular, we show that two time-delayed systems can realize synchronization if the average coupling strength is large enough.

The rest of this paper is organized as follows. In Section 2, we give the system formulation and some useful preliminaries. In Section 3, the sufficient conditions for synchronization are established. In Section 4, numerical simulations are provided to verify the effectiveness of the theoretical results. Finally, some conclusions are provided in Section 5.

Notations: Throughout this paper unless specified we let $\|\cdot\|$ be Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T.

2. SYSTEM FORMULATION AND PRELIMINARIES

Consider the following system:

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau(t))), \tag{1}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T \in R^n$ is the state vector of the system, $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuously differentiable nonlinear vector functions, $\tau(t)$ is the time-varying delay and satisfies $0 < \tau(t), \dot{\tau}(t) \le \varepsilon < 1$, ε is a positive constant.

To realize the synchronization of two time-delayed chaotic systems, we refer to model (1) as the drive system, and the response system is given by the following equation:

$$\dot{y}(t) = f(y(t)) + g(y(t - \tau(t))) + u(t), \tag{2}$$

where $y(t) = (y_1, \dots, y_n)^T \in R^n$ is the state vector of the response system (2), $u(t)$ is the control input to be designed. Let $e(t) = y(t) - x(t)$ is the synchronization error between the drive system (1) and the response system (2), then one gets the following error system:

$$\dot{e}(t) = f(y(t)) - f(x(t)) + g(y(t - \tau(t))) - g(x(t - \tau(t))) + u(t). \tag{3}$$

Definition 2.1. (Shi et al. [24]) Chaotic systems (1) and (2) are said to achieve complete synchronization if, for any initial states $x(0), y(0)$,

$$\lim_{t\to\infty} \|y(t, y(0)) - x(t, x(0))\| = 0.$$

The following assumption will be used throughout this paper in establishing our synchronization conditions.

Assumption 2.2. (Shi et al. [23]) For functions $f(\cdot)$, $g(\cdot)$ and $\forall x, y \in R^n$, there exist two nonnegative constants l_f and l_g such that

$$(y - x)^T[f(y) - f(x)] \le l_f(y - x)^T(y - x), \|g(y) - g(x)\| \le l_g\|y - x\|.$$

3. SUFFICIENT CONDITIONS FOR SYNCHRONIZATION

3.1. Complete synchronization with time-invariant delay

In this section, we will study the complete synchronization between systems (1) and (2) with time-invariant delay. The drive-response systems (1) and (2) can be rewritten as:

$$\dot{x}(t) = f(x(t)) + g(x(t - \tau)), \tag{4}$$

$$\dot{y}(t) = f(y(t)) + g(y(t - \tau)) + u(t), \tag{5}$$

where

$$u(t) = -k(t)e(t). \tag{6}$$

Here we choose the coupling strength $k(t)$ as the on-off periodic coupling,

$$k(t) = \begin{cases} k, & nT \le t < (n + \theta)T; \\ 0, & (n + \theta)T \le t < (n + 1)T, \end{cases} \quad (n = 0, 1, 2, \dots) \tag{7}$$

where k is a positive constant, $T > 0$ is called the on-off period, $\theta (0 < \theta < 1)$ is called the on-off rate. It is easy to see that (6) is a discontinuous coupling when $0 < \theta < 1$, while $\theta = 1$, corresponds to the continuous case. Then we can rewrite error system (3) in the following form:

$$\dot{e}(t) = f(y(t)) - f(x(t)) + g(y(t-\tau)) - g(x(t-\tau)) - k(t)e(t). \tag{8}$$

In order to get our main results on time-invariant delay, we need a lemma as follows:

Lemma 3.1. (Hmamed [8]) Consider the time-delayed system

$$\dot{W}(t) = BW(t) + CW(t-\tau), \tag{9}$$

where $W(t) \in R^n$, B and C are matrices in proper dimensions. The stability of system (9) is equivalent to the stability for the following system

$$\dot{H}(t) = (B + zC)H(t), \forall |z| = 1, \tag{10}$$

where $z = \exp(j\eta)$, $\eta \in (-\pi, \pi], j = \sqrt{-1}$.

One of the main theorems of the paper is presented here.

Theorem 3.2. Suppose that Assumption 2.2 holds and the following condition is satisfied:

$$\overline{k}(t) > k_c \triangleq l_f + l_g, \tag{11}$$

where $\bar{k}(t)$ is the time-average coupling strength defined by $\bar{k}(t) = \frac{1}{T}\int_0^T k(s)\mathrm{d}s$. Then, under the controller (6), systems (4) and (5) can achieve complete synchronization.

Proof. Based on the theory of differential equations, it is easy to see that differential equation (8) has a unique global solution on $t \geq 0$, denoted by $e(t, e(0))$ for any initial data $e(0) = y(0) - x(0)$. And $e(t, 0) \equiv 0$ is a trivial solution of the system (8). Obviously, if this trivial solution is globally asymptotic stable, then the linear time-delay differential equation (8) is asymptotically stable about their zero solution and systems (4) and (5) can achieve complete synchronization.

Without using the Krasovskii–Lyapunov method, this paper constructs a positively-defined function as follows.

$$V = \frac{1}{2}e^T(t)e(t).$$

Then the derivative of V along the trajectory of (8) is

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= e^T(t)\dot{e}(t) \\
&= e^T(t)[f(y(t)) - f(x(t)) + g(y(t-\tau)) - g(x(t-\tau)) - k(t)e(t)] \\
&\leq [l_f - k(t)]e^T(t)e(t) + e^T(t)[g(y(t-\tau)) - g(x(t-\tau))].
\end{aligned}$$

From Assumption 2.2, we have

$$\begin{aligned}
&e^T(t)[g(y(t-\tau)) - g(x(t-\tau))] \leq \|e^T(t)\|\|g(y(t-\tau)) - g(x(t-\tau))\| \\
&\leq l_g\|e^T(t)\|\|e(t-\tau)\| \leq \frac{l_g}{2}(\|e^T(t)\|^2 + \|e(t-\tau)\|^2) \\
&= \frac{l_g}{2}[e^T(t)e(t) + e^T(t-\tau)e(t-\tau)].
\end{aligned} \tag{12}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le \left[l_f - k(t) + \frac{l_g}{2}\right] e^T(t)e(t) + \frac{l_g}{2} e^T(t-\tau)e(t-\tau) \\ &= [2l_f - 2k(t) + l_g]V(t) + l_g V(t-\tau). \end{aligned} \tag{13}$$

By using the comparison theorem of the delayed equation[10], $V(t)$ satisfies

$$V(t) \le W(t), \forall t > 0, \tag{14}$$

where $W(t)$ is the solution of the following differential equation:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = [2l_f - 2k(t) + l_g]W(t) + l_g W(t-\tau), \tag{15}$$

with the initial condition $W(0) = V(0)$. If we can prove that $\lim_{t\to\infty} W(t) = 0$, then we get $\lim_{t\to\infty} V(t) = 0$, which further results in $\lim_{t\to\infty} e(t) = 0$.

Recalling Lemma 3.1, the stability of system (15) is equivalent to the stability of the following system:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = [2l_f - 2k(t) + l_g + z l_g]H(t), \tag{16}$$

where $z = \exp(j\eta)$, $\eta \in (-\pi, \pi], j = \sqrt{-1}$.

In the following, we will show that the trivial solution of Eq. (16) is globally exponentially stable. From (16) we get

$$\begin{aligned} H(t) &= H(0)\exp\left\{\int_0^t [2l_f - 2k(s) + l_g + z l_g]\mathrm{d}s\right\} \\ &= H(0)\exp\{(2l_f + l_g)t\}\exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\}\exp\{z l_g t\}. \end{aligned} \tag{17}$$

Taking the modulus of both sides of (17) yields

$$\begin{aligned} |H(t)| &= |H(0)\exp\{(2l_f + l_g)t\}\exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\}\exp\{z l_g t\}| \\ &\le \Big|H(0)\|\exp\{(2l_f + l_g)t\}\|\Big\|\exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\}\Big\|\exp\{z l_g t\}\Big|. \end{aligned} \tag{18}$$

It is easy to see that for any t in $[0, \infty)$ there exists a positive integer m such that $t \in [mT, (m+1)T)$, then we have

$$\int_0^t k(s)\mathrm{d}s \ge \int_0^{mT} k(s)\mathrm{d}s = m\int_0^T k(s)\mathrm{d}s = mT\bar{k}(t) \ge \bar{k}(t)(t-T), \tag{19}$$

which implies

$$\left|\exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\}\right| = \exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\} \le \exp\{-2\bar{k}(t)(t-T)\}. \tag{20}$$

From the definition of $k(t)$ we have

$$\bar{k}(t) = \frac{1}{T}\int_0^T k(s)\mathrm{d}s = \frac{1}{T}\int_0^{\theta T} k\mathrm{d}s = k\theta.$$

Note that $z = \exp(j\eta) = \cos\eta + j\sin\eta$, and $\exp\{zl_g t\} = \exp\{(\cos\eta + j\sin\eta)l_g t\}$, then

$$|\exp\{zl_g t\}| = \exp\{(\cos\eta)l_g t\} \le \exp\{l_g t\}. \tag{21}$$

Combining inequalities (18) – (21) results in

$$\begin{aligned} |H(t)| &\le |H(0)|\exp\{2\bar{k}(t)T\}\exp\{-2[\bar{k}(t) - l_f - l_g]t\} \\ &= M\exp\{-2[\bar{k}(t) - l_f - l_g]t\}, \end{aligned} \tag{22}$$

where $M = |H(0)|\exp\{2\bar{k}(t)T\} = |H(0)|\exp(2k\theta T)$.

Therefore, if the condition (11) is satisfied, then the trivial solution of Eq. (16) is globally exponentially stable, which means that $\lim_{t\to\infty} W(t) = 0$. So we get $\lim_{t\to\infty} V(t) = 0$, which further results in $\lim_{t\to\infty} e(t) = 0$. This means that complete synchronization between systems (4) and (5) could be achieved for every initial data. The proof is completed. □

Remark 3.3. Recently, the Krasovskii–Lyapunov method has been extensively used to analyze the synchronization problems of time-delayed systems. The Krasovskii–Lyapunov method requires that the derivative of a Lyapunov functional V is negative for all time. Taking the Lyapunov functional $V(t) = \frac{1}{2}e^T(t)e(t) + \int_{t-\tau}^t e^T(s)e(s)\mathrm{d}s$, one can easily obtain a sufficient condition for synchronization $k(t) > l_f + l_g$ for all time. However, for the on-off coupling defined in (7), this condition can not hold when $k(t) = 0$. Different from the Krasovskii–Lyapunov method, we use the comparison theorem of differential equations to derive sufficient conditions for the synchronization of time-delayed systems with periodic on-off coupling, which does not require the negativeness of the derivative of the positively-defined function. Compared with the results based on the Krasovskii–Lyapunov method, the sufficient conditions established in this paper are less restrictive.

Remark 3.4. As we all know, the synchronization speed is an important issue in chaotic synchronization. From the inequality (22), one can see that the speed of synchronization is proportional to the coupling strength k and the on-off rate θ. So we can accelerate the speed of synchronization by increasing k or θ, which will be confirmed by the numerical results in Section 4.

3.2. Complete synchronization with time-varying delay

In Theorem 3.2, we have introduced the complete synchronization between two chaotic systems with time-invariant delay. But many real-world systems are characterized instead by evolving, adaptive couplings which always vary in time according to different environmental conditions. For examples, swarms under varying environmental conditions; wireless sensor networks that gather and communicate data to a central base station. Therefore, establishing complete synchronization of on-off coupled chaotic systems with time-varying delay is a challenging task.

Consider coupled systems (1) and (2) with time-varying delay, the coupling function $u(t)$ of response system is defined by

$$u(t) = -k(t)e(t) - 2\mu k(t)\frac{e(t)}{|e(t)|^2}\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s, \tag{23}$$

where $k(t)$ is the coupling strength as noted above, μ is a positive constant.

Theorem 3.5. Consider coupled systems (1) and (2) with time-varying delay $\tau(t)$. Suppose that $0 < \tau(t), \dot{\tau}(t) \leq \varepsilon < 1$ and Assumption 2.2 holds. If there exist $\mu(> 0)$ and $\overline{k}(t)$ such that inequalities

$$\mu > \frac{l_g}{2(1-\varepsilon)}, \overline{k}(t) > l_f + \mu + \frac{l_g}{2} \tag{24}$$

hold, then under the controller (23), systems (1) and (2) can achieve complete synchronization.

P r o o f. The synchronization error between systems (1) and (2) can be written as:

$$\begin{aligned}\dot{e}(t) \quad = \quad & f(y(t)) - f(x(t)) + g(y(t-\tau(t)) - g(x(t-\tau(t))) - k(t)e(t) \\ & -2\mu k(t)\tfrac{e(t)}{|e(t)|^2}\textstyle\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s.\end{aligned} \tag{25}$$

We can see that if the origin of the error system (25) is asymptotically stable, then systems (1) and (2) can achieve complete synchronization.

Let

$$V = \frac{1}{2}e^T(t)e(t) + \mu\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s.$$

Then the derivative of V along the trajectory of (25) is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= e^T(t)\dot{e}(t) + \mu e^T(t)e(t) - \mu(1-\dot{\tau}(t))e^T(t-\tau(t))e(t-\tau(t)) \\ &= e^T(t)[f(y(t)) - f(x(t)) + g(y(t-\tau(t)) - g(x(t-\tau(t))) - k(t)e(t)] \\ &\quad -2\mu k(t)\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s + \mu e^T(t)e(t) - \mu(1-\dot{\tau}(t))e^T(t-\tau(t))e(t-\tau(t)).\end{aligned}$$

From Assumption 2.2, we have

$$\begin{aligned}e^T(t)[g(y(t-\tau(t))) - g(x(t-\tau(t)))] &\leq l_g\|e^T(t)\|\|e(t-\tau(t))\| \\ &\leq \frac{l_g}{2}[e^T(t)e(t) + e^T(t-\tau(t))e(t-\tau(t))]\end{aligned} \tag{26}$$

So one gets

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} \leq\ & [l_f + \mu - k(t) + \frac{l_g}{2}]e^T(t)e(t) - 2\mu k(t)\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s \\ & +(\frac{l_g}{2} - \mu + \mu\varepsilon)e^T(t-\tau(t))e(t-\tau(t)).\end{aligned} \tag{27}$$

Applying condition (24), we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le \left[l_f + \mu - k(t) + \frac{l_g}{2}\right] e^T(t)e(t) - 2\mu k(t)\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s \\ &\le [2l_f + 2\mu - 2k(t) + l_g]\left[\frac{1}{2}e^T(t)e(t) + \mu\int_{t-\tau(t)}^{t} e^T(s)e(s)\mathrm{d}s\right] \\ &= [2l_f + 2\mu - 2k(t) + l_g]V(t). \end{aligned} \tag{28}$$

By using the comparison theorem of the delayed equation, $V(t)$ satisfies

$$V(t) \le W(t), \forall t > 0, \tag{29}$$

where $W(t)$ is the solution of the following differential equation:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = [2l_f + 2\mu - 2k(t) + l_g]W(t), \tag{30}$$

with the initial condition $W(0) = V(0)$.

From (30) we get

$$\begin{aligned} W(t) &= W(0)\exp\left\{\int_0^t [2l_f + 2\mu - 2k(s) + l_g]\mathrm{d}s\right\} \\ &= W(0)\exp\{(2l_f + 2\mu + l_g)t\}\exp\left\{-2\int_0^t k(s)\mathrm{d}s\right\}. \end{aligned} \tag{31}$$

The rest procedure of the proof is similar to that in the proof of Theorem 1, hence we omit it. Finally, we get

$$\begin{aligned} |W(t)| &\le |W(0)|\exp\{2\bar{k}(t)T\}\exp\{[2l_f + 2\mu - 2\bar{k}(t) + l_g]t\} \\ &= M\exp\left\{-2\left[\bar{k}(t) - l_f - \mu - \frac{l_g}{2}\right]t\right\}, \end{aligned} \tag{32}$$

where $M = |W(0)|\exp(2\bar{k}(t)T) = |W(0)|\exp(2k\theta T)$.

Therefore, if the condition (24) is satisfied, then the trivial solution of Eq. (30) is globally exponentially stable, which means that $\lim_{t\to\infty} W(t) = 0$. So we get $\lim_{t\to\infty} V(t) = 0$, which further results in $\lim_{t\to\infty} e(t) = 0$. The proof is completed. □

Remark 3.6. From the criterion presented in Theorem 3.5, we can conclude that: on-off coupled systems (1) and (2) with time-varying delay can achieve complete synchronization, if the following inequality is satisfied:

$$\bar{k}(t) > k_c' \triangleq l_f + \frac{l_g}{2} + \frac{l_g}{2(1-\varepsilon)}. \tag{33}$$

When $\tau(t) = \tau(\varepsilon = 0)$, the inequality (33) is equivalent to (11), which also implies that Theorem 3.5 is the extension of Theorem 3.2.

Remark 3.7. In the pervious works [24], we have discussed the synchronization of chaotic systems with on-off periodic coupling, but time delay hasn't been mentioned. On the one hand, time delay extensively occurs in many biological or physical systems and should be taken into account. On the other hand, from Theorems 3.2 and 3.5, we can see that our results are valid for arbitrarily periodic coupling only if the time-average coupling strength $\overline{k}(t)$ is large enough. Therefore, the results in this paper complement and extend existing results.

4. SIMULATION RESULTS

In order to demonstrate the effectiveness of proposed approaches, several chaotic systems with different structures are used as examples to illustrate how to apply the results obtained.

Example 1. In the first example, we take the Ikeda system with time-invariant delay as the dynamic of the drive-response systems. The Ikeda system describing the dynamics of an optical bistable resonator was well known for delay-induced chaotic behavior and can be described as follows:

$$\dot{x}(t) = -\alpha x(t) + \beta \sin(x(t-\tau)) \triangleq f(x(t)) + g(x(t-\tau)). \tag{34}$$

From the simulations in Ref. [6], we can see that system (34) exhibits chaotic behavior, for suitable parameters α, β, τ.

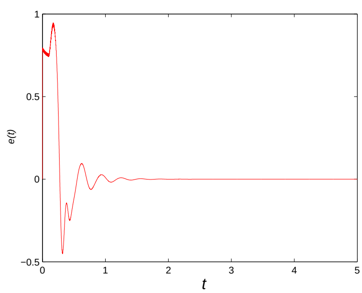

Fig. 1. Trajectory of the synchronization error between systems (1) and (2) with $\alpha = 5, \beta = 20$ and $k = 20, \theta = 0.4, \tau = 0.2$.

It is easy to compute that $(y-x)(-\alpha y + \alpha x) = -\alpha(y-x)^2$ and $|\beta \sin y - \beta \sin x| = |\beta||\sin y - \sin x| \le 2|\beta||\sin \frac{y-x}{2}||\cos \frac{y+x}{2}| \le |\beta||y-x|$. By taking $\alpha = 5, \beta = 20$, we can see that Assumption 1 is satisfied with $l_f = 0, l_g = 20$. When $k = 20, \theta = 0.4, \tau = 0.2$, the simulation result is exhibited in Figure 1. One can see that the complete synchronization is realized exponentially, and the simulation matches the theoretical results perfectly. It should be pointed out that the condition (11) is just a sufficient condition, systems (1) and (2) may achieve synchronization when $\overline{k}(t)$ is less than k_c, which can be confirmed by the fact that complete synchronization can be achieved with $\bar{k}(t) = 8 < k_c = 20$.

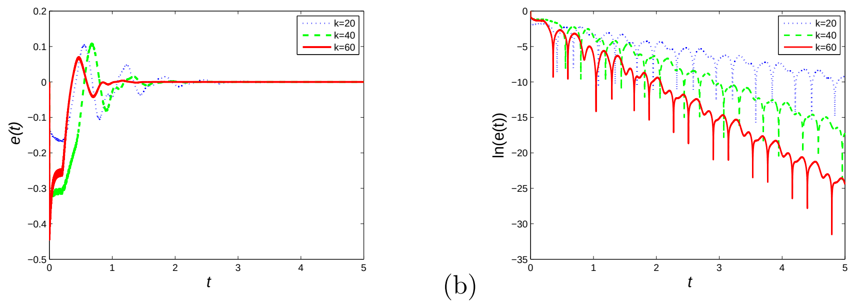

Fig. 2. (a) The variation of synchronization error $e(t)$ between chaotic systems (1) and (2) with $\alpha = 5, \beta = 20, \theta = 0.4$ and $k = 20, 40, 60$; (b)The corresponding logarithmic plot.

To study the effect of the average coupling strength $\bar{k}(t)$ on the synchronization speed, we simulate the evolution of two systems through taking different values of k or θ. Both the simulation results in Figures 2 and 3 show that systems with large $\bar{k}(t)$ converge faster than those with small $\bar{k}(t)$, which is consistent with the analysis of Remark 3.4.

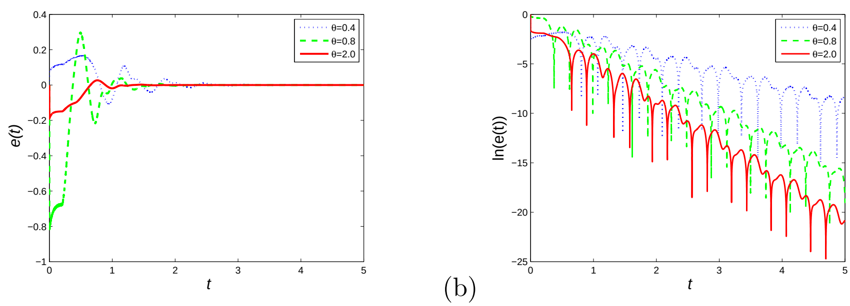

Fig. 3. (a) The variation of synchronization error $e(t)$ between chaotic systems (1) and (2) with $\alpha = 5, \beta = 20, k = 20$ and $\theta = 0.4, 0.8, 2.0$; (b)The corresponding logarithmic plot.

Although the theoretical analysis shows that the synchronization between two systems does not depend on time delay, we want to analyze the effect of time delay on the synchronization speed. Figure 4 indicates that the smaller time delay is, the faster synchronization speed converges, which is consistent with the reality.

Example 2. In order to demonstrate the effectiveness of the theoretical results of Theorem 3.5, we take the Mackey-Glass system as the dynamic of the drive-response

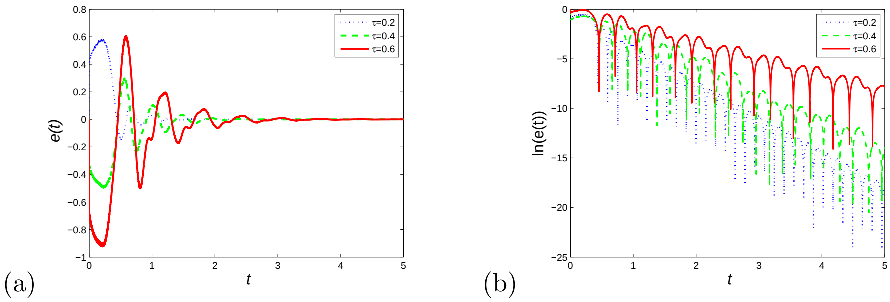

Fig. 4. (a) The variation of synchronization error $e(t)$ between chaotic systems (1) and (2) with $\alpha = 5, \beta = 20, k = 20, \theta = 0.4$ and $\tau = 0.2, 0.4, 0.6$; (b)The corresponding logarithmic plot.

systems. As a model for producing high-dimensional chaos to test various methods of chaotic time series analysis, controlling chaos, the Mackey-Glass system has been widely investigated. The drive-response systems can be described as follows:

$$\dot{x}(t) = -cx(t) + a\frac{x(t-\tau(t))}{1+x^b(t-\tau(t))} \triangleq f(x(t)) + g(x(t-\tau(t))), \tag{35}$$

$$\dot{y}(t) = -cy(t) + a\frac{y(t-\tau(t))}{1+y^b(t-\tau(t))} + u(t) \triangleq f(y(t)) + g(y(t-\tau(t))) + u(t), \tag{36}$$

where $a, b, c > 0$, $\tau(t)$ is a time-varying delay. Without loss of generality, we set $\tau(t) = 0.3 + 0.1\sin(2t)$; obviously, $0 < \tau(t) < 0.4, \dot{\tau}(t) \leq 0.2 = \varepsilon < 1$.

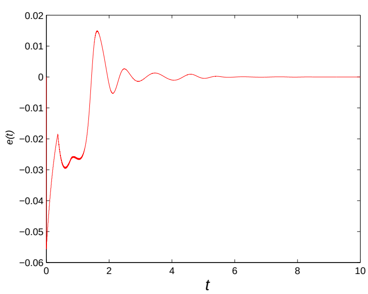

Fig. 5. Trajectory of the synchronization error between systems (35) and (36) with $a = 3, b = 10, c = 1$, $k = 20, \theta = 0.4, \tau = 0.3 + 0.1\sin(2t)$.

It is easy to see that $|g'(x)| \leq a(b-1)^2/4b$ and $|g(y)-g(x)| = |g'(\xi)||y-x|$. Therefore Assumption 1 is satisfied with $l_f = 0, l_g = |a(b-1)^2/4b|$. When we take $a = 3, b = 10, c =$

1, and $k = 20, \theta = 0.4$, the simulation result exhibited in Figure 5 indicates that the complete synchronization is realized exponentially. Figures 6 and 7 show that systems with large $\bar{k}(t)$ converge faster than those with small $\bar{k}(t)$, which also in accordance with the analysis of Remark 3.4.

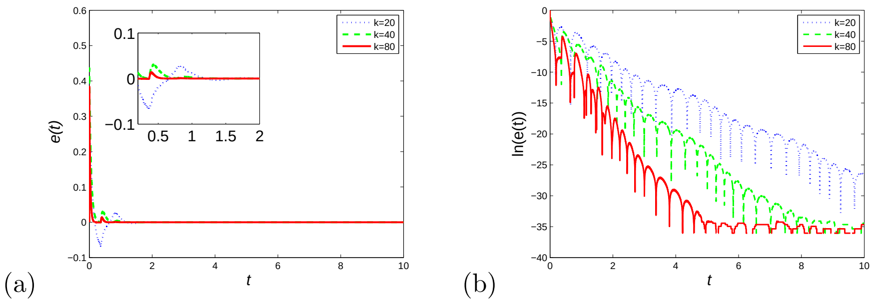

Fig. 6. (a) The variation of synchronization errors $e(t)$ between chaotic systems (35) and (36) with $\theta = 0.4$ and $k = 20, 40, 80$; (b)The corresponding logarithmic plot.

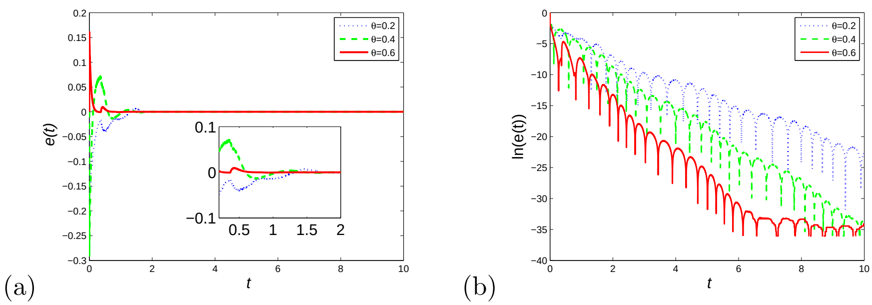

Fig. 7. (a) The variation of synchronization errors $e(t)$ between chaotic systems (35) and (36) with $k = 30$ and $\theta = 0.2, 0.4, 0.6$; (b)The corresponding logarithmic plot.

5. CONCLUSIONS

In conclusion, we have investigated the complete synchronization between two time-delayed systems with on-off periodic coupling. Sufficient conditions for the complete synchronization are obtained based on the stability theory and the comparison theorem of time-delayed differential equations. The theoretical results show that time-delayed chaotic systems with on-off coupling can achieve complete synchronization when the time-average coupling strength is large enough. Numerical simulations fully verify our main results.

We have developed a new approach to analyze the synchronization of time-delayed systems with periodic on-off coupling, which can derive less restrictive synchronization conditions than those resulting from the Krasovskii–Lyapunov theory. Our method has broad applications in synchronization problem of complex systems. For example, it is applicable to investigate the generalized synchronization of two different time-delayed systems with discontinuous coupling. Our method is as well applicable to synchronization of gene regulatory networks and neural networks with discontinuous coupling and time delays.

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