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CONE-TYPE CONSTRAINED RELATIVE CONTROLLABILITY OF SEMILINEAR FRACTIONAL SYSTEMS WITH DELAYS

Beata Sikora and Jerzy Klamka

The paper presents fractional-order semilinear, continuous, finite-dimensional dynamical systems with multiple delays both in controls and nonlinear function f. The constrained relative controllability of the presented semilinear system and corresponding linear one are discussed. New criteria of constrained relative controllability for the fractional semilinear systems with delays under assumptions put on the control values are established and proved. The conical type constraints are considered. The results are illustrated by an example.

Keywords: the Caputo derivative, semilinear fractional systems, relative controllability, delays in control, constraints

Classification: 93B05, 93C05, 93C10, 34G20

1. INTRODUCTION

Studying the controllability of dynamical systems is one of the main elements in their analysis. Due to a large number of mathematical models of dynamical systems with delays, the controllability problem for such systems is especially important. Delay differential equations are used in mathematical models of physical, biological, biochemical and medical phenomena [7, 8, 9, 31], and many others.

Recent few decades in the mathematical modeling show that linear, semilinear and nonlinear systems and processes have led to the use of not only the integer-order differential equation but also fractional-order differential equations. It becomes clear that fractional-order models reflect the behavior of many real-life processes more accurately then the integer-order ones. Depth discussions about fractional differential equations and their practical applications can be found in monographs [12, 19, 20, 22, 23, 25, 26]. Fractional differential equations occur, for example, in mathematical models of biological and biochemical models such as: cancer models [1], population growth models [16, 29], models of migration of interacting agents [18]. In [17], the fractional dynamics in DNA have been modeled.

The controllability is an important qualitative property reflecting behavior of a dynamical system. Kaczorek in [10] and [11] analyzed controllability problems for positive

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linear continuous-time fractional systems with delays in control, without constraints. Balachandran et al. in [3] studied relative controllability of linear fractional control systems with multiple delays in control. Controllability of linear fractional systems with one control delay without constraints has been studied by Wei in [30]. Controllability of time-delay fractional systems with multiple delays in control, with and without constraints, were studied by Sikora in [27]. All the mentioned works deal with constant delays. In [15], the controllability criteria for linear fractional systems with varying delays were proposed.

Controllability of linear systems with delays in the state without constraints was analyzed by Zhang in [32], while different type constraints for the systems were studied by Sikora in [28]. Moreover, controllability for a class of fractional neutral integrodifferential equations with unbounded delay and controllability of neutral fractional functional equations with impulses and infinite delay were discussed in [2].

Balachandran et al. in [4] proposed some controllability criteria for nonlinear fractional dynamical systems with time varying multiple delays and distributed delays in control defined in finite dimensional spaces. In [5] the global relative controllability of semilinear fractional dynamical systems with multiple delays in control without constraints for finite dimensional spaces was discussed. The results in [4] and [5] were obtained by using the Schauder fixed point theorem. Sufficient conditions for the controllability of nonlinear fractional delay systems obtained by using fixed point arguments were proposed also by Balachandran in [6]. A detailed analysis of fractional systems with and without control delays was presented in [21].

It should be stressed that papers on controllability of fractional-order systems address mainly controllability issues for unconstrained controls. However, in practice, control (input function) is not completely unlimited, it is usually constrained in various ways.

The majority of papers on semilinear and nonlinear fractional-order systems, both with and without delays, are based on various fixed point theorems. Our proposal is to use the Frechet derivative to solve some controllability problems for semilinear fractional systems. As it is shown in the illustrative example, the proposed criteria are easy in use. The aim of the paper is to give new controllability criteria for continuous-time semilinear fractional systems with multiple delays in control and nonlinear function f based on the Frechet derivative. We consider cone-type constraints of the control values, the ones most frequently appearing in practical applications.

The paper is organized as follows. Section 2 includes some preliminary definitions and formulas. Section 3 presents the mathematical model of the discussed semilinear fractional dynamical systems with delays. The existence theorem for solution of the discussed systems is formulated and proved in this section. Moreover, definitions of local and global constrained controllability for the systems are formulated there. Section 4 contains the main results of the paper – the criteria for relative controllability of the considered fractional systems with delays. Proofs of the theorems are provided in detail. In Section 5 the theoretical results are illustrated by a numerical example. Finally, some concluding remarks and future work are presented in Section 6.

2. MATHEMATICAL PRELIMINARIES

In this section we present basic definitions, properties and notation that is used in the paper. In the fractional calculus several definitions of derivatives can be found, e.g. the Riemann–Liouville, the Grünwald–Letnikov, the Caputo fractional derivatives [23]. In the paper, the Caputo fractional derivatives are used due to the fact that the Caputo definition makes it possible to apply the initial conditions for fractional differential equations in the same form as for the integer-order case.

The Caputo fractional derivative of an order α ($n < \alpha < n+1$, $n \in \mathbb{N}$) for a differentiable function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}} \, \mathrm{d}\tau,$$

where Γ is the gamma function.

For $\alpha > 0$, $\beta > 0$, the so-called Mittag–Leffler function [20, 23] is defined by the following formula

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; z \in \mathbb{C}.$$

Taking $\beta = 1$, for $\alpha > 0$ and an arbitrary nth order square matrix A, a pseudo-transition matrix $\Phi_0(t)$ of the linear fractional system $^{C}D^{\alpha}x(t) = Ax(t)$ is

$$\Phi_0(t) = E_{\alpha,1}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)}.$$

Moreover, taking $\beta = \alpha$, we have

$$E_{\alpha,\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{(At^{\alpha})^k}{\Gamma(\alpha k + \alpha)} = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

And we introduce the following denotation [10]

$$\Phi(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

We also provide formulas for the inverse Laplace transform that are needed in the paper [20]:

$$\mathcal{L}^{-1}[s^{\alpha-1}(s^{\alpha}I - A)^{-1}] = \Phi_0(t),$$

$$\mathcal{L}^{-1}[(s^{\alpha}I - A)^{-1}] = \Phi(t).$$

There are several methods used to compute the functions $\Phi_0(t)$ and $\Phi(t)$: the inverse Laplace transform method (applying the above formulas), the Jordan matrix decomposition method and the Cayley–Hamilton method. All the methods are presented in [20]. In the paper, in an example, we apply the method based on the Cayley–Hamilton theorem which states that a matrix A satisfies its own characteristic equation. That is (see also [10]), if

$$\det[s^\alpha I - A] = (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_1 s^\alpha + a_0,$$

then

$$A^n + a_{n-1}(A)^{n-1} + \cdots + a_1 A^\alpha + a_0 I = 0.$$

Finally, let $L^2([0,\infty),\, \mathbb{R}^m)$ denote the Hilbert spaces of square integrable functions with values in $\mathbb{R}^m$, $L^2_{loc}([0,\infty),\, \mathbb{R}^m)$ be the linear space of locally square integrable functions with values in $\mathbb{R}^m$, and $L_\infty([0,T],U)$ mean the Banach space of functions bounded almost everywhere, defined on $[0,T]$ with values in U.

3. SYSTEM DESCRIPTION

Let us consider the following semilinear fractional dynamical systems with multiple delays

$$^{C}D^\alpha x(t) = A\,x(t) + \sum_{i=0}^{M} B_i\,u(t-h_i) + f(x(t),u(t),u(t-h_1),\ldots,u(t-h_M)) \tag{1}$$

for $t \geq 0$, where the state $x(t) \in \mathbb{R}^n$ and the control values $u(t) \in \mathbb{R}^m$, A is a $(n \times n)$-dimensional matrix with constant elements, B_i are $(n \times m)$-dimensional matrices with constant elements for $i = 0, 1, \ldots, M$, $h_i \colon [0,T] \to \mathbb{R}$, $i = 1, 2 \ldots, M$ are constant delays in control that satisfy the inequalities:

$$0 = h_0 < h_1 < \ldots < h_i < \ldots < h_{M-1} < h_M.$$

Moreover, the function f is the nonlinear mapping $f \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}^n$, continuously differentiable near the origin in the space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ such that $f(0,0,0,\ldots,0) = 0$.

Let $\omega(0) = (x(0), u_0)$ be given initial conditions called the initial complete state.

Theorem 3.1. For the given initial conditions $\omega(0) = (x(0), u_0) \in \mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^m)$ and a control $u \in L^2_{\mathrm{loc}}([0,\infty), \mathbb{R}^m)$, there exists a unique solution $x(t) = x(t, \omega(0), u)$ of the semilinear system (1), for each $t \geq 0$, taking the form

$$x(t) = \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau) \sum_{i=0}^{M} B_i\,u(\tau - h_i)\,\mathrm{d}\tau \tag{2}$$

$$+ \int_0^t \Phi(t-\tau) f(x(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))\,\mathrm{d}\tau.$$

Proof. In the proof, the similar method is used as for the corresponding linear case derived in [27]. Applying the Laplace transform to the fractional equation (1), for any fixed $t \geq 0$, we have

$$s^\alpha \mathcal{L}[x(t)] - s^{\alpha-1}x(0) = A\mathcal{L}[x(t)] +$$
$$+\mathcal{L}\big[\sum_{i=1}^{M} B_i u(t-h_i)\big] + \mathcal{L}\big[f(x(t),u(t),u(t-h_1),\ldots,u(t-h_M))\big],$$

Hence

$$\begin{aligned}\mathcal{L}[x(t)] &= (s^\alpha I - A)^{-1}s^{\alpha-1}x(0) + (s^\alpha I - A)^{-1}\mathcal{L}\big[\sum_{i=1}^{M} B_i u(t-h_i)\big]\\ &\quad +(s^\alpha I - A)^{-1}\mathcal{L}\big[f(x(t),u(t),u(t-h_1),\ldots,u(t-h_M))\big]\\ &= \mathcal{L}[\Phi_0(t)x(0)] + \mathcal{L}\big[\Phi(t)\big]\mathcal{L}\big[\sum_{i=1}^{M} B_i u(t-h_i)\big]\\ &\quad +\mathcal{L}\big[\Phi(t)\big]\mathcal{L}\big[f(x(t),u(t),u(t-h_1),\ldots,u(t-h_M))\big].\end{aligned}$$

Using the convolution theorem for the Laplace transform, we have

$$\mathcal{L}[x(t)] = \mathcal{L}[\Phi_0(t)x(0)] + \mathcal{L}\Big[\int_0^t \Phi(t-\tau)\sum_{i=1}^{M} B_i u(\tau-h_i)\,\mathrm{d}\tau\Big]$$

$$+\mathcal{L}\Big[\int_0^t \Phi(t-\tau) f(x(\tau),u(\tau),u(\tau-h_1),\ldots,u(\tau-h_M))\,\mathrm{d}\tau\Big].$$

And finally, using the inverse Laplace transform, we obtain the solution (2). □

Let U be any non-empty subset of $\mathbb{R}^m$. We define a set of reachable states (the attainable set) for the semilinear system (1). The definition corresponds with the one for integer-order dynamical systems (see [13]).

Definition 3.2. The attainable set from the initial complete state $\omega(0) = (x(0), u_0)$ for the time-delay fractional system (1) is the set

$$K_T(U) = \Big\{x(T) \in \mathbb{R}^n : x(T) = \Phi_0(T)x(0) + \int_0^T \Phi(T-\tau)\sum_{i=0}^{M} B_i\, u(\tau-h_i)\,\mathrm{d}\tau \tag{3}$$

$$+\int_0^T \Phi(t-\tau) f(x(\tau),u(\tau),u(\tau-h_1),\ldots,u(\tau-h_M))\,\mathrm{d}\tau \colon u(t) \in U \text{ for } t \in [0,T]\Big\}.$$

Any control $u \in L_\infty([0,T],U)$ is called an admissible control for the system (1) on the interval $[0,T]$. Let the set of admissible controls be denoted as U_{ad}. Definitions of local and global constrained relative controllability for the semilinear system (1) on $[0,T]$ are presented below. In the paper, we consider constraints put on control values. Such constraints occur in practical problems concerning many industrial and biological processes.

Definition 3.3. The semilinear fractional system (1) is called locally relatively U-controllable on $[0,T]$ if the attainable set $K_T(U)$ contains a certain neighborhood of zero in the space $\mathbb{R}^n$.

Definition 3.4. The semilinear fractional system (1) is called (globally) relatively U-controllable on $[0,T]$ if it is relatively U-controllable on $[0,T]$ for every initial complete state $\omega(0) = (x(0), u_0)$.

Remark 3.5. Definition 3.4 implies that the system (1) is (globally) relatively U-controllable on $[0,T]$ if $K_T(U) = \mathbb{R}^n$, see [14].

4. MAIN RESULTS

In this section we provide constrained relative controllability criteria for the semilinear system with delays (1). We assume that the set $U \subset \mathbb{R}^n$ of values of admissible controls is closed and convex cone with nonempty interior and vertex at zero.

For this purpose we start with formulating controllability results for the corresponding linear system of the form

$$ {}^{C}D^{\alpha}x(t) = A\,x(t) + \sum_{i=0}^{M} B_i\,u(t-h_i). \tag{4}$$

The unique solution $x(t) = x(t,\omega(0),u)$ of the linear fractional system (4), for every $t \geq 0$, takes the form (see [27])

$$x(t) = \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau) \sum_{i=0}^{M} B_i\,u(\tau-h_i)\,\mathrm{d}\tau. \tag{5}$$

Let us introduce the following denotation

$$\widetilde{B}_k(t) = [B_0 \;\; B_1 \;\; \dots \;\; B_j \;\; \dots \;\; B_k],$$

where $\widetilde{B}_k(t)$ are $n \times m(k+1)$-dimensional constant matrices defined for $h_k < t \leq h_{k+1}$, k=0,1,…,M and $h_{M+1} = +\infty$. The matrices $\widetilde{B}_k(t)$ depend on time in a sense that the subscript k specifying the number of component matrices depends on t.

Lemma 4.1. The fractional dynamical system with delays (4) is relatively U-controllable on $[0,T]$ for $h_k < T \leq h_{k+1}$, $k = 0,1,2,\dots,M, h_{M+1} = +\infty$, if and only if the fractional dynamical system without delays in control

$$ {}^{C}D^{\alpha}x(t) = A\,x(t) + \widetilde{B}_k(t)\,v(t) \tag{6}$$

is V-controllable on $[0,T]$, where $v \in L_{\infty}([0,T],V)$ and $V = U \times U \times \cdots \times U \subset \mathbb{R}^{m(k+1)}$ is a given closed and convex cone with nonempty interior and vertex at zero.

P r o o f. Let us consider zero initial conditions $\omega(0) = (0,0)$ and transform the equality (5). By substitution and definite integral properties, the solution (5) of the linear system (4), for any $t \in [0,T]$, can be rewritten in the following form

$$x(t,z(0),u) = \Phi_0(t)x(0) + \sum_{i=0}^{M} \int_{-h_i}^{t-h_i} \Phi(t-\tau-h_i)B_i u(\tau)\,\mathrm{d}\tau$$

$$= \sum_{i=0}^{k} \int_{0}^{t-h_i} \Phi(t-\tau-h_i)B_i u(\tau)\,\mathrm{d}\tau.$$

for t satisfying inequalities $h_i < t \leq h_{i+1}$, $1 = 0,1,\dots,k-1$.

Since the matrices $\Phi(T-t-h_i)$ are nonsingular for any $t \in [0,T]$, they do not change controllability property of the dynamical system. Therefore, relative controllability of linear fractional system with delays in control (4) is equivalent to controllability of a linear fractional system without delays in control (6), which completes the proof. □

Lemma 4.1 is the generalization of constrained controllability results obtained in [14] for integer-order systems with delays in control. It follows from Lemma 4.1 that constrained relative controllability of the linear fractional system with delays in control (4) is equivalent to constrained controllability of the linear system without delays (6).

From Lemma 4.1 it follows also that a condition for relative controllability of the system (4) without any constraints is the following

$$\operatorname{rank}[B_0\ \ B_1\ \ \ldots\ \ B_k\ \ AB_0\ \ AB_1\ \ \ldots\ \ AB_k\ \ A^2B_0\ \ A^2B_1\ \ \ldots\ \ A^2B_k\ \ \ldots$$
$$\ldots A^{n-1}B_0\ \ A^{n-1}B_1\ \ \ldots\ \ A^{n-1}B_k] = n,$$

for $h_k < T \leq h_{k+1}$, $k = 0, 1, 2, \ldots, M$, $h_{M+1} = +\infty$.

Next, let X and Y be given Banach spaces, and $g\colon X \to Y$ be a nonlinear mapping continuously differentiable near the origin in the space X. Suppose $g(0) = 0$. It follows from the implicit-function theorem (see [24]) that if the Frechet derivative at zero $Dg(0)\colon X \to Y$ maps the space X onto the whole space Y, then the nonlinear mapping g transforms a neighborhood of zero in the space X onto some neighborhood of zero in the space Y. The following theorem provides a property of the nonlinear mapping g, which helps to prove our criterion.

Lemma 4.2. (Robinson [24]) Let X and Y be the Banach spaces, Ω be an open subset of X containing zero, and $U \subset X$ be a closed and convex cone with nonempty interior and vertex at zero. Let $g\colon \Omega \to Y$ be a nonlinear mapping which has the Frechet derivative Dg on Ω, continuous at 0. Moreover, assume that $g(0) = 0$ and linear mapping $Dg(0)$ maps U onto the whole space Y. Then there exist neighborhoods of zero $N_0 \subset Y$ and $M_0 \subset \Omega$ such that the nonlinear equation $x = g(u)$ has at least one solution $u \in M_0 \cap U$ for each $x \in N_0$, where $M_0 \cap U$ is the so-called conical neighborhood of zero in the space X.

Now we can formulate the constrained controllability criteria for the semilinear fractional system (1).

Theorem 4.3. Let all the following conditions be satisfied:

(i) $U \subset \mathbb{R}^m$ is a closed and convex cone with nonempty interior and vertex at zero;

(ii) $f(0, 0, \ldots, 0) = 0$;

(iii) $\operatorname{rank}[B_0\ \ B_1\ \ \ldots\ \ B_k\ \ AB_0\ \ AB_1\ \ \ldots\ \ AB_k\ \ A^2B_0\ \ A^2B_1\ \ \ldots\ \ A^2B_k\ \ \ldots$ $\ldots A^{n-1}B_0\ \ A^{n-1}B_1\ \ \ldots\ \ A^{n-1}B_k] = n$,
for $h_k < T \leq h_{k+1}$, $k = 0, 1, 2, \ldots, M$, $h_{M+1} = +\infty$;

(iv) there is no real eigenvector $v \in \mathbb{R}^n$ of the matrix A^* satisfying $v^*\widetilde{B}_k u \leq 0$ for all $u = u(t) \in U$, where * means the transpose.

Then the semilinear fractional system (1) is locally relatively U-controllable on $[0, T]$.

P r o o f . For the semilinear system (1) we define a nonlinear mapping $g\colon L_\infty([0,T],U) \to \mathbb{R}^n$ by $g(u) = x(T, \omega(0), u)$. For the corresponding linear system (5) we define a linear mapping $H\colon L_\infty([0,T],U) \to \mathbb{R}^n$ by $H\widetilde{u} = x(T, \omega(0), \widetilde{u})$.

If the conditions (i)-(iv) are satisfied, for the linear system (5) it is sufficient to reach all states in $\mathbb{R}^n$. In fact, since U is the cone with vertex at the origin and nonempty interior, for any admissible control u also $ku \in L^2([0,T],U)$ for all $k \geq 0$. The attainable set $K_T(U)$ is a convex set containing 0 in its interior (due to the third condition) and it is a cone with vertex at the origin (because it is linear with respect to $u(\cdot)$), hence it has to be a whole space $\mathbb{R}^n$. This means that the linear fractional system (4) is then (globally) relatively U-controllable. Moreover, it follows from Definition 3.4 that the linear operator H maps the cone U_{ad} onto the whole space $\mathbb{R}^n$, and by Lemma 4.2 we obtain $Dg(0) = H$.

By (i), $U \subset \mathbb{R}^m$ is a closed and convex cone with nonempty interior and vertex at zero, so the set of admissible controls U_{ad} is also a closed and convex cone in the function space $L_\infty([0,T],U)$. It follows that the nonlinear mapping g satisfies all the assumptions of Lemma 4.2. Therefore, the nonlinear mapping g transforms a conical neighborhood of zero in the set of admissible controls U_{ad} onto some neighborhood of zero in the space $\mathbb{R}^n$. Whereas this fact is, by Definition 3.3, equivalent to the local relative U-controllability of the semilinear fractional system (1) on the time interval $[0,T]$, which completes the proof. □

Corollary 4.4. Let all the following conditions be satisfied:

(i) $m = 1$ and $U = [0, +\infty)$;

(ii) $f(0, 0, \dots, 0) = 0$;

(iii) $\operatorname{rank}[B_0 \;\; B_1 \;\; \dots \;\; B_k \;\; AB_0 \;\; AB_1 \;\; \dots \;\; AB_k \;\; A^2B_0 \;\; A^2B_1 \;\; \dots \;\; A^2B_k \;\; \dots$
$\dots A^{n-1}B_0 \;\; A^{n-1}B_1 \;\; \dots \;\; A^{n-1}B_k] = n$,
for $h_k < T \leq h_{k+1}$, $k = 0, 1, 2, \dots, M$, $h_{M+1} = +\infty$;

(iv) the matrix A has only complex eigenvalues.

Then the semilinear fractional system (1) is locally relatively U-controllable on $[0,T]$.

P r o o f . The criterion follows immediately from Theorem 4.3 and the fact that in case of $m = 1$, the condition (iv) of the theorem is equivalent to the condition that the matrix A has only complex eigenvalues. □

5. ILLUSTRATIVE EXAMPLE

In this section we present a numerical example to illustrate the obtained theoretical results. As the below example shows, the proposed new criteria are easy to verify which is an undoubted advantage of our method.

Example 5.1. Let us be given the following semilinear fractional system with delays

$$
\begin{aligned}
{}^{C}D^{\frac{1}{2}}x(t) &= Ax(t) + B_0(t)u(t) + B_1(t)u(t-1) + B_2(t)u(t-2) \\
&\quad + f(x(t), u(t), u(t-1), u(t-2)), \qquad (7)
\end{aligned}
$$

for $t \in [0,3]$ and $u(t) \in [0,+\infty)$, with initial conditions $\omega(0) = (0,0)$, where

$$
A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix},\; B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\; B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\; x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

and

$$
f(x(t), u(t), u(t-1), u(t-2)) = \begin{bmatrix} 0 \\ \sin x_1(t) + \cos u(t-2) - 1 \end{bmatrix}.
$$

We have $\alpha = \frac{1}{2}, n = 2, m = 1, M = 2, h_0 = 0, h_1 = 1$ and $h_2 = 2$. The set of admissible control values $U = [0,+\infty)$ is a cone with vertex at zero and a nonempty interior in the one-dimensional space $\mathbb{R}$.

By the Cayley–Hamilton method we calculate the matrix $E_{\frac{1}{2},\frac{1}{2}}(At^{\frac{1}{2}})$.

$$
E_{\frac{1}{2},\frac{1}{2}}(At^{\frac{1}{2}}) = \sum_{k=0}^{1} \frac{A^k t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}(k+1))} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^0}{\sqrt{\pi}} + \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \frac{t^{\frac{1}{2}}}{1} = \begin{bmatrix} \frac{1}{\sqrt{\pi}} & -2t^{\frac{1}{2}} \\ t^{\frac{1}{2}} & \frac{1}{\sqrt{\pi}} \end{bmatrix}
$$

and then

$$
E_{\frac{1}{2},\frac{1}{2}}(A^*t^{\frac{1}{2}}) = \sum_{k=0}^{1} \frac{(A^*)^k t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}(k+1))} = \begin{bmatrix} \frac{1}{\sqrt{\pi}} & t^{\frac{1}{2}} \\ -2t^{\frac{1}{2}} & \frac{1}{\sqrt{\pi}} \end{bmatrix}.
$$

Since $h_2 < T$, we have

$$
\operatorname{rank}[B_0 \;\; B_1 \;\; B_2 \;\; AB_0 \;\; AB_1 \;\; AB_2] = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 2 = n. \qquad (8)
$$

Moreover, the matrix A has only complex eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}$ and f(0,0,. . . ,0)=0. Therefore, based on Corollary 4.4, we conclude that the semilinear fractional system (7) is locally relatively U-controllable on $[0,3]$ for $U = [0,+\infty)$.

6. CONCLUSIONS

The constrained relative controllability of the semilinear fractional systems with delays both in the control and nonlinear function f have been discussed in the paper. Constraints of the cone type imposed on the delay values have been considered. The formula for a solution of the discussed systems has been derived with the use of the Laplace transform (Theorem 3.1). Definitions of the local and global relative controllability in the case of constrained control values have been formulated. Lemma 4.1 has provided the necessary and sufficient conditions that reduce studying the relative U-controllability of the semilinear system (1) to control some linear system with cone-type constraints.

Theorem 4.3 and Corollary 4.4 have stated new controllability criteria for local relative U-controllability for the semilinear fractional systems described by the equation (1) provided that U is a cone. The Frechet derivative method has been used to prove the criteria. The numerical example has been presented to illustrate how to verify the local relative controllability of the discussed systems with the use of the established criteria.

Systems with delays and cone-type constraints of controls appear in many industrial processes. A possibility to use our method can be seen, for example, in modeling the process of steel rolling where thickness can only be measured at some distances from rolls which leads to measurement delays. Moreover, in networked control sampling, encoding, transmission and decoding need measurement and actuation delays as well as regenerative chatter in metal cutting modeling leads to delays depending on full rotation time.

Our future work will focus on solving controllability problems for semilinear and nonlinear fractional systems. We plan to propose a fractional model and find an optimal control for the steel rolling process, since fractional calculus provides more accurate models of systems under considerations. Moreover, our next proposal will be the use of the Rothe fixed point theorem to analyze the controllability of nonlinear fractional systems with delays.

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Beata Sikora, Institute of Applied Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice. Poland.
e-mail: beata.sikora@polsl.pl

Jerzy Klamka, Institute of Control Engineering, Silesian University of Technology, Akademicka 16, 44-100 Gliwice. Poland.
e-mail: jerzy.klamka@polsl.pl