ROBUST OBSERVER-BASED CONTROL OF SWITCHED NONLINEAR SYSTEMS WITH QUANTIZED AND SAMPLED OUTPUT

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This paper deals with the robust stabilization of a class of nonlinear switched systems with non-vanishing bounded perturbations. The nonlinearities in the systems satisfy a quasi-Lipschitz condition. An observer-based linear-type switching controller with quantized and sampled output signal is considered. Using a dwell-time approach and an extended version of the invariant ellipsoid method (IEM) sufficient conditions for stability in a practical sense are derived. These conditions are represented as Bilinear Matrix Inequalities (BMI's). Finally, two examples are given to verify the efficiency of the proposed method.

Keywords: switched systems, robust stabilization, quantization

Classification: 93D21, 93C57

1. INTRODUCTION

Nowadays, more than any other age, control systems are inherent to digital communications. The increasingly necessity of computer processing, embedded system and/or digital networks in the control loops has added a lot of complexity to their analysis using classical approaches. This scenario has led us to bring together three research areas: switched systems, limited information control and robust control.

Switched systems have been a research topic with a lot of activity lately. This enthusiasm comes from the fact that such systems are able to reproduce complex dynamical behaviours of actual phenomena (see e. g. [1, 3, 4, 7, 20]). Most research lines regarding this topic are focused on stability analysis and robust stabilization. For stability analysis, interesting results are presented in [14, 23, 26, 39, 40]. Also, some relevant results of robust stabilization can be found in [22, 32] ($\mathcal{H}_\infty$ approach) and [27] (Sliding modes approach).

By limited information control, we mean that the measurements being passed from the system output to the controller have some sort of data loss, in this specific case the output is being sampled and quantized [24, 25]. This two phenomena, sampling and quantization, have been revitalized with the popularization of networked control systems. Successful results were obtained on this subject, as issued in [31, 34, 41, 44].

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Addressing the quantization problem, approaches such as $\mathcal{H}_\infty$ ([13]) or the sector bound ([12]) have been adopted. It is worth pointing out the work of [11], where the sampling phenomena is treated as a particular case of delay. Also some interesting and more recent results on nonlinear quantized systems can be found in [28] and [29].

In order to make more realistic our stabilizing scheme, we device a robust controller based on the Invariant Ellipsoid Method ([2, 21, 35, 36]). This method is founded on the second Lyapunov method and the concept of invariant sets. An outstanding reference on invariant sets is [5]. The IEM allow us to deal with nonlinearities, norm-bounded uncertainties/disturbances, and even stochastic noises (see e. g. [30]).

In this paper, we study a particular family of nonlinear switched systems with non-vanishing disturbances and uncertainties. The nonlinearities in our contribution satisfy a "quasi-Lipschitz" condition [17]. Nonlinear quasi-Lipschitz systems were considered because many nonlinear models fulfil this condition and represent a considerable number of applications such as robotic manipulators [33] and space vehicles [16]. This latter condition lets us to represent the nonlinear system as a linear one, and then obtain some stability conditions through matrix inequalities. The IEM guarantees the convergence of the system state to a prescribed set in spite of uncertainties, this can be understood as a practical stability property. The stabilization (in a practical sense) of the switched system is based on the well-known dwell-time approach [19]. The effects of sampling and quantization are overcome in a similar way of ([11]), which allows us to use the continuous-time Lyapunov–Krasovskii approach, instead of considering that the system is already in the discrete-time form.

From the theoretical and computational points of view, we are interested in designing effective control algorithms that extends the robust control design schemes proposed in [38] to a class of systems that present switching, sampling and quantization phenomena. So, the objective of this paper was to design a robust switching linear-type controller based on a Luenberger-estimator for switched nonlinear systems with limited information, that guarantee the stability (in a practical sense) of the closed-loop system.

The outline of the paper is as follows. Section 2 contains the problem formulation and some basic assumptions. In section 3 an extended version of the attractive ellipsoid method is developed. There, we deal with the sampling and quantization issue trough Lyapunov–Krasovskii functionals approach. Next, we derive a dwell-time condition to get practical stability. At the end of this section, we present our main result by setting the robust controller design problem as an auxiliary (relaxed) BMI-constrained optimization problem. In Section 4, two numerical illustrative examples are presented in order to show the effectiveness of our method. Section 5 summarizes the paper.

2. PROBLEM FORMULATION

In this paper, we deal with a class of nonlinear switched systems described by

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(t, x(t)) + B_{\sigma(t)}u(t) + v_x(t) \\ x(0) &= x_0,\ \sigma(0) = \sigma_0 \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $v_x(t) \in \mathbb{R}^n$ are, respectively, the state vector, control input and exogenous disturbance at time $t \in \mathbb{R}_+$. Moreover, $\{f_i(\cdot,\cdot)\},\ i = 1, \dots, M \in \mathbb{N}$

is a family of quasi-Lipschitz functions $f_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ (see definition below). The switching signal in (1) is determined by a time-dependent piecewise-constant function $\sigma : \mathbb{R}_+ \mapsto \mathcal{I} = \{1, \ldots, M\}$ where $\mathcal{I}$ is the finite index set. Initial conditions are given by the pair $\{x_0, \sigma_0\} \in \mathbb{R}^n \times \mathcal{I}$. The transitions between the subsystems occurs at the switching times t_r, where $r \in \mathbb{N}$, i. e., $\sigma(t) = i \in \mathcal{I}$, $t \in [t_{r-1}, t_r)$. A constant $\tau_d > 0$, such that $t_r - t_{r-1} \geq \tau_d$, is called the *dwell-time*, because $\sigma(\cdot)$ dwells on each of its values for at least τ_d units of time.

We use the following model to describe a noisy, sampled and quantized output of above switched system:

$$\bar{\bar{y}}(t) = Cx(t) + \omega_y(t) \ , \tag{2a}$$

$$\bar{y}(t) = \sum_{\bar{t}_k} \bar{\bar{y}}(\bar{t}_k)\chi_{[\bar{t}_k, \bar{t}_{k+1})}(t) \ , \tag{2b}$$

$$y(t) = \pi(\bar{y}(t)) \ . \tag{2c}$$

The vector $\omega_y(t) \in R^q$ in (2a) is the deterministic noise. The symbol $\chi_{[\bar{t}_k, \bar{t}_{k+1})}$ in (2b) denotes the characteristic function of the time interval $[\bar{t}_k, \bar{t}_{k+1})$, i. e.,

$$\chi_{[\bar{t}_k, \bar{t}_{k+1})}(t) := \begin{cases} 1 & \text{if } t \in [\bar{t}_k, \bar{t}_{k+1}) \\ 0 & \text{otherwise} \end{cases} \quad , \quad k = 0, 1, 2, \ldots .$$

Thus, $\bar{y} : \mathbb{R}_+ \to \mathbb{R}^q$ is the piecewise constant function which is obtained by sampling and holding $\bar{\bar{y}}$ at the discrete instants $\bar{t}_k$ (the sample times). The measurable system output at time t is $y(t) \in \mathbb{R}^q$, and is obtained by quantizing the sampled signal $\bar{y}(t)$. Formally: Let $Y \subset \mathbb{R}^q$ be a countable set of possible output values or quantization levels. Then, $\pi : \mathbb{R}^q \to Y$ in (2c) is a function such that

$$\pi(\bar{y}(t)) := \underset{y(t) \in Y}{\operatorname{argmin}} \ \varrho(y(t), \bar{y}(t)) \ ,$$

with

$$\varrho(y(t), \bar{y}(t)) := \|y(t) - \bar{y}(t)\|^2_{Q_y} .$$

By $\{B_i, C\}$, $B_i \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$ we denote here a family of given systems matrices. A block diagram of the system is shown in Figure 1.

Let us now formulate our basic assumptions.

Assumption 2.1. (**A**)

1. The exogenous disturbance and noise are unknown but bounded. More precisely, there are known positive definite matrices $Q_x \in \mathbb{R}^{n \times n}$ and $Q_y \in \mathbb{R}^{q \times q}$ such that

$$\|\upsilon_x(t)\|^2_{Q_x} + \|\omega_y(t)\|^2_{Q_y} \leq 1 \quad \text{for all } t \in \mathbb{R}_+ \ . \tag{3}$$

Here, $\|s\|^2_Q = s^\intercal Q s$ is the weighted Euclidean norm of any vector $s \in \mathbb{R}^n$ given by a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$.

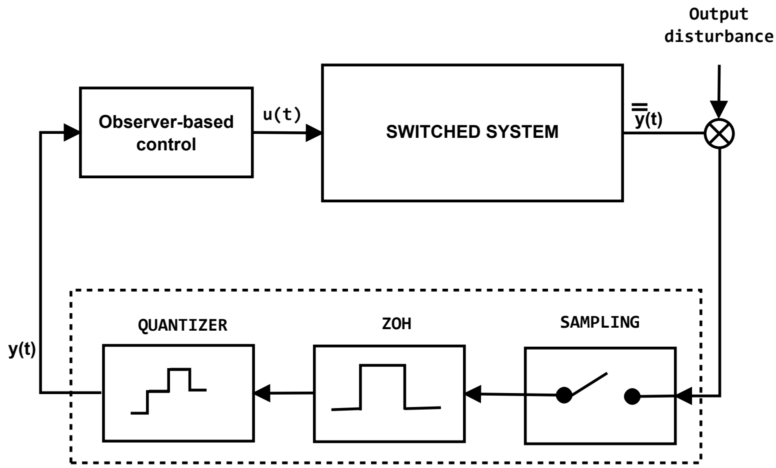


Fig. 1. Switched nonlinear system with sampled and quantizing output.

2. The functions f_i satisfies the quasi-Lipschitz bound
$$\|f_i(t,x) - A_i x(t)\|^2_{Q_x} \le \delta + ||x(t)||^2_{Q_i} \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \ , \tag{4}$$
where $\delta > 0$ is a scalar and $Q_i > 0$ and A_i are known $(n \times n)$-dimensional matrices.
3. The pairs (A_i, B_i) are controllable and (A_i, C) are observable.
4. The sampling intervals does not need to be regular, but there exists a maximum sampling interval
$$h := \max_k |\bar{t}_{k+1} - \bar{t}_k| \ .$$
5. The quantization error is bounded, i. e., the positive scalar
$$c := \max_{\bar{y} \in \mathbb{R}^q} \|\pi(\bar{y}) - \bar{y}\|^2_{Q_y} \tag{5}$$
is finite.
6. Quantization is uniform, this mean that all the quantization levels are equally spaced.
7. The "Zeno behavior" (infinite switchings in finite time) in $\sigma(t)$ is assumed to be excluded. Also, it is a natural consequence to impose a dwell-time scheme in the switching signal.

Notice that (4) is not restrictive and comprises a large class of unknown nonlinear functions [17]. By defining the auxiliary function $\omega_x(t) := \upsilon_x(t) + f_{\sigma(t)}(t, x(t)) - A_{\sigma(t)} x(t)$, we can rewrite (1) as
$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) + \omega_x(t) \ . \tag{6}$$

We propose a classical Luenberger observer (assumption (2.1.3) becomes natural) as an approach for the partial-information problem

$$\dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - C\hat{x}(t)) \tag{7}$$

where $L_i \in \mathbb{R}^{n\times q}$ are the observer gains. The control law is taken as

$$u(t) = K_{\sigma(t)}\hat{x}(t) \tag{8}$$

where $K_i \in \mathbb{R}^{m\times n}$ are the control gains.

Since the switching function $\sigma(t)$ is well-defined, i. e., there is no infinite switching in finite time, the solution of the unforced system ($u(t) = 0$) $x(t, x(t), \sigma(t), 0)$ in (1) is understood in the classical sense and it is well-defined in every interval $[t_{r-1}, t_r)$. The case of discontinuous right-hand side of (1) induced by the quasi-Lipschitz property is not discarded. In this case the solution $x(t, x(t), \sigma(t), 0)$ is understood in the sense of Filipov ([8]). Furthermore, in the feedback case using the quantized and sampled output, a nonlinear discontinuous right-hand side of (1) is induced by the observer-based control, even with continuous function $f_{\sigma(t)}(t, x(t))$. The solutions $x(t, x(t), \sigma(t), u(\hat{x}(t)))$ in this latter case are also understood in the sense of Filipov.

Now, introducing the estimation error vector $e(t) := x(t) - \hat{x}(t)$ and the auxiliary variable $\Delta y(t) := y(t) - \bar{\bar{y}}(t)$, it can be readily seen that $e(t)$ satisfies the dynamic equation

$$\begin{aligned} \dot{e}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + \omega_x(t) - \big(A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(\bar{\bar{y}} + \Delta y - C\hat{x}(t))\big), \\ \dot{e}(t) &= (A_{\sigma(t)} - L_{\sigma(t)}C)e(t) - L_{\sigma(t)}(\Delta y(t) + \omega_y(t)) + \omega_x(t). \end{aligned} \tag{9}$$

It is possible to write the closed-loop equations (7) and (9) more compactly as

$$\dot{z}(t) = \tilde{A}_{\sigma(t)}z(t) + F_{\sigma(t)}\omega(t) + \psi(t) \tag{10}$$

where we have defined the vectors

$$z(t) := \begin{pmatrix} \hat{x}(t) \\ e(t) \end{pmatrix}, \quad \omega(t) := \begin{pmatrix} \omega_x(t) \\ \omega_y(t) \end{pmatrix} \quad \text{and} \quad \psi(t) := \begin{pmatrix} L_{\sigma(t)} \\ -L_{\sigma(t)} \end{pmatrix} \Delta y(t)$$

and the matrices

$$\tilde{A}_{\sigma(t)} := \begin{pmatrix} A_{\sigma(t)} + BK_{\sigma(t)} & L_{\sigma(t)}C \\ 0 & A_{\sigma(t)} - L_{\sigma(t)}C \end{pmatrix} \quad \text{and} \quad F_{\sigma(t)} := \begin{pmatrix} 0 & L_{\sigma(t)} \\ I & -L_{\sigma(t)} \end{pmatrix}.$$

Because of the presence of ω and ψ, the convergence of $z(t)$ to the origin as $t \to \infty$ can not be reasonably expected. But, if K_i and L_i are properly chosen, it is reasonable to expect $z(t)$ to converge to a 'small' set containing the origin. First, our problem is construct a characterization of such a set, and then, find L_i and K_i that minimize (in particular sense to be defined later) its size.

Now, let us introduce a important concept concerning switched systems when a dwell-time approach is used.

Definition 2.2. (Liberzon [23]) For a switching signal $\sigma(\cdot)$ and any $T_2 > T_1 \geq 0$, let $N(T_1, T_2)$ be the switching number of $\sigma(t)$ over the interval $[T_1, T_2)$. If

$$N(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{\tau_{av}} \tag{11}$$

holds for $N_0 \geq 1$, $\tau_{av} > 0$, then τ_{av} is called *the average dwell-time* and N_0 the chatter bound.

3. EXTENDED INVARIANT ELLIPSOID METHOD

To estimate the region where the states of (10) converge, we use the ellipsoid method and propose an extension to deal with the sampling and the quantization of the output. Let us sketch the main idea first and let us recall a basic lemma about differential inequalities.

Lemma 3.1. Let a function $V : \mathbb{R}_+ \to \mathbb{R}$ satisfy the differential inequality

$$\dot{V}(t) \leq -\alpha V(t) + \beta\,. \tag{12}$$

Then, its solutions satisfy

$$V(t) \leq e^{-\alpha t} V(0) + \frac{\beta}{\alpha}(1 - e^{-\alpha t})\,. \tag{13}$$

Lemma 3.1 is a particular case of Theorem 4.1 [18, Ch. III]. Now, suppose that

$$V(t) := \mathcal{V} \circ z(t)$$

with $\mathcal{V} : \mathbb{R}^{2n} \to \mathbb{R}_+$ differentiable and $z(t)$ a solution of (10) evaluated at time t. Then, equation (13) with $\alpha > 0$ and $\beta \geq 0$ clearly implies that the sub-level set

$$\mathcal{V}_{\beta/\alpha} := \left\{ z \in \mathbb{R}^{2n} \ : \ \mathcal{V}(z) \leq \frac{\beta}{\alpha} \right\}$$

is invariant (i. e., $z(0) \in \mathcal{V}_{\beta/\alpha} \Rightarrow z(t) \in \mathcal{V}_{\beta/\alpha}$ for all $t \geq 0$) and attractive (i. e., $\limsup_{t\to\infty} V(t) \leq \beta/\alpha$).

3.1. A Lyapunov–Krasovskii-like functional

Considering that the sampling phenomenon involves a delay, we use a Lyapunov–Krasovskii-like functional instead of a regular function. Let $\mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n})$ be the space of all continuous functions of $\mathbb{R}$ into $\mathbb{R}^{2n}$, differentiable almost everywhere; let $R_i > 0$ and $P_i > 0$ be $(2n \times 2n)$-dimensional matrices and let $\alpha_i > 0$ be a scalar. We propose the functional $V_i : \mathbb{R} \times \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n}) \to \mathbb{R}_+$, $i \in \mathcal{I}$, defined as

$$V_i(t, z(\cdot)) := z^\top(t) P_i z(t) + h \int_{\theta=-h}^{0} \int_{s=t+\theta}^{t} e^{\alpha_i(s-t)} \dot{z}^\top(s) R_i \dot{z}(s) \mathrm{d}s\mathrm{d}\theta. \tag{14}$$

Our primary goal is to derive sufficient conditions for $V_i(t, z(\cdot))$ to satisfy (12) with $\alpha_i > 0$ and $\beta \geq 0$ when z is a solution of (10). Let us begin with the case when z is arbitrary.

Theorem 3.2. For any given

$$z(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n}) , \quad h, \alpha_i, b \in \mathbb{R} , \quad P_i, R_i \in \mathbb{R}^{2n\times 2n}, \, i \in \mathcal{I}$$

such that $h > 0$, $\alpha_i > 0$, $P_i > 0$, $R_i > 0$ and $\mathcal{I}$ is a finite index set, the time derivative of $V_i(t, z(\cdot))$ in (14) satisfies the bound

$$\dot{V}_i(t, z(\cdot)) \leq -\alpha_i V_i(t, z(\cdot)) + b\bar{\delta} + \eta(t, z(\cdot))^\top W_i \eta(t, z(\cdot)) , \tag{15}$$

where

$$\eta(t, z(\cdot)) := \begin{pmatrix} z(t) \\ \dot{z}(t) \\ z(t) - z(t_k) \\ \omega(t) \end{pmatrix} , \quad W_i := \begin{pmatrix} \alpha_i P_i + bQ_{zi} & P_i & 0 & 0 \\ * & h^2 R_i & 0 & 0 \\ * & * & -he^{-\alpha_i h} R_i & 0 \\ * & * & * & -b\bar{Q} \end{pmatrix} ,$$
$$\bar{Q} := \begin{pmatrix} Q_x & 0 \\ 0 & Q_y \end{pmatrix} , \quad Q_{zi} := \begin{pmatrix} I \\ I \end{pmatrix} Q_i \begin{pmatrix} I & I \end{pmatrix} \quad \text{and} \quad \bar{\delta} := \delta + 1 . \tag{16}$$

Before giving the proof of the theorem, let us state a pair of simple lemmas.

Lemma 3.3. The perturbation ω satisfies the bound

$$\|\omega(t)\|^2_{\bar{Q}} \leq \bar{\delta} + \|x(t)\|^2_{Q_i}. \tag{17}$$

Proof. Directly from the norms an upper bound can be obtained

$$\begin{aligned} \|\omega(t)\|^2_{\bar{Q}} = \|\omega_x(t)\|^2_{Q_x} + \|\omega_y(t)\|^2_{Q_y} &= \|\nu_x(t) + f_i(t, x(t)) - A_i x(t)\|^2_{Q_x} + \|\omega_y(t)\|^2_{Q_y} \\ &\leq \|\nu_x(t)\|^2_{Q_x} + \|f_i(t, x(t)) - A_i x(t)\|^2_{Q_x} + \|\omega_y(t)\|^2_{Q_y} . \end{aligned} \tag{18}$$

Substitution of (3) and (4) in (18) shows that

$$\|\omega(t)\|^2_{\bar{Q}} \leq 1 + \delta + \|x\|^2_{Q_i}.$$

□

Lemma 3.4. For any given $z(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n})$, $h > 0$, $\alpha_i > 0$, $R_i > 0$, we have

$$-h \int_{t-h}^{t} e^{\alpha_i(s-t)} \dot{z}^\top(s) R_i \dot{z}(s) \mathrm{d}s \leq -he^{-\alpha_i h} \int_{t_k}^{t} \dot{z}^\top(s) \mathrm{d}s R_i \int_{t_k}^{t} \dot{z}(s) \mathrm{d}s. \tag{19}$$

Proof. Since $e^{-\alpha_i h} \leq e^{\alpha_i(s-t)}$ for all $s \in [t-h, t]$ and R_i is positive definite, we have

$$-h \int_{t-h}^{t} e^{\alpha_i(s-t)} \dot{z}^\top(s) R_i \dot{z}(s) \mathrm{d}s \leq -he^{-\alpha_i h} \int_{t-h}^{t} \dot{z}^\top(s) R_i \dot{z}(s) \mathrm{d}s. \tag{20}$$

By splitting the integration interval at the time $t_k \in [t-h, t)$, we obtain

$$
\begin{aligned}
&-he^{-\alpha_i h}\int_{t-h}^{t}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s = -he^{-\alpha_i h}\int_{t-h}^{t_k}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s - he^{-\alpha_i h}\int_{t_k}^{t}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s\\
&\le -he^{-\alpha_i h}\int_{t_k}^{t}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s \le -he^{-\alpha_i h}\int_{t_k}^{t}\dot z^\top(s)\mathrm{d}sR_i\int_{t_k}^{t}\dot z(s)\mathrm{d}s,
\end{aligned}
\tag{21}
$$

where the first inequality follows from the fact that h is positive, and the second one follows from Jensen's inequality [37]. Combining (20) and (21), gives (19). □

Proof. (of Theorem 3.2) We begin by directly computing $\dot V$:

$$
\begin{aligned}
\dot V_i(t, z(\cdot)) = 2z^\top(t)P_i\dot z(t) + h^2\dot z^\top(t)R_i\dot z(t) - h\int_{t-h}^{t} e^{\alpha_i(s-t)}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s\\
- \alpha_i h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha_i(s-t)}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s\mathrm{d}\theta.
\end{aligned}
\tag{22}
$$

Adding and subtracting $\alpha_i V_i(t, z(\cdot))$ to the right-hand side of (22) we obtain

$$
\begin{aligned}
\dot V_i(t, z(\cdot)) = 2z^\top(t)P_i\dot z(t) + \alpha_i z^\top(t)P_i z(t) + h^2\dot z^\top(t)R_i\dot z(t)\\
- h\int_{t-h}^{t} e^{\alpha_i(s-t)}\dot z^\top(s)R_i\dot z(s)\mathrm{d}s - \alpha_i V_i(t, z(\cdot)).
\end{aligned}
\tag{23}
$$

The following upper bound for $\dot V_i$ can be easily obtained from (23) and (19):

$$
\dot V_i(t, z(\cdot)) \le -\alpha_i V_i(t, z(\cdot)) + b\|\omega(t)\|^2_{\bar Q} + \eta(t, z(\cdot))^\top W_{1i}\eta(t, z(\cdot)) \tag{24}
$$

where W_{1i} is a symmetric matrix defined by

$$
W_{1i} := \begin{pmatrix} \alpha_i P_i & P_i & 0 & 0\\ * & h^2 R_i & 0 & 0\\ * & * & -he^{-\alpha_i h}R_i & 0\\ * & * & * & -b\bar Q \end{pmatrix}.
$$

From (17), we have

$$
\dot V_i(t, z(\cdot)) \le -\alpha_i V_i(t, z(\cdot)) + b(\bar\delta + \|x(t)\|^2_{Q_i}) + \eta(t, z(\cdot))^\top W_{1i}\eta(t, z(\cdot)). \tag{25}
$$

Since

$$
\|x(t)\|^2_{Q_i} = \|\hat x(t) + e(t)\|^2_{Q_i} = \left\|\begin{pmatrix} I & I \end{pmatrix} z(t)\right\|^2_{Q_i} = z(t)^\top Q_{zi} z(t)\ ,
$$

we can finally rewrite (25) as (15). □

Now we will refine the bound given in Theorem 3.2 by restricting $z(\cdot)$ to the set of solutions of (10) on the interval $[t_{r-1}, t_r)$, $r \in \mathbb{N}$. In order to do so, we follow the idea presented in [9] and [10] which, originally devised for systems in descriptor form, consists in adding a term (the *descriptor term*) to the expression for $\dot{V}_i$. The descriptor term has to be zero for any solution z of the system. In our case, we will add the term

$$\mathcal{D}_i(t, z(\cdot)) := 2\left(z(t)^\top \Pi_{ai} + \dot{z}^\top(t)\Pi_{bi}\right) \times \left(\tilde{A}_i z(t) + F_i\omega(t) + \psi(t) - \dot{z}(t)\right) ,$$

where Π_{ai} and Π_{bi} are in $\mathbb{R}^{2n}$. Obviously, $\mathcal{D}_i$ is zero along the solutions of (10).

Theorem 3.5. Let ρ_1 be a positive scalar satisfying

$$L_i^\top L_i \leq \rho_1 I. \tag{26}$$

Then, for any

$$z(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n}) , \quad h, \alpha_i, b, \varepsilon \in \mathbb{R} , \quad P_i, R_i, \Pi_{ai}, \Pi_{bi} \in \mathbb{R}^{2n\times 2n},\ i \in \mathcal{I}$$

such that z is a solution of (10), $h > 0$, $\alpha_i > 0$, $P_i > 0$ and $R_i > 0$, the time derivative of $V_i(t, z(\cdot))$ in (14) satisfies

$$\dot{V}_i(t, z(\cdot)) \leq -\alpha_i V_i(t, z(\cdot)) + \beta + \xi(t, z(\cdot))^\top \Omega_i \xi(t, z(\cdot)) \tag{27}$$

for all $t \in [t_{r-1}, t_r),\ r \in \mathbb{N}$ and $\sigma(t) = i$, where

$$\Omega_i := \begin{pmatrix} \alpha_i P_i + bQ_{zi} + 2\Pi_{ai}\tilde{A}_i & P_i - \Pi_{ai} + \Pi_{bi}\tilde{A}_i & 0 & \Pi_{ai}F_i & \Pi_{ai} \\ * & h^2 R_i - 2\Pi_{bi} & 0 & \Pi_{bi}F_i & \Pi_{bi} \\ * & * & -he^{-\alpha_i h} R_i + \varepsilon\rho Q_c & 0 & 0 \\ * & * & * & -b\bar{Q} & 0 \\ * & * & * & * & -\varepsilon I \end{pmatrix} \tag{28}$$

and

$$\xi(t, z(\cdot)) := \begin{pmatrix} z(t) \\ \dot{z}(t) \\ z(t) - z(t_k) \\ \omega(t) \\ \psi(t) \end{pmatrix} , \quad Q_c := \begin{pmatrix} I \\ I \end{pmatrix} C^\top Q_y C \begin{pmatrix} I & I \end{pmatrix} , \quad \beta := b\bar{\delta} + \varepsilon\rho(2 + c) ,$$

$$\rho := 2\rho_1/\lambda_{\min}(Q_y) . \tag{29}$$

The following lemma will be needed before the proof of the theorem.

Lemma 3.6. The uncertainty resulting from noise, sampling and quantization is bounded by

$$\|\psi(t)\|^2 \leq \rho\left((z(t) - z(t_k))^\top Q_c (z(t) - z(t_k)) + 2 + c\right) . \tag{30}$$

Proof. We will begin by computing an upper bound for Δy (see p. 63). We have

$$\|\Delta y(t)\|^2_{Q_y} = \|y(t) - \bar{\bar{y}}(t)\|^2_{Q_y} \leq \|y(t) - \bar{y}(t)\|^2_{Q_y} + \|\bar{y}(t) - \bar{\bar{y}}(t)\|^2_{Q_y} . \tag{31}$$

Notice that

$$\begin{aligned}\bar{\bar{y}}(t)-\bar{y}(t)&=C(x(t)-x(t_k))+\omega_y(t)-\omega_y(t_k)\\&=C\begin{pmatrix}I & I\end{pmatrix}(z(t)-z(t_k))+\omega_y(t)-\omega_y(t_k)\ ,\end{aligned}$$

so

$$\|\bar{y}(t)-\bar{\bar{y}}(t)\|^2_{Q_y}\le(z(t)-z(t_k))^\top Q_c(z(t)-z(t_k))+2\ , \tag{32}$$

where we have used (17) to establish $\|\omega_y(t)\|^2_{Q_y}+\|\omega_y(t_k)\|^2_{Q_y}\le 2$. Substituting (32) and (5) in (31) gives

$$\|\Delta y(t)\|^2_{Q_y}\le(z(t)-z(t_k))^\top Q_c(z(t)-z(t_k))+2+c\ . \tag{33}$$

The norm of ψ then satisfies

$$\|\psi(t)\|^2=\left\|\begin{pmatrix}I\\-I\end{pmatrix}L_i\Delta y(t)\right\|^2=2\Delta y(t)^\top L_i^\top L_i\Delta y(t)\le 2\rho_1\|\Delta y(t)\|^2\le\frac{2\rho_1}{\lambda_{\min(Q_y)}}\|\Delta y(t)\|^2_{Q_y}, \tag{34}$$

from (34) and (33) we conclude (30).

□

Proof. (of Theorem 3.5) Adding the null term $\mathcal{D}_i(t,z(\cdot))+\varepsilon\|\psi(t)\|^2-\varepsilon\|\psi(t)\|^2$ to (15) gives

$$\begin{aligned}\dot{V}_i(t,z(\cdot))\le&-\alpha_iV_i(t,z(\cdot))+b\bar{\delta}+\varepsilon\|\psi(t)\|^2+\eta(t,z(\cdot))^\top W_{1i}\eta(t,z(\cdot))\\&+2\left(z(t)^\top\Pi_{ai}+\dot{z}(t)^\top\Pi_{bi}\right)\times\left(\tilde{A}_iz(t)+F_i\omega(t)+\psi(t)-\dot{z}(t)\right)-\varepsilon\|\psi(t)\|^2.\end{aligned} \tag{35}$$

Substituting (30) in (35) establishes

$$\begin{aligned}\dot{V}_i(t,z(\cdot))\le&-\alpha_iV_i(t,z(\cdot))+\beta+\varepsilon\rho(z(t)-z(t_k))^\top Q_c(z(t)-z(t_k))+\eta(t,z(\cdot))^\top W_{1i}\eta(t,z(\cdot))\\&+2\left(z(t)^\top\Pi_{ai}+\dot{z}(t)^\top\Pi_{bi}\right)\times\left(\tilde{A}_iz(t)+F_i\omega(t)+\psi(t)-\dot{z}(t)\right)-\varepsilon\|\psi(t)\|^2.\end{aligned} \tag{36}$$

Equation (27) is (36) rewritten in a compact form . □

3.2. Practical stability

We mean that the system (10) is practical stable if there exists a prescribed attractive set associated with the dynamics of the system. Considering the ellipsoidal sets as attractive, we may associate the property of the practical stability with the state vector $z(t)$ satisfying

$$\limsup_{t\to\infty} z^\top(t)Q_{\sigma(t)}z(t)\le 1$$

under the matrix constraints

$$Q_i\ge Q_0>0,\ i=1,\dots,M$$

for an *a priori* given matrix $Q_0 \in \mathbb{R}^{2n\times 2n}$.

We derive the practical stability property subject to an average dwell-time condition for the switching signal. We use the property given in Theorem 3.5 to construct a storage function for the switched system (10).

Theorem 3.7. Let

$$\mathbf{V}(t) = V_{\sigma(t)}(t, z(t)) \tag{37}$$

be a piecewise continuous function, where each $V_i(t, z(t))$ satisfies Theorem 3.5. Furthermore, we ask for $\Omega_i < 0$ and there exists a constant $\mu > 1$ such that

$$V_i(t,z) \le \mu V_j(t,z), \quad \forall\, i,\, j \in \mathcal{I}, t \in \mathbb{R}_+. \tag{38}$$

Then, for positive constants $(\gamma_0, \gamma_1, \alpha_{\min})$ there exists a finite constant $\tau_{av} = \frac{\log \mu}{\alpha_{\min}-\gamma_1}$ such that $\mathbf{V}(t)$ is a storage function for the switched system fulfilling

$$\mathbf{V}(t) \le \exp(\gamma_0 - \gamma_1(t-t_0))\mathbf{V}(t_0) + \frac{\beta}{\alpha_{\min}}\left(\frac{\mu^2}{\mu-1}\right)\Big(1-\exp(-N(t_0,t)\log\mu)\Big) \tag{39}$$

with $t_0 \ge 0$, decay rate γ_1 and average dwell-time τ_{av}. Moreover,

$$\limsup_{t\to\infty} \mathbf{V}(t) \le \frac{\beta}{\alpha_{\min}}\left(\frac{\mu^2}{\mu-1}\right) := \kappa. \tag{40}$$

P r o o f. The property (38) is fulfilled with the conditions

$$\begin{aligned} &P_i \le \mu P_j, \quad i \ne j \\ &e^{-\alpha_i \bar{h}} R_i \le \mu e^{-\alpha_j \bar{h}} R_j, \quad \bar{h} \in [0,h],\, i \ne j. \end{aligned}$$

These last conditions are satisfied, for example, with $\mu = \max\{\mu_P, \mu_R\}$ with

$$\mu_P = \sup_{a,b\in\mathcal{I}} \lambda_{\max}(P_a)/\lambda_{\min}(P_b) \ \text{ and } \ \mu_R = \sup_{c,d\in\mathcal{I}} \lambda_{\max}(R_c)/\lambda_{\min}(R_d),$$

where $\lambda_{\max}(X)(\lambda_{\min}(X))$ denotes the largest (smallest) eigenvalue of a matrix X. By using this condition we have that in the switching instants t_r

$$\mathbf{V}(t_r) \le \mu \lim_{t\to t_r^-} V_{\sigma(t)}(t, z(t)) = \mu \mathbf{V}(t_r^-), \quad r \in \mathbb{N}. \tag{41}$$

Consider that every $V_i(z(t))$ satisfies Theorem 3.5 and also $\Omega_i < 0$, $\forall\, i \in \mathcal{I}$, then

$$V_i(t, z(t)) \le V_i(t_{r-1}, z(t_{r-1})) \exp\big(-\alpha_i(t-t_{r-1})\big) + \frac{\beta}{\alpha_i}\Big(1-\exp\big(-\alpha_i(t-t_{r-1})\big)\Big)$$

for all $t \in [t_{r-1}, t_r)$. Let $N(t_0, t)$ be the number of switchings of $\sigma(\cdot)$ in the interval $[t_0, t)$, such that

$$0 \le t_0 < t_1 \cdots < t_{N(t_0,t)} < t < t_{N(t_0,t)+1} = T.$$

Denote $\bar{\alpha}_r := \alpha_{\sigma(t)} = \alpha_{\sigma(t_{r-1})}$, $t \in [t_{r-1}, t_r)$ and $\tau_r = t_r - t_{r-1}$. From the foregoing inequality and (41) it follows that by backwards iteration from t_0 to $t_{N(t_0,t)}$ we get (let us omit arguments of $N(t_0,t)$)

$$\begin{aligned}
\mathbf{V}(t_N) &\leq \mu \exp\big(-\bar{\alpha}_N (t_N - t_{N-1})\big)\mathbf{V}\big(t_{N-1}\big) + \frac{\beta}{\bar{\alpha}_N}\mu\Big[1 - \exp\big(-\bar{\alpha}_N (t_N - t_{N-1})\big)\Big] \\
&\leq \mu^2 \exp\left(-\sum_{k=0}^{1} \bar{\alpha}_{N-k}\tau_{N-k}\right)\mathbf{V}\big(t_{N-2}\big) + \frac{\beta}{\bar{\alpha}_N}\mu\Big[1 - \exp\big(-\bar{\alpha}_N\tau_N\big)\Big] \\
&+ \frac{\beta}{\bar{\alpha}_{N-1}}\mu^2\Big[1 - \exp\big(-\bar{\alpha}_{N-1}\tau_{N-1}\big)\Big]\exp\big(-\bar{\alpha}_N\tau_N\big) \\
&\leq \mu^3 \exp\left(-\sum_{k=0}^{2} \bar{\alpha}_{N-k}\tau_{N-k}\right)\mathbf{V}\big(t_{N-3}\big) + \frac{\beta}{\bar{\alpha}_N}\mu\Big[1 - \exp\big(-\bar{\alpha}_N\tau_N\big)\Big] \\
&+ \frac{\beta}{\bar{\alpha}_{N-1}}\mu^2\Big[1 - \exp\big(-\bar{\alpha}_{N-1}\tau_{N-1}\big)\Big]\exp\big(-\bar{\alpha}_N\tau_N\big) \\
&+ \frac{\beta}{\bar{\alpha}_{N-2}}\mu^3\Big[1 - \exp\big(-\bar{\alpha}_{N-2}\tau_{N-2}\big)\Big]\exp\big(-\bar{\alpha}_N\tau_N - \bar{\alpha}_{N-1}\tau_{N-1}\big) \\
&\leq \cdots\cdots \\
&\mu^{N(t_0,t)} \exp\left(-\sum_{k=0}^{N(t_0,t)} \bar{\alpha}_{N-k}\tau_{N-k}\right)\mathbf{V}(t_0) + \frac{\beta}{\bar{\alpha}_N}\mu\Big[1 - \exp\big(-\bar{\alpha}_N\tau_N\big)\Big] \\
&\beta \sum_{k=1}^{N(t_0,t)-1} \frac{\mu^{k+1}}{\bar{\alpha}_{N-k}}\Big[1 - \exp\big(-\bar{\alpha}_{N-k}\tau_{N-k}\big)\Big]\exp\left(-\sum_{l=0}^{k-1}\bar{\alpha}_{N-l}\tau_{N-l}\right).
\end{aligned}$$

Let $\alpha_{\min} = \min_{i\in\mathcal{I}} \alpha_i$ be, then the last inequality implies

$$\begin{aligned}
\mathbf{V}(t_N) &\leq \exp\big(N(t_0,t)\log\mu - \alpha_{\min}(t_N - t_0)\big)\mathbf{V}(t_0) \\
&+ \frac{\beta}{\bar{\alpha}_{min}}\mu\left[1 + \sum_{k=1}^{N(t_0,t)-1} \exp\left(k\log\mu - \alpha_{\min}\sum_{l=0}^{k-1}\tau_{N-l}\right)\right].
\end{aligned} \tag{42}$$

To guarantee a decay rate γ_1, for the first term of (42), it must be fulfilled that

$$N(t_0,t)\log\mu - \alpha_{\min}(t_N - t_0) \leq \gamma_0 - \gamma_1(t_N - t_0) \tag{43}$$

where $\gamma_0 > 0$, $\gamma_1 > 0$. This last expression is equivalent to (11) with $N_0 = \frac{\gamma_0}{\log\mu}$ and $\tau_{av} = \frac{\log\mu}{\alpha_{\min}-\gamma_1}$ subject to $0 < \gamma_1 < \alpha_{\min}$. For the second term of (42), there is no loss of generality if we consider

$$\sum_{l=0}^{k-1}\tau_{N-l} \geq k\tau_{av}.$$

So, we get

$$1 + \sum_{k=1}^{N(t_0,t)-1} \exp\left(k\log\mu - \alpha_{\min}\sum_{l=0}^{k-1}\tau_{N-l}\right) \leq \sum_{k=0}^{N(t_0,t)-1} \exp\big(k(\log\mu - \alpha_{\min}\tau_{av})\big).$$

Choosing $\gamma_1 = \frac{\alpha_{\min}}{2}$, which implies $\tau_{av} = \dfrac{2\log\mu}{\alpha_{\min}}$, this allow us to rewrite the right-hand side of the last inequality as

$$\sum_{k=0}^{N(t_0,t)-1} \exp\left(\frac{1}{\mu}\right)^k = \frac{1-\left(\frac{1}{\mu}\right)^{N(t_0,t)}}{1-\frac{1}{\mu}}.$$

Substituting this last expression into (42), and considering inequality (43), we obtain (39). □

3.2.1. Intersection of Ellipsoids

From the above procedure and considering

$$\begin{aligned} &t_r - t_{r-1} \geq \tau_{av} = \frac{2\log\mu}{\alpha_{\min}} \\ &V_{\sigma(t)}(x(t)) \geq \kappa,\ \forall\, x \in \mathcal{X} := \left(x : \dot{V}_i(x) \geq -\alpha_i V_i(x) + \beta\right) \end{aligned}$$

we have that

$$\begin{aligned} V_{\sigma(t_r)}(t_r) - V_{\sigma(t_{r-1})}(t_{r-1}) &\leq \mu V_{\sigma(t_{r-1})}(t_r) - V_{\sigma(t_{r-1})}(t_{r-1}) \\ &\leq -V_{\sigma(t_{r-1})}(t_{r-1})\left[\frac{\mu-1}{\mu}\right] + \mu\frac{\beta}{\alpha_{\min}}\left(1 - e^{-\alpha_{\min}\tau_r}\right) \\ &\leq -\mu\frac{\beta}{\alpha_{\min}} e^{-\alpha_{\min}\tau_r} < 0. \end{aligned}$$

Let t_{i_j}, $j \in \mathbb{N}$, be the switching times such that $\sigma(t_{i_j}) = i$, so the above inequality implies

$$V_i(t_{i_{j+1}}) - V_i(t_{i_j}) \leq 0.$$

Therefore, if we suppose that each one is active during the process infinitely many times we have that the subsequence $V_i(x(t_{i_1}))$, $V_i(x(t_{i_2}))$, ..., is decreasing and has a limit κ. The foregoing considerations imply that

$$\sum_{i=1}^{M}\left[\sqrt{V_i(t_{i_j})} - \kappa\right]^2 \underset{j\to\infty}{\longrightarrow} 0.$$

Finally, this means that any trajectory of the switched system converges to the intersection of the individuals ellipsoid, namely,

$$z(t) \underset{N(t_0,t)\to\infty}{\longrightarrow} \bigcap_{i=1}^{M} \mathcal{E}\left(0, \frac{1}{\kappa}P_i\right). \tag{44}$$

3.3. Main result

The next result follows from Theorem 3.5 and Theorem 3.7.

Theorem 3.8. Let

$$\{\alpha_i > 0, b > 0, \varepsilon > 0, \rho_1 > 0, \mu > 1, P_i > 0, R_i > 0, \Pi_{ai}, \Pi_{bi}, L_i, K_i\} \tag{45}$$

be a set of control parameters such that

$$\begin{aligned} &\Omega_i \leq 0 \ , \\ &L_i^\top L_i \leq \rho_1 \ , \\ &P_i \leq \mu P_j, \quad \forall\, i, j \in \mathcal{I}, \\ &\frac{\alpha_i}{\beta} P_i > Q_0 \end{aligned} \tag{46}$$

with Ω_i defined by (28); Q_z, Q_c, $\bar{Q}$ and ρ given by (29) and (16). The intersection set

$$\mathrm{Int}\mathcal{E} := \left\{ z \in \mathbb{R}^{2n} \ : \ z^\top P_i z \leq \kappa, \forall\, i \in \mathcal{I} \right\} \ ,$$

with β given by (29), κ by (40), and for a prescribed Q_0, is an attractive and invariant set.

4. EXAMPLES

The following examples are presented to illustrate the possible practical implementations of the previously introduced method. The first example is purely academic, however it is presented to show the applicability of our developed method to a strongly nonlinear system. The second example shows the results for a separately excited DC motor, considering a nonlinear model with two inputs.

Example 4.1. Consider the following nonlinear subsystems of the switched system (1):

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} \sin(x_2(t)) + \upsilon_{1x}(t) \\ (\lambda^2+1)x_1(t) - 2\lambda x_2(t) + u(t) + \upsilon_{2x}(t) \end{pmatrix} \\ \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} x_2(t) + \upsilon_{1x}(t) \\ (\lambda^2+4)\sin(x_1(t)) - 2\lambda x_2(t) + 2u(t) + \upsilon_{2x}(t) \end{pmatrix} \\ \bar{\bar{y}}(t) &= x_1(t) + 2x_2(t) + \omega_y(t) \end{aligned}$$

where $\lambda = 0.01$. Let us assume that $|\upsilon_{1x}(t)| \leq 0.5, |\upsilon_{2x}(t)| \leq 0.5$ and that $|\omega_y(t)| \leq 0.5$. Using the equivalent transformations discussed in Section 2, we can write the equivalent system (6) with the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ \lambda^2+1 & -2\lambda \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ \lambda^2+4 & -2\lambda \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The numerical treatment of the minimization problem was stated using the following parameters: the sample time interval is fixed at 0.01 seconds, so we can choose directly $h = 0.01$, the initial conditions for the dynamic system are $x_1(0) = x_2(0) = 10$ and the quantization constant selected was $c = 1$. The prescribed matrix we use is $Q_0 = I_{4\times 4}$. For the observer, the initial conditions were chosen as the origin. The observer and the controller gains obtained using the algorithm were

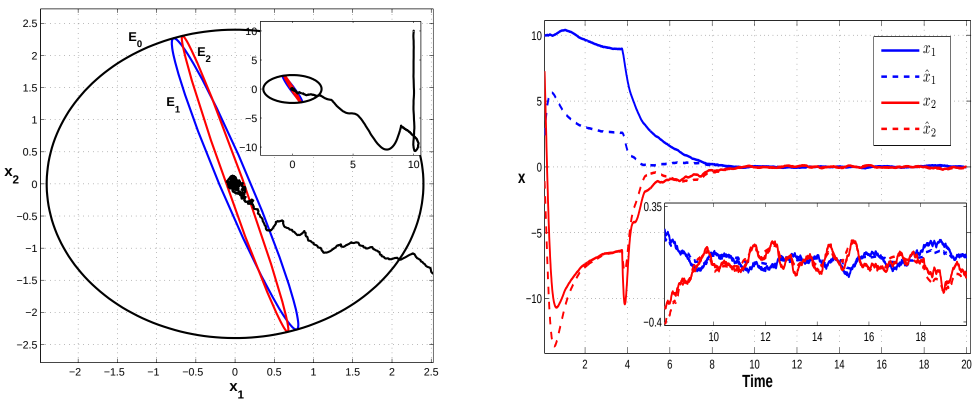
$$K_1 = \begin{pmatrix} -33.0001 & -11.9800 \end{pmatrix} \quad L_1 = \begin{pmatrix} 0.9934 & 0.9933 \end{pmatrix}^\top$$
$$K_2 = \begin{pmatrix} -32.0000 & -7.9900 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0.5399 & 0.8701 \end{pmatrix}^\top.$$

The ellipsoidal matrices P_i and other important parameters are

$$P_1 = \begin{pmatrix} 140.378 & 48.280 & 0 & 0 \\ 48.280 & 17.725 & 0 & 0 \\ 0 & 0 & 1.008 & 0.175 \\ 0 & 0 & 0.175 & 4.980 \end{pmatrix} \quad P_2 = \begin{pmatrix} 431.261 & 125.268 & 0 & 0 \\ 125.268 & 37.472 & 0 & 0 \\ 0 & 0 & 4.250 & -1.548 \\ 0 & 0 & -1.548 & 1.739 \end{pmatrix}$$

$$\alpha_{\min} = 0.8 \quad \beta = 0.7998 \quad \mu = 4.4770 \quad \tau_{av} = 3.7474.$$

The simulated trajectories are presented on Figure 2. In Figure 2a, the estimated ellipsoid region is shown. Also, it is shown that obtained ellipsoids $\mathcal{E}_1(0, \kappa^{-1}P_1)$ and $\mathcal{E}_2(0, \kappa^{-1}P_2)$ are inside of the prescribe ellipsoid $\mathcal{E}_0(0, Q_0)$. Figure 2b shows how the estimated states converge to the actual ones. Figure 3 shows the control input $u(t)$ and the measurable (Sampled and Quantized) output $y(t)$.



(a) (b)

Fig. 2. Simulation results for Example 1. (a) Estimated ellipsoid and system trajectories. (b) Simulated actual states (solid line) and estimated states (dashed lines).

Example 4.2. For the second example a separately excited DC motor is considered. The following model describes the dynamics of the motor with a switching inertia

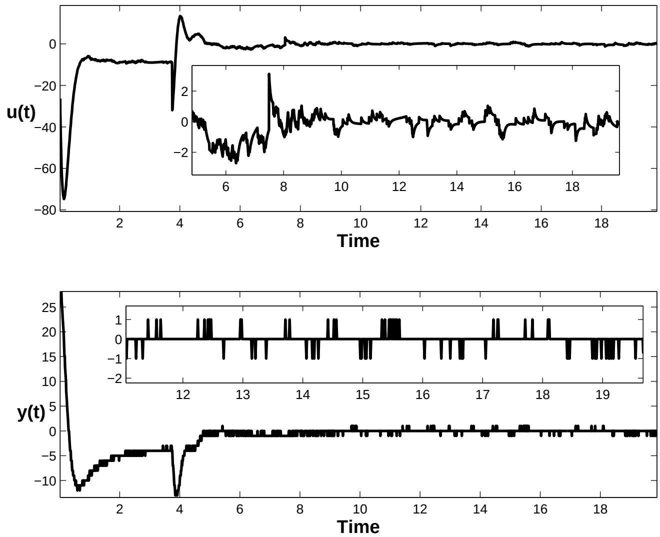


Fig. 3. Input and measured output signals for Example 1.

$$
\begin{aligned}
J_{\sigma(t)} \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} &= c_m \phi_s(t) i_r(t) - B_m \omega(t) - \eta_1(t) \\
L_r \frac{\mathrm{d}i_r(t)}{\mathrm{d}t} &= U_r(t) - R_r i_r(t) - c_m \phi_s(t)\omega(t) + \eta_2(t) \\
\frac{\mathrm{d}\phi_s(t)}{\mathrm{d}t} &= U_s(t) - R_s \phi_s(t) + \eta_3(t)
\end{aligned}
\tag{47}
$$

where $\omega(t)$ denotes the angular velocity of the shaft; $i_r(t)$ is the current of the rotor circuit, and R_r and R_s are the rotor and stator resistances, respectively. The rotor and stator voltages are expressed by $U_r(t)$ and $U_s(t)$. The rotor inductance is denoted here by L_r and $\phi_s(t)$ is the stator flux. The parameters $J_{\sigma(t)} \in \{J_1, J, 2\}$ and B_m in the above model express the moment of inertia of the rotor and the viscous friction coefficient, respectively. Finally, $\eta = (\eta_1, \eta_2, \eta_3)^\top$ denotes a parametrical uncertainty and c_m represents a constant parameter that depends on the spatial architecture of the drive.

We choose the states variables as $(x_1, x_2, x_3)^\top = (\omega, i_r, \phi_s)^\top$, an then let us apply the conventional linearization procedure to (47) around a given reference point $(\Omega^{ref}, I_r^{ref}, \Phi_s^{ref})$. The resulting linearized model satisfies the quasi-linear representation (6) with

$$A_i = \begin{bmatrix} -\frac{B_m}{J_i} & \frac{c_m \Phi_s^{ref}}{J_i} & \frac{c_m I_r^{ref}}{J} \\ -\frac{c_m \Phi_s^{ref}}{L_r} & -\frac{R_r}{L_r} & -\frac{c_m \Phi_s^{ref}}{L_r} \\ 0 & 0 & -R_s \end{bmatrix} \quad B_1 = B_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{L_r} & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where $i = 1, 2$, the values of the parameters are shown in Table 1. The sample time interval is fixed at 0.01 seconds, so we can choose directly $h = 0.01$, and the quantization constant selected was $c = 0.25$. The initial conditions for the dynamic system (47) are $x(0) = (1, 1, 1)^\top$. The prescribed matrix we use is a diagonal matrix $Q_0 = \mathrm{diag}\,(4, 400, 400, 4, 400, 400)$ an the initial conditions for (47) are selected as follows $(\omega^0, i_r^0, \phi_s^0)^\top = (1, 1, 1)^\top$.

Parameter	Value	Unit	Parameter	Value	Unit
c_m	0.03	Wb/rad	L_s	50	H
J_1	0.001	Kg/m^2	B_m	0.009	Nm/rad
J_2	0.004	Kg/m^2	Ω_{ref}	120	rad/s
R_r	0.5	Ohms	I_r^{ref}	0.1	A
R_s	85	Ohms	Φ_s^{ref}	15	Wb
L_r	8.9	mH			

Tab. 1. Parameters for the DC motor.

The observer and the controller gains obtained using the algorithm were

$$K_1 = \begin{pmatrix} -1.076 & -0.639 & -0.657 \\ 0.869 & -0.387 & -1.046 \end{pmatrix} \quad L_1 = \begin{pmatrix} -0.550 & -0.0627 & 1.301 \\ 0.802 & 1.096 & 0.392 \end{pmatrix}^\top$$
$$K_2 = \begin{pmatrix} -0.483 & -0.894 & -0.982 \\ 1.318 & -0.369 & -0.312 \end{pmatrix} \quad L_2 = \begin{pmatrix} -0.468 & 0.285 & 1.303 \\ 1.266 & 0.530 & 0.338 \end{pmatrix}^\top.$$

The ellipsoidal matrices P_i and other important parameters are (where $e_c = 10^{-c}$)

$$P_1 = \begin{bmatrix} 0.190 & 4.418 \cdot e_3 & -2.622 \cdot e_3 & -2.280 \cdot e_4 & -0.012 & -5.676 \cdot e_3 \\ 4.418 \cdot e_3 & 45.123 & -10.828 & 0.123 & 6.896 & -10.197 \\ -2.622 \cdot e_3 & -10.828 & 66.162 & -0.013 & -10.812 & 9.776 \\ -2.280 \cdot e_4 & 0.123 & -0.013 & 0.184 & -0.357 & -0.011 \\ -0.012 & 6.896 & -10.812 & -0.357 & 27.256 & -8.285 \\ -5.676 \cdot e_3 & -10.197 & 9.776 & -0.011 & -8.2857 & 29.841 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.157 & 3.545 \cdot e_4 & -8.303 \cdot e_5 & -1.265 \cdot e_5 & -0.002 & 0.004 \\ 3.545 \cdot e_4 & 37.827 & -38.072 & 0.015 & 2.989 & -7.421 \\ -8.303 \cdot e_5 & -38.072 & 112.66 & 0.003 & -3.617 & 11.119 \\ -1.265 \cdot e_5 & 0.015 & 0.003 & 0.156 & -0.116 & 0.032 \\ -0.002 & 2.989 & -3.617 & -0.116 & 16.385 & -1.172 \\ 0.004 & -7.421 & 11.119 & 0.032 & -1.172 & 18.372 \end{bmatrix}$$

$$\alpha_{\min} = 11 \quad \beta = 0.13106 \quad \mu = 2 \quad \tau_{av} = 0.12603.$$

The results of the system simulation are shown in Figures 4 − 6. Figure 4 contains the projection (the obtained attractive ellipsoid and the system trajectory) of the three-dimensional state space on the two-dimensional subspace (x_1, x_2), subspace (x_1, x_3) and subspace (x_2, x_3), respectively. Figure 5 shows how the estimated states converge around the origin. Figure 6a shows the control input $u(t)$. In Figure 6b the measurable (Sampled and Quantized) output $y(t)$ is shown.

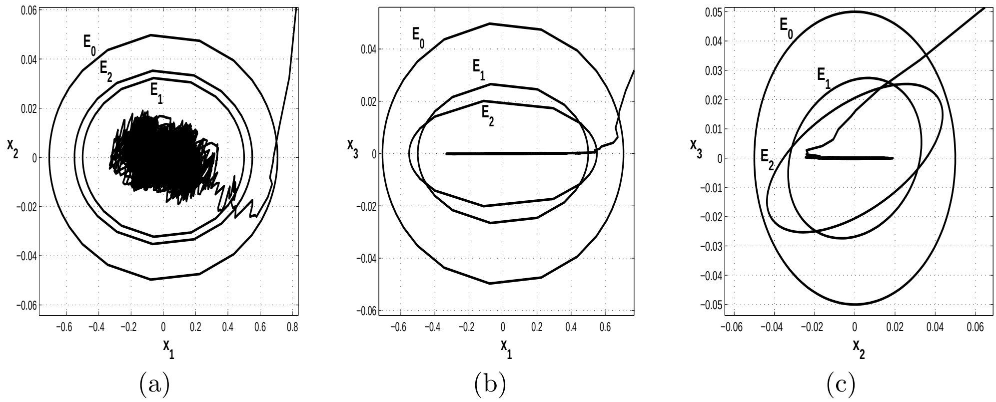


Fig. 4. Estimated ellipsoid and system trajectories for Example 2.
(a) x_1 vs. x_2. (b) x_1 vs. x_3. (c) x_2 vs. x_3.

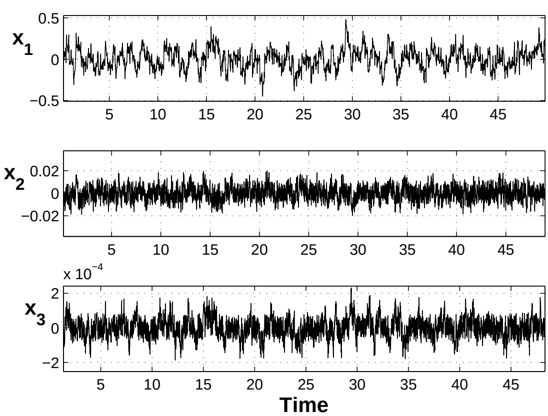
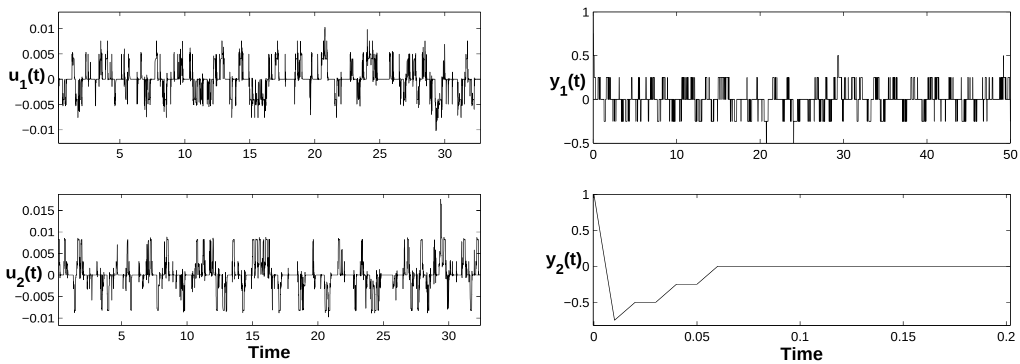


Fig. 5. States $x(t)$ for Example 2.



(a) (b)

Fig. 6. Control input and sample/quantized output signals. (a) $u_1(t)$ and $u_2(t)$. (b) $y_1(t)$ and $y_2(t)$.

5. CONCLUSIONS

In this contribution, we introduced an extension of the IEM for the robust control design of switched systems. Sampling and quantization at the output were considered to represent the result of a digitalization process. Also, the dwelling-time approach for switched systems was included in the development of this extended method. From the theoretical point of view the developed approach produced a feedback control law that not only ensures the existence, but defines an actual characterization of a minimal size ellipsoid for the corresponding closed-loop system trajectories.

The main result of this paper is presented in the form of a minimization problem with constraints represented as BMI's. The characterization of the ellipsoid was obtained from the numerical solution of the minimization problem, this ellipsoid has some minimal properties (minimal "size") that can be interpreted as a maximal robustness or practical stability of the closed-loop system.

The effectiveness of the proposed computational schemes and the associated control design was demonstrated by two illustrative examples, including a separately excited DC motor.

Additional conditions regarding the size of the ellipsoid respect to the size of the quantization levels need to be considered in order to avoid chattering. Addressing this issue can be an improvement to the results presented in this paper.

Finally, it is noteworthy that this approach can be easily extended to a broader class of nonlinear systems with complex discrete-continuous dynamical behaviours. Specifically it can be extended to systems with finite quantization levels, which implies an unbounded quantization error and saturation phenomena. Also, it seems possible to design control strategies that combine our method with well-known nonlinear design tools.

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