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PERONA–MALIK EQUATION: PROPERTIES OF EXPLICIT FINITE VOLUME SCHEME

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The Perona–Malik nonlinear parabolic problem, which is widely used in image processing, is investigated in this paper from the numerical point of view. An explicit finite volume numerical scheme for this problem is presented and consistency property is proved.

Keywords: Perona–Malik equation, finite volume method, consistency, stability monotonicity property

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1. INTRODUCTION

The modified Perona–Malik problem suggested by [2] is a nonlinear parabolic problem of the form

$$\partial_t u + F(x,u,Du,D^2u) = 0 \quad \text{in } Q_T \equiv [0,T]\times\Omega, \tag{1}$$
$$\partial_\nu u = 0 \quad \text{on } [0,T]\times\partial\Omega, \tag{2}$$
$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $F(x,u,Du,D^2u) = -\nabla.(g(|\nabla G_\sigma * u|)\nabla u)$, $\Omega\subset\mathbb{R}^2$ is a rectangular domain, $[0,T]$ is a scaling interval, and

$g(s)$ is a Lipschitz continuous decreasing function with Lipschitz constant L_g, (4)

and second derivative is bounded with constant L_{2g}, (5)

$g(0)=1, 0<g(s)\to 0$ for $s\to\infty$, (6)

$G_\sigma\in C^\infty(\mathbb{R}^d)$ is a smoothing kernel with compact support K_σ (7)

with $\displaystyle\int_{\mathbb{R}^d} G_\sigma(x)\,\mathrm{d}x = 1$

and $G_\sigma(x)\to\delta_x$ for $\sigma\to 0$, δ_x – Dirac function at point x, $u_0\in L_2(\Omega)$. (8)

2. FORMULATION OF THE FINITE VOLUME METHOD

Let τ_h be a uniform mesh of Ω with cells p of measure $m(p)$ and diameter $d(p)$ (we assume square cells here). Let us denote by $h = \max_{p\in\tau_h} d(p)$. For every cell p we consider a set of the neighbours $N(p)$ consisting of all cells $q \in \tau_h$ for which common interface of p and q, denoted by e_{pq}, is of non-zero measure $m(e_{pq})$. We denote the set of all these edges for all volumes $p \in \tau_h$ by $\mathcal{E}$ and by σ we denote the edge which connects the volumes p and q. (Clearly $e_{pq} = e_{qp} = e_{pq}I$.) It is assumed that for every p, there exists a representative point $x_p \in p$, such that for every pair $p, q \in N(p)$, the vector $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to a unit vector n_{pq} which is normal to e_{pq} and oriented from p to q. Let d_{pq} be defined as $d_{pq} := |x_p - x_q|$. In a simple case of a uniform grid x_p is just the center of the pixel. Then, let x_{pq} be the point of e_{pq} intersecting the segment $\overline{x_p x_q}$. We define

$$T_{pq} := \frac{m(e_{pq})}{d_{pq}} \leq T_0. \tag{9}$$

In this paper we will consider square finite volumes only, that means $m(p) = h^2, m(e_{pq}) = h, d_{pq} = h, T_{pq} = 1$.

Any discrete approximation of a solution of partial differential equation is considered to be piecewise constant on control volumes [3].

The finite volume explicit scheme on a uniform grid is then written as follows:

Let $0 = t_0 \leq t_1 \leq \cdots \leq t_n \cdots \leq t_N$, $N \cdot k = T$ *denote the scale discretization steps with* $t_l = t_{l-1} + k$, *where* k *is the discrete scale step,* $l = 1, 2, \ldots, N$.

For $n = 0, \ldots, N-1$ *we look for* u_p^{n+1}, $p \in \tau_h$, *satisfying the identities*

$$\left(u_p^{n+1} - u_p^n\right) m\left(p\right) = k \sum_{q\in N(p)} g_{pq}^{\sigma,n} T_{pq} \left(u_q^n - u_p^n\right), \tag{10}$$

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x)\, \mathrm{d}x, \tag{11}$$

$$g_{pq}^{\sigma,n} := g\left(|\nabla G_\sigma * \widetilde{u}^n\left(x_{pq}\right)|\right), \tag{12}$$

where $\widetilde{u}^n$ is a mirror extension of the discrete image computed in the nth scale step and u_p^n is a value of the numerical solution on the volume p in the nth scale step.

Let $\overline{u}_{h,k}$ denote the finite volume numerical solution for some fixed space mesh h and scale step k. In each time step this solution is piecewise constant on each finite volume as it is usual for finite volume numerical schemes of a parabolic type. The function piecewise constant on each finite volume at the lth scale step is denoted by $\bar{u}^l$.

3. STABILITY AND CONVERGENCE RESULTS

We briefly mention results of [7] obtained for the explicit finite volume scheme concerning the stability and convergence properties. For the semi-implicit scheme see [9].

First we make the following stability assumption:

$$k \leq (1-\eta)\frac{m(p)}{\sum\limits_{q\in N(p)} g_{pq}^{\sigma,n} T_{pq}} \quad \text{for all } p \in \tau_h \text{ and } \eta \in (0,1). \tag{13}$$

Stability estimates are of the following type (for the proof of Lemma A see [7], for Lemma B see [4]):

Lemma A. (A priori estimates in $L_2(Q_T)$) It holds that there exist positive constants C_1, C_2 such that

(i) $\max\limits_{0\leq l\leq N} \sum\limits_{p\in\tau_h} \left(u_p^l\right)^2 m(p) \leq C_1$

(ii) $\sum\limits_{l=0}^{N} k \sum\limits_{(p,q)\in\mathcal{E}} \frac{\left(u_p^l - u_q^l\right)^2}{d_{pq}} m\left(e_{pq}\right) \leq C_2$

and the constants C_1, C_2 do not depend on the mesh parameters h, k.

Lemma B. (L_∞ stability for a discrete solution) There exists positive constant C, such that for all $n = 1, 2 \ldots, N$ it holds:

$$\|\overline{u}^{n+1}\|_{L_\infty(\Omega)} \leq C. \tag{14}$$

Finally we present the convergence result for the proposed numerical scheme [7]:

Lemma C. (Convergence of $\overline{u}_{h,k}$) There exists $u \in L_2\left(Q_T\right)$ which is the weak solution of (1)–(3) such that

$$\overline{u}_{h,k} \to u \ \textit{in}\ L_2\left(Q_T\right)$$

as $h, k \to 0$. Furthermore, the convergence is pointwise.

4. PROPERTIES OF FV SCHEME

We want to prove some properties of our numerical scheme, so following the notations and results of [1] we first propose some notation and definitions.

We denote by $B(\overline{\Omega})$ the set of all uniformly bounded functions on domain $\overline{\Omega}$.

We consider the problem

$$\begin{aligned} u_t + F(D^2u) &= 0 \text{ on } [0,T]\times\Omega \\ u &= u_0 \text{ on } \Omega\times 0 \\ \partial_\nu u &= 0 \text{ on } [0,T]\times\Omega, \end{aligned} \tag{15}$$

where u and F are continuous functions of their arguments, D^2u denotes the second derivatives matrix of u with respect to x and F is elliptic.

A general scheme that is supposed to construct (15) can be denote as:

$$S(k) : B(\overline{\Omega}) \to B(\overline{\Omega}).$$

Definition 4.1. Approximation scheme $S(k)$ has the *monotonicity* property if it holds for all $u,\ v \in B(\overline{\Omega})$:

$$S(k)u \geq S(k)v \text{ if } \ u \geq v. \tag{16}$$

Quasi monotonicity property means:

$$S(k)v \leq S(k)u + o(k) \text{ if } \ u \geq v.$$

Definition 4.2. Approximation scheme $S(k) : B(\overline{\Omega}) \to B(\overline{\Omega})$ has the *stability* property if for all $K \in \mathbb{R},\ K > 0, u \in B(\overline{\Omega})$ it holds:

$$S(k)(u + K) = S(k)u + K. \tag{17}$$

Definition 4.3. Approximation scheme $S(k) : B(\overline{\Omega}) \to B(\overline{\Omega})$ has the *consistency* property if for all $\Phi \in C^{\infty}(\overline{\Omega})$ it holds:

$$\lim_{k\to 0} \frac{\Phi - S(k)\Phi}{k} = F(D^2\Phi). \tag{18}$$

For arbitrary function $u \in B(\overline{\Omega})$ the explicit scheme (10) can be rewritten in the following way:

$$S(k)u(x) = u(x) + \frac{k}{m\,(p)} \sum_{q\in N(p)} g_{pq}^{\sigma,u} T_{pq}\,(u(x_q) - u(x_p)) \quad \text{for every } x \in p, \tag{19}$$

where $g_{pq}^{\sigma,u} := g\,(|\nabla G_\sigma * \widetilde{u}\,(x_{pq})|)$ as in (12).

Then we obtain our approximate solution as in [1]:

$$u_N = \begin{cases} S(t - n\frac{T}{N})u_N(\cdot, n\frac{T}{N})(\cdot) & \text{if} \quad t \in \left(n\frac{T}{N}, (n+1)\frac{T}{N}\right),\ n = 1, 2, \ldots N-1 \\ u_0(\cdot) & \text{if} \quad t = 0. \end{cases} \tag{20}$$

For an arbitrary function $\Phi \in C^{\infty}(\Omega)$ we denote by $\|\Phi\|_{L_\infty}$ the norm in this functional space and $\|\Phi\|_k$ a norm of a functional space $C^k(\Omega)$.

Convergence theorem for problems with elliptic operator F under assumption of monotonicity, stability and consistency of the approximation solution, is proved in [1]. Although the convergence of the approximation scheme (10) has been proved in [7], the presented scheme does not posses the monotonicity property [4]:

Theorem 1. Let

$$k = C^* m(p), \tag{21}$$

where C^* is chosen in such a way that (13) is fulfilled for some $\eta \in (0, 1)$.Then the explicit approximation scheme (19) has the stability property, but it is not monotone.

Theorem 2. For the new relation between scale and space step in the form:

$$k = C \cdot m(p)^{2+\alpha}, \tag{22}$$

where $\alpha > 0$ is small, scheme (10) is quasi monotone.

Remark. Quasi monotonicity is sufficient to prove theorem of [1], monotonicity itself is not necessary.

Theorem 3. For the relation (22) from the previous theorem the consistency property holds for our scheme.

Proof. For $x \in \Omega$ and arbitrary $\Phi \in C^\infty(\overline{\Omega})$ we must prove the estimation

$$\left| \frac{\Phi(x) - S(k)\Phi(x)}{k} - F(x, \Phi, D\Phi, D^2\Phi) \right| \leq Ck^\eta \tag{23}$$

for some $\eta > 0$, where C is a generic constant independent on k and h. After applying the formula for $S(k)$ into (23) we can see that we must estimate the term:

$$\left| -\frac{1}{m(p)} \sum_{q \in N(p)} g_{pq}^{\sigma,\Phi} T_{pq} \left(\Phi_q - \Phi_p\right) - F(x, \Phi, D\Phi, D^2\Phi) \right|,$$

where we have used the notation

$$\Phi_q = \Phi(x_q),\ \Phi_p = \Phi(x_p) \quad \text{and} \quad g_{pq}^{\sigma,\Phi} = g\left(\left| \nabla G_\sigma * \widetilde{\Phi}\left(x_{pq}\right) \right| \right)$$

and $\widetilde{\Phi}$ has the same meaning as before. We can rewrite the previous term in the following way:

$$\left| -\frac{1}{m(p)} \sum_{q \in N(p)} \int_{e_{pq}} g_{pq}^{\sigma,\Phi} \frac{\Phi_q - \Phi_p}{\mathrm{d}_{pq}} \,\mathrm{d}s - F(x, \Phi, D\Phi, D^2\Phi) \right|,$$

or for our square finite volumes:

$$\left| \frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} g_{pq}^{\sigma,\Phi} \frac{\Phi_q - \Phi_p}{h} \,\mathrm{d}s + F(x, \Phi, D\Phi, D^2\Phi) \right|. \tag{24}$$

Now we use the same idea as in [6]: For the difference term $(\Phi_q - \Phi_p)/(h)$ let us use the Taylor expansion on each edge e_{pq} in a similar way as for deriving a usual central difference approximation. Let $x_p = (x_{1p}, x_{2p})$ and $x_{q_i} = (x_{1q_i}, x_{2q_i})$ for $i = 1, \ldots, 4$, $q_i \in N(p)$ where q_1 is the neighbor of volume p on the right, q_2

on the top, q_3 on the left and q_4 on the bottom. Let $s = (s_1, s_2)$ be a point on the boundary of a volume p. Then

$$\text{for a point } s \in e_{pq_1} \text{ we have } s = \left(x_{1p} + \frac{h}{2}, x_{2p} + t\frac{h}{2}\right), \; t \in \langle -1, 1\rangle, \tag{25}$$

$$\text{for a point } s \in e_{pq_2} \text{ we have } s = \left(x_{1p} + t\frac{h}{2}, x_{2p} + \frac{h}{2}\right), \; t \in \langle -1, 1\rangle, \tag{26}$$

$$\text{for a point } s \in e_{pq_3} \text{ we have } s = \left(x_{1p} - \frac{h}{2}, x_{2p} + t\frac{h}{2}\right), \; t \in \langle -1, 1\rangle, \tag{27}$$

$$\text{for a point } s \in e_{pq_4} \text{ we have } s = \left(x_{1p} + t\frac{h}{2}, x_{2p} - \frac{h}{2}\right), \; t \in \langle -1, 1\rangle. \tag{28}$$

Then for e_{pq_1} and e_{pq_3} we have

$$\frac{\Phi_q - \Phi_p}{h} = \frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{1q} - x_{1p})(x_{2q} - s_2) + O(h^2) \tag{29}$$

and for e_{pq_2} and e_{pq_4} similarly

$$\frac{\Phi_q - \Phi_p}{h} = \frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{2q} - x_{2p})(x_{1q} - s_1) + O(h^2). \tag{30}$$

We get

$$\begin{aligned}
& \frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} g_{pq}^{\sigma,\Phi} \frac{\Phi_q - \Phi_p}{h} \, \mathrm{d} \\
= \; & \frac{1}{h^2} \sum_{i=1,3} \int_{e_{pq_i}} g_{pq_i}^{\sigma,\Phi} \left(\frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{1q_i} - x_{1p})(x_{2q_i} - s_2) + O(h^2) \right) \mathrm{d}s \\
+ \; & \frac{1}{h^2} \sum_{i=2,4} \int_{e_{pq+i}} g_{pq_i}^{\sigma,\Phi} \left(\frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{2q_i} - x_{2p})(x_{1q_i} - s_1) + O(h^2) \right) \mathrm{d}s \\
= \; & \frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} g_{pq}^{\sigma,\Phi} \frac{\partial \Phi(s)}{\partial \nu} \, \mathrm{d}s \\
+ \; & \frac{1}{h^2} \sum_{i=1,3} \int_{e_{pq_i}} g_{pq_i}^{\sigma,\Phi} 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{1q_i} - x_{1p})(x_{2q_i} - s_2) \, \mathrm{d}s \\
+ \; & \frac{1}{h^2} \sum_{i=2,4} \int_{e_{pq_i}} g_{pq_i}^{\sigma,\Phi} 2\Phi_{xy}(s) \cdot \operatorname{sgn}(x_{1q_i} - x_{1p})(x_{2q_i} - s_2) \, \mathrm{d}s + C(\|\Phi\|_3)h \\
= \; & I_1 + I_2 + I_3 + C(\|\Phi\|_3)h.
\end{aligned}$$

So the term for estimation now has the form:

$$\begin{aligned}
& \left| I_1 + I_2 + I_3 + C(\|\Phi\|_3)h + F(x, \Phi, D\Phi, D^2\Phi) \right| \\
\le \; & \left| I_1 + F(x, \Phi, D\Phi, D^2\Phi) \right| + |I_2| + |I_3| + Ch.
\end{aligned}$$

Using parametrizations (25)–(28) we can rearrange the term I_2 (the term I_3 can be estimated analogously) on the edge e_{pq_1} into the following form

$$\frac{2h}{2h^2}\int_{-1}^{1} g^{\sigma,\Phi}_{pq_1}\Phi_{xy}\left(x_{1p}+\frac{h}{2}, x_{2p}+t\frac{h}{2}\right)\left(-t\frac{h}{2}\right)\,\mathrm{d}t$$

and on the edge e_{pq_3} similarly

$$-\frac{2h}{2h^2}\int_{-1}^{1} g^{\sigma,\Phi}_{pq_3}\Phi_{xy}\left(x_{1p}-\frac{h}{2}, x_{2p}+t\frac{h}{2}\right)\left(t\frac{h}{2}\right)\,\mathrm{d}t.$$

Collecting these two terms together, and using the fact that $\Phi\in C^\infty(Q)$ we obtain

$$|I_2| \le \left|\frac{h^2}{2h^2}\int_{-1}^{1} t\left(g^{\sigma,\Phi}_{pq_1}-g^{\sigma,\Phi}_{pq_3}\right)\Phi_{xy}\left(x_{1p}+\frac{h}{2}\, x_{2p}+t\frac{h}{2})\right)\,\mathrm{d}t+\right.$$

$$\left.\frac{h^2}{2h^2}\int_{-1}^{1} t g^{\sigma,\Phi}_{pq_3}\left(\Phi_{xy}(x_{1p}+\frac{h}{2}, x_{2p}+t\frac{h}{2})-\Phi_{xy}(x_{1p}-\frac{h}{2}, x_{2p}+t\frac{h}{2})\right)\,\mathrm{d}t\right|$$

$$\le \|\Phi\|_2\frac{g^{\sigma,\Phi}_{pq_1}-g^{\sigma,\Phi}_{pq_3}}{2}+C(\|\Phi\|_3)\frac{g^{\sigma,\Phi}_{pq_3}}{2}h.$$

Putting all together we obtain

$$|I_2|+|I_3| \le C(\|\Phi\|_2)h\left(g^{\sigma,\Phi}_{pq_3}+g^{\sigma,\Phi}_{pq_4}\right) \tag{31}$$

$$+\quad C(\|\Phi\|_3)\left(|g^{\sigma,\Phi}_{pq_1}-g^{\sigma,\Phi}_{pq_3}|+|g^{\sigma,\Phi}_{pq_2}-g^{\sigma,\Phi}_{pq_4}|\right). \tag{32}$$

Last term in this inequality can be estimated as in [5]:

$$\left|g^{\sigma,\Phi}_{pq_1}-g^{\sigma,\Phi}_{pq_3}\right| \le Ch\left(\|D^2G_\sigma\|_{L_\infty}\|\Phi\|_{L_\infty}+\|DG_\sigma\|_{L_\infty}\|D\Phi\|_{L_\infty}\right).$$

Finally we have

$$|I_2|+|I_3| \le C(\|\Phi\|_2)h.$$

Now for the first term it holds:

$$\begin{aligned} I_1 &= \frac{1}{h^2}\sum_{q\in N(p)}\int_{e_{pq}} g^{\sigma,\Phi}_{pq}\frac{\partial\Phi(s)}{\partial\nu}\,\mathrm{d}s = \frac{1}{h^2}\sum_{q\in N(p)}\int_{e_{pq}} g^{\sigma,\Phi}_{p}\frac{\partial\Phi(s)}{\partial\nu}\,\mathrm{d}s \\ &+ \frac{1}{h^2}\sum_{q\in N(p)}\int_{e_{pq}} (g^{\sigma,\Phi}_{pq}-g^{\sigma,\Phi}_{p})\frac{\partial\Phi(s)}{\partial\nu}\,\mathrm{d}s = II_1+II_2 \end{aligned}$$

So we must now estimate

$$|II_1 - F(x,\Phi,D\Phi,D^2\Phi)| + |II_2|.$$

First we can rearrange the term in II_2.

$$g_{pq}^{\sigma,\Phi} - g_p^{\sigma,\Phi} = g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x_{pq}\right)\right|\right) - g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x_p\right)\right|\right).$$

For simplicity we denote

$$s_{pq} = \left|\nabla G_\sigma * \tilde{\Phi}\left(x_{pq}\right)\right|,$$

$$s_p = \left|\nabla G_\sigma * \tilde{\Phi}\left(x_p\right)\right|.$$

Then

$$g(s_p) - g(s_{pq}) = g'\left(s_p\right) \cdot D\left(s_p\right)\left|x_{pq} - x_p\right| + O(h^2)$$

So

$$II_2 = \frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} (g_{pq}^{\sigma,\Phi} - g_p^{\sigma,\Phi}) \frac{\partial \Phi(s)}{\partial \nu}\, \mathrm{d}s =$$

$$\frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} (g'\left(s_p\right) \cdot D\left(s_p\right)\left|x_{pq} - x_p\right| + O(h^2)) \frac{\partial \Phi(s)}{\partial \nu}\, \mathrm{d}s =$$

$$\frac{1}{h^2} \sum_{q \in N(p)} \int_{e_{pq}} (g'\left(s_p\right) \cdot D\left(s_p\right)\left|x_{pq} - x_p\right|) \frac{\partial \Phi(s)}{\partial \nu}\, \mathrm{d}s + O(h).$$

Now we can apply Green's theorem on the first term and according to the properties of function g and G_σ for $|x_{pq} - x_p| = \frac{h}{2}$ we obtain:

$$II_2 = \frac{1}{2h} g'\left(s_p\right) \cdot D\left(s_p\right) \int_p \Delta\Phi(x)\, \mathrm{d}x + O(h).$$

$$|II_2| \le h\left(L_g \|D^2 G_\sigma\|_{L_\infty} \|\Phi\|_2\right) + O(h) \le Ch.$$

Finally for II_1 we can use again Green's theorem and we have:

$$II_1 = \frac{1}{h^2} \int_p \nabla \cdot \left(g_p^{\sigma,\Phi} \nabla\Phi(x)\right)\, \mathrm{d}x =$$

$$\frac{1}{h^2} \int_p \nabla \cdot \left(g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x\right)\right|\right) \nabla\Phi(x)\right) \mathrm{d}x + \frac{1}{h^2} \int_p \nabla \cdot \left(\left(g_p^{\sigma,\Phi} - g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x\right)\right|\right)\right) \nabla\Phi(x)\right) \mathrm{d}x.$$

So applying the mean value theorem to the first term we obtain

$$\begin{aligned} & |II_1 + F(x, \Phi, D\Phi, D^2\Phi)| \\ \le\ & \left|\nabla \cdot \left(g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(\xi\right)\right|\right) \nabla\Phi(\xi)\right) - \nabla \cdot \left(g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x\right)\right|\right) \nabla\Phi(x)|\right)\right| \\ +\ & \frac{1}{h^2} \left| \int_p \nabla \cdot \left(\left(g_p^{\sigma,\Phi} - g\left(\left|\nabla G_\sigma * \tilde{\Phi}\left(x\right)\right|\right)\right) \nabla\Phi(x)\right) \mathrm{d}x \right| = III_1 + III_2, \end{aligned}$$

for some $\xi \in p$ For the second III_2 term in this estimation we can use again the same argument as for the estimation of the term II_2 before and it is easy to see that this term is of $O(h)$. For the first term III_1 we use the properties of g, Φ and G_σ. Again we denote

$$s_\xi = \left|\nabla G_\sigma * \tilde{\Phi}(\xi)\right|, \quad s_x = \left|\nabla G_\sigma * \tilde{\Phi}(x)\right|$$

$$\begin{aligned} & |III_1| = |\nabla \cdot (g(s_\xi)\nabla\Phi(\xi)) - \nabla \cdot (g(s_x)\nabla\Phi(x)|)| \\ \leq \; & |\nabla g(s_\xi) \cdot \nabla\Phi(\xi) - \nabla g(s_x)\nabla\Phi(x)| + |g(s_\xi)\Delta\Phi(\xi) - g(s_x)\Delta\Phi(x)| \\ \leq \; & |\nabla g(s_\xi) \cdot \nabla\Phi(\xi) \pm \nabla g(s_\xi) \cdot \nabla\Phi(x) - \nabla g(s_x)\nabla\Phi(x)| \\ + \; & |g(s_\xi)\Delta\Phi(\xi) \pm g(s_\xi)\Delta\Phi(x) - g(s_x)\Delta\Phi(x)| \\ \leq \; & L_{2g}\|D^2G_\sigma\|_{L_\infty}\|D\Phi\|_{L_\infty} \cdot h + \|\Phi\|_3 \cdot h. \end{aligned}$$

If we now take into account the relations (13) and (22) we can conclude our proof with the estimation

$$\left|\frac{\Phi(x) - S(k)\Phi(x)}{k} - F(x, \Phi, D\Phi, D^2\Phi)\right| \leq Ck^{\frac{1}{2(1+\alpha)}}. \qquad \square$$

Remark to numerical experiments. Proposed numerical scheme of Perona–Malik equation can be used in a very effective way for problems of filtering in image processing. Many results, examples and numerical experiments can be seen for example in [4, 5, 7, 8].

5. CONCLUSION

The properties of numerical scheme for Perona–Malik equation was proved. These properties are sufficient for convergence result of the numerical solution to the exact one. The results are done for uniform mesh only, the generalization for non-uniform mesh brings some technical difficulties and calculations.

From computational point of view numerical method based on finite volume is very effective and natural for image processing, because the initial noisy image is also piecewise constant function on pixels, which can be used as a first meshing of an image domain. In practical computations the relation (22) is not so constrained, because the variable t is not a real time but scale variable and the whole scaling interval $[0, T]$ is not usually very long.

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REFERENCES

[1] G. Barles and P. E. Souganidis: Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic Analysis *4* (1991), 271–283.

[2] F. Catté, P. L. Lions, J. M. Morel, and T. Coll: Image selective smoothing and edge detection by nonlinear diffusion. SIAM J. Numer. Anal. *29* (1992), 182–193.
[3] R. Eymard, T. Gallouet, and R. Herbin: The Finite Volume Method. In: Handbook of Numerical Analysis, Vol. VII, 2000 (Ph. Ciarlet and J. L. Lions, eds.), pp. 715–1022.
[4] A. Handlovičová and Z. Krivá: Properties of explicit finite volume scheme for Perona–Malik Equation. In: Proc. Prastan 04, Kočovce 2004, pp. 133–140.
[5] A. Handlovičová and Z. Krivá: Error estimates for finite volume scheme for Perona–Malik Equation. Acta Math. Univ. Comenianae LXXIV *1* (2005), 79–94.
[6] A. Handlovičová and K. Mikula: Convergence of the semi-implicit co-volume scheme for the curvature driven level set equation. To appear.
[7] Z. Krivá: Explicit finite volume scheme for the Perona–Malik equation. Computational Methods in Applied Math. *5* (2005), 2, 170–200.
[8] Z. Krivá and K. Mikula: An adaptive finie volume scheme for solving nonlinear diffusion equations in image processing. J. Visual. Comm. and Image Representation *13* (2002), 22–35.
[9] K. Mikula and N. Ramarosy: Semi-implicit finite volume scheme for solving nonlinear diffusion equations in image processing. Numer. Math. *89* (2001), 3, 561–590.

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