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TEST OF LINEAR HYPOTHESIS IN MULTIVARIATE MODELS

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In regular multivariate regression model a test of linear hypothesis is dependent on a structure and a knowledge of the covariance matrix. Several tests procedures are given for the cases that the covariance matrix is either totally unknown, or partially unknown (variance components), or totally known.

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1. NOTATIONS AND AUXILIARY STATEMENTS

Let a model

$$\underline{\boldsymbol{Y}} \sim N_{nm}(\boldsymbol{X}\boldsymbol{B}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}) \tag{1}$$

be under consideration. Here $\underline{\boldsymbol{Y}}$ is an $n \times m$ normally distributed matrix with the mean value matrix $\mathrm{E}(\underline{\boldsymbol{Y}})$ equal to $\boldsymbol{X}\boldsymbol{B}$. The covariance matrix of the vector $\mathrm{vec}(\underline{\boldsymbol{Y}})$ (the vector composed of the columns of the matrix $\underline{\boldsymbol{Y}}$) is $\mathrm{Var}[\mathrm{vec}(\underline{\boldsymbol{Y}})] = \boldsymbol{\Sigma} \otimes \boldsymbol{I}$ ($\boldsymbol{I}$ is the $n \times n$ identity matrix). The model is regular if the rank $r(\boldsymbol{X})$ of the matrix $\boldsymbol{X}$ is $r(\boldsymbol{X}) = k < n$ and the $m \times m$ matrix $\boldsymbol{\Sigma}$ is positive definite (p.d.).

The linear hypothesis of the unknown $k \times m$ parameter matrix $\boldsymbol{B}$ is considered in the form

$$H_0: \quad \boldsymbol{H}\boldsymbol{B} + \boldsymbol{H}_0 = \boldsymbol{0}, \tag{2}$$

where $h \times k$ matrix $\boldsymbol{H}$ is assumed to be known. The $h \times m$ matrix $\boldsymbol{H}_0$ is also assumed to be known. The hypothesis is regular if $r(\boldsymbol{H}) = h < k$. The alternative hypothesis is

$$H_a: \quad \boldsymbol{H}\boldsymbol{B} + \boldsymbol{H}_0 \neq \boldsymbol{0}.$$

Lemma 1.1. The best linear unbiased estimator of the matrix $\boldsymbol{B}$ is

$$\widehat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\underline{\boldsymbol{Y}} \sim N_{km}[\boldsymbol{B}, \boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}].$$

Proof. Cf. [1]. □

Lemma 1.2. One of the test statistics for the regular hypothesis (2) in the case of the known matrix $\boldsymbol{\Sigma}$ is

$$T = \mathrm{Tr}\Big\{(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)\boldsymbol{\Sigma}^{-1}\Big\} \sim \chi^2_{mh}(\delta), \tag{3}$$

where

$$\delta = \mathrm{Tr}\Big\{(\boldsymbol{H}\boldsymbol{B}^* + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{B}^* + \boldsymbol{H}_0)\boldsymbol{\Sigma}^{-1}\Big\}.$$

The symbol $\chi^2_{mh}(\delta)$ means the noncentral chi-square random variable with mh degrees of freedom and with the parameter of noncentrality equal to δ, $\boldsymbol{B}^*$ means the actual value of the matrix $\boldsymbol{B}$.

Proof. The statement can be obtained from an univariate model $\mathrm{vec}(\underline{\boldsymbol{Y}}) \sim N_{nm}[(\boldsymbol{I} \otimes \boldsymbol{X})\mathrm{vec}(\boldsymbol{B}), \boldsymbol{\Sigma} \otimes \boldsymbol{I}]$ in a standard way by utilization of the relationship $\mathrm{vec}(\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{I} \otimes \boldsymbol{X})\mathrm{vec}(\boldsymbol{B})$. □

Lemma 1.3. The matrix $(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})$ is the $m \times m$ Wishart matrix with the $n-k$ degrees of freedom and with the covariance matrix $\boldsymbol{\Sigma}$, i.e. $(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}}) \sim W_m(n-k, \boldsymbol{\Sigma})$.

Proof. The matrix $\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}}$ is distributed as $N_{nm}(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{M}_X)$, where $\boldsymbol{M}_X = \boldsymbol{I} - \boldsymbol{P}_X$ and $\boldsymbol{P}_X$ is the Euclidean projector on the subspace $\mathcal{M}(\boldsymbol{X}) = \{\boldsymbol{X}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^k\}$. Thus for any generalized inverse (cf. [6]) $\boldsymbol{M}_X^-$ of the matrix $\boldsymbol{M}_X$ the matrix $(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'\boldsymbol{M}_X^-(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})$ has the Wishart distribution $W_m([r(\boldsymbol{M}_X), \boldsymbol{\Sigma}]$. One version of the matrix $\boldsymbol{M}_X^-$ is $\boldsymbol{I}$. □

Lemma 1.4. If $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{V}$ ($\boldsymbol{V}$ is p.d.), then the best estimator of σ^2 is

$$\widehat{\sigma}^2 = \frac{\mathrm{Tr}[(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})\boldsymbol{V}^{-1}]}{m(n-k)} \sim \sigma^2 \frac{\chi^2_{m(n-k)}(0)}{m(n-k)}.$$

This estimator is independent of the estimator $\widehat{\boldsymbol{B}}$.

Proof. The statement is a transcription of the well known statement from the theory of the univariate linear models (cf. e.g. [2]). □

Corollary 1.5. If $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{V}$, then one of the test statistics for the regular hypothesis (2) is

$$T = \frac{\mathrm{Tr}\Big\{(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)\boldsymbol{V}^{-1}\Big\}/(mh)}{\mathrm{Tr}[(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})\boldsymbol{V}^{-1}]/[m(n-k)]} \sim F_{mh,m(n-k)}(\delta),$$

where

$$\delta = \frac{\mathrm{Tr}\Big\{(\boldsymbol{H}\boldsymbol{B}^* + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{B}^* + \boldsymbol{H}_0)\boldsymbol{V}^{-1}\Big\}}{\sigma^2}$$

and $F_{mh,m(n-k)}(\delta)$ is the noncentral Fisher–Snedecor random variable with degrees of freedom equal to mh and $m(n-k)$ and with the noncentrality parameter equal to δ.

2. DIFFERENT STRUCTURES OF THE MATRIX $\mathbf{\Sigma}$

Let $\mathbf{\Sigma}$ be given. Then

$$(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)=\boldsymbol{Q}_1\sim W_m(h,\mathbf{\Sigma})$$

(possibly noncentral) and therefore, under the null hypothesis, for any nonzero $\boldsymbol{f}\in\mathbb{R}^m$ it is valid

$$\boldsymbol{f}'\boldsymbol{Q}_1\boldsymbol{f}/(\boldsymbol{f}'\mathbf{\Sigma}\boldsymbol{f})\sim\chi^2_h(0).$$

Let $\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0\neq\mathbf{0}$ ($\boldsymbol{B}^*$ is the actual value of the matrix $\boldsymbol{B}$) and let $\lambda_{\max}$ be the maximum solution of the equation

$$\det\left\{(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)-\lambda\mathbf{\Sigma}\right\}=0$$

and let $\boldsymbol{f}_{\max}$ satisfy the relationship

$$\left\{(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)-\lambda_{\max}\mathbf{\Sigma}\right\}\boldsymbol{f}_{\max}=\mathbf{0}.$$

Then

$$\delta=\boldsymbol{f}'_{\max}(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{B}^*+\boldsymbol{H}_0)\boldsymbol{f}_{\max}\Big/\boldsymbol{f}'_{\max}\mathbf{\Sigma}\boldsymbol{f}_{\max},$$

i. e. the parameter of noncentrality of the statistic

$$\chi^2_h(\delta)=\boldsymbol{f}'_{\max}\boldsymbol{Q}_1\boldsymbol{f}_{\max}/\boldsymbol{f}'_{\max}\mathbf{\Sigma}\boldsymbol{f}_{\max}\tag{4}$$

is for this vector $\boldsymbol{f}_{\max}$ maximum and therefore the chance to detect that H_0 is not true is also maximum.

It is of some importance to compare the power functions of the statistics (3) and (4).

Let

$$\underline{\boldsymbol{Y}}=\begin{pmatrix}-2, & 1, & 4\\ -1, & 2, & 2\\ 0, & 4, & -4\\ 1, & 2, & 2\\ 2, & 1, & 4\end{pmatrix}\boldsymbol{B}_{3,3}+\boldsymbol{\varepsilon}_{5,3},\quad \mathrm{Var}[\mathrm{vec}(\underline{\boldsymbol{Y}})]=\begin{pmatrix}1^2, & 0, & 0\\ 0, & 2^2, & 0\\ 0, & 0, & 3^2\end{pmatrix}\otimes\boldsymbol{I}_{5,5}$$

and the null hypothesis be $\begin{pmatrix}1, & 1, & 1\\ 0, & 1, & 1\end{pmatrix}\boldsymbol{B}=\mathbf{0}$. It means $h=2$, $m=3$, $n=5$, $k=3$. If

$$\begin{pmatrix}1, & 1, & 1\\ 0, & 1, & 1\end{pmatrix}\boldsymbol{B}=\begin{pmatrix}0.5, & -0.5, & 1.0\\ 0, & 0.5, & -0.5\end{pmatrix},$$

then $\boldsymbol{f}'_{\max}\boldsymbol{Q}_1\boldsymbol{f}_{\max}\Big/\boldsymbol{f}'_{\max}\mathbf{\Sigma}\boldsymbol{f}_{\max}\sim\chi^2_2(\delta_1),\delta_1=2.994$ and $T\sim\chi^2_6(\delta_2),\delta_2=6.603$ (cf. Lemma 1.2).

If $\chi^2_f(\delta)$ is approximated by $\frac{f+2\delta}{f+\delta}\chi^2_{\frac{(f+\delta)^2}{f+2\delta}}(0)$, then we obtain for $\alpha = 0.05$ $\mathrm{P}\left\{\chi^2_2(2.994) \geq 5.99\right\} = 21\,\%$ and $\mathrm{P}\left\{\chi^2_6(6.603) \geq 12.6\right\} = 44\,\%$. It shows a prevalence of the test (3) versus (4). However it can be utilized only in the case of the known matrix $\boldsymbol{\Sigma}$, or if its estimator is very precise.

If the matrix $\boldsymbol{\Sigma}$ is unknown and (2) is true, then the relationships

$$\begin{aligned}\boldsymbol{Q}_1 &= (\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0) \sim W_m(h,\boldsymbol{\Sigma}),\\ \boldsymbol{Q}_2 &= (\underline{\boldsymbol{Y}}-\boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}}-\boldsymbol{X}\widehat{\boldsymbol{B}}) \sim W_m(n-k,\boldsymbol{\Sigma})\end{aligned}$$

(it is to be remarked that $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are independent) can be utilized for a construction of different tests for the hypothesis (2). As and example can serve the statistic $\boldsymbol{g}'\boldsymbol{Q}_1\boldsymbol{g}/\boldsymbol{g}'\boldsymbol{Q}_2\boldsymbol{g} \sim F_{h,n-k}$, where

$$\frac{\boldsymbol{g}'\boldsymbol{Q}_1\boldsymbol{g}}{\boldsymbol{g}'\boldsymbol{Q}_2\boldsymbol{g}} = \max\left\{\frac{\boldsymbol{u}'\boldsymbol{Q}_1\boldsymbol{u}}{\boldsymbol{u}'\boldsymbol{Q}_2\boldsymbol{u}} : \boldsymbol{u}\in\mathbb{R}^m\right\}.$$

This statistic has the Fisher–Snedecor distribution $F_{h,n-k}(0)$ if the hypothesis H_0 is true and the distribution is independent of $\boldsymbol{g}$. However if H_0 is not true then the statistics has the largest realization and thus there is the greatest chance to recognize that H_0 is not true.

If $n-k$ tends to infinity, then $\widehat{\boldsymbol{\Sigma}} = (\underline{\boldsymbol{Y}}-\boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}}-\boldsymbol{X}\widehat{\boldsymbol{B}})/(n-k)$ tends to $\boldsymbol{\Sigma}$ in probability and thus $\mathrm{Tr}\left\{(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)\widehat{\boldsymbol{\Sigma}}^{-1}\right\}$ tends in distribution to χ^2_{mh}. This fact can be also utilized mainly in connection to a consideration at the beginning of this section. Other tests based on the matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, respectively, are analyzed in [4] and therefore they are omitted here.

Lemma 2.1. Let $\boldsymbol{\Sigma} = \sum_{i=1}^p \vartheta_i \boldsymbol{V}_i$, where ϑ_i, $i=1,\dots,p$, are unknown parameters, $\boldsymbol{\vartheta}\in\underline{\vartheta}\subset R^p$, and $\boldsymbol{V}_1,\dots,\boldsymbol{V}_p$, are known symmetric matrices. The set $\underline{\vartheta}$ is open and it is valid $\boldsymbol{\vartheta}\in\underline{\vartheta} \Rightarrow \sum_{i=1}^p \vartheta_i\boldsymbol{V}_i$ is p.d. Let the matrix $\boldsymbol{S}_{\Sigma_0^{-1}}$ be regular. Here

$$\left\{\boldsymbol{S}_{\Sigma_0^{-1}}\right\}_{i,j} = \mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_j), \quad i,j=1,\dots,p,$$

and $\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_i^{(0)}\boldsymbol{V}_i, \boldsymbol{\vartheta}^{(0)} = (\vartheta_1^{(0)},\dots,\vartheta_p^{(0)})'$ is an approximate value of he unknown parameter $\boldsymbol{\vartheta}$. Then the unbiased $\boldsymbol{\vartheta}^{(0)}$-locally minimum variance quadratic invariant estimator of the parameter $\boldsymbol{\vartheta}$ is

$$\widehat{\boldsymbol{\vartheta}} = \frac{1}{n-k}\boldsymbol{S}^{-1}_{\Sigma_0^{-1}}\begin{pmatrix}\mathrm{Tr}(\underline{\boldsymbol{Y}}'\boldsymbol{M}_X\underline{\boldsymbol{Y}}\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1})\\ \vdots \\ \mathrm{Tr}(\underline{\boldsymbol{Y}}'\boldsymbol{M}_X\underline{\boldsymbol{Y}}\boldsymbol{\Sigma}_0^{-1}\boldsymbol{V}_p\boldsymbol{\Sigma}_0^{-1})\end{pmatrix}, \quad \mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = \frac{2}{n-k}\boldsymbol{S}^{-1}_{\Sigma_0^{-1}}.$$

P r o o f. Cf. [5]. □

Now the problem arises whether the matrix $\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}) = \sum_{i=1}^p \widehat{\vartheta}_i\boldsymbol{V}_i$ can be used instead the matrix $\boldsymbol{\Sigma}$ in the statistic (3) without any essential deterioration of the inference.

In the following text a procedure for a construction of an insensitivity region is described. For the sake of simplicity only a problem of the risk α of the test is analyzed and problems of construction of the insensitivity region for the power function of the test is omitted.

Lemma 2.2. Let

$$T(\boldsymbol{\vartheta}) = \mathrm{Tr}\Big\{(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\Big\}.$$

Then

$$\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i} = -\mathrm{Tr}\Big\{(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\Big\},$$

thus $T(\boldsymbol{\vartheta} + \delta\boldsymbol{\vartheta}) \approx T(\boldsymbol{\vartheta}) + \sum_{i=1}^p \frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\delta\vartheta_i = T(\boldsymbol{\vartheta}) + \xi$ and

$$\xi \sim_1 (-h\boldsymbol{a}'\delta\boldsymbol{\vartheta}, 2h\delta\boldsymbol{\vartheta}'\boldsymbol{S}_{\Sigma^{-1}}\delta\boldsymbol{\vartheta}),$$

where $\boldsymbol{a}' = [\mathrm{Tr}(\boldsymbol{V}_1\boldsymbol{\Sigma}^{-1}), \dots, \mathrm{Tr}(\boldsymbol{V}_p\boldsymbol{\Sigma}^{-1})]$.

P r o o f. Since under the null hypothesis (2)

$$\begin{aligned}
\mathrm{E}\left(\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right) &= -\mathrm{E}\Big([\mathrm{vec}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)]'\Big\{(\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}) \otimes [\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}\Big\} \\
&\times \mathrm{vec}(\boldsymbol{H}\widehat{\boldsymbol{B}} + \boldsymbol{H}_0)\Big) = -\mathrm{Tr}\Big(((\boldsymbol{I} \otimes \boldsymbol{H})[\boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}](\boldsymbol{I} \otimes \boldsymbol{H}')\Big\{(\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}) \\
&\otimes [\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}\Big\}\Big) = -\mathrm{Tr}\Big((\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}) \otimes \Big\{\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}' \\
&\times [\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}\Big\}\Big) = -h\mathrm{Tr}(\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}),
\end{aligned}$$

we have $\mathrm{E}\left(\sum_{i=1}^p \frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\delta\vartheta_i\right) = -h\boldsymbol{a}'\delta\boldsymbol{\vartheta}$.

Further

$$\begin{aligned}
\mathrm{cov}\left(\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}, \frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right) &= 2\,\mathrm{Tr}\Big((\boldsymbol{I} \otimes \boldsymbol{H})[\boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}](\boldsymbol{I} \otimes \boldsymbol{H}')\Big\{(\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}) \\
&\otimes [(\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}\Big\}(\boldsymbol{I} \otimes \boldsymbol{H})[\boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}](\boldsymbol{I} \otimes \boldsymbol{H}')\Big\{(\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_j\boldsymbol{\Sigma}^{-1}) \\
&\otimes [(\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}\Big\}\Big) = 2\mathrm{Tr}\Big[(\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_i\boldsymbol{\Sigma}^{-1}\boldsymbol{V}_j) \otimes \boldsymbol{I}_{h,h}\Big] = 2h\,\{\boldsymbol{S}_{\Sigma^{-1}}\}_{i,j}\,, \\
&\qquad i,j = 1,\dots,p.
\end{aligned}$$

□

Theorem 2.3. If H_0 is true and $\delta\boldsymbol{\vartheta} \in \mathcal{N}_{\vartheta_0}$, where an insensitivity region is

$$\begin{aligned}
\mathcal{N}_{\vartheta_0} &= \left\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \boldsymbol{u}_0)'\boldsymbol{A}_0(\delta\boldsymbol{\vartheta} - \boldsymbol{u}_0) \le c^2\right\}, \boldsymbol{u}_0 = \boldsymbol{A}_0^{-1}h\delta_{\max}\boldsymbol{a}_0, \\
\boldsymbol{A}_0 &= 2t^2h\boldsymbol{S}_{\Sigma_0^{-1}} - h^2\boldsymbol{a}_0\boldsymbol{a}_0', \quad c^2 = \delta_{\max}^2 + h^2\delta_{\max}^2\boldsymbol{a}_0'\boldsymbol{A}_0^{-1}\boldsymbol{a}_0, \\
\boldsymbol{a}_0' &= [\mathrm{Tr}(\boldsymbol{V}_1\boldsymbol{\Sigma}_0^{-1}), \dots, \mathrm{Tr}(\boldsymbol{V}_p\boldsymbol{\Sigma}_0^{-1})],
\end{aligned}$$

then $\mathrm{P}_{H_0}\Big\{T(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta}) \geq \chi^2_{mh}(0;1-\alpha)\Big\} \leq \alpha+\varepsilon$. Here $\delta_{\max}$ is a solution of the equation $\mathrm{P}\left\{\chi^2_{mh}(0)+\delta \geq \chi^2_{mh}(0;1-\alpha)\right\} = \alpha+\varepsilon$ and t is sufficiently large real number.

P r o o f. If H_0 is true, then for a given $\delta\boldsymbol{\vartheta}$ and sufficiently large t the inequality

$$\xi < -h\boldsymbol{a}_0'\delta\boldsymbol{\vartheta} + t\sqrt{2h\delta\boldsymbol{\vartheta}'\boldsymbol{S}_{\Sigma_0^{-1}}\delta\boldsymbol{\vartheta}} \tag{5}$$

occurs with probability near to one. If

$$-h\boldsymbol{a}_0'\delta\boldsymbol{\vartheta} + t\sqrt{2h\delta\boldsymbol{\vartheta}'\boldsymbol{S}_{\Sigma_0^{-1}}\delta\boldsymbol{\vartheta}} < \delta_{\max}, \tag{6}$$

then $\mathrm{P}\left\{\chi^2_{mh}(0)+\xi \geq \chi^2_{mh}(0;1-\alpha)\right\} \leq \alpha+\varepsilon$. The inequality (5) is implied by the inequality $(\delta\boldsymbol{\vartheta}-\boldsymbol{u}_0)'\boldsymbol{A}_0(\delta\boldsymbol{\vartheta}-\boldsymbol{u}_0) \leq c^2$. □

Remark 2.4. The value t need not be larger than 4. In [3] an optimum choice of t was studied for some cases and it was found that the value $t=3$ can be sufficient large.

Corollary 2.5 If $p=1$, i.e. $\boldsymbol{\Sigma}=\sigma^2\boldsymbol{V}$, then the inequality (6) can be rewritten as

$$-h\frac{m}{\vartheta}\delta\vartheta + t\sqrt{2hm\frac{(\delta\vartheta)^2}{\vartheta^2}} < \delta_{\max}.$$

Since $\delta\vartheta$ can be negative in this case, it must satisfy the inequality $\left|\frac{\delta\vartheta}{\vartheta}\right| < \frac{\delta_{\max}}{hm+t\sqrt{2hm}}$, what can be approximated as $\left|\frac{\delta\sigma}{\sigma}\right| < \frac{1}{2}\frac{\delta_{\max}}{hm+t\sqrt{2hm}}$, where $\vartheta=\sigma^2$. From Lemma 2.1 we obtain $\sqrt{\mathrm{Var}(\widehat{\sigma})} = \frac{0.707\sigma}{\sqrt{m(n-k)}}$. In this case the value $\widehat{\vartheta}$, i.e. the matrix $\widehat{\boldsymbol{\Sigma}}=\widehat{\vartheta}\boldsymbol{V}$ can be used in the test (3) instead the actual value if the following inequality

$$\frac{1}{2}\frac{\delta_{\max}}{hm+t\sqrt{2hm}} \gg t\frac{0.707}{\sqrt{m(n-k)}}$$

is satisfied. If $\alpha=0.05$, $\varepsilon=0.05$, $m=5$, $h=4$, $t=3$, then $n-k \gg 617$. It is quite clear that a requirement on the accuracy of the estimator $\widehat{\vartheta}$ can be rigorous.

In the case $p=1$ obviously the test from Corollary 1.5 must be used. The example is given only for a demonstration how large the necessary number of observations can be.

Remark 2.6. If the matrix $2t^2h\boldsymbol{S}_{\Sigma_0^{-1}} - h^2\boldsymbol{a}_0\boldsymbol{a}_0'$ is not p.d., then from the practical purposes in the spectral decomposition $2t^2h\boldsymbol{S}_{\Sigma_0^{-1}} - h^2\boldsymbol{a}_0\boldsymbol{a}_0' = \sum_{i=1}^m \lambda_i\boldsymbol{f}_i\boldsymbol{f}_i'$ the negative eigenvalues λ_i are substituted by their absolute values $|\lambda_i|$. In this way the shape of the insensitivity region $\mathcal{N}_{\vartheta_0}$ is always ellipsoid.

Remark 2.7. If $p \geq 2$, and only $\widehat{\boldsymbol{\Sigma}} = \sum_{i=1}^{p} \widehat{\vartheta}_i \boldsymbol{V}_i$ is at our disposal, the matrix $\widehat{\boldsymbol{\Sigma}}$ can be used in the test (3) in such case only that $\widehat{\delta\boldsymbol{\vartheta}} \in \mathcal{N}_{\vartheta_0}$ with certainty. Thus a consideration on the basis of $\mathrm{Var}(\widehat{\boldsymbol{\vartheta}})$ from Lemma 2.1 must be made.

If the estimator $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n-k}(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})$ is at our disposal only and the test (3) is to be used, the analogous consideration as in Theorem 2.3 can be made.

Let $\boldsymbol{A}*\boldsymbol{B}$ means the Hadamard product of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, i. e. $\{\boldsymbol{A}*\boldsymbol{B}\}_{i,j} = A_{i,j}B_{i,j}$ and $\mathrm{diag}(\boldsymbol{\Sigma})$ means the vector composed of the entries of the diagonal of the matrix $\boldsymbol{\Sigma}$.

If $\boldsymbol{W} \sim W_m(n-k, \boldsymbol{\Sigma})$, then

$$\boldsymbol{K} = \frac{1}{n-k}\left\{\mathrm{diag}(\boldsymbol{\Sigma})[\mathrm{diag}(\boldsymbol{\Sigma})]' + \boldsymbol{\Sigma}*\boldsymbol{\Sigma}\right\} \tag{7}$$

is the matrix with the following property. Its (i,j)th entry is the dispersion of $\widehat{\sigma}_{i,j} = \{\boldsymbol{W}\}_{i,j}/(n-k)$.

If $\delta\boldsymbol{\Sigma}$ is a matrix of infinitesimal shifts of the entries of the matrix $\boldsymbol{\Sigma}$, it is valid under the null hypothesis H_0:

$$\begin{aligned} T(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma}) \approx \mathrm{Tr}\Big\{&(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0) \\ &\times(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1})\Big\} = \chi^2_{mh}(0) + \xi, \end{aligned}$$

where

$$\xi = -\mathrm{Tr}\Big\{(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\widehat{\boldsymbol{B}}+\boldsymbol{H}_0)\boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\Big\}.$$

Further

$$\xi \sim_1 \Big[-h\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\Sigma}), 2h\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\delta\boldsymbol{\Sigma})\Big].$$

Theorem 2.8. If H_0 is true and $\delta\boldsymbol{\Sigma} \in \mathcal{N}_{\Sigma_0}$, where

$$\begin{aligned} \mathcal{N}_{\Sigma_0} &= \Big\{\delta\boldsymbol{\Sigma} : \big[\mathrm{vec}(\delta\boldsymbol{\Sigma}) - \boldsymbol{u}_0\big]'\boldsymbol{A}_0\big[\mathrm{vec}(\delta\boldsymbol{\Sigma}) - \boldsymbol{u}_0\big] \leq c^2\Big\}, \\ \boldsymbol{u}_0 &= h\delta_{\max}\boldsymbol{A}_0^{-1}\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1}), \\ \boldsymbol{A}_0 &= 2t^2h(\boldsymbol{\Sigma}_0\otimes\boldsymbol{\Sigma}_0) - h^2\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})[\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})]', \\ c^2 &= \delta^2_{\max} + h^2\delta^2_{\max}[\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})]'\boldsymbol{A}_0^{-1}[\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})], \end{aligned}$$

$$\mathrm{P}\left\{\chi^2_{mh}(0) + \delta_{\max} \geq \chi^2_{mh}(0; 1-\alpha)\right\} = \alpha + \varepsilon,$$

then

$$\mathrm{P}\{T(\boldsymbol{\Sigma}_0 + \delta\boldsymbol{\Sigma}) \geq \chi^2_{mh}(0; 1-\alpha)\} \leq \alpha + \varepsilon.$$

Proof is analogous as in Theorem 2.3. □

Remark 2.9. Let $\boldsymbol{k} = \text{vec}(\boldsymbol{K})$ from (7) and $\sqrt{\{\boldsymbol{k}\}_i} = \{\boldsymbol{l}\}_i,\ i = 1, \dots, m^2$. The vector $\boldsymbol{l}$ is composed of the standard deviations $\sqrt{\text{Var}(\widehat{\sigma}_{i,j})} = l_{i,j}$ of the estimators $\frac{1}{n-k}\{(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})'(\underline{\boldsymbol{Y}} - \boldsymbol{X}\widehat{\boldsymbol{B}})\}_{i,j}$ of $\{\boldsymbol{\Sigma}\}_{i,j} = \sigma_{i,j}$. The vector $\boldsymbol{l}$ generates the class of 2^{m^2} vectors which have the same absolute values of their coordinates, however different signs, e. g.

$$\boldsymbol{r} = (+l_{1,1}, -l_{1,2}, \dots, +l_{1,m}, \dots, +l_{2,1}, \dots, +l_{2,m}, \dots, -l_{m,1}, \dots, -l_{m,m})'.$$

Now if the vectors $\boldsymbol{r}$ are sufficiently small with respect to the set $\mathcal{N}_{\Sigma_0}$, i. e.

$$-h[\text{vec}(\boldsymbol{\Sigma}_0^{-1})]'\boldsymbol{r} + t\sqrt{2h\boldsymbol{r}'(\boldsymbol{\Sigma}_0^{-1} \otimes \boldsymbol{\Sigma}_0^{-1})\boldsymbol{r}} \ll \delta_{\max},$$

then the estimator of $\boldsymbol{\Sigma}$ can be used in the test (3). This check is rather rough, nevertheless for the first orientation is sufficient.

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