KYBERNETIKA — VOLUME 34 (1998), NUMBER 5, PAGES 595–607

A REDUCTION PRINCIPLE FOR GLOBAL STABILIZATION OF NONLINEAR SYSTEMS

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The goal of this paper is to propose new sufficient conditions for dynamic stabilization of nonlinear systems. More precisely, we present a reduction principle for the stabilization of systems that are obtained by adding integrators. This represents a generalization of the well-known lemma on integrators (see for instance [3] or [16]).

1. INTRODUCTION

In this paper we address the following problem: Consider a smooth system on $\mathbb{R}^n$

$$\dot{x} = f(x, u) \tag{1}$$

with an equilibrium point $f(0,0) = 0$, which is globally stabilizable by a smooth feedback $\overline{u}(x)$ at the origin.

The problem is to determine a class of functions $g : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^+$ for which it is true that the extended system:

$$\begin{cases} \dot{x} = h(x,y)\, f(x, g(x,y)) \\ \dot{y} = u \end{cases} \tag{2}$$

is smoothly stabilizable.

Our goal is to provide some answers to this problem. We stress that the feedbacks we consider are smooth or analytic. Throughout the paper we use asymptotic stability to refer to global asymptotic stability.

When the function h, g satisfy $h(x,y) \equiv 1$ and $g(x,y) = y$ we have the very well-known classical lemma on integrators [3, 5, 6, 10, 11, 15, 16, 17] which says that if (1) is globally asymptotically stabilizable by a smooth feedback control then the system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = u \end{cases} \tag{3}$$

is also globally asymptotically stabilizable by a smooth feedback control.

Answers to our problem will provide conditions for which a system (3) is asymptotically stabilizable, or equivalently conditions for which a system (1) is stabilizable by a dynamic feedback. To be more precise if we have a system

$$\begin{cases} \dot{x} = F(x, y) \\ \dot{y} = u \end{cases} \tag{4}$$

we shall try to rewrite $F(x, y) = h(x, y)\, f(x, g(x, y))$ for some functions f, g, h. Then we shall deduce the stabilizability of (4) from the stabilizability of the reduced system $\dot{x} = f(x, u)$. This reduced system is different from the system $\dot{x} = F(x, u)$ obtained by the classical lemma on integrators.

Our method is designed to prove stabilizability of system with integrators. We illustrate the effectiveness of our technique by considering the Coron and Praly example (see [4]). They give an example of a system $\dot{x} = F(x, u)$ which is not stabilizable even by a continuous feedback. We prove by our "different reducing" technique that the corresponding system with integrator is globally stabilizable by a polynomial feedback. This is an improvement of the result of Coron–Praly which provide an almost analytic feedback.

We show further the relevance of our method by considering a class of planar systems studied by Hermes (see [7]). By taking advantage of more possibilities for reduction we exhibit polynomial stabilizing feedback instead of only C^1 as obtained by Hermes.

The paper is organized as follows: In Section 2 we give two main results. The first result pertains to the case $h(x, y) \equiv 1$. The Coron–Praly's example is of this kind. We demonstrate the advantage of this method by finding a polynomial feedback. The second result provides stability for a class of positive definite functions h. In Section 3 we discuss a class of planar systems which has been studied by Hermes in [7]:

$$\begin{cases} \dot{x} = y^k + cx^n y^m \\ \dot{y} = u \end{cases} \tag{5}$$

we give conditions on n, m, k under which the system (5) is stabilizable by polynomial feedback.

2. MAIN RESULTS

We begin by reminding the reader about the basic facts concerning Lyapunov functions. A Lyapunov function is said to be strict (see for instance [15]) for a system with a feedback $\overline{u}(x)$ if the derivative $\dot{V}$ of V along the trajectories of the closed-loop system is a negative definite function.

Throughout the paper we use $\mathcal{C}$ to denote the class of nonnegative smooth functions $\Phi : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}$ such that:

1. $\Phi(0, y) = 0$ if and only if $y = 0$.

2. The function $y \to \Phi(x,y)$ is radially unbounded for any x and $\frac{\partial\Phi}{\partial y}(0,y)=0$ implies $y=0$.

If f is a differentiable function we denote the action of a smooth vector field X, considered as a differential operator, on f by Xf. We define, by recurrence, $X^k f = X(X^{k-1}f)$ for $k>1$.

The following is our first main result:

Theorem 2.1. *Let a system (1)* $\dot{x}=f(x,u)$ be asymptotically stabilizable by a smooth feedback control $\overline{u}(x)$.

For any smooth function $g:\mathbb{R}^{m+n}\to\mathbb{R}^n$ and Φ in $\mathcal{C}$ such that:

1.
$$g(x,y)=\overline{u}(x)+\frac{\partial\Phi}{\partial y}(x,y) \tag{6}$$

2. There exists a smooth function $k(x,y)$ for which

$$\beta(x,y)=\left\langle f(x,\overline{u}(x)),\frac{\partial\Phi}{\partial x}(x,y)\right\rangle+\left\langle k(x,y),\frac{\partial\Phi}{\partial y}(x,y)\right\rangle\leq 0 \tag{7}$$

the system

$$\begin{cases}\dot{x}=f(x,g(x,y))\\ \dot{y}=u\end{cases} \tag{8}$$

is asymptotically stabilizable by a smooth feedback control.

Proof. Since $\dot{x}=f(x,u)$ is asymptotically stabilizable, there exists a strict Lyapunov function V such that $\langle f(x,\overline{u}(x)),\nabla V(x)\rangle<0$ if $x\neq 0$. We have

$$f(x,g(x,y))=f(x,\overline{u}(x))+G(x,y)\,(g(x,y)-\overline{u}(x))$$

where $G(x,y)$ is defineded by $G(x,y)=\int_0^1\frac{\partial f}{\partial u}(x,tg(x,y)+(1-t)\overline{u}(x))\,dt$. If we define now $W(x,y)=V(x)+\Phi(x,y)$, this function is a Lyapunov function and we have

$$\begin{aligned}\dot{W}(x,y) \;=\;& \langle f(x,\overline{u}(x)),\nabla V(x)\rangle+\left\langle G(x,y)\,\frac{\partial\Phi}{\partial y}(x,y),\nabla V(x)\right\rangle\\ &+\left\langle f(x,\overline{u}(x)),\frac{\partial\Phi}{\partial x}(x,y)\right\rangle+\left\langle G(x,y)\,\frac{\partial\Phi}{\partial y}(x,y),\frac{\partial\Phi}{\partial x}(x,y)\right\rangle\\ &+\left\langle u,\frac{\partial\Phi}{\partial y}(x,y)\right\rangle.\end{aligned}$$

So that if we choose

$$u(x,y)=-G^T(x,y)\,\left(\nabla V(x)+\frac{\partial\Phi}{\partial x}(x,y)\right)+k(x,y)-\frac{\partial\Phi}{\partial y}(x,y),$$

we have, with the notation of the theorem,

$$\dot{W}(x,y) = \langle f(x,\overline{u}(x)), \nabla V(x)\rangle + \beta(x,y) - \left\| \frac{\partial \Phi}{\partial y}(x,y) \right\|^2 < 0 \quad \text{for} \quad (x,y) \neq (0,0).$$

As a consequence of the Theorem 2.1 hypothesis this quantity is clearly definite negative. This ends the proof of the theorem. □

We now illustrate the use of Theorem 2.1 by means of two examples.

Example 2.1. (Coron–Praly [4]) Let F be the function $F : \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ defined by

$$F(x_1, x_2, y) = -\left[\left(\|x_1\|^2 + x_2^2\right)^3 - C^2\left(y^3 + x_2^3 - \|x_1\|^2 y\right)^2\right] x.$$

where $C > 0$ large enough.

It is shown in [4] that the extended system $\{\dot{x} = F(x,y); \dot{y} = u\}$ (4) is asymptotically stabilizable by an almost smooth feedback control, even though the reduced system $\dot{x} = F(x,u)$ is not even stabilizable by a continuous feedback.

Let write the above system as:

$$\begin{cases} \dot{x} = f(x, g(x,y)) \\ \dot{y} = u \end{cases} \tag{8}$$

with $g(x,y) = y^3 - \|x_1\|^2 y + x_2^3$ and $f(x,z) = -\left[\left(\|x_1\|^2 + x_2^2\right)^3 - C^2 z^2\right] x$.

Then, by Theorem 2.1, the reduced system becomes

$$\dot{x} = -\left[\left(\|x_1\|^2 + x_2^2\right)^3 - C^2 u^2\right] x\,. \tag{9}$$

It is clear that (9) is stabilizable by $\overline{u}(x) \equiv 0$. Let $V(x) = \frac{1}{4}\left(\|x_1\|^2 + x_2^2\right)^2$ and

$$\Phi(x,y) = \frac{1}{4}y^4 - \frac{1}{2}\|x_1\|^2 y^2 + x_2^3 y + \left(\|x_1\|^2 + x_2^2\right)^2\,.$$

It is easy to check that $g(x,y) = \frac{\partial \Phi}{\partial y}(x,y)$. If we set $k(x,y) = -\left(\|x_1\|^2 + x_2^2\right)^3 y$, by using the notation of Theorem 2.1 $\beta(x,y) = -4\left(\|x_1\|^2 + x_2^2\right)^3 \Phi(x,y)$, is clearly nonpositive.

All the requirements of Theorem 2.1 are satisfied, hence

$$\begin{aligned} u(x_1,x_2,y) &= -C^2\left(y^3 - \|x_1\|^2 y + x_2^3\right)\left(5\left(\|x_1\|^2 + x_2^2\right)^2 - \|x_1\|^2 y^2 + 3x_2^3 y + \frac{1}{C^2}\right) \\ &\quad - \left(\|x_1\|^2 + x_2^2\right)^3 y \end{aligned}$$

is a polynomial stabilizing feedback for the system.

Remark 2.2. The above example shows that a system which is not stabilizable even by means of a continuous state feedback can be stabilizable by a polynomial dynamic feedback.

Example 2.2. (Hu [8]) Consider the following planar system

$$\begin{cases} \dot{x}_1 = x_1^6 - x_1^4 x_2^5 - x_2^{10} \\ \dot{x}_2 = u\,. \end{cases} \tag{10}$$

It is shown in [8] that (10) is stabilizable locally by means of a continuous feedback. Here, we prove that (10) is stabilizable globally by means of a polynomial feedback.

Let write system (10) as:

$$\begin{cases} \dot{x}_1 = f(x_1, g(x_2)) \\ \dot{x}_2 = u\,. \end{cases}$$

with $g(y) = y^5$ and $f(x, z) = x_1^6 - x_1^4 z - z^2$.

Using Theorem 2.1, the reduced system is

$$\dot{x}_1 = x_1^6 - x_1^4 u - u^2\,. \tag{11}$$

Straightforward computations show that (11) is stabilizable by $\bar{u}(x_1) = x_1^3$.
Let

$$\Phi(x_1, x_2) = \frac{1}{6}x_2^6 - x_1^3 x_2 + \frac{a}{2}x_1^2 + \frac{b}{6}x_1^6\,.$$

where a and b are positive real numbers large enough.

Clearly $\Phi \in C$ and $g(x_2) = \bar{u}(x_1) + \frac{\partial \Phi}{\partial x_2}$.

For $k(x_1, x_2) = -x_1^8 x_2^2 (x_2^5 - x_1^3)$ we obtain

$$\beta(x_1, x_2) = x_1^8 \left(3x_1 x_2 - a - bx_1^4 - x_2^2(x_2^5 - x_1^3)^2\right)\,.$$

In order to show that $\beta(x_1, x_2) \leq 0$, we discuss two cases. First, assume that $|x_2|^5 \leq |x_1|^3 + 1$. Then, for a and b large enough $3|x_1 x_2| \leq a + bx_1^4$ which clearly implies that $\beta(x_1, x_2) \leq 0$. Now if $|x_2|^5 > |x_1|^3 + 1$ then

$$\beta(x_1, x_2) \leq -x_1^8 \left(-3x_1 x_2 + a + bx_1^4 + x_2^2\right) \leq 0\,.$$

Finally using the result of Theorem 2.1, with $V(x) = x^2$, we deduce that (10) is stabilizable by means of a polynomial feedback.

Remark 2.2. The system (10) if reduced by means of the classical lemma on integrators becomes $\dot{x}_1 = x_1^6 - x_1^4 u^5 - u^{10}$ which is not C^1 stabilizable.

Now, we state and prove our second main result.

Theorem 2.2. Assume that the system *(1)* $\dot{x} = f(x,u)$ be asymptotically stabilizable by a smooth feedback control $\overline{u}(x)$, with $V(x)$ a strict Lyapunov function for the closed-loop system. Then for any smooth functions $g : \mathbb{R}^m \to \mathbb{R}^m$ which vanishes only at the origin, $h : \mathbb{R}^m \to \mathbb{R}^+$ a definite function such that all the following properties are satisfied:

1.
$$h(y)(g(y) - \overline{u}(x)) = \frac{\partial \Phi}{\partial y}(x,y), \tag{12}$$
for a function Φ in $\mathcal{C}$, and

2. There exists a smooth function $k(x,y)$, such that the quantity:
$$\beta(x,y) = h(y)\left\langle f(x,\overline{u}(x)), \frac{\partial \Phi}{\partial x}(x,y)\right\rangle + \left\langle k(x,y), \frac{\partial \Phi}{\partial y}(x,y)\right\rangle \leq 0, \tag{13}$$
and

3. $\displaystyle\int_0^1 \frac{\partial f}{\partial u}^T (x, t\overline{u}(x))\, dt \left(\nabla V(x) + \frac{\partial \Phi}{\partial x}(x,0)\right) = 0$ implies $x = 0$

the system
$$\begin{cases} \dot{x} = h(y)\, f(x, g(y)) \\ \dot{y} = u \end{cases} \tag{14}$$
is asymptotically stabilizable by a smooth feedback control.

Proof. We consider $W(x,y) = V(x) + \Phi(x,y)$ as in the proof of Theorem 2.1. Using the fact that,
$$\begin{aligned} h(y)\, f(x,y) &= h(y) + G(x,y)h(y)(g(y) - \overline{u}(x)) \\ &= h(y)\, f(x,\overline{u}(x)) + G(x,y)\frac{\partial \Phi}{\partial y}(x,y) \end{aligned}$$
if we choose
$$u(x,y) = -G^T(x,y)\frac{\partial \Phi}{\partial x}(x,y) - G^T(x,y)\nabla V(x) + k(x,y) - \frac{\partial \Phi}{\partial y}(x,y). \tag{15}$$
The derivative of W along the trajectories of the closed-loop system is given by
$$\dot{W}(x,y) = h(y)\,\langle f(x,\overline{u}(x)), \nabla V(x)\rangle - \left\|\frac{\partial \Phi}{\partial y}(x,y)\right\|^2 + \beta(x,y).$$
This quantity is nonnegative according to the hypothesis of the theorem.

We shall use the LaSalle's invariance principle. We shall prove first by contradiction that if $\dot{W}(x,y) = 0$ then $y = 0$. Suppose $y \neq 0$, then $h(y) \neq 0$, therefore we have $\langle f(x,\overline{u}(x)), \nabla V(x)\rangle = 0$ which in turns implies $x = 0$, since $\frac{\partial \Phi}{\partial y}(0,y) = 0$ implies $y = 0$ we have a contradiction.

Then on the set $\{\dot{W} = 0\}$ we have $y = 0$. Now in the largest invariant set in $\{\dot{W} = 0\}$, since $\dot{y} = u$, we must have,

$$u(x,0) = -G^T(x,0)\left(\nabla V(x) + \frac{\partial \Phi}{\partial x}(x,0)\right) = 0$$

and since this relation implies $x = 0$ by the hypothesis 3 of the theorem, the proof is finished. □

Remark 2.3. In the preceding theorems we require for the reduced system that the Lyapunov function $V(x)$ is strict. All the preceding arguments extent to the situation which is covered by the Lasalle's invariance principle. The argument is as follows:

Let a smooth reduced system $\dot{x} = f(x,u)$ be stabilizable by means of a smooth feedback law $u = \bar{u}(x)$. Let V be a known Lyapunov function such that if we denote by X the closed-loop vector field $X(x) = f(x,\overline{u}(x))$, we have $XV(x) = \langle X(x), \nabla V(x)\rangle \leq 0$ for all x in $\mathbb{R}^n$ and so that

$$\{x \in \mathbb{R}^n | X^k V(x) = 0\,; k \in \mathbb{N}\} = \{0\}.$$

With this result it is clear that the proofs of the theorems are unchanged, and the same computed feedbacks stabilize the system. In fact, in this case we obtain similar, but more general, results than those established in [9]. Note that since, in general, it is not easy to find a strict Lyapunov function V even if the global asy mptotic stability of the origin for the closed-loop reduced system is proved, the advantage of this approach is that the stabilizing feedback, for the augmented system, is given explicitly in some situations when with the preceding theorems we prove only the existence.

Remark 2.4. Note that the feedback obtained is smooth. If we relax the condition (7) of Theorem 2.1 by the condition

$$\left(\frac{\partial \Phi}{\partial y}(x,y) = 0\right) \Longrightarrow \left\langle f(x,\overline{u}(x)), \frac{\partial \Phi}{\partial x}(x,y)\right\rangle \leq 0$$

we only obtain an almost smooth feedback (i.e. smooth excepted at the origin). The argument is as follows: This condition is a necessary condition for the existence of $k(x,y)$. The condition is also sufficient for the existence of an almost smooth function $k(x,y)$ as we shall see by using an argument analogous to the one used by Sontag in his proof of Arstein's Theorem (see [14]):

Let φ the function defined by

$$\varphi(\alpha,\beta) = \begin{cases} \frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta} & \text{if } \beta > 0 \\ 0 & \text{if } \beta = 0 \end{cases}$$

then

$$k(x,y) = -\varphi\left(f(x,\overline{u}(x)), \left\|\frac{\partial \Phi}{\partial y}(x,y)\right\|^2\right)\frac{\partial \Phi}{\partial y}(x,y)$$

satisfies condition (7). The corresponding result for condition (13) of Theorem 2.2 is obtained by a similar computation which we shall omit.

3. THE PLANAR SYSTEMS OF HERMES

In his interesting paper [7] Hermes consider the following planar systems

$$\begin{cases} \dot{x} = y^k + cx^n y^m \\ \dot{y} = u \qquad\qquad c \neq 0. \end{cases} \tag{5}$$

This author proves that when the nonnegative integers k, m, n satisfy

$$\begin{aligned} &\text{(i)} \qquad k \text{ is odd} \\ &\text{(ii)} \qquad \frac{n(k+1)}{2} \geq k - m \geq n \geq 1 \end{aligned}$$

the control system (5) admits a C^1 global feedback control. As an application of our theorems we shall prove that (5) is globally asymptotically stabilizable by polynomial feedback when

$$k \text{ is odd}$$

$$k \leq m \text{ or} \tag{16}$$

$$k > m \text{ and } (n \geq 2 \text{ or } m \text{ is odd}) \text{ or} \tag{17}$$

$$k > m \text{ and } n = 1 \text{ and } 2m + 1 \geq k. \tag{18}$$

In other words when $k < m$ or ($k > m$ and $n = 2$) we can relax the left side of the inequality (ii) of Hermes. Then our conditions are larger, moreover our feedback control is polynomial and explicitly exhibited. To prove this result we shall distinguish two cases.

3.1. $k \leq m$

In this case, and in order to stabilize system (5), we use the Jurdjevic–Quinn's method (see for instance [13] and references therein): After the preliminary feedback

$$u = -x\left(1 + y^{m-k}x^n\right) + v$$

it is easy to verify that the Lyapunov function

$$V(x,y) = \frac{1}{k+1}y^{k+1} + \frac{1}{2}x^2$$

is a first integral for the drift term and simple reasoning shows that

$$\left\{(x,y) \in \mathbb{R}^2 \mid X^k Y V(x,y) = 0; k \in \mathbb{N}\right\} = \{(0,0)\}$$

where X represents the drift term and Y is defined by $Y(x,y) = (0\ 1)^t$.

Finally the closed-loop system defined from (5) with

$$u(x,y) = -x(1 + cy^{m-k}x^n) - y^k,$$

is asymptotically stable at the origin.

3.2. $k > m$

In this case we rewrite the system (5):

$$\begin{cases} \dot{x} = y^m(y^p + cx^n) \\ \dot{y} = u \end{cases} \tag{19}$$

with $p = k - m$ a positive integer. The study of this case is divided in three sub-cases.

3.2.1. m is odd

Using similar arguments as those given in the case $k \leq m$ it is straightforward to check that with the Lyapunov function

$$V(x,y) = \frac{1}{m+1}y^{m+1} + \frac{1}{2}x^2$$

and with the feedback

$$u(x,y) = -x\,(y^p + cx^n) - y^m$$

the closed-loop system is asymptotically stable at the origin.

3.2.2. m is even and $n \geq 2$

The stabilizability of (19) will be shown using Theorem 2.2. In Theorem 2.2 appears a function $\Phi(x,y)$ with some properties. To obtain this kind of function we need a lemma.

Lemma 3.1. Let

$$\Phi(x,y) = \alpha y^{2r} + \beta x^n y^{2p+1} + ax^2 + bx^{2(p+1)n}$$

where r,n,p, are integers such that $n \geq 2$,$r > p$, α and β are given real with $\alpha > 0$.

For a and b positive, sufficiently large, Φ is a positive definite function.

The proof of this lemma is straightforward, and we shall omit it.

Now we shall say that a polynomial is odd (resp. even) if the corresponding polynomial function is odd (resp. even), i.e. all the monomial are of odd (resp. even) degree. By a finite repeated use of the preceding lemma we get the corollary:

Corollary 3.1. For any real $\alpha > 0$, any integers r, $n \geq 2$, any odd polynomial P of degree strictly less than $2r$, there exists an even polynomial Q of degree less than $(\deg(P)+1)n$ such that the function

$$\Phi(x,y) = \alpha y^{2r} + P(y)x^n + Q(x)$$

is positive definite.

From these results we know that for a, b, d, positive reals large enough

$$\begin{aligned}\Phi(x,y) &= \frac{1}{m+p+1}y^{m+p+1} + \frac{1}{m+1}cx^n y^{m+1} \frac{1}{m+1}x^3y^{m+1} \\ &\quad + ax^2 + bx^{m+2}n + dx^{3(m+2)}\end{aligned}$$

is positive definite. With the notation of Theorem 2.2, let $g(y) = y^p$ and $h(y) = y^m$, so that the reduced system is $f(x,u) = u + cx^n$.

The feedback $\overline{u}(x) = -cx^n - x^3$ stabilizes the reduced system and we have $\frac{\partial \Phi}{\partial y}(x,y) = y^m(y^p + cx^n + x^3)$. The conditions 1 and 2 of the Theorem 2.2 are satisfied. If we define $k(x,y) = -x^2y^{m+1}$ we have

$$\begin{aligned}\beta(x,y) &= -x^2y^m\left[y^{m+p+1} + c\left(1 - \frac{n}{m+1}\right)x^n y^{m+1} + \left(1 + \frac{3}{m+1}\right)x^3y^{m+1}\right. \\ &\quad \left. + 2ax^2 + (m+2)nbx^{(m+2)n} + 3(m+2)\,dx^{3(m+2)}\right].\end{aligned}$$

From the Corollary 3.1 we know that this quantity is negative (m is even) provided a,b,d are chosen large enough. The condition 3 of Theorem 2.2 is satisfied. The relation 4 is simply

$$(x + 2ax + (m+2)nbx^{(m+2)n-1} + 3(m+2)x^{3(m+2)-1} = 0) \Rightarrow x = 0\,.$$

Since all the conditions of Theorem 2.2 are satisfied, it is now easy to give the feedback, by using relation (15) in the proof of the theorem. The feedback control is polynomial.

3.2.3. m is even, $n = 1$ and $2m + 1 \geq k$

We shall prove the stabilization of (19) by means of Theorem 2.2. With the current hypothesis (19) is now

$$\begin{cases} \dot{x} = y^{m-p+1}y^{p-1}(y^p + cx) \\ \dot{y} = u\,. \end{cases} \tag{20}$$

We shall prove first that the system

$$\begin{cases} \dot{x} = y^{p-1}(y^p + cx) \\ \dot{y} = u \end{cases} \tag{21}$$

is stabilizable.

With the notation of Theorem 2.2, we let $g(y) = y^p$, $h(y) = y^{p-1}$, $f(x,u) = u + cx$, $\overline{u}(x) = -(c+1)x$, $\Phi(x,y) = \frac{1}{2p}y^{2p} + \frac{c+1}{p}y^p x + \frac{a}{2}x^2$, with $a > \frac{(c+1)^2}{p}$, and $k(x,y) = \frac{c+1}{p}x$.

It is straightforward to check that conditions 1 and 3 of Theorem 2.2 are satisfied. On the other hand, we have

$$\beta(x,y) = -x^2 y^{p-1}\left(a - \frac{(c+1)^2}{p^2}\right) \leq 0.$$

This implies that condition 2 is satisfied. Then, by Theorem 2.2, system (21) is stabilizable by a polynomial feedback.

In order to conclude that (20) is asymptotically stabilizable, we need a lemma.

Lemma 1. Assume that the system

$$\dot{x} = f(x) + ug(x) \tag{22}$$

is stabilizable. Then, for any nonnegative continuous scalar function h which is such that $h(x) = 0$ implies $f(x) = 0$, the system

$$\dot{x} = h(x)f(x) + ug(x) \tag{23}$$

is stabilizable.

Proof. Let $\bar{u}(x)$ be a stabilizing feedback for system (22). By the converse theorem of Lyapunov there exists a smooth Lyapunov function V with time derivative negative along the trajectories of the closed-loop system defined from (22) controlled $u = \bar{u}(x)$

$$\dot{V}_{(22)}(x) = \langle f(x), \nabla V(x)\rangle + \bar{u}(x)\langle g(x), \nabla V(x)\rangle$$

is negative definite. Consider now the following feedback

$$\bar{u}_1(x) = h(x)\bar{u}(x) - \langle g(x), \nabla V(x)\rangle . \tag{24}$$

The derivative of V along the trajectories of the closed-loop system (23), (24) is given by

$$\dot{V}_{(23)}(x) = h(x)\,\dot{V}_{(22)}(x) - \langle g(x), \nabla V(x)\rangle^2 .$$

Since $\dot{V}_{(22)}$ is negative definite we get

$$\left\{\dot{V}_{(23)}(x) = 0\right\} = \{h(x) = 0, \langle g(x), \nabla V(x)\rangle = 0\} .$$

Now, $\langle g(x), \nabla V(x)\rangle = 0$ implies that $\langle f(x), \nabla V(x)\rangle \neq 0$ or $x = 0$, then we have

$$\{\dot{V}_{(23)}(x) = 0\} \subset \{h(x) = 0, \langle f(x), \nabla V(x)\rangle \neq 0\} \cup \{0\}$$

which implies, since the hypothesis of the lemma, that $\{\dot{V}_{(23)}(x) = 0\}$ is reduced to $\{0\}$. □

Using the preceding lemma we get the following corollary:

Corollary 3.2. If the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = u \end{cases} \tag{25}$$

with $f(x, 0) = 0$, is stabilizable then the system

$$\begin{cases} \dot{x} = h(y)\, f(x, y) \\ \dot{y} = u \end{cases} \tag{26}$$

is stabilizable for any positive definite smooth function h.

The Corollary 3.2 proves that (20) is stabilizable when $p > 1$, since we need $f(x, 0) \equiv 0$. When $p = 1$, one can use similar arguments as those given in the case $k \leq m$ to show that with $u(x, y) = -y^m(cy + 2c^2x) - (y + cx)$, and using $V(x, y) = \frac{1}{2}y^2 + cxy + c^2x^2$ as Lyapunov function, the closed-loop system is asymptotically stable at the origin.

This ends the case 3.2.2.

(Received August 20, 1996.)

REFERENCES

[1] D. Aeyels: Stabilization of a class of non-linear systems by smooth feedback control. Systems Control Lett. *5* (1985), 289–29.

[2] Z. Arstein: Stabilization with relaxed controls. Nonlinear Anal. Theory Method Appl. *11* (1983), 1163–1173.

[3] C. Byrnes and A. Isidori: New results and examples in nonlinear feedback stabilization. Systems Control Lett. *12* (1989), 437–442.

[4] J. M. Coron and L. Praly: Adding an integrator for the stabilization problem. Systems Control Lett. *17* (1991), 84–104.

[5] W. P. Dayawansa and C. F. Martin: Asymptotic stabilization of two dimensional real analytic systems. Systems Control Lett. *12* (1989), 205–211.

[6] W. P. Dayawansa, C. F. Martin and G. Knowles: Asymptotic stabilization of a class of smooth two dimensional systems. SIAM J. Control Optim. *28* (1990), 1321–1349.

[7] H. Hermes: Asymptotic stabilization of planar systems. Systems Control Lett. *17* (1991), 437–443.

[8] X. Hu: Stabilization of planar nonlinear systems by polynomial feedback control. Systems Control Lett. *22* (1994), 177–185.

[9] A. Iggidr and G. Sallet: Nonlinear stabilization by adding integrators. Kybernetika *30* (1994), 5, 499–506.

[10] D. E. Koditschek: Adaptative techniques for mechanical systems. In: 5th Yale Workshop on Adaptative Systems, 1987, pp. 259-265.

[11] P. V. Kokotovic and H. J. Sussmann: A positive real condition for global stabilization of nonlinear systems. Systems Control Lett. *13* (1989), 125–133.

[12] J. P. Lasalle and S. Lefschetz: Stability by Lyapunov Method with Applications. Academic Press, New York 1961.

[13] R. Outbib and G. Sallet: Stabilizability of the angular velocity of a rigid body revisited. Systems Control Lett. *18* (1991), 93–98.

[14] E. D. Sontag: A universal construction of Arstein's theorem on nonlinear stabilization. Systems Control Lett. *13* (1989), 117–123.
[15] E. D. Sontag and H. J. Sussmann: Further comments on the stabilizability on the angular velocity of a rigid body. Systems Control Lett. *12* (1989), 213–217.
[16] J. Tsinias: Sufficient Lyapunov–like conditions for stabilization. Math. Control Signals Systems *18* (1989), 343–357.
[17] J. Tsinias: A local stabilization theorem for interconnected systems. Systems Control Lett. *18* (1992), 429–434.

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