Chapter 17

Measurements I

17.1 Introduction

I place a tape measure with one end on the floor next to a table, read the height of the table from the tape, and record the result in a notebook. What are the essential features of this *measurement process*? The key point is the establishment of a *correlation* between a *physical property* (the height) of a *measured system* (the table) and a suitable *record* (in the notebook), which is itself a physical property of some other system. It will be convenient in what follows to think of this record as part of the measuring apparatus, which consists of everything essential to the measuring process apart from the measured system. Human beings are not essential to the measuring process. The height of a table could be measured by a robot. In the modern laboratory, measurements are often carried out by automated equipment, and the results stored in a computer memory or on magnetic tape, etc. While scientific progress requires that human beings pay attention to the resulting data, this may occur a long time after the measurements are completed.

In this and the next chapter we consider measurements as physical processes in which a *property* of some quantum system, which we shall usually think of as some sort of "particle", becomes correlated with the *outcome* of the measurement, itself a property of another quantum system, the "apparatus". Both the measured system and the apparatus which carries out the measurement are to be thought of as parts of a single closed quantum-mechanical system. This makes it possible to apply the principles of quantum theory developed in earlier chapters. There are no special principles which apply to measurements in contrast to other quantum processes. We need an appropriate Hilbert space for the measured system plus apparatus, some sort of initial state, unitary time-development operators, and a suitable framework or consistent family of histories. There are, as always, many possible frameworks. A correct quantum description of the measuring process must employ a *single framework*; mixing results from incompatible frameworks will only cause confusion.

In practice it is necessary to make a number of idealizations and approximations in order to discuss measurements as quantum-mechanical processes. This should not be surprising, for the same is true of classical physics. For example, the motion of the planets in the solar system can be described to quite high precision by treating them as point masses subject to gravitational forces, but of course this is not an exact description. The usual procedure followed by a physicist is to first work out an approximate description of some situation in order to get an idea of the various magnitudes involved, and then see how this first approximation can be improved, if greater precision is needed, by including effects which have been ignored. We shall follow this approach in this and the following chapter, sometimes pointing out how a particular approximation can be improved upon, at least in principle. The aim is physical insight, not a precise formalism which will cover all cases.

Quantum measurements can be divided into two broad categories: nondestructive and destructive. In *nondestructive measurements*, also called nondemolition measurements, the measured property is preserved, so the particle has the same, or almost the same property after the measurement is completed as it had before the measurement. While it is easy to make nondestructive measurements on macroscopic objects, such as tables, nondestructive measurements of microscopic quantum systems are much more difficult. Even when a quantum measurement is nondestructive for a particular property, it will be destructive for many other properties, so that the term nondestructive can only be defined relative to some property or properties, and does not refer to all conceivable properties of the quantum system.

In *destructive measurements* the property of interest is altered during the measurement process, often in an uncontrolled fashion, so that after the measurement the particle no longer has this property. For example, the kinetic energy of an energetic particle can be measured by bringing it to rest in a scintillator and finding the amount of light produced. This tells one what the energy of the particle was before it entered the scintillator, whereas at the end of the measurement process the kinetic energy of the particle is zero. In this and other examples of destructive measurements it is clear that the correlation of interest is between a property the particle had *before* the measurement took place, and the state of the apparatus *after* the measurement, and thus involves properties at two *different* times. The absence of a systematic way of treating correlations involving different times, except in very special cases, is the basic reason why the theory of measurement developed by von Neumann, Sec. 18.2, is not very satisfactory.

17.2 Microscopic Measurement

The measurement of the spin of a spin-half particle illustrates many of the principles of the quantum theory of measurement, so we begin with this simple case, using a certain number of approximations to keep the discussion from becoming too complicated. Consider a neutral spin-half particle, e.g., a silver atom in its ground state, moving through the inhomogeneous magnetic field of a Stern-Gerlach apparatus, shown schematically in Fig. 17.1. We shall assume the magnetic field is such that if the z component S_z of the spin is $+1/2$, there is an upwards force on the particle, and it emerges from the magnet moving upwards, whereas if $S_z = -1/2$, the force is in the opposite direction, and the particle moves downward as it leaves the magnet.

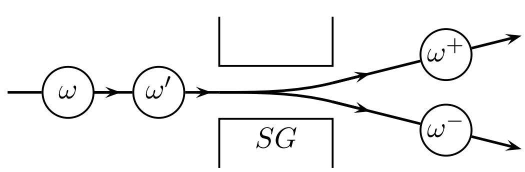

Figure 17.1: Spin-half particle passing through a Stern-Gerlach magnet.

This can be described in quantum mechanical terms as follows. The spin states of the particle corresponding to $S_z = \pm 1/2$ are $|z^+\rangle$ and $|z^-\rangle$ in the notation of Sec. 4.2. Let t_0 and t_1 be two successive times preceding the moment at which the particle enters the magnetic field, see Fig. 17.1, and t_2 a later time after it has emerged from the magnetic field. Assume that the unitary time development from t_0 to t_1 to t_2 is given by

$$\begin{aligned} |z^+\rangle|\omega\rangle &\mapsto |z^+\rangle|\omega'\rangle \mapsto |z^+\rangle|\omega^+\rangle, \\ |z^-\rangle|\omega\rangle &\mapsto |z^-\rangle|\omega'\rangle \mapsto |z^-\rangle|\omega^-\rangle, \end{aligned} \tag{17.1}$$

where $|\omega\rangle, |\omega'\rangle, |\omega^+\rangle, |\omega^-\rangle$ are wave packets for the particle's center of mass, at the locations indicated in Fig. 17.1. (One could also write these as $\omega(\boldsymbol{r})$, etc.)

One can think of the center of mass of the particle as the "apparatus". The two possible outcomes of the measurement are that the particle emerges from the magnet in one of the two spatial wave packets $|\omega^+\rangle$ or $|\omega^-\rangle$. It is important that the outcome wave packets be orthogonal,

$$\langle\omega^+|\omega^-\rangle = 0, \tag{17.2}$$

as otherwise we cannot speak of them as mutually exclusive possibilities. This condition will be fulfilled if the wave packets have negligible overlap, as suggested by the sketch in Fig. 17.1.

In calculating the unitary time development in (17.1) we assume that the Hamiltonian for the particle includes an interaction with the magnetic field, and this field is assumed to be "classical"; that is, it provides a potential for the particle's motion, but does not itself need to be described using an appropriate field-theoretical Hilbert space. Similarly, we have omitted from our quantum description the atoms of the magnet which actually produce this magnetic field. These "inert" parts of the apparatus could, in principle, be included in the sort of quantum description discussed in Sec. 17.3 below, but this is an unnecessary complication, since their essential role is included in the unitary time development in (17.1).

The process shown in Fig. 17.1 can be thought of as a measurement because the value of S_z before the measurement, the property being measured, is correlated with the spatial wave packet of the particle after the measurement, which forms the output of the measurement. It is also the case that S_z before the measurement is correlated with its value after the measurement, and this means the measurement is nondestructive for the properties $S_z = \pm 1/2$. One can easily imagine a destructive version of the same measurement by supposing that the wave packets emerging from the field gradient of the main magnet pass through some regions of uniform magnetic field, which do not affect the center of mass motion, but do cause a precession of the spin. Consequently, at the end the process the location of the wave packet for the center of mass will still serve to indicate the value of S_z before the measurement began, even though the final value of S_z need not be the same as the initial value.

Suppose that the initial spin state is not one of the possibilities $S_z = \pm 1/2$, but instead

$$|x^+\rangle = \big(|z^+\rangle + |z^-\rangle\big)/\sqrt{2} \tag{17.3}$$

corresponding to $S_x = +1/2$. What happens during the measuring process? The unitary time development of the initial state

$$|\psi_0\rangle = |x^+\rangle|\omega\rangle = \big(|z^+\rangle|\omega\rangle + |z^-\rangle|\omega\rangle\big)/\sqrt{2} \tag{17.4}$$

is obtained by taking a linear combination of the two cases in (17.1):

$$|\psi_0\rangle \mapsto |x^+\rangle|\omega'\rangle \mapsto \big(|z^+\rangle|\omega^+\rangle + |z^-\rangle|\omega^-\rangle\big)/\sqrt{2}. \tag{17.5}$$

The unitary history in (17.5) cannot be used to describe the measuring process, because the measurement outcomes, $|\omega^+\rangle$ and $|\omega^-\rangle$, are clearly incompatible with the final state in (17.5). A quantum mechanical description of a measurement with particular outcomes must, obviously, employ a framework in which these outcomes are represented by appropriate projectors, as in the consistent family whose support consists of the two histories

$$[\psi_0] \odot [x^+][\omega'] \odot \begin{cases} [z^+][\omega^+], \\ [z^-][\omega^-]. \end{cases} \tag{17.6}$$

The notation, see (14.12), indicates that the two histories are identical at the times t_0 and t_1, but contain different events at t_2. While this family contains the measurement outcomes $[\omega^+]$ and $[\omega^-]$, it is still not satisfactory for discussing the process in Fig. 17.1 as a *measurement*, because it does not allow us to relate these outcomes to the spin states $[z^+]$ and $[z^-]$ of the particle *before* the measurement took place. Since the properties $S_z = \pm 1/2$ are incompatible with a spin state $[x^+]$ at t_1, (17.6) does not allow us to say anything about S_z before the particle enters the magnetic field gradient. It is true that S_z at t_2 is correlated with the measurement outcome if we use (17.6). But this would also be true if the apparatus had somehow produced a particle in a certain spin state without any reference to its previous properties, and calling that a "measurement" would be rather odd.

A more satisfactory description of the process in Fig. 17.1 as a measurement is obtained by using an alternative consistent family whose support is the two histories

$$[\psi_0] \odot \begin{cases} [z^+][\omega'] \odot [z^+][\omega^+], \\ [z^-][\omega'] \odot [z^-][\omega^-]. \end{cases} \tag{17.7}$$

As both histories have positive weights, one sees that

$$\begin{aligned} \Pr(z_1^+ \,|\, \omega_2^+) = 1 = \Pr(z_1^- \,|\, \omega_2^-), \\ \Pr(\omega_2^+ \,|\, z_1^+) = 1 = \Pr(\omega_2^- \,|\, z_1^-), \end{aligned} \tag{17.8}$$

where we follow our usual convention that square brackets can be omitted and subscripts refer to times: e.g., z_1^+ is the same as $[z^+]_1$ and means $S_z = +1/2$ at t_1. (In addition, the initial state ψ_0 could be included among the conditions, but, as usual, we omit it.) These conditional probabilities tell us that if the measurement outcome is ω^+ at t_2, we can be certain that the particle had $S_z = +1/2$ at t_1, and vice versa; likewise, ω^- at t_2 implies $S_z = -1/2$ at t_1. (For an initial spin state $|z^+\rangle$ the conditional probabilities involving z^- and ω^- are undefined, and those involving z^+ and ω^+ are undefined for an initial $|z^-\rangle$.)

It is (17.8) which tells us that what we have been referring to as a *measurement process* actually deserves that name, for it shows that the result of this process is a correlation between specific outcomes and appropriate properties of the measured system before the measurement took place. If these probabilities were slightly less than one, it would still be possible to speak of an *approximate measurement*, and in practice all measurements are to some degree approximate.

In conclusion it is worth emphasizing that in order to describe a quantum process as a measurement it is necessary to employ a framework which includes *both* the measurement outcomes (pointer positions) *and* the properties of the measured system before the measurement took place, by means of suitable projectors. These requirements are satisfied by (17.7), whereas (17.6), even though it is an improvement over a unitary family, cannot be used to derive the correlations (17.8) that are characteristic of a measurement.

17.3 Macroscopic Measurement, First Version

If the results are to be of use to scientists, measurements of the properties of microscopic quantum systems must eventually produce macroscopic results visible to the eye or at least accessible to the computer. This requires devices that amplify microscopic signals and produce some sort of macroscopic record. These processes are thermodynamically irreversible, and this irreversibility contributes to the permanence of the resulting records. Thus even though the production of certain correlations, which is the central feature of the measuring process, can occur on a microscopic scale, as discussed in the previous section, macroscopic systems must be taken into account when quantum theory is used to describe practical measurements. A full and detailed quantum mechanical description of the processes going on in a macroscopic piece of apparatus containing 10^{23} particles is obviously not possible. Nonetheless, by making a certain number of plausible assumptions it is possible to explore what such a description might contain, and this is what we shall do in this and the next section, for a macroscopic version of the measurement of the spin of a spin-half particle.

Once again, assume that the particle passes through a magnetic field gradient, Fig. 17.1, which splits the center-of-mass wave packet into two pieces which are eventually separated by a macroscopic distance. The macroscopic measurement is then completed by adding particle detectors to determine whether the particle is in the upper or lower beam as it leaves the magnetic field. One could, for example, suppose that light from a laser ionizes a silver atom as it travels along one of the paths emerging from the apparatus, and the resulting electron is accelerated in an electric field and made to produce a macroscopic current by a cascade process. Detection of single atoms in this fashion is is technically feasible, though it is not easy. Of course one must expect that in such a measurement process the spin direction of the atom will not be preserved; indeed, the atom itself is broken up by the ionization process. Hence such a measurement is destructive.

Let us assume once again that three times t_0, t_1, and t_2 are used in a quantum description of the measurement process. The times t_0 and t_1 precede the entry of the particle into the magnetic field, Fig. 17.1, whereas t_2 is long enough after the particle has emerged from the magnetic field to allow its detection, and the result indicating the channel in which it emerged to be recorded in some macroscopic device, say a pointer easily visible to the naked eye. Assume that before the measurement takes place the pointer points in a horizontal direction, and at the completion of the measurement it either points upwards, indicating that the particle emerged in the upper channel corresponding to $S_z = +1/2$, or downwards, indicating that the particle emerged in the $S_z = -1/2$ channel. Of course, no one would build an apparatus in this fashion nowadays, but when discussing conceptual questions there is an advantage in using something easily visualized, rather than the direction of magnetization in some region on a magnetic tape or disk. The principles are in any case the same.

As a first attempt at a quantum description of such a macroscopic measurement, assume that at t_0 the apparatus plus the center of mass of the particle whose spin is to be measured is in a quantum state $|\Omega\rangle$. Then we might expect that the unitary time development of the apparatus plus particle would be similar to (17.1), of the form

$$\begin{aligned} |z^+\rangle|\Omega\rangle &\mapsto |z^+\rangle|\Omega'\rangle \mapsto |\Omega^+\rangle, \\ |z^-\rangle|\Omega\rangle &\mapsto |z^-\rangle|\Omega'\rangle \mapsto |\Omega^-\rangle, \end{aligned} \tag{17.9}$$

where $|\Omega^+\rangle$ is some state of the apparatus in which the pointer points upwards, and $|\Omega^-\rangle$ a state in which the pointer points downwards. The difference between $|\Omega\rangle$ and $|\Omega'\rangle$ reflects both the fact that the position of the center of mass of the particle changes between t_0 and t_1, and that the apparatus itself is evolving in time. The only assumption we have made is that this time evolution is not influenced by the direction of the spin of the particle, which seems plausible. In contrast to (17.1), the particle spin does not appear at time t_2 in (17.9). This is because we are dealing with a destructive measurement, and the value of the particle's spin at t_2 is irrelevant. Indeed, the concept may not even be well defined. Thus $|\Omega^+\rangle$ and $|\Omega^-\rangle$ are defined on a slightly different Hilbert space than $|\Omega\rangle$ and $|\Omega'\rangle$.

The counterpart of (17.2) is

$$\langle\Omega^+|\Omega^-\rangle = 0, \tag{17.10}$$

a consequence of unitary time development: since the two states in (17.9) at time t_0 are orthogonal to each other, those at t_2 must also be orthogonal. But (17.10) is also what one would expect on physical grounds for quantum states corresponding to distinct macroscopic situations, in this case different orientations of the pointer. The orthogonality in (17.2) was justified by assuming that the two emerging wave packets in Fig. 17.1 have negligible overlap. Two distinct pointer positions will mean that there are an enormous number of atoms whose wave packets have negligible overlap, and thus (17.10) will be satisfied to an excellent approximation.

It follows from (17.9) and our assumption about the way in which $|\Omega^+\rangle$ and $|\Omega^-\rangle$ are related to the pointer position that if the particle starts off with $S_z = +1/2$ at t_0, the pointer will be pointing upwards at t_2, while if the particle starts off with $S_z = -1/2$, the pointer will later point downwards. But what will happen if the initial spin state is not an eigenstate of S_z? Let us assume a spin state $|x^+\rangle$, (17.3), at t_0 corresponding to $S_x = +1/2$. The unitary time development of the initial state

$$|\Psi_0\rangle = |x^+\rangle|\Omega\rangle = \big(|z^+\rangle|\Omega\rangle + |z^-\rangle|\Omega\rangle\big)/\sqrt{2}, \tag{17.11}$$

the macroscopic counterpart of (17.4), is given by

$$|\Psi_0\rangle \mapsto |x^+\rangle|\Omega'\rangle \mapsto |\bar{\Omega}\rangle = \big(|\Omega^+\rangle + |\Omega^-\rangle\big)/\sqrt{2}. \tag{17.12}$$

The state $|\bar{\Omega}\rangle$ on the right side is a macroscopic quantum superposition (MQS) of states representing distinct macroscopic situations: a pointer pointing up and a pointer pointing down. It is incompatible with the measurement outcomes Ω^+ and Ω^- in the same way as the right side of (17.5) is incompatible with ω^+ and ω^-, so it cannot be used for describing the possible outcomes of the measurement. See the discussion in Sec. 9.6.

The measurement outcomes can be discussed using a family resembling (17.6) with support

$$[\Psi_0] \odot [x^+][\Omega'] \odot \begin{cases} [\Omega^+], \\ [\Omega^-]. \end{cases} \tag{17.13}$$

However, as pointed out in connection with (17.6), the presence of $[x^+]$ in the histories in this family at times preceding the measurement makes it impossible to discuss S_z. Thus one cannot employ (17.13) to obtain a correlation between the measurement outcomes and the value of S_z before the measurement took place.

Hence we are led to consider yet another family, the counterpart of (17.7), whose support is the two histories:

$$[\Psi_0] \odot \begin{cases} [z^+][\Omega'] \odot [\Omega^+], \\ [z^-][\Omega'] \odot [\Omega^-]. \end{cases} \tag{17.14}$$

From it we can deduce the conditional probabilities

$$\begin{aligned} \Pr(z_1^+ \,|\, \Omega_2^+) = 1 = \Pr(z_1^- \,|\, \Omega_2^-), \\ \Pr(\Omega_2^+ \,|\, z_1^+) = 1 = \Pr(\Omega_2^- \,|\, z_1^-), \end{aligned} \tag{17.15}$$

which are the analogs of (17.8). The initial state Ψ_0 can be thought of as one of the conditions, though it is not shown explicitly.

However, (17.15), while technically correct, does not really provide the sort of result one wants from a macroscopic theory of measurement. What one would like to say is: "Given the initial state and the fact that the pointer points up at the time t_2, S_z must have had the value $+1/2$ at t_1." While the state $|\Omega^+\rangle$ is, indeed, a state of the apparatus for which the pointer is up, it does not mean the same thing as "the pointer points up". There are an enormous number of quantum states of the apparatus consistent with "the pointer points up", and $|\Omega^+\rangle$ is just one of these, so it contains a lot of information in addition to the direction of the pointer. It provides a very precise description of the state of the apparatus, whereas what we would like to have is a conditional probability whose condition involves only a relatively coarse "macroscopic" description of the apparatus. One can also fault the use of the family (17.14) on the grounds that $|\Psi_0\rangle$ is itself a very precise description of the initial state of the apparatus. In practice it is impossible to set up an apparatus in such a way that one can be sure it is in such a precise initial state.

What we need are conditional probabilities which lead to the same conclusions as (17.15), but with conditions which involve a much less detailed description of the apparatus at t_0 and t_2. Such coarse-grained descriptions in classical physics are provided by statistical mechanics. Whereas quantum statistical mechanics lies outside the scope of this book, the histories formalism developed earlier provides tools which are adequate for the task at hand, and we shall use them in the next section to provide an improved version of macroscopic measurements.

17.4 Macroscopic Measurement, Second Version

Physical properties in quantum theory are associated with subspaces of the Hilbert space, or the corresponding projectors. Often these are projectors on relatively small subspaces. However, it is also possible to consider projectors which correspond to macroscopic properties of a piece of apparatus, such as "the pointer points upwards". We shall call such projectors "macro projectors", since they single out regions of the Hilbert space corresponding to macroscopic properties.

Let Z be a macro projector onto the initial state of the apparatus ready to carry out a measurement of the spin of the particle. It projects onto an enormous subspace $\mathcal{Z}$ of the Hilbert space, one with a dimension, $\mathrm{Tr}[Z]$, which is of the order of $\exp[S/k]$, where S is the (absolute) thermodynamic entropy of the apparatus, and k is Boltzmann's constant. Thus $\mathrm{Tr}[Z]$ could be 10 raised to the power 10^{23}. Such a macro projector is not uniquely defined, but the ambiguity is not important for the argument which follows. It is convenient to include in $\mathcal{Z}$ the information about the center of mass of the particle at t_0, but not its spin. Similarly, the apparatus after the measurement can be described by the macro projectors Z^+, projecting on a subspace $\mathcal{Z}^+$ for which the pointer points up, and Z^-, projecting on a subspace $\mathcal{Z}^-$ for which the pointer points down. For reasons indicated in Sec. 17.3, any state in which the pointer is directed upwards will surely be orthogonal to any state in which it is directed downwards, and thus

$$Z^+Z^- = 0. \tag{17.16}$$

Let $\{|\Omega_j\rangle\}, j = 1, 2, \dots$ be an orthonormal basis for $\mathcal{Z}$. We assume that the unitary time evolution from t_0 to t_1 to t_2 takes the form

$$\begin{aligned} |z^+\rangle|\Omega_j\rangle &\mapsto |z^+\rangle|\Omega'_j\rangle \mapsto |\Omega_j^+\rangle, \\ |z^-\rangle|\Omega_j\rangle &\mapsto |z^-\rangle|\Omega'_j\rangle \mapsto |\Omega_j^-\rangle, \end{aligned} \tag{17.17}$$

for $j = 1, 2, \dots$, and that for every j,

$$Z^+|\Omega_j^+\rangle = |\Omega_j^+\rangle, \quad Z^-|\Omega_j^-\rangle = |\Omega_j^-\rangle. \tag{17.18}$$

That is to say, whatever may be the precise initial state of the apparatus at t_0, if $S_z = +1/2$ at this time, then at t_2 the apparatus pointer will be directed upwards, whereas if $S_z = -1/2$ at t_0, the pointer will later be pointing downwards. Note that combining (17.16) with (17.18) tells us that for every j

$$Z^+|\Omega_j^-\rangle = 0 = Z^-|\Omega_j^+\rangle. \tag{17.19}$$

Since the $\{|\Omega_j^+\rangle\}$ are mutually orthogonal—(17.17) represents a unitary time development—they span a subspace of the Hilbert space having the same dimension, $\mathrm{Tr}[Z]$, as $\mathcal{Z}$. Hence (17.18) can only be true if the subspace $\mathcal{Z}^+$ onto which Z^+ projects has a dimension $\mathrm{Tr}[Z^+]$ at least as large as $\mathrm{Tr}[Z]$, and the same comment applies to $\mathcal{Z}^-$. We expect the process which results in moving the pointer to a particular position to be irreversible in the thermodynamic sense: the entropy of the apparatus will increase during this process. Since, as noted earlier, the trace of a macro projector is on the order of $\exp[S/k]$, where S is the thermodynamic entropy, even a modest (macroscopic) increase in entropy is enough to make $\mathrm{Tr}[Z^+]$ (and likewise $\mathrm{Tr}[Z^-]$) enormously larger than the already very large $\mathrm{Tr}[Z]$: the ratio $\mathrm{Tr}[Z^+]/\mathrm{Tr}[Z]$ will be 10 raised to a large power. There is thus no difficulty in supposing that (17.18) is satisfied, as there is plenty of room in $\mathcal{Z}^+$ and $\mathcal{Z}^-$ to hold all the states which evolve unitarily from $\mathcal{Z}$, and in this respect the unitary time development assumed in (17.17) is physically plausible.

Now let us consider various families of histories based upon an initial state represented by the projector

$$\Phi_0 = [x^+] \otimes Z, \tag{17.20}$$

which in physical terms means that the particle has $S_x = +1/2$ and the apparatus is ready to carry out the measurement. Note that Φ_0, in contrast to the pure state Ψ_0 used in Sec. 17.3, is a projector on a very large subspace, and thus a relatively imprecise description of the initial state of the apparatus.

Consider first the case of unitary time evolution starting with Φ_0 at t_0 and leading to a state

$$\Phi_2 = T(t_2, t_0)\Phi_0 T(t_0, t_2) = \sum_j |\bar{\Omega}_j\rangle\langle\bar{\Omega}_j|, \tag{17.21}$$

at t_2, where

$$|\bar{\Omega}_j\rangle = \left(|\Omega_j^+\rangle + |\Omega_j^-\rangle\right)/\sqrt{2} \tag{17.22}$$

is an MQS state. None of the terms in the sum in (17.21) commutes with Z^+ and Z^-, and it is easy to show that the same is true of the sum itself (that is, there are no cancellations). Since Φ_2 does not commute with Z^+ and Z^-, a history using unitary time evolution precludes any discussion of measurement outcomes. Another way of stating this is that whatever the initial apparatus state, unitary time evolution will inevitably lead to an MQS state in which the pointer positions have no meaning. Hence it is essential to employ a non-unitary history in order to discuss the measurement process; using macro projectors does not change our earlier conclusion in this respect.

Likewise the counterpart of the family (17.13), in which one can discuss measurement outcomes but not the value of S_z at t_1, is unsatisfactory as a description of a measurement process for the same reason indicated earlier. Thus we are led to consider a family analogous to that in (17.14), whose support consists of the two histories

$$\Phi_0 \odot \begin{cases} [z^+] \odot Z^+, \\ [z^-] \odot Z^- \end{cases} \tag{17.23}$$

involving events at the times t_0, t_1, t_2. Note that (in contrast to (17.14)) no mention is made of an apparatus state at t_1, and of course no mention is made of a spin state at t_2. The histories $\Phi_0 \odot [z^-] \odot Z^+$ and $\Phi_0 \odot [z^+] \odot Z^-$ have zero weight in view of (17.19).

As both histories in (17.23) have positive weight, it is clear that

$$\begin{aligned} \Pr(z_1^+ \mid Z_2^+) = 1 = \Pr(z_1^- \mid Z_2^-), \\ \Pr(Z_2^+ \mid z_1^+) = 1 = \Pr(Z_2^- \mid z_1^-), \end{aligned} \tag{17.24}$$

where the initial state Φ_0 can be thought of as one of the conditions, even though it is not shown explicitly. Thus if the pointer is directed upwards at t_2, then S_z had the value $+1/2$ at t_1, while a pointer directed downwards at t_2 means that S_z was $-1/2$ at t_1. These results are formally the same as those in (17.15), but (17.24) is more satisfactory from a physical point of view in that the conditions (including the implicit Φ_0) only involve "macroscopic" information about the measuring apparatus. Note that (17.14) is not misleading, even though its physical interpretation is less satisfactory than (17.24), and the former is somewhat easier to derive. It is often the case that one can model a macroscopic measurement process in somewhat simplistic terms, and nonetheless obtain a plausible answer. Of course, if there are any doubts about this procedure, it us a good idea to check it using macro projectors.

An alternative to the preceding discussion is an approach based upon statistical mechanics, which in its simplest form consists in choosing an appropriate basis $\{|\Omega_j\rangle\}$ for the subspace corresponding to the initial state of the apparatus (the space on which Z projects), and assigning a probability p_j to the state $|\Omega_j\rangle$. Assuming the correctness of (17.17) and (17.18), one can use the consistent family supported by the (enormous) collection of histories of the form

$$\begin{aligned} &[x^+] \otimes [\Omega_j] \odot [z^+] \odot Z^+, \\ &[x^+] \otimes [\Omega_j] \odot [z^-] \odot Z^-, \end{aligned} \tag{17.25}$$

with $j = 1, 2, \ldots$, to obtain (17.24). Note that consistency is ensured by $Z^+Z^- = 0$ along with the fact that the initial states for histories ending in Z^+ are mutually orthogonal, and likewise those ending in Z^-.

Yet another approach to the same problem is to describe the measuring apparatus at t_0 by means of a density matrix ρ thought of as a pre-probability, as discussed in Sec. 15.6. Since ρ describes an apparatus in an initial ready state, the probability, computed from ρ, that the apparatus will *not* be in this state must be zero:

$$\mathrm{Tr}[\rho(I - Z)] = 0. \tag{17.26}$$

Since both ρ and $I - Z$ are positive operators, (17.26) implies, see (3.92), that $\rho(I - Z) = 0$, or

$$Z\rho = \rho, \tag{17.27}$$

which means that the support of ρ (Sec. 3.9) falls in the subspace $\mathcal{Z}$ on which Z projects. Consequently, ρ may be written in the diagonal form

$$\rho = \sum_j p_j |\Omega_j\rangle\langle\Omega_j|, \tag{17.28}$$

where $\{|\Omega_j\rangle\}$ is an orthonormal basis of $\mathcal{Z}$. To be sure, this could be a different basis of $\mathcal{Z}$ from the one introduced earlier, but since the vectors in any basis can be expressed as linear combinations of vectors in the other, it follows that (17.17) and (17.18) will still be true.

Given ρ in the form (17.28), the measurement process can be analyzed using the procedures of Sec. 15.6, including (15.48) for consistency conditions and (15.50) for weights. Using these one can show that the two histories

$$[x^+] \odot \begin{cases} [z^+] \odot Z^+, \\ [z^-] \odot Z^- \end{cases} \tag{17.29}$$

form the support of a consistent family. This family resembles (17.23), except that the initial state $[x^+]$ at t_0 contains no reference to the apparatus, since the initial state of the apparatus is represented by a density matrix. The weights of the histories in (17.29) are the same as for their counterparts in (17.23), and once again lead to the conditional probabilities in (17.24).

17.5 General Destructive Measurements

The preceding discussion of the measurement of S_z for a spin-half particle can be easily extended to a schematic description of an idealized measuring process for a more complicated system $\mathcal{S}$ which interacts with a measuring apparatus $\mathcal{M}$. The measured properties will correspond to some orthonormal basis $\{|s^k\rangle\}$, $k = 1, 2, \ldots n$ of $\mathcal{S}$, and we shall assume that the measurement process corresponds to a unitary time development from t_0 to t_1 to t_2 of the form

$$|s^k\rangle \otimes |M_0\rangle \mapsto |s^k\rangle \otimes |M_1\rangle \mapsto |N^k\rangle, \tag{17.30}$$

where $|M_0\rangle$ and $|M_1\rangle$ are states of the apparatus at t_0 and t_1 before it interacts with $\mathcal{S}$, and the $\{|N^k\rangle\}$ are orthonormal states on $\mathcal{S} \otimes \mathcal{M}$ for which a measurement pointer indicates a definite outcome of the measurement. (The $\{|N^k\rangle\}$ are a *pointer basis* in the notation introduced at the end of Sec. 9.5.)

Assume that at t_0 the initial state of $\mathcal{S} \otimes \mathcal{M}$ is

$$|\Psi_0\rangle = |s_0\rangle \otimes |M_0\rangle, \tag{17.31}$$

where

$$|s_0\rangle = \sum_k c_k |s^k\rangle, \tag{17.32}$$

with $\sum_k |c_k|^2 = 1$, is an arbitrary superposition of the basis states of $\mathcal{S}$. Unitary time evolution will then result in a state

$$|\Psi_2\rangle = T(t_2, t_0)|\Psi_0\rangle = \sum_k c_k |N^k\rangle \tag{17.33}$$

at t_2. Using the two-time family

$$\Psi_0 \odot I \odot \{N^1, N^2, \ldots\}, \tag{17.34}$$

where square brackets have been omitted from $[N^k]$, and regarding $|\Psi_2\rangle$ as a pre-probability, one finds

$$\Pr(N_2^k) = |c_k|^2 \tag{17.35}$$

for the probability of the k'th outcome of the measurement at the time t_2.

One can refine (17.34) to a consistent family with support

$$\Psi_0 \odot \begin{cases} [s^1] \odot N^1, \\ [s^2] \odot N^2, \\ \ldots\ldots\ldots\ldots \\ [s^n] \odot N^n, \end{cases} \tag{17.36}$$

and from it derive the conditional probabilities

$$\Pr(s_1^j \,|\, N_2^k) = \delta_{jk} = \Pr(N_2^k \,|\, s_1^j), \tag{17.37}$$

assuming $\Pr(N_2^k) > 0$. That is, given the measurement outcome N^k at t_2, $\mathcal{S}$ was in the state $|s^k\rangle$ before the measurement took place. Thus the measurement interaction results in an appropriate correlation between the later apparatus output and the earlier state of the measured system.

The preceding analysis can be generalized to a measurement of properties which are not necessarily pure states, but form a decomposition of the identity

$$I_{\mathcal{S}} = \sum_k S^k \tag{17.38}$$

for system $\mathcal{S}$, where some of the projectors are onto subspaces of dimension greater than one. This might arise if one were interested in the measurement of a physical variable of the form

$$V = \sum_k v'_k S^k, \tag{17.39}$$

see (5.24), some of whose eigenvalues are degenerate.

Let us assume that the subspace onto which S^k projects is spanned by an orthonormal collection $\{|s^{kl}\rangle, l = 1, 2, \ldots\}$, so that

$$S^k = \sum_l |s^{kl}\rangle\langle s^{kl}|. \tag{17.40}$$

Assume that the counterpart of (17.30) is a unitary time development

$$|s^{kl}\rangle \otimes |M_0\rangle \mapsto |s^{kl}\rangle \otimes |M_1\rangle \mapsto |N^{kl}\rangle, \tag{17.41}$$

where $\{|N^{kl}\rangle\}$ is an orthonormal collection of states on $\mathcal{S} \otimes \mathcal{M}$ labeled by both k and l, and

$$N^k = \sum_l |N^{kl}\rangle\langle N^{kl}| \tag{17.42}$$

represents a property of $\mathcal{M}$ corresponding to the k'th measurement outcome.

The counterpart of (17.36) is the consistent family

$$\Psi_0 \odot \begin{cases} S^1 \odot N^1, \\ S^2 \odot N^2, \\ \ldots\ldots\ldots \\ S^n \odot N^n, \end{cases} \tag{17.43}$$

where $|\Psi_0\rangle$ is given by (17.31), with

$$|s_0\rangle = \sum_{kl} c_{kl} |s^{kl}\rangle \tag{17.44}$$

the obvious counterpart of (17.32). Corresponding to (17.37) one has

$$\Pr(S_1^j \mid N_2^k) = \delta_{jk} = \Pr(N_2^k \mid S_1^j), \tag{17.45}$$

with the physical interpretation that a measurement outcome N^k at t_2 implies that $\mathcal{S}$ had the property S^k at t_1, and vice versa.

The measurement schemes discussed in this section can be extended to a genuinely macroscopic description of the measuring apparatus in a straightforward manner using either of the approaches discussed in Sec. 17.4.