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The Dialectica categories

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The Dialectica Categories

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Abstract

This work consists of two main parts. The first one, which gives it its name, presents an internal categorical version of Gödel's "Dialectica interpretation" of higher-order arithmetic. The idea is to analyse the Dialectica interpretation using a category **DC** where objects are relations on objects of a basic category **C** and maps are pairs of maps of **C** satisfying a certain pullback condition. If **C** is finitely complete, **DC** exists and has a very natural symmetric monoidal structure. If **C** is locally cartesian closed then **DC** is symmetric monoidal closed. If we assume **C** with stable and disjoint coproducts, **DC** has cartesian products and weak-coproducts and satisfies a weak form of distributivity. Using the structure above, **DC** is a categorical model for Intuitionistic Linear Logic.

Moreover, if **C** has free monoids then **DC** has cofree comonoids and the corresponding comonad "!" on **DC**, which has some special properties, can be used to model the exponential "of course!" in Intuitionistic Linear Logic. The category of "!"-coalgebras is isomorphic to the category of comonoids in **DC** and, if we assume *commutative* monoids in **C**, the "!"-Kleisli category, which is cartesian closed, corresponds to the Diller-Nahm variant of the Dialectica interpretation.

The second part introduces the categories **GC**. The objects of **GC** are the same objects of **DC**, but morphisms are easier to handle, since they are maps in **C**, in opposite directions. If **C** is finitely complete, the category **GC** exists. If **C** is cartesian closed, we can define a symmetric monoidal structure and if **C** is locally cartesian closed as well, we can define internal homs in **GC**, that make it a symmetric monoidal closed category. Supposing **C** with stable and disjoint coproducts, we can define cartesian products and coproducts in **GC** and, more interesting, we can define a dual operation to the tensor product bifunctor, called "par". The operation "par" is a bifunctor and has a unit "$\perp$", which is a dualising object. Using the internal hom and $\perp$ we define a contravariant functor "$(-)^{\perp}$", which behaves like a negation and thus it is used to model linear negation. We show that the category **GC**, with all the structure above, is a categorical model for Linear Logic, but not exactly the classical one.

In the last chapter, a comonad and a monad are defined to model the exponentials "!" and "?". To define those endofunctors, we use Beck's *distributive laws*, in a interesting way. Finally, we show that the the Kleisli category $\mathbf{GC}_!$ is cartesian closed and that the categories **DC** and **GC** are related by a Kleisli construction.

Introduction

This work grew from the idea of providing an internal categorical version of the "Dialectica Interpretation" of higher order arithmetic.

Gödel's "Dialectica Interpretation" - as it came to be known - based on his "System T", was first published in the journal "Dialectica" in 1958. A very elucidating translation, by W. Hodges and B. Watson, can be found in the Journal of Philosophical Logic, where there is also an extensive bibliography of work resulting from it, compiled by J.R. Hindley. This bibliography does not mention Scott's *"The Dialectica Interpretation and Categories"* [Sco], a first categorical version of the Dialectica.

The work presented here, however, is completely different from the work of Scott. There, a syntactical categorical characterization of HA^w - a version of intuitionistic arithmetic in all finite types - is given and the Dialectica Interpretation appears as a functor preserving the logical structure. Here we have a much more internal characterization, in the sense that morphisms in the category in question, correspond to Dialectica interpretations of implication.

The original idea, as suggested to me by Hyland, was to consider the interpretation in a way now familiar from the "propositions as types" school of categorical proof-theory. As usual the objects of the category are well-determined, and in our case they represent essentially the Φ^D, where Φ is a formula in higher-order arithmetic and $(\ \ )^D$ is the Dialectica translation, see [Tro]. The maps are more problematic, however - looked at from the proof-theoretic point of view they should represent normalisation classes of proofs, but more abstractly a map from Φ^D to Ψ^D can be taken to be some kind of realisation of the formula "$\Phi^D \to \Psi^D$". Hyland's observation was that in the case of the Dialectica Interpretation this realisation could be given very abstractly, leading to the notion of a Dialectica category **DC** for an arbitrary category **C** with limits, which we shall discuss in Chapter 1.

The objects of the category **DC** are relations in the base category **C**, which we write as $(U \overset{\alpha}{\nleftrightarrow} X)$ and the maps from an object $(U \overset{\alpha}{\nleftrightarrow} X)$ to another $(V \overset{\beta}{\nleftrightarrow} Y)$ are pairs of maps $f\colon U \to V$ and $F\colon U \times Y \to X$ in **C**, satisfying a certain condition. The motivation behind such an odd definition of maps can be found in the Dialectica translation of implication. Implication, by far the most interesting rule in the Dialectica translation, is described by Troelstra [p.231] as:

$$\begin{aligned}
(A \Rightarrow B)^D &= (\exists u \forall x A_D \Rightarrow \exists v \forall y B_D)^D \\
&\equiv [\,\forall u(\forall x A_D \Rightarrow \exists v \forall y B_D)\,]^D \\
&\equiv [\,\forall u \exists v(\forall x A_D \Rightarrow \forall y B_D)\,]^D \\
&\equiv [\,\forall u \exists v \forall y(\forall x A_D \Rightarrow B_D)\,]^D \\
&\equiv [\,\forall u \exists v \forall y \exists x(A_D \Rightarrow B_D)\,]^D \\
&\equiv \exists \mathbf{V}\mathbf{X} \forall u y [A_D(u, \mathbf{X}(u,y)) \Rightarrow B_D(\mathbf{V}(u), y)]
\end{aligned}$$

Hence to translate the logical connective implication we need the functionals $\mathbf{V}\colon U \to V$ and $\mathbf{X}\colon U \times Y \to X$, which correspond to morphisms (f, F) in **C** in our definition.

A Dialectica category, however, differs from conventional proof-theoretic categories in that it is not cartesian closed. In fact, in a Dialectica category, there are two constructions that seem to correspond to the interpretation of conjunction, a tensor and a categorical product. Rather than the product it is the tensor, that, along with the interpretation of implication, provides a "good" (i.e a monoidal closed) categorical structure.

New input came when we received accounts of Girard's work on Linear Logic, [Gir] 1986, and realised that many aspects of it seemed close to the categorical behaviour of the Dialectica categories. Indeed, it became clear that we had a categorical version of the intuitionistic fragment of linear logic (cf.[G-L]).

The interpretation of the operator "!", called by Girard the modality "of course" turned out to pose a problem. This was solved by looking at cofree comonoid structures in **DC**, and as a spin-off from this categorical setting we got another category $\mathbf{DC}_!$, the !-Kleisli category, which corresponds to the variant of the Dialectica Interpretation described by Diller and Nahm [D/N].

The work above on the Dialectica categories was presented at the A.M.S Conference on Categories in Computer Science and Logic, Boulder 1987, and there I met Girard, who suggested a new category, which should be a model of Classical Linear Logic. This new construction is presented in Chapter 3 and for obvious reasons is called **GC**, since it is again functorial for finitely complete categories **C**.

One of the interesting points about the categories **GC** is that when we describe the categorical constructions modelling the linear logic connectives, we find that the units for "tensor" and "par" do not collapse into a single object, as they do in most other models.

But for the **GC** categories the exponential connectives "!" and "?" proved more elusive. They were eventually found by composing the connective "!" of **DC** with the comonad, respectively monad, suggested by Girard for "!" and "?" in **GC**. That gave rise to some interesting uses of *"distributive laws of monads"*, a concept proposed by J. Beck in 1973, but not very often used.

Another interesting point raised by the categories **GC** is that, one might wish to talk about a linear logic which has an "almost" involutive negation, instead of a really involutive one. As R.A.G Seely points out in [See] 1987, the problem of finding a categorical model for Classic Linear Logic, was answered, long before it was posed, by M. Barr, who wrote a book on the subject of $*$-autonomous categories. But using an analogy not completely out of place, in the same way that general vector spaces are more interesting than finite-dimensional ones, categories with a "half-involutive" negation may prove to be more exciting than $*$-autonomous categories.

I should perhaps mention that when Girard explained his ideas about these new categories **GC**, he suggested that they could probably be connected with Henkin quantifiers ([Hen] and [B/G]), in a interesting way. In the year or so, that has elapsed between the meeting in Boulder and the writing up of this work, I have not had time to pursue this idea as I would have liked. There is clearly some connection. We can try to understand the meaning of an object in **GC**, $(U \overset{\alpha}{\leftrightarrow} X)$ as a kind of "game" between players "U" and "X", where "U" picks an element "u" and "X" picks "x's". If "u" is such that for every "x" the second player chooses, it happens that $u\alpha x$, then "U" wins. On the other hand, if for every "x" it is true that $\neg(u\alpha x)$ then "X" wins. With this kind of interpretation in mind, the connection of **GC** with Henkin quantifiers may be unveiled, but at the moment it seems very opaque to me. Of course, this interpretation, does raise several interesting questions about the natural meaning of linear logic and related issues, but that is a new research project in itself.

This work consists of 4 chapters. In Chapter 1 we describe our basic construction, the Dialectica categories **DC**, for **C** finitely complete. In Chapter 2 we define a comonad $(!, \varepsilon_!, \delta_!)$, which is used to model the modality "of course!" in Intuitionistic Linear Logic. In Chapter 3 we describe the categories **GC** with all their structure and show they are a model of Linear Logic, but not exactly the Linear Logic presented in [Gir]. In Chapter 4 to define the comonad "!" we use distributive laws and conclude that **DC** is a Kleisli category for **GC**.

[We take for granted some basic concepts of Category Theory, which can be found for instance in MacLane's book.]

Chapter 1

The Dialectica categories DC

This chapter presents the general construction of a Dialectica category **DC**, for any category **C** finitely complete, and describes some of their categorical structure. The chapter contains 5 sections. The first explains our basic construction **DC** and shows it is really a category. The second shows a symmetric monoidal closed structure in **DC**. The third defines cartesian products and weak-coproducts in **DC**, while the fourth observes that the construction $D(-)\colon \mathbf{Cat} \to \mathbf{Cat}$ is functorial. In the fifth and last section we show the very nice connection with Intuitionistic Linear Logic.

1.1 The general construction

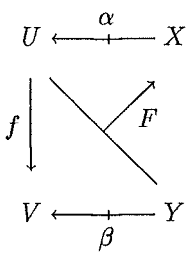
In this section we describe the general construction of the Dialectica category **DC** associated to a basic category **C** with finite limits. Martin Hyland's idea was to build a category of relations on objects of the basic category **C**, with rather special maps.

A typical object of **DC** is a subobject of the product $U \times X$, thus a monomorphism $A \overset{\alpha}{\rightarrowtail} U \times X$, where U, X and A are objects in **C**. We write this object as $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$ and call it simply A or sometimes "α", meaning the (equivalence class of the) monic.

A map between two such objects $A \overset{\alpha}{\rightarrowtail} U \times X$ and $B \overset{\beta}{\rightarrowtail} V \times Y$ consists of a pair of maps of **C**, (f, F) $f\colon U \to V$, $F\colon U \times Y \to X$ such that a non-trivial condition is satisfied. Namely, pulling back $A \overset{\alpha}{\rightarrowtail} U \times X$ along $U \times Y \overset{(\pi_1, F)}{\to} U \times X$ and $B \overset{\beta}{\rightarrowtail} V \times Y$ along $U \times Y \overset{f \times Y}{\to} V \times Y$, as the diagram shows, the first subobject $A' \overset{\alpha'}{\rightarrowtail} U \times Y$ is smaller than the second $B' \overset{\beta'}{\rightarrowtail} U \times Y$.
Thus, there is a (unique) map $k\colon A' \to B'$ in **C** making a comutative triangle in the diagram:

$$
\begin{array}{ccccc}
 & & A' & \longrightarrow & A \\
 & & {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\
B' & \overset{\beta'}{\rightarrowtail} & U \times Y & \overset{(\pi_1, F)}{\longrightarrow} & U \times X \\
\downarrow & & \downarrow{\scriptstyle f \times Y} & & \\
B & \underset{\beta}{\rightarrowtail} & V \times Y & &
\end{array}
$$

If we write $(U \stackrel{\alpha}{\leftarrow\!\!\!\!+} X)$ for $A \stackrel{\alpha}{\rightarrowtail} U \times X$, $(V \stackrel{\beta}{\leftarrow\!\!\!\!+} Y)$ for $B \stackrel{\beta}{\rightarrowtail} V \times Y$ and $(-)^{-1}$ for the pullback functor, then a map in **DC** can be represented as the pair (f, F) in the diagram below



where the condition reads as

$$(\pi_1, F)^{-1}(\alpha) \leq (f \times Y)^{-1}(\beta). \qquad (*)$$

The intuition here is to think of an object $(U \stackrel{\alpha}{\leftarrow\!\!\!\!+} X)$ as a set-theoretic relation between U and X, so that for some u's and some x's we have $u\alpha x$, for others we do not. Thus there is a map $A \stackrel{(f,F)}{\to} B$ in **DC** iff whenever $u\alpha F(u, y)$ then $f(u)\beta y$.

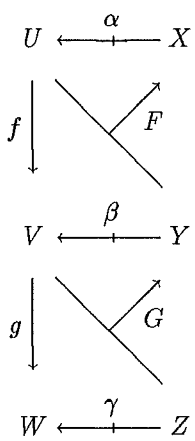
Since $(*)$ is not a straightforward categorical condition it is not obvious that **DC** is a category and we have our first proposition.

Proposition 1 *Given a category* **C** *with finite limits we can define a category* **DC** *using the construction above.*

Proof: Composition needs checking. Given two maps $(f, F): A \to B$ and $(g, G): B \to C$ their composition $(g, G) \cdot (f, F)$ is $gf: U \to W$ in the first coordinate and $G \circ F: U \times Z \to X$ given by

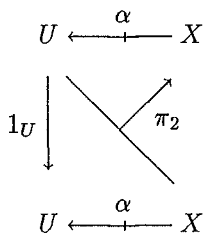
$$U \times Z \stackrel{\Delta \times Z}{\to} U \times U \times Z \stackrel{U \times f \times Z}{\to} U \times V \times Z \stackrel{U \times G}{\to} U \times Y \stackrel{F}{\to} X$$

in the second. Using diagrams we have:



To verify that the new map $(gf, G \circ F): A \to C$ satisfies condition $(*)$ we use pullback patching. It is easy to see that composition is associative and that identities are $(1_U, \pi_2)$ where $1_U: U \to U$

is the identity and $\pi_2: U \times X \to X$ is the canonical second projection in **C**.



Notice that, of the finite limits required of **C**, only finite products and pullbacks have been used to define the categorical structure of **DC**.

1.2 A monoidal closed structure in DC

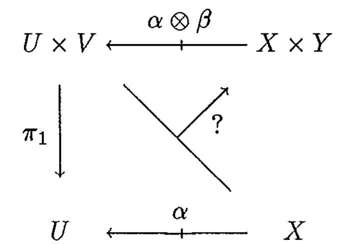
The category **DC** has a natural symmetric monoidal structure and we can define an internal hom functor (cf.[Kel]) to make it monoidal closed.

Definition 1 *For objects* $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$ *and* $B = (V \overset{\beta}{\leftarrow\!\!\!+} Y)$ *in* **DC**, *define their tensor product* $A \otimes B$ *as the object*

$$(U \times V \overset{\alpha\otimes\beta}{\leftarrow\!\!\!+} X \times Y)$$

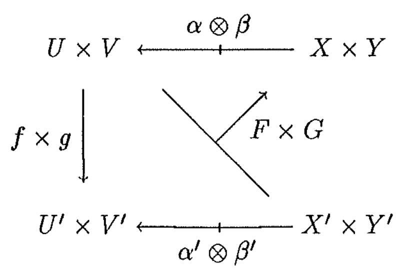
The relation "$\alpha\otimes\beta$*" is defined by straightforward product of the morphisms* α *and* β *and intuitively it says* $(u,v)\alpha\otimes\beta(x,y)$ *iff* $u\alpha x$ *and* $v\beta y$.

Notice that the tensor product above does not define a product, since we, in general, do not have projections. For example, a projection $p_1: A \otimes B \to A$,



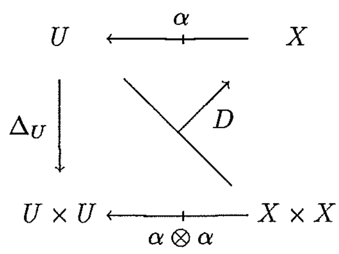
would imply the existence of a canonical map $U \times V \times X \xrightarrow{?} Y$.

The operation "$\otimes$" is a bifunctor. Given maps $A \xrightarrow{(f,F)} A'$ and $B \xrightarrow{(g,G)} B'$ we have a correspondent map $A \otimes B \to A' \otimes B'$ given by $(f \times g, F \times G)$ in the diagram



One can easily check that the bifunctor "$\otimes$" defines a symmetric monoidal structure on **DC** with $I = (1 \overset{1}{\nleftarrow} 1)$ as its unit.

Another remark is that there is no natural diagonal map with respect to $\otimes$, $A \longrightarrow A \otimes A$, since neither of the natural maps in the second coordinate satisfies condition $(*)$. Using diagrams,



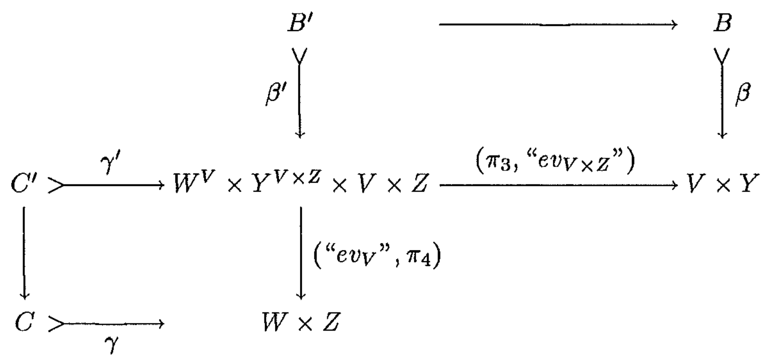
if we take D above as any of the canonical projections, the condition $(*)$ is not satisfied, since $u\alpha x$ does not imply $u\alpha x$ and $u\alpha x'$.

Assuming **C** finitely complete, we defined **DC** and verified that it has a symmetric monoidal structure. Next, assuming **C** is *cartesian closed*, we want to define internal homs, or function spaces, cf. [Kel] page 33, in **DC**.

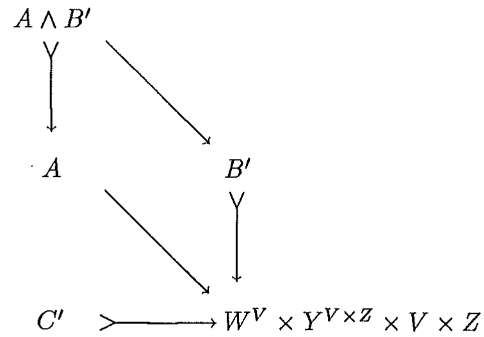
To define the internal homs in **DC** recall that, intuitively, $[B, C]_{\mathbf{DC}}$ should represent "the set of pairs of maps in **C**, $f: V \longrightarrow W$, $F: V \times Z \longrightarrow Y$ satisfying the $(*)$ condition". Therefore it is reasonable to start with $[B, C]_{DC}$ as a subobject of $W^V \times Y^{V \times Z} \times V \times Z$, or better, an object of the form

$$(W^V \times Y^{V \times Z} \nleftarrow V \times Z).$$

Definition 2 *Given the objects* $B = (V \overset{\beta}{\nleftarrow} Y)$ *and* $C = (W \overset{\gamma}{\nleftarrow} Z)$, *to define the internal hom* $[B, C]_{DC}$ *consider the following diagram, where the first square is a pullback of* $B \overset{\beta}{\rightarrowtail} V \times Y$ *along* $W^V \times Y^{V \times Z} \times V \times Z \overset{(\pi_3, \text{“}ev_{V \times Z}\text{”})}{\longrightarrow} V \times Y$, *the second square is a pullback of* $C \overset{\gamma}{\rightarrowtail} W \times Z$ *along* $W^V \times Y^{V \times Z} \times V \times Z \overset{(\text{“}ev_V\text{”}, \pi_4)}{\longrightarrow} W \times Z$,



and the objects B' and C' are defined by these pullbacks. Then define $[B,C]_{DC}$ as the greatest subobject $A \overset{\alpha}{\rightarrowtail} W^V \times Y^{V\times Z} \times V \times Z$ such that $A \wedge B' \leq C'$,



where the symbol "$\wedge$" means pullback over $W^V \times Y^{V\times Z} \times V \times Z$, as the diagram shows.

Note that this is the usual categorical translation of Heyting implication. Intuitively, the relation "γ^β" in the definition

$$[B,C]_{\mathbf{DC}} = (W^V \times Y^{V\times Z} \overset{\gamma^\beta}{\leftarrow} V \times Z)$$

says $(f,F)\gamma^\beta(v,z)$ iff $v\beta F(v,z) \Rightarrow f(v)\gamma z$.

To guarantee the existence of the greatest subobject we ask for $\mathbf{C}$ locally cartesian closed as well as cartesian closed. By that we mean that for any object A of $\mathbf{C}$, the slice category $\mathbf{C}/A$ is cartesian closed, cf [See] 1984.

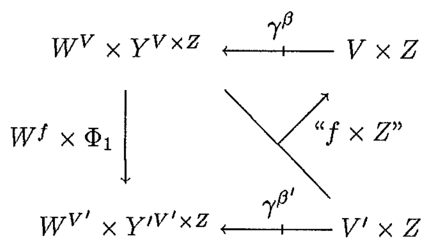
[But notice that we are not, at the moment, taking in consideration minimality of assumptions, so taking $\mathbf{C}$ locally cartesian closed is certainly enough to make the definitions work, but we could probably have a weaker assumption.]

Proposition 2 *The construction above defines an internal hom bifunctor*

$$[(-),(-)]_{\mathbf{DC}}: \mathbf{DC}^{op} \times \mathbf{DC} \to \mathbf{DC},$$

contravariant in the first coordinate and covariant in the second coordinate.

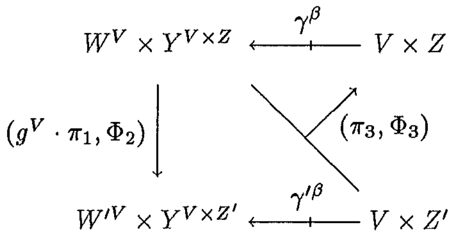
Given a map $(f,F): B' \to B$, it induces a map $[B,C]_{\mathbf{DC}} \to [B',C]_{\mathbf{DC}}$ shown in the diagram below, where



- the map $W^f: W^V \to W^{V'}$ is simply pre-composition with $f: V' \to V$;
- by "$f \times Z$" we mean $W^Y \times Y^{V\times Z} \times V' \times Z \xrightarrow{(\pi_3,\pi_4)} V' \times Z \xrightarrow{f\times Z} V \times Z$ and
- the map Φ_1 is given by the exponential transpose of the long composition,

$$Y^{V\times Z}\times V'\times Z \xrightarrow{\Delta} Y^{V\times Z}\times V'\times V'\times Z \xrightarrow{f} Y^{V\times Z}\times V\times V'\times Z \xrightarrow{\text{“}ev\text{”}} V'\times Y \xrightarrow{F} Y'.$$

Also, given a morphism $(g, G): C \to C'$ it induces a map $[B, C]_{\mathbf{DC}} \to [B, C']_{\mathbf{DC}}$ as shown by the diagram below, where



- the map $g^V: W^V \to W'^V$ is post-composition with $g: W \to W'$;
- the morphism Φ_2 is the exponential transpose of

$$Y^{V\times Z}\times W^V\times V\times Z' \xrightarrow{\text{“}ev_V\text{”}} Y^{V\times Z}\times W\times Z'\times V \xrightarrow{G} Y^{V\times Z}\times V\times Z \xrightarrow{ev} Y,$$

  where $G: W\times Z' \to Z$;
- the map “Φ_3” is given by

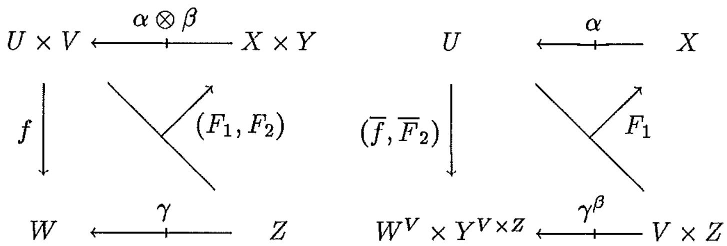
$$Y^{V\times Z}\times W^V\times V\times Z' \xrightarrow{1\times\text{“}ev\text{”}\times 1} Y^{V\times Z}\times W\times Z' \xrightarrow{(\pi_2,\pi_3)} W\times Z' \xrightarrow{G} Z.$$

Proposition 3 *The adjunction* $(-)\otimes B \dashv [B,(-)]_{\mathbf{DC}}$ *makes* **DC** *a monoidal closed category.*

Proof: We check the natural isomorphism

$$\mathrm{Hom}_{\mathbf{DC}}(A\otimes B, C) \cong \mathrm{Hom}_{\mathbf{DC}}(A, [B, C]_{DC}),$$

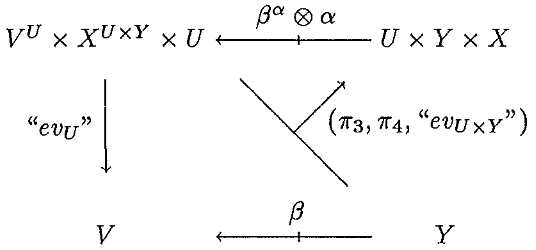
which corresponds to the following diagram



Firstly, to see the bijective correspondence take a morphism in $\mathrm{Hom}_{\mathbf{DC}}(A\otimes B, C)$. It is of the form $(f, (F_1, F_2))$ where $f: U\times V \to W$, and on the second coordinate, we have two components, $F_1: U\times V\times Z \to X$ and $F_2: U\times V\times Z \to Y$. The map f is bijectively associated (by exponential transpose in **C**) to $\overline{f}: U \to W^V$ and analogously F_2 is associated to $\overline{F_2}: U \to Y^{V\times Z}$.

Therefore the mapping $(f, (F_1, F_2)) \mapsto ((\overline{f}, \overline{F_2}), F_1)$ has appropriate domain and codomain and is clearly bijective. Also a long and tedious arrow-chasing proves that $(f, (F_1, F_2))$ is a map in **DC** if and only if $((\overline{f}, \overline{F_2}), F_1)$ is one as well. □

As usual with monoidal closed categories we have isomorphisms $A \cong [I, A]_{DC}$ and natural transformations $ev_A: [A, B]_{\mathbf{DC}} \otimes A \to B$, given by the maps



1.3 Products and weak-coproducts in DC

In this section we consider **C** a locally cartesian closed category with *stable* and *disjoint* coproducts. [As mentioned before we are not interested, for the time being, in minimality of assumptions, but it has been pointed out to us by P.T. Johnstone that coproducts in lcc categories are stable and quasi-disjoint.]

Using these extra hypotheses, we shall define categorical products and weak-coproducts in **DC** and show a weak form of distributivity of product over weak-coproducts. Before doing that, we recall the notions of *stable* and *disjoint* coproducts.

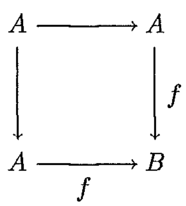
Say that a coproduct in **C**, $A = \coprod_{\alpha \in \Lambda} A_\alpha$ is *disjoint* if each of the canonical injections $j_\alpha: A_\alpha \to A$ is a monomorphism and for each pair of distinct indices α, α' the pullback of $j_\alpha, j_{\alpha'}$ is the initial object.

Say also that the coproduct A above is *stable under pullbacks* if, given any map $f: B \to A$, if we take the pullbacks of each of the canonical injections j_α along $f: B \to A$ and call them $f^{-1}A_\alpha$, then $B \cong \coprod_{\alpha \in \Lambda} f^{-1}A_\alpha$ cf. [Makkai-Reyes] page 49.

Proposition 4 *Given* disjoint *monics* $A \overset{m}{\rightarrowtail} X$ *and* $B \overset{n}{\rightarrowtail} X$ *in a category* **C** *finitely complete with stable and disjoint coproducts, the canonical map* $A + B \xrightarrow{\binom{m}{n}} X$ *is monic.*

Proof: We say that monics $A \overset{m}{\rightarrowtail} X$ and $B \overset{n}{\rightarrowtail} X$ are disjoint if their pullback is the initial object.

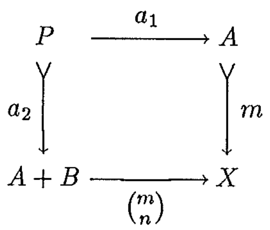
Recall that in a category **C** with pullbacks $A \xrightarrow{f} B$ is monic iff the square below



is a pullback.

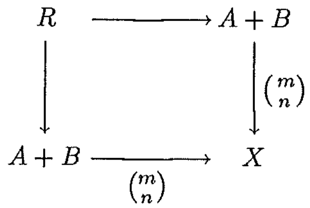
It is a simple calculation to check that the pullback of $A \overset{m}{\rightarrowtail} X$ along $A + B \xrightarrow{\binom{m}{n}} X$ is simply A.

We form the pullback square



Call it P and verify that, using identity $A \to A$ and canonical inclusion $j_1 : A \to A+B$, the object A makes the outer square commute, so there is a unique map $A \xrightarrow{x} P$ such that the composition $A \xrightarrow{x} P \xrightarrow{a_1} A$ is the identity in A. Using that a_2 is monic we show that the composite $P \xrightarrow{a_1} A \xrightarrow{x} P$ is the identity on P, so A and P are isomorphic. Similarly, the pullback of n along $\binom{m}{n}$ is simply B.

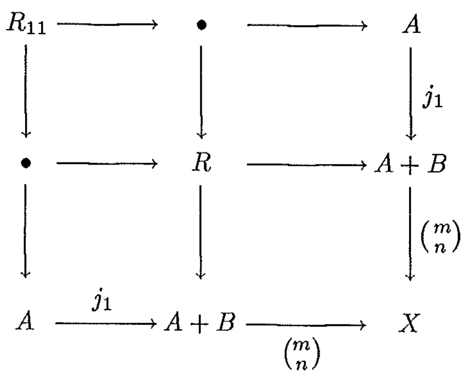
Now to show $A+B \xrightarrow{\binom{m}{n}} X$ is monic, we simply calculate the pullback



In other words, we calculate the kernel-pair of $\binom{m}{n}$.

Using the stability of coproducts we decompose R as $R_{11} + R_{12} + R_{21} + R_{22}$.

And if we draw the big pullback diagrams, e.g



they show that R_{11} is simply the pullback of m along m, R_{22} is the pullback of n along n and both R_{12} and R_{21} are pullbacks of m along n. But since the monics are disjoint, R_{12} and R_{21} turn out to be 0 so $R \cong A+B$ and we have the result.

$\square$

Corollary 1 *Given monics $A \overset{m}{\rightarrowtail} X$ and $B \overset{n}{\rightarrowtail} Y$ in* **C**, *with stable and disjoint coproducts, the canonical morphism $A+B \xrightarrow{m+n} X+Y$ is a monic.*

As the canonical injections are monic in **C**, so are $A \overset{i_1 \cdot m}{\rightarrowtail} X+Y$ and $B \overset{i_2 \cdot n}{\rightarrowtail} X+Y$, also these monics are disjoint by construction, which implies that $A+B \overset{m+n}{\rightarrowtail} X+Y$ is a monic.

Definition 3 *Consider the product*

$$A\&B = (U \times V \overset{\alpha\&\beta}{\nleftarrow} X + Y)$$

of two objects $A = (U \overset{\alpha}{\nleftarrow} X)$ *and* $B = (V \overset{\beta}{\nleftarrow} Y)$ *of* **DC**, *obtained by "adding up" the subobjects* $U \times B \overset{U\times\beta}{\rightarrowtail} U \times V \times Y$ *and* $A \times V \overset{\alpha\times V}{\rightarrowtail} U \times V \times X$ *in* **C**. *Thus,*

$$A\&B = A \times V + U \times B \overset{\alpha\times V + U\times\beta}{\rightarrowtail} U \times X \times V + U \times V \times Y \cong U \times V \times (X + Y),$$

or using diagrams

$$\begin{array}{ccccc}
A \times V & \xrightarrow{j_1} & A \times V + U \times B & \xleftarrow{j_2} & U \times B \\
\big\downarrow{\scriptstyle \alpha \times V} & & \big\downarrow{\scriptstyle \alpha\&\beta} & & \big\downarrow{\scriptstyle U \times \beta} \\
U \times X \times V & \xrightarrow{j_1} & U \times V \times (X + Y) & \xleftarrow{j_2} & V \times Y \times U
\end{array}$$

Notice that we use the corollary of Proposition 4 to say that $A\&B$ is a monic, thus an object of **DC**.

This determines a bifunctor $\&\colon \mathbf{DC} \times \mathbf{DC} \to \mathbf{DC}$. Given morphisms $(f, F)\colon A \to A'$ and $(g, G)\colon B \to B'$, we have a map $A\&B \to A'\&B'$ given by

$$\begin{array}{ccc}
U \times V & \overset{\alpha\&\beta}{\nleftarrow} & X + Y \\
{\scriptstyle f \times g}\big\downarrow & \nwarrow\!\!\!\!\searrow \; {\scriptstyle F + G} & \\
U' \times V' & \underset{\alpha'\&\beta'}{\nleftarrow} & X' + Y'
\end{array}$$

The bifunctor "&" is a symmetric monoidal structure on the category **DC**, with unit given by the object $\mathbf{1} = (1 \overset{e}{\nleftarrow} 0)$. Intuitively, $(u, v)\alpha\&\beta\binom{x,0}{y,1}$ reads as either $u\alpha x$ or $v\beta y$.

Proposition 5 *The category* **DC** *has cartesian products.*

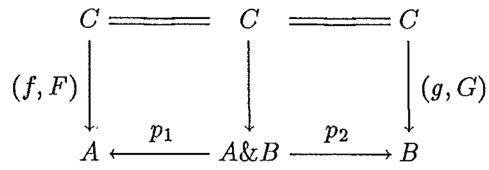
Proof: To check that $A\&B$ does give a categorical product, note that:

- There are canonical projections $p_1\colon A\&B \to A$ and $p_2\colon A\&B \to B$. The map $p_1\colon A\&B \to A$ consists of $(\pi_1, j_1 \cdot \pi_3)$, where $U \times V \xrightarrow{\pi_1} U$ and $j_1 \cdot \pi_3$ is the composition $U \times V \times X \xrightarrow{j_1 \cdot \pi_3} X + Y$. Using diagrams,

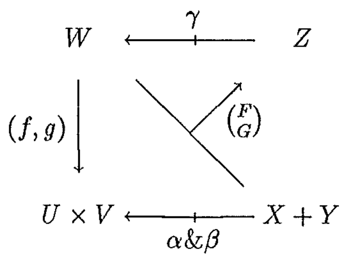
$$\begin{array}{ccc}
U \times V & \overset{\alpha\&\beta}{\nleftarrow} & X + Y \\
{\scriptstyle \pi_1}\big\downarrow & \nwarrow\!\!\!\!\searrow \; {\scriptstyle j_1 \cdot \pi_3} & \\
U & \overset{\alpha}{\nleftarrow} & X
\end{array}$$

Similarly, $p_2\colon A\&B \to B$ consists of $(\pi_2, j_2 \cdot \pi_3)$, where $j_1\colon X \to X + Y$ and $j_2\colon Y \to X + Y$ are canonical injections in **C**.

- The object $A\&B$ has the universal property. Given morphisms in **DC**, $(f,F)\colon C \to A$ and $(g,G)\colon C \to B$ there is a unique map in **DC**, namely $((f,g),\binom{F}{G})\colon C \to A\&B$, making the diagram

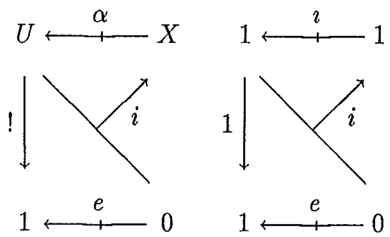


commute. Notice that the universal map corresponds to the diagram below



Again some pullback patching is needed to show $((f,g),\binom{F}{G})$ is a map in **DC**. □

As expected $\mathrm{Hom}_{\mathbf{DC}}(A,\mathbf{1})$ is a singleton, since there is a unique map $U \to 1$ and a unique map $U \times 0 \to X$. Moreover, $[A,\mathbf{1}]_{DC} = (1^U \times X^{U\times 0} \overset{1^\alpha}{\leftrightarrow} U \times 0) \cong \mathbf{1}$, cf. left diagram below. Notice as well that there is a map $I \to \mathbf{1}$, from the unit of the tensor $\otimes$ into the unit of "&", (right diagram)

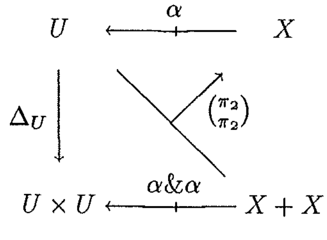


but not conversely, as that would imply $1 \cong 0$ in **C**.

Another remark is that we have a diagonal map with respect to "&",

$$\Delta_A\colon A \to A\&A$$

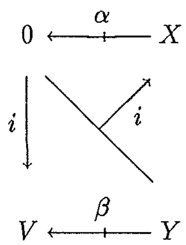
given by:



We do not seem to have all coproducts in **DC**, but we have some special ones, e.g. for $(U \overset{\alpha}{\nleftrightarrow} X)$ and $(V \overset{\beta}{\nleftrightarrow} X)$ the object $(U + V \overset{\alpha+\beta}{\nleftrightarrow} X)$ is a coproduct, where the relation "$\alpha + \beta$" is defined in the obvious way. We also have the following proposition.

Proposition 6 *There is an initial object in the category* **DC**, *given by* $0 = (0 \nleftrightarrow 0)$. *Actually any object of the form* $(0 \overset{\alpha}{\nleftrightarrow} X)$ *is an initial object.*

It is clear that there is a unique map from $(0 \overset{\alpha}{\nleftrightarrow} X)$ into any object $(V \overset{\beta}{\nleftrightarrow} Y)$,



More importantly, we always have *weak-coproducts*, and by that we mean that there is an operation "$\oplus$", not a bifunctor, which satisfies some of the properties of coproducts, namely:

- There are canonical injections $i_1 \colon A \to A \oplus B$, $i_2 \colon B \to A \oplus B$.
- If there are maps $(f, F) \colon A \to D$ and $(g, G) \colon B \to D$, then there is a map $A \oplus B \to D$, but that is not necessarily unique.

Definition 4 *For objects* $(U \overset{\alpha}{\nleftrightarrow} X)$ *and* $(V \overset{\beta}{\nleftrightarrow} Y)$, *the operation* $\oplus$ *is defined by first taking the pullback of* $A \overset{\alpha}{\rightarrowtail} U \times X$ *(respectively* $B \overset{\beta}{\rightarrowtail} V \times Y$*) along* $U \times X^U \overset{(\pi_1, ev_U)}{\longrightarrow} U \times X$ *(respectively* $V \times Y^V \overset{(\pi_1, ev_V)}{\longrightarrow} V \times Y$*), multiplying the new* α' *(resp.* β'*) by* Y^V *(resp.* X^U*),*

$$\begin{array}{ccccccc}
A' & \longrightarrow & A & & B' & \longrightarrow & B \\
\alpha' \downarrow & & \downarrow \alpha & & \beta' \downarrow & & \downarrow \beta \\
U \times X^U & \xrightarrow{(\pi_1, ev)} & U \times X & & V \times Y^V & \xrightarrow{(\pi_1, ev)} & V \times Y
\end{array}$$

and adding it to the correspondent new β', *as the diagram shows.*

$$\begin{array}{ccccc}
A' \times Y^V & \longrightarrow & A' \times Y^V + B' \times X^U & \longleftarrow & B' \times X^U \\
\downarrow \alpha' \times Y^V & & \downarrow \alpha + \beta & & \downarrow \beta' \times X^U \\
U \times X^U \times Y^V & \longrightarrow & (U + V) \times X^U \times Y^V & \longleftarrow & V \times Y^V \times X^U
\end{array}$$

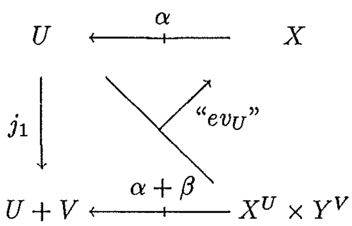
Therefore,

$$(A \oplus B) = (U + V \overset{\alpha \oplus \beta}{\nleftrightarrow} X^U \times Y^V)$$

and the relation "$\alpha \oplus \beta$" reads intuitively as

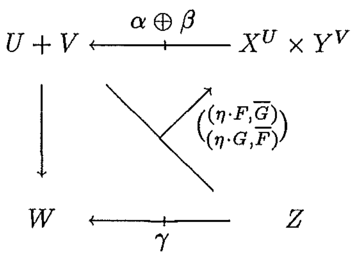
$$\begin{pmatrix} u, 0 \\ v, 1 \end{pmatrix} \alpha \oplus \beta (f, g) \text{ iff } u\alpha f(u) \text{ or } v\beta g(v)\,.$$

The operation $\oplus$ clearly satisfies the conditions above definition 4. For the canonical injections $i_1 \colon A \to A \oplus B$ and $i_2 \colon B \to A \oplus B$, use canonical injections and evaluations in $\mathbf{C}$. For example, to get $i_1 \colon A \to A \oplus B$,



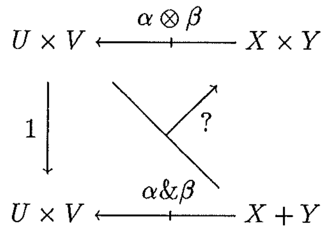
use $j_1 \colon U \to U + V$ and projection followed by evaluation $U \times X^U \times Y^V \overset{\text{"}ev_U\text{"}}{\longrightarrow} X$.

Given maps $A \xrightarrow{(f,F)} C$ and $A \xrightarrow{(g,G)} B$ to get a morphism $A \oplus B \to D$ use the natural map $\binom{f}{g} \colon U + V \to W$ and any of the possible maps in the second coordinate, for example

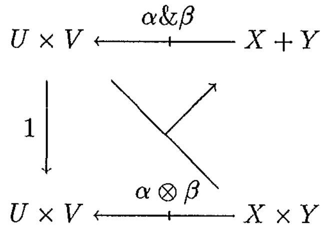


where $U \times Z \xrightarrow{F} X \xrightarrow{\eta} X^U$; $V \times Z \xrightarrow{G} Y$ and $\overline{G}$ is the composition $U \times Z \xrightarrow{\pi_2} Z \xrightarrow{G_t} Y^V$.

Notice that despite the formal similarities between $A \& B$ and $A \otimes B$, there are no natural morphisms between those objects in the category $\mathbf{DC}$. In one direction $A \otimes B \to A \& B$,



because, e.g. there is no necessary map $U \times V \times X \to Y$ and in the other direction,

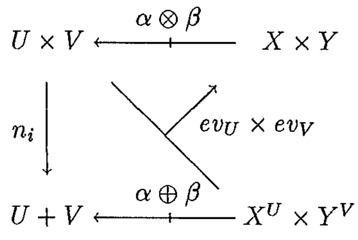


- because even if there are maps of right domain and codomain - the relations are not satisfied, since $u\alpha x$ or $v\beta y$ does not imply $u\alpha x$ and $v\beta u$.

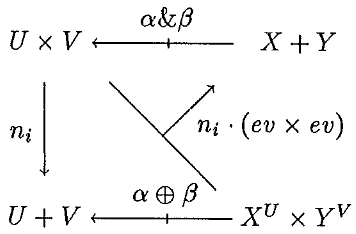
Now we want to relate the bifunctors "&" and "$\otimes$" to the operation "$\oplus$". To do that we use morphisms $X \times Y \to X + Y$. There are two natural maps to consider here, namely $n_1: X \times Y \xrightarrow{\pi_1} X \xrightarrow{j_1} X + Y$ and $n_2: X \times Y \xrightarrow{\pi_2} Y \xrightarrow{j_2} X + Y$. We use either of these maps "n_i" in the following.

We have the following morphisms, in the category **DC**:

- $A \otimes B \to A \oplus B$, given by

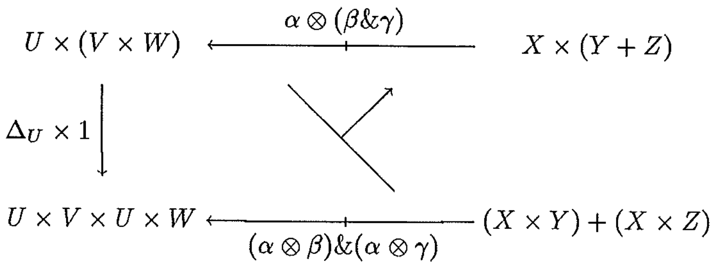


- $A\&B \to A \oplus B$, given by

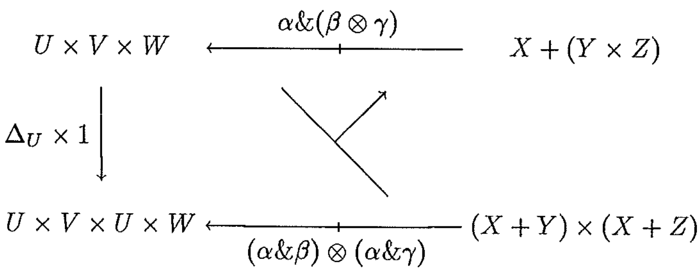


Distributivity Laws

Looking for distributivity laws, we have a natural morphism $A \otimes (B\&C) \to (A \otimes B)\&(A \otimes C)$, as the diagram shows, but not conversely.



Similarly, we have a natural map $A\&(B \otimes C) \to (A\&B) \otimes (A\&C)$, but not conversely.



Another point to mention is that we do not have distributivity of tensor product $\otimes$ over weak-coproduct $\oplus$, nor of categorical product "&" over weak-coproduct $\oplus$. But we do have maps going from $A \otimes (B \oplus C)$ to $(A \otimes B) \oplus (A \otimes C)$ and conversely.

Proposition 7 *The natural morphisms*

$$(i, I): A \otimes (B \oplus C) \leftrightarrow (A \otimes B) \oplus (A \otimes C): (j, J)$$

form a retraction in **DC**, *which means that* $(j, J) \cdot (i, I) = 1_{A \otimes (B \oplus C)}$.

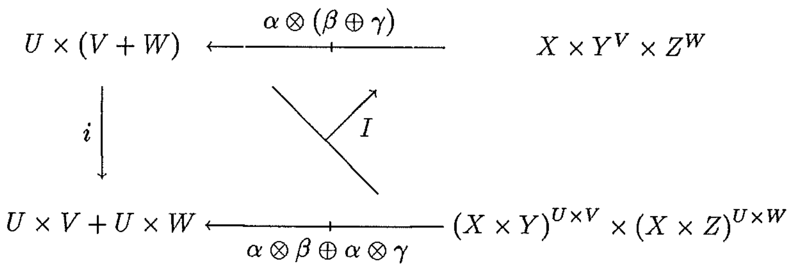
We describe explicitly the map (i, I) which is going to be mentioned again. In the first coordinate, it consists of the usual isomorphism

$$i: U \times (V + W) \to U \times V + U \times W\,.$$

In the second coordinate the map

$$I: U \times (V + W) \times (X \times Y)^{U \times V} \times (X \times Z)^{U \times W} \to X \times Y^V \times Z^W$$

can be decomposed as (H, M, N) where $H = H_1 + H_2$, all of them consisting of evaluations and projections.

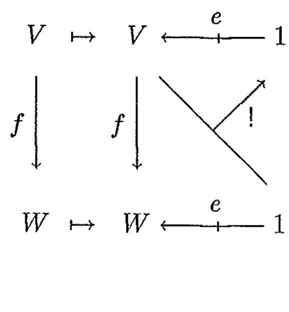


Recapitulating:

C	*finitely complete*	$\Rightarrow$	**DC** *exists*
C	*locally cartesian closed*	$\Rightarrow$	**DC** *monoidal closed*
C	*stable and disjoint coproducts*	$\Rightarrow$	**DC** *has products and weak – coproducts*

1.4 The relationship between C and DC

We can consider a natural "forgetful" functor U: **DC** $\to$ **C**. The functor U: **DC** $\to$ **C** takes an object $(V \overset{\beta}{\nleftarrow} Y)$ to V and a map $(f, F): B \to C$ to $f: V \to W$. This functor U has a left-adjoint, called E: **C** $\to$ **DC**. The functor E: **C** $\to$ **DC** is given by $U \mapsto (U \overset{e}{\nleftarrow} 1)$ where the relation e on $U \times 1$ is the empty relation. The functor E acts on maps $V \to W$ as the diagram shows,



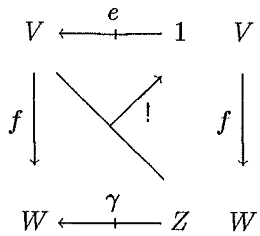
thus, $E(f) = (f, !): EV \to EW$.

Proposition 8 *The functor* $E: \mathbf{C} \to \mathbf{DC}$ *is left-adjoint to the forgetful functor* $U: \mathbf{DC} \to \mathbf{C}$.

It is enough to check the natural isomorphism

$$\mathrm{Hom}_{\mathbf{DC}}(E(V), C) \cong \mathrm{Hom}_{\mathbf{C}}(V, U(C)).$$

This is immediate from the diagram



There are some other "inclusions" of $\mathbf{C}$ into $\mathbf{DC}$ to consider. For example, we could consider the functor $G: \mathbf{C} \to \mathbf{DC}$ which takes U to the object $(U \overset{t}{\nleftrightarrow} U)$, where the relation "$t$" is the total or identity relation in $U \times U$. But G does not seem to have an easy adjoint.

Besides that, as usual, we have a diagonal functor $\Delta: \mathbf{DC} \to \mathbf{DC} \times \mathbf{DC}$ which has a right adjoint, corresponding to the existence of cartesian products "&" in $\mathbf{DC}$.

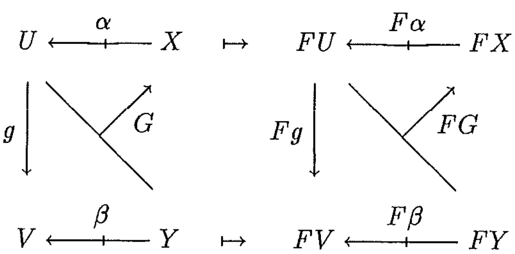
A more interesting observation is that if $\mathbf{B}$ and $\mathbf{C}$ are finitely complete categories and $F: \mathbf{B} \to \mathbf{C}$ is a lex-functor, that is a functor which preserves finite limits, then the construction

$$\mathbf{D}(-): \mathbf{Cat} \to \mathbf{Cat}$$

is functorial. Note that $\mathbf{Cat}$ means the category of (small) categories with functors as morphisms.

Proposition 9 *Given a lex-functor* $F: \mathbf{B} \to \mathbf{C}$, *where* $\mathbf{B}$ *and* $\mathbf{C}$ *are finitely complete categories, we have an induced functor* $\mathbf{D}F: \mathbf{DB} \to \mathbf{DC}$.

The functor $\mathbf{D}F$ *acts on objects* $(U \overset{\alpha}{\nleftrightarrow} X)$ *as* $(FU \overset{F\alpha}{\nleftrightarrow} FX)$ *and on morphisms* $(g, G): A \to B$ *as* $(Fg, FG): FA \to FB$, *as the diagram shows.*



Notice that, as F preserves all finite limits, in particular it preserves monics and pullbacks. Thus, $F(U \overset{\alpha}{\nleftrightarrow} X) = F(A \overset{\alpha}{\rightarrowtail} U \times X) = (FA \overset{F\alpha}{\rightarrowtail} FU \times FX)$ is a monic and a well-defined object in $\mathbf{DC}$, $(FU \overset{F\alpha}{\nleftrightarrow} FX)$. Since F also preserves pullbacks, morphisms in $\mathbf{DB}$ are taken to morphisms in $\mathbf{DC}$.

1.5 Intuitionistic Linear Logic and DC

Linear Logic was recently introduced by Girard in [Gir] 1986. The key idea is to decompose the logical connectives into more primitive ones. Thus the usual implication "$\Rightarrow$" is decomposed into two operations, a binary "$\multimap$", called linear implication and a unary "!", called by Girard the modality "of course!". The Intuitionistic version, or rather the propositional part of the Intuitionistic version of Linear Logic, was described in [G/L] and this section presupposes some acquaintance with these two papers.

In [G/L] categorical models for the propositional fragment of Intuitionistic Linear Logic are briefly considered and *linear categories* are defined, for this purpose, as symmetric monoidal closed categories with cartesian products and coproducts. Those categories should have units for product (**1**), for tensor (**I**) and for coproduct (**0**). An observation is that in [See] 1987, a richer notion of "linear category" - including an involution - is considered.

The aim of this section is to show that **DC** can be considered a categorical model for (the propositional fragment of) Intuitionistic Linear Logic. As we have remarked before, the category **DC** is symmetric monoidal closed and has cartesian products, but it does not have all coproducts, only weak-coproducts. Thus, the aim is to show that the constructions in **DC** satisfy the rules of the Gentzen style system for (propositional) Intuitionistic Linear Logic.

Intuitionistic Linear Logic

We shall not repeat here the motivations behind Linear Logic or Intuitionistic Linear Logic, since these are thoroughly discussed in [Gir] 1986 and [G/L]. But for the sake of self-containment, we describe briefly, the logical system referred to as Intuitionistic Linear Logic.

This system can be conveniently explained using sequents a la Gentzen, but some comments are in order. The main difference between Intuitionistic Linear Logic and usual Intuitionistic Logic, from a proof-theoretic point-of-view, is that in Linear Logic, one is not allowed to use the contraction or weakening (structural) rules, cf. below, when giving a proof.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ (weakening) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (contraction)$$

One of the consequences of the lack of the rules above is that the two possible ways of introducing conjunction, namely,

$$\frac{\Gamma \vdash B \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

are not provably equivalent and thus we have two different kinds of conjunction, respectively, "$\otimes$" - or tensor product - and "&", direct or cartesian product.

The very concise presentation below introduces the linear logic binary connectives $\otimes$, &, $\oplus$ and $\multimap$, and the constants **I**, **1** and **0**, and at the same time describes their behaviour.

Recall the Gentzen style presentation of the rules of Intuitionistic Linear Logic, from [G/L]. Structural Rules:

$$1.\frac{}{A \vdash A}\ (id) \quad 2.\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ (cut) \quad 3.\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ (exc)$$

Logical Rules:

$$1.\ \frac{}{\vdash I} \qquad 2.\ \frac{\Gamma \vdash A}{\Gamma, I \vdash A} \qquad 3.\ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$4.\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad 5.\ \frac{}{\Gamma \vdash \mathbf{1}} \qquad 6.\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$$

$$7.\ \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \qquad 8.\ \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C} \qquad 9.\ \frac{}{\Gamma, 0 \vdash A}$$

$$10.\ \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad 11.\ \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \qquad 12.\ \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$$

$$13.\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad 14.\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

where $\Gamma = (G_1, ..., G_n)$ and $\Delta = (D_1, \ldots, D_k)$ are strings of formulae and Γ, Δ is juxtaposition.

A remark about notation. In [G/L] the constant I is written **1** and **1** is written as "*t*", but I is a far more traditional notation for the unit of the tensor product in category theory. Moreover, **1** is usually kept for the terminal object, thus we make these two modifications.

Notice that some rules might as well be considered as axioms, for example the structural rule (*id*) could be given as the axiom $A \vdash A$.

We call the logical system above *I.L.L*, since later on we want to compare it with Intuitionistic Logic (*I.L*) and Classical Linear Logic (*C.L.L*).

Categorical Interpretation of Propositions-as-Types

Now we want to recapitulate some basics of Categorical Model Theory, or rather, the categorical interpretation of "***propositions-as-types***", cf. [How] or [Gir] 1988. The idea here is to amalgamate two steps, hence we consider "propositions as types" and "types as objects of a category", which gives us "propositions as objects of a (suitable) category". To do that it is by now traditional to define an ***interpretation*** of (the propositional part) of a (any) ***logical system*** $\mathcal{L}$ ***in a category*** $\mathbf{C}$ as a map $| - |_0$ which associates to each atomic formula A of $\mathcal{L}$ an object of the category $\mathbf{C}$.

This definition is only useful if one can extend the interpretation function to all the formulae of the logical system $\mathcal{L}$ considered. This is done by associating the logical connectives in $\mathcal{L}$ to categorical constructions in $\mathbf{C}$. Usually, the categorical constructions involved are limits and colimits. Thus we have a function $| - |$: *Formulae of* $\mathcal{L} \to \mathbf{C}$.

But the point here is to use the structure that exists in the collection of proofs in $\mathcal{L}$ - for example, it makes sense to compose proofs - to establish a correspondence between proofs in $\mathcal{L}$ and morphisms in $\mathbf{C}$. In the syntax, the structure on the collection of proofs is expressed by the rules of the logical system, thus we read the rules as recipes for basic proofs. Then a "deduction" in the logical system corresponds to the existence of a morphism, made out of the composition of basic morphisms, in the category $\mathbf{C}$.

Finally, we say that a category $\mathbf{C}$ is a ***categorical model*** of $\mathcal{L}$, if every entailment in the logical system $\mathcal{L}$, $\Gamma \vdash_{\mathcal{L}} A$, corresponds to the existence of a morphism $|\Gamma| \to_{\mathbf{C}} |A|$ in the category $\mathbf{C}$.

Intuitionistic Linear Logic and DC

In our case we want to propose **DC** as a model for (the propositional part of) Intuitionistic Linear Logic. Therefore we suppose that we are given an interpretation function which maps atomic formulae of *I.L.L* to objects of **DC**, $|A|_0 = (U \overset{\alpha}{\nleftarrow} X)$. We extend that interpretation to the sets of formulae by setting $|\Gamma| = |G_1, ..., G_n| = |G_n| \otimes \ldots \otimes |G_1|$ and by interpreting the connectives $\otimes$, &, $\multimap$, $\oplus$ as the corresponding constructions - the bifunctors $\otimes$, & and $[-,-]_{\mathbf{DC}}$ and the operation $\oplus$ of weak-coproducts, in **DC**.

Then it is straightforward to check that the structures defined for **DC** satisfy the rules above for *I.L.L.*, when we read the rules downwards.

Theorem 1 *The category* **DC** *is a categorical model for (propositional) Intuitionistic Linear Logic and if* $\Gamma \vdash_{I.L.L} A$ *then there exists a morphism in* **DC**, $(f, F): |\Gamma| \to |A|$.

Proof: We have only to check each of the rules presented and they are trivially verified, but for rule 4, which is actually the origin of the interpretation of sets of formulae.

As a notational simplification we write G for $|G_n| \otimes \ldots \otimes |G_1|$, the tensor product of the objects $|G_i|$ in the category **DC** and similarly we write D for $|D_k| \otimes \ldots \otimes |D_1|$.

The structural rule 1 is ensured by the existence of identities in **DC**. Rule 2 is obtained by tensoring and composing maps. Thus $\Gamma \vdash A$ and $\Delta, A \vdash B$ imply that there are morphisms $(f, F): G \to A$ and $(h, H): A \otimes D \to B$ and then the composition $G \otimes D \stackrel{(f,F)\otimes 1}{\longrightarrow} A \otimes D \stackrel{(h,H)}{\longrightarrow} B$ gives $\Gamma, \Delta \vdash B$. Rule 3 comes from the symmetry of the monoidal closed structure, since we have $A \otimes B \cong B \otimes A$.

Actually, the symmetric monoidal structure "$\otimes$" in **DC** ensures logical rules 1 to 4. Rule 1 states the existence of the unit for tensor product, $|I| = (1 \stackrel{i}{\leftrightarrow} 1)$. Rule 2 states a property of the unit I of tensor, namely that if there is a map $G \to A$ then there is a map $G \otimes I \to A$. Rule 3 only says that the tensor product is a bifunctor; if $G \stackrel{(f,F)}{\longrightarrow} A$ and $D \stackrel{(h,H)}{\longrightarrow} B$ then $G \otimes D \stackrel{(f,F)\otimes(h,H)}{\longrightarrow} A \otimes B$.

Rules 5 to 8 are obtained by interpreting $|A\&B|$ as the categorical product of objects $|A|$ and $|B|$, $|A|\&|B|$ in **DC**. Rule 5 says there is always a morphism from any object $|G_n| \otimes \ldots \otimes |G_1|$ into $\mathbf{1} = |1| = (1 \stackrel{e}{\leftrightarrow} 0)$, which is obvious, since $\mathbf{1}$ is a terminal object. Rule 6 corresponds to the existence of a morphism, given by the universal property of products. Thus if $G \stackrel{(f,F)}{\longrightarrow} A$ and $G \stackrel{(h,H)}{\longrightarrow} B$, then $G \stackrel{((f,h),\binom{F}{H}))}{\longrightarrow} A\&B$. Rules 7 and 8 correspond to the existence of canonical projections in **DC**, since e.g., if $G \otimes A \stackrel{(f,F)}{\longrightarrow} C$, then $G \otimes (A\&B) \stackrel{G\otimes p_1}{\longrightarrow} G \otimes A \stackrel{(f,F)}{\longrightarrow} C$.

In addition logical rules 9 to 12 correspond to the weak-coproduct. Notice that the weak-coproduct does not have a unit, but every object of the form $(0 \stackrel{\alpha}{\leftrightarrow} X)$ is an initial object. In particular, in rule 9, the object $\mathbf{0} = (0 \stackrel{\alpha}{\leftrightarrow} 0)$ works, since $G \otimes \mathbf{0} \cong \mathbf{0}$ and there is a morphism into any object A in **DC** from $\mathbf{0}$, given by the initial map in **C**, $0 \to U$ and the identity on 0, $0 \times X \to 0$. Rules 10 and 11 correspond to the existence of canonical injections into the weak-coproduct, since, e.g. if $G \stackrel{(f,F)}{\longrightarrow} A$ then $G \stackrel{(f,F)}{\longrightarrow} A \stackrel{i_1}{\longrightarrow} A \oplus B$.

As we are reading the rules only downwards, Rule 12 corresponds to the weak form of distributivity in **DC**. The weak-distributivity gives the map $(i, I): G \otimes (A \oplus B) \to (A \otimes B) \oplus (A \otimes C)$ mentioned in section 2. If $G \otimes A \stackrel{(f,F)}{\longrightarrow} C$ and $G \otimes B \stackrel{(h,H)}{\longrightarrow} C$ then

$$G \otimes (A \oplus B) \stackrel{(i,I)}{\longrightarrow} (G \otimes A) \oplus (G \otimes B) \stackrel{((\substack{f\\h}), F \times H)}{\longrightarrow} C.$$

Finally, rules 13 and 14 reflect the monoidal closed structure. Thus if $G \otimes A \stackrel{(f,F)}{\longrightarrow} B$, using the adjunction we get $G \stackrel{\overline{(f,F)}}{\longrightarrow} [A, B]_{\mathbf{DC}}$ or $\Gamma \vdash A \multimap B$. Similarly, if $G \stackrel{(f,F)}{\longrightarrow} A$ and $D \otimes B \stackrel{(h,H)}{\longrightarrow} C$ then we have a morphism $G \otimes D \otimes [A, B]_{\mathbf{DC}} \to C$ using the long composition

$$G \otimes D \otimes [A, B]_{\mathbf{DC}} \stackrel{(f,F)\otimes 1}{\to} A \otimes D \otimes [A, B]_{\mathbf{DC}} \cong D \otimes A \otimes [A, B]_{\mathbf{DC}} \stackrel{D\otimes ev}{\to} D \otimes B \stackrel{(h,H)}{\to} C$$

which corresponds to $\Gamma, \Delta, A \multimap B \vdash C$. □

As we said earlier, the proof of Theorem 1 is very easy, but the theorem itself has a very interesting meaning. It says that the intrinsic logic of the "Dialectica Interpretation" is not intuitionistic, which is quite surprising. It can be made so, if one takes, as Gödel did, "decidable" atomic propositions, non-empty types and forgets about the different proofs of a proposition. But in its more general form it is "linear" in Girard's terminology. More about that at the end of Chapter 2.

Chapter 2

The linear connective "!" in DC

The logical idea behind the connective "!" in linear logic is that it should give you the possibility of using the same hypothesis as many times as you wish. Thus, even if there is no diagonal map in **DC** (with respect to $\otimes$) we would like to have a natural map $!A \rightarrow !A \otimes !A$. From that to develop the idea that "!" should be, not only an endofunctor, but a comonad in **DC** is not, perhaps, the most natural thought, but it seems to work.

In the first section we recall some basic facts about comonads and comonoid objects in monoidal categories. Then we describe monoids and comonoids in **C** and **DC**. In the next sections, assuming **DC** with all the structure described in chapter 1, we discuss the comonad "!" (section 3), its basic properties (section 4), a very useful variant of "!" (section 5) and some logical consequences (section 6).

2.1 Preliminaries

Monoids and comonoids in categories

Recall that if $(\mathbf{B}, \Box, I)$ is any monoidal category, where "$\Box$" is the associative bifunctor and I a left and right unit for "$\Box$", we can consider the category **Mon B**, consisting of monoid objects in D with respect to this monoidal structure, cf. page 166 [CWM].

The category **Mon B** consists of triplets $(M, \mu: M \Box M \rightarrow M, \eta: I \rightarrow M)$, where M is an object of **B**, and μ and η are morphisms in **B**, μ the monoid multiplication and η its unit. These maps make the following diagrams commute:

$$\begin{array}{ccc} M \Box M \Box M & \xrightarrow{\mu \Box M} & M \Box M \\ {\scriptstyle M \Box \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ M \Box M & \xrightarrow{\mu} & M \end{array}$$

$$
\begin{array}{ccccc}
M\Box I & \xrightarrow{M\Box\eta} & M\Box M & \xleftarrow{\eta\Box M} & I\Box M \\
\downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow \\
M & = & M & = & M
\end{array}
$$

Morphisms of monoids $f\colon (M,\mu,\eta) \to (M',\mu',\eta')$ are maps $f\colon M \to M'$, which preserve the monoidal structure, thus they make the following diagrams commute:

$$
\begin{array}{ccc}
I & \xrightarrow{\eta} & M \\
\| & & \downarrow{\scriptstyle f} \\
I & \xrightarrow{\eta'} & M'
\end{array}
\qquad
\begin{array}{ccc}
M\Box M & \xrightarrow{\mu} & M \\
{\scriptstyle f\Box f}\downarrow & & \downarrow{\scriptstyle f} \\
M'\Box M' & \xrightarrow{\mu'} & M'
\end{array}
$$

Recall that we say that $(\mathbf{B},\Box,I)$ is a *symmetric* monoidal category if we have "twist" isomorphisms $\tau_{X,Y}\colon X\Box Y \to Y\Box X$, natural in X and Y in $\mathbf{B}$, satisfying some coherence equations, cf. page 180 [CWM].

Notice that, one can also define the category $\mathbf{Comon}\,\mathbf{B}$ of comonoids on $\mathbf{B}$, whose objects are triplets,

$$(C, \delta\colon C \to C\Box C, \varepsilon\colon C \to I)$$

where C is an object in $\mathbf{B}$, δ and ε are morphisms in $\mathbf{B}$ as above, δ called comultiplication and ε the counit. These maps satisfy diagrams dual to the ones for monoids, thus

$$
\begin{array}{ccc}
C & \xrightarrow{\delta} & C\Box C \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle C\Box\delta} \\
C\Box C & \xrightarrow[\delta\Box C]{} & C\Box C\Box C
\end{array}
$$

$$
\begin{array}{ccccc}
C & = & C & = & C \\
\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow \\
C\Box I & \xleftarrow{C\Box\varepsilon} & C\Box C & \xrightarrow{\varepsilon\Box C} & I\Box C
\end{array}
$$

Basic Comonad Theory I

We recall some results from chapter 6 in [CWM], using comonads instead of monads. The proofs are only easy dualisations of MacLane's, so we omit them.

Recall that a comonad $\mathbf{G} = (G,\varepsilon,\delta)$ in a monoidal category $\mathbf{C}$, consists of an endofunctor G and two natural transformations $\varepsilon\colon G \dot{\to} I$ and $\delta\colon G \dot{\to} G^2$, which make the following diagrams commute.

$$
\begin{array}{ccc}
G & \xrightarrow{\delta} & G^2 \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle G\delta} \\
G^2 & \xrightarrow{\delta_G} & G^3
\end{array}
\qquad
\begin{array}{ccccc}
G & = & G & = & G \\
\| & & \downarrow{\scriptstyle\delta} & & \| \\
IG & \xleftarrow{\varepsilon_G} & G^2 & \xrightarrow{G\varepsilon} & GI
\end{array}
$$

Recall as well that every adjunction $< F, U, \eta, \varepsilon >: \mathbf{D} \rightharpoonup \mathbf{C}$ gives rise to a monad in the category $\mathbf{D}$ and a comonad in $\mathbf{C}$. The functor part of the comonad is given by the endofunctor FU, the co-unit of the comonad ε by the co-unit of the adjunction $\varepsilon: FUX \to X$ and the unit of the adjunction $\eta: I \to UF$ yields by composition a natural transformation δ, where $\delta = F\eta U: FUX \to FUFUX$.

Also every comonad $G: \mathbf{C} \to \mathbf{C}$ gives rise to two categories, the category $\mathbf{C}^G$ of G-coalgebras (or Eilenberg-Moore category) and the G-Kleisli category, $\mathbf{C}_G$. The category $\mathbf{C}^G$ has as objects G-coalgebras, that is pairs $(X, h: X \to GX)$, where X is an object of $\mathbf{C}$ and h is a morphism, called the structure map of the coalgebra, which makes both diagrams commute:

$$
\begin{array}{ccc}
X & \xrightarrow{h} & GX \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gh} \\
GX & \xrightarrow{\delta_G} & G^2X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{h} & GX \\
\| & & \downarrow{\scriptstyle\varepsilon_G} \\
X & = & X
\end{array}
$$

A morphism of G-coalgebras is an arrow $f: X \to X'$ of $\mathbf{C}$ which renders commutative the diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{h_X} & GX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gf} \\
X' & \xrightarrow{h_{X'}} & GX'
\end{array}
$$

The G-Kleisli category $\mathbf{C}_G$, has the same objects as $\mathbf{C}$, but $\mathrm{Hom}_{\mathbf{C}_G}(X, Y)$ is, by definition, $\mathrm{Hom}_{\mathbf{C}}(GX, Y)$. Composition of $f: GX \to Y$ and $g: GY \to Z$ is given by:

$$GX \xrightarrow{\delta} G^2X \xrightarrow{Gf} GY \xrightarrow{g} Z.$$

Finally, let $< F, U, \eta, \varepsilon >: \mathbf{D} \rightharpoonup \mathbf{C}$ be an adjunction, $\mathbf{G} = (FU, \varepsilon, \delta)$ the comonad it defines in $\mathbf{C}$. Then there are unique functors $K: \mathbf{D} \to \mathbf{C}^G$ and $L: \mathbf{C}_G \to \mathbf{D}$ making the following diagram commute:

$$
\begin{array}{ccccc}
\mathbf{C}_G & \xrightarrow{L} & \mathbf{D} & \xrightarrow{K} & \mathbf{C}^G \\
{\scriptstyle F_G}\uparrow \dashv \downarrow{\scriptstyle U_G} & & {\scriptstyle F}\uparrow \dashv \downarrow{\scriptstyle U} & & {\scriptstyle F^G}\uparrow \dashv \downarrow{\scriptstyle U^G} \\
\mathbf{C} & = & \mathbf{C} & = & \mathbf{C}
\end{array}
$$

The functors L and K are called "comparison functors". Under certain hypotheses the (unique) functor $K: \mathbf{D} \to \mathbf{C}^G$ can be an equivalence as we shall discuss.

A proposition about comonoids

This proposition is just an application of Beck's Theorem, as well as being part of the folklore of symmetric monoidal closed categories. For the proof we need Beck's Theorem for comonads, which we quote, adapting from [CWM], page 147.

Theorem 2 (Beck) . *Let* $< F, U, \eta, \varepsilon >: \mathbf{C} \rightharpoonup \mathbf{D}$ *be an adjunction,* $\mathbf{G} = (G, \varepsilon, \delta)$ *the comonad it determines in* $\mathbf{D}$*,* $\mathbf{D}^G$ *the category of coalgebras for this comonad. Then the following conditions are equivalent:*

- *The (unique) comparison functor* $K: \mathbf{C} \to \mathbf{D}^G$ *is an equivalence of categories.*
- *The functor* $F: \mathbf{C} \to \mathbf{D}$ *creates equalisers for those parallel pairs* f, g *in* $\mathbf{C}$ *for which* Ff, Fg *has an absolute equaliser in* $\mathbf{D}$*.*

Notice that in the dualisation, one asks for the left-adjoint to create equalisers.

Proposition 10 *If* $\mathbf{B}$ *is any monoidal category and* $U: \mathbf{Comon\,B} \to \mathbf{B}$ *has a right-adjoint* R*, then* U *is comonadic. The functor* U *being comonadic means that* $\mathbf{Comon\,B} \cong \mathbf{B}^G$*=*$G$*-coalgebras, where* G *is the comonad defined in* $\mathbf{B}$ *by the adjunction* $U \dashv R$*.*

Proof: To show the proposition we use the theorem for the adjunction $U \dashv R$. We verify that the forgetful functor U satisfies the condition required, namely that $U: \mathbf{Comon\,B} \to \mathbf{B}$ creates equalisers for those parallel pairs f, g in $\mathbf{Comon\,B}$ for which Uf, Ug has an absolute equaliser in $\mathbf{B}$.

Thus we have to show that, if, in the following diagram, (E, e) is an absolute equaliser in $\mathbf{B}$, it induces an equaliser in $\mathbf{Comon\,B}$ as well.

$$E \rightarrowtail^{e} U(A, \varepsilon_A, \delta_A) \underset{Ug}{\overset{Uf}{\rightrightarrows}} U(B, \varepsilon_B, \delta_B)$$

As the functor U is simply the forgetful functor, we can write the equaliser in $\mathbf{B}$ as

$$E \overset{e}{\rightarrowtail} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

But since the equaliser $E \overset{e}{\rightarrowtail} A \rightrightarrows B$ is an absolute equaliser we have that $(E \times E, e \times e)$ in the diagram below is an equaliser too.

Using $(E \times E, e \times e)$ is an equaliser in $\mathbf{B}$, we define a comultiplication in E, δ_E induced by the comultiplication in A, $\delta_A: A \times A \to A$. Similarly, we can define a co-unit for E, $\varepsilon_E: E \to I$.

$$\begin{array}{ccccc}
E \times E & \overset{e \times e}{\rightarrowtail} & A \times A & \underset{g \times g}{\overset{f \times f}{\rightrightarrows}} & B \times B \\
\uparrow{\scriptstyle \delta_E} & & \uparrow{\scriptstyle \delta_A} & & \uparrow{\scriptstyle \delta_B} \\
E & \overset{e}{\rightarrowtail} & A & \underset{g}{\overset{f}{\rightrightarrows}} & B \\
\downarrow{\scriptstyle \varepsilon_E} & & \downarrow{\scriptstyle \varepsilon_A} & & \downarrow{\scriptstyle \varepsilon_B} \\
I & = & I & = & I
\end{array}$$

Then E with the induced structure is a comonoid object, the map $E \overset{e}{\rightarrowtail} A$ is a comonoid homomorphism and it is easy to see that (E, e) is an equaliser in $\mathbf{Comon\,B}$.

2.2 Monoids and comonoids in C and DC

Recall that to define the category **DC** appropriate to our purposes in Chapter 1, we have considered **C**, a finitely complete, locally cartesian closed category, with stable and disjoint coproducts. We want to describe monoids and comonoids in **C** and **DC**, but before doing that, we prove another folklore proposition about T-algebras on cartesian closed categories.

A proposition about T-algebras

A *strong* functor (also called a $\mathcal{V}$-functor), is usually defined for ***enriched categories***, cf. [Kel] or [E/K], page 444-445. But as any symmetric monoidal closed category (thus any cartesian closed category) can be seen as enriched over itself, the definition of a *strong* functor can be given solely in terms of the symmetric monoidal closed structure.

Notice that we use the exponential notation for internal homs in a cartesian closed category, thus we say, generally, X^Y instead of $[Y,X]_{\mathbf{C}}$. But sometimes the second notation - without the subscript - is more convenient, as it is for example, in the next definition.

Given a symmetric monoidal closed category **C**, a *strong functor* $T\colon \mathbf{C} \to \mathbf{C}$ is a functor such that, for every pair of objects (X,Y) in **C**, there is a map $st_{(X,Y)}\colon [X,Y] \to [TX,TY]$ making the following diagrams commute, cf. [Kel] page 24.

$$\begin{array}{ccc} I & = & I \\ \downarrow & & \downarrow \\ [X,X] & \xrightarrow[st]{} & [TX,TX] \end{array}$$

$$\begin{array}{ccc} [X,Y]\otimes[Y,Z] & \xrightarrow{M} & [X,Z] \\ {\scriptstyle st\otimes st}\downarrow & & \downarrow{\scriptstyle st} \\ [TX,TY]\otimes[TY,TZ] & \xrightarrow[M]{} & [TX,TZ] \end{array}$$

A *strong monad* is a monad $\mathbf{T} = (T,\eta,\mu)$ whose functor part T is a strong functor and whose natural transformations η and μ are strong natural transformations. The natural transformations μ, η being strong make the following diagrams commute, cf. [Koc].

$$\begin{array}{ccc} [X,Y] & = & [X,Y] \\ {\scriptstyle st}\downarrow & & \downarrow{\scriptstyle [1,\eta_Y]} \\ [TX,TY] & \xrightarrow[[\eta_X,1]]{} & [X,TY] \end{array}$$

$$\begin{array}{ccccc} [X,Y] & \xrightarrow{st} & [TX,TY] & \xrightarrow{st} & [T^2X,T^2Y] \\ {\scriptstyle st}\downarrow & & & & \downarrow{\scriptstyle [1,\mu_Y]} \\ [TX,TY] & & \xrightarrow{[\mu_X,1]} & & [T^2X,TY] \end{array}$$

Proposition 11 *If $\mathbf{T} = (T,\eta,\mu)$ is a strong monad on a cartesian closed category* **C**, *then the induced category of T-algebras, $\mathbf{C}^T$ is closed under exponentiation by objects of* **C**. *Thus, if $(X, \theta{:}TX \to X)$ is a T-algebra, then X^Y, where Y is any object in* **C**, *has a natural T-algebra structure.*

Proof: We have to define a natural T-algebra structure for X^Y, thus a map $\theta'{:}T(X^Y) \to X^Y$, using the T-algebra structure of X, $\theta{:}TX \to X$. Since we have a map $\theta^Y{:}(TX)^Y \to X^Y$, what is needed is a map $\alpha{:}T(X^Y) \to (TX)^Y$. But using the fact that T is strong, we have the following transformations:

$$\frac{\dfrac{X^Y \otimes Y \xrightarrow{ev} X}{Y \xrightarrow{\overline{ev}} X^{(X^Y)} \xrightarrow{st} TX^{T(X^Y)}}}{T(X^Y) \xrightarrow{\alpha} TX^Y}$$

We should show that $\theta'{:}T(X^Y) \xrightarrow{\alpha} (TX)^Y \xrightarrow{\theta^Y} X^Y$ is a T-algebra structure, but the commutativity of the necessary diagrams is simply given by η, μ being strong natural transformations. □

A remark is that if **C** is cartesian closed, then any monad whose functor part preserves products, has a unique structure as a strong monad.

Notice that if (X,θ) and (Y,θ') are T-algebras, their tensor product $X \otimes Y$ has a natural T-algebra structure, provided there is a natural transformation $T(X \otimes Y) \xrightarrow{\alpha} TX \otimes TY$, satisfying some conditions. If so, the composition

$$T(X \otimes Y) \xrightarrow{\alpha} TX \otimes TY \xrightarrow{\theta\otimes\theta'} X \otimes Y$$

gives a structure map for the tensor product.

Monoids and comonoids on **C**

Recall that we consider **C** a cartesian closed category, with coproducts. Thus we have two symmetric monoidal structures in **C**, "$\times$" and "$+$". They give rise to four categories, namely, $\mathbf{Mon}_\times\mathbf{C}$, $\mathbf{Mon}_+\mathbf{C}$, $\mathbf{Comon}_\times\mathbf{C}$ and $\mathbf{Comon}_+\mathbf{C}$.

We can easily show that $\mathbf{Mon}_+\mathbf{C} \cong \mathbf{C}$, since every object in **C** has a unique monoidal structure with respect to "$+$". This monoidal structure is given by the (unique) initial map $0 \to X$ and the multiplication, by the folding map $\binom{id}{id}{:}X + X \to X$. The uniqueness can easily be checked from the diagrams.

Dually, we have $\mathbf{Comon}_\times\mathbf{C} \cong \mathbf{C}$ since every object in **C** has a unique comonoid structure. The co-unit is given by the (unique) terminal map $X \to 1$ and the co-unit laws imply that $\delta{:}X \to X \times X$ is really the diagonal map $\Delta_X = (id, id)$.

Moreover, $\mathbf{Comon}_+\mathbf{C} \cong \mathbf{0}$, the degenerate category using the fact that if there is a map $X \to 0$ in a cartesian closed category **C** with coproducts, then $X \cong 0$, [Lambek-Scott] page 67.

The only interesting category is $\mathbf{Mon}_\times\mathbf{C}$, which we write as $\mathbf{Mon}\,\mathbf{C}$. It consists of triplets (X, η_X, μ_X), where $\eta_X{:}1 \to X$ and $\mu_X{:}X \times X \to X$ and the appropriate equations are satisfied.

Free monoids on $\mathbf{C}$

Now consider a category $\mathbf{C}$, cartesian closed, with stable and disjoint coproducts and with *free monoid* structures. By that we mean that there exists a functor $F: \mathbf{C} \to \mathbf{Mon\,C}$, which is left-adjoint to the forgetful functor $U: \mathbf{Mon\ C} \to \mathbf{C}$. In other words, we suppose that we are given an adjunction $< F, U, \eta, \varepsilon >: \mathbf{C} \rightharpoonup \mathbf{Mon\,C}$.

If $\mathbf{C}$ has countable coproducts, such left-adjoint F does exist and it is given by

$$F(X) = (X^* = \coprod_{i \in N} X^i, \eta_{X^*}, \mu_{X^*})$$

cf. [CWM] page 168. The intuition here is to think of X^* as finite sequences of elements of X, thus $\eta_{X^*}: 1 \to X^*$ is the "empty sequence" in X, $1 \mapsto \Lambda$ and $\mu_{X^*}: X^* \times X^* \to X^*$ means "concatenation of sequences".

The adjunction says that every map on $\mathbf{C}$, $X \xrightarrow{f} U(Y, \eta_Y, \mu_Y)$, corresponds by natural isomorphism, to a monoid homomorphism

$$(X^*, \eta_{X^*}, \mu_{X^*}) \xrightarrow{\bar{f}} (Y, \eta_Y, \mu_Y).$$

Notice that the unit of the adjunction $F \dashv U$, $\eta_{X^*}: X \to UFX = X^*$, is given by the canonical injection of X into X^* or intuitively the "singleton sequence". The counit of the adjunction $\varepsilon: FU(Y, \eta_Y, \mu_Y) \to (Y, \eta_Y, \mu_Y)$ which maps

$$(Y^*, \eta_{Y^*}, \mu_{Y^*}) \mapsto (Y, \eta_Y, \mu_Y)$$

is given by multiple applications of the map μ_Y, thus e.g the sequence (y_1, y_2, y_3) is taken by ε to $\mu_Y(y_1, \mu_Y(y_2, y_3))$.

The composite $* = U \cdot F: \mathbf{C} \to \mathbf{C}$ is a monad, its unit is given by the singleton sequence, $\eta^*_X: X \to X^*$ and the multiplication $(\mu^*)_X: X^{**} \to X^*$ by "forgetting parenthesis".

We check that the monad $(*, \eta_*, \mu_*)$ in $\mathbf{C}$, defined by the adjunction $F \dashv U$ is a strong monad. There is clearly a family of maps $st_{X,Y}: X^Y \to X^{*Y^*}$, where $\overline{st}: X^Y \times Y^* \to X^*$ is given, intuitively, by $(f, [y_1, \ldots, y_k]) \mapsto [f(y_1), \ldots, f(y_k)]$. The maps "$st$" satisfy the following diagrams:

$$\begin{array}{ccc} I & = & I \\ \downarrow & & \downarrow \\ [X,X] & \xrightarrow{st} & [X^*, X^*] \end{array}$$

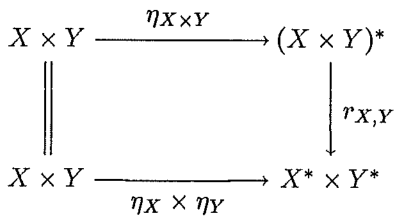
$$\begin{array}{ccc} [X,Y] \times [Y,Z] & \xrightarrow{M} & [X,Z]] \\ {\scriptstyle st \times st}\downarrow & & \downarrow{\scriptstyle st} \\ [X^*,Y^*] \times [Y^*,Z^*] & \xrightarrow[M]{} & [X^*, Z^*] \end{array}$$

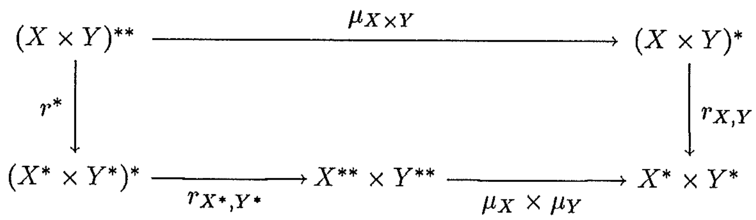
But notice that the functor part of the monad "$*$" does not commute with products, that is, we have maps going both ways $m: X^* \times Y^* \leftrightarrow (X \times Y)^*: r$, but they do not form an isomorphism, nor a retraction. Intuitively, the map $r: (X \times Y)^* \to X^* \times Y^*$ transforms a sequence $[x_1 y_1 \ldots x_k y_k]$

into pairs of sequences $([x_1 \dots x_k], [y_1 \dots y_k])$. The map $m: X^* \times Y^* \to (X \times Y)^*$ transforms the pair of sequences $([x_1, \dots, x_k], [y_1, \dots, y_m])$ into the sequence

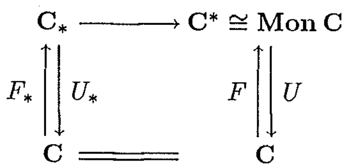
$$(x_1[y_1, y_2, \dots, y_m], \dots, x_k[y_1, \dots, y_m]) = [x_1y_1, \dots, x_1y_m, \dots, x_ky_1, \dots, x_ky_m].$$

Another remark is that the maps $r: (X \times Y)^* \to X^* \times Y^*$ form a natural transformation and moreover, they make the following diagrams commute





Finally, we have the following diagram and the isomorphism of categories $\mathbf{C}^* \cong \mathbf{Mon}\,\mathbf{C}$.

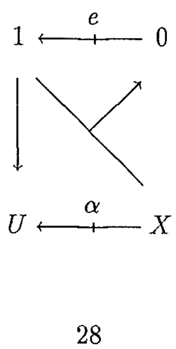


Monoids and comonoids in DC

Now considering the category **DC**, we have two symmetric monoidal structures given by the bifunctors "&" and "⊗". Therefore we can consider categories $\mathbf{Mon}_\otimes\mathbf{DC}$, $\mathbf{Mon}_\&\mathbf{DC}$, $\mathbf{Comon}_\otimes\mathbf{DC}$ and $\mathbf{Comon}_\&\mathbf{DC}$.

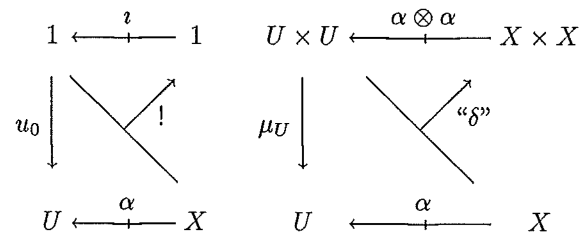
The category $\mathbf{Comon}_\&\mathbf{DC}$ is equivalent to the category **DC**. To see that, just use the result that $\mathbf{Comon}_\times\mathbf{C} \cong \mathbf{C}$, since "&" is the cartesian product in **DC**.

The category $\mathbf{Mon}_\&\mathbf{DC}$ consists of triplets $(A, \eta_A: \mathbf{1} \to A, \mu_A: A\&A \to A)$, where A is an object of **DC**, and η, μ are morphisms in **DC**. The existence of the morphism, $\eta_A: \mathbf{1} \to A$



implies that A is of the form $(U \overset{\alpha}{\nleftarrow} 0)$ and that U is a monoid object in $\mathbf{C}$, thus a triplet $(U, u_0: 1 \to U, \mu_U: U \times U \to U)$. Notice that there are no conditions imposed on the relation "α" and $\mathbf{Mon_{\&}DC} \cong \mathbf{Mon\,C}$.

The category $\mathbf{Mon_{\otimes}DC}$ consists of triplets $(A, \eta_A: I \to A, \mu_A: A \otimes A \to A)$, where A is the object $(U \overset{\alpha}{\nleftarrow} X)$, and η_A and μ_A are morphisms in $\mathbf{GC}$ given by



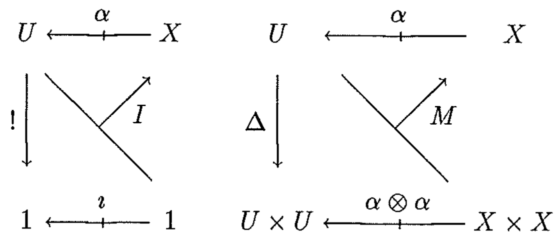
Thus, U is a monoid object in $\mathbf{C}$, $(U, u_0: 1 \to U, \mu_U: U \times U \to U)$. The unit η_A consists of $(u_0, !_X)$ and the multiplication μ_A is given by $(\mu_U, \text{"}\delta\text{"})$. The relation "$\alpha$" satisfies:

- If u_0 is the unit of the monoid U, $u_0 \alpha x$, for all $x \in X$.
- If $u \alpha x$ and $u' \alpha x$ then $\mu_U(u, u') \alpha x$.

The category $\mathbf{Comon_{\otimes}DC}$ consists of triplets

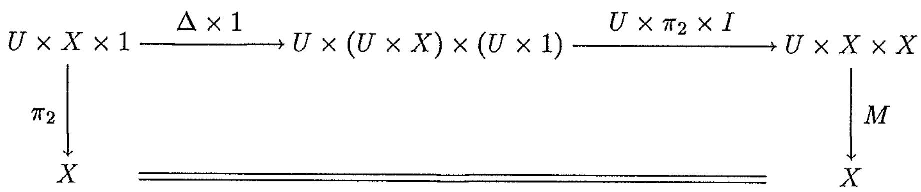
$$(A, \varepsilon_A: A \to I, \delta_A: A \to A \otimes A),$$

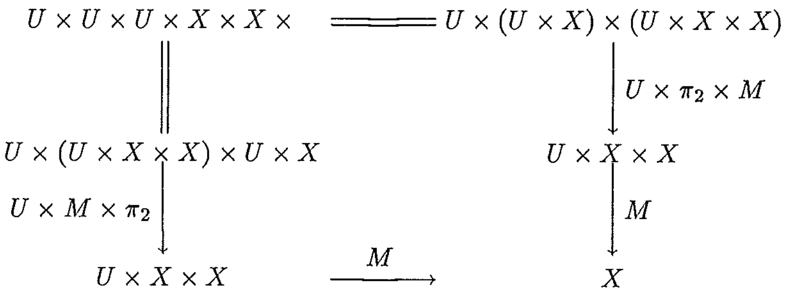
where $A = (U \overset{\alpha}{\nleftarrow} X)$ is an object in $\mathbf{GC}$ and the morphisms, ε_A and δ_A are given by



As the diagram shows, the co-unit of A, $\varepsilon_A: A \to I$ consists of the terminal map $U \xrightarrow{!} 1$ and a morphism $I: U \times 1 \to X$. The comultiplication map $\delta_A: A \to A \otimes A$ consists of the diagonal map $U \xrightarrow{\Delta} U \times U$ and a morphism $M: U \times X \times X \to X$.

Notice that $I: U \times 1 \to X$ and $M: U \times X \times X \to X$ define an U-indexed monoidal structure on X, since they make the following diagrams commute.





The relation "α" satisfies:

- $\forall u \in U,\ u\alpha I(u) = x_0$,
- $u\alpha M(u, x, x') \Rightarrow u\alpha x$ and $u\alpha x'$.

From now on we drop the subscript $\otimes$, since we are only interested in **Comon DC** and **Mon DC** and no confusion can arise.

2.3 A co-free comonad in DC

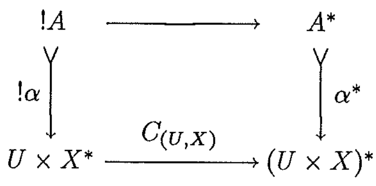
The aim in this section is to define an endofunctor "!: **DC** $\to$ **DC**", a comonad to model the linear connective "!". First, we need to define special maps $C_{(-,-)}$ for pairs of objects V, Y of **C** and to define those maps $C_{(-,-)}$ we use Proposition 2 of last section.

Definition 5 *We can define auxiliary maps $C_{V,Y}$ using the following transformations:*

$$\begin{array}{c} V \times Y \xrightarrow{\eta(V\times Y)} (V \times Y)^* \\ \hline Y \xrightarrow{\overline{\eta}} (V \times Y)^{*V} \\ \hline Y^* \xrightarrow{\overline{C}} (V \times Y)^{*V} \\ \hline V \times Y^* \xrightarrow{C_{(V,Y)}} (V \times Y)^* \end{array}$$

Some explaining is needed on those transformations. The first line is just the unit of the adjunction $F \dashv U$ in **C**. The second line is its exponential transpose. To go from line 2 to line 3, we use Proposition 2, because as $(V \times Y)^*$ has a natural monoid structure, so does $(V \times Y)^{*V}$. Thus a morphism in **C**, $Y \xrightarrow{\overline{\eta}} U((V \times Y)^{*V})$ corresponds naturally, by the adjunction $F \dashv U$ in **C**, to a monoid homomorphism $(Y^*, \eta_*, \mu_*) \to ((V \times Y)^{*V}, \eta, \mu)$, which we write simply as $\overline{C}$. The last line is its exponential transpose.

Definition 6 *The endofunctor "!" is defined on objects of* **DC**, *which are monomorphisms in* **C** *of the form $A \overset{\alpha}{\rightarrowtail} U \times X$. An object $(U \overset{\alpha}{\leftrightarrow} X)$ is taken by the endofunctor "!" to $(U \overset{!\alpha}{\leftrightarrow} X^*)$, where the relation "$!\alpha$" is given by the pullback of $A^* \overset{\alpha^*}{\rightarrowtail} (U \times X)^*$ along $U \times X^* \xrightarrow{C_{(U,X)}} (U \times X)^*$ as the diagram shows:*



The functor "!" acts on morphisms in **DC** as, $!(f,F) = (f, !F): !A \to !B$, where $!F: U \times Y^* \to X^*$ is the composite

$$U \times Y^* \overset{C_{(U,Y)}}{\to} (U \times Y)^* \overset{F^*}{\to} X^*.$$

Intuitively, the relation "$u\alpha x$" is transformed into the relation "$!\alpha$", which is given by

$$\text{"}u(!\alpha)[x_1 \ldots x_k] \text{ iff } u\alpha x_1 \text{ and } u\alpha x_2 \text{ and } \ldots \text{ and } u\alpha x_k\text{"}$$

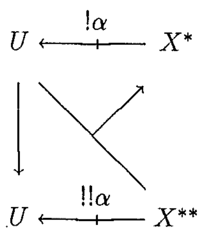
It is easy to check that $(f, !F)$ is a map in **DC** and that "!" does define an endofunctor.

Proposition 12 *The functor "$!: \mathbf{DC} \to \mathbf{DC}$" has a natural comonad $(!, \varepsilon, \delta)$ structure.*

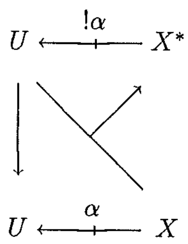
Proof: To describe the comonad we have to exhibit two natural transformations, $\delta: !A \to !!A$ and $\varepsilon: !A \to A$, which make the comonad diagrams below commute.

$$\begin{array}{ccc} !A & \xrightarrow{\delta} & !!A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle !\delta} \\ !!A & \xrightarrow[\delta_{!A}]{} & !!!A \end{array} \qquad \begin{array}{ccccc} !A & = & !A & = & !A \\ \| & & \downarrow{\scriptstyle\delta} & & \| \\ !A & \xleftarrow{\varepsilon_{!A}} & !!A & \xrightarrow{!\varepsilon} & !A \end{array}$$

The natural transformation $\delta_A: !A \to !!A$, maps $(U \overset{!\alpha}{\leftrightarrow} X^*) \mapsto (U \overset{!!\alpha}{\leftrightarrow} X^{**})$, using diagrams



and it is given by the identity in the first coordinate $U \to U$ and U-indexed "forgetting parenthesis" $\mu_*: U \times X^{**} \to X^*$ in the second coordinate. The natural transformation $\varepsilon_A: !A \to A$ or



is given by identity in U and the U-indexed "singleton sequence" on the second coordinate, $\eta_*: U \times X \to X^*$.

An easy manipulation shows that the two natural transformations $\varepsilon_!$ and $\delta_!$ satisfy the comonad equations:

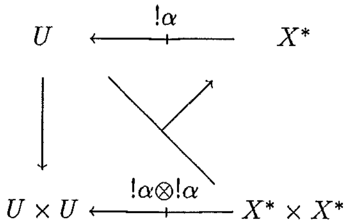
- $!\delta_A \cdot \delta_A = \delta_{!A} \cdot \delta_A$

- $\varepsilon_{!A} \cdot \delta_A = id_{!A} = !\varepsilon \cdot \delta_A$ $\square$

Moreover, the object $!A$ has a natural comonoid structure with respect to $\otimes$, as the next proposition shows.

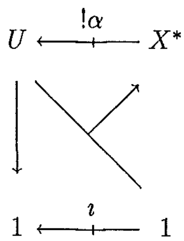
Proposition 13 *There is a functor $R: \mathbf{DC} \to \mathbf{Comon\,DC}$, such that the composition $U \cdot R \cong$ "!".*

Proof: To define R we just check that objects $!A = (U \overset{!\alpha}{\leftrightarrow} X^*)$ admit a natural comonoid structure. Indeed, there is a natural transformation, a comultiplication, $\delta_{!A}: !A \to !A \otimes !A$. That natural transformation maps the object $(U \overset{!\alpha}{\leftrightarrow} X^*) \mapsto (U \times U \overset{!\alpha\otimes!\alpha}{\leftrightarrow} X^* \times X^*)$ or using diagrams



The natural transformation δ is given by diagonal in $\mathbf{C}$, $\Delta: U \to U \times U$ and U-indexed "concatenation of sequences" $C: U \times X^* \times X^* \to X^*$.

Also a co-unit $\varepsilon_{!A}: !A \to I$, or



is easily seen as the canonical unique map to terminal object, $U \to 1$ and U-indexed canonical injection into coproduct, $U \times 1 \to X^*$. So $!A$ has a natural comonoid structure and we call R the functor from $\mathbf{DC}$ to $\mathbf{Comon\,DC}$ which takes A to $(!A, \delta_{!A}: !A \to !A \otimes !A, \varepsilon_{!A}: !A \to I)$.

The next step is to show that "R" provides us with good categorical structure.

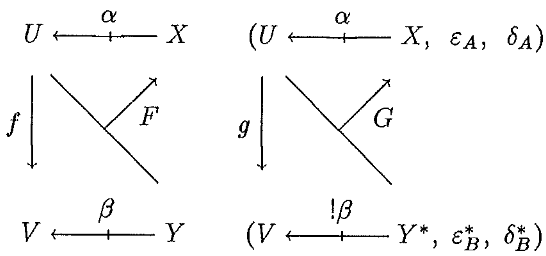
Proposition 14 *The functor $R: \mathbf{DC} \to \mathbf{Comon\,DC}$ is right-adjoint to the forgetful functor,*

$$U: \mathbf{Comon\,DC} \to \mathbf{DC}$$

Proof: We have to show $U \dashv R$, that is

$$\mathrm{Hom}_{\mathbf{DC}}(UA, B) \cong \mathrm{Hom}_{\mathbf{Comon\,DC}}(A, RB).$$

Thus we have to check that for every map $(f, F): A \to B$ in $\mathbf{DC}$ there is a natural comonoid homomorphism $(g, G): (A, \varepsilon_A, \delta_A) \to (!B, \varepsilon_{!B}, \delta_{!B})$ and conversely. Or using diagrams,



But notice that if $(A, \varepsilon_A, \delta_A)$ is a comonoid in $\mathbf{DC}$, where A is an object of the form $(U \overset{\alpha}{\nleftarrow} X)$ then X is a U-indexed monoid in $\mathbf{C}$,

$$X = (X, I: U \times 1 \to X, M: U \times X \times X \to X).$$

Then, using the "monoid" structure of X, we want to show a natural comonoid homomorphism $\tau_A: (A, \varepsilon_A, \delta_A) \to (!A, \varepsilon_{!A}, \delta_{!A})$. If we can show such comonoid homomorphism, given $(f, F): UA \to B$ in $\mathbf{DC}$ we get $(g, G): A \to !B$ via composition,

$$(g, G) = !(f, F) \cdot \tau_A: (A, \varepsilon, \delta) \xrightarrow{\tau_A} (!A, \varepsilon_{!A}, \delta_{!A}) \xrightarrow{!(f,F)} (!B, \varepsilon, \delta).$$

The natural morphism $\tau_A: A \to !A$, is given by identity $U \to U$ in the first coordinate and $T: U \times X^* \to X$ in the second. Formally, T is given by the co-unit ε of the adjunction $F \dashv U$ in $\mathbf{C}$. Intuitively T is obtained using the U-indexed multiplication on X, $M: U \times X \times X \to X$ as many times as necessary to transform the sequence $[x_1, x_2, \ldots, x_k]$ into a single element of X.

Conversely, given $(t, T): A \to RB$ to get $(s, S): UA \to B$ we simply compose (t, T) with the co-unit of the comonad $(!, \varepsilon_!, \delta_!)$, that is, the natural transformation $!B \to B$.

Of course the comonad induced by the adjunction $U \vdash R$, send us back to the described comonad in $\mathbf{DC}$ given by $(!, \varepsilon_!, \delta_!)$.

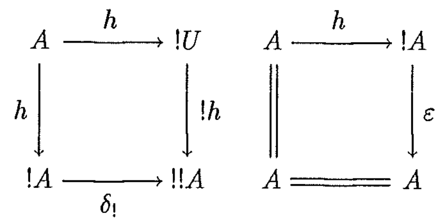
Notice that all this section is very similar to the subsection *"Free monoids in* $\mathbf{C}$*"*, with the obvious differences that $\otimes$ is not a cartesian product and $\mathbf{DC}$ is not cartesian closed.

2.4 Properties of the comonad "!"

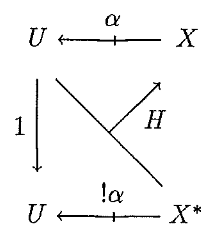
We want to discuss some of the properties of the rather special comonad "!" in $\mathbf{DC}$.

We start by recalling the categories it gives rise to, respectively, $\mathbf{DC}^!$ the !-coalgebras and $\mathbf{DC}_!$, the !-Kleisli category.

The objects of $\mathbf{DC}^!$ are pairs $(A, h: A \to !A)$, where h satisfies the !-coalgebra diagrams below:



Notice that a map $h: A \to !A$ corresponds to a morphism $U \to U$ and a morphism $H: U \times X^* \to X$, as follows



such that $u\alpha H(u, x_1, \ldots, x_k) \Rightarrow u\alpha x_1$ and ... and $u\alpha x_k$.

Morphisms are maps in $\mathbf{DC}$, which preserve the coalgebra structure.

The category $\mathbf{DC}_!$, on the other hand, has the same objects as $\mathbf{DC}$ but

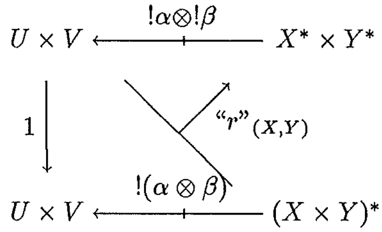
$$\mathrm{Hom}\,\mathbf{DC}_!(A, B) = \mathrm{Hom}\,\mathbf{DC}(!A, B).$$

Recall as well that Propositions 1 and 4, from previous sections, have shown that:

Corollary 2 *The "comparison functor" K: **Comon DC** $\rightarrow$ $\mathbf{DC}^!$ is an equivalence of categories.*

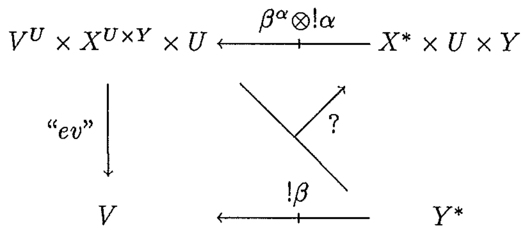
This equivalence can be seen concretely by noticing that given a T-algebra structure map $h: A \rightarrow !A$, its second coordinate $H: U \times X^* \rightarrow X$ necessarily implies the existence of maps $I = H_0: U \times I \rightarrow X$ and $M = H_2: U \times X \times X \rightarrow X$, which satisfy the conditions making A a comonoid. Conversely, if $(A, \varepsilon_A, \delta_A)$ is a comonoid in **DC**, a structure map $h: A \rightarrow !A$ can be built using I and M from ε_A and δ_A, as we did in Proposition 4. The map $h: A \rightarrow !A$ is of the form $(1_U, H)$ and $H: U \times X^* \rightarrow X$ is given by and similarly for any k.

The functor part of the comonad $!: \mathbf{DC} \rightarrow \mathbf{DC}$ has one characteristic of a "monoidal" functor cf. page 473 of [E/K], namely, we have natural transformations $R_{A,B}: !A \otimes !B \rightarrow !(A \otimes B)$, given by



where the maps $r_{X,Y}: (X \times Y)^* \rightarrow X^* \times Y^*$ were described in the subsection *"Free monoids in* **C***"*.

Notice, however, that the comonad "!" is not strong, since there are no natural morphisms $st: [A, B]_{\mathbf{DC}} \rightarrow [!A, !B]_{\mathbf{DC}}$. These, if they existed, would correspond to $\overline{st}: [A, B]_{\mathbf{DC}} \otimes !A \rightarrow !B$, thus to a map in the diagram



On the other hand, it is easy to see that the !-Kleisli category $\mathbf{DC}_!$ inherits the cartesian products from **DC**.

Proposition 15 *The !-Kleisli category $\mathbf{DC}_!$ has products "lifted" from the category* **DC**.

Proof: Actually it is a general fact, in particular, we have the following

$$\mathrm{Hom}_{\mathbf{DC}_!}(C, A\&B) = \mathrm{Hom}_{\mathbf{DC}}(!C, A\&B) \cong \mathrm{Hom}_{\mathbf{DC}}(!C, A) \times \mathrm{Hom}_{\mathbf{DC}}(!C, B)$$
$$= \mathrm{Hom}_{\mathbf{DC}_!}(C, A) \times \mathrm{Hom}_{\mathbf{DC}_!}(C, B).$$

We can also easily verify that the tensor product of two comonoids in **DC** is a comonoid, using the symmetry of the tensor product $\otimes$ bifunctor.

Proposition 16 *The category* **Comon DC** *is closed under tensor products and the comonoid*

$$(A \otimes B, \varepsilon_{A \otimes B}, \delta_{A \otimes B})$$

has natural projections to A and B.

Given $(A, \varepsilon_A, \delta_A)$ and $(B, \varepsilon_B, \delta_B)$ comonoids in **DC**, we have a natural comonoid structure for $A \otimes B$. The co-unit $\varepsilon_{A\otimes B}: A \otimes B \to I$ is given by the map $A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} I \otimes I \cong I$ and the comultiplication morphism $\delta_{A\otimes B}$ is given by

$$A \otimes B \xrightarrow{\delta_A \otimes \delta_B} (A \otimes A) \otimes (B \otimes B) \xrightarrow{A \otimes \tau_{A,B} \otimes B} A \otimes B \otimes A \otimes B,$$

where τ is the symmetry isomorphism associated with $\otimes$.

Projections are given by $A \otimes B \xrightarrow{\varepsilon_A \otimes B} I \otimes B \cong B$ and $A \otimes B \xrightarrow{A \otimes \varepsilon_B} A \otimes I \cong A$.

Using the equivalence $\mathbf{Comon\,DC} \cong \mathbf{DC}^!$, we have the following proposition.

Proposition 17 *The category of !-coalgebras* $\mathbf{DC}^!$ *is closed under tensor products.*

Given !-coalgebras $(A, h: A \to !A)$ and $(B, h': B \to !B)$, their tensor product $A \otimes B$ has a natural coalgebra structure, given by the composition

$$A \otimes B \xrightarrow{h \otimes h'} !A \otimes !B \xrightarrow{R} !(A \otimes B)$$

Notice that the tensor product in **DC** does not become a cartesian product in $\mathbf{Comon}_{\mathbf{DC}}$. If A, B and C are comonoids in **DC**, there are natural maps

$$Hom_{\mathbf{DC}}(C, A) \times Hom_{\mathbf{DC}}(C, B) \leftrightarrow Hom_{\mathbf{DC}}(C, A \otimes B),$$

In the direction "$\to$", the map is given by the composition

$$((f, F), (g, G)) \mapsto C \xrightarrow{\delta_C} C \otimes C \xrightarrow{(f,F)\otimes(g,G)} A \otimes B.$$

Conversely, we use the projections just described, thus $C \xrightarrow{(h,H)} A \otimes B \xrightarrow{p_1} A$ and similarly for B. But neither δ_C nor ε_A need to be a comonoid homomorphism.

A remark is that if $(A, \varepsilon_A, \delta_A)$ and $(B, \varepsilon_B, \delta_B)$ are comonoids in **DC**, their cartesian product $A \& B$ need not be a comonoid, since there is no natural map $\delta_{A\&B}: (A\&B) \to (A\&B) \otimes (A\&B)$.

Another remark is that even if the category **Comon DC** were cartesian, it would not be necessarily closed, since the internal hom of two comonoids $[A, B]_{DC}$ need not be a comonoid.

Notice that we have a natural morphism $!(A\&B) \to !A \otimes !B$ given by the composition

$$!(A\&B) \xrightarrow{\delta} !(A\&B) \otimes !(A\&B) \xrightarrow{!p_1 \otimes !p_2} !A \otimes !B$$

It would be very useful, if we could have an isomorphism between objects $!(A\&B) \cong !A \otimes !B$. To have that we need a natural transformation in **DC**, $!A \otimes !B \to !(A\&B)$, which implies a natural transformation in **C** of the form $U \times V \times (X+Y)^* \to X^* \times Y^*$. If we only consider free *commutative* monoids in **C**, we do have a morphism $(X+Y)^* \to X^* \times Y^*$. Thus, we ask for *commutative* monoid structures in **C**.

2.5 Commutative comonoids in DC

Recall that given a symmetric monoidal structure in **C**, we can consider *commutative* monoid objects in **C**, as well as monoid objects in **C**. By that we mean triplets as before, satisfying the extra condition that the diagram below commutes

$$\begin{array}{ccc} M \times M & \xrightarrow{\tau} & M \times M \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ M & = & M \end{array}$$

Dually, we can consider (co)commutative comonoids in $\mathbf{DC}$, which make the dual diagram commute.

$$\begin{array}{ccc} C & = & C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ C \times C & \xrightarrow{\tau} & C \times C \end{array}$$

From now on we consider the category $\mathbf{C}$ locally cartesian closed, with stable and disjoint coproducts and with *free commutative* monoid structures. Then there is a functor $F_c\colon \mathbf{C} \to \mathbf{Mon}_c\mathbf{C}$ left-adjoint to $U\colon \mathbf{Mon}_c\mathbf{C} \to \mathbf{C}$, where $U\colon \mathbf{Mon}_c\,\mathbf{C} \to \mathbf{C}$ is the forgetful functor. So we suppose we are given an adjunction $< F_c, U, \eta, \varepsilon >\colon \mathbf{C} \rightharpoonup \mathbf{Mon_c}\,\mathbf{C}$. As before, that adjunction induces a monad $\star\colon \mathbf{C} \to \mathbf{C}$, which gives us free commutative monoids in $\mathbf{C}$.

The main difference from the situation we had before is the isomorphism

$$X^\star \times Y^\star \cong (X+Y)^\star,$$

whereas before we only had $X^* \times Y^* \to (X+Y)^*$.

It is still possible to define a comonad, as before, called, $!\colon \mathbf{DC} \to \mathbf{DC}$, taking $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$ to $(U \overset{!\alpha}{\leftarrow\!\!\!+} X^\star)$, where $X^\star$ is the free commutative monoid on X.

The functor "!" naturally induces a functor $F\colon \mathbf{DC} \to \mathbf{Comon}_c\mathbf{DC}$, where the subscript "c" serves to remind us that we are only considering commutative comonoids structures in $\mathbf{DC}$. The functor $F\colon \mathbf{DC} \to \mathbf{Comon}_c\mathbf{DC}$ takes A to $(!A, \varepsilon_!\colon !A \to I, \delta_!\colon !A \to !A \otimes !A)$. Also it naturally induces categories $\mathbf{DC}_!$ and $\mathbf{DC}^!$.

Proposition 18 *We have the adjunction $U \vdash F$, with* $\mathbf{Comon}_c\mathbf{DC} \cong \mathbf{DC}^!$.

Proposition 19 *The endofunctor "!" gives us the isomorphism* $!(A\&B) \cong !A \otimes !B$ *in* $\mathbf{DC}$.

Proof: This can be seen directly, using the isomorphism mentioned above, $X^* \times Y^* \cong (X+Y)^*$. But it also says that the category $\mathbf{DC}^!$ has some cartesian products, induced by the tensor product in $\mathbf{DC}$.

Since F is a right-adjoint it preserves products, $(A\&B)$ is the product in $\mathbf{DC}$, so $F(A\&B) = (!(A\&B), \varepsilon, \delta)$ is isomorphic to the product of comonoids $!A \otimes !B$.

That takes us to the last result in this section. Let the Kleisli category for the commutative comonad "!", $\mathbf{DC}_!$, be called $\mathbf{DNC}$, then we have:

Proposition 20 *The category* $\mathbf{DNC}$*, the Kleisli category associated with the comonad "!" is cartesian closed.*

Proof: We have to verify the natural isomorphism

$$\mathrm{Hom}_{\mathbf{DC}_!}(A\&B, C) \cong \mathrm{Hom}_{\mathbf{DC}_!}(A, [B, C]_{\mathbf{DC}_!}).$$

To check that we look at the following series of equivalences

$$\mathrm{Hom}_{\mathbf{DC}_!}(A\&B, C) = \mathrm{Hom}_{\mathbf{DC}}(!(A\&B), C) \cong \mathrm{Hom}_{\mathbf{DC}}(!A \otimes !B, C)$$

$$\cong \mathrm{Hom}_{\mathbf{DC}}(!A, [!B, C]_{\mathbf{DC}}) = \mathrm{Hom}_{\mathbf{DC}_!}(A, [!B, C]_{\mathbf{DC}}) = \mathrm{Hom}_{\mathbf{DC}_!}(A, [B, C]_{\mathbf{DC}_!}).$$

2.6 Intuitionistic Linear Logic with modality "!".

The aim of this section is to tie up the Linear Logic aspects with the category theory presented in this chapter. So we show that $(\mathbf{DC}, !)$ corresponds to a model of Intuitionistic Linear Logic with modality and that $\mathbf{DC}_!$, the Kleisli category of $\mathbf{DC}$, which is cartesian closed, can be related to $\mathbf{DC}$ in a very interesting way.

Denote by $I.L.L_!$ the logical system consisting of (the propositional part) of Intuitionistic Linear Logic plus the modality "!" rules, which we read off from Girard's paper. These are:

$$\text{I.}\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (\textit{dereliction}) \qquad \text{II.}\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\textit{weakening})$$

$$\text{III.}\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\textit{contraction}) \qquad \text{IV.}\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!)$$

Note that in rule IV the notation $!\Gamma \vdash A$ means, in fact, $!G_1, \ldots, !G_n \vdash A$.

Let $(\mathbf{DC}, !)$ denote the category $\mathbf{DC}$ with the comonad $! = (!, \varepsilon_!, \delta_!)$ described in the last section. Then it is easy, cf. section **5** of Chapter 1, to check that $(\mathbf{DC}, !)$ is a model for Intuitionistic Linear Logic with modality or $I.L.L_!$.

Proposition 21 *The category* $\mathbf{DC}$ *with the comonad "!" is a model of* $I.L.L_!$. *Thus for each* $\Gamma \vdash_{I.L.L_!} A$*, we have a correspondent morphism* $|\Gamma| \rightarrow_{\mathbf{DC}} |A|$ *in* $\mathbf{DC}$.

Proof: We have only to check the rules for the modality.

To have rule I it is enough to have a map $!A \xrightarrow{\varepsilon} A$, since if $G \otimes A \xrightarrow{(f,F)} B$ and $!A \xrightarrow{\varepsilon} A$ we can compose these maps to get $G \otimes !A \xrightarrow{G \otimes \varepsilon} G \otimes A \xrightarrow{(f,F)} B$. But as "!" is a comonad, there is a natural transformation $!A \xrightarrow{\varepsilon} A$.

To have rules II and III it is enough to have maps $!A \rightarrow I$ and $!A \rightarrow !A \otimes !A$ which we have since $!A$ is a comonoid object in $\mathbf{DC}$.

Finally, we do have rule IV. First recall that $!\Gamma$ corresponds to

$$!G_n \otimes \ldots \otimes !G_1 \cong !(G_n \& \ldots \& G_1) = !H$$

by the isomorphism in Proposition 6 in section **5**. Since we have the adjunction, cf. prop. 5, $< U, R, \eta, \varepsilon >$: $\mathbf{Comon\,DC} \rightarrow \mathbf{DC}$, any map $!H \rightarrow A$ corresponds, bijectively by the adjunction, to a map $!H \rightarrow !A$.

Observe that, since "!" is a comonad, given a map $!H \xrightarrow{(f,F)} A$, it is very easy to get another map $!H \rightarrow !A$, by simply composing morphisms $!H \xrightarrow{\delta} !!H \xrightarrow{!(f,F)} !A$. The adjunction allows you to interpret the rule upwards as well as downwards.

Relationship between Intuitionistic Logic and Intuitionistic Linear Logic

The modality "!" was introduced by Girard to recover the strength of Intuitionistic Logic, by means of the following translation, cf. [Gir] 1986.

$$\begin{array}{rcl} A' & = & A \quad \textit{for } A \textit{ atomic} \\ (A \wedge B)' & = & A' \& B' \\ (A \vee B)' & = & !(A') \oplus !(B') \\ (A \rightarrow B)' & = & !(A') \multimap (B') \\ (\neg A)' & = & !(A') \multimap \mathbf{0} \end{array}$$

Using this translation we want to show the proposition below, which is slightly stronger than the corresponding Proposition 4 in [G-L]. Before stating the proposition, we recall the rules for the (propositional) fragment of Intuitionistic Logic.

Structural Rules

$$1.\ \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad 2.\ \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad 3.\ \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

$$4.\ \frac{}{A \vdash A} \qquad 5.\ \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

Logical Rules

$$1.\ \frac{}{\vdash t} \qquad 2.\ \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \qquad 3.\ \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$4.\ \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad 5.\ \frac{}{f \vdash A} \qquad 6.\ \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$7.\ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad 8.\ \frac{\Gamma, A \vdash C \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad 9.\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

$$10.\ \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C}$$

For clarity, in the proposition below, we shall write "$\vdash_{Lin}$" for $\vdash_{I.L.L}$ and "$\vdash_{Int}$" for $\vdash_{I.L}$.

Proposition 22 $\Gamma \vdash_{Int} A$ *iff* $!\Gamma' \vdash_{Lin} A'$

Proof: Notice that $!\Gamma' \vdash_{Lin} A'$ means $!G'_1, \ldots, !G'_n \vdash_{Lin} A'$.

We show the direct implication by structural induction on the deduction $\Gamma \vdash_{Int} A$. Thus, we look at the last application of any of the rules of $I.L$ and check that, if the premises have been translated using $I.L.L$ plus modality rules, then we can get a translation of the consequence. For instance, the structural rule 5, the CUT rule, namely

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

becomes the following deduction:

$$\frac{\dfrac{!\Gamma' \vdash_{\text{Lin}} A'}{!\Gamma' \vdash_{\text{Lin}} !A'} \qquad !\Delta', !A' \vdash_{\text{Lin}} B'}{!\Gamma', !\Delta' \vdash_{\text{Lin}} B'}$$

Another example, structural rule 2, the contraction rule becomes,

$$\frac{!\Gamma', !A', !A' \vdash B'}{!\Gamma, !A' \vdash_{\text{Lin}} B'}$$

where the point is that the use of contraction and weakening is only allowed for formulae of the form "$!A$".

The other structural rules are straightforward applications of the modality rules, but for the exchange rule which is simply linear exchange.

For the logical rules we have to add some steps, in general very easy ones as in the introduction of conjunction on the left, logical rule 3, which becomes

$$\frac{\dfrac{\dfrac{!\Gamma', !A' \vdash_{\mathrm{Lin}} C'}{!\Gamma', !A', !B' \vdash_{\mathrm{Lin}} C'}}{!\Gamma', !A' \otimes !B' \vdash_{\mathrm{Lin}} C'}}{!\Gamma', !(A'\&B') \vdash_{\mathrm{Lin}} C'}$$

where we use the fact that $!(A\&B) \cong !A \otimes !B$ and recall that $A'\&B' \cong (A \wedge B)'$.

The only slightly complicated case is the introduction of "$\rightarrow$" on the left, rule 10, which requires the lemma $!A \multimap !B \vdash_{Lin} !(!A \multimap B)$.

The lemma is easily given by :

$$\frac{\dfrac{!A' \multimap !B' \qquad !B' \multimap B'}{!A' \multimap B'}}{!(!A' \multimap B')}$$

Thus we have, for rule 10,

$$\frac{\dfrac{\dfrac{!\Gamma' \vdash_{\mathrm{Lin}} A'}{!\Gamma' \vdash_{\mathrm{Lin}} !A'} \qquad !\Delta', !B' \vdash_{\mathrm{Lin}} C'}{!\Gamma', !\Delta', !A' \multimap !B' \vdash_{\mathrm{Lin}} C'}}{!\Gamma', !\Delta', !(!A' \multimap B') \vdash_{\mathrm{Lin}} C'}$$

and all the other rules are similar to the ones above.

To show the converse we follow the suggestion in [G-L] and look at the translation which takes linear logic into intuitionistic logic via:

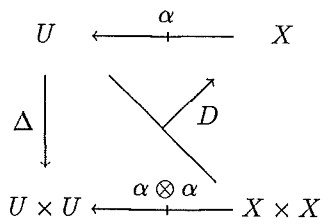
$$\begin{array}{rcl} |!A| & = & A \quad \textit{for } A \textit{ atomic} \\ |A \otimes B| & = & |A| \wedge |B| \\ |A \oplus B| & = & |A| \vee |B| \\ |A \multimap B| & = & |A| \rightarrow |B| \end{array}$$

Then it is trivial to check that for A an intuitionistic formula $|A'| = A$. Moreover, if $\Delta \vdash_{Lin} B$ then $|\Delta| \vdash_{Int} |B|$. Thus if $!(\Gamma') \vdash_{Lin} A'$ then we obtain $|!\Gamma'| \vdash_{Int} |A'|$ which implies $\Gamma \vdash_{Int} A$.

Conceptually, the proposition above reflects the fact that in the same way as **DC** is a model for Intuitionistic Linear Logic, its Kleisli category $\mathbf{DC}_! = \mathbf{DNC}$ is a model for Intuitionistic Logic. It was shown in section 5 that the category **DNC** is cartesian closed, thus if one takes its poset reflection, one gets a Heyting algebra, cf. [Fre] page 18.

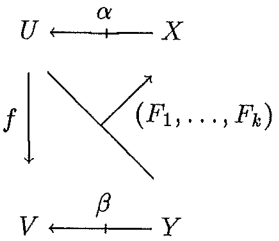
On the other hand, taking the poset reflection of **DC** one ends up with a very odd-looking algebraic structure, unless one assumes that all the relations on **C**, that is, that all the objects in **DC** are decidable and that all objects in **C** are inhabited. If these two conditions hold, then the tensor product and the cartesian product collapse into one, in the poset reflection. If we have that every U in **C** is inhabited, we can provide projections from the tensor product into its components. Decidability is used in a subtler manner.

Recall that the original Dialectica assumed decidability of the atomic formulae. Decidability was essential to prove the consistency of $A \rightarrow A \wedge A$ and the soundness of the whole system depended upon it, cf. Troelstra's comments, [Tro] page 230. The categorical model gives us a glimpse of why that happens. There we do not have, in general, maps $A \rightarrow A \otimes A$, but if A is decidable we can use the following trick to get a map in **DC** from A to $A \otimes A$

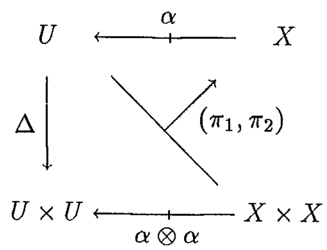


where $D(u,x,x') = \begin{cases} x' & \text{if } u\alpha x \\ x & \text{otherwise} \end{cases}$ makes (Δ, D) a map of the category. Thus, using the map above, we have that A and $A \otimes A$ are equivalent in the poset reflection, which means that the logic is allowing contraction and weakening again.

The last interesting remark is that the Kleisli category $\mathbf{DC}_! = \mathbf{DNC}$ corresponds to the Diller-Nahm variant of the Dialectica Interpretation, cf. [Tro]. For the Diller-Nahm variant of the Dialectica, one does not assume decidability of atomic formulae, but there is a bound on the scope of the quantifier. This bound should correspond, loosely speaking, to a finite number of functions in the second coordinate of objects in the category $\mathbf{DC}$. This would give rise to a category whose objects would be like the objects of $\mathbf{DC}$, but where a morphism from $(U \overset{\alpha}{\nleftarrow} X)$ to $(V \overset{\beta}{\nleftarrow} Y)$ corresponds to a map in $\mathbf{C}$, $f\colon U \to V$ in the first coordinate and finitely many maps, say k, on the second coordinate, $F_1, F_2, \ldots, F_k\colon U \times V \to X$ such that if $u\alpha F_1(u,v)$ and $\ldots$ and $u\alpha F_k(u,v)$ then $f(u)\beta v$.



For instance, in the example above, if one is allowed the use of many maps on the second coordinate, one can certainly get a function from A to $A \otimes A$, by saying, cf. the diagram,



that if $u\alpha(\pi_1(x,x'), \pi_2(x,x'))$ then $(u,u)\alpha \otimes \alpha(x,x')$. The comonad "!" does this "multiplication" of functions in a uniform way.

Chapter 3

The categories GC

This chapter is very similar in essence to Chapter 1. Here we define the categories **GC** and describe some of their categorical structure. The main differences are that, in one hand, the morphisms in **GC** are much easier to handle than those in **DC**; on the other hand there is much more structure to describe in **GC**. We have 6 sections, the first shows that **GC** is a monoidal closed category, with respect to bifunctors tensor "$\oslash$" and internal hom $[-,-]_{\mathbf{GC}}$. The second section describes the bifunctors par or "$\Box$", cartesian product or "&" and coproduct or "$\oplus$". The third section deals with distributivity between those bifunctors. The fourth defines linear negation, while the fifth shows that the construction $\mathbf{G}(-)\colon \mathbf{Cat} \to \mathbf{Cat}$ is functorial and that, under special circumstances, we can define left and right adjoints to the functor $\Delta\colon \mathbf{GC} \to \mathbf{GD}$, induced by the functor $F\colon \mathbf{C} \to \mathbf{D}$. The sixth section makes explicit the connections with Linear Logic.

3.1 Basic definitions

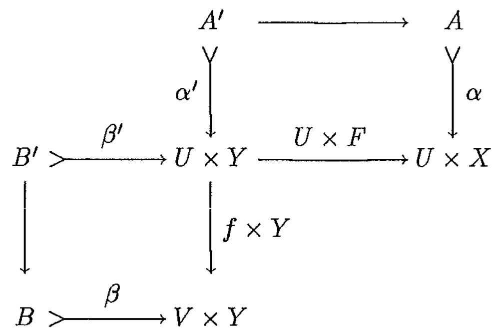
We start with a finitely complete category **C**. Then to describe **GC** say that its objects are relations on objects of **C**, that is monics $A \overset{\alpha}{\rightarrowtail} U \times X$, which we usually write $(U \overset{\alpha}{\leftrightarrow} X)$.

Given two such objects, $(U \overset{\alpha}{\leftrightarrow} X)$ and $(V \overset{\beta}{\leftrightarrow} Y)$, which we call simply A and B, a morphism from A to B consists of a pair of maps in **C**, $f\colon U \to V$ and $F\colon Y \to X$, such that a pullback condition is satisfied, namely that

$$(U \times F)^{-1}(\alpha) \leq (f \times Y)^{-1}(\beta), \tag{1}$$

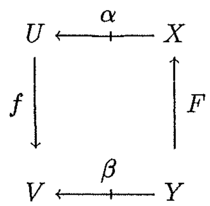
where $(-)^{-1}$ represents pullbacks. Notice that condition (1) above is a simplification of condition $(*)$ in Chapter 1.

We say (f, F) is a morphism in **GC** if there is a (unique) map in **C** $k\colon A' \to B'$ making a commutative triangle in the diagram,



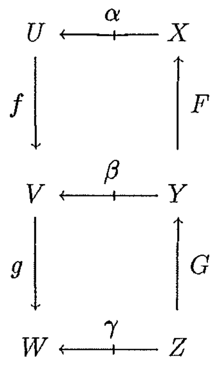
where A' is the pullback of α along $U \times F$ and B' the pullback of β along $f \times Y$. Note that we refer to the object $(U \overset{\alpha}{\nleftarrow} X)$ as "α", meaning the (equivalence class of the) monic, as well as A.

The intuition here is that, if we consider α and β set-theoretic relations, there is a morphism from α to β



iff whenever $u\, \alpha\, F(y)$ then $f(u)\, \beta\, y$.

This time is easy to see **GC** is a category, since composition is just composition in each 'coordinate', thus if $(f, F): A \to B$ and $(g, G): B \to C$ then $(g, G) \circ (f, F) = (gf, FG): A \to C$.



Clearly, if $u\alpha F(y) \Rightarrow f(u)\beta y$ and $v\beta G(z) \Rightarrow g(v)\gamma z$ then

$$u\alpha FG(z) \Rightarrow f(u)\beta G(z) \Rightarrow gf(u)\gamma z.$$

Proposition 23 *Given a finitely complete category* **C**, *the description above gives a category* **GC**.

If we assume that **C** is cartesian closed, the category **GC** has a symmetric monoidal structure (tensor product) denoted by $\oslash$ that makes it symmetric monoidal closed. This tensor bifunctor -

next definition - seems somewhat involved and not very intuitive, but it is exactly what is needed to show that **GC** is monoidal closed. Actually, while in **DC** the tensor product was very natural and the internal hom was contrived to make the category monoidal closed, in **GC** the internal hom is natural and tensor is defined to obtain monoidal closedness.

Definition 7 *Assuming* **C** *is cartesian closed, consider the tensor product in* **GC** *given by the operation* $\oslash\colon \mathbf{GC}\times\mathbf{GC}\to\mathbf{GC}$ *which takes the pair of objects* (A,B) *to*

$$A\oslash B=(U\times V \overset{\alpha\oslash\beta}{\longleftrightarrow} X^V\times Y^U). \tag{2}$$

To describe the relation $\alpha\oslash\beta$ *notice that, pulling back the monic* $A\overset{\alpha}{\rightarrowtail}U\times X$ *along the map* $U\times X^V\times V\overset{(\pi_0,\text{“}ev\text{”})}{\longrightarrow}U\times X$ *we get a new relation* $A'\overset{\alpha'}{\rightarrowtail}U\times X^V\times V$*, (similarly for B)*

$$\begin{array}{ccccccc}
A' & \longrightarrow & A & \quad & B' & \longrightarrow & B\\
{\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha} & & {\scriptstyle\beta'}\downarrow & & \downarrow{\scriptstyle\beta}\\
U\times V\times X^V & \underset{(\pi_0,\text{“}ev\text{”})}{\longrightarrow} & U\times X & & U\times Y^U\times V & \underset{(\pi_3,\text{“}ev\text{”})}{\longrightarrow} & V\times Y
\end{array}$$

and then define $\alpha\oslash\beta$ *as the pullback of* $\alpha'\times Y^U$ *along* $\beta'\times X^V$*. Intuitively,* $(u,v)\alpha\oslash\beta(f,g)$ *iff* $u\alpha f(v)$ *and* $v\beta g(u)$*.*

$$\begin{array}{ccc}
A\oslash B & \rightarrowtail & A'\times Y^U\\
\downarrow & & \downarrow{\scriptstyle\alpha'\times Y^U}\\
B'\times X^V & \underset{\beta'\times X^V}{\rightarrowtail} & U\times V\times X^V\times Y^U
\end{array}$$

To see that the operation $\oslash$ defined above is really a bifunctor, consider morphisms in **GC** $A\overset{(f,F)}{\longrightarrow}A'$ and $B\overset{(g,G)}{\longrightarrow}B'$. Then we have induced maps $A\oslash B\to A'\oslash B$

$$\begin{array}{ccc}
U\times V & \overset{\alpha\oslash\beta}{\longleftrightarrow} & X^V\times Y^U\\
{\scriptstyle f\times V}\downarrow & & \uparrow{\scriptstyle F^V\times Y^f}\\
U'\times V & \underset{}{\overset{\alpha'\oslash\beta}{\longleftrightarrow}} & X'^V\times Y^{U'}
\end{array}$$

and similarly, $A\oslash B\to A\oslash B'$,

$$\begin{array}{ccc}
U\times V & \overset{\alpha\oslash\beta}{\longleftrightarrow} & X^V\times Y^U\\
{\scriptstyle U\times g}\downarrow & & \uparrow{\scriptstyle X^g\times G^Y}\\
U\times V' & \overset{\alpha\oslash\beta'}{\longleftrightarrow} & X^{V'}\times Y'^U
\end{array}$$

Thus we have a morphism $A \oslash B \to A' \oslash B'$ given by $(f \times g, F(-)g \times G(-)f)$, using diagrams,

$$\begin{array}{ccc} U \times V & \xleftarrow{\quad \alpha \oslash \beta \quad} & X^V \times Y^U \\ {\scriptstyle f \times g}\Big\downarrow & & \Big\uparrow{\scriptstyle F(-)g \times G(-)f} \\ U' \times V' & \xleftarrow{\quad \alpha' \oslash \beta' \quad} & X'^{V'} \times Y'^{U'} \end{array}$$

Intuitively, it is easy to see that this is a map in **GC**; if $(u,v)\alpha \oslash \beta(Fh_1g, Gh_2f)$ then $u\alpha F(h_1gv)$ and $v\beta G(h_2fu)$. But since (f,F) and (g,G) are maps in **GC**, then $u\alpha F(x') \Rightarrow f(u)\alpha' x'$ and $v\beta G(y') \Rightarrow g(v)\beta y'$. Thus we have $u\alpha F(h_1gv) \Rightarrow f(u)\alpha' h_1g(v)$ and $v\beta G(h_2fu) \Rightarrow g(v)\beta h_2f(u)$, which corresponds exactly to $(fu, gv)\alpha' \oslash \beta'(h_1gv, h_2fu)$.

The functor "$\oslash$" is not a categorical product, for example projections do not exist necessarily.

$$\begin{array}{ccc} U \times V & \xleftarrow{\quad \alpha \oslash \beta \quad} & X^V \times Y^U \\ {\scriptstyle \pi_1}\Big\downarrow & & \Big\uparrow{\scriptstyle ?} \\ U & \xleftarrow{\quad \alpha \quad} & X \end{array}$$

The object $I = (1 \overset{t}{\leftrightarrow} 1)$ is the unit for the tensor product "$\oslash$", which is associative and symmetric.

Another tensor product, similar to the tensor bifunctor in **DC** can be defined, but it is not very useful, since it is not left-adjoint to the internal hom.

Definition 8 *The bifunctor* $\otimes$: **GC** $\to$ **GC**, *which takes* (A,B) *to*

$$A \otimes B = (U \times V \overset{\alpha \otimes \beta}{\leftrightarrow} X \times Y) \tag{3}$$

is associative and symmetric. It has the same unit $I = (1 \overset{t}{\leftrightarrow} 1)$ *as the bifunctor "$\oslash$".*

Notice that there is a natural transformation $\tau_{(A,B)}: A \oslash B \to A \otimes B$, given by the morphism $(1_{U \times V}, (k_1, k_2))$,

$$\begin{array}{ccc} U \times V & \xleftarrow{\quad \alpha \oslash \beta \quad} & X^V \times Y^U \\ {\scriptstyle 1_{U \times V}}\Big\downarrow & & \Big\uparrow{\scriptstyle (k_1, k_2)} \\ U \times V & \xleftarrow{\quad \alpha \times \beta \quad} & X \times Y \end{array}$$

where $(k_1, k_2): X \times Y \to X^V \times Y^U$ consists of constant maps - or exponential transposes of the projections - in each coordinate. But note that there is no natural transformation in the opposite direction.

Definition 9 *There is an internal hom bifunctor in* **GC**,

$$[-,-]_{\mathbf{GC}}: \mathbf{GC}^{\mathbf{op}} \times \mathbf{GC} \to \mathbf{GC}$$

given by

$$[A,B]_{GC} = (V^U \times X^Y \overset{\beta^\alpha}{\leftrightarrow} U \times Y) \tag{4}$$

where intuitively the relation β^α reads as $(f,F)\beta^\alpha(u,y)$ iff whenever $u\alpha F(y)$ then $f(u)\beta y$.

Formally, we define β^α as the greatest subobject E of $V^U \times X^Y \times U \times Y$ such that $E \wedge A' \leq B'$, where A' is the pullback of A along the map

$$V^U \times X^Y \times U \times Y \overset{(\pi_3, \text{“}ev_Y\text{”})}{\longrightarrow} U \times X,$$

B' is the pullback of B along $V^U \times X^Y \times U \times Y \overset{(\text{“}ev_U\text{”}, \pi_4)}{\longrightarrow} V \times Y$ as the diagram shows,

$$\begin{array}{ccccc}
 & & A' & \longrightarrow & A \\
 & & \alpha' \big\downarrow & & \big\downarrow \alpha \\
B' & \overset{\beta'}{\rightarrowtail} & V^U \times X^Y \times U \times Y & \xrightarrow{(\pi_3,\, \text{“}ev\text{”})} & U \times X \\
\big\downarrow & & \big\downarrow (\text{“}ev\text{”}, \pi_4) & & \\
B & \overset{\beta}{\rightarrowtail} & V \times Y & &
\end{array}$$

and “$\wedge$” means pullback again, cf. Chapter 1.

To guarantee the existence of such greatest subobject, we insist on $\mathbf{C}$ being *locally cartesian closed.* Note that $\mathbf{C}$ being locally cartesian closed, the pullback functors do preserve function spaces.

To show $[-,-]_{\mathbf{GC}}$ is a contravariant functor in the first coordinate and a covariant one in the second coordinate, look at maps $A' \overset{(f,F)}{\to} A$ and $B \overset{(g,G)}{\to} B'$. They induce maps as follows, $[A,B]_{\mathbf{GC}} \to [A',B]_{\mathbf{GC}}$, given by $[(f,F), 1_B]$ or diagrammatically

$$\begin{array}{ccc}
V^U \times X^Y & \overset{\beta^\alpha}{\longleftarrow\!\!\!\!\!+\!\!\!-} & U \times Y \\
V^f \times F^Y \big\downarrow & & \big\uparrow f \times Y \\
V^{U'} \times X'^Y & \overset{\beta'^\alpha}{\longleftarrow\!\!\!\!\!+\!\!\!-} & U' \times Y
\end{array}$$

and $[A,B]_{\mathbf{GC}} \to [A,B']_{\mathbf{GC}}$ given by $[1_A, (g,G)]$ or

$$\begin{array}{ccc}
V^U \times X^Y & \overset{\beta^\alpha}{\longleftarrow\!\!\!\!\!+\!\!\!-} & U \times Y \\
g^U \times X^G \big\downarrow & & \big\uparrow U \times G \\
V'^U \times X^{Y'} & \overset{\beta'^\alpha}{\longleftarrow\!\!\!\!\!+\!\!\!-} & U \times Y'
\end{array}$$

Thus we have an induced map $[A,B]_{\mathbf{GC}} \to [A',B']_{\mathbf{GC}}$ given by

$$\begin{array}{ccc}
V^U \times X^Y & \xleftarrow{\ \beta^\alpha\ } & U \times Y \\
{\scriptstyle \Phi_1 \times \Phi_2}\downarrow & & \uparrow{\scriptstyle f \times G} \\
V'^{U'} \times X'^{Y'} & \xleftarrow{\ \beta'^{\alpha'}\ } & U' \times Y'
\end{array}$$

where Φ_1 is the exponential transpose of the composition

$$V^U \times U' \xrightarrow{V^U \times f} V^U \times U \xrightarrow{ev} V \xrightarrow{g} V',$$

and dually, Φ_2 is the transpose of $X^Y \times Y' \xrightarrow{X^Y \times G} X^Y \times Y \xrightarrow{ev} X \xrightarrow{F} X'$.

Proposition 24 *The category* **GC** *is a symmetric monoidal closed category.*

Proof: It's enough to see the natural isomorphism

$$\mathrm{Hom}_{\mathbf{GC}}(A \oslash B, C) \cong \mathrm{Hom}_{\mathbf{GC}}(A, [B,C]_{\mathbf{GC}}). \tag{5}$$

Using diagrams,

$$\begin{array}{ccc}
U \times V & \xleftarrow{\ \alpha \oslash \beta\ } & X^V \times Y^U \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle (F_1, F_2)} \\
W & \xleftarrow{\ \gamma\ } & Z
\end{array}
\qquad
\begin{array}{ccc}
U & \xleftarrow{\ \alpha\ } & X \\
{\scriptstyle (\overline{f}, \overline{F}_2)}\downarrow & & \uparrow{\scriptstyle \overline{F}_1} \\
W^V \times Y^Z & \xleftarrow{\ \gamma^\beta\ } & V \times Z
\end{array}$$

That is very similar to what we have done for **DC** and so we skip the details. □

As usual with monoidal closed categories we have $A \cong [I,A]_{\mathbf{GC}}$ and evaluation natural transformations, $ev_A: [A,B]_{\mathbf{GC}} \oslash A \to B$ given by morphisms $(f,(F_1,F_2))$,

$$\begin{array}{ccc}
V^U \times X^Y \times U & \xleftarrow{\ \alpha^\beta \times \alpha\ } & (U \times Y)^U \times X^{(V^U \times X^Y)} \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle (F_1, F_2)} \\
V & \xleftarrow{\ \beta\ } & Y
\end{array}$$

where $f: V^U \times X^Y \times U \to V$ is "evaluation" at U and in the second coordinate

$$(F_1, F_2): Y \to (U \times Y)^U \times X^{(V^U \times X^Y)}$$

is given by exponential transposes in **C**. The first component $F_1: Y \to (U \times Y)^U$ is the transpose of $1_{U \times Y}$, and the second component $F_2: Y \to X^{(V^U \times X^Y)}$ the transpose of $V^U \times X^Y \times Y \overset{\text{“}ev_Y\text{”}}{\longrightarrow} X$.

For symmetry reasons that will be apparent later, we want to introduce yet another bifunctor, to be called "□", which is, in some sense, dual to the tensor product "⊘" bifunctor. Note though, that the duality mentioned above cannot be made precise yet, since the operation of "swapping coordinates of an object" - very natural if you think of symmetric relations on **C** - is not a functor in **GC**.

To define the bifunctor "□" categorically we need some extra hypotheses on the category **C**, which up to now had to be finitely complete and locally cartesian closed.

3.2 More structure in GC

For the following definition and propositions we have to require additional categorical structure on **C**, the same way we had to require it in Chapter 1. From now on, we consider categories **C** finitely complete, locally cartesian closed with *stable* (under pullbacks) and *disjoint coproducts*. [see observation about minimality of assumptions in Chapter 1]

Recall that to say a coproduct $A = \coprod_{\alpha\in\Lambda} A_\alpha$ is disjoint means that each of the canonical injections $j_\alpha\colon A_\alpha \to A$ is a monomorphism and for each pair of distinct indices α, α' the pullback of $j_\alpha, j_{\alpha'}$ is the initial object.

Again, we say that A as above is *stable under pullbacks* if given any map $f\colon B \to A$, if we take the pullbacks of each of the canonical injections j_α along $f\colon B \to A$ and call them $f^{-1}A_\alpha$, then $B \cong \coprod_{\alpha\in\Lambda} f^{-1}A_\alpha$ cf. [Makkai-Reyes].

Definition 10 *Consider the bifunctor □ that takes (A, B) to*

$$A\Box B = (U^Y \times V^X \overset{\alpha\Box\beta}{\longleftrightarrow} X \times Y). \qquad (6)$$

The relation defining $A\Box B$ says that $(f,g)\alpha\Box\beta(x,y)$ iff $f(y)\alpha x$ or $g(x)\beta y$. Categorically, the object $A\Box B$ is defined, like the tensor $A\oslash B$, by pulling back α along $U^Y \times X \times Y \overset{(\text{"ev"},\pi_2)}{\longrightarrow} U \times X$, resp. β along $V^X \times X \times Y \overset{(\text{"ev"},\pi_3)}{\longrightarrow} V \times Y$, multiplying the new object $(A' \rightarrowtail U^Y \times X \times Y)$ by V^X, again respectively $(B' \rightarrowtail V^X \times X \times Y)$ by U^Y,

$$\begin{array}{ccccccc}
A' & \longrightarrow & A & & B' & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle\beta} \\
U^Y\times X\times Y & \xrightarrow{(\text{"ev"},\,\pi_2)} & U\times X & & V^X\times X\times Y & \xrightarrow{(\text{"ev"},\,\pi_3)} & V\times Y
\end{array}$$

and then taking the "coproduct map" $A' \times V^X + B' \times U^Y \rightarrowtail U^Y \times X \times Y \times V^X$, as the diagram shows

$$\begin{array}{ccccc}
B'\times U^Y & \xrightarrow{j_2} & A'\times V^X + B'\times U^Y & \xleftarrow{j_1} & A'\times V^X \\
\downarrow & & \downarrow{\scriptstyle\binom{\alpha'\times V^X}{\beta'\times U^Y}} & & \downarrow \\
V^X\times U^Y\times X\times Y & = & U^Y\times V^X\times X\times Y & = & V^X\times U^Y\times X\times Y
\end{array}$$

To see that the operation "□" really determines a bifunctor, we take maps $A \overset{(f,F)}{\longrightarrow} A'$ and

$B \stackrel{(g,G)}{\longrightarrow} B'$ and check the induced morphism $A\Box B \to A'\Box B'$ given by

$$\begin{array}{ccc}
U^V \times V^X & \stackrel{\alpha\Box\beta}{\longleftarrow\!\!\!+\!\!\!-} & X \times Y \\
{\scriptstyle f(-)G \times g(-)F}\downarrow & & \uparrow{\scriptstyle F \times G} \\
U'^{V'} \times V'^{X'} & \stackrel{\alpha'\Box\beta'}{\longleftarrow\!\!\!+\!\!\!-} & X' \times Y'
\end{array}$$

Note that the object $\bot = (1 \stackrel{0}{\leftarrow\!\!\!+} 1)$, where 0 is the empty relation on 1×1, is the unit for the operation "$\Box$" and that there is a natural map $\bot \to I$, given by,

$$\begin{array}{ccc}
1 & \stackrel{0}{\longleftarrow\!\!\!+\!\!\!-} & 1 \\
{\scriptstyle 1}\downarrow & & \uparrow{\scriptstyle 1} \\
1 & \stackrel{\imath}{\longleftarrow\!\!\!+\!\!\!-} & 1
\end{array}$$

but not conversely.

Another remark is that there is a natural transformation of bifunctors $\tau'_{(A,B)}: A \otimes B \to A\Box B$ given by

$$\begin{array}{ccc}
U \times V & \stackrel{\alpha\otimes\beta}{\longleftarrow\!\!\!+\!\!\!-} & X \times Y \\
{\scriptstyle (k_1, k_2)}\downarrow & & \uparrow{\scriptstyle 1} \\
U^Y \times V^X & \stackrel{\alpha\Box\beta}{\longleftarrow\!\!\!+\!\!\!-} & X \times Y
\end{array}$$

where $(k_1, k_2): U \times V \to U^Y \times V^X$ consists of the constant map in each coordinate. Note that this natural transformation can be composed with $\tau: A \oslash B \to A \otimes B$ and so we have a natural transformation $\tau' \circ \tau_{(A,B)}: A \oslash B \to A\Box B$.

Notice that tensor "$\oslash$" and its dual "$\Box$" are very similar, but duality here is transforming the metalanguage "and" into "or".

It is not surprising that **GC** has cartesian products, analogous to the products in **DC**.

Proposition 25 *The category* **GC** *has categorical products.*

Proof: Categorical products are given by the bifunctor $\&: \mathbf{GC} \times \mathbf{GC} \to \mathbf{GC}$, which takes the pair of objects (A, B) in **GC** to

$$A\&B = (U \times V \stackrel{\alpha\&\beta}{\leftarrow\!\!\!+} X + Y) \tag{7}$$

and the relation "$\alpha\&\beta$" is given, intuitively, by $(u, v)\ \alpha\&\beta\ \binom{x,0}{y,1}$ iff either $u\alpha x$ or $v\beta v$.

Categorically, we take the natural coproduct map induced by the morphisms $A \times V \stackrel{\alpha\times V}{\rightarrowtail\!\!\!+}$

$U \times X \times V$ and $B \times U \overset{\beta \times U}{\rightarrowtail} V \times Y \times U$ as the diagram shows,

$$\begin{array}{ccccc}
B \times U & \xrightarrow{j_2} & A \times V + B \times U & \xleftarrow{j_1} & A \times V \\
{\scriptstyle \beta \times U}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha \times V} \\
V \times Y \times U & \xrightarrow{j_2} & U \times V \times (X+Y) & \xleftarrow{j_1} & U \times X \times V
\end{array}$$

Use the corollary of Proposition 4 to say that the coproduct map

$$A \times V + B \times U \overset{(\alpha \times V + \beta \times U)}{\longrightarrow} U \times V \times (X+Y)$$

is monic.

To check "&" is a bifunctor look at maps $A \overset{(f,F)}{\longrightarrow} A'$ and $B \overset{(g,G)}{\longrightarrow} B'$. They induce a map $A\&B \to A'\&B'$ as the diagram shows

$$\begin{array}{ccc}
U \times V & \xleftarrow{\alpha\&\beta} & X + Y \\
{\scriptstyle f \times g}\downarrow & & \uparrow{\scriptstyle F+G} \\
U' \times V' & \xleftarrow{\alpha'\&\beta'} & X' + Y'
\end{array}$$

The object $A\&B$ is a cartesian product, as can easily be checked by noting that

- Projections $p_1: A\&B \to A$ and $p_2: A\&B \to B$ are given by $p_1 = (\pi_1, j_1)$ and $p_2 = (\pi_2, j_2)$, for example,

$$\begin{array}{ccc}
U \times V & \xleftarrow{\alpha\&\beta} & X + Y \\
{\scriptstyle \pi_1}\downarrow & & \uparrow{\scriptstyle j_1} \\
U & \xleftarrow{\alpha} & X
\end{array}$$

  where $\pi_1: U \times V \to U$ and $\pi_2: U \times V \to V$ are canonical projections and $j_1: X \to X + Y$ and $j_2: Y \to X + Y$ canonical injections in $\mathbf{C}$.

- If there are maps $C \overset{(f,F)}{\longrightarrow} A$ and $C \overset{(g,G)}{\longrightarrow} B$, then there is a (unique) morphism

$$C \overset{((f,g),\binom{F}{G})}{\longrightarrow} A\&B$$

  with the universal property, cf. diagram.

$$\begin{array}{ccc}
W & \xleftarrow{\gamma} & Z \\
{\scriptstyle (f,g)}\downarrow & & \uparrow{\scriptstyle \binom{F}{G}} \\
U \times V & \xleftarrow{\alpha\&\beta} & X + Y
\end{array}$$

Also note that the object $\mathbf{1} = (1 \overset{e}{\leftarrow\!\!\!+} 0)$ is the unit for the cartesian product and so it is a terminal object in **GC**, meaning that there is a unique map $A \to \mathbf{1}$, given by the terminal map on U, $U \xrightarrow{!} 1$ and the initial map to X, $0 \xrightarrow{i} X$. Moreover, $[A, \mathbf{1}]_{\mathbf{GC}} = (1^U \times X^0 \overset{1^\alpha}{\leftarrow\!\!\!+} u \times 0) \cong \mathbf{1}$.

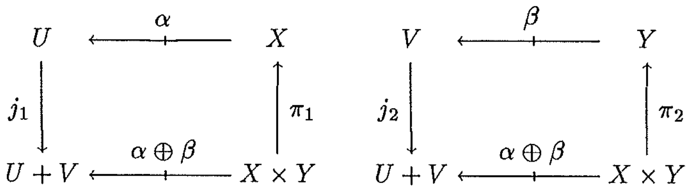
Proposition 26 *The category* **GC** *has categorical coproducts.*

It is easy to see that the construction above can be dualised in the first coordinate. Thus, if we take the coproduct map of $A \times Y \overset{\alpha \times Y}{\rightarrowtail} U \times X \times Y$ and $B \times X \overset{\beta \times X}{\rightarrowtail} V \times Y \times X$ that gives us

$$A \oplus B = (U + V \overset{\alpha \oplus \beta}{\leftarrow\!\!\!+} X \times Y) \tag{8}$$

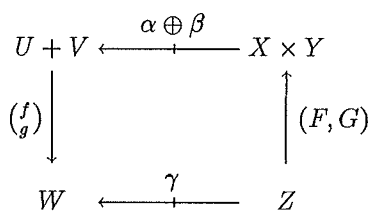
where the natural relation reads as $\binom{u,0}{v,1}\, \alpha \oplus \beta\, (x, y)$ iff either $u\alpha x$ or $v\beta y$.

Clearly we have an endofunctor $\oplus$ and that defines coproducts in **GC**. Canonical injections $i_1 : A \to A \oplus B$ and $i_2 : B \to A \oplus B$ are given by



where $j_1 : U \to U + V$ and $j_2 : V \to U + V$ are canonical injections and $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ canonical projections in **C**.

If $(f, F) : A \to C$ and $(g, G) : B \to C$ then the map $(\binom{f}{g}, (F, G)) : A \oplus B \to C$ has the couniversal property, cf. diagram.



□

The object $\mathbf{0} = (0 \overset{e}{\leftarrow\!\!\!+} 1)$ is the unit for this construction and the initial object. Moreover, $[0, A]_{\mathbf{GC}} \cong \mathbf{1}$. Another remark is that the "or" in the definitions of & and $\oplus$ are given by the coproducts, while the one in the definition of $\Box$ is a real "or".

We now turn our attention to the maps $X \times Y \to X + Y$. There are two very natural maps to consider here, namely $n_1 : X \times Y \xrightarrow{\pi_1} X \xrightarrow{j_1} X + Y$ and $n_2 : X \times Y \xrightarrow{\pi_2} Y \xrightarrow{j_2} X + Y$.

If you think of products as conjunctions and coproducts as disjunctions, the existence of the maps "n_i" tells you that $A \wedge B$ entails $A \vee B$. Note that the nullary version of this implication does not hold, since $T \not\vdash \bot$. Using maps like "n_i" above we have:

Proposition 27 *There are "natural" maps $A \otimes B \to A \oplus B$, $A \& B \to A \Box B$ and $A \& B \to A \oplus B$.*

Those are given by

- $(n_i, id_{X\times Y})\colon A\otimes B \to A\oplus B$,

$$
\begin{array}{ccc}
U\times V & \xleftarrow{\;\alpha\otimes\beta\;} & X\times Y \\
{\scriptstyle n_i}\big\downarrow & & \big\uparrow{\scriptstyle 1_{X\times Y}} \\
U+V & \xleftarrow{\;\alpha\oplus\beta\;} & X\times Y
\end{array}
$$

- $((k_1,k_2),n_i)\colon A\&B \to A\Box B$, where $(k_1,k_2)\colon U\times V \to U^Y\times V^X$ are constant maps,

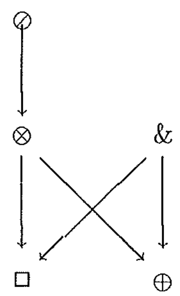
$$
\begin{array}{ccc}
U\times V & \xleftarrow{\;\alpha\&\beta\;} & X+Y \\
{\scriptstyle (k_1,k_2)}\big\downarrow & & \big\uparrow{\scriptstyle n_i} \\
U^Y\times V^X & \xleftarrow{\;\alpha\Box\beta\;} & X\times Y
\end{array}
$$

- and $(n_i,n_i)\colon A\&B \to A\oplus B$,

$$
\begin{array}{ccc}
U\times V & \xleftarrow{\;\alpha\&\beta\;} & X+Y \\
{\scriptstyle n_i}\big\downarrow & & \big\uparrow{\scriptstyle n_i} \\
U+V & \xleftarrow{\;\alpha\oplus\beta\;} & X\times Y
\end{array}
$$

Notice that whenever we say "n_i", we mean choose one of the maps n_1 or n_2.

Summarizing all the remarks on the connectives in **GC** we have the following diagram:



3.3 Distributivity in GC

It is easy to verify that the tensor product "$\oslash$" of **GC** distributes over the coproduct $\oplus$ and dually, that the bifunctor par "$\Box$" distributes over the cartesian product "&".

Proposition 28 *The following isomorphisms hold in* **GC**

$$A \oslash (B \oplus C) \cong (A \oslash B) \oplus (A \oslash C) \quad \textit{and} \quad A\Box(B\&C) \cong (A\Box B)\&(A\Box C).$$

The proof uses isomorphisms in **C**, $i: U\times(V+W) \cong (U\times V)+(U\times W)$ and $j: X^{V+W}\times(Y\times Z)^U \cong X^V \times Y^U \times X^W \times Z^U$, as the diagram shows.

$$\begin{array}{ccc}
U\times(V+W) & \xleftarrow{\alpha\oslash(\beta\oplus\gamma)} & X^{(V+W)}\times(Y\times Z)^U \\
{\scriptstyle i}\downarrow & & \uparrow{\scriptstyle j} \\
U\times V+U\times W & \xleftarrow{(\alpha\oslash\beta)\oplus(\alpha\oslash\gamma)} & X^V\times Y^U\times X^W\times Z^U
\end{array}$$

Dually, of course, we have the isomorphism $A\Box(B\&C) \cong (A\Box B)\&(A\Box C)$.

$$\begin{array}{ccc}
U^{(Y+Z)}\times(V\times W)^X & \xleftarrow{\alpha\Box(\beta\&\gamma)} & X\times(Y+Z) \\
{\scriptstyle j}\downarrow & & \uparrow{\scriptstyle i} \\
U^Y\times V^X\times U^Z\times W^X & \xleftarrow{(\alpha\Box\beta)\&(\alpha\Box\gamma)} & X\times U+X\times Z
\end{array}$$

□

Notice that 'multiplicatives' distribute over 'additives', thus we do not have distributivity of "$\oslash$" over "$\Box$", as in $A\oslash(B\Box C) \cong (A\oslash B)\Box(A\oslash C)$, nor do we have $A\&(B\oplus C) \cong (A\&B)\oplus(A\&C)$.

But there are natural morphisms of the form

$$(A\oslash A')\oslash(B\Box C) \xrightarrow{k} (A\oslash B)\Box(A'\oslash C)$$

or using diagrams,

$$\begin{array}{ccc}
U\times U'\times V^Z\times W^Y & \longleftarrow & (X^{U'}\times X'^{U})^{V^Z\times W^Y}\times(Y\times Z)^{U\times U'} \\
\downarrow & & \uparrow \\
(U\times V)^{X'^W\times Z^{U'}}\times(U'\times W)^{X^V\times Y^U} & \longleftarrow & X^V\times Y^U\times X'^W\times Z^{U'}
\end{array}$$

And symmetrically

$$(A\Box B)\oslash(C\oslash C') \xrightarrow{k'} (A\oslash C)\Box(B\oslash C').$$

They reduce, k, if $A' = I$, respectively k' if $C = I$, to the morphisms $A\oslash(B\Box C) \xrightarrow{i} (A\oslash B)\Box C$ and $(A\Box B)\oslash C' \xrightarrow{i'} A\Box(B\oslash C')$. Those are given by the following diagrams:

$$\begin{array}{ccc}
U\times V^Z\times W^Y & \xleftarrow{\alpha\oslash(\beta\Box\gamma)} & X^{(V^Z\times W^Y)}\times(Y\times Z)^U \\
\downarrow & & \uparrow \\
(U\times V)^Z\times W^{(X^V\times Y^U)} & \xleftarrow[(\alpha\oslash\beta)\Box\gamma]{} & X^V\times Y^U\times Z
\end{array}$$

and

$$\begin{array}{ccc}
U^Y \times V^X \times W' & \overset{(\alpha\Box\beta)\oslash\gamma}{\longleftarrow\!\!\!+\!\!\!-} & (X \times Y)^{W'} \times (Z')^{(U^Y \times V^X)} \\
\downarrow & & \uparrow \\
U^{(Y^{W'} \times Z'^V)} \times (V \times W')^X & \underset{\alpha\Box(\beta\oslash\gamma}{\longleftarrow\!\!\!+\!\!\!-} & X \times (Y^{W'} \times Z'^V)
\end{array}$$

There are also natural morphisms in **GC**, $(A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C)$, and $A \& B \oplus A \& C \to A \& (B \oplus C)$, but not conversely.

3.4 Linear negation in GC

We shall define in **GC** a strong contravariant functor, which induces an involution on a subcategory of **GC**.

Recall that, given a symmetric monoidal closed category **C**, a *contravariant strong functor* $T: \mathbf{C}^{\mathbf{op}} \to \mathbf{C}$ is a functor such that, for every pair of objects (A, B) in **C**, there is a family of maps $st_{(A,B)}: [A, B]_{\mathbf{C}} \to [TB, TA]_{\mathbf{C}}$ making the following diagrams commute.

$$\begin{array}{ccc}
I & = & I \\
\downarrow & & \downarrow \\
[X, X] & \underset{st}{\longrightarrow} & [TX, TX]
\end{array}$$

$$\begin{array}{ccc}
[X, Y] \otimes [Y, Z] & \overset{M}{\longrightarrow} & [X, Z] \\
{\scriptstyle st \otimes st}\downarrow & & \downarrow{\scriptstyle st} \\
[TY, TX] \otimes [TZ, TY] & \underset{M}{\longrightarrow} & [TZ, TX]
\end{array}$$

Definition 11 *Consider the internal hom bifunctor evaluated at* $\perp = (1 \overset{0}{\leftarrow\!\!\!+} 1)$ *in the second coordinate, that is consider* $[-, \perp]_{\mathbf{GC}}$. *This defines a contravariant functor* $(-)^{\perp}: \mathbf{GC}^{op} \to \mathbf{GC}$.

More precisely to each object $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$, the functor $(-)^{\perp}$ associates the object $(X \overset{\perp^{\alpha}}{\leftarrow\!\!\!+} U)$ where the relation "$\perp^{\alpha}$" intuitively says $x \perp^{\alpha} u$ iff whenever $u\alpha x$ then $\perp$. As "$\perp$" is the empty relation, it is never the case, so if we are dealing with decidable relations in **Sets**, $x \perp^{\alpha} u$ iff it is not the case that $u\alpha x$. Hence the name *linear negation.*

Proposition 29 *The functor* $(-)^{\perp}: \mathbf{GC}^{op} \to \mathbf{GC}$ *is a strong contravariant functor.*

To see $(-)^{\perp}$ *is strong, we need to show that there is a family of maps* $st_{(A,B)}$, *where*

$$st_{(A,B)}: [A, B]_{GC} \to [B^{\perp}, A^{\perp}]_{GC}$$

goes from $(V^U \times X^Y \overset{\beta^{\alpha}}{\leftarrow\!\!\!+} U \times Y)$ *to* $(X^Y \times V^U \overset{\perp^{\alpha^{\perp^{\beta}}}}{\leftarrow\!\!\!+} Y \times U)$. *The morphisms st are given by the symmetry* τ *of* **C** *in both coordinates and the diagrams commute without much effort.* $\Box$

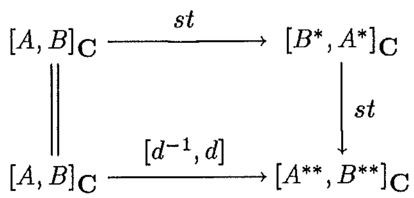
Note that there is always a map in **GC** of the form, $A \to A^{\perp\perp} \cong [[A, \perp], \perp]$, but not conversely.

Now we want to consider the subcategory "Dec GC", whose objects are the decidable objects in **GC**, that is decidable relations on **C**.

Definition 12 *By a decidable object on* **GC** *we mean that* $(U \overset{\alpha}{\nleftrightarrow} X)$ *is such that the canonical map from* $(U \overset{\alpha}{\nleftrightarrow} X)$ *to* $(U \overset{\perp\perp^{\alpha}}{\nleftrightarrow} X)$ *is an isomorphism.*

Our next proposition is to give names to structures. Following Barr, cf. [Bar] page 13, we say that a $*$-autonomous category comprises:

- A symmetric monoidal closed category **C**.
- A strong (contravariant) functor $*: \mathbf{C}^{op} \to \mathbf{C}$, thus the functor $*$ and a family of maps $st^*: [A, B]_{\mathbf{GC}} \to [B^*, A^*]_{\mathbf{GC}}$.
- An isomorphism $d = dA: A \to A^{**}$ such that the following diagram commutes



Proposition 30 *The subcategory* Dec **GC** *is an* $*$*-autonomous category, for* $* = (-)^{\perp}$.

□

3.5 Relationship between C and GC

We can consider at least three natural "forgetful" functors from **GC**. Not surprisingly two of them induce adjunctions.

Besides that, as usual, we have a diagonal functor $\Delta: \mathbf{GC} \to \mathbf{GC} \times \mathbf{GC}$, $A \mapsto (A, A)$ which has left and right adjoints, corresponding to cartesian products and coproducts in **GC**.

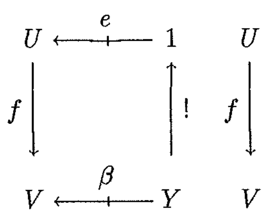
Let us denote by $F_1: \mathbf{GC} \to \mathbf{C}$ the forgetful functor which takes an object $(U \overset{\alpha}{\nleftrightarrow} X)$ to U and a map $(f, F): A \to B$ to $f: U \to V$. Consider also the functor $G_l: \mathbf{C} \to \mathbf{GC}$, given by $V \mapsto (V \overset{e}{\nleftrightarrow} 1)$ where the relation e on $V \times 1$ is the empty relation.

Proposition 31 *The functor* G_l *is left-adjoint to the functor* F_1.

It is enough to check the natural isomorphism

$$\mathrm{Hom}_{\mathbf{GC}}(G_l(U), B) \cong \mathrm{Hom}_{\mathbf{C}}(U, F_1(B)),$$

which is trivial from the diagram



On the same lines we can consider the functor $G_r\colon \mathbf{C} \to \mathbf{GC}$, which takes V to $(V \overset{\bullet}{\leftarrow\!\!\!\!+} 0)$, where the relation "$\bullet$" corresponds to the identity relation on $V \times 0$.

Proposition 32 *The functor F_1 is left-adjoint to the functor G_r.*

It is easy to check the natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(F_1(A), V) \cong \mathrm{Hom}_{\mathbf{GC}}(A, G_r(V)),$$

or using diagrams

$$\begin{array}{ccccc} U & & U & \overset{\alpha}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle !} \\ V & & V & \overset{\bullet}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & 0 \end{array}$$

Combining the results that we have obtained above, we have the 3-adjoint situation

$$G_l \dashv F_1 \dashv G_r$$

Using the symmetry of $\mathbf{GC}$, denote by F_2 the forgetful functor which takes the object $(U \overset{\alpha}{\leftarrow\!\!\!\!+} X)$ to X and the map $(f, F)\colon A \to B$ to $F\colon Y \to X$. Note that F_2 is a contravariant functor. By symmetry we have functors H_l and H_r which satisfy $H_l \dashv F_2 \dashv H_r$.

We could also consider the forgetful functor $F_3\colon \mathbf{GC} \to \mathbf{C} \times \mathbf{C}^{\mathbf{op}}$, which forgets the relations.

Notice that if F is a lex-functor, that is if F preserves all finite limits, we have that the construction $G(-)\colon \mathbf{Cat} \to \mathbf{Cat}$ is functorial.

Proposition 33 *Given a lex-functor $F\colon \mathbf{C} \to \mathbf{D}$, between finitely complete, locally cartesian closed categories $\mathbf{C}$ and $\mathbf{D}$, it induces a functor*

$$\mathbf{G}F\colon \mathbf{GC} \to \mathbf{GD}.$$

Proof: If F preserves all finite limits, in particular it preserves monics and pullbacks. Thus, given an object $(U \overset{\alpha}{\leftarrow\!\!\!\!+} X)$ in $\mathbf{GC}$, simply applying F to it, gives us a monic $FA \overset{F\alpha}{\rightarrowtail} FU \times FX$ in $\mathbf{GD}$ or $(FU \overset{F\alpha}{\leftarrow\!\!\!\!+} FX)$. Also $\mathbf{G}F$ acts on maps $(g, G)\colon A \to B$ as $(Fg, FG)\colon FA \to FB$.

$$\begin{array}{ccccccc} U & \overset{\alpha}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & X & \mapsto & FU & \overset{F\alpha}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & FX \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle G} & & {\scriptstyle Fg}\downarrow & & \uparrow{\scriptstyle FG} \\ V & \overset{\beta}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & Y & \mapsto & FV & \overset{F\beta}{\longleftarrow\!\!\!\!\!\!\!+\;\;} & FY \end{array}$$

Notice that (Fg, FG) being a map in $\mathbf{GD}$ is a immediate consequence of the pullback preservation property of F.

As the last result in this section we want to show the following proposition.

Proposition 34 *Given a 3-adjoint situation $\mathbf{C} \underset{\leftarrow}{\rightleftarrows} \mathbf{D}$ where $\Sigma \dashv \Delta \dashv \Pi$ and Σ is a faithful functor, that gives rise to a 3-adjoint situation $\mathbf{GC} \underset{\leftarrow}{\rightleftarrows} \mathbf{GD}$, where $L \dashv \Delta \dashv R$.*

Proof: The functor Δ is a right-adjoint, thus it preserves limits and so it defines a functor, still called $\Delta\colon \mathbf{GC} \to \mathbf{GD}$, as in Proposition 11. We show that Δ which maps $(U \overset{\alpha}{\leftarrow\!\!\!\!+} X)$ to $(\Delta U \overset{\Delta\alpha}{\leftarrow\!\!\!\!+} \Delta X)$ has a right-adjoint $R\colon \mathbf{GC} \to \mathbf{GD}$.

For an object $(V \overset{\beta}{\leftarrow\!\!\!\!+} Y)$ in $\mathbf{GD}$, define $R(V \overset{\beta}{\leftarrow\!\!\!\!+} Y) = (\Pi V \overset{R\beta}{\leftarrow\!\!\!\!+} \Sigma Y)$. As usual the difficulty lies in defining the new relation "$R\beta$".

Recall that, as we have two adjunctions $\Sigma \vdash \Delta \vdash \Pi$, we have two units and two co-units, as follows $\eta_1\colon Y \to \Delta\Sigma Y$, $\varepsilon_1\colon \Sigma\Delta X \to X$, $\eta_2\colon U \to \Pi\Delta U$ and $\varepsilon_2\colon \Delta\Pi V \to V$. Since Σ is a faithful functor, η_1 is monic for every Y in $\mathbf{C}$.

To define the relation "$R\beta$" compose $B \overset{\beta}{\rightarrowtail} V \times Y$ with $V \times Y \overset{V\times\eta_1}{\rightarrowtail} V \times \Delta\Sigma Y$ and pullback the composite along the map $\Delta\Pi V \times \Delta\Sigma Y \overset{\varepsilon_2 \times \Delta\Sigma Y}{\longrightarrow} V \times \Delta\Sigma Y$, as the diagram shows.

$$\begin{array}{ccccc}
B' & \rightarrowtail & \Delta\Pi V \times Y & \overset{\Delta\Pi V \times \eta_1}{\rightarrowtail} & \Delta\Pi V \times \Delta\Sigma Y \\
\downarrow & & \downarrow{\scriptstyle \varepsilon_2 \times Y} & & \downarrow{\scriptstyle \varepsilon_2 \times \Delta\Sigma Y} \\
B & \overset{\beta}{\rightarrowtail} & V \times Y & \overset{V \times \eta_1}{\rightarrowtail} & V \times \Delta\Sigma Y
\end{array}$$

Then apply the functor $\Pi\colon \mathbf{D} \to \mathbf{C}$, which preserves monics, to the big pullback diagram above and pullback the result along η_2, as follows

$$\begin{array}{ccccc}
R\beta & \rightarrowtail & & \longrightarrow & \Pi V \times \Sigma Y \\
\downarrow & & & & \downarrow{\scriptstyle \eta_2} \\
\Pi B' & \rightarrowtail & \Pi(\Delta\Pi V \times Y) & \rightarrowtail & \Pi\Delta(\Pi V \times \Sigma Y) \\
\downarrow & & \downarrow & & \downarrow \\
\Pi B & \rightarrowtail & \Pi(V \times Y) & \rightarrowtail & \Pi(V \times \Delta\Sigma Y)
\end{array}$$

The functor R acts on maps as $R(f, F) = (\Pi f, \Sigma F)$ and it is easy to check the adjunction $\Delta \vdash R$, since in the diagram,

$$\begin{array}{ccc}
\Delta U & \overset{\Delta\alpha}{\leftarrow\!\!\!\!+} & \Delta X \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle F} \\
V & \overset{\beta}{\leftarrow\!\!\!\!+} & Y
\end{array}
\qquad
\begin{array}{ccc}
U & \overset{\alpha}{\leftarrow\!\!\!\!+} & X \\
{\scriptstyle \overline{f}}\downarrow & & \uparrow{\scriptstyle \overline{F}} \\
\Pi V & \overset{R\beta}{\leftarrow\!\!\!\!+} & \Sigma Y
\end{array}$$

f corresponds to $\overline{f}$, using the first adjunction and F corresponds to $\overline{F}$ by the second adjunction in $\mathbf{C}$. Dually, we can define $L\colon \mathbf{GD} \to \mathbf{GC}$ using the same construction on the first coordinate of objects in $\mathbf{GC}$. Thus we have the 3-adjoint situation $L \dashv \Delta \dashv R$, between categories $\mathbf{GD}$ and $\mathbf{GC}$.

3.6 Classical Linear Logic and GC.

The category **GC** came into existence aiming to be a categorical model of Classic Linear Logic. It stems from a suggestion of Girard in Boulder 87, to whom I am very grateful, and to a great extent it fulfils its promise. In particular, the category **GC** is a very interesting model of Classical Linear Logic, since it does not collapse the units of "tensor" and "par" into a single object.

But to show that **GC** is in fact a model of C.L.L is not as straightforward as it was before. Because of the huge symmetries of C.L.L, it does not fit as nicely into a pattern of directed morphisms as does the Intuitionistic fragment. We have 2 equivalent presentations of Classical Linear Logic with slight variations in notation.

The original one, cf. [Gir] 1986 page 22, presented below is very sleek and elegant, but it is hard to read off a categorical model from it.

Identity rule :

$$1.\ \frac{}{\vdash \mathrm{A}, \mathrm{A}^{\perp}}$$

Cut rule :

$$2.\ \frac{\vdash \mathrm{A}, \Gamma \qquad \vdash \mathrm{A}^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

Exchange rule :

$$3.\ \frac{\vdash \Gamma}{\vdash \Gamma'}$$

where Γ' is obtained by permuting the formulas of Γ

Additive rules:

$$\frac{}{\vdash t, A} \qquad \frac{\vdash A, \Gamma \qquad B \vdash \Gamma}{\vdash A \& B, \Gamma}$$

$$\frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}$$

Multiplicative rules:

$$\frac{}{\vdash 1} \qquad \frac{\vdash A}{\vdash 1, A}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash A, B, \Gamma}{\vdash A \Box B, \Gamma}$$

Note that there are no rules for $(-)^{\perp}$, **0** and $\perp$. Again, following the categorical tradition, we shall replace "t" by **1** and 1 by I. [Also due to typographical reasons we use □ for Girard's *par*, usually written as an upside down ampersand.]

Seely in [See] 1987, on the other hand, gives a presentation, which is geared towards the symmetries and thus more helpful. In his presentation a sequent has the form

$$G_1, G_2, \ldots, G_n \vdash D_1, D_2, \ldots, D_m,$$

where the commas on the left should be thought as some kind of conjunction and those on the right, some kind of disjunction.

A (propositional) Classic Linear Logic consists of formulae and sequents. Formulae are generated by the binary connectives $\otimes$, □, &, $\oplus$ and $\multimap$ and by the unary operation $(-)^{\perp}$, from a set of constants including I, $\perp$, **1** and **0** and from variables.

The sequents are generated by the following rules, from initial sequents or axioms.

$$\begin{array}{ccl c ccl}
A & \vdash & A & (\textit{identity}) & & & \\
 & \vdash & I & & \perp & \vdash & \\
\Gamma & \vdash & 1, \Delta & & \Gamma, 0 & \vdash & \Delta \\
A & \vdash & A^{\perp\perp} & & A^{\perp\perp} & \vdash & A \quad (\textit{negation})
\end{array}$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \tau\Delta}\ (permutation) \qquad \frac{\Gamma \vdash A, \Delta \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ (cut)$$

Logical Rules:

$$(var_r)\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} \qquad (var_l)\frac{\Gamma \vdash B, \Delta}{\Gamma, B^{\perp} \vdash \Delta}$$

Multiplicatives:

$$(unit_l)\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta} \qquad (unit_r)\frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}$$

$$(\otimes_l)\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad (\otimes_r)\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

$$(\Box_l)\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Box B \vdash \Delta, \Delta'} \qquad (\Box_r)\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \Box B, \Delta}$$

$$(\multimap_l)\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta', \Delta} \qquad (\multimap_r)\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}$$

Additives:

$$(\&_r)\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \qquad (\&_l)\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}$$

$$(\oplus_l)\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad (\oplus_r)\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

A remark on notation. Seely writes in his paper "$\times$" for "&", "+" for "$\oplus$", $\odot$ for $\Box$ and $\neg$ for $(-)^{\perp}$, but we want to keep, as much as possible, the original notation from [Gir]. [Seely also has a single rule for negation]

We would like **GC** with all the structure defined in Chapter 3, to be a categorical model of Classical Linear Logic. But it is clear that we do not have morphisms of the form $A^{\perp\perp} \to A$ for all objects A in **GC**. So, not all the objects are equivalent to their double linear negations, $A \cong A^{\perp\perp}$.

Thus, we omit from the system just presented the negation axiom $A^{\perp\perp} \vdash A$. Negation, in our model, is the same as linear implication into $\perp$, so $A \vdash A^{\perp\perp}$ is trivially true. But the negation axioms and rules are not essential when describing linear logic, cf for example [Sch].

Also only the rule (var_l) is satisfied in **GC**. That happens because the logic we are dealing with is really intuitionistic, at the bottom level. Thus, for example, in the model, the objects $(A \oslash B)^{\perp}$ and $(A^{\perp} \Box B^{\perp})$ "look" exactly the same; they are both of the form

$$(X^V \times Y^U \nleftrightarrow U \times V)$$

But taking in consideration the relations, we only have a morphism in one direction

$$(A^{\perp} \Box B^{\perp}) \vdash (A \oslash B)^{\perp}$$

This is just as in Intuitionistic Logic, thinking of "$\Box$" as "*or*" and "$\oslash$" as "*and*". Also from $A^{\perp} \Box D$ we can prove intuitionistically, $A \multimap D$, but not conversely. Thus we have to change the rule for $\multimap$-introduction on the right. The new rule, not surprisingly is

$$(\multimap_r)\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

Let the new logical system, with restricted $(\multimap_r)$, be called $L.L_{\star}$ or sometimes just $L.L.$ [Thanks to H. Schellinx, who made me think a bit more clearly about the logical system I was modelling.]

Theorem 3 *The symmetric monoidal closed category* **GC**, *with bifunctors tensor product* $\oslash$; *"par"* $\Box$; *internal hom* $[-,-]_{\mathbf{GC}}$; *cartesian product* &; *coproduct* $\oplus$ *and contravariant functor* $(-)^{\perp}$ *for linear negation, is a model of* $L.L_{\star}$. *Thus to each entailment* $\Gamma \vdash_{L.L_{\star}} A$ *corresponds the existence of a morphism in* **GC**,$(f,F)\colon |\Gamma| \to |A|$.

Proof: We check each of the axioms and rules, as we did before for $I.L.L.$

Notice that rules $\oslash_l$ and $\Box_r$ are fundamental, since they indicate how we should interpret the sequents in the category **GC**. They show that

$$G_1, G_2, \ldots, G_n \vdash D_1, D_2, \ldots, D_m$$

should be read as there exists a morphism in **GC**,

$$|G_n| \oslash \ldots \oslash |G_1| \to |D_k| \Box \ldots \Box |D_1|$$

As before we write G for $|G_n| \oslash \ldots \oslash |G_1|$, the tensor product of the objects $|G_i|$ in **GC** and D for $|D_k|\Box \ldots \Box|D_1|$, the "par" of objects $|D_j|$ in **GC**. Apologies for the notation $\oslash$, for tensor product, which is normally written $\otimes$, but in **GC** we need two different names for distinct tensor products.

The first axiom only says we have identities in **GC**. The second pair of axioms states the existence of constants I and $\perp$. The third pair of axioms is satisfied since there are isomorphisms $0 \oslash G \cong 0$ and $1 \Box D \cong 1$.

The rule (*permutation*) holds since both monoidal structures $\oslash$ and $\Box$ are symmetric.

To check the rule (*cut*) we use the natural morphism

$$i\colon A \oslash (B \Box C) \to (A \oslash B) \Box C$$

given in section 3. Thus we say that given morphisms $(f,F)\colon G \to A \Box D$ and $(g,G)\colon A \oslash G' \to D'$, which correspond to $\Gamma \vdash A, \Delta$ and $\Gamma', A \vdash \Delta'$, we look at the composition

$$G \oslash G' \xrightarrow{(f,F)} (A \Box D) \oslash G' \cong G' \oslash (A \Box D) \xrightarrow{i} (G' \oslash A) \Box D \xrightarrow{(g,G)} D' \Box D,$$

and that corresponds to $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

The rules for unit, ($unit_l$) and ($unit_r$) are satisfied since I is the unit for the tensor and $\perp$ the unit for par. Thus $G \oslash I \cong G$ and $\perp \Box D \cong D$.

Rule ($\oslash_r$) uses again the natural morphism "i". If we have morphisms $G \xrightarrow{(f,F)} A \Box D$ and $G' \xrightarrow{(g,G)} B \Box D'$, which correspond to $\Gamma \vdash A, \Delta$ and $\Gamma' \vdash B, \Delta'$, composition gives us

$$G \oslash G' \xrightarrow{(f,F)\oslash(g,G)} (A \Box D) \oslash (B \Box D') \xrightarrow{i} ((A \Box D) \oslash B) \Box D' \cong$$

$$(B \oslash (A \Box D)) \Box D' \xrightarrow{i} (B \oslash A) \Box D \Box D' \cong (A \oslash B) \Box D \Box D'$$

Rule ($\Box_l$) uses the natural morphism "k" of section 3. Given maps $(f,F)\colon G \oslash A \to D$ and $(g,G)\colon G' \oslash B \to D'$ we have

$$(G \oslash G') \oslash (A \Box B) \xrightarrow{k} (G \oslash A) \Box (G' \oslash B) \xrightarrow{(f,F)\Box(g,G)} D \Box D',$$

which corresponds to $\Gamma, \Gamma', A \Box B \vdash \Delta, \Delta'$.

Rule ($\multimap_l$) uses "i" and "ev" as follows. If $G \xrightarrow{(f,F)} A \Box D$ and $G' \oslash B \xrightarrow{(g,G)} D'$, then the long composition gives

$$G \oslash (G' \oslash (A \multimap B)) \xrightarrow{(f,F)\oslash 1} (A \Box D) \oslash (G' \oslash (A \multimap B)) \cong$$

$$(G' \oslash (A \multimap B)) \oslash (A \Box D) \xrightarrow{i} (G' \oslash (A \multimap B) \oslash A) \Box D \xrightarrow{ev} (G') \Box D \xrightarrow{(g,G)\Box D} D' \Box D,$$

which says $\Gamma, \Gamma', A \multimap B \vdash \Delta', \Delta$.

Rule ($\multimap_r$) uses the adjunction. If $G \oslash A \to B$, then, by adjunction, $G \to (A \multimap B)$.

Finally, the rules for the additives are satisfied using the distributivity laws and canonical injections and projections.

Rule ($\&_l$) says that, if e.g. $G \oslash A \to D$, then $G \oslash (A\&B) \xrightarrow{G \oslash p_1} G \oslash A \to D$ and rule ($\oplus_r$) says if e.g. $G \to A \Box D$, then $G \to A \Box D \xrightarrow{i_1 \Box D} (A \oplus B) \Box D$.

Rule ($\&_r$) uses the distributive law. Given $G \to A \Box D$ and $G \to B \Box D$ then

$$G \to (A \Box D)\&(B \Box D) \cong (A\&B) \Box D$$

Rule ($\oplus_l$) uses the other distributivity law, if $G \oslash A \to D$ and $G \oslash B \to D$ then $G \oslash (A \oplus B) \cong G \oslash A \oplus G \oslash B \to D$. □

We want to check the symmetries involved and summarize the properties of the connectives in **GC**.

Proposition 35 *We have the following properties of the connectives in the category* **GC**.

Properties of the multiplicatives:

1. *The bifunctors "par" $\Box$ and "tensor" $\oslash$ are commutative and associative. They have neutral elements $\bot$ and I respectively.*

2. *Linear implication $\multimap$ satisfies:*
   - $I \multimap A \cong A$
   - $A \multimap \bot \cong A^\bot$
   - $(A \oslash B) \multimap C \cong A \multimap (B \multimap C)$
   - $(A \multimap B) \vdash B^\bot \multimap A^\bot$
   - $A \multimap (B \Box C) \vdash (A \multimap B) \Box C$

3. *We have the following natural morphisms in* **GC**:
   - $A \oslash B \to A^{\bot\bot} \oslash B^{\bot\bot} \cong (A^\bot \Box B^\bot)^\bot$
   - $A \Box B \to A^{\bot\bot} \Box B^{\bot\bot} \to (A^\bot \oslash B^\bot)^\bot$
   - $A \Box B \to A^{\bot\bot} \Box B \to (A^\bot \multimap B)$
   - $A^\bot \Box B \to A \multimap B$

   Properties of the additives:

4. *The bifunctors "plus" $\oplus$ and "with" $\&$ (or cartesian product) are commutative and associative. They have neutral elements,* $\mathbf{0}$ *and* $\mathbf{1}$*, respectively, which satisfy:*
   - $\mathbf{0} \oslash A \cong \mathbf{0}$
   - $\mathbf{1} \Box A \cong \mathbf{1}$
   - $\mathbf{0} \multimap A \cong \mathbf{1}$
   - $A \multimap \mathbf{1} \cong \mathbf{1}$

5. *We have semi "de Morgan" principles:*
   - $A\&B \to A^{\bot\bot}\&B^{\bot\bot} \cong (A^\bot \oplus B^\bot)^\bot$
   - $A \oplus B \to (A^{\bot\bot}) \oplus (B^{\bot\bot}) \to (A^\bot \& B^\bot)^\bot$.

   Distributivity properties:

6. *We have the following distributivity rules:*
   - $A \oslash (B \oplus C) \cong (A \oslash B) \oplus (A \oslash C)$

- $A\Box(B\&C) \cong (A\Box B)\&(A\Box C)$
- $A \multimap (B\&C) \cong (A \multimap B)\&(A \multimap C)$
- $(A \oplus B) \multimap C \cong (A \multimap C)\&(B \multimap C)$

7. *We have the following natural morphisms:*

   - $A \oslash (B\&C) \to (A \oslash B)\&(A \oslash C)$
   - $(A\Box B) \oplus (A \oslash C) \to A\Box(B \oplus C)$
   - $(A \multimap B) \oplus (A \multimap C) \to (A \multimap (B \oplus C)$

Proposition 36 *The subcategory "*Dec **GC***" is a model of C.L.L.*

Proof: Just have to check that the subcategory Dec **GC** is closed under the logical operations, represented by the bifunctors $\oslash$, $\Box$, $[-,-]_{\mathbf{GC}}$, & and $\oplus$.

Proposition 37 *We have the following isomorphisms in the category* Dec **GC**

1. $A \cong A^{\perp\perp}$.

2. $A \multimap B \cong B^{\perp} \multimap A^{\perp}$.

3. $A \multimap B \cong A^{\perp}\Box B$.

   De Morgan principles:

4. $(A \oslash B)^{\perp} \cong A^{\perp}\Box B^{\perp}$.

5. $(A\Box B)^{\perp} \cong A^{\perp} \oslash B^{\perp}$.

6. $(A\&B)^{\perp} \cong A^{\perp} \oplus B^{\perp}$.

7. $(A \oplus B)^{\perp} \cong A^{\perp}\&B^{\perp}$.

Chapter 4

Modalities in GC

Interpretations of the modal, or exponential, operators "!" and "?" of Linear Logic, in a categorical set-up, should correspond to a comonad and a monad, respectively, satisfying certain conditions.

We shall discuss several possible endofunctors on **GC**, which could play the role of the connective "!" in Classical Linear Logic.

We have 6 sections, the first is another recapitulation of monad theory. Then we define two monads in **C** and induced comonads, T and S in **GC**. The third section recall distributive laws - following Beck - while the fourth discusses some properties of T. The fifth section defines the comonad "!" and the sixth relates the categorical results to Linear Logic.

4.1 More Preliminaries

Recall that given a monad $\mathbf{T} = (T, \eta: 1 \to T, \mu: T^2 \to T)$ on a cartesian closed category **C** we can describe its category of algebras (or Eilenberg-Moore category) usually denoted $\mathbf{C}^T$ and its T-Kleisli category, $\mathbf{C}_T$. The category $\mathbf{C}^T$ of algebras consists of pairs (X, θ) where X is an object and θ is a morphism in **C**, $\theta: TX \to X$, called the structure map of the algebra, which makes the following diagrams commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & TX \\
\| & & \downarrow{\scriptstyle \theta} \\
X & = & X
\end{array}
\qquad
\begin{array}{ccc}
T^2X & \xrightarrow{T\theta} & TX \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \theta} \\
TX & \xrightarrow[\theta]{} & X
\end{array}
$$

A map of T-algebras is a morphism of $\mathbf{C}$, $f: X \to Y$ such that the square below commutes

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The T-Kleisli category $\mathbf{C}_T$ has the same objects as $\mathbf{C}$, but morphisms are, by definition, given by $\mathrm{Hom}_{\mathbf{C}_{\mathbf{T}}}(A, B) \cong \mathrm{Hom}_{\mathbf{C}}(A, TB)$.

Also, given the monad $\mathbf{T} = (T, \eta, \mu)$, from abstract nonsense only, we can draw the following diagram, where the horizontal map Φ is called the "comparison functor"

$$\begin{array}{ccc} \mathbf{C}_T & \xrightarrow{\Phi} & \mathbf{C}^T \\ {\scriptstyle F_T}\uparrow\downarrow{\scriptstyle U_T} & & {\scriptstyle F^T}\uparrow\downarrow{\scriptstyle U^T} \\ \mathbf{C} & = & \mathbf{C} \end{array}$$

Given two monads on the same category $\mathbf{C}$, it is usual to define a (monad) morphism $\alpha: \mathbf{T} \to \mathbf{T}'$ (cf. [Barr-Wells] page 125) between the monads $\mathbf{T} = (T, \eta, \mu)$ and $\mathbf{T}' = (T', \eta', \mu')$ in $\mathbf{C}$, as a natural transformation $\alpha: T \to T'$ making diagrams below commute.

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & T' \\ {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \eta'} \\ I & = & I \end{array} \qquad \begin{array}{ccc} T^2 & \xrightarrow{\alpha^2} & T'^2 \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ T & \xrightarrow[\alpha]{} & T' \end{array}$$

It is easy to verify that $\alpha: \mathbf{T} \to \mathbf{T}'$ a monad morphism induces a morphism in the opposite direction, $\hat{\alpha}: \mathbf{C}^{T'} \to \mathbf{C}^T$ between the T-algebra categories. Given a T'-algebra $(X, \theta': T'X \to X)$, $\hat{\alpha}(X, \theta')$ is the composition induced T-algebra θ given by

$$\begin{array}{ccc} TX & \xrightarrow{\theta} & X \\ {\scriptstyle \alpha_X}\downarrow & & \| \\ T'X & \xrightarrow[\theta']{} & X \end{array}$$

Similarly, the monad morphism $\alpha: \mathbf{T} \to \mathbf{T}'$ induces a morphism $\overline{\alpha}: \mathbf{C}_T \to \mathbf{C}_{T'}$ - in the same direction as α - between the Kleisli categories. Given a map $X \to Y$ in $\mathbf{C}_T$, which corresponds to a map $X \to TY$ in $\mathbf{C}$, we can compose it with α_Y, to get $X \to TY \xrightarrow{\alpha_Y} T'Y$ in $\mathbf{C}$ and this corresponds to a morphism $X \to Y$ in $\mathbf{C}_{T'}$.

Another remark is that monads on $\mathbf{C}$, with monad morphisms, form a category which we call $\mathbf{MON_C}$. Observe that capitals are used to make $\mathbf{MON_C}$ different from $\mathbf{Mon_C}$, which means the category of monoid objects in $\mathbf{C}$.

All the above can be dualised, simply inverting arrows and adding the prefix "co", where appropriate. Thus, given a comonad $\mathbf{G} = (G, \varepsilon: G \to 1, \delta: G \to G^2)$ we can form the category of coalgebras, denoted $\mathbf{C}^G$ and the G-Kleisli category, denoted $\mathbf{C}_G$. Note that $h: X \to GX$ a coalgebra structure map, makes the following diagram commute,

$$\begin{array}{ccc} X & \xrightarrow{h} & GX \\ \| & & \downarrow{\scriptstyle \varepsilon} \\ X & = & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{h} & GX \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \delta} \\ GX & \xrightarrow[Gh]{} & G^2X \end{array}$$

but not necessarily

$$\begin{array}{ccc} GX & \xrightarrow{\varepsilon} & X \\ \| & & \downarrow{\scriptstyle h} \\ GX & = & GX \end{array}$$

Given two comonads in $\mathbf{C}$ a comonad map is a natural transformation $\alpha: \mathbf{G} \to \mathbf{G}'$ such that the following diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon'} \\ I & = & I \end{array} \qquad \begin{array}{ccc} G^2 & \xrightarrow{\alpha^2} & G'^2 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta'} \\ G & \xrightarrow[\alpha]{} & G' \end{array}$$

But notice that a comonad morphism $\alpha: \mathbf{G} \to \mathbf{G}'$ induces a morphism of coalgebras $\hat{\alpha}: \mathbf{C}^G \to \mathbf{C}^{G'}$ in the same direction as "α". Thus, given a G-coalgebra $(X, h: X \to GX)$, $\hat{\alpha}(X, h)$ is the G'-coalgebra $(X, h': X \to G'X)$, where h' is given by the composition

$$X \xrightarrow{h} GX \xrightarrow{\alpha_X} G'X.$$

Dually, $\alpha: G \to G'$ induces a morphism $\overline{\alpha}: \mathbf{C}_{G'} \to \mathbf{C}_G$ between the Kleisli categories. Given a map $X \to Y$ in $\mathbf{C}_{G'}$, which corresponds to $G'X \to Y$ in $\mathbf{C}$, precompose it with α_X to get $GX \xrightarrow{\alpha_X} G'X \longrightarrow Y$, which corresponds to $X \to Y$ in $\mathbf{C}_G$. We can consider the category $\mathbf{COMON_C}$, whose objects are comonads on $\mathbf{C}$ and maps are comonad morphisms.

Monoids and comonoids in GC

Recall from Chapter 2 that if we consider $\mathbf{C}$ cartesian closed with (stable and disjoint) coproducts, then $\mathbf{Comon_+C} \cong \mathbf{0}$, $\mathbf{Mon_+C} \cong \mathbf{C}$ and $\mathbf{Comon_\times C} \cong \mathbf{C}$.

We now turn our attention to $\mathbf{GC}$, which is only symmetric monoidal closed. Since we have five symmetric monoidal structures in $\mathbf{GC}$, we can consider $\mathbf{Mon_\oslash GC}$, $\mathbf{Mon_\otimes GC}$, $\mathbf{Mon_\& GC}$,

$\mathbf{Mon}_{\oplus}\mathbf{GC}$ and $\mathbf{Mon}_{\Box}\mathbf{GC}$. Dually, we can consider categories $\mathbf{Comon}_{\oslash}\mathbf{GC}$, $\mathbf{Comon}_{\otimes}\mathbf{GC}$, $\mathbf{Comon}_{\&}\mathbf{GC}$, $\mathbf{Comon}_{\oplus}\mathbf{GC}$ and $\mathbf{Comon}_{\Box}\mathbf{GC}$.

From these ten categories, using the argument above, we deduce the isomorphisms of categories $\mathbf{Mon}_{\oplus}\mathbf{GC} \cong \mathbf{GC}$ and $\mathbf{Comon}_{\&}\mathbf{GC} \cong \mathbf{GC}$. Notice that to say $\mathbf{Comon}_{+}\mathbf{C} \cong \mathbf{0}$ we have used the cartesian closed structure of $\mathbf{C}$, thus now we have, instead, $\mathbf{Comon}_{\oplus}\mathbf{GC} \cong \mathbf{MonC}$ and $\mathbf{Mon}_{\&}\mathbf{GC} \cong \mathbf{MonC}$.

To describe the two categories induced by the "old" tensor product $\otimes$ we recall:

- The category $\mathbf{Mon}_{\otimes}\mathbf{GC}$ consists of triplets (A, η_A, μ_A), where A is an object $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$, and the maps $\eta_A: I \to A$ and $\mu_A: A \otimes A \to A$ satisfy monoid equations. This means that U is a monoid object in $\mathbf{C}$, $(U, 1 \overset{u_0}{\to} U, \mu_U: U \times U \to U)$ and the relation "α" satisfies

  - If u_0 is the unit of U, $u_0 \alpha x$ for all $x \in X$.
  - If $u\alpha x$ and $u'\alpha x$ then $\mu_U(u, u')\alpha x$.

  Notice that the existence of maps in the second coordinate is not a problem, since X has a natural comonoid structure given by the terminal map $X \overset{!}{\to} 1$ and the diagonal map in $\mathbf{C}$, $X \overset{\Delta}{\to} X \times X$.

- Dually, the category $\mathbf{Comon}_{\otimes}\mathbf{GC}$ consists of triplets (A, η_A, μ_A) where $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$ and X is a monoid object in $\mathbf{C}$, $(X, 1 \overset{x_0}{\to} X, \mu_X: X \times X \to X)$. The relation "$\alpha$" satisfies

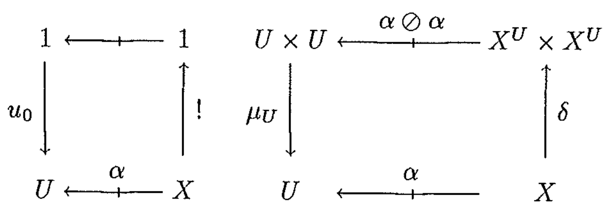
  - If x_0 is the unit of X, $u\alpha x_0$ for all $u \in U$.
  - If $u\alpha\mu_X(x, x')$ then $u\alpha x$ and $u\alpha x'$.

  Again the existence of maps in the first coordinate is no problem, since U has a natural comonoid structure given by the terminal map $U \overset{!}{\to} 1$ and the diagonal map $U \overset{\Delta}{\to} U \times U$.

There are still four categories left, which we describe as follows:

- The category $\mathbf{Mon}_{\oslash}\mathbf{GC}$ consists of triplets $(A, \eta_A: I \to A, \mu_A: A \oslash A \to A)$ where A is an object of $\mathbf{GC}$ $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$ and U is a monoid in $\mathbf{C}$,

$$(U, 1 \overset{u_0}{\to} U, \mu_U: U \times U \to U).$$



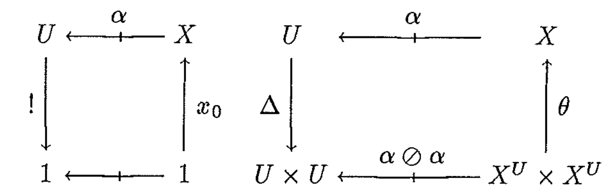
  The natural transformation η_A consists of the unit of U, $u_0: 1 \to U$ and the terminal map in X, $X \overset{!}{\to} 1$. The relation "α" satisfies $u_0\alpha x$ for all $x \in X$. The multiplication $\mu_A: A \oslash A \to A$ is given by the multiplication on U, μ_U, and the existence of a morphism $\delta: X \overset{<\delta_1,\delta_2>}{\longrightarrow} X^U \times X^U$ such that $u\alpha\delta_1(x)(u')$ and $u'\alpha\delta_2(x)(u)$ implies that $\mu_U(u, u')\alpha x$.

- The category $\mathbf{Comon}_{\oslash}\mathbf{GC}$ consists of triplets

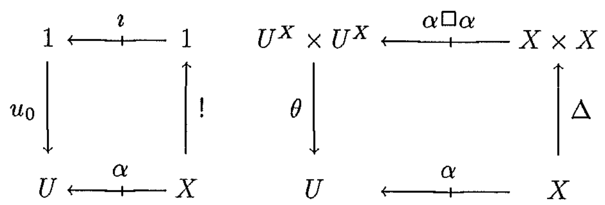
$$(A, \eta_A: A \to 1, \delta_A: A \to A \oslash A),$$

  where $A = (U \overset{\alpha}{\leftarrow\!\!\!+} X)$ is an object in $\mathbf{GC}$ and X has a point $1 \overset{x_0}{\to} X$. The natural transformation η_A is given by the terminal map in U, $U \overset{!}{\to} 1$ and the point of X, $x_0: 1 \to X$. The relation

"α" is such that $u\alpha x_0$ for all $u \in U$. More importantly , there is a map $\theta: X^U \times X^U \to X$, such that the comultiplication $\delta_A: A \to A \oslash A$ is given by (Δ_U, θ). This says that the relation "α" satisfies $u\alpha\theta(f,g) \Rightarrow (u\alpha fu$ and $u\alpha gu)$.



- The category $\mathbf{Mon}_\Box\mathbf{GC}$ consists of $(A, \eta_A: \perp \to A, \mu_A: A\Box A \to A)$ where $A = (U \overset{\alpha}{\leftrightarrow} X)$ and U has a point $1 \overset{u_0}{\to} U$. The natural transformation η_A consists of the point $1 \overset{u_0}{\to} U$ and the terminal map in X, $X \to 1$.

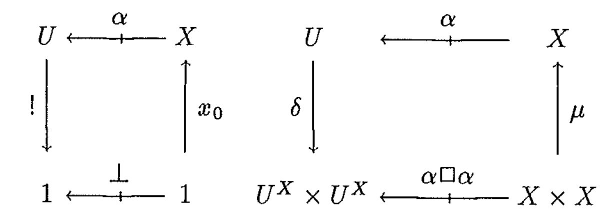


There exists a map $\theta: U^X \times U^X \to X$, which is the first coordinate of the multiplication on A and the relation α satisfies $fx\alpha x$ or $gx\alpha x$ implies $\theta(f,g)\alpha x$.

- Finally, the category $\mathbf{Comon}_\Box\mathbf{GC}$ consists of triplets,

$$(A, \eta_A: A \to \perp, \delta_A: A \to A\Box A),$$

where $A = (U \overset{\alpha}{\leftrightarrow} X)$ and X is a monoid in C, $(X, 1 \overset{x_0}{\to} U, \mu_X: X \times X \to X)$.



There exists a map $\delta: U \overset{<\delta_1,\delta_2>}{\to} U^X \times U^X$, such that $u\alpha\mu_X(x, x')$ implies that $\delta_1(u)(x')\alpha x$ or $\delta_2(u)(x)\alpha x$.

The point of going through these lengthy descriptions is to show that, with respect to "$\oslash$", which is the tensor product in **GC**, comonoids do not look familiar at all.

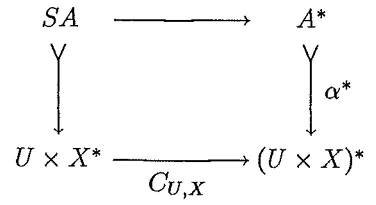
4.2 The comonads T and S

In this section we shall consider monads in the category **C**, two "induced" comonads in **GC** and describe the categories they give rise to. In the next sections we consider some relationship among those categories .

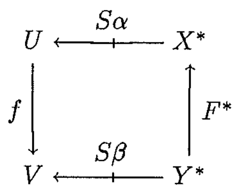
The first monad we consider in $\mathbf{C}$ is the one described in Chapter 2, called there $*: \mathbf{C} \to \mathbf{C}$. This monad, which will be called S_0 in this Chapter, gives free monoids on $\mathbf{C}$. Thus, as before, we suppose we are given is an adjunction $F \dashv U: \mathbf{C} \to \mathbf{Mon\,C}$.

Recall that $S_0(X) = X^*$, $S_0(Y) = Y^*$ and $S_0(f) = f^*$. Intuitively X^* stands for "finite sequences of elements of X" and f^* for "f applied to each element of the sequence". Also S_0 is clearly a monad and it does not preserve products. Despite that, we can still, as in $\mathbf{DC}$, define an induced endofunctor $S: \mathbf{GC} \to \mathbf{GC}$ and "S" has a natural structure as a comonad.

Definition 13 *The endofunctor S on $\mathbf{GC}$ is given by $S(U \overset{\alpha}{\leftarrow\!\!\!\!+} X) = (U \overset{S\alpha}{\leftarrow\!\!\!\!+} X^*)$ on objects, where the relation "$S\alpha$ " is given by the pullback below*



and $S(f, F) = (f, F^*)$ on maps, as the diagram shows

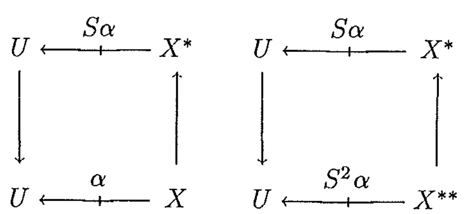


The relation "$S\alpha$" reads intuitively as

$$\text{“}u(S\alpha)(x_1, \ldots, x_k) \text{ iff } u\alpha x_1 \text{ and } \ldots \text{ and } u\alpha x_k\text{”}.$$

To define the relation "$S\alpha$" we pullback the auxiliary map $C_{(-,-)}$ - discussed in Chapter 2 - along the image of $A \overset{\alpha}{\rightarrowtail} U \times X$ by S_0, $A^* \overset{\alpha^*}{\rightarrowtail} (U \times X)^*$, as we did before.

Proposition 38 *The functor $S: \mathbf{GC} \to \mathbf{GC}$ has a natural comonad structure, induced by the monad structure of S_0. Namely, the counit $(\varepsilon_S)_A: SA \to A$ is given by identity on U and the singleton map $X \xrightarrow{\eta} X^*$ and the comultiplication $(\delta_S)_A: SA \to S^2A$ is given by identity on U and "forgetting brackets" $X^{**} \xrightarrow{\mu} X^*$ on the second coordinate, as the diagram shows*



Both are easily natural transformations and they make the respective diagrams commute.

Alas, this comonad has not the nice categorical properties it had before, due to the fact that the tensor product in $\mathbf{GC}$ is much more complicated than then one in $\mathbf{DC}$.

There are other very natural monads to consider in $\mathbf{C}$, if $\mathbf{C}$ is cartesian closed.

Definition 14 *For each U in $\mathbf{C}$, a cartesian closed category, let $T_U : \mathbf{C} \to \mathbf{C}$ be the endofunctor which takes $X \mapsto X^U$, $Y \mapsto Y^U$ and $f \in X^U$ to $fg \in Y^U$.*

That is clearly a monad in bfC with unit $(\eta_{T_U})_X : X \to X^U$ given by the "constant map", and multiplication $(\mu_{T_U})_X : X^{U \times U} \to X^U$, simply "precomposition with diagonal". We now turn our attention to defining a comonad "induced by the monads T_U" in $\mathbf{GC}$.

Definition 15 *Consider the endofunctor $T : \mathbf{GC} \to \mathbf{GC}$ which takes the object $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$ to the object $(U \overset{T\alpha}{\leftarrow\!\!\!+} X^U)$ and the object $(V \overset{\beta}{\leftarrow\!\!\!+} Y)$ to $(V \overset{T\beta}{\leftarrow\!\!\!+} Y^V)$. The relation "$T\alpha$" is defined by the pullback of the object $A \overset{\alpha}{\rightarrowtail} U \times X$ along the map $U \times X^U \overset{\text{"}ev_U\text{"}}{\to} U \times X$.*

$$\begin{array}{ccc} TA & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \alpha \\ U \times X^U & \xrightarrow[(\pi_1, ev)]{} & U \times X \end{array}$$

Intuitively, that says that "$u(T\alpha)f$ iff $u\alpha f(u)$". To complete the definition say that T applied to a map $(f, F) : A \to B$ is $(f, F(-)f) : TA \to TB$ as the diagram shows

$$\begin{array}{ccc} U & \overset{T\alpha}{\leftarrow\!\!\!+} & X^U \\ f \big\downarrow & & \big\uparrow F(-)f \\ V & \overset{T\beta}{\leftarrow\!\!\!+} & Y^V. \end{array}$$

It is easy to show that T is a comonad, but it is interesting to see that to describe its comonad structure only the natural transformations given by the monads T_U need to be considered.

Proposition 39 *The endofunctor T is the functor part of a comonad $(T, \varepsilon_T, \delta_T)$ with co-unit $\varepsilon_T : TA \to A$ given by identity on U and the natural "constant" map $\eta_U : X \to X^U$ in the second coordinate. The comultiplication $(\delta_T)_A : TA \to T^2 A$ is given by identity on U and "restriction to the diagonal" $\mu_U : X^{U \times U} \to X^U$ in the second coordinate.*

$$\begin{array}{ccc} U & \overset{T\alpha}{\leftarrow\!\!\!+} & X^U \\ \big\downarrow & & \big\uparrow \eta_U \\ U & \overset{\alpha}{\leftarrow\!\!\!+} & X \end{array} \qquad \begin{array}{ccc} U & \overset{T\alpha}{\leftarrow\!\!\!+} & X^U \\ \big\downarrow & & \big\uparrow \mu_U \\ U & \overset{T^2\alpha}{\leftarrow\!\!\!+} & X^{U \times U} \end{array}$$

It is easy to check that both natural transformations are really maps in $\mathbf{GC}$ and they do make the usual comonad diagrams commute.

Moreover, the monads T_U relate to S_0 in a very special way, described by Beck [69] as a "distributive law".

Beck defines a *distributive law of* $\mathbf{S}$ *over* $\mathbf{T}$, where we are given monads $\mathbf{T} = (T, \eta_T, \mu_T)$ and $\mathbf{S} = (S, \eta_S, \mu_S)$ on a category $\mathbf{C}$, as a natural transformation $\lambda: ST \to TS$ such that the diagrams

$$\begin{array}{ccc} T & = & T \\ {\scriptstyle \eta_S}\downarrow & & \downarrow{\scriptstyle T\eta_S} \\ ST & \xrightarrow{\lambda} & TS \end{array} \qquad \begin{array}{ccc} S & = & S \\ {\scriptstyle S\eta_T}\downarrow & & \downarrow{\scriptstyle \eta_T} \\ ST & \xrightarrow{\lambda} & TS \end{array}$$

and

$$\begin{array}{ccccc} STT & \xrightarrow{\lambda_T} & TST & \xrightarrow{T\lambda} & TTS \\ {\scriptstyle S\mu_T}\downarrow & & & & \downarrow{\scriptstyle \mu_T} \\ ST & & \xrightarrow{\lambda} & & TS \end{array}$$

$$\begin{array}{ccccc} SST & \xrightarrow{S\lambda} & STS & \xrightarrow{\lambda_S} & TSS \\ {\scriptstyle \mu_S}\downarrow & & & & \downarrow{\scriptstyle T\mu_S} \\ ST & & \xrightarrow{\lambda} & & TS \end{array}$$

commute.

Now we want to see that, for each U in $\mathbf{C}$, we have a natural transformation $\lambda: S_0T_U \to T_US_0$ or $\lambda_X: S_0T_UX \to T_US_0X$, corresponding to $(X^U)^* \to (X^*)^U$, satisfying those diagrams. Intuitively, such λ exists. Given a sequence of functions $(\phi_i)_{i\in I}$, each $\phi_i: U \to X$, take the product function $\phi_1 \times \ldots \times \phi_k: U^k \to X^k$ and precompose it with the diagonal map $\Delta: U \to U^k$, to get a map $U \to X^*$.

Definition 16 *There is a natural transformation in* $\mathbf{C}$, $\lambda: S_0T_U \to T_US_0$ *such that at the object* X, $(\lambda)_X: S_0T_UX \to T_US_0X$ *is given by* $(\lambda)_X: (X^U)^* \to (X^*)^U$. *To define the map* λ, *it is enough to define its exponential transpose* $\overline{\lambda}$, *which is given by the composition of (an instance of) the map* $C_{(-,-)}$ *in Chapter 2 and evaluation, as follows:*

$$(X^U)^* \times U \xrightarrow{C_{(U,X^U)}} (X^U \times U)^* \xrightarrow{ev^*} X^*$$

Proposition 40 *The natural transformation* λ *satisfies the conditions for a "distributive law" in* $\mathbf{C}$.

More interesting is the fact that λ above, induces a new distributivity law Λ, this time between the comonads T and S in $\mathbf{GC}$.

Proposition 41 *There is a natural transformation* $\Lambda: TS \to ST$, *at each object* A, $\Lambda_A: TSA \to STA$ *is given by* $(1_U, (\lambda)_X)$ *where* $(\lambda)_X: (X^U)^* \to (X^*)^U$ *is the distributive law above in* $\mathbf{C}$. *This natural transformation* Λ *satisfies the conditions for a "distributive law of comonads".*

Using the monads T_U and S_0 in $\mathbf{C}$ and checking the definitions in the previous section, we can consider:

- The category $\mathbf{C}^{T_U}$ of T_U-algebras, that is pairs (X, θ_X) where the structure map of the algebra X, $\theta_X : X^U \to X$ is such that the composition of the morphisms $X \xrightarrow{\eta_{T_U}} X^U \xrightarrow{\theta_X} X$ is the identity on X and also such that the two morphisms $(X^U)^U \xrightarrow{(h_X)^U} X^U \xrightarrow{h_X} X$ and $(X^U)^U \cong X^{U \times U} \xrightarrow{\mu_{T_U}} X^U \xrightarrow{h_X} X$ are equal.
- The category $\mathbf{C}^{S_0}$ of S_0-algebras, that is pairs (Y, j_Y) where $j_Y : Y^* \to Y$ makes Y a monoid object in C. Thus we have the equivalence of categories $\mathbf{C}^{S_0} \cong \mathbf{Mon}\,\mathbf{C}$.
- The category $\mathbf{C}_{T_U}$, the T_U-Kleisli category, with the same objects as $\mathbf{C}$, but for morphisms we have

$$\mathrm{Hom}_{\mathbf{C}_{T_U}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, T_U Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y^U) \cong \mathrm{Hom}_{\mathbf{C}}(U \times X, Y).$$

- The category $\mathbf{C}_{S_0}$, the S_0-Kleisli category, with the same objects as $\mathbf{C}$, but morphisms are given by $\mathrm{Hom}_{\mathbf{C}_{S_0}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, S_0 Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y^*)$.

On the other hand, looking at the comonads we have described in $\mathbf{GC}$ we have the following categories:

- The category $\mathbf{GC}^T$ of T-coalgebras, that is pairs (A, h_A) where A is an object in $\mathbf{GC}$ and $h_A : A \to TA$, the structure map of the coalgebra, consists of identity on U and $\theta_X : X^U \to X$, a structure map for T_U in X.
- The category $\mathbf{GC}^S$ of S-coalgebras that is pairs (A, h_A) where $h_A : A \to SA$ the structure map of the coalgebra, consists of identity on U and a morphism $j_X : X^* \to X$ making X a monoid object in $\mathbf{C}$, so $\mathbf{GC}^S \cong \mathbf{Comon}_{\otimes}\mathbf{GC}$.
- The category $\mathbf{GC}_T$, the T-Kleisli category of $\mathbf{GC}$, with the same objects as $\mathbf{GC}$, but maps given by $\mathrm{Hom}_{\mathbf{GC}_T}(A, B) \cong \mathrm{Hom}_{\mathbf{GC}}(TA, B)$, cf. section 4.
- The category $\mathbf{GC}_S$, the S-Kleisli category, with the same objects as $\mathbf{GC}$, and morphisms given by $\mathrm{Hom}_{\mathbf{GC}_S}(A, B) \cong \mathrm{Hom}_{\mathbf{GC}}(SA, B)$.

Since we have the distributive laws λ and Λ, it makes sense to adapt Beck's paper on "Distributive Laws" to our monads and comonads and to check some conclusions that can be drawn from it.

4.3 Using Distributive Laws

It is widely known that the composition of monads is not always a monad, but given a distributive law λ, we can define the *composite monad* defined by λ, cf. [Beck]. We can also define the "lifting" of one of the monads and several relationships among the categories of algebras and Kleisli categories involved.

Recall that $\lambda : S_0 T_U \to T_U S_0$ is a distributive law in $\mathbf{C}$. Thus, if we define η_0 as the diagonal in the first diagram below, and μ_0 as the composition of $T_U \lambda S_0$ with the diagonal of the second diagram below, then $T_U S_0$ with these two natural transformations comprises the *composite monad* induced by λ, cf. [Bec]

$$\begin{array}{ccc} X & \xrightarrow{\eta_{S_0}} & S_0 X \\ {\scriptstyle \eta_{T_U}}\downarrow & & \downarrow{\scriptstyle \eta_{T_U}} \\ T_U X & \xrightarrow[T_U \eta_{S_0}]{} & T_U S_0 X \end{array}$$

$$
\begin{array}{ccccc}
T_US_0T_US_0X & \xrightarrow{T_U\lambda S_0} & T_UT_US_0S_0X & \xrightarrow{T_UT_U\mu_{S_0}} & T_UT_US_0X \\
 & & \downarrow{\scriptstyle \mu_{T_U}} & & \downarrow{\scriptstyle \mu_{T_U}} \\
 & & T_US_0S_0X & \xrightarrow[T_U\mu_{S_0}]{} & T_US_0X
\end{array}
$$

Definition 17 *The composite monad $(T_US_0)_{\lambda_0}$ in $\mathbf{C}$, takes $X \mapsto (X^*)^U$ and its unit η_0 is given by the composition $X \overset{\eta_{S_0}}{\to} X^* \overset{\eta_{T_U}}{\to} (X^*)^U$ or equivalently by $X \overset{\eta_{T_U}}{\to} X^U \overset{T_U(\eta_{S_0})}{\to} (X^*)^U$. Multiplication $\mu_0: (T_US_0)^2 \to T_US_0$ is given by the exponential transpose of the long composition*

$$(((X^*)^U)^*)^U \times U \overset{ev \times U}{\to} [(X^*)^U]^* \times U \overset{\lambda \times U}{\to} ((X^*)^*)^U \times U \overset{ev}{\to} X^{**} \overset{\mu_S}{\to} X^*,$$

which corresponds to the composite in the second diagram above.

To show that (T_US_0, η_0, μ_0) is indeed a monad is just a long naturality calculation.
Similarly, we have the *composite comonad,* induced by Λ and given by (TS): $\mathbf{GC} \to \mathbf{GC}$.

Definition 18 *The composite comonad (TS) takes $(U \overset{\alpha}{\leftarrow\!\!\!+} X) \mapsto (U \overset{TS\alpha}{\leftarrow\!\!\!+} (X^*)^U)$ and has counit $(\varepsilon_{TS})_A: TSA \to A$ given by identity on U and η_0 - the unit of T_US_0 - on the second coordinate. The comultiplication $(\delta_{TS})_A: TSA \to (TS)^2A$ is given by identity on U and μ_0 the multiplication of T_US_0 on the second coordinate.*

$$
\begin{array}{ccc}
U & \xleftarrow{TS\alpha} & (X^*)^U \\
\downarrow & & \uparrow{\scriptstyle \eta_0} \\
U & \xleftarrow{\alpha} & X
\end{array}
\qquad
\begin{array}{ccc}
U & \xleftarrow{TS\alpha} & (X^*)^U \\
\downarrow & & \uparrow{\scriptstyle \mu_0} \\
U & \xleftarrow{(TS)^2\alpha} & ((X^*)^U)^{*U}
\end{array}
$$

Besides the "composite monad", a distributive law provides a "lifting" of one of the monads to the category of algebras for the other monad.

Proposition 42 *The monad $T_U: \mathbf{C} \to \mathbf{C}$ "lifts" to the category of S_0-algebras, which means that we can describe a monad $\widetilde{T_U}: \mathbf{C}^{S_0} \to \mathbf{C}^{S_0}$. Given an S_0-algebra $(X, j_X: S_0X \to X)$, the endofunctor $\widetilde{T_U}$ is given by*

$$\widetilde{T_U}(X, j_X) = (T_UX, h_{T_UX}),$$

where $T_UX = X^U$. The new structural map $h_{(X^U)}: (X^U)^* \to X^U$ is given by the composition $(X^U)^* \overset{\lambda}{\to} (X^*)^U \overset{(j_X)^U}{\to} X^U$.

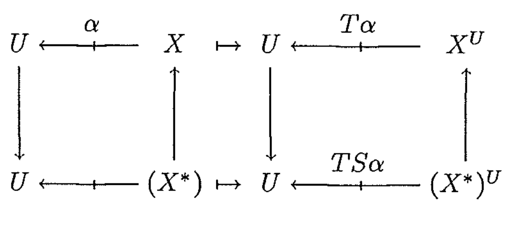
The endofunctor $\widetilde{T_U}$ acts on maps as T_U. The unit for $\widetilde{T_U}$ is the unit for T_U and the multiplication for $\widetilde{T_U}$ is μ_{T_U}. Beck shows that $h_{(X^U)}$ is a structure map and that the unit and multiplication are S_0-algebra maps.

Dually, you can lift one of the comonads to the category of coalgebras for the other comonad.

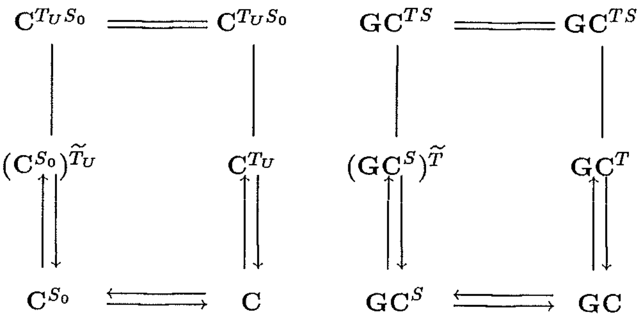
Proposition 43 *The comonad T in $\mathbf{GC}$ lifts to the category of S-coalgebras. The endofunctor $\widetilde{T}: (\mathbf{GC})^S \to (\mathbf{GC})^S$ has a comonad structure given by the monad structure of $\widetilde{T_U}$. Namely, if $(A, \beta_A: A \to SA)$ is an S-coalgebra, then $\widetilde{T}(A, \beta_A) = (TA, \alpha_{\widetilde{T}A})$, where $TA = (U \overset{T\alpha}{\leftarrow\!\!\!+} X^U)$ and the new structure map $\alpha_{\widetilde{T}A}: TA \to STA$ is given by identity on U and the structure map*

$$h_{(X^U)}: (X^U)^* \overset{(j_X)^U \cdot \lambda}{\to} X^U$$

given in Proposition 5.

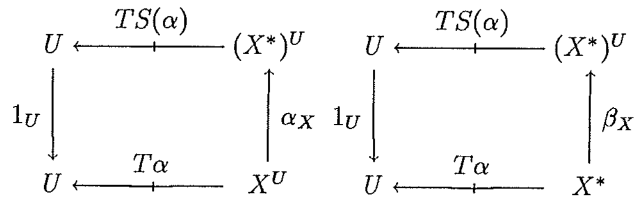


Using the categories above we can draw the following diagrams, and each of them has three sides consisting of adjoint-pairs.



It is easy to check that the maps $T_U \overset{T\eta_{S_0}}{\longrightarrow} T_U S_0$ and $S_0 \overset{(\eta_{T_U})}{\longrightarrow} T_U S_0$ are monad maps. To make notation less cumbersome we write α for $T_U \to T_U S_0$, thus the natural transformation $\alpha_X : T_U X \to T_U S_0 X$ takes $X^U \mapsto (X^*)^U$ and is given by T_U applied to the natural transformation $(\eta_{S_0}) : I \to S_0$. Also write β for $S_0 \to T_U S_0$, which is (η_{T_U}) applied to the object $S_0 X$, thus taking $X^* \to (X^*)^U$.

Similarly, there are comonad morphisms $\delta : TS \to T$ and $\kappa : TS \to S$, where $\delta_A : TSA \to TA$ is given by $\delta_A = (1_U, \alpha_X)$ and $\kappa_A : TSA \to SA$ by $\kappa_A = (1_U, \beta_X)$ as the diagram shows.



Proposition 44 *The monad and comonad morphisms above induce:*

- *maps in the categories of algebras,* $\overline{\alpha} : \mathbf{C}^{T_U S_0} \to \mathbf{C}^{T_U}$ *and* $\overline{\beta} : \mathbf{C}^{T_U S_0} \to \mathbf{C}^{S_0}$*;*
- *maps in the categories of coalgebras* $\overline{\delta} : \mathbf{GC}^T \to \mathbf{GC}^{TS}$ *and* $\overline{\kappa} : \mathbf{GC}^S \to \mathbf{GC}^{TS}$*.*

Thus we now know the direction of two of the morphisms in the diagrams on the previous page. Our next aim is to relate the categories $(\mathbf{C}^{S_0})^{\widetilde{T_U}}$ and $\mathbf{C}^{S_0 T_U}$ - dually $(\mathbf{GC}^S)^{\widetilde{T}}$ and $\mathbf{GC}^{ST}$.

Proposition 45 *There is an equivalence of categories of algebras,*

$$\Phi_0 : (\mathbf{C}^{S_0})^{\widetilde{T_U}} \to \mathbf{C}^{S_0 T_U}$$

and respectively, of categories of coalgebras $\Phi : (\mathbf{GC}^S)^{\widetilde{T}} \to \mathbf{GC}^{ST}$*.*

The proof for algebras in Beck's paper translates exactly to the coalgebras case, thus we omit it. □

Clearly the monad S_0 does not lift to the category of T_U-algebras, since we cannot define the T_U-structural map for S_0X using λ, but it seems to lift to the T_U-Kleisli category, $\mathbf{C}_{T_U}$. Here we have to be a little more careful since the results have not been proved by Beck, who was only interested in algebras. Clearly also, we are talking about duality once more, but that is a more subtle case.

Proposition 46 *The monad S_0 "lifts" to the Kleisli category C_{T_U}. That is the endofunctor $\widetilde{S_0}: C_{T_U} \to C_{T_U}$ that takes X to $S_0(X) = X^*$ and a morphism in $\mathbf{C}_{T_U}$, $X \to Y$ - corresponding to a map $f: X \to T_UY$ in $\mathbf{C}$ - to the composition $S_0(X) \stackrel{S_0(f_0)}{\to} S_0T_U(Y) \stackrel{\lambda}{\to} T_US_0(Y)$ which corresponds to $\widetilde{S_0}(f): S_0X \to S_0Y$ in $\mathbf{C}_{T_U}$ has a natural comonad structure.*

Proof: This is a general consequence of the existence of the distributive law. The unit

$$\eta_{\widetilde{S_0}}: X \to S_0X$$

in $\mathbf{C}_{T_U}$ corresponds to the map $\eta_0: X \to T_US_0X$ in $\mathbf{C}$. The multiplication of $\widetilde{S_0}$, $\mu_{\widetilde{S_0}}: X^{**} \to X^*$ in $\mathbf{C}_{T_U}$ corresponds to a map $\delta: X^{**} \to (X^*)^U$ in $\mathbf{C}$.

There are more easy calculations to come. The important point here is that they all "could" be read off from Street's paper *"The formal theory of monads"*, by a very clever 2-categorically minded reader. We will not go into the 2-categorical aspects of the theory here, but for one observation:

Recall that if $\mathcal{K}$ is a 2-category, one can write $\mathcal{K}^{op}$ for "reverse the 1-cells" and $\mathcal{K}_{co}$ for "reverse the 2-cells". Recall as well, from [Str] that a monad (A, S) in $\mathcal{K}$ consists of an object A and a 1-cell $S: A \to A$, together with two 2-cells $\eta: 1 \Rightarrow S$ and $\mu: S^2 \Rightarrow S$ such that the usual diagrams commute. In this context a morphism of monads $(f, \phi): (A, S) \to (B, T)$ consists of a 1-cell $f: A \to B$ and a 2-cell $\phi: Tf \Rightarrow fS$, such that appropriate diagrams

$$\begin{array}{ccc} f & = & f \\ {\scriptstyle \eta f}\downarrow & & \downarrow{\scriptstyle f\eta} \\ Tf & \xrightarrow{\phi} & fS \end{array} \qquad \begin{array}{ccc} TfS & = & TfS \\ {\scriptstyle T\phi}\downarrow & & \downarrow{\scriptstyle \phi S} \\ T^2f & & fS^2 \\ {\scriptstyle \mu f}\downarrow & & \downarrow{\scriptstyle f\mu} \\ Tf & \xrightarrow{\phi} & fS \end{array}$$

commute. Also, a transformation (of monad morphisms) consists of a 2-cell $\alpha: (f, \phi) \to (g, \psi)$ such that the diagram

$$\begin{array}{ccc} Tf & \xrightarrow{T\alpha} & Tg \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ fS & \xrightarrow{\alpha S} & gS \end{array}$$

commutes.

Thus, we have a 2-category of monads in $\mathcal{K}$, denoted by $\underline{Mnd}(\mathcal{K})$ and by inserting or not "op" and "co" in the appropriate positions, marked with a "+" in

$$\underline{Mnd}(\mathcal{K}\,{}^{+}_{+})\,{}^{+}_{+},$$

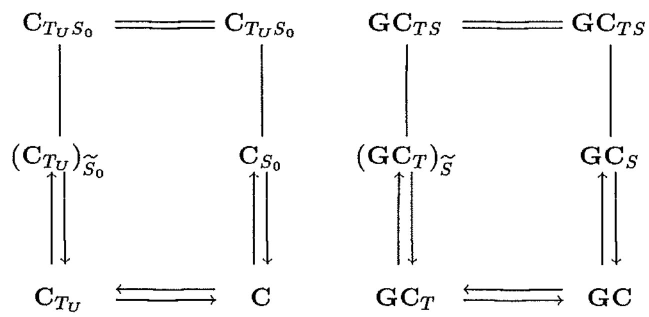
one obtains 16 distinct 2-categories!!

The point we want to make here is that a distributive law, between monads $\mathbf{T} = (T, \eta_T, \mu_T)$ and $\mathbf{S} = (S, \eta_S, \mu_S)$ on $\mathbf{C}$, as defined by Beck, that is a natural transformation $\lambda: ST \to TS$, such that the 4 mentioned diagrams commute, consists of $(T, \lambda): (C, S) \to (C, S)$, a standard morphism of monads in $\underline{Mnd}(\mathbf{CAT})$ and $(S, \lambda): (C, T) \to (C, T)$ a non-standard morphism in $\underline{Mnd}(\mathbf{CAT}^{\mathrm{op}})$.

Back to more pedestrian issues, we have

Proposition 47 *The comonad S lifts to the Kleisli category $\mathbf{GC}_T$. The endofunctor $\widetilde{S}$ takes A to SA. Given $(f, F): A \to B$ a map in $\mathbf{GC}_T$, corresponding to $(f_0, F_0): TA \to B$ a map in $\mathbf{GC}$, define $\widetilde{S}(f, F)$ as the composition of the maps $TSA \xrightarrow{\Lambda} STA \xrightarrow{S(f_0, F_0)} SB$, which corresponds to a map $SA \to SB$ in $\mathbf{GC}_T$.*

Using propositions 9 and 10, we can draw two new diagrams as follows:

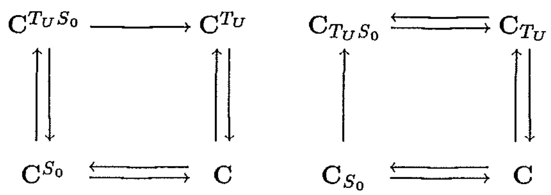


Proposition 48 *The monad maps in $\mathbf{C}$, $\alpha: T_U \to T_U S_0$ and $\beta: S_0 \to T_U S_0$ and the comonad maps in $\mathbf{GC}$, $\delta: TS \to T$ and $\kappa: TS \to S$ induce morphisms between the Kleisli categories $\hat{\alpha}: \mathbf{C}_{T_U} \to \mathbf{C}_{T_U S_0}$ and $\hat{\beta}: \mathbf{C}_{S_0} \to \mathbf{C}_{T_U S_0}$ over $\mathbf{C}$, and $\hat{\delta}: \mathbf{GC}_{TS} \to \mathbf{GC}_T$ and $\hat{\kappa}: \mathbf{GC}_{TS} \to \mathbf{GC}_S$ over $\mathbf{GC}$.*

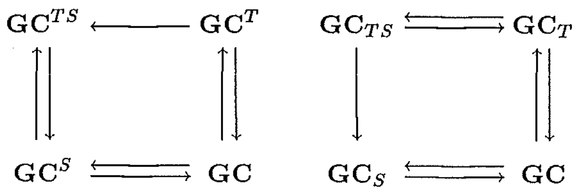
To finish the "dualisations" the next proposition relates with an equivalence, two vertices of the "pentagons" above.

Proposition 49 *There is an equivalence of categories $\Psi_0: (\mathbf{C}_{T_U})_{\widetilde{S}_0} \to \mathbf{C}_{T_U S_0}$. Dually, there is an equivalence $\Psi: (\mathbf{GC}_T)_{\widetilde{S}} \to (\mathbf{GC})_{TS}$.*

Summary: We can sum up the results of this section in the four "squares" below. Each square has three sides consisting of adjoint-pairs and the last side given by a very natural morphism. In $\mathbf{C}$, relating algebras and Kleisli categories,



and in **GC** relating coalgebras and Kleisli categories,



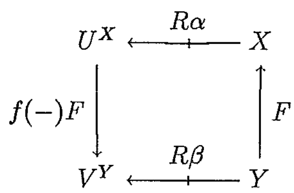
Note that if we ask for **C** with equalisers then, the two top squares are totally composed of adjoint-pairs, but we do not pursue it here, since it is not clear that equalisers in **C** would imply equalisers in **GC**.

4.4 Properties of the comonad T

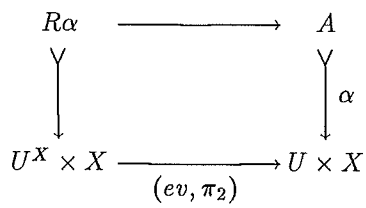
In the last section the endofunctor "T" was defined in **GC** taking the object $(U \overset{\alpha}{\leftrightarrow} X)$ to the object $(U \overset{T\alpha}{\leftrightarrow} X^U)$ and with relation "$T\alpha$" defined by the pullback of the object $A \overset{\alpha}{\rightarrowtail} U \times X$ along the map $U \times X^U \overset{U \times ev_U}{\rightarrow} U \times X$.

This endofunctor seems a reasonable candidate to represent the connective "!". For a start it has a "dual" endofunctor, to be denoted R, described in the next paragraph.

Definition 19 *The endofunctor R takes the object $(U \overset{\alpha}{\leftrightarrow} X)$ to $(U^X \overset{R\alpha}{\leftrightarrow} X)$, the object $(V \overset{\beta}{\leftrightarrow} Y)$ to $(V^Y \overset{R\beta}{\leftrightarrow} Y)$ and the map $(f, F): A \rightarrow B$ to the map $(f(-)F, F): RA \rightarrow RB$, using a diagram,*



Similarly to "$T\alpha$", the relation "$R\alpha$" is defined using the pullback of $A \overset{\alpha}{\rightarrowtail} U \times X$ along the evaluation morphism $U^X \times X \overset{(ev, \pi_2)}{\rightarrow} U \times X$ and intuitively it says "$g(R\alpha)x$ iff $g(x)\alpha x$".



Again using the monad properties of T_U we have the following proposition.

Proposition 50 *The functor R is the functor part of a monad, with unit $\eta_A: A \rightarrow RA$ given by the constant map $\eta_U: U \rightarrow U^X$ in the first coordinate and identity on X. Multiplication $\mu: R^2 A \rightarrow RA$*

is given by "restriction to the diagonal" $\mu_U\colon U^{X\times X} \to U^X$ in the first coordinate and identity on X, using diagrams,

$$
\begin{array}{ccc}
U & \xleftarrow{\ \alpha\ } & X \\
\downarrow & & \uparrow \\
U^X & \xleftarrow{\ R\alpha\ } & X
\end{array}
\qquad
\begin{array}{ccc}
(U^X)^X & \xleftarrow{\ R^2\alpha\ } & X \\
\downarrow & & \uparrow \\
U^X & \xleftarrow{\ R\alpha\ } & X
\end{array}
$$

We would like to have in **GC** results for "T" analogous to the ones for "!" in **DC**. For example the isomorphism $!(A\&B) \cong !A \otimes !B$ would be nice. But there is no obvious relationship between $T(A\&B) = (U \times V \leftrightarrow (X+Y)^{U\times V})$ and $T(A) \oslash T(B) = (U \times V \leftrightarrow X^{U\times V} \times Y^{U\times V})$. What we do have is a relation between the tensor products in **GC**.

Proposition 51 *There is a natural isomorphism in* **GC**, $T(A \otimes B) \cong TA \oslash TB$.
Proof: The result needed is

$$
\begin{array}{ccc}
U \times V & \xleftarrow{\ T(\alpha \otimes \beta)\ } & (X \times Y)^{U\times V} \\
\downarrow & & \uparrow \\
U \times V & \xleftarrow{\ T\alpha \oslash T\beta\ } & X^{U\times V} \times Y^{V\times U}
\end{array}
$$

That means isomorphisms in both coordinates, which is trivially the case.

For a far more interesting result, analogous to the ones in Chapter 2, recall that the T-Kleisli category $\mathbf{GC}_T$ has as objects the objects of **GC** but as maps from A to B, maps in **GC** from TA to B.

Proposition 52 *The maps from A to B in the T-Kleisli category* $\mathbf{GC}_T$*, are in 1-1 correspondence with maps from A to B in the category* **DC**.

Proof: We want to check

$$\mathrm{Hom}_{\mathbf{GC}_T}(A,B) = \mathrm{Hom}_{\mathbf{GC}}(TA,B) \approx \mathrm{Hom}_{\mathbf{DC}}(A,B).$$

The second equivalence holds, since a map $(f,F)\colon TA \to B$ in **GC**, corresponds to $f\colon U \to V$ and $F\colon Y \to X^U$, satisfying the condition

$$(U \times F)^{-1}(\alpha) \leq (f \times Y)^{-1}(\beta). \tag{1}$$

in Chapter 3. That corresponds to $f\colon U \to V$ and, by exponential transpose, to $\overline{F}\colon U \times Y \to X$, satisfying the correspondent condition $(*)$ in Chapter 1, that is a map $(f,\overline{F})\colon A \to B$ in **DC**.

Since objects are the same in both categories **GC** and **DC**, Proposition 15 implies that there is an equivalence between categories $\mathbf{GC}_T$ and **DC**.

4.5 The comonad "!"

In this section we consider the composite comonad TS defined in the last section, with the difference that now S_0 denotes free *commutative* monoids in **C**. Thus the composition US_0 corresponds to "$\star$" in Chapter 2 and S on objects, correspond to the functor "!" of that Chapter. Thus

$S(U \overset{\alpha}{\leftarrow\!\!\!\!+} X) = (U \overset{S\alpha}{\leftarrow\!\!\!\!+} X^*)$ and S takes a morphism $(f, F): A \longrightarrow B$ to $(f, F^*): SA \longrightarrow SB$, as the diagram shows:

$$\begin{array}{ccc} U & \overset{S\alpha}{\longleftarrow\!\!\!\!+\!\!\!\!\longrightarrow} & X^* \\ f \big\downarrow & & \big\uparrow F^* \\ V & \overset{S\beta}{\longleftarrow\!\!\!\!+\!\!\!\!\longrightarrow} & Y^* \end{array}$$

Let the comonad TS be called "!". The functor part of "!" acts on objects as $!(U \overset{\alpha}{\leftarrow\!\!\!\!+} X) = (U \overset{!\alpha}{\leftarrow\!\!\!\!+} (X^*)^U)$ and on maps $!(f, F) = (f, F^*(-)f)$, or using a diagram

$$\begin{array}{ccc} U & \overset{TS\alpha}{\longleftarrow\!\!\!\!+\!\!\!\!\longrightarrow} & (X^*)^U \\ f \big\downarrow & & \big\uparrow F^*(-)f \\ V & \overset{TS\beta}{\longleftarrow\!\!\!\!+\!\!\!\!\longrightarrow} & (Y^*)^V \end{array}$$

As we have shown in section 3 "!" is the functor part of the composite comonad

$$(!, \varepsilon_!: !A \longrightarrow A, \delta_!: !A \longrightarrow !!A)$$

and we can consider the categories $\mathbf{GC}^!$ of !-coalgebras and $\mathbf{GC}_!$, the !-Kleisli category.

Moreover, the objects "$!A$" have a natural comonoid-like structure, with respect to "$\oslash$".

Proposition 53 *There are natural morphisms in* **GC** *as follows*

- *From the object $!A$ to I, given by the terminal map on U and the natural map $1 \longrightarrow (X^*)^U$.*
- *From $!A$ to $!A \oslash !A$, which is given by the diagonal map in* **C**, *$\Delta: U \longrightarrow U \times U$ and the map $\theta: (X^*)^{U\times U} \times (X^*)^{U\times U} \longrightarrow (X^*)^U$. The map θ is given, intuitively, by taking a pair of functions (ϕ, ψ), each of them of the form $U \times U \longrightarrow X^*$, to the product map $\phi \times \psi$ precomposing it with the diagonal in U and post-composing it with the multiplication on X^*, as follows,*

$$U \xrightarrow{\Delta} U \times U \xrightarrow{<\phi,\psi>} X^* \times X^* \xrightarrow{\mu_*} X^*.$$

Proposition 54 *We have the following natural isomorphisms for all objects A and B in* **GC**,

$$!(A\&B) \cong !A \oslash !B$$

Proof: Look at the following series of equivalences and recall that S on objects is the same as "!" in Chapter 2.

$$!(A\&B) = TS(A\&B) \cong T(SA \otimes SB) \cong TSA \oslash TSB = !A \oslash !B.$$

Then the first equivalence comes from Proposition 9 and the second from Proposition 14.

Proposition 55 *The Kleisli category* $\mathbf{GC}_!$ *is cartesian closed.*

That is an easy corollary of the above, since

$$\mathrm{Hom}_{\mathbf{GC}}(TS(A \oslash B), C) \cong \mathrm{Hom}_{\mathbf{GC}}(!A \oslash !B, C) \cong$$

$$\cong \mathrm{Hom}_{\mathbf{GC}}(!A, [!B, C]_{\mathbf{GC}}) \cong \mathrm{Hom}_{\mathbf{GC}_!}(A, [B, C]_{\mathbf{GC}_!}).$$

Proposition 56 *The morphisms from A to B in the category* $\mathbf{GC}_!$*, correspond naturally to morphisms in the category* **DNC** *from A to B.*

We have to go through the series of equivalences:

$$\mathrm{Hom}_{\mathbf{GC}_!}(A, B) \cong \mathrm{Hom}_{\mathbf{GC}}(TSA, B) \cong \mathrm{Hom}_{\mathbf{GC}_T}(SA, B)$$

$$\cong \mathrm{Hom}_{\mathbf{DC}}(SA, B) \cong \mathrm{Hom}_{\mathbf{DNC}}(A, B).$$

The notation is a bit unfortunate here, since we have two very different functors, one in **DC** the other in **GC**, with the same name "!".

4.6 Linear Logic with modalities.

In this section the situation is slightly more complicated than the one in Chapter 2. As we have stressed we have a composite comonad, which satisfies the rules for the modality "!", but we would like also a monad "?" satisfying the rules for the dual connective, called by Girard "why not ?".

We start by recalling the rules for the modality "!". These are:

$$\text{I.}\frac{\Gamma, \mathrm{A} \vdash \mathrm{B}}{\Gamma, !\mathrm{A} \vdash \mathrm{B}}\ (\textit{dereliction}) \qquad \text{II.}\frac{\Gamma \vdash \mathrm{B}}{\Gamma, !\mathrm{A} \vdash \mathrm{B}}\ (\textit{weakening})$$

$$\text{III.}\frac{\Gamma, !\mathrm{A}, !\mathrm{A} \vdash \mathrm{B}}{\Gamma, !\mathrm{A} \vdash \mathrm{B}}\ (\textit{contraction}) \qquad \text{IV.}\frac{!\Gamma \vdash \mathrm{A}}{!\Gamma \vdash !\mathrm{A}}\ (!)$$

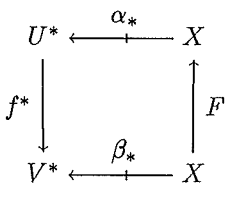
Then it is clear that the category **GC** with modality "!" , defined in section 4, is a model for Linear Logic enriched with modality "!".

Proposition 57 *The category* **GC** *with the composite comonad "!" defined in section 4 is a model for Linear Logic enriched with modality "!".*

We have already checked that the composite comonad satisfies the necessary conditions in Proposition 16, previous section.

Now we turn our attention to the monad "?". As we have seen in definition **7**, the comonad T has a dual functor, called R, which is a monad in **GC**. Composing R with the monad in **GC**, induced by the monad $U \mapsto U^*$ in **C**, we get a composite monad, which satisfies all the necessary conditions.

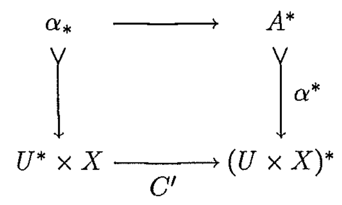
Definition 20 *Consider the monad* $S_1\colon \mathbf{GC} \to \mathbf{GC}$*, given by* $S_1(U \overset{\alpha}{\leftrightarrow} X) = (U^* \overset{\alpha_*}{\leftrightarrow} X)$ *on objects and* $S_1(f, F) = (f^*, F)$ *on maps, as the diagram shows:*



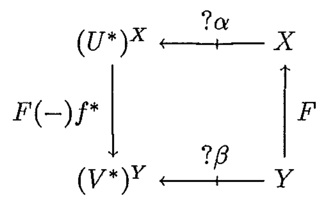
The relation "$S_1\alpha$", which is written a_*, is defined in the same way as "$S\alpha$". Thus we use morphisms like the morphisms $C_{(-,-)}$ of Chapter 2, as follows:

$$\frac{\dfrac{\dfrac{V \times Y \overset{\zeta_{(V\times Y)}}{\longrightarrow} (V \times Y)^*}{V \overset{\overline{\zeta}}{\longrightarrow} (V \times Y)^{*Y}}}{V^* \longrightarrow (V \times Y)^{*Y}}}{V^* \times Y \overset{C'_{(V,Y)}}{\longrightarrow} (V \times Y)^*}$$

And we define the relation "$S_1\alpha$" using the pullback

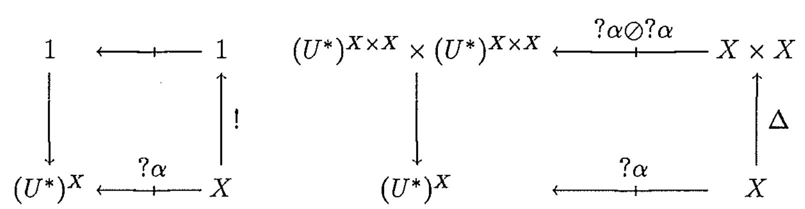


Definition 21 *Consider the endofunctor "?" in* **GC** *given by* $?(U \overset{\alpha}{\leftrightarrow} X) = ((U^*)^X \overset{?\alpha}{\leftrightarrow} X)$ *on objects and* $?(f, F) = (f^*(-)F, F)$ *on morphisms, as the diagram shows.*



This endofunctor is the composition of R and S_1 in **GC**.

Proposition 58 *The endofunctor "?" has a natural structure as a monad in* **GC**. *Moreover, it satisfies dual conditions to the ones on "!". Namely, we have natural morphisms* $\eta: I \longrightarrow ?A$ *and* $?A\Box?A \longrightarrow ?A$, *given by*



We repeat the rules for the modalities, using the modality "?".

$$\text{I.}\frac{\Gamma, ?A \vdash B}{\Gamma, A \vdash B}\ (\textit{dereliction}) \qquad \text{II.}\frac{\Gamma \vdash B}{\Gamma, ?A \vdash B}\ (\textit{weakening})$$

$$\text{III.}\frac{\Gamma, ?A \vdash B}{\Gamma, ?A, ?A \vdash B}\ (\textit{contraction}) \qquad \text{IV.}\frac{\Gamma \vdash ?A}{?\Gamma \vdash ?A}\ (?)$$

Proposition 59 *The category* **GC** *with the monad "?" is a model of* $L.L_?$. □

Concluding remarks

To conclude it is perhaps worth mentioning some of the several questions that the work on the categories **DC** and **GC** prompts, apart from the ones already mentioned in the introduction.

1. Is there an interesting connection between the categorical models **DC** and **GC** and Girard's new work on the Geometry of Interactions ?

2. Since we think of maps in **DC** and **GC** as "linear morphisms", in opposition to the more usual morphisms in the Kleisli categories, can we characterize bilinear maps in this context ? There is some interesting work of Kock on categorical bilinearity, but the obvious approach does not work, due to the fact that the comonad "!", or rather, its functor part, is not a strong functor.

3. We have shunned away from the 2-categorical aspects of everything discussed in the previous 4 chapters, but that is not, probably, the best policy, as was indicated by the need of distributive laws in this chapter. More to the point, there is a very interesting question of using "spans" instead of relations in the construction of **DC** and **GC**, which was suggested by Aurelio Carboni.

4. We have worked only with commutative versions of the connectives, that is with symmetric tensor products, "par" bifunctors etc. There is a interesting case to look at, if this commutativity condition is dropped. Along these lines there is some connection with Joyal and Street's work on braided monoidal categories. In particular there is also a preprint by D. N. Yetter on "Quantales and (Non-commutative) Linear Logic".

5. Finally, there is the very promising, but as yet very vague idea of connecting Linear Logic with Concurrency and Parallelism. The idea being that Linear Logic may provide an *integrated logic*, where one would hope to model computational processes in a less *ad hoc* fashion than it has been up to now. In particular, Petri Nets have been proposed as a model for Linear Logic, cf. [Gir] 1987 and the references therein.

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