Homotopy Theoretic Aspects of Constructive Type Theory

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Contents

Abstract

In this dissertation I prove several results which serve to relate Martin-Löf's intensional type theory to certain structures arising in homotopy theory and higher-dimensional category theory. First, I describe a general semantics for type theory utilizing Quillen's model categories and study the coherence issues arising in this setting. Secondly, I introduce the notion of an interval I in a category. I show that, when $\mathcal{E}$ possesses an interval, there exists a distinguished collection of maps in $\mathcal{E}$, called *split fibrations*, which give rise to a model of type theory. This model, moreover, avoids the coherence problems related to the interpretation of the elimination terms for identity types. This result allows us, for example, to obtain models of type theory using internal groupoids. Finally, I extend the groupoids model of type theory, due to Hofmann and Streicher, to the setting of strict ω-groupoids. In particular, I prove that strict ω-groupoids soundly model intensional type theory. As a consequence I obtain new independence results for type theory relating to the higher-dimensional structure induced by the intensional identity types.

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Introduction

The principal aim of this dissertation is to establish several connections between Martin-Löf's intensional type theory and homotopy theory. In particular, we obtain new models of intensional type theory using structures that arise in homotopy theory and higher-dimensional category theory.

Martin-Löf's type theory was introduced as a foundation for constructive mathematics and can also be regarded as extending the propositions-as-types paradigm — under which Church's [**14**] simple type theory corresponds to propositional intuitionistic logic — to (a form of) intuitionistic logic (cf. [**63, 60, 62, 61, 67**]). For us, the most distinctive feature of this theory are the *identity types.* Identity types are intended to correspond, under the propositions-as-types idea [**19, 37**], to the equality relation. Explicitly, given a type A together with terms a and b both of type A, there exists a new type $\mathrm{Id}_A(a,b)$ called the *identity type of A at a and b*. This type can be thought of as the proposition which states that a and b are identical terms of type A. In the *extensional* forms of the theory these identity types are trivial in the sense that if there exists any term p of type $\mathrm{Id}_A(a,b)$, then $a = b$, where $=$ denotes the "definitional" or "real" equality between terms. In the *intensional* form of the theory this is not the case and the identity types possess a richer structure. Moreover, type-checking is decidable in the intensional, but not extensional theory (cf. [**60, 32**]). The question which naturally arises, and which is one of the motivations for the research contained herein, is, "What kind of structure and properties do the identity types in intensional type theory possess?"

A significant step toward an answer to this question was provided by Hofmann and Streicher [**35**], who suggested that the identity type endows its base type with a certain algebraic structure like a category. Thus, we think of terms a and b of type A as *objects* and the identity type $\mathrm{Id}_A(a,b)$ as a kind of "hom-set" of arrows. Then, for example, the reflexivity terms $r_A(a)$ of type $\mathrm{Id}_A(a,a)$, which are guaranteed to exist by the introduction rule for identity types, can be thought of as being like "identities" 1_a. At the time, it was an important open question whether the existence of terms f and g both of type $\mathrm{Id}_A(a,b)$ implies either that $f = g$ or (even) that there exists a term of type $\mathrm{Id}_{\mathrm{Id}_A(a,b)}(f,g)$. This problem of the *uniqueness of identity proofs* — which from the perspective just described amounts to the question whether these algebraic gadgets behave like preorders — was solved in the negative by Hofmann and Streicher by constructing a particular model of intensional type theory which refuted these principles. In particular, they interpreted contexts, and so also closed types, as groupoids and types in context as functors $\mathcal{C} \longrightarrow \mathbf{Gpd}$, where $\mathcal{C}$ is the groupoid denoting the context. Identity types in this model are then interpreted as actual (discrete) hom-groupoids $A(a,b)$, when A is a closed type.

The clue which led *idem* to arrive at groupoids was the fact that, under the aforementioned view of $\mathrm{Id}_A(a, b)$ as a kind of hom-set, the types themselves satisfy forms of the familiar groupoid laws. For example, given terms f of type $\mathrm{Id}_A(a, b)$ and g of type $\mathrm{Id}_A(b, c)$, there exists a "composite" $(g \cdot f)$ of type $\mathrm{Id}_A(a, c)$. However, this composition and the identities mentioned above fail to satisfy the actual category axioms "on-the-nose", but only up to the existence of terms of further "higher-dimensional" identity types. Thus, given f and g as above together with a further term h of type $\mathrm{Id}_A(c, d)$, the type

$$\mathrm{Id}_{\mathrm{Id}_A(a,d)}\big(h \cdot (g \cdot f),\ (h \cdot g) \cdot f\big)$$

is inhabited; but it is not in general the case that $h\cdot(g\cdot f) = (h\cdot g)\cdot f$. The additional laws governing groupoids are likewise satisfied up to the existence of further terms of identity type. Thus, we are naturally led by these observations to regard this algebraic structure on types as being a kind of "higher-dimensional groupoid". In particular, the syntax of Martin-Löf type theory appears to impart on the types themselves a variety of *weak* higher-dimensional structure (cf. [**54**] for some of the proposed definitions of weak higher-dimensional categories).

Algebraic topology can be regarded as (among other things) that branch of mathematics which is concerned with studying the connection between topological spaces, on the one hand, and (higher-dimensional) algebraic structures, on the other. The attempt to understand the algebraic structure of spaces — specifically, the attempt to classify homotopy types — leads naturally, via the early work of Eilenberg, Mac Lane and Whitehead, also to the category of groupoids (cf. [**13**]). Specifically, regular (1-dimensional) groupoids classify homotopy 1-types (those connected CW-complexes whose homotopy groups vanish above dimension 1). I.e., the category of groupoids is equivalent to the category of 1-types. In order to classify higher homotopy types, it has been similarly necessary to consider weak higher-dimensional generalizations of groupoids (cf. [**6**]). Indeed, regarding spaces as algebraic "gadgets" in much the same way as we did above for types (i.e., a space has "objects" points and "arrows" paths, *et cetera*) yields, intuitively, a weak ω-groupoid (cf. [**29, 5, 49**]). This situation is strikingly similar to the type theoretic situation and it therefore suggests that it may be profitable to search for novel models of type theory among the kinds of structures homotopy theorists and higher-dimensional category theorists have developed for these purposes. On the one hand, such models would clearly be beneficial to type theory since they would surely lead to new independence results. On the other hand, the existence of such models should also be beneficial to homotopy theory and higher-dimensional category theory. For instance, knowing that the categories (and related structures) employed in homotopy theory admit models of type theory indicates that the theory itself may be employed as an "internal language" for the categories in question. In topos theory, to take one example, the existence of a logically rich internal language has the potential to yield new results which would not be tractable using the "naïve" diagrammatic reasoning [**59, 41**]. It is the aim of the present work to provide one step in the direction of such a connection between homotopy theory and type theory by providing several new models of type theory using structures and techniques from homotopy theory and higher-dimensional category theory.

We now turn to a summary of the four chapters of this dissertation. Each chapter itself begins with a more detailed summary than that given here.

Chapter 1: Forms of Type Theory. This preliminary chapter introduces the versions of type theory with which the latter chapters will be concerned. Specifically, we recall the rules governing identity types and then introduce a *hierarchy of theories* obtained by augmenting the basic form of type theory $\mathbb{T}_\omega$ with various *truncation rules* which serve to restrict the behavior of the identity types in certain ways. We then offer some basic observations regarding the comparison of these theories. Intuitively, this hierarchy of theories can be thought of as the type theoretic analogue of the hierarchy of (categories of) homotopy n-types:

$$\text{1-Types} \subseteq \text{2-Types} \subseteq \cdots \subseteq n\text{-Types} \subseteq \cdots \subseteq \omega\text{-Types}.$$

Chapter 2: Homotopical Semantics of Type Theory. The semantics of extensional Martin-Löf type theory have been thoroughly studied and it is known, for example, that its models correspond, in an appropriate sense, to locally cartesian closed categories [**72, 38, 33**]. In addition to the groupoid model of Hofmann and Streicher mentioned above, the only other models of the intensional theory that we know of are syntactic, realizability and domain-theoretic models (cf. [**80, 32**]). These models, however, are not well suited, in the way that the groupoid model is, to studying the higher-dimensional structure of identity types. In Chapter 2, following a suggestion of Moerdijk, we describe a general semantics for type theory using weak factorization systems and Quillen's model categories [**70**]. Model categories arose out of an attempt to axiomatize, based on a number of examples such as the category of simplicial sets or chain complexes, the relevant features of those categories in which it is possible to develop a homotopy theory. The resulting axiomatization has proven to be very successful and these tools now predominate the field. Indeed, they have been used by Voevodsky and others in algebraic geometry [**66, 82**]. Model categories have also been employed in the work of Joyal [**45, 44**] on quasi-categories and in the related work of Lurie [**56**] on higher-dimensional topos theory. As such, the general approach to the semantics of intensional type theory offered here has potential to yield further interesting models.

Regarding the specific contents of the chapter, after reviewing the basic definitions and examples, we discuss the interpretation of types as *fibrations* in a weak factorization system (or model category). Interpreting types in this way *always* gives rise to a model of a form of type theory. However, the type theory of which it is a model may nonetheless fail to validate all of the rules governing identity types. Yet, these models do always offer a "hint" of how to interpret the identity types. Namely, in weak factorization systems we may form what are called *path objects*, which we can think of as the fibration consisting of paths in a given space. Such path objects, provided they are stable under pullback in a sense which we make precise, will always very nearly support the interpretation of identity types. Voevodsky [**84**] has considered, in a somewhat different setting, a similar interpretation of identity types in the particular case of simplicial sets. In particular, they will necessarily satisfy (up to isomorphism) all of the rules governing identity types except for the *coherence* (or *Beck-Chevalley*) rule for the elimination J terms. We call such structures (which almost model identity types in this way) *quasi-models* of identity types. Our first main result, Theorem 2.29, is that every simplicial model category in which the cofibrations are the monomorphisms is a quasi-model of identity types.

The final sections are concerned with providing an answer to the question, "When does a weak factorization system give rise to an *actual* model (and not just a quasi-model) of type theory?" Specifically, we describe conditions on a weak factorization system under which the interpretation of types as fibrations and identity types as path objects yields a model (up to isomorphism) of intensional Martin-Löf type theory. Such models still suffer from the familiar coherence problems afflicting models of (extensional) type theory in locally cartesian closed categories (cf. [**18**]). As such, our next task is to establish that genuine *split* models of intensional type theory can be obtained, as for extensional type theory, by applying Bénabou's [**7**] fibred Yoneda lemma. This is indeed the case and extends the familiar result due to Hofmann [**33**] to the intensional setting. This is Theorem . We mention that Gambino and Garner [**23**] have recently constructed in the category of contexts of intensional type theory a weak factorization system, thereby showing the completeness of the corresponding fragment of type theory with respect to the semantics developed here.

Chapter 3: Cocategories and Intervals. In Chapter 3 we study one general class of models of type theory which includes the original groupoid model of Hofmann and Streicher discussed above. Namely, we describe a way in which models can be obtained using certain cocategory objects which we call *intervals*. Using such an interval I it is possible to define many of the notions from homotopy theory such as *homotopy*, *strong deformation retract*, *Hurewicz fibration*, *et cetera*. The first several sections of the chapter are concerned with studying these and related notions in the setting of a category equipped with such an interval. We mention that related techniques, also employing cocategory objects, have been employed in the setting of the homotopy theory of categories enriched in simplicial modules (and related structures) by Stanculescu [**74**].

An interval I in $\mathcal{E}$ necessarily induces a 2-category structure on $\mathcal{E}$, with 2-cells homotopies, and our next task is to investigate this higher-dimensional structure. In particular, we describe necessary and sufficient conditions for such an interval to induce a *representable* (or *finitely complete*) 2-categorical structure. In the case where this structure is representable there exists, by a result due to Lack [**51**], a model structure on the underlying category wherein fibrations are isofibrations (defined representably using the notion of isofibration in **Cat**).

When $\mathcal{E}$ possesses an interval it is possible, by adapting an approach due to Street [**75**] to this setting, to define a reasonable notion of *split fibration* in $\mathcal{E}$. In particular, the split fibrations are (strict) algebras for a 2-monad on $\mathcal{E}$ defined using the interval I. The first main type theoretic result of this chapter, Theorem 3.47 is that, given an interval I in a finitely bicomplete category $\mathcal{E}$ which is cartesian closed, the split fibrations in $\mathcal{E}$ defined in this way are a model of intensional type theory which satisfies all of the required stability properties from Chapter 2. Finally, we apply Theorem 3.47 to the special case of internal groupoids $\mathbf{Gpd}(\mathcal{E})$ in a category $\mathcal{E}$ and show also that when $\mathcal{E}$ is locally cartesian closed the resulting model using split fibrations supports the interpretation of dependent products (all of the models described in this dissertation soundly model dependent sums). This essentially recovers the original Hofmann-Streicher model as a special case.

Chapter 4: ω-Groupoids. Although Chapter 3 develops techniques for obtaining models of intensional type theory like the original groupoid model of Hofmann and Streicher [**35**] the resulting models will only be 1-dimensional in the sense that they validate the truncation principles from Chapter 1 for dimensions $n > 1$. E.g., in such 1-dimensional models, if α and β are both of type $\mathrm{Id}_{\mathrm{Id}_A(a,b)}(f, g)$, then it follow that $\alpha = \beta$. I.e., in such models all identity types of identity types satisfy the uniqueness of identity proofs. This is related to the fact that the intervals considered in Chapter 3 are required only to be 1-dimensional cogroupoids rather than higher-dimensional ones. In Chapter 4, we take a page out of the homotopy theorist's book and turn our attention to ω-groupoids (cf. [**49**]).

In particular, we extend the original groupoid model of Hofmann and Streicher [**35**] to higher dimensions by proving that, interpreting closed types as *strict* ω-groupoids (for a suitable notion of strict ω-groupoid), and types in context as (strict ω-)functors $\mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$, yields a sound model of intensional type theory (Theorem 4.25). Moreover, this model is truly higher-dimensional in the sense that it refutes all of the truncation principles introduced in Chapter 1. For example, it refutes all higher-dimensional generalizations of the principle of "uniqueness of identity proofs". The rather technical proof of Theorem 4.25 proceeds according to the following steps. First, we describe, for a functor $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Cat}$ of strict ω-categories (with $\mathcal{C}$ small), two kinds of Grothendieck construction $\int A$ and $\int^* A$ which yield in turn ω-categories. When $\mathcal{C}$ is a (small) strict ω-groupoid and $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$, both $\int A$ and $\int^* A$ are also ω-groupoids and we prove that there exists a functor $\neg : \int A \longrightarrow \int^* A$ which acts by turning certain triangles occurring in the construction of $\int A$ "inside-out". Using $\neg$ we show that it is then possible to define the interpretation of identity types in this setting. In particular, when A is a type in the empty context with a and b terms (objects) of A, the identity type $\mathrm{Id}_A(a, b)$ is interpreted as expected as the hom-ω-groupoid $A(a, b)$. Moreover, all of the constructions can be truncated at any $n \geq 1$ and therefore also yield corresponding models using n-groupoids. We obtain in this way new models of type theory and corresponding novel independence results. These independence results — that the truncation principles from Chapter 1 are not derivable in intensional type theory — provide us with a much better picture of the behavior of identity types and also confirm the suspicion that the algebraic structure of identity types is genuinely higher-dimensional.

Finally, it is our hope that these constructions can be generalized to yield models using other kinds of higher-dimensional structures such as, e.g., the weak ω-groupoids of Kapranov and Voevodsky [**49**].

CHAPTER 1

Forms of Type Theory

In this chapter we introduce the forms of Martin-Löf type theory with which we will be concerned. We assume that the reader is familiar with the syntax of such theories and refer the reader to Appendix B for an overview of the syntax of such theories. Further details can be found in the literature [**60, 34, 80, 38, 67**]. Part of the purpose of this section is to exhibit a hierarchy of type theories, related to the higher-dimensional structure introduced into the general setting by the intensional identity types, with which we will be concerned. This hierarchy of theories is analogous to the kind of "dimensional" hierarchies arising in higher-dimensional category theory and homotopy theory.

Remark 1.1. One feature of our presentation of type theory which is perhaps worth mentioning is that we do not work explicitly in a logical framework. This is essentially a pragmatic decision and does not reflect any deeper preference for one formulation of the theory over the other. The reader may assume that we are implicitly using the logical framework from [**67**].

1.0.1. Identity types. The basic forms of type theory with which we will be concerned all have six forms of judgement as summarized in Figure 1.1.

Figure 1.1 Forms of Judgement

$$\begin{aligned} \vdash\ \Gamma &: \text{context} \\ \vdash\ \Gamma = \Delta &: \text{context} \\ \Gamma \vdash\ A &: \text{type} \\ \Gamma \vdash\ A = B &: \text{type} \\ \Gamma \vdash\ a &: A \\ \Gamma \vdash\ a = b &: A \end{aligned}$$

Moreover, all of the theories we study possess dependent products and dependent sums. We denote by $\mathbb{T}_-$ the basic form of Martin-Löf type theory which is given only by the rules for dependent products and sums in addition to the basic structural rules. The specific rules we adopt for dependent products and sums are given in Appendix B. All of the other theories we consider are obtainable by adding to $\mathbb{T}_-$ various rules governing identity types. The first rules for identity types we consider are the standard (intensional) rules. Because of their importance in what follows we review these rules explicitly here. First, the formation rule for identity types is stated as follows:

$$\frac{\Gamma \vdash A : \mathrm{type} \qquad \Gamma \vdash a, b : A}{\Gamma \vdash \mathrm{Id}_A(a,b) : \mathrm{type}}\ \text{Id formation}$$

Several remarks about this rule are in order. To begin with, we adopt the convention of omitting some judgements when they are understood. For example, this rule, properly stated, should include the additional judgement $\vdash \Gamma : \mathrm{context}$. Similarly, rather than stating two typing judgements $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$ we adopt the convention of condensing using the obvious abbreviation $\Gamma \vdash a, b : A$ as above. Additionally, we often write rules in the empty context and it is always assumed that these rules apply also in non-empty contexts. Thus, with these conventions the introduction rule for identity types is stated as follows:

$$\frac{\vdash a : A}{\vdash r_A(a) : \mathrm{Id}_A(a,a)}\ \text{Id introduction}$$

The term $r_A(a)$ is referred to as the **reflexivity term** for a. Next we have the elimination rule:

$$\frac{\begin{array}{c} x : A, y : A, z : \mathrm{Id}_A(x,y) \vdash B(x,y,z) : \mathrm{type} \\ u : A \vdash b(u) : B\big(u,u,r_A(u)\big) \\ \vdash p : \mathrm{Id}_A(a,a') \end{array}}{\vdash J_{A,B}([u:A]b(u), a, a', p) : B(a,a',p)}\ \text{Id elimination}$$

Here the presence of the expression $[u : A]b(u)$ indicates that the variable u occurring in $b(u)$ is bound in the subexpresion $b(u)$ of the **elimination term** $J_{A,B}([u : A]b(u), a, a', p)$. Sometimes we omit the $[u : A]$ and the subscripts and simply write $J(b, a, a', p)$ when no confusion will result. The conversion rule, which describes the behavior of the elimination term when applied to the reflexivity term, is as follows:

$$\frac{\begin{array}{c} x : A, y : A, z : \mathrm{Id}_A(x,y) \vdash B(x,y,z) : \mathrm{type} \\ u : A \vdash b(u) : B\big(u,u,r_A(u)\big) \\ \vdash a : A \end{array}}{\vdash J_{A,B}\big([u:A]b(u), a, a, r_A(a)\big) = b(a) : B\big(a,a,r_A(a)\big)}\ \text{Id conversion}$$

Finally, there are coherence (or "Beck-Chevalley") rules governing the behavior of identity types, reflexivity terms and elimination terms with respect to substitution. Technically these are meta-theoretic rules and are therefore really trivial. However, we state them here "for the record". It is perhaps convenient for the reader to regard them as rules in the logical framework. The semantic understanding of these rules is important and will play a significant part in Chapter 2. First we have the coherence condition for the identity types themselves:

$$\frac{x : C \vdash A(x) : \mathrm{type} \qquad x : C \vdash a(x), b(x) : A(x) \qquad \vdash c : C}{\vdash \big(\mathrm{Id}_{A(x)}(a(x), b(x))\big)[c/x] = \mathrm{Id}_{A(c))}(a(c), b(c))}\ \text{Id coherence}$$

Under the same hypotheses the coherence conditions for reflexivity terms is given by the following judgement of definitional equality:

$$\vdash \big(r_{A(x)}(a(x))\big)[c/x] = r_{A[c/x]}\big(a(c)\big) : \mathrm{Id}_{A(c)}\big(a(c), a(c)\big).$$

Finally, the coherence condition for elimination terms is as follows:

$$\frac{\begin{array}{c} x : C, v : A(x), w : A(x), z : \mathrm{Id}_{A(x)}(v, w) \ \vdash \ B(x, v, w, z) : \mathrm{type} \\ x : C, u : A(x) \ \vdash \ b(x, u) : B\big(x, u, u, r_{A(x)}(u)\big) \\ x : C \ \vdash \ p(x) : \mathrm{Id}_{A(x)}(a(x), a'(x)) \qquad\qquad \vdash \ c : C \end{array}}{\Big(J\big([u : A(x)]b(x, u), a(x), a'(x), p(x)\big)\Big)[c/x] \ = \ J\big([u : A(c)]b(c, u), a(c), a'(c), p(c)\big)}$$

We will sometimes refer to rules for identity types just given as the **categorical** rules for identity types. However, the rules may also profitably be formulated in a **hypothetical** form. Namely, the hypothetical version of the formation rule is given as follows:

$$\frac{\vdash \ A : \mathrm{type}}{x : A, y : A \ \vdash \ \mathrm{Id}_A(x, y) : \mathrm{type}} \text{ Id form. (hypothetical)}$$

Similarly, the hypothetical versions of the introduction, elimination and conversion rules are listed in Section B.2.3 of Appendix B. In general (assuming the basic structural rules governing type theory) these two forms of rules for identity types are equivalent. Nonetheless, the hypothetical versions are convenient as they are sometimes easier to verify in particular models. These matters are discussed in more detail in Appendix B.

1.0.2. Truncation rules. In order to most efficiently (and readably) state some of the additional principles for identity types which we will consider it will be useful to introduce an alternate notation for identity types. Letting a type A in some ambient context be given, we introduce the (at this stage superfluous) notation

$$\underline{A}^0 := A.$$

When we are given terms a and b of type A we then define

$$\underline{A}^1(a, b) := \mathrm{Id}_A(a, b).$$

In this case, we sometimes omit the superscript and simply write $\underline{A}(a, b)$ for the identity type. This notation is suggestive of the connection we have in mind between identity types and hom-sets in higher-dimensional category theory. But more on this later. At the next stage, given a_1 and b_1 of type A together with

$$\vdash \ a_2, b_2 : \underline{A}(a_1, b_1),$$

we define

$$\underline{A}^2(a_1, b_1; a_2, b_2) := \mathrm{Id}_{\underline{A}(a_1, b_1)}(a_2, b_2).$$

In general, assuming given

$$\vdash \ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots ; a_n, b_n),$$

we define

$$\underline{A}^{n+1}(a_1, b_1; \cdots; a_n, b_n; a_{n+1}, b_{n+1}) := \mathrm{Id}_{\underline{A}^n(a_1,b_1;\cdots;a_n,b_n)}(a_{n+1}, b_{n+1}).$$

With this notation to hand, we recall that the **reflection rule** for identity types is stated as follows:

$$\frac{\vdash\ a, b : A \qquad \vdash\ p : \underline{A}(a, b)}{\vdash\ a = b : A}\ \text{Reflection}$$

When the reflection rule is assumed identity types are trivial. This is one of the distinguishing features between intensional and extensional type theory. Namely, a theory $\mathbb{T}$ extending $\mathbb{T}_{-}$ with identity types is **extensional** if the reflection rule is derivable. On the other hand, we will often say that a theory is **intensional** or has intensional identity types if the reflection rule is not derivable. However, we should mention that there are additional criteria which have been proposed as constituting the notion of intensionality and, as such, it would be perhaps better to speak of a theory being *non-extensional* (for more on these matters we refer the reader to [**80**]).

Although we will not be concerned here with extensional type theory, we will consider type theories satisfying additional "truncation" principles related to the reflection principle. The most significant such principle is precisely the higher-dimensional generalization of the reflection rule, which we call the n**-truncation rule**:

$$\frac{\vdash\ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n) \qquad \vdash\ p : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, b_{n+1})}{\vdash\ a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}\ \mathrm{TR}_n$$

Thus, the 0-truncation rule is exactly the usual reflection rule.

1.0.3. Uniqueness of identity proofs. One of the principles for identity types which as been considered is the following principle of **(definitional) uniqueness of identity proofs**:

$$\frac{\vdash\ a_2, b_2 : \underline{A}(a_1, b_1)}{\vdash\ a_2 = b_2 : \underline{A}(a_1, b_1)}\ \mathrm{UIP}_1$$

This principle was shown by Hofmann and Streicher [**35**] to be independent of the basic rules for identity types from Section 1.0.1 by constructing a model of the theory using groupoids. We will discuss the results of [**35**] in more detail below once we have introduced the various theories under consideration.

As the subscript UIP_1 indicates, there are generalizations of uniqueness of identity proofs to higher dimensions. For $n \geq 1$, the principle of **(definitional)** n**-dimensional uniqueness of identity proofs** is

$$\frac{\vdash\ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}{\vdash\ a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}\ \mathrm{UIP}_n$$

Informally, thinking of a type as a kind of higher-dimensional groupoid, these principles state that types are like preorders and hence discrete above certain dimensions.

We denote by $\mathrm{UIP}_n^{\simeq}$ the corresponding **propositional n-dimensional uniqueness of identity proofs** principle

$$\frac{\vdash\ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}{\vdash\ a_{n+1} \simeq b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}\ \mathrm{UIP}_n^{\simeq}$$

where we write $\vdash\ d \simeq e : D$ to indicate that the identity type $\underline{D}(d, e)$ is inhabited. We will discuss these principles further below. Obviously UIP_n implies $\mathrm{UIP}_n^{\simeq}$.

1.0.4. Connection with the truncation rules. One consequence of the reflection rule is that all identity proofs $p : \underline{A}(a, b)$ are reflexivity terms. I.e., the **(definitional) ordinary unit principle**

$$\frac{\vdash\ a : A \qquad \vdash\ p : \underline{A}(a, a)}{\vdash\ p\ =\ r_A(a) : \underline{A}(a, a)}\ \mathrm{OUP}_0$$

follows from the reflection rule and also implies UIP_1. Indeed, in the presence of the reflection rule OUP_0 and UIP_1 are equivalent (cf. Appendix B). Similarly, we define higher-dimensional generalizations of the ordinary principle as follows:

$$\frac{\vdash\ a_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n) \qquad \vdash\ p : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, a_{n+1})}{\vdash\ p\ =\ r_{\underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}(a_{n+1}) : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, a_{n+1})}\ \mathrm{OUP}_n$$

There are also propositional versions of these rules stated in the obvious way and denoted by $\mathrm{OUP}_n^{\simeq}$.

Whereas the various uniqueness of identity proofs principles can be thought of as requiring that the identity types are discrete above certain dimensions, the ordinary unit rules indicate that all loops (above certain dimensions) are necessarily identities. For strict ω-groupoids the principles corresponding to UIP_{n+1} and OUP_n are easily seen to be equivalent using the fact that inverses are unique. In the present situation we instead obtain the following:

SCHOLIUM 1.2. *For any $n \geq 0$, the following hold:*

(1) UIP_{n+1} *implies* OUP_n.
(2) OUP_n *implies* $\mathrm{UIP}_{n+1}^{\simeq}$.
(3) $\mathrm{UIP}_{n+1}^{\simeq}$ *is equivalent to* $\mathrm{OUP}_n^{\simeq}$.

PROOF. (1) is trivial and (2) follows, using the idea of the proof sketched above for groupoids, from the propositional forms of the groupoid identities from [**35**]. The proof of (3) is straightforward. □

The ordinary unit principles also allow us to relate the truncation rules and uniqueness of identity proofs (the idea for this proof comes essentially from results, which are not "stratified" in the way considered here, from [**80**]).

LEMMA 1.3. *Assuming the rules of $\mathbb{T}_-$ and the usual rules for identity types from Section 1.0.1, the following implications hold:*

(1) TR_n *implies* OUP_n.
(2) TR_n *implies* UIP_{n+1}.
(3) UIP_n *implies* TR_n.

for $n \geq 0$.

PROOF. For the proof of (1) we will use Streicher's [**80**] K elimination rule which is stated in Appendix B). Recall that in the presence of reflection, the usual rules for identity types imply the K rule. This fact justifies the use of this rule made below in the presence of TR_n. First, let a term a_{n+1} of type $\underline{A}^n(a_1, b_1; \cdots ; a_n, b_n)$ and a "loop" p of type $\underline{A}^{n+1}(a_1, b_1; \cdots ; a_{n+1}, a_{n+1})$ be given. Then, by TR_n it suffices to show that

$$\vdash\ p\ \simeq\ r(a_{n+1})\ :\underline{A}^{n+1}(a_1, b_1; \cdots ; a_{n+1}, a_{n+1}).$$

To this end, define the type

$$D(x, y) := \underline{A}^{n+2}\big(y, r(x)\big)$$

in the context $\big(x : \underline{A}^n(a_1, b_1; \cdots ; a_n, b_n),\ y : \underline{A}^{n+1}(a_1, b_1; \cdots ; a_n, b_n; x, x)\big)$. Clearly,

$$x : \underline{A}^n(a_1, b_1; \cdots ; a_n, b_n)\ \vdash\ r\big(r(x)\big) : D\big(x, r(x)\big),$$

and therefore the K elimination rule yields the required term of type

$$D(a_{n+1}, p) = \underline{A}^{n+2}(p, r(a_{n+1})),$$

as required.

Suppose, for the proof of (2), that we are given terms a_{n+2} and b_{n+2} of type $\underline{A}^{n+1}(a_1, b_1; \cdots\ a_{n+1}, b_{n+1})$. Then, by TR_n, $a_{n+1} = b_{n+1}$. By (1) it follows that OUP_n holds and therefore we obtain

$$a_{n+2}\ =\ r(a_{n+1})\ =\ b_{n+2},$$

as required.

Finally, (3) holds trivially. □

1.0.5. The hierarchy of type theories. We are now in a position to describe the type theories we will be concerned with. First, we denote by $\mathbb{T}_\omega$ the theory obtained by adding to $\mathbb{T}_-$ the usual rules for identity types from Section 1.0.1. $\mathbb{T}_\omega$ is the principal theory which we aim to study. The justification for this nomenclature, and that adopted below, is that we think of the types in $\mathbb{T}_\omega$ as being, in virtue of the structure imparted by the identity types, weak ω-groupoids. Of course, weak ω-groupoids are not terribly easy things with which to deal — indeed, the issue of which definition to adopt is perhaps not entirely settled — and so it will be convenient at times to consider theories whose identity types are somewhat more manageable. From this perspective, the theory which is simultaneously the most well-behaved and least interesting is ordinary extensional Martin-Löf type theory [**62, 61**]. This theory, the types of which we regard as being simply discrete sets, is denoted by $\mathbb{T}_0$. The additional theories we consider are obtained by truncating, in various ways, the identity types at certain dimensions and constitute a spectrum or hierarchy of theories fitting between $\mathbb{T}_\omega$ on the one hand and $\mathbb{T}_0$ on the other.

To take one example, we may consider the theory obtained by forcing all (iterated) identity types of the form $\underline{A}^k(a_1, b_1; \cdots ; a_k, b_k)$ to be "preorders" for $k \geq n$. Explicitly, this theory is obtained by setting

$$\mathbb{P}_n := \mathbb{T}_\omega + \mathrm{UIP}_n.$$

The original issue of the independence of UIP_1 which motivated the groupoids model of Hofmann and Streicher [**35**] can the be stated as the question whether $\mathbb{P}_1$ and $\mathbb{T}_\omega$ are identical. On the other hand, in the same way we may consider the theories obtained by adding to $\mathbb{T}_\omega$ the truncation principles and in this way we obtain the theories

$$\mathbb{T}_n := \mathbb{T}_\omega + \mathrm{TR}_n$$

for $n \geq 0$. By the foregoing discussion of the relation between the truncation rules and uniqueness of identity proofs we obtain:

PROPOSITION 1.4. *The following inclusions hold:*

$$\mathbb{T}_{n+1} \subseteq \mathbb{P}_{n+1} \subseteq \mathbb{T}_n$$

for $n \geq 0$.

PROOF. By definition and Lemma 1.3. □

In light of Proposition 1.4 we obtain a hierarchy of theories as indicated in Figure 1.2. In order to better understand the connection between the theories occurring in this hierarchy we will turn next to study the semantics of such theories. Ultimately, it follows as a corollary of Theorem 4.25 that all of the theories occuring in this hierarchy are distinct.

Figure 1.2 The Hierarchy of Theories

$$\mathbb{T}_\omega \subseteq \cdots \subseteq \mathbb{T}_{n+1} \subseteq \mathbb{T}_n \subseteq \cdots \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_0$$

CHAPTER 2

Homotopical Semantics of Type Theory

In this chapter we introduce and study a homotopical semantics for intensional type theory utilizing *weak factorization system* and (closed) *model categories*. The principal feature of a weak factorization system in a category $\mathcal{C}$ which makes such a semantics possible is the existence of a distinguished class of maps called *fibrations* which possess several nice properties. In particular, fibrations are stable under pullback and therefore, by a familiar result from the semantics of type theory (cf. [**38, 79, 81, 34**]), give rise to a model of type theory (albeit one which may not support the interpretation of many type formers). As is well-known, weak factorization systems admit the construction of what are called *path objects* and it turns out that these path objects exhibit many of the features of Martin-Löf's identity types. The idea behind this interpretation is to think of a type A as a(n) (abstract) space and the type $\mathrm{Id}_A(a, b)$ as the corresponding space of "paths" from a to b. The aim of this chapter is to provide a precise mathematical articulation of this idea.

In summary, weak factorization systems and the resulting homotopical semantics are introduced in Section 2.1 and are related to the well-known semantics for type theory using *comprehension categories* (cf. Section B.3 of Appendix B for the definition and basic details of this semantics). In particular, we will see that every weak factorization system gives rise to a comprehension category. Although weak factorization systems admit the construction of path objects, these path objects need not be well-behaved in the ways necessary to interpret type theory. In practical terms this defect exhibits itself by the presence of certain "coherence" issues. Indeed, at least two *distinct* kinds of coherence issue arise for the interpretation of intensional type theory (both in the homotopical semantics and in general). The first issue amounts to the question whether the coherence rules governing identity types, reflexivity terms and the elimination J terms for identity types are satisfied. I.e., whether the structures interpreting these syntactic constructs are stable under pullback. E.g., in an arbitrary weak factorization system path objects themselves need not be stable under pullback and, in order to validate the coherence conditions for the identity type and reflexivity term, it is necessary to restrict to those weak factorization systems which possess this additional property. Such weak factorization systems are said to have *stable path objects*. This notion is made precise in Section 2.1 and it is the aim of Section 2.2 to show that a particularly broad class of examples of weak factorization systems — namely, those arising from *simplicial model categories* — possess stable path objects. Even if path objects, and therefore also the interpretations of identity types and reflexivity terms, are stable under pullback it does not immediately follow that the elimination terms are stable under pullback. This further issue is considered in Section 2.3 where the abstract conditions corresponding to this property are described (the actual examples of models which have "coherent" elimination terms are considered in the subsequent chapters). Finally, in Section 2.4 we address the other coherence problem affecting the homotopical interpretation of type theory. This is the problem of interpreting substitution in general and is quite well-known as it also arises for the interpretation of extensional type theory in locally cartesian closed categories. For locally cartesian closed categories, Hofmann [**33**] has shown that Bénabou's [**7**] theorem which states that every Grothendieck fibration is equivalent to a split fibration can be used to turn "non-split" models into "split" ones and thereby solved this coherence problem in the extensional case. The principal result of Section 2.4 is to show that this construction also yields "split" models of intensional type theory when applied to those weak factorization systems which satisfy certain conditions (given in Section 2.3). This useful technical result will be employed in Chapter 3 to obtain "split" models.

REMARK 2.1. Throughout we attempt to adhere to terminological convention which should ensure that the various coherence problems addressed above are not confused. Namely, we generally employ the adjective *stable* to refer to the satisfaction up to isomorphism of the coherence rules for identity types and reflexivity terms (but not elimination terms); *coherent* is used to refer to the case where the coherence condition for elimination J terms is also satisfied up to isomorphism; and *split* refers to the case where whatever "up to isomorphism" structure is under consideration is actually "on the nose".

REMARK 2.2. Some of the results of this chapter occur originally (albeit in a somewhat condensed form) in the joint paper [**3**] with Steve Awodey.

REMARK 2.3. We state here some of our category theoretic notation for the record. Given an arrow $f : B \longrightarrow A$ in a category $\mathcal{C}$ with pullbacks we write $\Delta_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ or f^* for the pullback functor. Similarly, the left-adjoint to Δ_f is written as $\Sigma_f : \mathcal{C}/B \longrightarrow \mathcal{C}/A$ and the right adjoint to Δ_f, if it exists, is denoted by $\Pi_f : \mathcal{C}/B \longrightarrow \mathcal{C}/A$.

We often employ "ordered pair" notation to denote induced maps into pullbacks. I.e., given a pullback square

$$\begin{array}{ccc} B' & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[g]{} & A \end{array}$$

and an object X together with maps $x_0 : X \longrightarrow A'$ and $x_1 : X \longrightarrow B$ such that $f \circ x_1 = g \circ x_0$, we will sometimes denote the induced map $X \longrightarrow B'$ by $\langle x_0, x_1 \rangle$. Similar notation $[x_0, x_1]$ will also be employed for the induced map out of a pushout.

2.1. Homotopical semantics

In this section we introduce the basic homotopy theoretic semantics using *weak factorization systems* and *Quillen model categories*. For more on weak factorization systems and Quillen model categories in homotopy theory the reader should consult the references [**70, 10, 36, 31, 48, 20, 2**].

2.1.1. Weak factorization systems and model categories. In any category $\mathcal{C}$, given maps $f : A \longrightarrow B$ and $g : C \longrightarrow D$, we write

$$f \pitchfork g$$

to indicate that f possesses **left-lifting property** (**LLP**) with respect to g. I.e. for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

there exists a map $l : B \longrightarrow C$ such that $g \circ l = k$ and $l \circ f = h$. In this situation we also say that g possesses the **right-lifting property** (**RLP**) with respect to F. Similarly, if $\mathfrak{M}$ is any collection of maps we denote by ${}^{\pitchfork}\mathfrak{M}$ the collection of maps in $\mathcal{C}$ having the LLP with respect to all maps in $\mathfrak{M}$. The collection of maps $\mathfrak{M}^{\pitchfork}$ is defined similarly.

Definition 2.4. A **weak factorization system** $(\mathfrak{L}, \mathfrak{R})$ in a category $\mathcal{C}$ consists of two collections $\mathfrak{L}$ (the "left-class") and $\mathfrak{R}$ (the "right-class") of maps in $\mathcal{C}$ such that

(WFS0): Every map $f : A \longrightarrow B$ has a factorization as

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & C \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

where i is a member of $\mathfrak{L}$ and p is a member of $\mathfrak{R}$.

(WFS1): $\mathfrak{L}^{\pitchfork} = \mathfrak{R}$ and $\mathfrak{L} = {}^{\pitchfork}\mathfrak{R}$.

Definition 2.5. A (**closed**) **model category** [**70**] is a bicomplete category $\mathcal{C}$ equipped with subcategories $\mathfrak{F}$ (**fibrations**), $\mathfrak{C}$ (**cofibrations**) and $\mathfrak{W}$ (**weak equivalences**) satisfying the following two conditions:

(MC0): ("Three-for-two") Given a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & C & \end{array}$$

if any two of f, g, h are weak equivalences, then so is the third.

(MC1): Both $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ are weak factorization systems.

The form of Definition 2.5 given here is, although equivalent to Quillen's original definition, due in this form to Adámek *et al* [**2**]. A map f is an **acyclic cofibration** if it is in $\mathfrak{C} \cap \mathfrak{W}$, i.e. both a cofibration and a weak equivalence. Similarly, an **acyclic fibration** is a map in $\mathfrak{F} \cap \mathfrak{W}$, i.e. which is simultaneously a fibration and a weak equivalence. An object A is said to be **fibrant** if the canonical map $A \longrightarrow 1$ is a fibration. Similarly, A is **cofibrant** if $0 \longrightarrow A$ is a cofibration. We denote by $\mathcal{C}_{\mathrm{f}}$ the full subcategory of $\mathcal{C}$ with objects the fibrant objects.

Because it is convenient to have a name for maps in the classes $\mathfrak{L}$ and $\mathfrak{R}$, and since many of the examples of weak factorization systems we consider arise from model categories, we will often refer to maps in the left-class $\mathfrak{L}$ of a weak factorization system as acyclic cofibrations and those in the right class $\mathfrak{R}$ as fibrations when no confusion will arise.

EXAMPLE 2.6. Examples of model categories include the following:

(1) The category **Top** of topological spaces with fibrations the Serre fibrations, weak equivalences the weak homotopy equivalences and cofibrations those maps which have the LLP with respect to acyclic fibrations. The cofibrant objects in this model structure are retracts of spaces constructed, like CW-complexes, by attaching cells.
(2) The category **SSet** of simplicial sets with cofibrations the monomorphisms, fibrations the Kan fibrations and weak equivalences the weak homotopy equivalences. The fibrant objects for this model structure are the Kan complexes. This example, like that of **Top** is due to Quillen [**70**].
(3) The category **Gpd** of (small) groupoids with cofibrations the functors injective on objects, fibrations the Grothendieck fibrations and weak equivalences the categorical equivalences. Here all objects are both fibrant and cofibrant. This example follows from the natural model structure on **Cat** due to Joyal and Tierney [**47**].

This brief list is far from being exhaustive and we will encounter additional examples of both model categories and weak factorization systems later. For now though we simply mention an example due to Gambino and Garner [**23**] that is especially relevant here.

EXAMPLE 2.7 (Gambino and Garner). It is shown in [**23**] that there exists a weak factorization system in $\mathcal{C}(\mathbb{T}_\omega)$ in which the left-class $\mathfrak{L}$ is defined to be ${}^{\pitchfork}\mathfrak{D}$ and the right-class is $\mathfrak{L}^{\pitchfork}$. Here $\mathfrak{D}$ is the set of the dependent projections. Moreover, it is shown in *ibid* that the classes $\mathfrak{L}$ and $\mathfrak{R}$ of maps admit explicit descriptions as the *type theoretic injective equivalences* and *type theoretic normal isofibrations*, respectively. We refer the reader to Section B.4 of Appendix B for more on the two notions of dependent projection.

As far as we know it is not possible to extend Example 2.7 to provide a "model structure" on $\mathcal{C}(\mathbb{T}_\omega)$ (where the quotes emphasize the fact that, officially, a category must be bicomplete in order to possess a model structure).

2.1.2. The interpretation of dependent types. Whereas the idea of the Curry-Howard correspondence is often summarized by the slogan "Propositions as Types", the idea underlying the homotopical semantics is

Fibrations as Types.

In order to make this idea precise we first introduce some additional notation. Assume $\mathcal{C}$ is a category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$. Then the category $\mathcal{C}_{\mathfrak{R}}$ is defined to be the full subcategory of the arrow category $\mathcal{C}^{\rightarrow}$ with objects elements of $\mathfrak{R}$. Similarly, we obtain a category $\mathcal{C}_{\mathfrak{L}}$. The restriction of the codomain map $\partial_1 : \mathcal{C}^{\rightarrow} \longrightarrow \mathcal{C}$ to $\mathcal{C}_{\mathfrak{R}}$ is denoted also by ∂_1.

LEMMA 2.8. *Assume $\mathcal{C}$ is a finitely complete category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$, then the codomain projection $\partial_1 : \mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}$ together with the inclusion $\mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}^{\rightarrow}$ gives $\mathcal{C}$ the structure of a comprehension category.*

Proof. When $(\mathfrak{L}, \mathfrak{R})$ is a weak factorization system the right class $\mathfrak{R}$ is necessarily stable under pullback, therefore $\partial_1 : \mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}$ is a Grothendieck fibration and $\mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}^{\rightarrow}$ is fibred. □

By Lemma 2.8 it follows that any finitely complete category $\mathcal{C}$ equipped with a weak factorization system gives rise to a (non-split) model of dependent type theory. Note that Lemma 2.8 also holds when we restrict to the full subcategory of $\mathcal{C}$ consisting of fibrant objects. It is this case of the result which we will employ below.

2.1.3. Path objects and cellular resolutions. We begin by recalling the following standard definition for model categories:

Definition 2.9. Given an object A of a model category $\mathcal{C}$, a **path object for** A consists of a factorization

$$(1) \qquad \begin{array}{ccc} A & \xrightarrow{\;r\;} & P \\ & {\scriptstyle\Delta}\searrow \quad \swarrow{\scriptstyle p} & \\ & A \times A & \end{array}$$

of the diagonal map $\Delta : A \longrightarrow A \times A$ as a weak equivalence r followed by a fibration p. Such a path object is said to be **very good** if r is also a cofibration.

When $\mathcal{C}$ is only assumed to possess a weak factorization system $(\mathfrak{L}, \mathfrak{R})$, the definition of very good path object still makes sense by regarding the left class $\mathfrak{L}$ as the acyclic cofibrations and the right class $\mathfrak{R}$ as the fibrations. It follows then from the the factorization axiom for weak factorization systems that every object of $\mathcal{C}$ possesses a very good path object.

We would like to interpret identity types using very good path objects. For example, given a fibrant object A which we think of as a closed type, we would like to be able to view the map $p : \mathrm{Path}(A) \longrightarrow A \times A$ as the judgement

$$x, y : A \;\vdash\; \mathrm{Id}_A(x, y) : \mathrm{type}\,.$$

Of course, because the axioms for weak factorization systems ensure only the existence of very good path objects and not the existence of "distinguished" or functorial path objects, it follows that we must restrict to those weak factorization systems which have such a choice of very good path objects. In order to make this precise we will first introduce some auxiliary notions.

Given a category $\mathcal{C}$ with finite limits the category $\mathbf{RGraphs}_{\mathcal{C}}$ of **reflexive graphs in** $\mathcal{C}$ has as objects pairs of objects (V, E) of $\mathcal{C}$ together with arrows

$$V \underset{s}{\overset{t}{\leftleftarrows}} \!\!\xrightarrow{\;r\;}\! E \qquad (t, s : E \to V,\; r : V \to E)$$

such that

$$s \circ r \;=\; 1_V \;=\; t \circ r.$$

Homomorphisms of reflexive graphs are then defined in the obvious way.

Definition 2.10. When $\mathcal{C}$ is a finitely complete category with a distinguished class of arrows $\mathfrak{R}$ in $\mathcal{C}$ which are stable under pullback, the category $\mathbf{CRes}^1_{\mathfrak{R}}$ of **1-cellular resolutions** (with respect to $\mathfrak{R}$) has as objects tuples (A, V, E) where A is an object of $\mathcal{C}$ and (V, E) is a reflexive graph in $\mathcal{C}/A$ such that the maps $V \longrightarrow A$ and $\langle s, t\rangle : E \longrightarrow V \times_A V$ are in $\mathfrak{R}$. We sometimes refer to the map $V \longrightarrow A$ as the **augmentation map**. Homomorphisms of 1-cellular resolutions $(B, V', E') \longrightarrow (A, V, E)$ are tuples (f, f_0, f_1), where $f : B \longrightarrow A$, $f_0 : V' \longrightarrow V$ and $f_1 : E' \longrightarrow E$, such that these maps make the following diagram commutes

$$\begin{array}{ccccc} E' & \overset{s}{\underset{t}{\rightrightarrows}} & V' & \longrightarrow & B \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ E & \overset{s}{\underset{t}{\rightrightarrows}} & V & \longrightarrow & A \end{array}$$

Although there is a more general notion of cellular resolution for $n > 1$ we will only be dealing with the case where $n = 1$. Accordingly, we often omit the prefix "1-" and simply refer to *cellular resolutions*. Similarly, when $\mathcal{C}$ is a category equipped with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ and this is understood, we will often write $\mathbf{CRes}_{\mathcal{C}}$. When (A, V, E) is an object of $\mathbf{CRes}_{\mathcal{C}}$ we say that (V, E) is a *cellular resolution of* A. Henceforth, when discussing cellular resolutions we always assume that the resolutions are with respect to a weak factorization system on $\mathcal{C}$ unless otherwise stated.

Example 2.11. Given an object A of $\mathcal{C}$, any path object factorization of the diagonal $\Delta : A \longrightarrow A \times A$ is a cellular resolution of the terminal object.

Example 2.11 generalizes to slices $\mathcal{C}/A$. I.e., a path object factorization of the diagonal $B \longrightarrow B \times_A B$ of a map $f : B \longrightarrow A$ gives a cellular resolution of A. The converse — that every cellular resolution of A is a path object of an object in $\mathcal{C}/A$ — does *not* hold since the map $r : V \longrightarrow E$ is not necessarily a weak equivalence (or element of $\mathfrak{L}$). The reason for not requiring that the maps r are in $\mathfrak{L}$ is the following lemma (which would not hold under this additional condition):

Lemma 2.12. *The projection* $\pi : \mathbf{CRes}_{\mathcal{C}} \longrightarrow \mathcal{C}$ *which sends a cellular resolution* (V, E) *of* A *to* A *and a map* (f, f_0, f_1) *to* f *is a Grothendieck fibration.*

Proof. It is straightforward to verify that cellular resolutions are stable under pullback and that therefore the projection is a Grothendieck fibration. □

There exists also a functor $\chi : \mathbf{CRes}_{\mathcal{C}} \longrightarrow \mathcal{C}^{\rightarrow}$ which sends a cellular resolution (V, E) of A to the augmentation map $V \longrightarrow A$ and sends an arrow (f, f_0, f_1) of cellular resolutions to the commutative square

$$\begin{array}{ccc} V' & \xrightarrow{f_0} & V \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & A \end{array}$$

We note that this functor is fibred from $\pi : \mathbf{CRes}_{\mathcal{C}} \longrightarrow \mathcal{C}$ to $\partial_1 : \mathbf{CRes}_{\mathcal{C}} \longrightarrow \mathcal{C}$ and therefore $\mathbf{CRes}_{\mathcal{C}}$ itself has the structure of a comprehension category.

With these definition at our disposal we may precisely state the conditions under which it is possible to interpret (the formation and introduction rules governing) identity types in a category which possesses a weak factorization system. Henceforth, as we will be predominately concerned with fibrant objects, we denote by $\mathcal{C}_{\mathfrak{R}}$ the full subcategory of the arrow category *of* $\mathcal{C}_{\mathrm{f}}$ with objects fibrations between fibrant objects. Similarly, $\mathbf{CRes}_{\mathcal{C}}$ will denote the category of cellular resolutions *of* fibrant objects.

Definition 2.13. Assume $\mathcal{C}$ is a finitely complete category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$. We say that this weak factorization system **has stable path objects** if there exists a fibred functor

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{R}} & \overset{\iota}{\dashrightarrow} & \mathbf{CRes}_{\mathcal{C}} \\ & {\scriptstyle\partial_1}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathcal{C}_{\mathrm{f}} & \end{array}$$

such that the following conditions are satisfied:

- The augmentation of $\iota(f)$ is f itself. When this condition is satisfied we denote by $I_A(f)$ the object of "edges" of $\iota_A(f)$. I.e., $I(f)$ is the codomain of the reflexivity map of $\iota(f)$.
- For any object $f : B \longrightarrow A$ of $\mathcal{C}_{\mathfrak{R}}$, the reflexivity map $r : B \longrightarrow I_A(f)$ is in $\mathfrak{L}$.
- If

$$\begin{array}{ccc} B' & \xrightarrow{\sigma'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\sigma]{} & A \end{array} \tag{2}$$

  is an arrow in $\mathcal{C}_{\mathfrak{R}}$ which is a pullback in $\mathcal{C}_{\mathrm{f}}$, then

$$\begin{array}{ccc} I_{A'}(f') & \longrightarrow & I_A(f) \\ \downarrow & & \downarrow \\ A' & \xrightarrow[\sigma]{} & A \end{array}$$

  is a pullback in $\mathcal{C}_{\mathrm{f}}$.

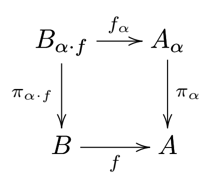
Given an arrow of the form (2) in $\mathcal{C}_{\mathfrak{R}}$ we denote by $I(\sigma, \sigma')$ the third component of the resulting map $\iota(\sigma, \sigma') = \big(\sigma, \sigma', I(\sigma, \sigma')\big)$ of cellular resolutions.

Example 2.14. The category of contexts, equipped with the weak factorization system from Example 2.7 has stable path objects. Indeed, this is essentially a modification of the proof of the familiar observation that the category of contexts gives rise to a (split) comprehension category which models identity types (cf. [**32**]) using the results of [**23**].

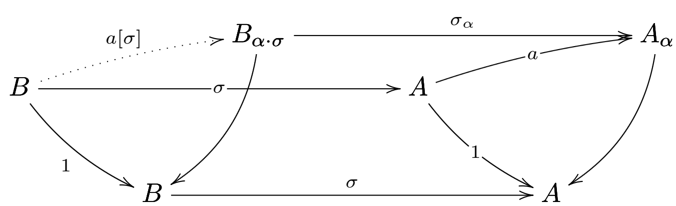
2.1.4. Coherence and quasi-models. Weak factorization systems also suggest a method for interpreting the elimination rule for intensional identity types. In particular, interpreting identity types using path objects, given a fibration $g : B \longrightarrow \mathrm{Path}(A)$ together with a map $b : \mathrm{Path}(A) \longrightarrow B$ for which the composite $g \circ b$ is identical to the acyclic cofibration $r : A \longrightarrow \mathrm{Path}(A)$, we may use the fact that

r possesses the left-lifting property with respect to g to obtain an "elimination" term $J : \mathrm{Path}(A) \longrightarrow B$. However, because, as defined, the diagonal fillers featuring in the definition of weak factorization systems are not assumed to be given by an *operation* (and are not, in particular, assumed to be given functorially) the corresponding "interpretations" of the elimination terms must be chosen arbitrarily and so fail to satisfy the coherence conditions for identity types. Nonetheless, the resulting notion of a *quasi-model* of type theory is of some interest in that the internal language associated to a category with a weak factorization system is always a quasi-model and, moreover, although the chosen representatives of the elimination terms are not operations "on-the-nose", they are coherent up to the existence of (right) homotopies. The issue of coherence of elimination terms, and the *distinct* issue of existence of split models, will be discussed in Section 2.3. The subsequent chapters are concerned exclusively with the construction of models which satisfy the coherence rule. However, we will now introduce and study quasi-models in more detail. We begin by describing our notational conventions for comprehension categories.

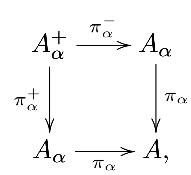
Remark 2.15. Given a comprehension category $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ with comprehension χ and an object A of $\mathcal{C}$, we denote by $\mathcal{P}(A)$ the fibre of $\mathbf{P}(-)$ over A. If $\alpha \in \mathcal{P}(A)$, we denote by $\pi_\alpha : A_\alpha \longrightarrow A$ the arrow $\chi(\alpha)$ in $\mathcal{C}$. Assuming a cleavage for $\mathbf{P}(-)$, when $f : B \longrightarrow A$, we denote by $(\alpha \cdot f)$ the domain of the cartesian lift $f_\alpha : (\alpha \cdot f) \longrightarrow \alpha$ of f. We will sometimes abuse notation and denote also by f_α the map indicated in the following pullback square $\chi(f_\alpha)$



Given an arrow $\sigma : B \longrightarrow A$ in $\mathcal{C}$ together with an element α of $\mathcal{P}(A)$ and a section $a : A \longrightarrow A_\alpha$ of $\pi_\alpha : A_\alpha \longrightarrow A$, we denote by $a[\sigma]$ the canonical map $B \longrightarrow B_{\alpha\cdot\sigma}$ indicated in the following diagram:



For our final bit of notation, given an object A of $\mathcal{C}$ and $\alpha \in \mathcal{P}(A)$, there exists a pullback square



where

$$
\begin{aligned}
A^+_\alpha &:= \big(A_\alpha\big)_{\alpha\cdot\pi_\alpha},\\
\pi^-_\alpha &:= \big(\pi_\alpha\big)_{\alpha},\\
\pi^+_\alpha &:= \pi_{\alpha\cdot\pi_\alpha}.
\end{aligned}
$$

Further details regarding comprehension categories can be found in Section B.3 of Appendix B.

Definition 2.16. A comprehension category $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ is a **quasi-model (of identity types)** if for every object Γ of $\mathcal{C}$ and every object α of $\mathcal{P}(\Gamma)$, there exists an object

$$\imath(\alpha) \in \mathcal{P}(\Gamma^+_\alpha)$$

such that the following conditions are satisfied:

- Writing I_α as an abbreviation for $(\Gamma^+_\alpha)_{\imath(()\alpha)}$, there exists a map $\rho_\alpha : \Gamma_\alpha \longrightarrow I_\alpha$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma_\alpha & \xrightarrow{\rho_\alpha} & I_\alpha \\
& {\scriptstyle\Delta}\searrow \quad \swarrow{\scriptstyle\pi_{\imath(\alpha)}} & \\
& \Gamma^+_\alpha &
\end{array}
$$

  where Δ is the diagonal.
- Given any $\beta \in \mathcal{P}(I_\alpha)$ and any map $b : \Gamma_\alpha \longrightarrow (I_\alpha)_\beta$ making the following square commute:

$$
\begin{array}{ccc}
\Gamma_\alpha & \xrightarrow{b} & \big(I_\alpha\big)_\beta \\
{\scriptstyle\rho_\alpha}\downarrow & & \downarrow{\scriptstyle\pi_\beta} \\
I_\alpha & \xrightarrow[1_{I_\alpha}]{} & I_\alpha
\end{array}
$$

  there exists a diagonal filler $J(\alpha, \beta, b) : I_\alpha \longrightarrow (I_\alpha)_\beta$ which makes both of the resulting triangle commute. I.e.,

$$
\begin{aligned}
\pi_\beta \circ J(\alpha, \beta, b) &= 1_{I_\alpha},\\
J(\alpha, \beta, b) \circ \rho_\alpha &= b.
\end{aligned}
$$

A quasi-model is **split** if the underlying comprehension category is a split Grothendieck fibration.

Although they are not quite models of type theory in its usual form, the existence of a quasi-model is useful insofar as it sometimes suggests the existence of an underlying (genuine) model.

Assume that $\mathcal{C}$ is a finitely complete category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$. We will now show that, if $\mathcal{C}$ has stable path objects, then the comprehension category structure from Lemma 2.8 is a quasi-model. The following theorem represents an important first step towards relating homotopy theory and intensional type theory:

Theorem 2.17. *Let $\mathcal{C}$ be a finitely complete category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ and stable path objects, then, as a comprehension category, $\mathcal{C}_{\mathfrak{R}}$ is a quasi-model.*

Proof. Given a fibrant object A of $\mathcal{C}$ and fibration $f : B \longrightarrow A$ we define $\imath(f)$ to be the fibration $\langle s, t\rangle : I_A(f) \longrightarrow B \times_A B$ obtained using the stable path objects of $\mathcal{C}$. I.e., explicitly, $I_A(f)$ is the object of edges of $\imath(f)$. The reflexivity term ρ_f is then defined to be $r : B \longrightarrow I_A(f)$.

To see that the elimination and conversion rules are satisfied suppose we are given a fibration $g : D \longrightarrow I_A(f)$ together with a map $d : B \longrightarrow D$ such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{\;b\;} & D \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle g} \\ I_A(f) & \xrightarrow[1_{I_A(f)}]{} & I_A(f) \end{array}$$

Then, because r is in $\mathfrak{L}$ and g is in $\mathfrak{R}$ there exists a diagonal filler $J : I_A(f) \longrightarrow D$, as required.

$$\begin{array}{ccc} B & \xrightarrow{\;b\;} & D \\ {\scriptstyle r}\downarrow & \overset{J}{\nearrow} & \downarrow{\scriptstyle g} \\ I_A(f) & \xrightarrow[1_{I_A(f)}]{} & I_A(f) \end{array}$$

Selecting a filler J for every such elimination rule situation (g, d) yields interpretations of all elimination J terms. □

There are numerous examples of categories satisfying the hypotheses of Theorem 2.17 including the category **Gpd** of groupoids and the category **SSet** of simplicial sets. We now turn to consider one important source of examples of categories satisfying the hypotheses of Theorem 2.17.

2.2. Simplicial model categories

Simplicial model categories play an important role in homotopy theory providing, as they do, a setting for the development not only of the homotopy theory of spaces, but also the theory of higher-stacks. In this section we will show that many simplicial model categories — and, indeed, the most interesting examples of such — possess stable path objects in the sense of Definition 2.13.

2.2.1. Simplicial categories. In the literature on homotopy theory there are several different definitions, not all of which are even equivalent, of what are called *simplicial categories* (cf. [**70, 11, 26, 31**]). Indeed, the definition we give below is not as common in the literature as one would expect. Yet, from the perspective of enriched category theory, the definition which we give is certainly the most natural. This difference will cause no confusion since the notion of simplicial *model* category will agree with the standard definition given in the literature (cf. *ibid*).

Definition 2.18. A **simplicial category** $\mathcal{C}$ is a category enriched in the category **SSet** of simplicial sets.

When $\mathcal{C}$ is a simplicial category, and A and B are objects of $\mathcal{C}$ we denote by $\underline{\mathcal{C}}(A, B)$ the simplicial set of arrows from A to B. By definition, any simplicial category is also has an underlying category with the hom-set

$$\begin{aligned}\mathcal{C}(A,B) &:= \underline{\mathcal{C}}(A,B)_0 \\ &\cong \mathbf{SSet}\big(\Delta[0], \underline{\mathcal{C}}(A,B)\big).\end{aligned}$$

Given an arrow $f : X \longrightarrow Y$ in $\mathcal{C}(X,Y)$ we write f^* for the induced map

$$\underline{\mathcal{C}}(Y,W) \xrightarrow{\underline{\mathcal{C}}(f,W)} \underline{\mathcal{C}}(X,W)$$

and f_* for

$$\underline{\mathcal{C}}(W,X) \xrightarrow{\underline{\mathcal{C}}(W,f)} \underline{\mathcal{C}}(W,Y),$$

when the object W of $\mathcal{C}$ is understood. When emphasizing that we are regarding a simplicial category $\mathcal{C}$ as a mere category we will sometimes write $|\mathcal{C}|$. I.e., $|\mathcal{C}|$ denotes the underlying category of $\mathcal{C}$. We refer the reader to [**50**] for further details regarding enriched category theory.

2.2.2. Simplicial model categories. We now define what it means for a simplicial category $\mathcal{C}$ to be a simplicial model category. We will then mention several important examples.

Definition 2.19. A simplicial category $\mathcal{C}$ is a **simplicial model category** if the following conditions are satisfied:

- **(SMC0):** The underlying category $|\mathcal{C}|$ of $\mathcal{C}$ is a model category.
- **(SMC1):** $\mathcal{C}$ is tensored and cotensored over **SSet**.
- **(SMC2):** Given a cofibration $i : X \longrightarrow Y$ and a fibration $p : E \longrightarrow B$ in $|\mathcal{C}|$, the canonical map

$$\underline{\mathcal{C}}(Y,E) \xrightarrow{\langle p_*, i^* \rangle} \underline{\mathcal{C}}(Y,B) \times_{\underline{\mathcal{C}}(X,B)} \underline{\mathcal{C}}(X,E) \tag{3}$$

in **SSet** is a Kan fibration which is trivial if either of i or p is a weak equivalence.

In general the cotensor of a simplicial set K with an object X of a simplicial model category $\mathcal{C}$ is denoted by $(K \pitchfork X)$, although we sometimes write X^K instead. The tensor product is denoted $(K \otimes X)$. As a consequence of the requirement that $\mathcal{C}$ be tensored and cotensored over **SSet**, there exist natural isomorphisms

$$\begin{aligned}\underline{\mathcal{C}}(K \otimes X, Y) &\cong \big[K, \underline{\mathcal{C}}(X,Y)\big] \\ &\cong \underline{\mathcal{C}}(X, K \pitchfork Y)\end{aligned}$$

in **SSet**, where $[-,-]$ denotes the internal hom in **SSet**, and

$$\begin{aligned}\mathcal{C}(K \otimes X, Y) &\cong \mathbf{SSet}\big(K, \underline{\mathcal{C}}(X,Y)\big) \\ &\cong \mathcal{C}(X, K \pitchfork Y)\end{aligned}$$

in **Set**.

Example 2.20. The following are a few examples of simplicial model categories (cf. [**70, 31, 26**]):

1. The categories **SSet** of simplicial sets and **Top** of compactly generated spaces (with their "standard" model structures) are both examples of simplicial model categories. These examples, as well as the case of simplicial groups, were studied already by Quillen [**70**].

(2) Given a category $\mathcal{E}$, the category of simplicial objects in $\mathcal{E}$ is denoted by $S(\mathcal{E})$. When $\mathcal{E}$ is a Grothendieck topos there are two significant model structures on $S(\mathcal{E})$. In one — called the **projective model structure** — the weak equivalences and fibrations are defined pointwise and the cofibrations are given by the lifting property. In the other — called the **injective model structure** — it is the cofibrations and weak equivalences which are pointwise. When $\mathcal{E}$ is the category of sheaves on a space the injective model structure on $S(\mathcal{E})$ was verified by Brown and Gersten [**12**]. The injective model structure on $S(\mathcal{E})$, for $\mathcal{E}$ an arbitrary Grothendieck topos, was demonstrated by Joyal [**43**] (cf. also [**47**] and [**40**]). The proof of this result should be of particular interest to logicians due to the fact that it crucially employs the internal language of the topos $\mathcal{E}$ in conjunction with Barr's theorem [**4**]. When $\mathcal{E}$ is a Grothendieck topos the projective model structure is due to Jardine [**39**]. These examples are of particular interest due in part to their connection with the theory of higher-stacks (cf. [**82**]).

2.2.3. Enriched slicing. We will now prove several basic lemmata regarding slices in the setting of categories enriched over a cartesian closed category. Throughout this section we assume that $\mathcal{V}$ is a finitely complete cartesian closed category.

LEMMA 2.21. *If $\mathcal{C}$ is a $\mathcal{V}$-category and X is an object of $\mathcal{C}$, then $|\mathcal{C}|/X$ can also be given the structure of a $\mathcal{V}$-category.*

PROOF. Given arrows $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ in $|\mathcal{C}|$, define the object $(\underline{\mathcal{C}/X})(f,g)$ of $\mathcal{V}$ to be the following equalizer

$$(\underline{\mathcal{C}/X})(f,g) \xrightarrow{e_{f,g}} \underline{\mathcal{C}}(Y,Z) \underset{\ulcorner f \urcorner}{\overset{g_*}{\rightrightarrows}} \underline{\mathcal{C}}(Y,X)$$

taken in $\mathcal{V}$. Moreover, because the map

$$(\underline{\mathcal{C}/X})(f,g) \times (\underline{\mathcal{C}/X})(g,h) \xrightarrow{e_{f,g}\times e_{g,h}} \underline{\mathcal{C}}(Y,Z) \times \underline{\mathcal{C}}(Z,W) \xrightarrow{\mu} \underline{\mathcal{C}}(Y,W)$$

equalizes the maps h_* and $\ulcorner f \urcorner : \underline{\mathcal{C}}(Y,W) \rightrightarrows \underline{\mathcal{C}}(Y,X)$, where $h : W \longrightarrow X$, it follows that there exists a canonical multiplication map

$$(\underline{\mathcal{C}/X})(f,g) \times (\underline{\mathcal{C}/X})(g,h) \xrightarrow{\mu_{f,g,h}} (\underline{\mathcal{C}/X})(f,h)$$

such that $\mu_{f,g,h} \circ (e_{f,g} \times e_{g,h}) = h_* \circ \mu_{Y,Z,W}$. It is then straightforward to verify that gives $\mathcal{C}/X$ the structure of a $\mathcal{V}$-category. ☐

SCHOLIUM 2.22. *If $\mathcal{C}$ is a tensored (cotensored) $\mathcal{V}$-category, then for any object X of $\mathcal{C}$,*

$$1 \otimes X \cong X$$
$$\Big(1 \pitchfork X \cong X\Big).$$

LEMMA 2.23. *Assume that $\mathcal{C}$ is a $\mathcal{V}$-category which is tensored and cotensored over $\mathcal{V}$ and for which $|\mathcal{C}|$ is finitely complete. Then $\mathcal{C}/X$ is also tensored and cotensored.*

PROOF. Given an arrow $f : Y \longrightarrow X$ in $\mathcal{C}$ and an object K of $\mathcal{V}$, define $(K \otimes f)_X$, where the subscript indicates that this is not the arrow $(K \otimes f) : (K \otimes Y) \longrightarrow (K \otimes X)$ in $\mathcal{C}$, to be the composite

$$K \otimes Y \xrightarrow{1 \otimes f} K \otimes X \xrightarrow{! \otimes 1} 1 \otimes X \cong X.$$

With this definition it is routine to show that there exists a $\mathcal{V}$-natural isomorphism:

$$\mathrm{Hom}\big(K \otimes f, g\big) \cong \mathrm{Hom}\big(K, \mathcal{C}/A[f, g]\big).$$

The cotensor $(K \pitchfork f)_X$ is defined as the map p_f indicated in the following pullback diagram:

$$\begin{array}{ccc} [K, f] & \xrightarrow{q_f} & (K \pitchfork Y) \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle (K \pitchfork f)} \\ X & \xrightarrow[r]{} & (K \pitchfork X) \end{array}$$

where r is the arrow of $|\mathcal{C}|$ obtained as the transpose of the map

$$K \longrightarrow 1 \xrightarrow{1_X} \underline{\mathcal{C}}(X, X)$$

in $\mathcal{V}$. With these definitions it is routine to verify that the required $\mathcal{V}$-natural transformations exist. □

2.2.4. Stability of simplicial model categories under slicing. We now show that simplicial model categories are stable under slicing.

PROPOSITION 2.24. *If $\mathcal{C}$ is a simplicial model category and A is an object of $\mathcal{C}$, then $\mathcal{C}/A$ is a simplicial model category.*

PROOF. By Lemma 2.23, $\mathcal{C}/A$ is a simplicial category. As such, it suffices to show that condition **(SMC2)** from Definition 2.19 is satisfied. To this end, let objects $e : E \longrightarrow A$, $b : B \longrightarrow A$, $x : X \longrightarrow A$ and $y : Y \longrightarrow A$ of $\mathcal{C}/A$ be given together with maps $p : e \longrightarrow b$ and $i : x \longrightarrow y$ such that p is a fibration and i is a cofibration.

Form the usual pullback, which is here denoted by $(i, p)_A$,

$$\begin{array}{ccc} (i, p)_A & \longrightarrow & \mathcal{C}/A(x, e) \\ \downarrow & & \downarrow{\scriptstyle p_*^A} \\ \mathcal{C}/A(y, b) & \xrightarrow[i_A^*]{} & \mathcal{C}/A(x, b) \end{array}$$

where i_A^* and p_*^A denote the usual action of the representable functors and the subscript and superscript distinguish these from the representable functors for $\mathcal{C}$.

The following square is a pullback

$$\begin{array}{ccc} \mathcal{C}/A(y, e) & \xrightarrow{q'} & (i, p)_A \\ \downarrow & & \downarrow \\ \mathcal{C}(Y, E) & \xrightarrow[q]{} & (i, p) \end{array}$$

As such, if $\mathcal{C}$ is a simplicial model category, then q' is a fibration which is acyclic when either i or p is. □

2.2.5. Path objects in simplicial model categories. We now consider path objects in simplicial model categories and whether these are, in general, stable in the sense of Definition 2.13. The "unit interval" $\Delta[1]$ of **SSet** is here denoted by I. Note the following standard result (cf. [**26**]):

Lemma 2.25. *If X is a fibrant object of a simplicial model category $\mathcal{C}$, then*

$$X^I \xrightarrow{\langle \partial_0, \partial_1 \rangle} X \times X$$

is a fibration.

Proof. It suffices to observe that, up to composition with an isomorphism, $\langle \partial_0, \partial_1 \rangle$ is

$$X^I \longrightarrow X^{\partial(I)},$$

where $\partial(I)$ is the usual simplicial boundary of I. I.e., $\partial(I)$ is the union of the faces of I.

Since the inclusion $\partial(I) \longrightarrow I$ is a cofibration and X is fibrant it follows by standard results that this map is a fibration (cf. Proposition 9.3.9 of [**31**]). □

We recall for the reader the notion of simplicial homotopy equivalence.

Definition 2.26. A map $f : X \longrightarrow Y$ in **SSet** is a **simplicial homotopy equivalence** if there exists a map $f' : Y \longrightarrow X$ together with maps $\alpha : X \times I \longrightarrow X$ and $\beta : Y \times I \longrightarrow Y$ such that the following diagrams commute

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1_X, \partial_1 \rangle} & X \times I & \xleftarrow{\langle 1_X, \partial_0 \rangle} & X \\ & {\scriptstyle 1_X} \searrow & \downarrow {\scriptstyle \alpha} & \swarrow {\scriptstyle f' \circ f} & \\ & & X & & \end{array} \qquad \begin{array}{ccccc} Y & \xrightarrow{\langle 1_Y, \partial_1 \rangle} & Y \times I & \xleftarrow{\langle 1_Y, \partial_0 \rangle} & Y \\ & {\scriptstyle 1_Y} \searrow & \downarrow {\scriptstyle \beta} & \swarrow {\scriptstyle f \circ f'} & \\ & & Y & & \end{array}$$

Note that we have the following (cf. Lemma 9.5.16 of [**31**]):

Lemma 2.27. *Let $\mathcal{C}$ be a simplicial model category. If $f : X \longrightarrow Y$ is a simplicial homotopy equivalence (i.e., a homotopy equivalence with respect to I), then f is a weak equivalence in $\mathcal{C}$.*

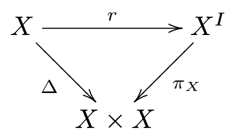
Lemma 2.28. *In a simplicial model category $\mathcal{C}$, the "constant loop" map $r : X \longrightarrow X^I$ is, for any object X, a weak equivalence.*

Proof. In fact, r is seen to be a simplicial strong deformation retract of $X^{d_0} : X^I \longrightarrow X$ using the multiplication map $\overline{\wedge} : I \times I \longrightarrow I$ to construct the required homotopy. Here $\overline{\wedge} : I \times I \longrightarrow I$ is given in simplices by

$$(f \,\overline{\wedge}_n\, g)(x) := \min\{f(x), g(x)\},$$

when f, g are in I_n and x is in $[n]$. □

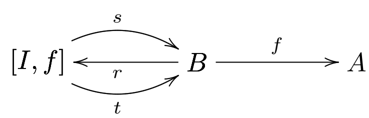
Although we have, under the rather general assumption that $\mathcal{C}$ is a simplicial model category, been able to show that the factorization



gives a good path object whenever X is fibrant, it is a bit more delicate to obtain a very good path object in this way. Nonetheless, for a great many interesting simplicial model categories r will be a cofibration. In particular, for all simplicial model categories in which the cofibrations are exactly the monomorphisms. This class of examples contains, among others, simplicial sheaves and simplicial presheaves. Accordingly we ask whether the construction of path objects in this way is stable.

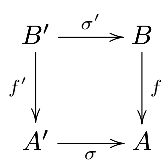
Theorem 2.29. *Assume $\mathcal{C}$ is a simplicial model category in which the cofibrations are exactly the monomorphisms, then $\mathcal{C}$ has stable path objects.*

Proof. Given a fibration $f : B \longrightarrow A$ in $\mathcal{C}$, we define $\iota(f)$ to be the cellular resolution given by

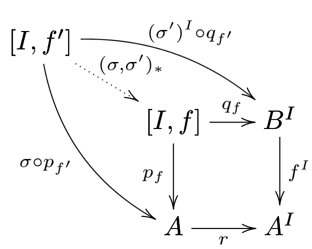


where $[I, f]$ denotes the domain of $(I \pitchfork f)_A$ as in the proof of Lemma 2.23. By Lemmata 2.25 and 2.28 and the fact that r is a monomorphism it follows that in $\mathcal{C}/A$ this data constitutes a very good path object. Thus, it is an object of $\mathbf{CRes}_{\mathcal{C}}$ and the condition that r be an acyclic cofibration is satisfied. To see that this assignment is functorial let a commutative square

(4)



in $\mathcal{C}$ be given with f' and f fibrations. Then there exists a canonical map as indicated in the following diagram



We define

$$\iota(\sigma, \sigma') := \big(\sigma, \sigma', (\sigma, \sigma')_*\big).$$

This assignment is functorial by the fact that $(I \pitchfork -)$ is functorial. That it is a homomorphism of cellular resolutions follows from the universal property of pullbacks.

To see that this functor $\iota : \mathcal{C}_{\mathfrak{F}} \longrightarrow \mathbf{CRes}_{\mathcal{C}}$ is fibred assume that (4) is a pullback and let a homomorphism $(h, h_0, h_1) : (C, V, E) \longrightarrow (A, B, [I, f])$ of cellular resolutions such that $h = \sigma \circ g$ for some $g : C \longrightarrow A'$ be given. Because (4) is a pullback there exists a canonical map $g_0 : V \longrightarrow B'$ such that $f' \circ g_0 = g \circ \varepsilon$ and $\sigma' \circ g_0 = h_0$, where ε is the augmentation $V \longrightarrow C$. Because $(I \pitchfork -)$ is a right adjoint and therefore preserves limits, it follows that

$$\begin{array}{ccc} [I, f'] & \xrightarrow{(\sigma,\sigma')_*} & [I, f] \\ {\scriptstyle p_{f'}}\downarrow & & \downarrow{\scriptstyle p_f} \\ A' & \xrightarrow[\sigma]{} & A \end{array}$$

is a pullback (and therefore the third condition of Definition 2.13 is satisfied). Thus, there also exists a canonical map g_1 as indicated in the following diagram:

$$\begin{array}{ccccc} & & E & & \\ & \swarrow{\scriptstyle g\circ\varepsilon\circ s} & \vdots{\scriptstyle g_1} & {\scriptstyle h_1}\searrow & \\ A' & \xleftarrow[p_{f'}]{} & [I, f'] & \xrightarrow[(\sigma,\sigma')_*]{} & [I, f] \end{array}$$

It follows from the definitions and the universal property of pullbacks that (g, g_0, g_1) is a homomorphism such that

$$\big(\sigma, \sigma', (\sigma, \sigma')_*\big) \circ (g, g_0, g_1) = (h, h_0, h_1).$$

Moreover, it is, by definition, the canonical homomorphism with this property. Thus, we have shown that $\mathcal{C}$ possesses stable path objects. $\square$

COROLLARY 2.30. *Any simplicial model category $\mathcal{C}$ satisfying the hypotheses of Theorem 2.29 is a quasi-model of type theory.*

2.3. Coherence of elimination terms

Although we have shown that using arbitrary weak factorization systems it is possible to *choose* maps J which can be used to interpret the elimination terms, because these choices are entirely arbitrary they need not be compatible with pullback. I.e., the coherence condition

(5)

$$\frac{\begin{array}{c} x : C, y : A(x), z : A(x), v : \mathrm{Id}_{A(x)}(y, z) \ \vdash \ D(x, y, z, v) : \mathrm{type} \\ x : C, u : A(x) \ \vdash \ d(u) : D\big(x, u, u, r_{A(x)}(u)\big) \\ x : C \ \vdash \ p(x) : \mathrm{Id}_{A(x)}(a(x), b(x)) \qquad\qquad \vdash \ c : C \end{array}}{\Big(J_{A(x),D}\big(d(x), a(x), b(x), p(x)\big)\Big)[c/x] \ = \ J_{A(c),D}\big(d(c), a(c), b(c), p(c)\big)} \ J \text{ coherence}$$

need not be satisfied by the choice of maps made in an arbitrary weak factorization system. It is the aim of this section to investigate this condition in more detail. Examples of models which satisfy this condition will be given in Chapter 3.

2.3.1. The coherence condition. Although, as we have seen, weak factorization systems are in general capable of satisfying some of the axioms of type theory, when it comes to obtaining models of type theory which possess pullback stable elimination terms it is often convenient to restrict the interpretation of types to certain well-behaved fibrations. In such situations one is then able to simply ignore the left-class $\mathfrak{L}$ of the factorization system and work with the special fibrations. As such, it will be convenient to phrase the discussion of the coherence of elimination terms in this more general setting. This task involves first reformulating the discussion of cellular resolutions from above in terms of arbitrary comprehension categories. This generalization is necessary because the examples we present in Chapter 3 fit in this setting and not in the setting of models derived from a weak factorization system.

Definition 2.31. Given a comprehension category $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$, the category $\mathbf{CRes}(\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C})$ of **cellular resolutions with respect to** $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ has objects tuples (α, α', r) such that α is an object of $\mathcal{P}$, α' is in the fibre $\mathcal{P}(A^+_\alpha)$ of $\mathbf{P}(-)$ over A^+_α, where $A = \mathbf{P}(\alpha)$, and r is as indicated in the following diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{r} & (A^+_\alpha)_{\alpha'} \\ & {\scriptstyle\Delta}\searrow \quad \swarrow{\scriptstyle\pi_{\alpha'}} & \\ & A^+_\alpha & \end{array}$$

An arrow $(\alpha, \alpha', r) \longrightarrow (\beta, \beta', r)$ in $\mathbf{CRes}(\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C})$ consists of a pair (f, f') such that $f : \alpha \longrightarrow \beta$ and $f' : \alpha' \longrightarrow \beta'$ are arrows in $\mathcal{P}$ such that f' lies over the induced map $f_1 \times_{f_0} f_1 : A^+_\alpha \longrightarrow B^+_\beta$ where

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f_1} & B_\beta \\ {\scriptstyle\pi_\alpha}\downarrow & & \downarrow{\scriptstyle\pi_\beta} \\ A & \xrightarrow[f_0]{} & B \end{array}$$

is $\chi(f)$. Finally, we require that

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f_1} & B_\beta \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ (A^+_\alpha)_{\alpha'} & \xrightarrow[f'_1]{} & (B^+_\beta)_{\beta'} \end{array}$$

commutes, where f'_1 as indicated in the following commutative square $\chi(f')$:

$$\begin{array}{ccc} (A^+_\alpha)_{\alpha'} & \xrightarrow{f'_1} & (B^+_\beta)_{\beta'} \\ {\scriptstyle\pi_{\alpha'}}\downarrow & & \downarrow{\scriptstyle\pi_{\beta'}} \\ A^+_\alpha & \xrightarrow[f_1 \times_{f_0} f_1]{} & B^+_\beta \end{array}$$

The category of cellular resolutions from Definition 2.31 is more general than Definition 2.10 in that it makes sense for arbitrary comprehension categories. We note that it is an actual generalization in the sense indicated in the following Scholium:

SCHOLIUM 2.32. *When the comprehension category* $\mathbf{P}(-)$ *is the codomain map* $\partial_1 : \mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}_{\mathrm{f}}$ *associated to a weak factorization system* $(\mathfrak{L}, \mathfrak{R})$ *on* $\mathcal{C}$*, then* $\mathbf{CRes}_{\mathcal{C}}$ *is isomorphic to* $\mathbf{CRes}(\partial_1 : \mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}_{\mathrm{f}})$.

We also note that, via the projection $\pi : \mathbf{CRes}(\mathbf{P}(-)) \longrightarrow \mathcal{C}$ which sends (α, α', r) to $A = \mathbf{P}(\alpha)$ is a Grothendieck fibration. The following definition generalizes the notion of stable path objects to the setting of an arbitrary comprehension category.

DEFINITION 2.33. Assume $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ is a comprehension category with comprehension χ. Then $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ is said to have **stable identity types** if there exists a fibred functor

$$\begin{array}{ccc} \mathcal{P} & \overset{\iota}{\dashrightarrow} & \mathbf{CRes}(\mathbf{P}(-)) \\ & {\scriptstyle \mathbf{P}(-)} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathcal{C} & \end{array}$$

such that:

- ι is a section of the first projection $\pi_0 : \mathbf{CRes}(\mathbf{P}(-)) \longrightarrow \mathcal{P}$ which sends (α, α', r) to α and similarly for maps; and
- If an arrow $\varphi : \alpha \longrightarrow \beta$ in $\mathcal{P}$ is cartesian over $f : A \longrightarrow B$, then the second component φ' of $\iota(\varphi) = (\varphi, \varphi')$ is cartesian over $(f_1 \times_f f_1) : A^+_\alpha \longrightarrow B^+_\beta$.

When $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ has stable identity types, we denote by $i(\alpha)$ the second component of $\iota(\alpha)$. I.e., $\iota(\alpha) = \big(\alpha, i(\alpha), r\big)$. Similarly, $I(\alpha)$ denotes the object $(A^+_\alpha)_{i(\alpha)}$, for $\alpha \in \mathcal{P}(A)$. Similar notation is employed for the result of applying ι to arrows. We also write $\langle s, t \rangle$ for the map $\pi_{\alpha'} : I(\alpha) \longrightarrow A^+_\alpha$.

Clearly when $\mathcal{P}$ is $\mathcal{C}_{\mathfrak{R}}$ for a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ in $\mathcal{C}$, the existence of stable identity types follows from the existence of stable path objects in the sense of Definition 2.13. The converse, however, need not hold since the reflexivity maps need not be in $\mathfrak{L}$. In particular, this further condition from Definition 2.13 will be unnecessary in light of the additional condition governing the existence of elimination terms which we now consider.

EXAMPLE 2.34. The motivating example of a comprehension category with stable identity types is the syntactic comprehension category $\mathbf{P}(-) : \mathcal{C}(\mathbb{T}_\omega)_{\mathfrak{D}} \longrightarrow \mathcal{C}(\mathbb{T}_\omega)$, where $\mathfrak{D}$ is the collection of dependent projections (cf. Section B.4 of Appendix B).

Assume $\alpha \in \mathcal{P}(A)$ and let Υ_α denote the functor

$$\mathcal{P}\big(I(\alpha)\big) \longrightarrow \mathbf{Set}$$

which sends an object β in the fiber $\mathcal{P}\big(I(\alpha)\big)$ to the hom-set $\big(\mathcal{C}/I(\alpha)\big)(r, \pi_\beta)$, where $\pi_\beta : I(\alpha)_\beta \longrightarrow I(\alpha)$, as usual and $r : A_\alpha \longrightarrow I(\alpha)$ is the "reflexivity" term associated to $\iota(\alpha)$, as above. There is also a functor $\Gamma_\alpha : \mathcal{P}\big(I(\alpha)\big) \longrightarrow \mathbf{Set}$ which sends a β to the set $\mathcal{P}\big(I(\alpha)\big)[1, \pi_\beta]$ of sections of $\pi_\beta : I(\alpha)_\beta \longrightarrow I(\alpha)$. Precomposition with $r : A_\alpha \longrightarrow I(\alpha)$ yields a natural transformation

$$\Gamma_\alpha \overset{\vartheta_{\alpha,-}}{\Longrightarrow} \Upsilon_\alpha.$$

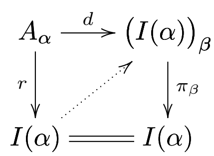
DEFINITION 2.35. A **choice of elimination terms** for a comprehension category $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ with stable identity types consists of a family of sections

$$\Upsilon_\alpha(\beta) \xrightarrow{J_{\alpha,\beta}} \Gamma_\alpha(\beta)$$

of the maps (of sets) $\vartheta_{\alpha,\beta}$ for all objects α of $\mathcal{P}$ and β of $\mathcal{P}\big(I(\alpha)\big)$.

It is straightforward to verify that a choice of elimination terms amounts exactly to, as the nomenclature suggests, a choice of diagonal fillers for diagrams of the following form:

(6)



REMARK 2.36. We emphasize that the maps $J_{\alpha,\beta}$ are, if they exist, arrows in **Set** and *not* arrows in $\mathcal{C}$. Such a function $J_{\alpha,\beta}$ between sets *is* a choice of diagonal fillers: it assigns to every commutative square (6) a corresponding diagonal filler $J_{\alpha,\beta}(d)$. Thus, when, as in Definition 2.35, we say that a comprehension category $\mathbf{P}(-)$ *has a choice of elimination terms* we mean that $\mathbf{P}(-)$ comes equipped with a fixed choice of sections $J_{\alpha,\beta}$ for all α and β. I.e., a choice of elimination terms is additional *structure* on $\mathbf{P}(-)$ and not merely a *property* of $\mathbf{P}(-)$.

In particular, the fact that, given a map d as indicated in the diagram, $J_{\alpha,\beta}(d)$ is in $\Gamma_\alpha(\beta)$ means that the bottom triangle commutes. Likewise the fact that $J_{\alpha,\beta}$ is a section of $\vartheta_{\alpha,\beta}$ means precisely that the upper triangle commutes.

REMARK 2.37. The reason for requiring "pointwise" sections $J_{\alpha,\beta}$ in Definition 2.35 rather than stipulating the existence of a section $J_{\alpha,-}$ of $\vartheta_{\alpha,-}$ in the functor category $\big[\mathcal{P}\big(I(\alpha)\big), \mathbf{Set}\big]$ is that, syntactically, naturality of $J_{\alpha,-}$ is too strong of a requirement. To see this observe that naturality of $J_{\alpha,-}$ can be stated syntactically as the requirement that, when $D(x,y,z)$ and $E(x,y,z)$ are both types in the context $\big(x,y : A, z : \underline{A}(x,y)\big)$,

$$\frac{\begin{array}{c} x,y : A, z : \mathrm{Id}_A(x,y), w : D(x,y,z) \ \vdash \ h(x,y,z,w) : E(x,y,z) \\ x : A \ \vdash \ d(x) : D(x,x,r_A(x)) \end{array}}{\begin{array}{c} x,y : A, z : \mathrm{Id}_A(x,y) \ \vdash \\ h\big(x,y,z,J(d,x,y,z)\big) \ = \ J\big([x:A]h(x,x,r_A(x),d(x)),x,y,z\big) : E(x,y,z) \end{array}} \ \text{Nat.}$$

where we have, for ease of presentation, omitted the ambient context Γ. However, this principle is implies the η-rule for identity types:

$$\frac{\begin{array}{c} \Gamma, x,y : A, z : \mathrm{Id}_A(x,y) \ \vdash \ D(x,y,z) : \mathrm{type} \\ \Gamma, x,y : A, z : \mathrm{Id}_A(x,y) \ \vdash \ e(x,y,z) : D(x,y,z) \end{array}}{\Gamma, x,y : A, z : \mathrm{Id}_A(x,y) \ \vdash \ e(x,y,z) \ = \ J\big([x:A]e(x,x,r_A(x)),x,y,z\big) : D(x,y,z)} \ \eta$$

which Streicher [**80**] has shown to be equivalent to the reflection rule. To see this, observe that (Nat) trivially implies (η) by taking $E(x,y,z)$ to be $D(x,y,z)$ and $h(x,y,z,w)$ to be $e(x,y,z)$ with the variable w weakened in.

Definition 2.38. Let $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ be a comprehension category with stable identity types as above and assume given $\alpha \in \mathcal{P}(A)$, $\beta \in \mathcal{P}(B)$ and an arrow $\varphi : \alpha \longrightarrow \beta$ in $\mathcal{P}$ which is cartesian over $f : A \longrightarrow B$ in $\mathcal{C}$. Then, when $\delta \in \mathcal{P}\big(I(\beta)\big)$, there exists an arrow $I(\varphi)_\delta : \delta \cdot I(\varphi) \longrightarrow \delta$ in $\mathcal{P}$ which is cartesian over $I(\varphi) : I(\alpha) \longrightarrow I(\beta)$. Applying the comprehension to this map yields the following pullback square:

$$\begin{array}{ccc} I(\alpha)_{\delta\cdot I(\varphi)} & \xrightarrow{\psi} & I(\beta)_\delta \\ \downarrow & & \downarrow \\ I(\beta) & \xrightarrow[I(\varphi)]{} & I(\alpha) \end{array}$$

We say that $\mathcal{P}$ has **coherent identity types** if it comes equipped with a choice of elimination terms such that if, for any element d of $\Upsilon_\beta(\delta)$,

$$\begin{array}{ccc} I(\alpha) & \xrightarrow{I(\varphi)} & I(\beta) \\ {\scriptstyle J_{\alpha,\delta\cdot I(\varphi)}(d')}\downarrow & & \downarrow{\scriptstyle J_{\beta,\delta}(d)} \\ I(\alpha)_{\delta\cdot I(\varphi)} & \xrightarrow[\psi]{} & I(\beta)_\delta \end{array} \tag{7}$$

commutes, where d' is the canonical element of $\Upsilon_\alpha\big(\delta \cdot I(\varphi)\big)$ obtained from d via pullback.

When $\mathcal{C}$ has a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ such that $\mathcal{C}_\mathfrak{R} \longrightarrow \mathcal{C}_\mathrm{f}$ has coherent identity types in the sense of Definition 2.38 we sometimes say that $\mathcal{C}$ has **coherent path objects**.

In general, it is unreasonable to expect a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ in a category $\mathcal{C}$ to possess coherent path objects since the lifts are not often given functorially. One possible method, which has been suggested by Richard Garner, for resolving this difficulty is to consider instead what are called *natural weak factorization systems* [**27, 24**]. The approach we pursue here, motivated by the groupoids model of Hofmann and Streicher [**35**], is instead to consider cases where it is possible to restrict the interpretation of types to certain well-behaved fibrations. For now we simply state the following definition and mention that the examples of such categories can be found in Chapter 3.

Definition 2.39. Assume $\mathcal{C}$ is a category equipped with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$. A **coherent restriction of** $(\mathfrak{L}, \mathfrak{R})$ consists of a category $\mathcal{S}$ together with a fibred functor k as indicated in the following diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{k} & \mathcal{C}_\mathfrak{R} \\ & {\scriptstyle \mathbf{P}(-)}\searrow \quad \swarrow{\scriptstyle \partial_1} & \\ & \mathcal{C}_\mathrm{f} & \end{array}$$

such that k is faithful and the resulting comprehension category $\mathbf{P}(-) : \mathcal{S} \longrightarrow \mathcal{C}_\mathrm{f}$ has coherent identity types. In this situation we think of the objects (in the image of) of $\mathcal{S}$ as "special" distinguished fibrations.

In some cases we can say more about $\mathcal{S}$. In particular, it will sometimes occur that the inclusion functor $k : \mathcal{S} \longrightarrow \mathcal{C}_\mathfrak{R}$ possesses one or both adjoints. Simply having a comprehension category equipped with coherent identity types is not enough to model on the nose all rules of type theory. We now turn to the issue of turning a coherent model of type theory (one where the rules are satisfied up to isomorphism) into a split model of type theory (one where the rules are satisfied on the nose).

2.4. Split models via the Bénabou construction

Although we have seen that categories $\mathcal{C}$ possessing weak factorization systems $(\mathfrak{L}, \mathfrak{R})$ provide *non-split* models of type theory it is desirable to have models which are also split. For extensional type theory Hofmann [**33**] observed that given a locally cartesian closed category $\mathcal{C}$, it is possible to obtain a split fibration equivalent to the codomain fibration $\mathcal{C}^{\to} \longrightarrow \mathcal{C}$ using a construction due to Jean Bénabou [**7, 25, 38**]. In this section we study the behavior of this construction for those comprehension categories resulting from the discussion above.

2.4.1. Identity types in the associated split fibration. We assume that the reader is familiar with the technique, due to Jean Bénabou [**7**], for constructing an equivalent split fibration from a given fibration. When $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ is a comprehension category with stable identity types as above, we would like to investigate the properties of the split fibration $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$ which corresponds in this way to $\mathbf{P}(-)$. First, recall that when A is an object of $\mathcal{C}$, $\mathbf{P}_s(-)$ is given in fibers by

$$\mathcal{P}_s(A) := \mathbf{Fib}(\mathcal{C})\big(\mathcal{C}/A \xrightarrow{\partial_0} \mathcal{C},\ \mathcal{P} \xrightarrow{\mathbf{P}(-)} \mathcal{C}\big),$$

where $\mathbf{Fib}(\mathcal{C})$ is the usual category of Grothendieck fibrations over $\mathcal{C}$, and, when $\sigma : A' \longrightarrow A$ is an arrow between fibrant objects and $\alpha \in \mathcal{P}_s(A)$,

$$(\alpha \cdot \sigma) := \big(\mathcal{C}/A' \xrightarrow{\Sigma_\sigma} \mathcal{C}/A \xrightarrow{\alpha} \mathcal{P}\big) \ \in \mathcal{P}_s(A').$$

The comprehension for $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$, which we leave nameless, sends an object $\alpha \in \mathcal{P}_s(A)$ to the arrow $\chi\big(\alpha(1_A)\big)$ in $\mathcal{C}$. Similarly, given $\beta \in \mathcal{P}_s(B)$, an arrow $\alpha \longrightarrow \beta$ — which, recall, is given by a pair (f, f') where $f : A \longrightarrow B$ is an arrow in $\mathcal{C}$ and f' is a fibred natural transformation from α to $(\beta \cdot f)$ — is sent to $\chi(f'_{1_A})$.

Remark 2.40. We mention that the construction given below of the identity types for the associated split fibration requires that we make a choice of pullbacks in $\mathcal{C}$. Because this choice is only required for the definition of the action of the identity types on objects the entire construction remains functorial. (This is the same as for the treatment of identity types by Hofmann in [**33**].)

Given $\alpha \in \mathcal{P}_s(A)$, we will abbreviate the element $\alpha(1_A) \in \mathcal{P}(A)$ by $\tilde{\alpha}$ in order to simplify some of the notation below. In this notation, applying the comprehension χ of $\mathbf{P}_s(-)$ to α yields the arrow $\pi_{\tilde{\alpha}} : A_{\tilde{\alpha}} \longrightarrow A$ in $\mathcal{C}$. For sections $a, b : A \rightrightarrows A_{\tilde{\alpha}}$ of $\pi_{\tilde{\alpha}} : A_{\tilde{\alpha}} \longrightarrow A$, the identity type $\imath(\alpha, a, b) \in \mathcal{P}_s(A)$ is defined as follows.

Objects: Given an object $f : B \longrightarrow A$ of $\mathcal{C}/A$, we have $\alpha(f) \in \mathcal{P}(B)$. Applying ι therefore yields an object $\iota\big(\alpha(f)\big) = \big(\alpha(f), i(\alpha), r\big)$ of $\mathbf{CRes}(\mathbf{P}(-))$. In particular, $i(\alpha(f)) \in \mathcal{P}(B^+_{\alpha(f)})$. Moreover, in $\mathcal{C}$ we have the arrow $\langle a[f], b[f]\rangle : B \longrightarrow B^+_{\alpha(f)}$. Therefore, we define $\imath(\alpha, a, b)(f)$ to be the domain of the (chosen) cartesian lift of $\langle a[f], b[f]\rangle$ with respect to $i(\alpha(f))$. I.e.,

$$\imath(\alpha, a, b)(B \xrightarrow{f} A) := i(\alpha(f)) \cdot \langle a[f], b[f]\rangle,$$

which is an object of $\mathcal{P}$.

Arrows: To define the action of $\imath(\alpha, a, b)$ on arrows let an arrow

$$\begin{array}{ccc} C & \xrightarrow{h} & B \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

in $\mathcal{C}/A$ be given. Applying α yields an arrow $\alpha(h) : \alpha(g) \longrightarrow \alpha(f)$ in $\mathcal{P}$ which is cartesian over h. Therefore, applying ι yields an arrow $\iota\big(\alpha(h)\big) : \iota\big(\alpha(g)\big) \longrightarrow \iota\big(\alpha(f)\big)$ in $\mathbf{CRes}(\mathbf{P}(-))$. In particular, $\iota\big(\alpha(h)\big) = \big(\alpha(h), i(\alpha(h))\big)$ and the arrow

$$\imath(\alpha, a, b)(g) \xrightarrow{\langle a[g], b[g]\rangle_{i(\alpha(g))}} i\big(\alpha(g)\big) \xrightarrow{i(\alpha(h))} i\big(\alpha(f)\big)$$

in $\mathcal{P}$ lies over $(h_{\tilde{\alpha}} \times_h h_{\tilde{\alpha}}) \circ \langle a[g], b[g]\rangle : C \longrightarrow B^+_{\alpha(f)}$. Now,

$$\begin{array}{ccc} C & \xrightarrow{\langle a[g], b[g]\rangle} & C^+_{\alpha(g)} \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h_{\tilde{\alpha}} \times_h h_{\tilde{\alpha}}} \\ B & \xrightarrow[\langle a[f], b[f]\rangle]{} & B^+_{\alpha(f)} \end{array}$$

commutes and therefore, using the fact that $\langle a[f], b[f]\rangle_{i(\alpha(f))}$ is cartesian, we define $\imath(\alpha, a, b)(h) : \imath(\alpha, a, b)(g) \longrightarrow \imath(\alpha, a, b)(f)$ to be the canonical map over h for which

$$\langle a[f], b[f]\rangle_{i(\alpha(f))} \circ \imath(\alpha, a, b)(h) = i\big(\alpha(h)\big) \circ \langle a[g], b[g]\rangle_{i(\alpha(g))}.$$

LEMMA 2.41. *As defined, $\imath(\alpha, a, b) : \mathcal{C}/A \longrightarrow \mathcal{P}$ is a fibred functor from ∂_0 to* $\mathbf{P}(-)$.

PROOF. Functoriality is routine since the action of $\imath(\alpha, a, b)$ on arrows is defined in a canonical way. That $\imath(\alpha, a, b)$ is fibred follows from the second condition from Definition 2.33. □

2.4.2. Properties of the identity types. We now turn to investigating the properties of the identity types just defined.

LEMMA 2.42. *The identity types for $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$ defined above satisfy the coherence condition for identity types.*

PROOF. Given any $\sigma : A' \longrightarrow A$ in the base and any object $f : B \longrightarrow A'$ of $\mathcal{C}/A'$ we have

$$\begin{aligned} \big(\imath(\alpha, a, b) \cdot \sigma\big)(f) &= \imath(\alpha, a, b)(\sigma \circ f) \\ &= i\big(\alpha(\sigma \circ f)\big) \cdot \langle a[\sigma \circ f], b[\sigma \circ f]\rangle \\ &= i\big((\alpha \cdot \sigma)(f)\big) \cdot \langle a[\sigma][f], b[\sigma][f]\rangle \\ &= \imath(\alpha \cdot \sigma, a[\sigma], b[\sigma])(f), \end{aligned}$$

as required. The case of arrows is similarly straightforward. □

Given $\alpha \in \mathcal{P}_s(A)$ as above and a section a of $A_{\tilde{\alpha}} \longrightarrow A$, we will now describe the reflexivity term $\rho(\alpha, a)$. First, note that applying $\imath(\alpha, a, a)$ to the object $1_A : A \longrightarrow A$ of $\mathcal{C}/A$ yields a cartesian arrow

$$\imath(\alpha, a, a)(1_A) \xrightarrow{\langle a,a\rangle_{i(\tilde{\alpha})}} i(\tilde{\alpha})$$

over $\langle a, a\rangle : A \longrightarrow A^+_{\tilde{\alpha}}$. Therefore, applying the comprehension χ to this data yields a pullback square

$$\begin{array}{ccc} P_{1_A} & \xrightarrow{m} & I(\tilde{\alpha}) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ A & \xrightarrow[\langle a,a\rangle]{} & A^+_{\tilde{\alpha}} \end{array}$$

where we have written P_{1_A} instead of the more cumbersome $A_{\imath(\alpha,a,a)(1_A)}$ and l in place of $\pi_{\imath(\alpha,a,a)(1_A)}$. The interpretation of the reflexivity term is then the section $\rho(\alpha, a) : A \longrightarrow P_{1_A}$ of $l : P_{1_A} \longrightarrow A$ induced the composite $r \circ a : A \longrightarrow I(\tilde{\alpha})$ where r is the reflexivity map $r : A_{\tilde{\alpha}} \longrightarrow I(\tilde{\alpha})$. I.e., r is the final component of $\imath(\tilde{\alpha}) = (\tilde{\alpha}, i(\tilde{\alpha}), r)$.

Lemma 2.43. *The reflexivity terms for $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$ satisfy the coherence condition for reflexivity terms.*

Proof. Suppose given $\sigma : A' \longrightarrow A$. Then,

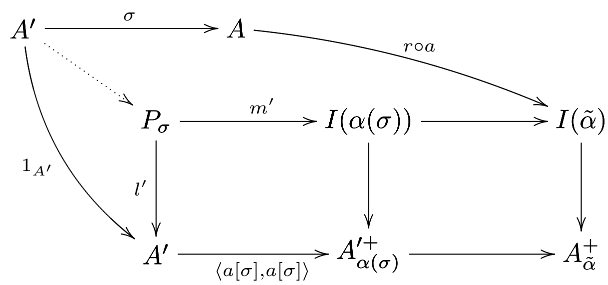
$$\begin{array}{ccc} A' & \xrightarrow{\sigma} & A \\ & {\scriptstyle \sigma}\searrow \quad \swarrow{\scriptstyle 1_A} & \\ & A & \end{array}$$

is an arrow in $\mathcal{C}/A$ and applying $\chi \circ \imath(\alpha, a, a)$ yields a pullback square

$$\begin{array}{ccc} P_\sigma & \xrightarrow{\hat{\sigma}} & P_{1_A} \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ A' & \xrightarrow[\sigma]{} & A \end{array}$$

in $\mathcal{C}$, where we have made similar abbreviations to those mentioned above. By definition, $\rho(\alpha, a)[\sigma]$ is then the canonical section $A' \longrightarrow P_\sigma$ of $l' : P_\sigma \longrightarrow A'$ for which $\hat{\sigma} \circ \rho(\alpha, a)[\sigma] = \rho(\alpha, a) \circ \sigma$ where

On the other hand, $\rho(\alpha \cdot \sigma, a[\sigma])$ is the canonical section of l' for which $m' \circ \rho(\alpha \cdot \sigma, a[\sigma]) = r \circ a[\sigma]$ where $m' : P_\sigma \longrightarrow I\big(\alpha(\sigma)\big)$ is the map obtained, for σ, as m was for 1_A above. In this case it is straightforward to verify that both $\rho(\alpha \cdot \sigma, a[\sigma])$ and $\rho(\alpha, a)[\sigma]$ make the following diagram commute when inserted as the dotted arrow:



To see that this is the case we emphasize that it must be used that $I(\alpha(\sigma)) : I(\alpha(\sigma)) \longrightarrow I(\tilde{\alpha})$ commutes with r maps and that $m \circ \hat{\sigma} = I(\alpha(\sigma)) \circ m'$, which follows from the definition of $\imath(\alpha, a, a)(\sigma : \sigma \longrightarrow 1_A)$. □

With these lemmata at our disposal we obtain a preliminary result regarding homotopical models.

Proposition 2.44. *Assume $\mathcal{C}$ if a finitely complete, locally cartesian closed category equipped with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ which has stable path objects for which pullback σ^* along a fibration $\sigma : A' \longrightarrow A$ between fibrant objects preserves maps in $\mathfrak{L}$. Then the split fibration $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}_{\mathrm{f}}$ associated to the fibration $\mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}_{\mathrm{f}}$ is a model of $\mathbb{T}_{-}$ together with the formation and introduction rules governing identity types. Moreover, the coherence conditions for identity types and reflexivity terms are satisfied, and $\mathbf{P}_s(-)$ is a split quasi-model of identity types.*

Proof. Under these conditions it follows by a standard argument relating adjunctions and lifting properties that, since σ^* preserves maps in $\mathfrak{L}$, the corresponding dependent product Π_σ preserves elements of $\mathfrak{R}$. Thus the formation rule for dependent products is valid. The underlying model always possesses dependent sums since $\mathfrak{R}$ is stable under composition. As Hofmann [**33**] has already shown that these operations are preserved, and satisfy all of the corresponding rules on the nose, in the associated split fibration $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}_{\mathrm{f}}$, it suffices to show that $\mathbf{P}_s(-)$ validates the rules governing quasi-models of identity types. By Lemmata 2.42 and 2.43 it suffices to check that the introduction, elimination and conversion rules are satisfied in the quasi-model sense. However, in the discussion above we have already seen that the introduction rule is valid. Finally, validity of the elimination and conversion rules follows from the argument given in the proof of Theorem 2.17. □

Remark 2.45. Note that the assumption that pullback along a fibration between fibrant objects preserves $\mathfrak{L}$ maps is required only for the interpretation of dependent products.

The hypothesis of Proposition 2.44 relating to dependent products is satisfied in many (locally cartesian closed) model categories. Recall that a model category $\mathcal{C}$ is **right proper** if and only if weak equivalences are stable under pullback along fibrations. Many model categories, including both the category **SSet** of simplicial sets and the category **Top** of spaces, are right proper (cf. [**31, 36**]).

Corollary 2.46. *Assume $\mathcal{C}$ is a right-proper simplicial model category in which the cofibrations are the monomorphisms. Then if $\mathcal{C}$ is locally cartesian closed,* *the associated split fibration* $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}_{\mathrm{f}}$ *is a model of* $\mathbb{T}_{-}$ *as well as the formation and introduction rules for identity types, and the coherence conditions on identity types and reflexivity terms. Moreover, it is a split quasi-model of identity types.*

2.4.3. Elimination terms. Assume given an object A of $\mathcal{C}$ and α in $\mathcal{P}_s(A)$. We would like to see that elimination terms are stable under substitution along a map $\sigma : A' \longrightarrow A$ in the base. In this situation, when there exists δ in $\mathcal{P}_s\big(I(\tilde{\alpha})\big)$ and a map

$$\begin{array}{ccc} A_{\tilde{\alpha}} & \xrightarrow{\;d\;} & D \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle g} & \\ & I(\tilde{\alpha}) & \end{array}$$

where we have written D for $\big(I(\tilde{\alpha})\big)_{\delta(1_{I(\tilde{\alpha})})}$ and g for $\pi_{\delta(1_{I(\tilde{\alpha})})}$, it follows that there is a distinguished section $J : I(\tilde{\alpha}) \longrightarrow D$ of g defined to be $J_{\tilde{\alpha},\delta}(d)$, where this notation is as in Section 2.3.1. There also exists a map $J' : I(\alpha(\sigma)) \longrightarrow D'$ obtained by applying $J_{\alpha(\sigma),\delta(\sigma)}$ to the map d' indicated in the following diagram:

$$\begin{array}{ccccc} & & D' & \xrightarrow{\;\sigma'\;} & D \\ & {\scriptstyle d'}\nearrow & & {\scriptstyle d}\nearrow & \\ A'_{\alpha(\sigma)} & \longrightarrow & & A_{\tilde{\alpha}} & \\ & \searrow & \downarrow{\scriptstyle g'} & \searrow & \downarrow{\scriptstyle g} \\ & & I(\alpha(\sigma)) & \xrightarrow[I(\alpha(\sigma))]{} & I(\tilde{\alpha}) \end{array}$$

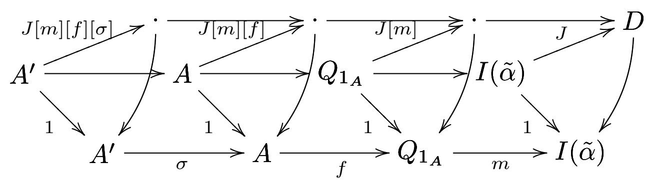
where D' is $\big(I(\alpha(\sigma))\big)_{\delta \cdot I(\alpha(\sigma))}$ and g' is the associated projection. Because the path objects in $\mathcal{C}$ are assumed to be coherent the following diagram commutes:

$$\begin{array}{ccc} I(\alpha(\sigma)) & \xrightarrow{\;I(\alpha(\sigma))\;} & I(\tilde{\alpha}) \\ {\scriptstyle J'}\downarrow & & \downarrow{\scriptstyle J} \\ D' & \xrightarrow[\sigma']{} & D \end{array} \tag{8}$$

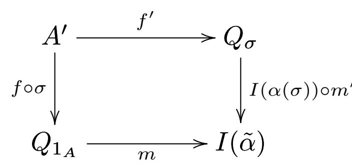
Now, let sections $a, b : A \longrightarrow A_{\tilde{\alpha}}$ of $\pi_{\tilde{\alpha}}$ be given together with a section $f : A \longrightarrow Q_{1_A}$ of the projection $l : Q_{1_A} \longrightarrow A$ where the notation Q_{1_A}, *et cetera* corresponds to the P_{1_A} notation from above as indicated in the following pullback diagrams:

$$\begin{array}{ccc} Q_{1_A} & \xrightarrow{\;m\;} & I(\tilde{\alpha}) \\ {\scriptstyle l}\downarrow & & \downarrow \\ A & \xrightarrow[\langle a,b\rangle]{} & A^{+}_{\tilde{\alpha}} \end{array} \qquad\qquad \begin{array}{ccc} Q_{\sigma} & \xrightarrow{\;m'\;} & I(\alpha(\sigma)) \\ {\scriptstyle l'}\downarrow & & \downarrow \\ A' & \xrightarrow[\langle a[\sigma],b[\sigma]\rangle]{} & (A')^{+}_{\alpha(\sigma)} \end{array}$$

We would now like to compare two different sections which arise given this data. On the one hand, we have $J[m][f][\sigma]$ which is obtained by repeatedly substituting into J as indicated in the following diagram (see Remark 2.15 for the definition of this notation):



The section $J'[m'][f']$ is constructed in an analogous way, where f' is the canonical section $A' \longrightarrow Q_\sigma$ of the projection $Q_\sigma \longrightarrow A'$ such that the following diagram commutes:



One of the first steps to verifying the coherence condition for the elimination terms is to compare this section with $J[m][f][\sigma]$.

Lemma 2.47. *Given the data as above,*

$$J[m][f][\sigma] = J'[m'][f']$$

Proof. It suffices to prove that

$$\sigma' \circ J' \circ m' \circ f' = J \circ m \circ f \circ \sigma,$$

which is seen to obtain as follows:

$$\begin{aligned} J \circ m \circ f \circ \sigma &= J \circ I(\alpha(\sigma)) \circ m' \circ f' \\ &= \sigma' \circ J' \circ m' \circ f' \end{aligned}$$

where the first equation is by definition of f' and the second equation is by (8). $\square$

In the case where the data above comes from the interpretation of judgements in type theory additional care is required. Namely, when the object A is the interpretation $[\![\Delta]\!]$ of a context and α is the interpretation $[\![\Delta \vdash T : \mathrm{type}]\!]$, the object $I(\tilde{\alpha})$ need not denote the context $(\Delta, x : T, y : T, z : \mathrm{Id}_T(x, y))$. In particular, to see how this context is interpreted, note that, in order to obtain the judgement

$$\Delta, x, y : T \vdash \mathrm{Id}_T(x, y) : \mathrm{type}$$

we must first regard T itself as a type in context $(\Delta, x, y : T)$ and then form the identity type with respect to this context and the weakened term judgements $\Delta, x, y : T \vdash \xi : T$ for $\xi = x, y$. Now, where $\hat{\pi} : A^+_{\tilde{\alpha}} \longrightarrow A$ is the projection, we have

$$\begin{gathered} [\![\Delta, x, y : T \vdash T : \mathrm{type}]\!] = (\alpha \cdot \hat{\pi}), \text{ and} \\ [\![\Delta, x, y, z : T]\!] = B, \end{gathered}$$

where B is $(A^+_{\tilde{\alpha}})_{\tilde{\alpha}}$ as indicated in the following pullback square:

$$\begin{array}{ccc} B & \xrightarrow{\hat{\pi}_{\tilde{\alpha}}} & A_{\tilde{\alpha}} \\ {\scriptstyle \pi_{\alpha(\hat{\pi})}}\downarrow & & \downarrow{\scriptstyle \pi_{\tilde{\alpha}}} \\ A^+_{\tilde{\alpha}} & \xrightarrow[\hat{\pi}]{} & A \end{array}$$

As such, there exist canonical sections $p_\xi : A^+_{\tilde{\alpha}} \rightrightarrows B$, for $\xi = +, -$, of the resulting projection $B \longrightarrow A^+_{\tilde{\alpha}}$ such that

$$\hat{\pi}_{\tilde{\alpha}} \circ p_\xi = \pi^\xi_{\tilde{\alpha}},$$

in the notation of Remark 2.15. These interpret the weakened judgements $\Delta, x, y : T \ \vdash \ x : T$ and $\Delta, x, y : T \ \vdash \ y : T$, respectively. In light of the interpretation of identity types given above, the context $(\Delta, x : T, y : T, z : \mathrm{Id}_T(x, y))$ is interpreted as the object $\tilde{I}(\tilde{\alpha})$ obtained by applying $\imath(\tilde{\alpha} \cdot \hat{\pi}, p_+, p_-)$ to $1_{A_{\tilde{\alpha}+}}$. In particular, $\tilde{I}(\tilde{\alpha})$ fits into the following pullback square

$$\begin{array}{ccc} \tilde{I}(\tilde{\alpha}) & \xrightarrow{q} & I(\alpha(\hat{\pi})) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ A^+_{\tilde{\alpha}} & \xrightarrow[\langle p_+, p_-\rangle]{} & B^+ \end{array}$$

Because this construction involves making a choice of cartesian lifts, it will not in general be the case that $\tilde{I}(\tilde{\alpha})$ is equal to $I(\tilde{\alpha})$. Nonetheless, it is straightforward to prove that, as objects of $\mathbf{CRes}(\mathbf{P}(-))$, $\imath(\tilde{\alpha}) = \big(\tilde{\alpha}, i(\tilde{\alpha}), r\big)$ and $\big(\tilde{\alpha}, i(\alpha(\hat{\pi})) \cdot \langle p_+, p_-\rangle, r\big)$ are isomorphic via a canonical isomorphism. Namely, we have

$$i\big(\alpha(\hat{\pi})\big) \cdot \langle p_+, p_1\rangle \xrightarrow{\langle p_+, p_-\rangle_{i(\alpha(\hat{\pi}))}} i\big(\alpha(\hat{\pi})\big) \xrightarrow{i(\alpha(\hat{\pi}))} i(\tilde{\alpha})$$

cartesian over $1_{A^+_{\tilde{\alpha}}}$, where we use the fact that $\hat{\pi} : \hat{\pi} \longrightarrow 1_A$ in $\mathcal{C}/A$. Let us denote this map by φ. It is easily shown, using the fact that φ is cartesian, that there exists an inverse $\varphi^{-1} : i(\tilde{\alpha}) \longrightarrow i\big(\alpha(\hat{\pi})\big) \cdot \langle p_+, p_1\rangle$, and that φ and its inverse commute with the required structure to be maps in $\mathbf{CRes}(\mathbf{P}(-))$. Bearing this isomorphism in mind we may now correctly interpret elimination terms in the split fibration $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$.

Theorem 2.48. *If $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$ is a comprehension category with coherent identity types and fibred dependent products and sums, then the split fibration $\mathbf{P}_s(-) : \mathcal{P}_s \longrightarrow \mathcal{C}$ associated to $\mathbf{P}(-)$ is a split model of $\mathbb{T}_\omega$.*

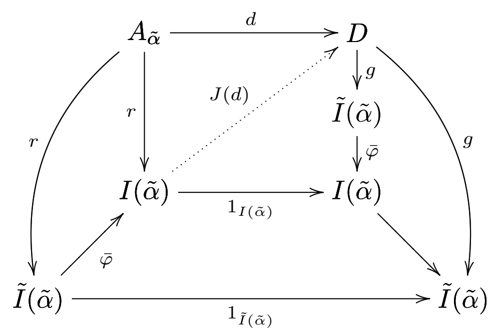
Proof. By the Corollary to Proposition 2.44 it suffices to show that the elimination, conversion and coherence rules for elimination terms is satisfied. Explicitly, the hypotheses of the elimination rule give us an element δ of $\mathbf{P}_s(\tilde{I}(\tilde{\alpha}))$ together with a commutative triangle:

$$\begin{array}{ccc} A_{\tilde{\alpha}} & \xrightarrow{d} & D \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle g} & \\ & \tilde{I}(\tilde{\alpha}) & \end{array}$$

where D now abbreviates $\big(\tilde{I}(\tilde{\alpha})\big)_{\delta(1_{\tilde{I}(\tilde{\alpha})})}$ and g denotes the associated projection. Thus, the isomorphism $\bar{\varphi} : \tilde{I}(\tilde{\alpha}) \longrightarrow I(\tilde{\alpha})$, obtained by applying χ to φ, yields the the appropriate data from which we have a section $J_{\tilde{\alpha},\bar{\varphi}\circ g}(d) : I(\tilde{\alpha}) \longrightarrow D$ of the composite $\bar{\varphi} \circ g : D \longrightarrow I(\tilde{\alpha})$. Precomposing with $\bar{\varphi}$ yields the interpretation of the elimination term:

$$[\![\Delta, x, y : T, z : \mathrm{Id}_T(x, y) \ \vdash\ J_{T,D}(d, x, y, z) : D(x, y, z)]\!] := J_{\tilde{\alpha},\bar{\varphi}\circ g}(d) \circ \bar{\varphi}.$$

Diagrammatically, we have



With this definition the elimination and conversion rules are clearly satisfied. The coherence rule then follows, taking into account the interpretation of the J terms just given, from Lemma 2.47.

Finally, although the proof in [**33**] does not consider the case of the split fibration associated to an arbitrary comprehension category with fibred dependent sums and products, the arguments given there, for the specific case of the codomain fibration of a locally cartesian closed category, generalize directly to this setting. □

COROLLARY 2.49. *If $\mathcal{C}$ is a locally cartesian closed category with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ and coherent identity types such that the pullback σ^* along a fibration $\sigma : A' \longrightarrow A$ between fibrant objects preserves maps in $\mathfrak{L}$, then the split fibration associated to $\partial_1 : \mathcal{C}_{\mathfrak{R}} \longrightarrow \mathcal{C}_{\mathrm{f}}$ is a split model of $\mathbb{T}_\omega$.*

In particular, we remark that Theorem 2.48 implies that, when $\mathbf{P}(-) : \mathcal{S} \longrightarrow \mathcal{C}$ is a coherent restriction of a weak factorization system in $\mathcal{C}$ which has coherent identity types and satisfies the pullback stability condition of Corollary 2.49, the associated split fibration is a split model of $\mathbb{T}_\omega$.

CHAPTER 3

Cocategories and Intervals

Approached from an abstract perspective, one of the most fundamental features of the category of spaces which makes a homotopy theory possible is the presence of an object I by which the notions of *paths* and appropriate *deformations* thereof may be defined. When dealing with topological spaces I is most naturally taken to be the unit interval; but there are other instances where the homotopy theory of a category is, in an appropriate sense, determined by a suitable interval. For example, the simplicial interval $I = \Delta[1]$ determines (in an appropriate sense) the classical model structure on the category of simplicial sets. The sense in which this holds has been recently clarified by the work of Cisinski [**15, 16**]. Another example comes from the natural model structure on **Cat** due to Joyal and Tierney [**47**], in which the role of I is played by the category **2** which is the free category on the graph consisting of two vertices and one edge between them. Finally, we also mention the work of Berger and Moerdijk [**8**] who employ Hopf intervals in order to study the homotopy theory of operads.

In this chapter, we introduce and study one notion of interval — namely, cocategory objects endowed with additional structure — which yields in the ambient category a useful notion of homotopy. The leading example is the category **Gpd** of small groupoids where the appropriate interval is the free connected groupoid with two distinct objects. Every such interval I gives rise to a 2-category structure on its ambient category and the initial sections of this chapter are devoted to introducing and studying the 2-categories which arise in this way. Along these lines, Section 3.2 is concerned with studying *Hurewicz fibrations* in a category equipped with an interval. In particular, whether certain maps are Hurewicz fibrations is related to the existence of additional algebraic structure on the interval. In Section 3.3 we address the question of when the 2-category structure induced by an interval is *representable* or *finitely complete* (in the 2-categorical sense). The principal result is Theorem 3.36 which provides a characterization of those intervals which give rise to representable 2-category structures. By a theorem due to Lack [**51**], any representable 2-category possesses a Quillen model structure. As a corollary of this result it follows that, whenever an interval I gives rise to a representable 2-category, the ambient category possesses a model structure in which the weak equivalences are exactly the "I-homotopy equivalences". Section 3.4 contains the main type theoretic results of this chapter. In particular, modifying a construction due to Street [**75**] to the present setting, whenever I possesses an interval there result (2-)monads on all of the slice (2-)categories of the ambient category. The strict algebras for these monads may be regarded as *split fibrations* induced by the interval. The main result of this chapter, Theorem 3.47, is that, when I is "invertible" in a suitable sense, restricting to these split fibrations yields a coherent model of type theory. Finally, in Section 3.5 we apply these results to show that categories of internal groupoids always admit coherent models of type theory. Moreover, when the ambient category is locally cartesian closed, the resulting model validates also the rules governing dependent products.

The results of this chapter regarding intervals should be of independent interest in homotopy theory and 2-dimensional category theory. Those regarding type theory extend the original Hofmann-Streicher [**35**] model to cover groupoids internal in categories other than the category of sets and also promise to yield further, more exotic, models of type theory.

Henceforth, unless otherwise stated, we assume that the ambient category $\mathcal{E}$ is a (finitely) bicomplete category which is also cartesian closed.

3.1. Cocategory objects

The definition of *internal cocategory* (or *cocategory object*) in $\mathcal{E}$ is exactly dual to the definition of internal categories. However, in order to fix notation and provide a bit more motivation for this concept we will rehearse the definition in full. For us, the principal motivation of the definition of cocategories is that a cocategory in $\mathcal{E}$ provides (more than) sufficient data to define a reasonable notion of homotopy in $\mathcal{E}$. In thinking about cocategory objects it is often instructive to view them as analogous to the unit interval in the category of topological spaces. However, we should emphasize that the unit interval is *not* a cocategory object (e.g., cocomposition is only associative up to homotopy) as the reader can easily verify.

3.1.1. The definition. Rather than rehearse the definition of internal categories and force the reader to dualize, we state the definition of cocategory object directly.

Definition 3.1. An **internal cocategory** or **cocategory object** $\mathbb{C}$ in a category $\mathcal{E}$ with pushouts consists of the following data.

- **Objects:** C_0 (**object of coobjects**), C_1 (**object of coarrows**) and C_2 (**object of cocomposable coarrows**).
- **Arrows:** $\bot, \top : C_0 \rightrightarrows C_1$ (**bottom** and **top**), $i : C_1 \longrightarrow C_0$ (**coidentities**), $\downarrow, \uparrow : C_1 \rightrightarrows C_2$ (**initial segment** and **final segment**), and $\star : C_1 \longrightarrow C_2$ (**cocomposition**).

Satisfying the following list of requirements.

- The following square is a pushout:

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \bot\ } & C_1 \\ {\scriptstyle\top}\big\downarrow & & \big\downarrow{\scriptstyle\uparrow} \\ C_1 & \xrightarrow[\ \downarrow\ ]{} & C_2. \end{array} \tag{9}$$

- The following diagram commutes:

$$\begin{array}{ccccc} C_0 & \xrightarrow{\ \bot\ } & C_1 & \xleftarrow{\ \top\ } & C_0 \\ & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle i} & \diagup\!\!\diagup & \\ & & C_0. & & \end{array} \tag{10}$$

- The following diagrams commute:

$$\begin{CD} C_0 @>{\bot}>> C_1 \\ @V{\bot}VV @VV{\star}V \\ C_1 @>>{\downarrow}> C_2, \end{CD} \qquad \text{and} \qquad \begin{CD} C_0 @>{\top}>> C_1 \\ @V{\top}VV @VV{\star}V \\ C_1 @>>{\uparrow}> C_2. \end{CD} \tag{11}$$

- The following **co-unit laws** hold:

$$\begin{array}{ccccc} & & C_1 & & \\ & \swarrow\!\!= & \downarrow{\scriptstyle \star} & =\!\!\searrow & \\ C_1 & \xleftarrow[{[\bot\circ i, 1_{C_1}]}]{} & C_2 & \xrightarrow[{[1_{C_1}, \top\circ i]}]{} & C_1. \end{array} \tag{12}$$

- Finally, let the object C_3 (**the object of cocomposable triples**) be defined as the the following pushout:

$$\begin{CD} C_1 @>{\uparrow}>> C_2 \\ @V{\downarrow}VV @VV{q_1}V \\ C_2 @>>{q_0}> C_3, \end{CD}$$

and observe that (by the dual of the "two-pullbacks" lemma) C_3 may be alternatively described as the following pushout:

$$\begin{CD} C_0 @>{\downarrow\circ\bot}>> C_2 \\ @V{\top}VV @VV{q_0}V \\ C_1 @>>{r_0}> C_3, \end{CD}$$

where $r_0 := q_1\circ\downarrow$ or as the pushout of $\uparrow\circ\top$ along $\bot$:

$$\begin{CD} C_0 @>{\uparrow\circ\top}>> C_2 \\ @V{\bot}VV @VV{q_1}V \\ C_1 @>>{r_1}> C_3, \end{CD}$$

where $r_1 := q_0\circ\uparrow$.

The **coassociative law** then states that the following diagram commutes:

$$\begin{CD} C_1 @>{\star}>> C_2 \\ @V{\star}VV @VV{[r_0, q_0\circ\star]}V \\ C_2 @>>{[q_1\circ\star, r_1]}> C_3. \end{CD} \tag{13}$$

Several comments on this definition are in order. Although some of the nomenclature employed is at this point unfamiliar it is justified below when we explain our intended interpretation (also, it allows us to avoid such repugnant locutions as "cocodomain"). In particular, $\bot$ is the dual of a domain map, $\top$ is the dual of a codomain map, and $\downarrow$ and $\uparrow$ are dual to the first and second projections, respectively.

Proposition 3.2. *If $\mathbb{C}$ is a cocategory object in $\mathcal{E}$, then, for any object D of $\mathcal{E}$, the slice category $\mathcal{E}/D$ also possesses a cocategory object $\mathbb{C}_D$. Moreover, if $f : B \longrightarrow D$ is an arrow in $\mathcal{E}$, then $\Delta_f : \mathcal{E}/D \longrightarrow \mathcal{E}/B$ preserves the cocategory structure.*

Proof. The cocategory object $\mathbb{C}_D$ is given by forming the product with D. I.e., the object of coobjects is simply the projection $\Delta_D(C_0)$ given by $D \times C_0 \longrightarrow D$. Since $\mathcal{E}$ is cartesian closed all of the relevant pushout diagrams are preserved. Since all of the other data is equational it is clear that this is a cocategory object in $\mathcal{E}/D$. It is also clear that this structure is preserved by pullback. □

In general, if $\mathbb{C} = (C_0, C_1, C_2)$ is a cocategory object and A is any object of $\mathcal{E}$, then $A \times \mathbb{C} = (A \times C_0, A \times C_1, A \times C_2)$ is also a cocategory object in $\mathcal{E}$. Moreover, if $\mathbb{C}$ is a cocategory object in $\mathcal{E}$ and A is any object, then one obtains an internal category $A^{\mathbb{C}}$ by exponentiation.

3.1.2. Cogroupoids. We are often interested in cocategory objects which possess additional structure. In particular, the cocategories with which we will be predominately concerned are all examples of cogroupoids.

Definition 3.3. A cocategory $\mathbb{C}$ in $\mathcal{E}$ is a **cogroupoid** if there exists a map $\rho : C_1 \longrightarrow C_1$ such that the following diagrams commute:

$$\begin{array}{ccccc} C_0 & \xrightarrow{\bot} & C_1 & \xleftarrow{\top} & C_0 \\ & {\scriptstyle\top}\searrow & \downarrow{\scriptstyle\rho} & \swarrow{\scriptstyle\bot} & \\ & & C_1 & & \end{array}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{\star} & C_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle[\rho, 1_{C_1}]} \\ C_0 & \xrightarrow[\top]{} & C_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_1 & \xrightarrow{\star} & C_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle[1_{C_1}, \rho]} \\ C_0 & \xrightarrow[\bot]{} & C_1 \end{array}$$

The map $\rho : C_1 \longrightarrow C_1$ is called the **coinverse map**.

In particular, if $\mathbb{C}$ is a cogroupoid in $\mathcal{E}$ and A is an object of $\mathcal{E}$, then A^{C_0} is the object of objects of an internal groupoid taking its structure from $\mathbb{C}$.

Definition 3.4. A cocategory object $\mathbb{C}$ in a category $\mathcal{E}$ is **pointed** if the object C_0 of coobjects is the terminal object of $\mathcal{E}$. $\mathbb{C}$ is **symmetric** if there exists a map $\sigma : C_1 \longrightarrow C_1$ (**the symmetry map**) such that $\sigma(\bot) = \top$ and $\sigma(\top) = \bot$. Finally, $\mathbb{C}$ is a **(strict) interval object** if it is both pointed and symmetric. When $\mathbb{C}$ is an interval object we write I for C_1 and I_2 for C_2. When an interval I is a cogroupoid and its coinverse map is also its symmetry, I is said to be **invertible**.

The reader should see Appendix C for a "schematic" illustration of the definition of interval object.

Example 3.5. (1) Every object A of a category $\mathcal{E}$ determines a cocategory object given by setting $A_i := A$ for $i = 0, 1, 2$ and defining all of the structure maps to be the identity 1_A. This is said to be the **discrete**

cocategory on A. When A is the terminal object 1 of $\mathcal{E}$ this cocategory is the terminal object in the categories $\mathbf{Cocat}(\mathcal{E})$ and $\mathbf{Cocat}_\bullet(\mathcal{E})$ of cocategory objects in $\mathcal{E}$ and pointed cocategory objects in $\mathcal{E}$, respectively (which have as arrows tuples of maps commuting with all of the cocategory structure).

(2) Assuming $\mathcal{E}$ possesses a terminal object and all finite coproducts, then there is a cocategory object $\mathbb{C}$ in $\mathcal{E}$ obtained by setting $\mathbb{C}_0 := 1$ and $\mathbb{C}_1 := 1+1$, with $\bot$ and $\top$ the coproduct injections. This is said to be the **codiscrete interval in** $\mathcal{E}$. This is the initial object in $\mathbf{Cocat}_\bullet(\mathcal{E})$. As a special case of this, we note that a topos $\mathcal{E}$ is Boolean if and only if its subobject classifier Ω is (the object of coarrows of) an invertible interval.

(3) In $\mathbf{Cat}$ the category $\mathbf{2}$ which is the free category on the graph consisting of two vertices and one edge between them is a cocategory object. Similarly, the free groupoid $\mathbf{I}$ on $\mathbf{2}$ is an invertible interval in $\mathbf{Cat}$ and in $\mathbf{Gpd}$ with the following structure:

$$\bot \underset{u}{\overset{d}{\rightleftarrows}} \top$$

such that u and d are inverse and where $\bot, \top : \mathbf{1} \rightrightarrows \mathbf{I}$ are the obvious functors. $\mathbf{I}_2$ is then the result of gluing $\mathbf{I}$ to itself along the top and bottom:

$$\bot \underset{u_\downarrow}{\overset{d_\downarrow}{\rightleftarrows}} \mu \underset{u_\uparrow}{\overset{d_\uparrow}{\rightleftarrows}} \top.$$

Cocomposition $\star : \mathbf{I} \longrightarrow \mathbf{I}_2$ is the functor given by $\star(\bot) := \bot$ and $\star(\top) := \top$, and the initial and final segment functors are defined in the evident way. Finally, $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is defined by $\sigma(\bot) := \top$ and $\sigma(\top) := \bot$. In fact, we will see below that if $\mathcal{E}$ is any finitely bicomplete category, then $\mathbf{Gpd}(\mathcal{E})$ possesses an invertible interval object (which is essentially described as above).

(4) Let $\mathbf{Ch}_+$ be the category of (non-negatively graded) chain complexes of abelian groups, then there exists a cocategory object $\mathbb{I}$ in $\mathbf{Ch}_+$ which we now describe. $\mathbb{I}^0$ is the chain complex which consists of $\mathbb{Z}$ in degree 0 and is 0 in all other degrees. $\mathbb{I}^1$ is given by

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & 0 & \xrightarrow{d} & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ & & & & x & \longmapsto & (x, -x), \end{array}$$

where x is an arbitrary integer. $\mathbb{I}^2$ consists of

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & 0 & \xrightarrow{d} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\ & & & & (x, y) & \longmapsto & (x, y - x, -y), \end{array}$$

for integers x and y. $\downarrow$ and $\uparrow$ are the left and right inclusions (in both non-trivial degrees), respectively. Similarly, $\bot$ and $\top$ are the left and right inclusions, respectively. $i : \mathbb{I}^1 \longrightarrow \mathbb{I}^0$ is the zero-map. Finally, cocomposition $\star : \mathbb{I}^1 \longrightarrow \mathbb{I}^2$ is given defined as follows:

$$\star_1(x) := (x, x)$$
$$\star_0(x, y) := (x, 0, y),$$

for integers x and y. There is also a symmetry map $\sigma : \mathbb{I}^1 \longrightarrow \mathbb{I}^1$ defined in the obvious way. These structures describe the homological structure of the unit interval together with the result of pasting the unit interval to itself (cf. [**57**]).

The topological unit interval I in **Top** fails to satisfy the co-associativity and co-unit laws on the nose (they *are* satisfied up to homotopy) and is therefore not an interval in this sense.

3.1.3. Homotopy. The first way in which we make use of the existence of an interval object in $\mathcal{E}$ is to define the notion of homotopy. The notion of homotopy we obtain is ubiquitous in homotopy theory (cf. the classical notion of simplicial homotopy [**22**] or its various generalizations [**66, 16**]).

Definition 3.6. Let $\mathbb{C}$ be a cocategory object in $\mathcal{E}$. A **homotopy with respect to** $\mathbb{C}$ (or $\mathbb{C}$-**homotopy**) $\eta : f \Longrightarrow g$ between two maps $f, g : A \rightrightarrows B$ in $\mathcal{E}$ is a map $\eta : A \times C_1 \longrightarrow B$ such that the following triangles commute:

$$\begin{array}{ccccc} A \times C_0 & \xrightarrow{A \times \bot} & A \times C_1 & \xleftarrow{A \times \top} & A \times C_0 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle g} & \\ & & B & & \end{array}$$

When $\mathbb{C}$ is pointed we often write I for $\mathbb{C}_1$ and write A_0 for $\langle 1_A, \bot \rangle$ and A_1 for $\langle 1_A, \top \rangle$ so that the above becomes

$$\begin{array}{ccccc} A & \xrightarrow{A_0} & A \times I & \xleftarrow{A_1} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle g} & \\ & & B & & \end{array}$$

When an object A is fixed, we also often write $\partial_0, \partial_1 : A^I \rightrightarrows A$ for the maps induced by $\bot : 1 \longrightarrow I$ and $\top : 1 \longrightarrow I$, respectively.

Example 3.7. The cocategory objects from Example 3.5 give rise to the following notions of homotopy:

(1) The terminal cocategory object 1 generates the most coarse notion of homotopy. I.e., there exists a homotopy between maps f and g with respect to this cocategory if and only if f and g are identical.
(2) The initial pointed cocategory object $1 + 1$ generates the finest relation of homotopy: all maps are homotopic. Indeed, given maps f and g there exists, with respect to this cocategory, a unique homotopy $f \Longrightarrow g$.
(3) In **Cat**, homotopies $f \Longrightarrow g$ are in bijective correspondence with natural transformations $f \Longrightarrow g$ and similarly in **Gpd** with respect to **I**.
(4) In $\mathbf{Ch}_+$, $\mathbb{I}$ induces the usual notion of chain homotopy.

REMARK 3.8. Notice that, when $\mathbb{C}$ is symmetric, any homotopy $\eta : f \Longrightarrow g$ between maps $f, g : A \rightrightarrows B$ induces a homotopy $\bar{\eta} : g \Longrightarrow f$ by composing with the symmetry:

$$\bar{\eta} := \eta \circ (1_A \times \sigma).$$

In what follows we will often assume, for the sake of presentational clarity, that the cocategory objects with which we deal are pointed. Note though that in many cases this assumption can be dropped without affecting the validity of the claims made. Nonetheless, nearly all of the examples we consider are pointed and every cocategory gives rise to a pointed one in the slice category.

3.1.4. Induced 2-categorical structure. Assume that $\mathcal{E}$ possesses a pointed cocategory object I. Then $\mathcal{E}$ can be equipped with the structure of a 2-category as follows. First, the 0-cells of $\mathcal{E}$ are simply the objects of $\mathcal{E}$ and the 1-cells are the arrows of $\mathcal{E}$. We then define

$$\mathcal{E}(A, B)_1 := \mathcal{E}(A \times I, B),$$

which endows $\mathcal{E}(A, B)$ with the structure of a category since B^I is an internal category in $\mathcal{E}$. Explicitly, given α in $\mathcal{E}(A, B)_1$, the domain of α is the arrow $\alpha \circ A_0 : A \longrightarrow B$ and the codomain is the arrow $\alpha \circ A_1 : A \longrightarrow B$. Given arrows $\eta : f \Longrightarrow g$ and $\gamma : g \Longrightarrow h$ in $\mathcal{E}(A, B)$, the vertical composite $f \Longrightarrow h$ is defined as follows. Since $\mathcal{E}$ is cartesian closed the following square is a pushout:

$$\begin{array}{ccc} A \times 1 & \xrightarrow{1_A \times \top} & A \times I \\ {\scriptstyle 1_A \times \bot}\downarrow & & \downarrow{\scriptstyle 1_A \times \downarrow} \\ A \times I & \xrightarrow[1_A \times \uparrow]{} & A \times I_2. \end{array}$$

Because $\eta \circ A_1 = \gamma \circ A_0$, there exists a canonical map $\delta : A \times I_2 \longrightarrow B$ such that:

$$\delta \circ (1_A \times \uparrow) = \gamma, \text{ and}$$
$$\delta \circ (1_A \times \downarrow) = \eta.$$

Recalling the third clause from the definition of cocategory object, it is easily verified that $\delta \circ (1_A \times \star)$ is the required vertical composite $(\gamma \cdot \eta) : f \Longrightarrow h$. It is convenient to introduce notation for the "mediating map" δ. As such, we write $c[\gamma, \eta]$ instead of δ and observe that $(\gamma \cdot \eta) = c[\gamma, \eta] \circ (1_A \times \star)$. I.e., $c[\gamma, \eta]$ is the composition $(\gamma \cdot \eta)$ prior to being "fused" or "merged" by precomposition with $(1_A \times \star)$.

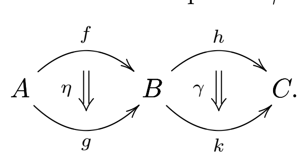
REMARK 3.9. Given homotopies $\alpha, \beta : A \times I \longrightarrow B$ for which the vertical composite $(\beta \cdot \alpha)$ exists and a map $g : D \longrightarrow A$, the following equation holds:

$$c[\beta, \alpha] \circ (g \times 1_{I_2}) = c[\beta \circ (g \times 1_I), \alpha \circ (g \times 1_I)].$$

The induced composition functor

$$\mathcal{E}(A, B) \times \mathcal{E}(B, C) \longrightarrow \mathcal{E}(A, C)$$

is then given by defining the horizontal composite $\gamma * \eta$ of a pair of 2-cells



to be the composite

$$A \times I \xrightarrow{1_A \times \Delta} A \times I \times I \xrightarrow{\eta \times 1_I} B \times I \xrightarrow{\gamma} C,$$

where $\Delta : I \rightarrowtail I \times I$ is the diagonal. This is clearly a homotopy $h \circ f \Longrightarrow k \circ g$. The proof of the following proposition is essentially well known (cf. [**78**]) and follows from the Yoneda lemma:

Proposition 3.10. *Suppose I is an interval object in $\mathcal{E}$. Then $\mathcal{E}$ is a 2-category with the same objects and arrows, and with 2-cells the homotopies.*

Remark 3.11. Note that, by Yoneda, *every* 2-category embedds fully into one in which the 2-category structure is given in this way by an interval.

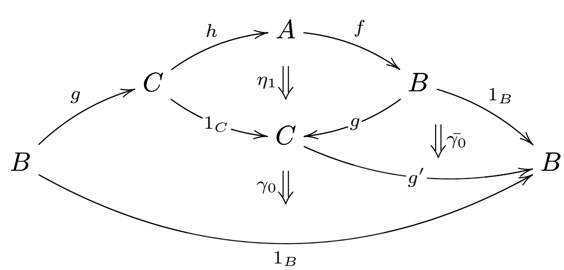
In light of this 2-categorical structure on $\mathcal{E}$ we can define a reasonable notion of "homotopy equivalence" as follows.

Definition 3.12. A map $f : A \longrightarrow B$ is a **lax homotopy equivalence (with respect to I)** if and only if there exists a map $f' : B \longrightarrow A$ together with homotopies $f \circ f' \Longrightarrow 1_B$ and $f' \circ f \Longrightarrow 1_A$. A map $f : A \longrightarrow B$ is a **homotopy equivalence (with respect to I)** if it is a lax homotopy equivalence for which the associated homotopies are invertible.

I.e., the homotopy equivalences are defined to be precisely the usual (strong) categorical equivalences in $\mathcal{E}$ regarded as a 2-category. With these definitions we have the following corollary to Proposition 3.10.

Corollary 3.13. *If $\mathcal{E}$ is a finitely bicartesian cartesian closed category with an interval object I, then the (lax) homotopy equivalences satisfy the "three-for-two" axiom.*

Proof. Let maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be given. First, assume $g \circ f$ and g are lax homotopy equivalences. As such, there exist maps $g' : B \longrightarrow C$ and $h : C \longrightarrow A$ together with homotopies $\gamma_0 : g' \circ g \Longrightarrow 1_B$, $\gamma_1 : g \circ g' \Longrightarrow 1_C$, $\eta_0 : h \circ (g \circ f) \Longrightarrow 1_A$ and $\eta_1 : (g \circ f) \circ h \Longrightarrow 1_C$. Define $f' := h \circ g$ and observe that $\eta_0 : f' \circ f \Longrightarrow 1_A$. To construct the other required 2-cell $f \circ f' \Longrightarrow 1_B$ note that, since $(\mathcal{E}, I)$ is a 2-category, we need only provide a pasting-diagram as follows:



where $\bar{\gamma_0} : 1_B \Longrightarrow g' \circ g$ is the "reverse homotopy" as discussed in Remark 3.8. The other two cases are similarly verified. □

Using the categories $\mathcal{E}(A, B)$ it is possible to provide an alternative characterization of when a pointed cocategory is a cogroupoid.

Proposition 3.14. *Assume I is a pointed cocategory object in $\mathcal{E}$, then the following are equivalent:*

(1) *For every object A and B of $\mathcal{E}$, the category $\mathcal{E}(A, B)$ is a groupoid.*
(2) *I is an invertible interval object.*

PROOF. Suppose that (1) holds to prove (2). Observe that the identity map 1_I is an arrow $\bot \Longrightarrow \top$ in $\mathcal{E}(I_0, I)$. Therefore there exists an inverse $\rho : \top \Longrightarrow \bot$. It is straightforward to verify that, with this definition, ρ is a coinverse map for I. The converse is trivial. □

3.2. Join, Meet and Hurewicz Fibrations

For topological spaces there exists a useful notion of fibration due to Hurewicz (cf. [**73**]) which is formulated in terms of a lifting condition with respect to the unit interval I. Namely, a map $f : X \longrightarrow Y$ of spaces is a Hurewicz fibration provided that, for any space Z, if

$$\begin{array}{ccc} Z \times \{0\} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z \times I & \longrightarrow & Y \end{array}$$

commutes, then there exists a diagonal filler. In this section we will consider a notion of Hurewicz fibration in $\mathcal{E}$ formulated as an analogous lifting property with respect to the interval object I. Because one of the examples we have in mind is **Cat** we will, however, require instead that f possess the lifting property with respect to the inclusion of the opposite end $Z \times \{1\}$ of the cylinder. We will also introduce in this section operations of "join" and "meet" on the interval I which will arise later in connection with the interpretation of type theory. The main result of this section is Proposition 3.19 which establishes an equivalence between the existence of joins and, for all A, the map $A^I \longrightarrow A \times A$ induced by ∂_0 and ∂_1 being a Hurewicz fibration.

3.2.1. Hurewicz fibrations. Explicitly, Hurewicz fibrations with respect to the interval object I are defined as follows:

DEFINITION 3.15. A map $p : E \longrightarrow B$ in $\mathcal{E}$ is a **Hurewicz fibration** for the interval I if for any object A, and maps $f : A \longrightarrow E$ and $h : A \times I \longrightarrow B$ there exists a diagonal filler:

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle A_\top}\downarrow & \nearrow & \downarrow p \\ A \times I & \xrightarrow[h]{} & B. \end{array}$$

I.e., $\mathfrak{I} \pitchfork p$ where $\mathfrak{I}$ is the collection of all maps of the form $A_\top$ for A an object of $\mathcal{E}$. A map p which possesses the analogous lifting property with respect to maps of the form $A_\bot$ is said to be a **Hurewicz opfibration**.

SCHOLIUM 3.16. *The collection of Hurewicz fibrations in $\mathcal{E}$ has the following properties:*

(1) *Hurewicz fibrations are stable under composition. I.e., if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are both Hurewicz fibrations, then so is the composite $g \circ f$. Moreover, all isomorphisms are Hurewicz fibrations.*
(2) *The collection of Hurewicz fibrations is stable under retracts.*

(3) *The collection of Hurewicz fibrations is stable under pullback along arbitrary maps. I.e., in any pullback square:*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \longrightarrow & B \end{array}$$

if f is a Hurewicz fibration, then so is f'.

(4) *For every object A, the canonical map $A \longrightarrow 1$ is a Hurewicz fibration.*

As Lemma 3.16 suggests the Hurewicz fibrations in this abstract setting already possess useful properties. The Hurewicz fibrations can alternative be charactered in terms of the 2-categorical structure of $\mathcal{E}$.

Scholium 3.17. *A map $f : A \longrightarrow B$ is a Hurewicz fibration if and only if, for every object E, the induced functor*

$$\mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)$$

has the property that, for every arrow $\phi : g \Longrightarrow f_*(h)$ in $\mathcal{E}(E, B)$, there exists an arrow $\phi' : g' \Longrightarrow h$ in $\mathcal{E}(E, A)$ such that $f_*(\phi') = \phi$.

3.2.2. Join. We now establish necessary and sufficient conditions under which the factorization

$$\begin{array}{ccc} A & \xrightarrow{\quad r \quad} & A^I \\ & {\scriptstyle \Delta}\searrow \quad \swarrow{\scriptstyle \iota} & \\ & A \times A & \end{array}$$

of the diagonal induced by the interval object I in $\mathcal{E}$ consists of a homotopy equivalence followed by a Hurewicz fibration

Definition 3.18. An object A of $\mathcal{E}$ is **lax contractible** if and only if the canonical map $!_A : A \longrightarrow 1$ is a lax homotopy equivalence. A subobject $m : S \rightarrowtail A$ is a **lax strong deformation retract of** A if there exists a retraction $r : A \longrightarrow S$ and a homotopy $\eta : m \circ r \Longrightarrow 1_A$ such that the following diagram commutes:

$$\begin{array}{ccc} S \times I & \xrightarrow{m \times 1_I} & A \times I \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \eta} \\ S & \xrightarrow[m]{} & A. \end{array}$$

We obtain the ordinary (non-lax) versions of these notions by requiring that the homotopies in question be invertible.

Proposition 3.19. *The following are equivalent:*

(1) *For any object A of $\mathcal{E}$, the map $\iota : A^I \longrightarrow A \times A$ defined by $\iota := \langle A^{\perp}, A^{\top} \rangle$ is a Hurewicz fibration.*

(2) *The interval I is lax contractible in the strong sense that the map $\top : 1 \longrightarrow I$ is a lax strong deformation retract of I.*

(3) *There exists a "binary operation"* $\veebar : I \times I \longrightarrow I$ *such that the following equations hold in the internal language:*

$$\begin{aligned} x \veebar \top &= \top \\ &= \top \veebar x, \\ x \veebar \bot &= x, \end{aligned}$$

for $x : I$*. I.e.,* $\veebar$ *is such that the following diagrams commute:*

$$\begin{array}{ccc} I & \xrightarrow{I_\top} & I \times I \\ {\scriptstyle I^\top}\big\downarrow & \searrow^{\top\circ !} & \big\downarrow{\scriptstyle \veebar} \\ I \times I & \xrightarrow[\veebar]{} & I, \end{array}$$

and:

$$\begin{array}{ccc} I & \xrightarrow{I_\bot} & I \times I \\ & {\scriptstyle 1_I}\searrow \quad \swarrow{\scriptstyle \veebar} & \\ & I, & \end{array}$$

where $I^\top := \langle \top\circ !, 1_I \rangle$*.*

Proof. (2) and (3) are clearly equivalent. To see that (1) implies (3) notice that since ι is a fibration there exists a lift $\lambda : I \longrightarrow I^I$ as indicated in the following diagram:

$$\begin{array}{ccc} 1 & \xrightarrow{r\circ\top} & I^I \\ {\scriptstyle \top}\big\downarrow & \overset{\lambda}{\nearrow} & \big\downarrow{\scriptstyle \iota} \\ I & \xrightarrow[I_\top]{} & I \times I. \end{array}$$

The desired map $\veebar$ is then defined to be the exponential transpose of λ.

To see that (3) implies (1), assume given maps $\varphi : X \longrightarrow A^I$ and $f : X \longrightarrow A \times A$ together with a 2-cell $\gamma : f \Longrightarrow \iota_*(\varphi)$. Diagrammatically:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & A^I \\ {\scriptstyle X_\top}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ X \times I & \xrightarrow[\gamma]{} & A \times A \end{array} \tag{14}$$

Let $\alpha : X \times I \times I \longrightarrow A$ be the following composite

$$X \times I \times I \xrightarrow{1_X \times \veebar} X \times I \xrightarrow{\pi_0 \circ \gamma} A.$$

Similarly, let β and δ be the following composites:

$$X \times I \times I \xrightarrow{1_X \times 1_I \times \sigma} X \times I \times I \xrightarrow{1_X \times \veebar} X \times I \xrightarrow{\pi_1 \circ \gamma} A,$$

$$X \times I \times I \xrightarrow{\langle \pi_0, \pi_2 \rangle} X \times I \xrightarrow{\varphi} A,$$

respectively. Observe that *qua* 2-cells in $\mathcal{E}(X \times I, A)$, these three arrows are composable in the sense that $(\beta \cdot \delta \cdot \alpha)$ exists. To see that this is the case it is convenient to argue using the internal language as follows:

$$\begin{aligned}\alpha(x,t,\top) &= \pi_0 \circ \gamma(x, t \veebar \top)\\ &= \pi_0 \circ \gamma(x,\top)\\ &= \varphi(x,\bot)\\ &= \delta(x,t,\bot),\end{aligned}$$

for $t : I$ and $x : X$. Similarly,

$$\begin{aligned}\beta(x,t,\bot) &= \pi_1 \circ \gamma(x, t \veebar \sigma(\bot))\\ &= \pi_1 \circ \gamma(x,\top)\\ &= \varphi(x,\top)\\ &= \delta(x,t,\top).\end{aligned}$$

Define $\varphi' : X \times I \longrightarrow A^I$ to be the exponential transpose of $(\beta \cdot \delta \cdot \alpha)$. We claim that φ' is the required lift.

First, that $\iota_*(\varphi') = \gamma$ is straightforward using the definition of δ. Secondly, to see that $\varphi' \circ X_\top = \varphi$ notice that

$$\big((\beta \cdot \delta) \cdot \alpha\big) \circ (X_\top \times 1_I) = c[(\beta \cdot \delta) \circ (X_\top \times 1_I), \alpha \circ (X_\top \times 1_I)] \circ (1_X \times \star).$$

Moreover,

$$\alpha \circ (X_\top \times 1_I) = \pi_0 \circ \gamma \circ (1_X \times \top \circ !_I),$$

which is the identity 2-cell $1_{\pi_0 \circ \gamma \circ X_\top} : \pi_0 \circ \gamma \circ X_\top \Longrightarrow \pi_0 \circ \gamma \circ X_\top$. A similar calculation shows that $\beta \circ (X_\top \times 1_I)$ is the identity 2-cell $1_{\pi_1 \gamma \circ X_\top}$. Combining this with the foregoing we obtain:

$$\begin{aligned}\big((\beta \cdot \delta) \cdot \alpha\big) \circ (X_\top \times 1_I) &= (\beta \cdot \delta) \circ (X_\top \times 1_I)\\ &= c[\beta \circ (X_\top \times 1_I), \delta \circ (X_\top \times 1_I)] \circ (1_X \times \star)\\ &= \delta \circ (X_\top \times 1_I)\\ &= \varphi.\end{aligned}$$

Therefore $\varphi' \circ X_\top = \varphi$, as required. □

Observe that the proof of Proposition 3.19 uses the fact that the interval is *strict* in the sense that all of the cocategory equations commute "on the nose" and not up to the existence of higher dimensional isomorphisms. The intuition behind this proof is that $\veebar : I \times I \longrightarrow I$ is a sort of **join** or maximum operation on the interval. I.e., we think of the action of $\veebar$ as taking the maximum:

$$x \veebar y := \max\{x, y\},$$

for x, y real numbers in the closed unit interval. Of course, this intuition should not be taken too seriously since $\veebar$ need not be commutative. If I is an interval which satisfies the equivalent conditions from Proposition 3.19, then we say that I has **joins**.

COROLLARY 3.20. *If the interval I in $\mathcal{E}$ has joins then, for any object A, the "constant loop" (or "reflexivity") map $r : A \longrightarrow A^I$ is a lax strong deformation retract of A^I.*

PROOF. Clearly r is a section of $A^{\top} : A^I \longrightarrow A$. The required homotopy $\eta : r \circ A^{\top} \Longrightarrow 1_{A^I}$ is constructed as the transpose of the composite:

$$A^I \times I \times I \xrightarrow{1_{A^I} \times \underline{\vee}} A^I \times I \xrightarrow{\text{ev}} A.$$

Then η is a homotopy $r \circ A^{\top} \Longrightarrow 1_{A^I}$ by definition of $\underline{\vee}$. Finally, η is a strong deformation retract since $\text{ev} \circ (r \times 1_I) = \pi_A$. □

3.2.3. Meet. There is a dual development to that of Section 3.2.2 for Hurewicz opfibrations. Namely, the map $\iota : A^I \longrightarrow A \times A$ being a Hurewicz opfibration is equivalent to the existence of a **meet** or minimum operation $\overline{\wedge} : I \times I \longrightarrow I$. Explicitly, we have the following proposition, the proof of which is dual to that of Proposition 3.19:

PROPOSITION 3.21. *The following are equivalent:*

1. *For any object A of $\mathcal{E}$, the map $\iota : A^I \longrightarrow A \times A$ is a Hurewicz opfibration.*
2. *The map $\bot : 1 \longrightarrow I$ is a lax strong deformation retract of I.*
3. *There exists a map $\overline{\wedge} : I \times I \longrightarrow I$ such that the following equations hold in the internal language:*

$$\begin{aligned} x \mathbin{\overline{\wedge}} \bot &= \bot \\ &= \bot \mathbin{\overline{\wedge}} x, \\ x \mathbin{\overline{\wedge}} \top &= x, \end{aligned}$$

for $x : I$.

When I is itself a cogroupoid existence of such a meet operation is equivalent to the existence of a join.

SCHOLIUM 3.22. *If I is invertible, then a map $f : X \longrightarrow Y$ is a Hurewicz fibration if and only if it is a Hurewicz opfibration.*

3.3. Representability

In this section we study the important 2-categorical notion of *representability* (*finite completeness*) [**75, 28**] in the context of the 2-category structure on $\mathcal{E}$ induced by an interval I. When $\mathcal{E}$ is representable, the interval I can be shown to possess additional useful structure. For example, such an I comes equipped with distinguished meet and join operations which satisfy additional equations. The main result of this section, Theorem 3.36 provides a characterization of those intervals I for which the induced 2-category structure on $\mathcal{E}$ is representable.

3.3.1. Conical limits and representability. Recall (cf. [**76, 50**]) that, when $\mathcal{K}$ is a 2-category and $F : \mathcal{C} \longrightarrow \mathcal{K}$ is a functor such that $\mathcal{C}$ is itself a category — regarded as a 2-category in which the only 2-cells are identities, the **conical limit of** F is given by an object L of $\mathcal{E}$ such that there exists an isomorphism

$$\mathcal{K}(X, L) \cong [\mathcal{C}, \mathcal{K}](\Delta(X), F) \tag{15}$$

in **Cat** which is 2-natural in X. I.e., L is the weighted limit of F with weight the functor $\Delta(\mathbf{1}) : \mathcal{C} \longrightarrow \mathbf{Cat}$. If the 2-limit of F exists, the object L is isomorphic to the usual 1-dimensional limit $\varprojlim F$ in $\mathcal{K}$ (cf. [**50**, Section 3.8]). Accordingly, we will employ the same notation for both the conical limit and the regular 1-dimensional limit. When $\mathcal{C}$ is the free category on

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

we say that $\varprojlim F$, if it exists, is a **2-pullback**. By dualizing, the notion of **conical colimit** is similarly obtained.

The following definition is due to Gray [**28**] and Street [**75**]:

Definition 3.23. A 2-category $\mathcal{K}$ is said to be **representable** if and only if $\mathcal{K}$ has 2-pullbacks and, for each object A of $\mathcal{K}$, the cotensor $\mathbf{2} \pitchfork A$ with the category $\mathbf{2}$ exists.

In much of the literature on 2-category theory representability is also called *finite completeness* (cf. [**76**]). We would like to make some observations regarding the connection between the 2-categorical structure induced by a cocategory object $\mathbb{C}$ and this notion of representability. As a first step, we investigate the existence of certain weighted limits in our ambient category $\mathcal{E}$.

Lemma 3.24. *If $\mathcal{C}$ is a (small) category, then every functor $F : \mathcal{C} \longrightarrow \mathcal{E}$ possesses a conical limit.*

Proof. As remarked above, if the conical limit of F exists, then it will be isomorphic to $\varprojlim F$. As such, let L be $\varprojlim F$. Then, in **Cat**, there exists a functor $\mathcal{E}(X, L) \longrightarrow [\mathcal{C}, \mathcal{E}](\Delta(X), F)$ defined by sending an object $f : X \longrightarrow L$ to the 2-natural transformation $\hat{f}$ which has as its component at an object C of $\mathcal{C}$ the composite

$$X \xrightarrow{\ f\ } L \xrightarrow{\ p_C\ } FC,$$

where p_{-} is the cone $\Delta(L) \Longrightarrow F$. Because $\mathcal{C}$ possesses only trivial 2-cells this is 2-natural. Likewise, a 2-cell $\alpha : f \Longrightarrow g$ is sent to $(p_C * \alpha)$ at the object C of $\mathcal{C}$. This yields a modification $\hat{\alpha}$ since

$$\begin{aligned} Fh * \hat{\alpha}_C &= Fh * p_C * \alpha \\ &= p_D * \alpha \ = \ \hat{\alpha}_D \end{aligned}$$

for any $h : C \longrightarrow D$ in $\mathcal{C}$. Functoriality is by the interchange law.

Going the other way, a 2-natural transformation $\gamma : \Delta X \longrightarrow F$ is sent to the induced map $\check{\gamma} : X \longrightarrow L$ and a modification $t : \gamma \longrightarrow \delta$ is sent to the canonical map $\check{t} : X \times I \longrightarrow L$ such that $p_C \circ \check{t} = t_C$ for each object C of $\mathcal{C}$.

These processes are trivially seen to be inverse to one another. □

Indeed, the analogue of Lemma 3.24 for conical colimits also holds.

3.3.2. Meet and join for representable intervals. In order to show that the 2-category structure on $\mathcal{E}$ induced by an interval I is representable it suffices, by Lemma 3.24, to prove that cotensor with the category $\mathbf{2}$ exists.

Lemma 3.25. *If the cotensor $(\mathbf{2} \pitchfork A)$ exists, then it is isomorphic to A^I.*

Proof. The 2-natural isomorphism:

$$\mathcal{E}(X, \mathbf{2} \pitchfork A) \cong \mathcal{E}(X, A)^{\rightarrow} \tag{16}$$

of categories restricts to a natural isomorphism of their respective collections of objects:

$$\mathcal{E}(X, \mathbf{2} \pitchfork A) \cong \mathcal{E}(X \times I, A).$$

□

Note that it does not follow that A^I is $(\mathbf{2} \pitchfork A)$. This remark should be compared with the fact, mentioned above, that a 2-category which possesses all 1-dimensional conical limits need not possesses all 2-dimensional conical limits. This leads to the following definition.

Definition 3.26. An interval I in $\mathcal{E}$ is **representable** if cotensors with $\mathbf{2}$ exist with respect to the 2-category structure on $\mathcal{E}$ induced by I.

The reason for the nomenclature of Definition 3.26 is that when I is representable, the induced 2-category structure on $\mathcal{E}$ is also representable in the sense of Definition 3.23.

We will make use of the following general result about arbitrary 2-categories (this result has nothing to do with intervals and we therefore none of the categories in question are assumed to possess intervals).

Scholium 3.27. *Let an arbitrary 2-category $\mathcal{E}$ be given together with categories C and D and a functor $f : C \to D$. If both cotensor products $(C \pitchfork A)$ and $(D \pitchfork A)$ exist, then there exists a canonical arrow $(f \pitchfork A) : (D \pitchfork A) \to (C \pitchfork A)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}(B, D \pitchfork A) & \xrightarrow{\cong} & [D, \mathcal{E}(B, A)] \\ {\scriptstyle (f \pitchfork A)_*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{E}(B, C \pitchfork A) & \xrightarrow[\cong]{} & [C, \mathcal{E}(B, A)] \end{array}$$

for any object B of $\mathcal{E}$.

Proof. The required map is easily seen to be given by applying the following composite to $1_{D \pitchfork A}$:

$$\mathcal{E}(D \pitchfork A, D \pitchfork A) \xrightarrow{\cong} [D, \mathcal{E}(D \pitchfork A, A)] \xrightarrow{f^*} [C, \mathcal{E}(D \pitchfork A, A)] \xrightarrow{\cong} \mathcal{E}(D \pitchfork A, C \pitchfork A).$$

To see that the square commutes when given a 2-cell it is necessary to use the fact that (16) is required to be 2-natural. □

Lemma 3.28. *If I is representable, then, for all objects A and B of $\mathcal{E}$, the following diagram in* **Cat** *commutes:*

$$\begin{array}{ccc} \mathcal{E}(B, A^I) & \xrightarrow{\cong} & \mathcal{E}(B, A)^{\to} \\ & {\scriptstyle \mathcal{E}(B, \partial_i)}\searrow \quad \swarrow{\scriptstyle \partial_i} & \\ & \mathcal{E}(B, A) & \end{array}$$

when $i = 0, 1$.

Proof. It suffices, by Scholium 3.27, to show that $\partial_i : A^I \longrightarrow A$ is, *qua* an object of $\mathcal{E}(A^I, A)$, the same as the result of applying

$$\mathcal{E}(A^I, A^I) \xrightarrow{\cong} \mathcal{E}(A^I, A)^{\to} \xrightarrow{\partial_i} \mathcal{E}(A^I, A)$$

to 1_{A^I}. This is a trivial verification in light of the fact, which follows from Lemma 3.25, that the isomorphisms (16) must act on objects by exponential transpose. $\square$

Using Lemma 3.28 it will be possible to obtain additional structure on a representable interval I. In particular, we will see that such an I is equipped with both meets and joins, and that these meets and joins satisfy additional special properties.

Proposition 3.29. *If I is representable, then it possesses distinguished meets $\bar{\wedge} : I \times I \longrightarrow I$ and joins $\veebar : I \times I \longrightarrow I$ satisfying the further equations*

$$\top \,\bar{\wedge}\, x = x \tag{17}$$

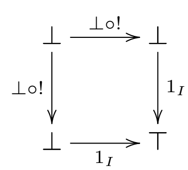
$$\bot \veebar x = x \tag{18}$$

for $x : I$

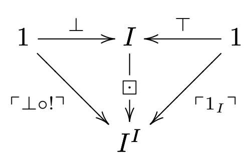
Proof. Because I is representable it follows from Lemma 3.25 that there exists a 2-natural isomorphism

$$\mathcal{E}(1, I)^{\to} \xrightarrow{\cong} \mathcal{E}(1, I^I) \tag{19}$$

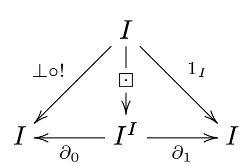
of categories which is given at the level of objects by exponential transpose. In $\mathcal{E}(1, I)$ the following diagram commutes



Thus, by Lemma 3.25, applying (19) to this arrow of $\mathcal{E}(1, I^I)^{\to}$ yields a map $\boxdot : I \longrightarrow I^I$ such that



commutes. On the other hand, by Lemma 3.28,



also commutes. It then follows that the exponential transpose $\bar{\wedge} : I \times I \longrightarrow I$ of $\boxdot$ is a meet operation which satisfies (17).

In the same way, applying the isomorphism (19) to the arrow

$$\begin{array}{ccc} \bot & \xrightarrow{1_I} & \top \\ {\scriptstyle 1_I}\big\downarrow & & \big\downarrow{\scriptstyle \top\circ !} \\ \top & \xrightarrow[\top\circ !]{} & \top \end{array}$$

of $\mathcal{E}(1, I)^{\rightarrow}$ yields a map $\boxplus : I \longrightarrow I^I$ with exponential transpose $\veebar : I \times I \longrightarrow I$ a join operation satisfying (18). $\square$

Definition 3.30. We say that I has **unital meets** if equation (17) is satisfied and, similarly, that I has **unital joins** if (18) is satisfied.

Henceforth, when I is representable, we refer to the operations $\barwedge$ and $\veebar$ defined in Proposition 3.29 as *the* meet and join for I. In addition to the existence of meets and joins, knowing that I is representable tells us much more. For example, we may apply (19) to the following additional arrows in $\mathcal{E}(1, I)^{\rightarrow}$:

$$\begin{array}{ccc} \bot & \xrightarrow{1_I} & \top \\ {\scriptstyle \bot\circ !}\big\downarrow & & \big\downarrow{\scriptstyle \top\circ !} \\ \bot & \xrightarrow[1_I]{} & \top \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \bot & \xrightarrow{\bot\circ !} & \bot \\ {\scriptstyle 1_I}\big\downarrow & & \big\downarrow{\scriptstyle 1_I} \\ \top & \xrightarrow[\top\circ !]{} & \top \end{array}$$

to obtain operations $I \times I \longrightarrow I$. Indeed, in the case of these commutative squares, we obtain the projections $\pi_0, \pi_1 : I \times I \longrightarrow I$, respectively. We establish this fact for π_0 in the following Lemma and the proof for the projection π_1 is essentially identical.

Lemma 3.31. *The image of the commutative diagram*

$$\begin{array}{ccc} \bot & \xrightarrow{1_I} & \top \\ {\scriptstyle \bot\circ !}\big\downarrow & & \big\downarrow{\scriptstyle \top\circ !} \\ \bot & \xrightarrow[1_I]{} & \top \end{array} \tag{20}$$

under the isomorphism (19) is the reflexivity map $r : I \longrightarrow I^I$ regarded as a 2-cell $\ulcorner \bot \urcorner \Longrightarrow \ulcorner \top \urcorner$.

Proof. By Lemmata 3.25 and 3.28 it follows that the result of applying the isomorphism (19) to the reflexivity map $r : I \longrightarrow I^I$ is (20). $\square$

Proposition 3.32. *Given a representable interval I, the meet and join operations for I satisfy the following absorption law:*

$$x \veebar (x \barwedge y) = x \tag{21}$$

for $x, y : I$.

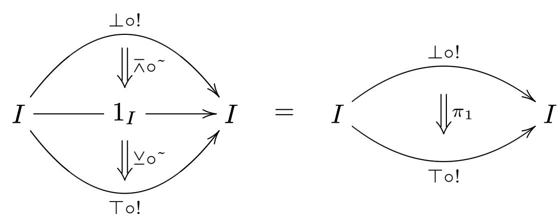
Proof. In $\mathcal{E}(1,I)^{\rightarrow}$ the following identity holds:

$$\begin{array}{ccccc} \bot & \xrightarrow{\bot\circ !} & \bot & \xrightarrow{1_I} & \top \\ {\scriptstyle \bot\circ !}\downarrow & & \downarrow{\scriptstyle 1_I} & & \downarrow{\scriptstyle \top\circ !} \\ \bot & \xrightarrow[1_I]{} & \top & \xrightarrow[\top\circ !]{} & \top \end{array} \quad = \quad \begin{array}{ccc} \bot & \xrightarrow{1_I} & \top \\ {\scriptstyle \bot\circ !}\downarrow & & \downarrow{\scriptstyle \top\circ !} \\ \bot & \xrightarrow[1_I]{} & \top \end{array}$$

By functoriality of the natural isomorphism (19) and Lemma 3.31, this gives the following equation between arrows in the category $\mathcal{E}(1,I^I)$:

$$\boxplus \cdot \boxdot = r.$$

Transposing yields



where $\tilde{}\ : I\times I \longrightarrow I\times I$ is the "twist" map. By the unit and interchange laws we obtain,

$$(\veebar \circ \tilde{}\,)\cdot(\barwedge \circ \tilde{}\,) = (\veebar \circ \tilde{}\,) * (\barwedge \circ \tilde{}\,).$$

By the definition of horizontal composition, this map acts as follows:

$$(x,y) \longmapsto \big(y \veebar (y \barwedge x)\big),$$

for $x, y : I$. Thus, we have shown that (21) holds, as required. $\square$

3.3.3. Parameterized squares. When A and B are objects of $\mathcal{E}$, we say that a map $\alpha : B\times I\times I \longrightarrow A$ is a **square in A parameterized by B**. The **boundary** of such a square, written $\partial(\alpha)$, is the tuple $(\alpha_0,\alpha_1,\alpha^0,\alpha^1)$ where $\alpha_0,\alpha^0 : B\times I \rightrightarrows A$ are the maps defined by setting

$$\alpha_0(x,s) := \alpha(x,\bot,s)$$
$$\alpha^0(x,s) := \alpha(x,s,\bot),$$

for $x : B$ and $s : I$, and similarly for α_1, α^1.

Lemma 3.33. *If I is representable, then, for all objects A and B of $\mathcal{E}$, squares in A parameterized by B are completely determined by their boundaries. I.e., when α and β are such squares, $\partial(\alpha) = \partial(\beta)$ implies that $\alpha = \beta$.*

Proof. Let squares α and β in A parameterized by B be given. Both of these determine arrows $\tilde{\alpha}$ and $\tilde{\beta}$ in the category $\mathcal{E}(B,A^I)$. Moreover, because they agree on their boundaries, they share a common domain $f : B \longrightarrow A^I$ and a common codomain $g : B \longrightarrow A^I$. It suffices to prove that the functor

$$\mathcal{E}(B,A^I) \xrightarrow{\Phi} \mathcal{E}(B,A)^{\rightarrow}$$

acts on arrows by projecting such a transposed square $\tilde{\alpha}$ to its boundary

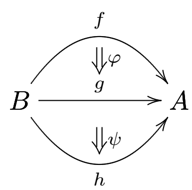
$$
\begin{array}{ccc}
\partial_0 f & \xrightarrow{\partial_0 * \tilde{\alpha}} & \partial_0 g \\
{\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle \hat{g}} \\
\partial_1 f & \xrightarrow[\partial_1 * \tilde{\alpha}]{} & \partial_1 g
\end{array}
\tag{22}
$$

which commutes by the interchange law. This is an immediate consequence of Lemma 3.25 and Lemma 3.28. $\square$

Throughout the remainder of this section we assume that I possesses meets and joins. Given a map $\alpha : B \times I \longrightarrow A$ we can construct the squares $\alpha^\flat, \alpha^\sharp$ and $\alpha^\natural$ in A parameterized by B defined as follows:

$$
\begin{aligned}
\alpha^\flat(x, s, t) &:= \alpha(x, s \mathbin{\overline{\wedge}} t), \\
\alpha^\sharp(x, s, t) &:= \alpha(x, s \mathbin{\underline{\vee}} t), \text{ and} \\
\alpha^\natural(x, s, t) &:= \alpha(x, t),
\end{aligned}
$$

where $x : B$ and $s, t : I$. Assume henceforth that the meets and joins are both unital (I.e., they satisfy (17) and (18), respectively). Given a composable pair of arrows



in $\mathcal{E}(B, A)$, it follows that both composites $(\psi^\natural \cdot \varphi^\sharp)$ and $(\psi^\flat \cdot \varphi^\natural)$ are defined. For example,

$$
\begin{aligned}
\varphi^\natural(x, s, \top) &= \varphi(x, \top) \\
&= g(x) \\
&= \psi(x, \bot) \\
&= \psi(x, s \mathbin{\overline{\wedge}} \bot) \\
&= \psi^\flat(x, s, \bot),
\end{aligned}
$$

for $x : B$ and $s : I$. Moreover, the exponential transpose $\widetilde{(\psi^\flat \cdot \varphi^\natural)} : B \times I \longrightarrow A^I$ is itself an arrow $\tilde{\varphi} \Longrightarrow \widetilde{(\psi \cdot \varphi)}$ in $\mathcal{E}(B, A^I)$. To see that this is the case observe that

$$
\begin{aligned}
(\psi^\flat \cdot \varphi^\natural) \circ (B_\bot \times 1_I) &= c[\psi^\flat \circ (B_\bot \times 1_I), \varphi^\natural \circ (B_\bot \times 1_I)] \circ (1_B \times \star) \\
&= c[1_{\psi \circ B_\bot}, \varphi] \circ (1_B \times \star) \\
&= \varphi.
\end{aligned}
$$

A similar calculation, and the fact the assumption that $\overline{\wedge}$ is unital, shows that

$$
\begin{aligned}
(\psi^\flat \cdot \varphi^\natural) \circ (B_\top \times 1_I) &= c[\psi, \varphi] \circ (1_B \times \star) \\
&= (\psi \cdot \varphi).
\end{aligned}
$$

In order to simplify notation, we will denote the map $(\widetilde{\psi^\flat \cdot \varphi^\natural})$ by $\tau_{\varphi,\psi} : \tilde{\varphi} \Longrightarrow \widetilde{\psi \cdot \varphi}$. By a dual argument it follows that $(\widetilde{\psi^\natural \cdot \varphi^\sharp})$ is an arrow $(\widetilde{\psi \cdot \varphi}) \Longrightarrow \tilde{\psi}$ and we will denote it by $\upsilon_{\varphi,\psi}$.

LEMMA 3.34. *Assume I satisfies the conclusion of Lemma 3.33 and let arrows $\varphi : f \Longrightarrow g$, $\psi : g \Longrightarrow h$ and $\chi : h \Longrightarrow k$ in $\mathcal{E}(B, A)$ be given. Then $\tau_{-,-}$ and $\upsilon_{-,-}$ satisfy the following* "cocycle conditions"*:*

$$\begin{aligned} \tau_{\varphi,1_g} &= 1_{\tilde{\varphi}} \\ &= \upsilon_{1_f,\varphi} \end{aligned}$$

$$\tau_{(\psi\cdot\varphi),\chi} \cdot \tau_{\varphi,\psi} = \tau_{\varphi,(\chi\cdot\psi)} \tag{23}$$

$$\upsilon_{\psi,\chi} \cdot \upsilon_{\varphi,(\chi\cdot\psi)} = \upsilon_{(\psi\cdot\varphi),\chi} \tag{24}$$

PROOF. It suffices to test (the exponential transposes of) these maps on their boundaries. To see that they agree on the boundaries is a straightforward calculation. For example, where τ' is the exponential transpose of the left-hand side of (23):

$$\begin{aligned} (\tau')^0 &= c[\widetilde{\tau_{(\psi\cdot\varphi),\chi}}, \widetilde{\tau_{\varphi,\psi}}] \circ (1_B \times \star \times 1_I) \circ (1_B \times 1_I \times \bot) \\ &= c[\widetilde{\tau_{(\psi\cdot\varphi),\chi}} \circ (1_B \times 1_I \times \bot), \widetilde{\tau_{\varphi,\psi}} \circ (1_B \times 1_I \times \bot)] \circ (1_B \times \star) \\ &= c[(\psi \cdot \varphi)^\natural \circ (1_B \times 1_I \times \bot), \varphi^\natural \circ (1_B \times 1_I \times \bot)] \circ (1_B \times \star) \\ &= c[\varphi \circ (1_B \times \bot \circ !), \varphi \circ (1_B \times \bot \circ !)] \circ (1_B \times \star) \\ &= (1_f \cdot 1_f) \\ &= 1_f \\ &= \varphi^\natural \circ (1_B \times 1_I \times \bot) \\ &= \big((\chi \cdot \psi)^\flat \cdot \varphi^\natural\big)^0. \end{aligned}$$

All of the other boundaries, as well as those for (24), are by similar calculations. □

LEMMA 3.35. *Assume I satisfies the conclusion of Lemma 3.33 and let a commutative diagram*

$$\begin{array}{ccccc} f & \xrightarrow{\alpha} & f' & \xrightarrow{\alpha'} & f'' \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \chi} \\ g & \xrightarrow[\beta]{} & g' & \xrightarrow[\beta']{} & g'' \end{array}$$

be given in $\mathcal{E}(B, A)$, then

$$\upsilon_{\alpha,(\chi\cdot\alpha')} \cdot \tau_{(\beta\cdot\varphi),\beta'} = \tau_{\psi,\beta'} \cdot \upsilon_{\alpha,\psi}. \tag{25}$$

PROOF. As with the proof of Lemma 3.34 it suffices to test the boundaries of these two maps, and these are straightforward calculations. □

3.3.4. Characterization of representable intervals. We have already seen that a representable interval I in $\mathcal{E}$ will possess additional properties which *a priori* an arbitrary interval in $\mathcal{E}$ need not possess. The following theorem establishes precisely which additional structure on I is required in order for it to be representable.

Theorem 3.36. *An interval I in $\mathcal{E}$ is representable if and only if the following conditions are satisfied:*

(1) *I possesses meets and joins which are both unital; and*
(2) *for any objects A and B of $\mathcal{E}$, squares in A parameterized by B are completely determined by their boundaries in the sense of Lemma 3.33.*

Proof. It follows from Proposition 3.29 and Lemma 3.33 that a representable interval possesses the required properties.

For the other direction of the equivalence, by Lemmata 3.24 and 3.25, it suffices to prove that there exist 2-natural isomorphisms

$$\mathcal{E}(B, A^I) \cong \mathcal{E}(B, A)^{\rightarrow}$$

of categories. Moreover, we have already seen that the functor $\Phi : \mathcal{E}(B, A^I) \longrightarrow \mathcal{E}(B, A)^{\rightarrow}$ should send an object $f : B \longrightarrow A^I$ to its exponential transpose

$$B \underset{\partial_1 f}{\overset{\partial_0 f}{\Downarrow \hat{f}}} A,$$

and an arrow $\alpha : f \Longrightarrow g$ in $\mathcal{E}(B, A^I)$ to the "boundary diagram" (22). Functoriality of Φ follows, and the 2-naturality of this construction, is a trivial consequence of the definitions.

The inverse $\Psi : \mathcal{E}(B, A)^{\rightarrow} \longrightarrow \mathcal{E}(B, A^I)$ of Φ is defined as follows. An arrow $\varphi : f \Longrightarrow f'$ in $\mathcal{E}(B, A)$ is sent to its exponential transpose $\tilde{\varphi} : B \longrightarrow A^I$. Next, let a commutative diagram

$$\begin{array}{ccc} f & \xrightarrow{\alpha} & f' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ g & \xrightarrow[\beta]{} & g' \end{array} \tag{26}$$

be given in $\mathcal{E}(B, A)$ and denote by ζ the composite $(\beta \cdot \varphi) = (\psi \cdot \alpha)$. We also write $\tilde{\zeta}$ for the exponential transpose of ζ. By the discussion in Section 3.3.3, we have that $\tau_{\varphi,\beta} : B \times I \longrightarrow A^I$ is an arrow an arrow $\tilde{\varphi} \Longrightarrow \tilde{\zeta}$ in $\mathcal{E}(B, A^I)$. Similarly, $\upsilon_{\alpha,\psi} : B \times I \longrightarrow A^I$ is an arrow $\delta : \tilde{\zeta} \Longrightarrow \tilde{\psi}$ in $\mathcal{E}(B, A^I)$. Let Ψ send the arrow (26) to to composite $(\upsilon_{\alpha,\psi} \cdot \tau_{\beta,\varphi})$. Functoriality of Ψ follows from Lemmata 3.34 and 3.35.

Φ and Ψ are easily seen to be inverse on objects. For arrows, let an arrow (26) be given. We must show that, where δ and γ are as above, $\partial_0 * (\upsilon_{\alpha,\psi} \cdot \tau_{\varphi,\beta}) = \alpha$ and $\partial_1 * (\upsilon_{\alpha,\psi} \cdot \tau_{\beta,\varphi}) = \beta$. For the first equation, observe that

$$\begin{aligned} c[\alpha, 1_f] &= c[\alpha^\sharp \circ (1_B \times 1_I \times \bot), \varphi^\natural \circ (1_B \times 1_I \times \bot)] \\ &= c[\widetilde{\upsilon_{\alpha,\psi}} \circ (1_B \times 1_I \times \bot), \widetilde{\tau_{\beta,\varphi}} \circ (1_B \times 1_I \times \bot)] \end{aligned}$$

Thus, $\partial_0 * (\delta \cdot \gamma) = \alpha$ and, by a similar calculation, $\partial_1 * (\delta \cdot \gamma) = \beta$.

Going the other direction, let an arrow $\alpha : f \Longrightarrow g$ in $\mathcal{E}(B, A^I)$ be given. It suffices, by the hypotheses of the theorem, to prove that $\Psi \circ \Phi(\alpha)$ has the same boundary as α. But this follows from the fact that, by what we have just proved,

$$\Phi \circ \Psi \circ \Phi(\alpha) = \Phi(\alpha).$$

□

3.3.5. The isofibration model structure. Recall that, when $\mathcal{E}$ is an arbitrary 2-category, a map $f : A \longrightarrow B$ is an **equivalence** if there exists a map $f' : B \longrightarrow A$ such that both $f \circ f' \cong 1_B$ and $f' \circ f \cong 1_A$. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ in **Cat** is said to be an isofibration when isomorphisms in $\mathcal{D}$ whose codomains lie in the image of F can be lifted to isomorphisms in $\mathcal{C}$. This notion also makes sense in arbitrary 2-categories $\mathcal{E}$. In particular, we define a map $f : A \longrightarrow B$ in $\mathcal{E}$ to be an **isofibration** if, for any object E of $\mathcal{E}$, the induced map

$$\mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)$$

is an isofibration in **Cat**. With these definitions, Lack [**51**] proved the following theorem:

THEOREM 3.37 (Lack). *If $\mathcal{E}$ is a representable 2-category, then there exists a model structure on $\mathcal{E}$ in which the weak equivalences are the equivalences, the fibrations are the isofibrations and the cofibrations are those maps having the left-lifting property with respect to maps which are simultaneously fibrations and weak equivalences.*

Lack refers to this as the **trivial model structure** on $\mathcal{E}$. However, we will refer to it as the **isofibration model structure** on $\mathcal{E}$. Every object is both fibrant and cofibrant in this model structure. This model structure is *not* in general cofibrantly generated [**51**].

It is an immediate consequence of Theorem 3.37 that, when $\mathcal{E}$ is a bicomplete cartesian closed category with a representable interval I, there is *always* a model structure on $\mathcal{E}$ in which the weak equivalences are exactly the homotopy equivalences.

PROPOSITION 3.38. *When I is an invertible interval, there exists a model structure on $\mathcal{E}$ in which the weak equivalences are the homotopy equivalences and the fibrations are the Hurewicz fibrations.*

PROOF. By Theorem 3.37 it suffices to observe that, when I is invertible, isofibrations and Hurewicz fibrations coincide. This, in turn, is a consequence of Proposition 3.14 and Scholium 3.17. □

3.4. Split fibrations

In this section we introduce a definition, inspired by the work of Street [**75**], of **split fibration** for categories $\mathcal{E}$ possessing an interval I. The setting of [**75**] is a representable 2-category. However, we will see that, when dealing with the 2-category structure induced by an interval I, the definitions, and many results regarding them, make sense independent of the assumption that I is representable. Accordingly, unless otherwise stated, we assume throughout this section only that I is an interval. The main result of this section is Theorem 3.47 which states that, when I is invertible, $\mathcal{E}$ supports the structure of a coherent model of type theory.

3.4.1. The split fibration monad. Before giving the definition of split fibrations it will be convenient to introduce some notation. Given an object A and a map $f : B \longrightarrow A$ in $\mathcal{E}$ we denote by $(A \downarrow f)$ the following pullback:

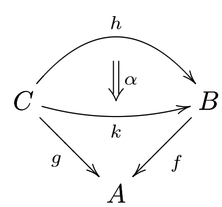
$$\begin{array}{ccc} (A \downarrow f) & \xrightarrow{\pi_B} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A^I & \xrightarrow[\partial_1]{} & A \end{array}$$

Intuitively, $(A \downarrow f)$ is like the comma category: it consists of paths in A whose codomains lie in the image of f. The composite

$$(A \downarrow f) \xrightarrow{f'} A^I \xrightarrow{\partial_0} A$$

is denoted by π_A. The map $(A \downarrow g) \longrightarrow (A \downarrow f)$, induced by an arrow $h : g \longrightarrow f$ in $\mathcal{E}/A$, is written as h_*.

Recall that each slice category $\mathcal{E}/A$ has the structure of a 2-category induced by the 2-category structure of $\mathcal{E}$. Namely, a 2-cell $\alpha : h \Longrightarrow k$ as indicated in the following diagram

(27) 

consists of a 2-cell $\alpha : h \Longrightarrow k$ in $\mathcal{E}$ such that $f * \alpha = g$. With this structure, the comma construction induces a 2-monad $S_A : \mathcal{E}/A \longrightarrow \mathcal{E}/A$ on each slice $\mathcal{E}/A$ which we will now describe.

Given an arrow $f : B \longrightarrow A$ in $\mathcal{E}$, $S_A(f)$ is defined to be the projection

$$(A \downarrow f) \xrightarrow{\pi_A} A.$$

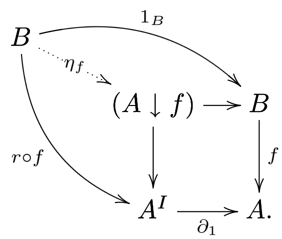
As such, S_A clearly has a functorial action on arrows. Now, given a 2-cell (27), there exists a canonical map $(A \downarrow g) \times I \longrightarrow (A \downarrow f)$ induced by the commutativity of the following diagram:

$$\begin{array}{ccc} (A \downarrow g) \times I & \xrightarrow{\alpha \circ (\pi_C \times 1_I)} & B \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle f} \\ (A \downarrow g) & \xrightarrow[\partial_1 \circ g']{} & A \end{array}$$

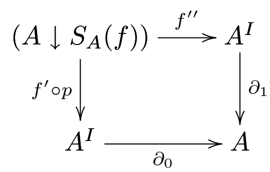
In particular, this map is a 2-cell $(A \downarrow h) \Longrightarrow (A \downarrow k)$. Defining $S_A(\alpha)$ in this way, it is easily seen that S_A is a 2-functor.

PROPOSITION 3.39. *As defined, $S_A : \mathcal{E}/A \longrightarrow \mathcal{E}/A$ is the 2-functor part of a 2-monad on $\mathcal{E}/A$.*

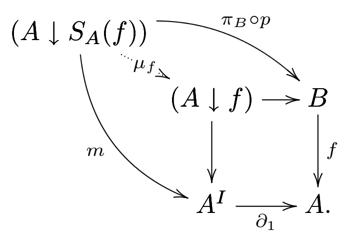
Proof. Given a map $f : B \longrightarrow A$, the unit $\eta_f : f \longrightarrow S_A(f)$ is defined to be the canonical map indicated in the following diagram:



For the multiplication $\mu_f : S_A^2(f) \longrightarrow S_A(f)$, let us first fix some notation. We abbreviate the map $(A \downarrow S_A(f)) \longrightarrow A^I$ by f'' and the map $(A \downarrow S_A(f)) \longrightarrow (A \downarrow f)$ by p. Then the maps $f'', f' \circ p : (A \downarrow S_A(f)) \rightrightarrows A^I$ are composable in the sense that



commutes. Define $m : (A \downarrow S_A(f)) \longrightarrow A^I$ to be the composite $A^\star \circ \langle f'', f' \circ p \rangle$, where $\star$ is the internal cocomposition map so that $\widetilde{A^\star} : A^I \times_A A^I \longrightarrow A^I$. I.e., m is the exponential transpose of the composite $((\widetilde{f' * p}) \cdot \tilde{f}'')$ in $\mathcal{E}((A \downarrow S_A(f)), A)$. Finally, $\mu_f : (A \downarrow S_A(f)) \longrightarrow (A \downarrow f)$ is defined follows:



With these definitions, it is straightforward to verify that η_- and μ_- are natural. The unit laws are then a consequence of the co-unit laws I and the multiplication law follows from the co-associativity law for I. □

Remark 3.40. Although S_A is a 2-monad, as far as obtaining models of type theory is concerned it will only be necessary to consider the 1-dimensional aspect of S_A. Accordingly, in our discussion of S_A-algebras below we assume these to be only ordinary (strict) algebras for a (1-dimensional) monad.

Definition 3.41. S_A-algebras are called **split fibrations over** A **with respect to** I. The Eilenberg-Moore category is denoted by $\mathbf{Sp}(A)$.

3.4.2. Properties of split fibrations. We now exhibit some of the useful properties of split fibrations which we will need in order to interpret type theory. To begin with, we will show that split fibrations are stable under pullback. The argument given here is identical to the argument given to establish the same fact for split fibrations in **Cat**.

Lemma 3.42. *Split fibrations are stable under pullback. I.e., given a map $\sigma : D \longrightarrow A$, if $f : B \longrightarrow A$ is a S_A-algebra, then $\Delta_\sigma(f)$ is a S_D-algebra.*

Proof. Suppose $f : B \longrightarrow A$ has an action $\alpha : S_A(f) \longrightarrow f$ and form the pullback

$$\begin{array}{ccc} E & \xrightarrow{\sigma'} & B \\ {\scriptstyle \Delta_\sigma(f)}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow[\sigma]{} & A. \end{array}$$

Let $\xi : (D \downarrow g) \longrightarrow (A \downarrow f)$ be the induced map. Then the action $\beta : S_D(g) \longrightarrow g$ is the canonical map $(D \downarrow g) \longrightarrow E$ such that

$$\sigma' \circ \beta = \alpha \circ \xi$$
$$g \circ \beta = \pi_D.$$

It is straightforward to verify that, with these definitions, β is an action. □

Lemma 3.43. *For any arrow $\sigma : D \longrightarrow A$, the functor $\Delta_\sigma : \mathcal{E}/A \longrightarrow \mathcal{E}/D$ restricts to a functor $\mathbf{Sp}(\sigma) : \mathbf{Sp}(A) \longrightarrow \mathbf{Sp}(D)$.*

Proof. By Lemma 3.42 it suffices to show that, given split fibrations $f : B \longrightarrow A$ and $g : C \longrightarrow A$, if $h : B \longrightarrow C$ is a S_A-algebra homomorphism $f \longrightarrow g$, then $\Delta_\sigma(h)$ is a S_D-algebra homomorphism. This, however, is a straightforward calculation using the description of the actions induced by pullback from Lemma 3.42. □

3.4.3. The interpretation of type theory. It is an immediate consequence of the results of Section 3.4.2 that the subcategory of the arrow category $\mathcal{E}^\rightarrow$ with objects split fibrations and arrows the algebra homomorphisms determines a comprehension category. I.e., the inclusion χ indicated in the following diagram is a fibered functor:

$$\begin{array}{ccc} \mathbf{Sp}(\mathcal{E}) & \xrightarrow{\chi} & \mathcal{E}^\rightarrow \\ & {\scriptstyle \mathbf{P}(-)}\searrow \quad \swarrow{\scriptstyle \partial_1} & \\ & \mathcal{E} & \end{array}$$

where $\mathbf{Sp}(\mathcal{E})$ is the subcategory of $\mathcal{E}^\rightarrow$ with objects split fibrations. This data therefore determines a (non-split) model of type theory. We will now show that this model possesses *coherent* identity types as defined in Chapter 2.

Given an object A of $\mathcal{E}$, an object in the fiber $\mathbf{P}(A)$ is precisely a split fibration with codomain A. Given such a split fibration $f : B \longrightarrow A$ with action β we may form the weakened context

$$\begin{array}{ccc} B \times_A B & \xrightarrow{\pi_0} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[f]{} & A \end{array}$$

and ask whether there exists an identity type in $\mathbf{P}(B \times_A B)$. It is to this question which we now turn.

Although $\mathcal{E}$ will not in general be locally cartesian closed, we may nonetheless form the exponential $[I, f]$ of an arrow $f : B \longrightarrow A$ by the interval I. This is the object of $\mathcal{E}/A$ defined, as in Section 2.2 of Chapter 2, by the following pullback:

$$\begin{array}{ccc} [I,f] & \xrightarrow{q} & B^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f^I} \\ A & \xrightarrow[r]{} & A^I \end{array}$$

Intuitively, $[I, f]$ consists of pairs (x, φ) such that x is in A and φ is a path in B which never leaves the fiber B_x. Indeed, this process is functorial and $[I, -] : \mathcal{E}/A \longrightarrow \mathcal{E}/A$ witnesses the exponentiability of $\Delta_A(I)$ in $\mathcal{E}/A$.

There exist domain and codomain maps as indicated in the following diagram:

$$\begin{array}{ccc} [I,f] & \xrightarrow{\partial_i} & B \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

with $i = 0, 1$. Accordingly, these induce a map $\iota : [I, f] \longrightarrow B \times_A B$ in $\mathcal{E}/A$.

Lemma 3.44. *Let I be an invertible interval, then, for any arrow $f : B \longrightarrow A$ in $\mathcal{E}$, the induced map $\iota : [I, f] \longrightarrow B \times_A B$ is a split fibration.*

Proof. Denote by $\hat{B}$ the object $(B \times_A B \downarrow \iota)$. I.e., $\hat{B}$ is given by the following pullback:

$$\begin{array}{ccc} \hat{B} & \xrightarrow{\pi_{[I,f]}} & [I,f] \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \iota} \\ B^I \times_{A^I} B^I & \xrightarrow[\partial_1 \times_{\partial_1} \partial_1]{} & B \times_A B \end{array}$$

where we have used the fact that $(-)^I$ preserves limits. Write φ, ψ, ξ for the arrows in $\mathcal{E}(\hat{B}, B)$ obtained by transposing the maps $\pi_i \circ \iota'$, for $i = 0, 1$, and $q_f \circ \pi_{[I,g]}$, respectively. Let $v : \hat{B} \longrightarrow B$ denote the domain of φ, x its codomain, and w the domain of ψ. By construction,

$$v \xrightarrow{\varphi} x \xrightarrow{\xi} y \xleftarrow{\psi} w$$

in $\mathcal{E}(\hat{B}, B)$. Moreover, we also have by construction that

$$f * \xi = 1_{f(x)}, \text{ and}$$
$$f * \varphi = f * \psi.$$

Let $\alpha : \hat{B} \longrightarrow B^I$ be the exponential transpose of $(\psi^{-1} \cdot \xi \cdot \varphi)$. Then, since

$$\begin{aligned} f * \big(\psi^{-1} \cdot (\xi \cdot \varphi)\big) &= (f * \psi^{-1}) \cdot \big((f * \xi) \cdot (f * \varphi)\big) \\ &= (f * \psi)^{-1} \cdot (f * \varphi) \\ &= 1_{f(v)}, \end{aligned}$$

it follows that

$$\begin{array}{ccc} \hat{B} & \xrightarrow{\alpha} & B^I \\ {\scriptstyle f\circ\partial_0\circ\pi\circ\iota'}\big\downarrow & & \big\downarrow{\scriptstyle f^I} \\ A & \xrightarrow[r]{} & A^I \end{array}$$

commutes, where π is the projection $B^I \times_{A^I} B^I \longrightarrow A^I$. We claim that the induced map $\hat{\beta} : \hat{G} \longrightarrow [I, g]$ is an action for S_A. To begin with, observe that, by definition, the square

$$\begin{array}{ccc} \hat{B} & \xrightarrow{\quad\hat{\beta}\quad} & [I,f] \\ & {\scriptstyle \pi_{B\times_A B}}\searrow \quad \swarrow{\scriptstyle \iota} & \\ & B \times_A B & \end{array}$$

commutes.

For the unit law, note that, by a straightforward calculation, $p \circ \hat{\beta} \circ \eta$ is equal to p. On the other hand, we must show that $\alpha \circ \eta$ is q. To see that this is the case notice that, since $\xi \circ (\eta \times 1_I) = \tilde{q}$, $\psi \circ (\eta \times 1_I) = 1_z$ and $\varphi \circ (\eta \times 1_I) = 1_y$,

$$\begin{aligned} \big(\psi^{-1} \cdot (\xi \cdot \varphi)\big) \circ (1_{\hat{B}} \times \eta) &= c\big[1_z, c[\xi \circ (\eta \times 1_I), 1_y] \circ (1_{[I,f]} \times \star)\big] \circ (1_{[I,f]} \times \star) \\ &= \big(1_z \cdot (\tilde{q} \cdot 1_y)\big) \\ &= \tilde{q}. \end{aligned}$$

For the multiplication law, we abbreviate $S_A^2(\iota)$ by D as indicated in the following pullback square:

$$\begin{array}{ccc} D & \xrightarrow{\quad\pi_{\hat{B}}\quad} & \hat{B} \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{B\times_A B}} \\ B^I \times_{A^I} B^I & \xrightarrow[\partial_1 \times_{\partial_1} \partial_1]{} & B \times_A B \end{array}$$

Denoting by $\varepsilon_i : D \times I \longrightarrow B$, for $i = 0, 1$, the exponential transposes of the maps $\pi_i \circ \iota''$, we see that $\iota' \circ \mu : D \longrightarrow B^I \times_{A^I} B^I$ has components δ_0 and δ_1 which are the exponential transposes of $(\varphi * \pi_{\hat{B}}) \cdot \varepsilon_0$ and $(\psi * \pi_{\hat{B}}) \cdot \varepsilon_1$, respectively. To begin with, we have that $p \circ \hat{\beta} \circ S_A(\hat{\beta})$ is equal to $p \circ \hat{\beta} \circ \mu$ since,

$$f * \big((\varphi * \pi_{\hat{B}}) \cdot \varepsilon_0\big) = (f * \varphi * \pi_{\hat{B}}) \cdot (f * \varepsilon_0),$$

which has the same domain, $f \circ u$, as $(f * \varepsilon_0)$.

Next, standard calculations show that the transpose of $q \circ \hat{\beta} \circ S_A(\hat{\beta})$ is the composite

$$\varepsilon^{-1} \cdot \big((\psi^{-1} \cdot \xi \cdot \varphi) * \pi_{\hat{B}}\big) \cdot \varepsilon_0,$$

and that the transpose of $q \circ \hat{\beta} \circ \mu$ is

$$\big((\psi * \pi_{\hat{B}}) \cdot \varepsilon_1\big)^{-1} \cdot (\xi * \pi_{\hat{B}}) \cdot \big((\varphi * \pi_{\hat{B}}) \cdot \varepsilon_0\big).$$

However, both of these composites are equal, by the interchange law, and therefore $\hat{\beta} \circ S_A(\hat{\beta})$ is equal to $\hat{\beta} \circ \mu$, as required. $\square$

REMARK 3.45. Note that we have crucially used the cogroupoid structure of I in the proof of Lemma 3.44.

Assume that the interval I in $\mathcal{E}$ possesses joins satisfying equation (18). Then there exists an induced operation

$$[I,f] \xrightarrow{\mathbb{W}_f} [I,f]^I$$

To construct $\mathbb{W}_f$, observe that, where $\upsilon_f : [I,f] \times I \longrightarrow B^I$ is the exponential transpose of the composite

$$[I,f] \times I \times I \xrightarrow{q \times \veebar} B^I \times I \xrightarrow{\mathrm{ev}} B$$

the following diagram commutes:

$$\begin{array}{ccc} [I,f] \times I & \xrightarrow{\upsilon_f} & B^I \\ {\scriptstyle p\circ\pi_0}\downarrow & & \downarrow{\scriptstyle f^I} \\ A & \xrightarrow[r]{} & A^I \end{array}$$

by definition of r and the fact that $f^I \circ q = r \circ p$. Thus, we may define $\mathbb{W}_f$ to be the transpose of the induced map $[I,f] \times I \longrightarrow [I,f]$.

LEMMA 3.46. *The map* $\mathbb{W}_f : [I,f] \longrightarrow [I,f]^I$ *satisfies the following equations:*

$$\partial_0 \circ \mathbb{W}_f = 1_{[I,f]}, \tag{28}$$

$$\partial_1 \circ \mathbb{W}_f = r_f \circ \partial_1, \tag{29}$$

$$\mathbb{W}_f \circ r_f = r \circ r_f, \tag{30}$$

where $r_f : B \longrightarrow [I,f]$ *is the canonical map induced by the reflexivity map* $r_B : B \longrightarrow B^I$ *and* $r : [I,f] \longrightarrow [I,f]^I$ *is the usual reflexivity term. I.e.,* r_f *is the canonical map such that* $p \circ r_f = f$ *and* $q \circ r_f = r_B$.

PROOF. Equations (28) and (29) are direct consequences of the corresponding equations for $\veebar$. (30) is by the commutativity of the following square:

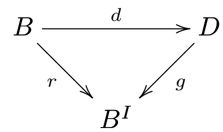
$$\begin{array}{ccc} B \times I \times I & \xrightarrow{r \times 1_I \times 1_I} & B^I \times I \times I \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle 1_{B^I} \times \veebar} \\ B & \xleftarrow[\mathrm{ev}]{} & B^I \times I \end{array}$$

commutes. ☐

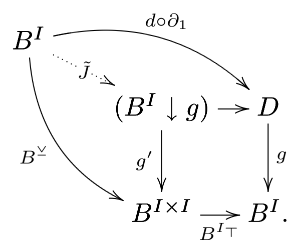
THEOREM 3.47. *If I is an invertible interval in $\mathcal{E}$ which is equipped with a fixed join operation satisfying (18), then* $(\mathbf{Sp}(\mathcal{E}), \chi, \mathbf{P}(-))$ *is a comprehension category with coherent identity types.*

PROOF. Given a split fibration $f : B \longrightarrow A$, the identity type $I(f)$ in $\mathbf{P}(B \times_A B)$ is defined to be the map $\iota : [I,f] \longrightarrow B \times_A B$, which is a split fibration by Lemma 3.44. The reflexivity map $r : B \longrightarrow [I,f]$ is as described above and clearly yields a factorization of the diagonal. The identity types are easily seen to be stable using an argument essentially identical to the argument given in the proof of Theorem 2.29. As such, it remains only to construct and verify the coherence of the elimination terms.

We first describe the construction of the elimination term for the case where A is the terminal object (i.e., in the empty context). Suppose given a split fibration $g : D \longrightarrow B^I$ together with a map $d : B \longrightarrow D$ such that the triangle

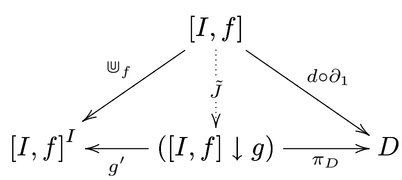
(31)


commutes. Then there exists a canonical map $\tilde{J} : B^I \longrightarrow (B^I \downarrow g)$ as indicated in the following diagram:



where we make evident use of the fact that $(B^I)^I \cong B^{I\times I}$. Now, where γ denotes the action of g, we define $J(d, g)$ to be the composite $\gamma \circ \tilde{J}$.

In the general case, where A is not assumed to be the empty context, the appropriate map $\tilde{J} : [I, f] \longrightarrow ([I, f] \downarrow g)$ is obtained using the map $\mathbb{U}_f : [I, f] \longrightarrow [I, f]^f$ constructed above. It follows from (29) that the induced map $\tilde{J}$ exists. I.e., $\tilde{J}$ is the canonical map $[I, f] \longrightarrow ([I, f] \downarrow g)$ such that the following diagram commutes:



Again, defining $J(d, g)$ to be $\gamma \circ \tilde{J}$ it follows that $J(d, g)$ is a section of g since

$$\begin{aligned} g \circ J(d, g) &= g \circ \gamma \circ \tilde{J} \\ &= \partial_0 \circ \mathbb{U}_f \\ &= 1_{[I,f]}, \end{aligned}$$

where the final equation is by (28). Next, it follows from (30) that $\tilde{J} \circ r_f = \eta_g \circ d$, where $\eta_g : D \longrightarrow ([I, f] \downarrow g)$ is the unit of the monad at g. Thus,

$$\begin{aligned} J(d, g) \circ r_f &= \gamma \circ \eta_g \circ d \\ &= d, \end{aligned}$$

since γ is an action.

Finally, to see that the elimination terms are coherent, observe that all of the structure employed in the construction of these terms is itself stable under pullback. □

The following corollary relates Theorem 3.47 to the homotopical semantics from Chapter 2:

COROLLARY 3.48. *If I is representable, then $(\mathbf{Sp}(\mathcal{E}), \chi, \mathbf{P}(-))$ is a coherent restriction with respect to the isofibration model structure from Section 3.3.5.*

COROLLARY 3.49. *When I is an interval satisfying the hypotheses of Theorem 3.47, there exists coherent model of type theory consisting only of the cloven fibrations (i.e., pointed algebras for the endofunctors S_A).*

PROOF. An examination of the proof of Theorem 3.47 reveals that the compatibility of actions with multiplication is not required in order to construct a coherent model of type theory. Lemma 3.44 ensures that identity types exist when restricting to cloven fibrations. □

LEMMA 3.50. *If I is invertible and parameterized squares are completely determined by their boundaries in the sense of Lemma 3.33, then the resulting model $(\mathbf{Sp}(\mathcal{E}), \chi, \mathbf{P}(-))$ is 1-dimensional. I.e., UIP_2 is valid in all such models.*

PROOF. Suppose given a split fibration $f : B \longrightarrow A$ together with sections a, b of f and maps φ_i as indicated in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_i} & [I, f] \\ & {\scriptstyle\langle a,b\rangle}\searrow \quad \swarrow{\scriptstyle\iota} & \\ & B \times_A B & \end{array}$$

for $i = 0, 1$. Let $\mathrm{Id}_A(a, b)$ be the following object

$$\begin{array}{ccc} \mathrm{Id}_A(a,b) & \xrightarrow{\mu} & [I, f] \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\iota} \\ A & \xrightarrow[\langle a,b\rangle]{} & B \times_A B \end{array}$$

and assume that there exist maps α_i

$$\begin{array}{ccc} A & \xrightarrow{\alpha_i} & [I, \nu] \\ & {\scriptstyle\langle \hat{\varphi}_0,\hat{\varphi}_1\rangle}\searrow \quad \swarrow{\scriptstyle\iota_\nu} & \\ & \mathrm{Id}_A(a,b) \times_A \mathrm{Id}_A(a,b) & \end{array}$$

for $i = 0, 1$, where $\hat{\varphi}_i : A \longrightarrow \mathrm{Id}_A(a, b)$ is the obvious map induced by φ_i. There are squares $\hat{\alpha}_i$ in B parameterized by A obtained via the composites

$$A \xrightarrow{\alpha_i} [I, \nu] \longrightarrow \mathrm{Id}_A(a,b)^I \xrightarrow{\mu^I} [I, f]^I \xrightarrow{q^I} (B^I)^I,$$

for $i = 0, 1$. In order to see that $\alpha_0 = \alpha_1$ it suffices to show that $\hat{\alpha}_0$ and $\hat{\alpha}_1$ agree on their boundaries. It is straightforward to confirm that both squares possess the same boundary. Namely, they both have boundary $(\widetilde{q \circ \varphi}, \widetilde{q \circ \psi}, a \circ \pi_0, b \circ \pi_0)$, where $\widetilde{q \circ \varphi}$ denotes the exponential transpose of $q \circ \varphi$. □

Lemma 3.50 implies, in particular, that whenever an invertible interval I is representable, the resulting model of type theory is 1-dimensional. There is a partial converse of Lemma 3.50 which we may obtain by restricting attention to certain parameterized squares.

DEFINITION 3.51. A square $\alpha : A \times I \times I \longrightarrow B$ in B parameterized by A is **globular** if its faces α^i, for $i = 0, 1$, are constant. I.e., there exist maps $a, b : A \rightrightarrows B$ such that $\alpha^0 = a \circ \pi_0$ and $\alpha^1 = b \circ \pi_0$.

Note that the proof of Lemma 3.50 requires only the hypothesis that globular squares are completely determined by their boundaries in the sense of Lemma 3.33. This is of course due to the fact that parameterized squares are cubical in shape, whereas the parameterized squares arising from this form of type theory are globular.

PROPOSITION 3.52. *Let I be an interval in $\mathcal{E}$ which satisfies the hypotheses of Theorem 3.47. Under these conditions, $(\mathbf{Sp}(\mathcal{E}), \chi, \mathbf{P}(-))$ is 1-dimensional if and only if, for any objects A and B of $\mathcal{E}$, globular squares in B parameterized by A are completely determined by their boundaries.*

PROOF. By Lemma 3.50 and the foregoing observation about its proof, it suffices to prove that 1-dimensionality implies the corresponding property of parameterized squares. Let globular squares $\alpha, \beta : A \times I \times I \rightrightarrows B$ which agree on their boundaries be given. Let us denote by $a, b : A \rightrightarrows B$ the maps occurring as the boundaries α^i for $i = 0, 1$, respectively. I.e., α^0 is $a \circ \pi_0$, et cetera. The projection $A \times B \longrightarrow A$ is a split fibration. Define sections $\hat{a}$ and $\hat{b}$ of this projection to be $\langle 1_A, a\rangle$ and $\langle 1_A, b\rangle$, respectively. Define arrows φ and ψ from a to b in $\mathcal{E}(A, B)$ as follows:

$$\begin{aligned} \varphi(x, s) &:= \alpha(x, \bot, s), \text{ and} \\ \psi(x, s) &:= \alpha(x, \top, s), \end{aligned}$$

for $x : A$ and $s : I$. In type theoretic notation, these maps induce terms $\hat{\varphi}$ and $\hat{\psi}$:

$$x : A \ \vdash \ \hat{\varphi}(x), \hat{\psi}(x) : \mathrm{Id}_{A\times B}\big(\hat{a}(x), \hat{a}(x)\big).$$

Similarly, α and β themselves induce terms

$$x : A \ \vdash \ \hat{\alpha}(x), \hat{\beta}(x) : \mathrm{Id}_{\mathrm{Id}_{A\times B}(\hat{a}(x), \hat{b}(x))}\big(\hat{\varphi}(x), \hat{\psi}(x)\big).$$

Therefore, since this model is 1-dimensional, these terms $\hat{\alpha}$ and $\hat{\beta}$ are identical. It follows from the construction of $\hat{\alpha}$ and $\hat{\beta}$ that $\alpha = \beta$. □

REMARK 3.53. Although we have shown that the identity types are modelled soundly in the abstract setting of a cartesian closed category with an invertible interval I, we have said nothing about dependent products and sums. Indeed, it does not appear to be possible to interpret these in this setting without requiring additional structure. Even the assumption that $\mathcal{E}$ is locally cartesian closed does not seem to suffice on its own. We turn now to one natural source of examples of categories with invertible intervals which *do* support the interpretation of these additional type formers.

3.5. Internal groupoids

The aim of this section is to develop our principal application of the results from Section 3.4.3. Namely, we show that whenever $\mathcal{E}$ is a finitely bicomplete cartesian closed category the category $\mathbf{Gpd}(\mathcal{E})$ of internal groupoids possesses a representable and invertible interval $\mathbf{I}$ and therefore yields a coherent model of Martin-Löf type theory. Moreover, we show that when $\mathcal{E}$ is itself locally cartesian closed, the category of internal groupoids also supports the interpretation of dependent products. In the case where $\mathcal{E}$ is the category **Set** of sets the resulting model is equivalent (in an appropriate categorical sense) to the Hofmann-Streicher groupoids model [**35**] using the familiar equivalence, via the Grothendieck construction, between split fibrations of groupoids and functors from small groupoids into **Gpd**, together with the fact, due to Street [**75**], that split fibrations of categories (and consequently also groupoids) are algebras for the 2-monad described in Section 3.4.1. By the results of Chapter 2 it follows that the split Grothendieck fibrations associated to these models are genuine models of type theory which are split in the sense of Remark 2.1 from Chapter 2. Unless otherwise stated $\mathcal{E}$ is assumed to be a (finitely) bicomplete category which is cartesian closed. We refer the reader to Appendix A for further details regarding internal groupoids.

3.5.1. The unit interval. A useful feature of the category $\mathbf{Gpd}(\mathcal{E})$ is that it possesses a strict unit interval $\mathbf{I}$. $\mathbf{I}$ is defined to be $\nabla(1+1)$, where ∇ is the right-adjoint to the forgetful functor $\mathbf{Gpd}(\mathcal{E}) \longrightarrow \mathcal{E}$ as described in Appendix A. $\mathbf{I}$ is given the structure of a strict interval as follows. First, the object of coarrows is defined to be $\mathbf{I}$ and the object $\mathbf{I}^2$ of cocomposable arrows is then obtained as the following pushout:

$$\begin{array}{ccc} 1 & \xrightarrow{\top} & \mathbf{I} \\ {\scriptstyle\perp}\downarrow & & \downarrow{\scriptstyle\downarrow} \\ \mathbf{I} & \xrightarrow[\uparrow]{} & \mathbf{I}^2 \end{array}$$

where $\perp$ and $\top$ are the internal functors induced by the respective coproduct injections $1 \rightrightarrows 1+1$.

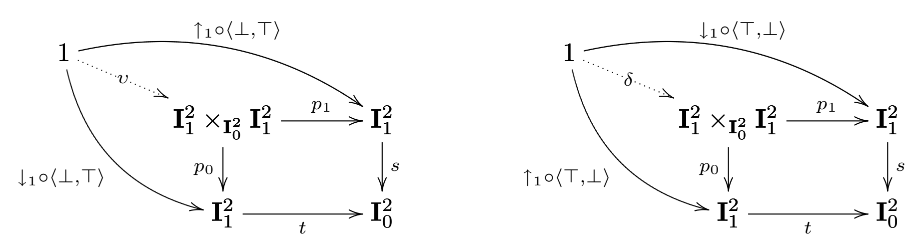
The cocomposition map $\star : \mathbf{I} \longrightarrow \mathbf{I}^2$ is constructed by first noting that

$$\mathbf{I}_1 \cong \mathbf{I}_0 + \mathbf{I}_0.$$

Accordingly, the arrow part $\star_1 : \mathbf{I}_1 \longrightarrow \mathbf{I}_1^2$ is described completely in terms of the four distinguished global sections of $\mathbf{I}_1$ denoted by $(\perp,\perp)$, $(\top,\perp)$, $(\perp,\top)$ and $(\top,\top)$ of $\mathbf{I}_1$. In particular we define

$$\star_1(x,y) := \begin{cases} i\circ \downarrow_0 \circ\perp & \text{if } x = y = \perp \\ c\circ\upsilon & \text{if } x = \perp,\ y = \top \\ c\circ\delta & \text{if } x = \top,\ y = \perp \\ i\circ \uparrow_0 \circ\top & \text{if } x = y = \top, \end{cases}$$

where $\upsilon, \delta : 1 \rightrightarrows \mathbf{I}_1^2 \times_{\mathbf{I}_0^2} \mathbf{I}_1^2$ are the canonical maps indicated in the following diagrams:



It is straightforward to show that these maps are nicely related in the sense that

$$\bar{r} \circ \upsilon = \delta \quad \text{and} \quad \bar{r} \circ \delta = \upsilon \tag{32}$$

where $\bar{r}$ is as in A.1.2 above.

Similarly, the object part $\star_0 : \mathbf{I}_0 \longrightarrow \mathbf{I}_0^2$ can be described in terms of the global sections $\bot$ and $\top$ of $\mathbf{I}_0$ as follows:

$$*_0(x) := \begin{cases} \downarrow_0 \circ \bot & \text{if } x = \bot \\ \uparrow_0 \circ \top & \text{if } x = \top. \end{cases}$$

I.e., $*_0$ is $[\downarrow_0 \circ \bot, \uparrow_0 \circ \top]$. It is easily shown that the source, target and identity conditions for $* : \mathbf{I} \longrightarrow \mathbf{I}^2$ to be an internal functor are met. It remains only to show that $*$ behaves functorially with respect to composition. I.e., that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{I}_1 \times_{\mathbf{I}_0} \mathbf{I}_1 & \xrightarrow{\gamma} & \mathbf{I}_1^2 \times_{\mathbf{I}_0^2} \mathbf{I}_1^2 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbf{I}_1 & \xrightarrow[*_1]{} & \mathbf{I}_1^2 \end{array}$$

where $\gamma : \mathbf{I}_1 \times_{\mathbf{I}_0} \mathbf{I}_1 \longrightarrow \mathbf{I}_1^2 \times_{\mathbf{I}_0^2} \mathbf{I}_1^2$ is map induced by $*$. To see that this is the case we use the fact that

$$\mathbf{I}_1 + \mathbf{I}_1 \cong \mathbf{I}_0 \times \mathbf{I}_1.$$

In fact, using the definition of $\mathbf{I}$, it is easily seen that

$$\mathbf{I}_0 \times \mathbf{I}_1 \cong \mathbf{I}_1 \times_{\mathbf{I}_0} \mathbf{I}_1.$$

This is routine to verify and the isomorphism $\phi : \mathbf{I}_0 \times \mathbf{I}_1 \longrightarrow \mathbf{I}_1 \times_{\mathbf{I}_0} \mathbf{I}_1$ is described informally as follows:

$$\begin{aligned} \phi\big(x, (y, z)\big) &:= \big((y, x), (x, z)\big) \\ \phi^{-1}\big((x, y), (y, z)\big) &:= \big(y, (x, z)\big). \end{aligned}$$

That cocomposition $*$ behaves functorially with respect composition follows easily from the fact that, by this observation, $\mathbf{I}_1 \times_{\mathbf{I}_0} \mathbf{I}_1$ is a coproduct. In particular, one may test on elements of the form $\big((x, y), (y, z)\big)$ for $x, y, z \in \{\bot, \top\}$. The only cases which are not straightforward are those of $\big((\bot, \top), (\top, \bot)\big)$ and $\big((\top, \bot), (\bot, \top)\big)$. For example, the first is seen to hold as follows:

$$\begin{aligned} *_1 \circ c\big((\bot, \top), (\top, \bot)\big) &= *_1(\bot, \bot) \\ &= i \circ \downarrow_0 \circ \bot \\ &= i \circ s \circ \downarrow_1 \langle \bot, \top \rangle \\ &= i \circ s \circ p_0 \circ \upsilon \\ &= i \circ s \circ c \circ \upsilon \\ &= c(c \circ \upsilon, r \circ c \circ \upsilon) \\ &= c(c \circ \upsilon, c \circ \bar{r} \circ \upsilon) \\ &= c(c \circ \upsilon, c \circ \delta) \\ &= c \circ \gamma\big((\bot, \top), (\top, \bot)\big). \end{aligned}$$

The other case is dual using the other half of (32). Therefore $* : \mathbf{I} \longrightarrow \mathbf{I}^2$ is an internal functor.

PROPOSITION 3.54. *With these definitions* $\mathbf{I}$ *is an interval in* $\mathbf{Gpd}(\mathcal{E})$*.*

PROOF. The "cosource" and "cotarget" equations

$$\begin{aligned} \downarrow \circ \bot &= * \circ \bot \\ \uparrow \circ \top &= * \circ \top \end{aligned}$$

are trivial by definition of the cocomposition map. The "counit" and "coassociativity" equations, because they have domain $\mathbf{I}$, can be tested on "elements" as above in both the "object" and "arrow" cases. For instance, where i^0 is the canonical map $\mathbf{I}^2 \longrightarrow \mathbf{I}$ such that $i^0 \circ \downarrow = \bot$ and $i^0 \circ \uparrow = 1_{\mathbf{I}}$, the case of the "arrow" $(\top, \bot)$ is as follows:

$$\begin{aligned} i^0_1 \circ *_1(\top, \bot) &= i^0_1 \circ c \circ \delta \\ &= c\big(i^0_1 \circ \uparrow_1 \circ (\top, \bot), i^0_1 \circ \downarrow_1 \circ (\top, \bot)\big) \\ &= c\big((\top, \bot), (\bot, \bot)\big) \\ &= (\top, \bot), \end{aligned}$$

where the second equation is by functoriality of i^0. All of the additional cases are of a similar elementary nature. □

3.5.2. Representability of I. We now establish that, with the 2-category structure induced by $\mathbf{I}$, $\mathbf{Gpd}(\mathcal{E})$ is a representable 2-category. This follows from the well-known fact that $\mathbf{Gpd}(\mathcal{E})$ is representable with respect to its usual 2-category structure together with the following observation:

SCHOLIUM 3.55. *The 2-category structure on* $\mathbf{Gpd}(\mathcal{E})$ *induced by* $\mathbf{I}$ *coincides with the usual 2-category structure on* $\mathbf{Gpd}(\mathcal{E})$*.*

Nonetheless, it is instructive to establish the representability of $\mathbf{Gpd}(\mathcal{E})$ using Theorem 3.36 since we will the meets and joins used to establish this will be required later.

LEMMA 3.56. $\mathbf{Gpd}(\mathcal{E})$ *possesses unital meets and joins.*

PROOF. The map $\bar{\wedge}_0 : \mathbf{I}_0 \times \mathbf{I}_0 \longrightarrow \mathbf{I}_0$ is specified by:

$$\bar{\wedge}_0(s,t) := \begin{cases} \bot & \text{if } s = \bot \text{ or } t = \bot \\ \top & \text{otherwise,} \end{cases}$$

for $s, t : \mathbf{I}_0$. This definition clearly extends to $\bar{\wedge}_1$. Similarly, $\underline{\vee}_0$ is given by

$$\underline{\vee}_0(s,t) := \begin{cases} \top & \text{if } s = \top \text{ or } t = \top \\ \bot & \text{otherwise,} \end{cases}$$

for $s, t : \mathbf{I}_0$. □

PROPOSITION 3.57. *The interval* $\boldsymbol{I}$ *in* $\mathbf{Gpd}(\mathcal{E})$ *is invertible and representable.*

PROOF. By Theorem 3.36 it suffices together with Lemma 3.56 above, it suffices to prove that parameterized squares in $\mathbf{Gpd}(\mathcal{E})$ are completely determined by their boundaries in the sense of Lemma 3.33. To see that this is the case suppose given squares α and β in A parameterized by B. Clearly $\alpha_0 = \beta_0$ by the definition of $\mathbf{I}$. To see that $\alpha_1 = \beta_1$ it suffices to test on the elements $(\bot, \top)$ and $(\top, \bot)$ of $\mathbf{I}_1$. Denoting these elements by u and d, respectively, this is easily verified by the following calculation in the internal language:

$$\begin{aligned}\alpha_1(f,u,u) &= \alpha_1\big(f, c(1_\bot, u), c(u, 1_\top)\big)\\ &= c\big(\alpha_1(f,1_\bot,u), \alpha_1(f,u,1_\top)\big)\\ &= c\big(\alpha_1(f,1_\bot,u), \beta_1(f,u,1_\top)\big)\\ &= \beta_1(f,u,u),\end{aligned}$$

where $f : B_1$ and the final equation follows from the hypotheses. Similarly we see that $\alpha_1(f,g,h) = \beta_1(f,g,h)$ for $g, h = u, d$. □

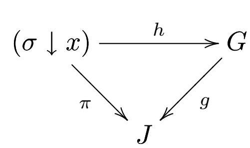
COROLLARY 3.58. $\mathbf{Gpd}(\mathcal{E})$ *is a coherent (non-split) model of type theory which is 1-dimensional.*

PROOF. By Proposition 3.57, Theorem 3.47 and Proposition 3.52. □

REMARK 3.59. In this setting it is trivial to prove that split fibrations compose and therefore that models of the form $\mathbf{Gpd}(\mathcal{E})$ support the interpretation of dependent sums.

3.5.3. Dependent products. It is well known that $\mathbf{Gpd}$ is not locally cartesian closed (cf. [**42, 68**]). However, the reindexing functor $\Delta_\sigma : \mathbf{Sp}(A) \longrightarrow \mathbf{Sp}(B)$ for $\sigma : B \longrightarrow A$ does possess a right-adjoint. We will now show that this is true also in the internal setting provided we assume $\mathcal{E}$ is itself locally cartesian closed. Because the construction is quite involved we make significantly more use of the (traditional) internal logic of $\mathcal{E}$ *qua* locally cartesian closed category. The following construction is inspired by both the construction of dependent products due to Hofmann and Streicher [**35**] and Palmgren's [**68**] 2-dimensional dependent products. That dependent products exist can alternatively be seen using the theory of Kan extensions.

Let a split fibration $g : G \longrightarrow J$ of internal groupoids, with action γ, and an internal functor $\sigma : J \longrightarrow K$ be given. We describe a new internal groupoid $P(\sigma, g)$ together with a split fibration $\pi : P(\sigma, g) \longrightarrow K$ as follows. Intuitively, the object $P(\sigma, g)_0$ should be thought of as consisting of pairs (x, h) such that x is an object of K and, where $(\sigma \downarrow x)$ is the comma category, h is a homomorphism $(\sigma \downarrow x) \longrightarrow G$ of split fibrations as indicated in the following commutative triangle:

(33)


This description is internalized using the locally cartesian closed structure of $\mathcal{E}$. In particular, we begin by observing that, in $\mathbf{Gpd}(\mathcal{E})$, we may construct the object

$(\sigma \downarrow i)$ as the following pullback:

$$\begin{array}{ccc} (\sigma \downarrow i) & \xrightarrow{i'} & (\sigma \downarrow K) \\ {\scriptstyle \pi_{K_0}} \downarrow & & \downarrow {\scriptstyle \pi_K} \\ \Delta(K_0) & \xrightarrow[i]{} & K \end{array}$$

taken in $\mathbf{Gpd}(\mathcal{E})$, where i is the insertion of identity arrows.

LEMMA 3.60. *The map* $\langle \pi_J \circ i', \pi_{K_0} \rangle : (\sigma \downarrow i) \longrightarrow J \times \Delta(K_0)$ *is a split fibration.*

PROOF. Let us denote $\langle \pi_J \circ i', \pi_{K_0} \rangle$ by ξ and consider the following pullback used to define $S_{J \times \Delta_0(K)}(\xi)$:

$$\begin{array}{ccc} (J \downarrow \xi) & \xrightarrow{\pi_{(\sigma \downarrow i)}} & (\sigma \downarrow i) \\ {\scriptstyle \xi'} \downarrow & & \downarrow {\scriptstyle \xi} \\ J^I \times \Delta(K_0)^I & \xrightarrow[\partial_1 \times_{\partial_1} \partial_1]{} & J \times \Delta(K_0) \end{array}$$

In $\mathbf{Gpd}(\mathcal{E})\big((J \downarrow \xi), K\big)$ there exist composable maps α and β defined by letting α be the transpose of the composite

$$(J \downarrow \xi) \xrightarrow{\pi_{(\sigma \downarrow i)}} (\sigma \downarrow i) \xrightarrow{i'} (\sigma \downarrow K) \xrightarrow{\sigma'} K^I,$$

and letting β be the transpose of

$$(J \downarrow \xi) \xrightarrow{\pi_0 \circ \xi'} J^I \xrightarrow{\sigma^I} K^I.$$

It is straightforward to verify that the composite $(\alpha \cdot \beta)$ is defined. Let υ denote its exponential transpose. Then there exists an induced map υ' making the following diagram commute:

$$\begin{array}{ccccc} & & (J \downarrow \xi) & & \\ & {\scriptstyle \upsilon} \swarrow & \vdots {\scriptstyle \upsilon'} & \searrow {\scriptstyle \partial_0 \circ \pi_0 \circ \xi'} & \\ K^I & \xleftarrow[\sigma']{} & (\sigma \downarrow K) & \xrightarrow[\pi_J]{} & J \end{array}$$

This map in turn induces the action $\hat{\upsilon} : (J \downarrow \xi) \longrightarrow (\sigma \downarrow i)$ in the evident way. □

Now, we regard $g : G \longrightarrow J$ as a split fibration $G \times \Delta(K_0) \longrightarrow J \times \Delta(K_0)$ and form the object $P(\sigma, g)_0 \longrightarrow K_0$ of $\mathcal{E}/K_0$ which has as its fiber over x in K_0 the collection of all homomorphisms (33). All of the subsequent arguments we give may be "internalized" in a similar way and, as such, we argue henceforth using the internal language.

Given objects (x, h) and (y, k) of $P(\sigma, g)$, an arrow $\theta : (x, h) \longrightarrow (y, k)$ consists of an arrow $\theta : x \longrightarrow y$ of K together with a 2-cell η as indicated in the following diagram:

$$\begin{array}{ccc} (\sigma \downarrow x) & \xrightarrow{(\sigma \downarrow \theta)} & (\sigma \downarrow y) \\ & \overset{\eta}{\Rightarrow} & \\ {\scriptstyle h} \searrow & & \swarrow {\scriptstyle k} \\ & G & \end{array}$$

Due to the presence of **I** in $\mathbf{Gpd}(\mathcal{E})$, and the equational nature of the definition of 2-cells, this definition makes sense. Given arrows a composable pair of arrows (θ, η) and (θ', η') the composite $(\theta', \eta') \circ (\theta, \eta)$ is given by $\theta' \circ \theta$ together with the 2-cell

$$(\eta' * (\sigma \downarrow \theta)) \cdot \eta.$$

It is easily seen that these definitions yield a groupoid in $\mathcal{E}$.

LEMMA 3.61. *The projection $\pi : P(\sigma, g) \longrightarrow K$ functor is a split fibration.*

PROOF. Given an object h in the fiber $P(\sigma, p)_x$ over x and an arrow $\phi : y \longrightarrow x$ in A, define $k : (\sigma \downarrow y) \longrightarrow E$ to be the composite

$$(\sigma \downarrow y) \xrightarrow{(\sigma \downarrow \phi)} (\sigma \downarrow x) \xrightarrow{\;h\;} E.$$

Taking $\phi : (y, k) \longrightarrow (x, h)$ to be the (cartesian) lift of $\phi : y \longrightarrow x$, with 2-cell the identity, this trivially defines an action. □

We may now use $\pi : P(\sigma, g) \longrightarrow K$ to interpret dependent products in $\mathbf{Gpd}(\mathcal{E})$ as the following Theorem shows:

THEOREM 3.62. *If $\sigma : J \longrightarrow K$ is a map in $\mathbf{Gpd}(\mathcal{E})$, then the induced functor $\Delta_\sigma : \mathbf{Sp}(K) \longrightarrow \mathbf{Sp}(J)$ possesses a right adjoint $P(\sigma, -)$.*

PROOF. Let split fibrations $p : E \longrightarrow J$ and $q : F \longrightarrow K$ be given. First, assume we have a map $f : J \times_K F \longrightarrow E$ in $\mathbf{Sp}(J)$. Given an object z of F we define $\hat{f}(z)$ to consist of the pair consisting of $x = q(z)$ and the map $\hat{f}(z) : (\sigma \downarrow x) \longrightarrow E$ over J defined by

$$\hat{f}(z)\big(j, \phi\big) := f(j, z \cdot \phi),$$

where $\phi_z : (z \cdot \phi) \longrightarrow z$ is the cartesian lift of ϕ given by the fact that q is a split fibration. Similarly, given an arrow $\psi : (j, \phi) \longrightarrow (j', \phi')$ in $(\sigma \downarrow x)$, $\hat{f}(z)(\psi)$ is defined as follows:

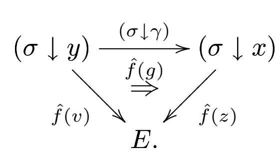
$$\hat{f}(z)\big(\psi\big) := f\big(\psi, \sigma(\psi)_{(z \cdot \phi')}\big).$$

$\hat{f}(z)$ is a homomorphism of split fibrations since, given (j, ϕ) in $(\sigma \downarrow x)$ and $\gamma : j' \longrightarrow j$ in B, we have

$$\begin{aligned} \hat{f}(z)\big(\gamma_{(j,\phi)}\big) &= \hat{f}(z)\big(\gamma : (j', \phi \circ \sigma(\gamma)) \to (j, \phi)\big) \\ &= f\big(\gamma, \sigma(\gamma)_{(z \cdot \phi)}\big) \\ &= \gamma_{f(j, z \cdot \phi)} \\ &= \gamma_{\hat{f}(z)(j,\phi)}, \end{aligned}$$

where the third equation is by the fact that f is a homomorphism of split fibrations.

Now, suppose given a map $g : v \longrightarrow z$ in F with $q(g)$ the map $\gamma : y \longrightarrow x$ in A. Then $\hat{f}(g) : \hat{f}(v) \longrightarrow \hat{f}(z)$ is the pair consisting of γ itself together with a 2-cell which we also call $\hat{f}(g)$ as indicated in the following diagram:



Here we define the component of $\hat{f}(g)$ at an object $\phi : \sigma(j) \longrightarrow y$ of $(\sigma \downarrow y)$ as follows

$$\hat{f}(j)_{(j,\phi)} := f\big(1_j, (\gamma \circ \phi)_z^{-1} \circ g \circ \phi_v\big).$$

With these definitions it is routine to verify both naturality of $\hat{f}(g)$ and that the resulting map $\hat{f}$ is a homomorphism of split fibrations.

Going the other way, assume given a homomorphism $g : F \longrightarrow P(\sigma, p)$ of split fibrations over K. Then, for an object j of J and an object z of F such that $\sigma(j) = q(z) = x$, we have, by definition, a functor $g(z) : (\sigma \downarrow x) \longrightarrow E$ over J. As such, $\check{g} : J \times_K F \longrightarrow E$ is defined on objects as follows:

$$\check{g}(j, z) := g(z)(j, 1_{\sigma(j)}).$$

Let an arrow $(h, k) : (j, z) \longrightarrow (j', z')$ in $J \times_K F$ be given and define $x := q(z)$, $y := q(z')$ and $\delta := q(k)$. Observe that $h : (j, \delta) \longrightarrow (j', 1_y)$ in $(\sigma \downarrow y)$ and therefore applying $g(z')$ yields a map

$$g(z')(j, \delta) \xrightarrow{g(z')(h)} g(z')(j', 1_y) = \check{g}(j', z').$$

Additionally, applying g to k yields a natural transformation

$$\begin{array}{ccc} (\sigma \downarrow x) & \xrightarrow{(\sigma\downarrow\delta)} & (\sigma \downarrow y) \\ & \overset{g(k)}{\Rightarrow} & \\ {\scriptstyle g(z)} \searrow & & \swarrow {\scriptstyle g(z')} \\ & E. & \end{array}$$

In particular, we have

$$\check{g}(j, z) = g(z)(j, 1_x) \xrightarrow{g(k)_{(j,1_x)}} g(z')(j, \delta).$$

Therefore we define $\check{g}(h, k)$ as follows:

$$\check{g}(h, k) := g(z')(h) \circ g(k)_{(j,1_x)}.$$

Functoriality is a routine verification. To see that $\check{g}$ is a homomorphism, suppose given (j, z) in $J \times_K F$ and $\gamma : c \longrightarrow j$ in J. Then we have

$$\begin{aligned} \check{g}\big(\gamma_{(j,z)}\big) &= g(z)(\gamma) \circ g\big(\sigma(\gamma)_z\big)_{(j',1_{\sigma(j')})} \\ &= g(z)(\gamma) \circ \big(\sigma(\gamma)_{g(z)}\big)_{(j',1_{\sigma(j')})} \\ &= g(z)(\gamma) \\ &= g(z)(\gamma_{(j,1_{\sigma(j)})}) \\ &= \gamma_{g(z)(j,1_{\sigma(j)})} \\ &= \gamma_{\check{g}(j,z)}, \end{aligned}$$

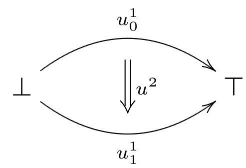
where the second equation is by the fact that g is a homomorphism, the third is by definition of lifts for $P(\sigma, p)$ and the fifth is by the fact that $g(z)$ is a homomorphism.

The processes $(\hat{-})$ and $(\check{-})$ are easily seen to be inverse to one another. □

Corollary 3.63. *If $\mathcal{E}$ is a (finitely) bicomplete category which is locally cartesian closed, then $\mathbf{Gpd}(\mathcal{E})$ is a coherent (non-split) model of $\mathbb{T}_2$.*

Proof. By Corollary 3.58 and Theorem 3.62. □

3.5.4. Higher-dimensional groupoids. It is well known (cf. [**78**]) that the categories n-**Gpd** and ω-**Gpd** as well as their internal variants n-$\mathbf{Gpd}(\mathcal{E})$ and ω-$\mathbf{Gpd}(\mathcal{E})$ possess distinguished higher-dimensional intervals. For example, the 2-dimensional interval $\mathbf{I}^2$ in 2-**Gpd** is the free 2-groupoid generated by the following 2-globular set:



Just as the usual interval **I** in **Gpd** can be regarded as (the object of co-arrows of) a co-groupoid, so too $\mathbf{I}^2$ can be regarded as (the object of co-2-cells of) a co-2-groupoid. In general, let $\mathbf{I}^n$ be the free n-groupoid generated by the n-globular set consisting of the following data:

- two 0-cells $u_0^0 = \bot$ and $u_1^0 = \top$;
- two parallel m-cells $u_0^m, u_1^m : u_0^{m-1} \rightrightarrows u_1^{m-1}$ for $1 \leq m < n$; and
- a single n-cell $u^n : u_0^{n-1} \longrightarrow u_1^{n-1}$.

Then $\mathbf{I}^n$ is a co-n-groupoid in the category of n-groupoids. Similarly, by carrying on this process — with two parallel n-cells at every dimension n — *ad infinitum* we obtain a co-omega-groupoid $\mathbf{I}^\omega$ in the category of ω-groupoids. One would expect these objects also to give rise to interpretations of the identity types in the corresponding categories. Indeed, exponentiation by $\mathbf{I}^\omega$ (or at least its "cousin" in ω-**Cat**) has been studied by Métayer [**64**] and used by Lafont, Métayer and Worytkiewicz [**52**] as a path object in their verification of the "folk" model structure on ω-**Cat**.

Indeed, it is already possible, using the results above, to obtain models of type theory in n-**Gpd** (and ω-**Gpd**) by truncating $\mathbf{I}^n$ (or $\mathbf{I}^\omega$) at any given dimension $k \leq n$. In particular, fixing $n \geq 2$, we obtain an invertible interval $\mathbf{I}(n, k)$ by taking as the object of co-objects the terminal object (as usual) and the object of co-arrows $\mathbf{I}^n_k$. Just as any n-category $\mathcal{C}$ gives rise a regular category $\mathcal{C}(n, k)$ by taking objects to be 0-cells and arrows $x \longrightarrow y$ to be k-cells α bounded at dimension 0 by x and y, so too $\mathbf{I}^n_k$ is an invertible interval. While this construction does provide novel models of type theory in n-**Gpd** (and similarly in ω-**Gpd**), it is easily seen that these models are 1-dimensional in the type theoretic sense of validating UIP_2. In order to obtain truly higher-dimensional models in this way it is necessary to generalize also the notion of split fibration to higher-dimensions. The notion of Grothendieck 2-fibration has been studied by Hermida [**30**] and, for 2-groupoids, by Moerdijk and Svennson [**65**]. However, we are unaware of similar work in higher-dimensions. As a first step toward developing this approach, we turn in Chapter 4 to consider models of type theory in categories of higher-dimensional groupoids using a "functorial" approach, which has its origins in Lawvere's notion of hyperdoctrine [**53**], which generalizes directly the approach taken in the original groupoids model of Hofmann and Streicher [**35**]. The "internalization" of such higher-dimensional models — analogous to the present treatment of the basic 1-groupoid model — can then proceed from that basis.

CHAPTER 4

ω-Groupoids

The aim of this chapter is to construct models of type theory which are genuinely higher-dimensional —in the sense that they refute such truncation principles as UIP_n, *et cetera*— and which interpret type theory in a split and coherent way. In particular, we prove that it is possible to interpret intensional type theory using (strict) ω-groupoids. The resulting model directly generalizes the (1-)groupoid model due to Hofmann and Streicher [**35**] and, by truncation, also yields models using n-groupoids. In the interpretation, contexts and therefore also closed types are interpreted as ω-groupoids. In particular, when the ω-groupoid $\mathcal{A}$ interprets a closed type and objects a and b of $\mathcal{A}$ correspond to terms, the ω-groupoid $\mathcal{A}(a,b)$ provides the interpretation of the identity type. Because identity types are interpreted in this way using ω-groupoids, we are able to refute all higher-dimension versions of the principle of uniqueness of identity proofs. Similarly, it follows that all of the truncation principles from Chapter 1 are also refuted in this model. We now turn to a summary of the chapter.

We have seen that closed types are to be interpreted as ω-groupoids. Types $\Gamma \vdash A : \mathsf{type}$ in context, on the other hand, will be interpreted as functors $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$ where $\mathcal{C}$ is the ω-groupoid interpreting the context Γ. Note that here and henceforth, unless otherwise stated, *functor* refers to strict ω-functors and similarly for "natural transformations" and "transformations". We refer the reader to Section A.3 of Appendix A for a description of the (large) ω-categorical structure of ω-**Gpd** and related notions. In [**35**], the extended context $(\Gamma, x : A)$ is interpreted using the familiar Grothendieck construction $\int A$ which takes a functor $A : \mathcal{C} \longrightarrow \mathbf{Gpd}$ to the associated split op-fibration. Accordingly, in order to generalize the interpretation from *ibid* to the present setting it is necessary to describe a corresponding Grothendieck construction for functors $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$. To that end, in Section 4.1 we introduce the corresponding Grothendieck construction in full generality for functors $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Cat}$ of ω-categories. As far as we know, this is the first place that such a construction has been explicitly given in the literature on ω-categories and, as such, should be of general interest. The combinatorics involved here (as well as throughout) is reminiscent of that occurring in to work of Street and Verity on (simplicial) nerves of ω-categories [**77, 83**].

In Section 4.2 we shift our attention to ω-groupoids with the aim of obtaining a "duality" functor $\neg$ which is required in order to describe the identity types in higher-dimensions. Given $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$, there is, in addition to the Grothendieck construction $\int A$, also an ω-groupoid $\int^* A$ which we call the *dual Grothendieck construction of* A obtained by orienting the Grothendieck construction in the opposite direction. For example, $\int^* A$ has the same objects as $\int A$, but arrows are given the opposite orientation. Although $\int^* A$ exists in general, when we are dealing with ω-groupoids there is a distinguished functor $\neg : \int A \longrightarrow \int^* A$ which will prove useful.

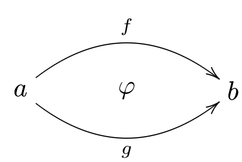
With the duality functor at our disposal we arrive in Section 4.3 at the description of the functors used to interpret identity types. Section 4.4 is then concerned with the description of the maps which interpret the elimination J terms. Finally, Section 4.5 contains a description of the interpretation, as well as a discussion of additional type formers. We end with the main results of the chapter: Theorem 4.25 and its corollaries.

4.1. The Grothendieck construction

In this section we introduce the basic combinatorial structure which makes higher-dimensional models of type theory (of the form we are considering) possible. This structure is a generalization to higher-dimensions of the familiar Grothendieck construction [**25**].

4.1.1. Notational conventions. Given a small ω-category $\mathcal{C}$ and a functor $A : \mathcal{C} \longrightarrow \omega$-**Cat** we denote by A_x the ω-category obtained by applying A to an object x of $\mathcal{C}$ and, when $f : x \longrightarrow y$ is an arrow in $\mathcal{C}$, $A_f : A_x \longrightarrow A_y$ denotes the induced functor. We employ similar notation for higher-dimensional cells. This convention will later allow us to avoid excessive use of parentheses. When z is any 0-cell of A_x we denote $A_f(z)$ by $(z.f)$. Similarly, by the definition of $(n+1)$-cells in ω-**Cat**, A_γ, for γ a $(n+1)$-cell with $n \geq 1$, is given by a family of n-cells parameterized by 0-cells of its domain category (say) A_x and we denote by $(z.\gamma)$ the n-cell $\left(A_\gamma\right)_z$ where z is an object of A_x, as above.

It will be convenient to introduce some conventions governing diagrams in higher-dimension. In particular, we will often want to describe the various boundaries of m-cells φ in ω-categories. In particular, we may wish to indicate diagrammatically the n-cells bounding such a φ even when $m > n+2$ so that drawing the details of the diagram would be cumbersome. As such, we will instead often include diagrams such as the following:



where f and g are n-cells. Such diagrams are oriented from "top-to-bottom" unless otherwise stated. I.e., the diagram indicates that

$$s^{m-n}(\varphi) = f, \text{ and}$$
$$t^{m-n}(\varphi) = g.$$

In the few cases where there is no "top-to-bottom" option available, the cells should be oriented "left-to-right". In this section, and throughout this chapter, we will often be dealing with "hom-ω-categories" of the form $\mathcal{A}(a,b)$ where $\mathcal{A}$ is an ω-category and a, b are parallel cells of $\mathcal{A}$. In this setting, or similar ones, the index of a composition $(\gamma *_n \delta)$ always refers to the dimension in $\mathcal{A}$ and not in $\mathcal{A}(a,b)$.

4.1.2. The underlying globular set. Assume given a small ω-category $\mathcal{C}$ and a functor $A : \mathcal{C} \longrightarrow \omega$-**Cat**. We describe here the **Grothendieck construction** $\int A$ **of** A which is itself a small ω-category. In the first two dimensions $\int A$ is the familiar Grothendieck construction (or category of elements) of A.

- **(0-Cells):** The 0-cells of $\int A$ are pairs (x, x_-) such that x is an object of $\mathcal{C}$ and x_- is an object of A_x.
- **(1-Cells):** The 1-cells $(x, x_-) \longrightarrow (y, y_-)$ are pairs (f, f_-) consisting of a 1-cell $f : x \longrightarrow y$ in $\mathcal{C}$ and a 1-cell $f_- : (x_-.f) \longrightarrow y_-$ in A_y.

Remark 4.1. We often employ vector notation $\vec{x} = (x, x_-)$, $\vec{f} = (f, f_-)$, ... for cells of the Grothendieck construction. The reason for the notation x_-, and related notation, will become clear later when we consider the construction of identity types.

Already at this low dimension there are several features of the definition which should be emphasized. First, in order to define the component f_- of arrows in $\int A$ we have made a choice of "weight" or "orientation". Namely, we have determined that the codomain of f_- should be y_- where we could have as easily determined that its domain should be this same object of A_y. Secondly, fixing objects $\vec{x}$ and $\vec{y}$ of $\int A$, there exists a functor

$$\mathcal{C}_1(x, y) \xrightarrow{\mathfrak{d}^1_{\vec{x},\vec{y}}} A_y$$

defined by

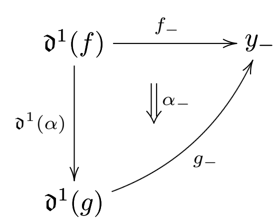
$$\mathfrak{d}^1_{\vec{x},\vec{y}}(\gamma) := (x_-.\gamma),$$

for any m-cell γ of $\mathcal{C}(x, y)$. Although this functor depends on $\vec{x}$ and $\vec{y}$ we often write $\mathfrak{d}^1$ when no confusion will result. With this definition we observe that an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ has

$$s(f_-) = \mathfrak{d}^1(f).$$

In this situation, the object $\mathfrak{d}^1(f)$ is said to be the **weighted face of** f_-. We will see that the higher-dimensional cells resulting from the construction of $\int A$ also possess suitably "weighted" faces.

- **(2-Cells):** Given 1-cells $\vec{f}$ and $\vec{g}$ with common source and target $\vec{x}$ and $\vec{y}$, respectively, a 2-cell $\vec{f} \longrightarrow \vec{g}$ consists of a 2-cell $\alpha : f \longrightarrow g$ in $\mathcal{C}$ together with a 2-cell α_- of A_y as indicated in the following diagram:



  Now, holding $\vec{f}$ and $\vec{g}$ fixed, there exists a functor

$$\mathcal{C}_2(f, g) \xrightarrow{\mathfrak{d}^2_{\vec{f},\vec{g}}} (A_y)_1\big(\mathfrak{d}^1_{\vec{x},\vec{y}}(f), y_-\big)$$

defined by

$$\mathfrak{d}^2_{\vec{f},\vec{g}}(\gamma) := g_- *_0 \mathfrak{d}^1_{\vec{x},\vec{y}}(\gamma)$$

where γ is a n-cell of $\mathcal{C}_2(f,g)$. Note that in this case $\mathfrak{d}^1(\gamma)$ is a $(n+1)$-cell of A_y so that this definition makes sense. It is straightforward to verify that this is functorial. Note that an arrow $\vec{\alpha} : \vec{f} \longrightarrow \vec{g}$ has

$$t(\alpha_-) = \mathfrak{d}^2_{\vec{f},\vec{g}}(\alpha).$$

As above, $\mathfrak{d}^2(\alpha)$ is the **weighted face of** α_-. In general, we will see that $(n+1)$-cells of $\int A$ are given by pairs (φ, φ_-) and that each component φ_- comes equipped with a weighted face. Namely, the weighted face of φ_- is $s(\varphi_-)$ if $(n+1)$ is even and it is $t(\varphi_-)$ if $(n+1)$ is odd. At each stage n we will construct, along with the definition of the n-cells, a functor $\mathfrak{d}^n(-)$ which gives an explicit description of the weighted faces of n-cells.

In general, $(\int A)_n$ is defined by induction on n alternating between even and odd steps in such a way that the following conditions are satisfied:

(1) At each stage n an element of $(\int A)_n$ is a pair (f, f_-) such that f is an n-cell of $\mathcal{C}$ and f_- is an n-cell of A_y.

(2) At each stage $(n+1)$, for $n \geq 1$, there is also constructed, for each pair $\vec{\alpha}, \vec{\beta}$ of parallel n-cells with source $\vec{f}$ and target $\vec{g}$, a functor $\mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta}}$ such that

$$\mathcal{C}_{n+1}(\alpha,\beta) \xrightarrow{\mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta}}} (A_y)_n\big(\mathfrak{d}^n_{\vec{f},\vec{g}}(\alpha), g_-\big)$$

if $(n+1)$ is even and

$$\mathcal{C}_{n+1}(\alpha,\beta) \xrightarrow{\mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta}}} (A_y)_n\big(f_-, \mathfrak{d}^n_{\vec{f},\vec{g}}(\beta)\big)$$

if $(n+1)$ is odd. The functor $\mathfrak{d}^{n+1}$ is called the **weighted face functor (in dimension $(n+1)$ determined by $\vec{\alpha}$ and $\vec{\beta}$)**.

(3) At each stage $(n+1)$, for $n \geq 0$, it is required that the following **weighted face conditions** are satisfied:

$$s(\varphi_-) = \begin{cases} \alpha_- & \text{if } (n+1) \text{ is even, and} \\ \\ \mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta}}(\varphi) & \text{if } (n+1) \text{ is odd;} \end{cases}$$

and:

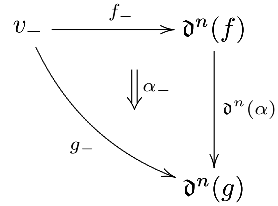
$$t(\varphi_-) = \begin{cases} \mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta}}(\varphi) & \text{if } (n+1) \text{ is even, and} \\ \\ \beta_- & \text{if } (n+1) \text{ is odd,} \end{cases}$$

when $\vec{\varphi}$ is an $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \longrightarrow \vec{\beta}$.

By the discussion above, the base cases $n = 0, 1, 2$ satisfy these conditions. We now consider the induction steps.

($(n+1)$ is odd): Fix parallel n-cells $\vec{f}$ and $\vec{g}$ of $\int A$. A $(n+1)$-cell $\vec{f} \Longrightarrow \vec{g}$ consists of a pair (α, α_-) with $\alpha : f \Longrightarrow g$ an $(n+1)$-cell in $\mathcal{C}$ and α_- is a $(n+1)$-cell of A_y subject to conditions which we will now describe.

Let $\vec{v} = s(\vec{f})$ and $\vec{w} = t(\vec{f})$. Then $\vec{v}$ and $\vec{w}$ are $(n-1)$-cells of $\int A$ and therefore, by the induction hypothesis, there exists a weighted face functor $\mathcal{C}_n(v,w) \xrightarrow{\mathfrak{d}^n_{\vec{v},\vec{w}}} (A_y)_{n-1}\big(\mathfrak{d}^{n-1}(v), w_-\big)$. As such, α_- is required to be a $(n+1)$-cell of A_y as indicated in the following diagram:

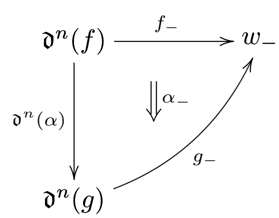


For the weighted functor, we hold $\vec{f}$ and $\vec{g}$ fixed and define

$$\mathfrak{d}^{(n+1)}_{\vec{f},\vec{g}}(\gamma) := \mathfrak{d}^n(\gamma) *_{(n-1)} f_-,$$

for γ a m-cell of $\mathcal{C}_{(n+1)}(f,g)$. The weighted face conditions are then trivially satisfied.

($(n+1)$ is even): Given parallel n-cells $\vec{f}$ and $\vec{g}$ of $\int A$, a $(n+1)$-cell $\vec{f} \Longrightarrow \vec{g}$ consists, as above, of a pair (α, α_-) with $\alpha : f \Longrightarrow g$ in $\mathcal{C}$ and α_- a $(n+1)$-cell of A_y as indicated in the following diagram:



Finally, we define:

$$\mathfrak{d}^{(n+1)}_{\vec{f},\vec{g}}(\gamma) := g_- *_{(n-1)} \mathfrak{d}^n(\gamma),$$

for γ a m-cell of $\mathcal{C}_{(n+1)}(f,g)$. The weighted face conditions are then trivial.

Putting the foregoing together we obtain the following basic fact:

LEMMA 4.2. *If $A : \mathcal{C} \longrightarrow \omega$-**Cat** is a functor, then $\int A$ is a reflexive globular set.*

Before moving on to discuss composition it will be convenient to mention here a useful fact regarding the behavior of the weighted face functors which is an immediate consequence of the construction given above:

LEMMA 4.3. *Given a m-cell $\vec{\varphi}$ together with n-cells $\vec{\alpha}$ and $\vec{\beta}$ of $\int A$ with $m > n$, if*

$$s^{(m-n)}(\vec{\varphi}) = \vec{\alpha}, \text{ and}$$
$$t^{(m-n)}(\vec{\varphi}) = \vec{\beta},$$

then

$$s(\alpha_-) = s^{(m-n+1)}(\varphi_-), \text{ and}$$
$$t(\beta_-) = t^{(m-n+1)}(\varphi_-).$$

4.1.3. Horizontal composition. Suppose we are given a pair of composable arrows $\vec{f}:\vec{x}\longrightarrow\vec{y}$ and $\vec{h}:\vec{y}\longrightarrow\vec{z}$ in $\int A$. Then we obtain

$$A_h\big(\mathfrak{d}^1_{\vec{x},\vec{y}}(f)\big)\xrightarrow{A_h(f_-)}\mathfrak{d}^1_{\vec{y},\vec{z}}(h)\xrightarrow{h_-}z_-$$

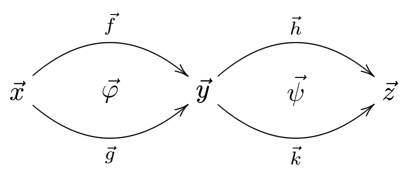
in A_z. Recall that $\mathfrak{d}^1_{\vec{y},\vec{z}}(h)=A_h(y_-)$ so that this makes sense. Moreover,

$$\begin{aligned}A_h(\mathfrak{d}^1_{\vec{x},\vec{y}}f)&=A_hA_f(x_-)\\&=\mathfrak{d}^1_{\vec{x},\vec{z}}(h\circ f),\end{aligned}$$

and therefore we define

$$(\vec{h}*_0\vec{f})_-:=h_-*_0A_h(f_-).$$

This is the familiar definition of composition in the (1-dimensional) category of elements. Now, suppose that we are given m-cells $\vec{\varphi}$ and $\vec{\psi}$ of $\int A$, for $m>1$, which are bounded by 0- and 1-cells as indicated in the following diagram:



Then, it follows from Lemma 4.3 and functoriality of A_h that

$$\begin{aligned}t^mA_h(\varphi_-)&=A_h(tg_-)\\&=A_h(y_-)\\&=\mathfrak{d}^1_{\vec{y},\vec{z}}(h).\end{aligned}$$

Thus, the following definition makes sense (i.e., the composite involved is indeed well-defined):

$$(\vec{\psi}*_0\vec{\varphi})_-:=\psi_-*_0A_h(\varphi_-).$$

The aim of the following lemma is to show that, with the definition just given, $(\vec{\psi}*_0\vec{\varphi})$ is a m-cell with the correct source and target. I.e., that the source and target conditions for horizontal composition $(-*_0-)$ from Section A.3.2 of Appendix A are satisfied.

LEMMA 4.4. *Suppose given m-cells $\vec{\varphi}$ and $\vec{\psi}$ as above, then*

$$s(\vec{\psi}*_0\vec{\varphi})_-=\begin{cases}(s\vec{\psi}*_0s\vec{\varphi})_- & \text{if } m \text{ is even, and}\\[2ex] \mathfrak{d}^m_{s\vec{\psi}*_0s\vec{\varphi},t\vec{\psi}*_0t\vec{\varphi}}(\psi*_0\varphi) & \text{if } m \text{ is odd,}\end{cases}$$

and

$$t(\vec{\psi}*_0\vec{\varphi})_-=\begin{cases}(t\vec{\psi}*_0t\vec{\varphi})_- & \text{if } m \text{ is odd, and}\\[2ex] \mathfrak{d}^m_{s\vec{\psi}*_0s\vec{\varphi},t\vec{\psi}*_0t\vec{\varphi}}(\psi*_0\varphi) & \text{if } m \text{ is even,}\end{cases}$$

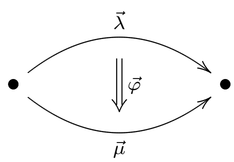
when $m > 1$*, and when* $m = 1$,

$$s(\vec{\psi} *_0 \vec{\varphi})_- = \mathfrak{d}^1_{\vec{x},\vec{z}}(h *_0 f), \text{ and}$$
$$t(\vec{\psi} *_0 \vec{\varphi})_- = t(\psi_-).$$

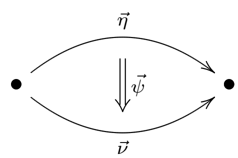
Proof. First, assume $m > 1$. By definition,

$$\ell(\psi_- *_0 A_h(\varphi_-)) = \ell\psi_- *_0 A_h(\ell\varphi_-)$$

for $\ell = s, t$. Thus, when m is even the source condition is trivially satisfied, as is the target condition when m is odd. For the other cases we will need to prove a stronger fact about the behavior of the weighted face functors. Let the $(m-1)$-boundaries of $\vec{\varphi}$ and $\vec{\psi}$ be given as follows:



and



We prove simultaneously by induction on $m > 1$ the following facts:

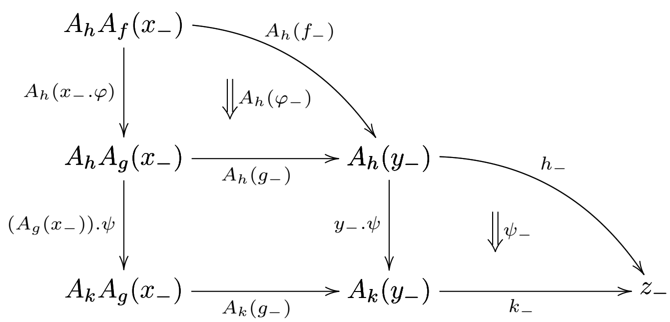
(1) If δ is a k-cell of $\mathcal{C}_m(\lambda, \mu)$ and ϵ is a k-cell of $\mathcal{C}_m(\eta, \nu)$, then

$$\mathfrak{d}^m_{\vec{\eta}*_0\vec{\lambda},\vec{\nu}*_0\vec{\mu}}(\epsilon \ *_0 \ \delta) = \mathfrak{d}^m_{\vec{\eta},\vec{\nu}}(\epsilon) \ *_0 \ A_h\big(\mathfrak{d}^m_{\vec{\lambda},\vec{\mu}}(\delta)\big). \tag{34}$$

(2) The boundary conditions from the statement of the lemma hold.

Note that, in order for the first condition to even make sense at stage m we must already have verified that the second condition holds at stage $(m-1)$.

In the base case $m = 2$, we have $\vec{\lambda} = \vec{f}$, $\vec{\mu} = \vec{g}$, $\vec{\eta} = \vec{h}$ and $\vec{\nu} = \vec{k}$. To see that the boundary condition (2) is satisfied consider the following diagram:

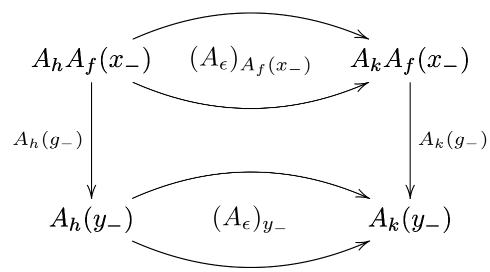


Here the square commutes by naturality of $A_\psi : A_h \Longrightarrow A_k$. Moreover,

$$\begin{aligned}\mathfrak{d}^1_{\vec{x},\vec{z}}(\psi *_0 \varphi) &= x_-.(\psi *_0 \varphi)\\ &= (A_\psi)_{A_g(x_-)} *_0 A_h(x_-.\varphi)\end{aligned}$$

by the definition of horizontal composition of 2-cells in ω-**Cat**. For (34) we note that the following diagram commutes since A_ϵ is a $(k+2)$-cell of $\mathcal{C}$:

(35)



As such,

$$\begin{aligned}\mathfrak{d}^2_{\vec{h},\vec{k}}(\epsilon) \ *_0\ A_h\big(\mathfrak{d}^2_{\vec{f},\vec{g}}(\delta)\big) &= k_- \ *_0\ (A_\epsilon)_{y_-} \ *_0\ A_h(g_-) \ *_0\ A_h\big((A_\delta)_{x_-}\big)\\ &= k_- \ *_0\ A_k(g_-) \ *_0\ (A_\epsilon)_{A_f(x_-)} \ *_0\ A_h\big((A_\delta)_{x_-}\big)\\ &= \mathfrak{d}^2_{\vec{h}*_0\vec{f},\vec{k}*_0\vec{g}}(\epsilon *_0 \delta).\end{aligned}$$

Here the first equation is by definition of $\mathfrak{d}^2(-)$ and functoriality of A_h, the second equation is by (35) and the final equation is by the definition of horizontal composition of k-cells in ω-**Cat**.

For the induction step when m is even we have,

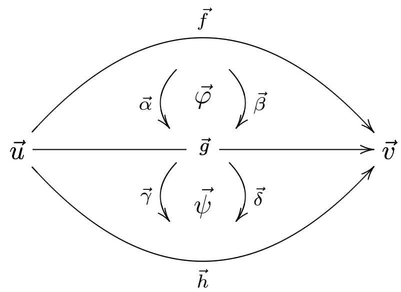
$$\begin{aligned}\mathfrak{d}^m_{\vec{\eta}*_0\vec{\lambda},\vec{\nu}*_0\vec{\mu}}(\epsilon *_0 \delta) &= (\vec{\nu} *_0 \vec{\mu})_- \ *_{(m-2)}\ \mathfrak{d}^{m-1}_{s\vec{\eta}*_0 s\vec{\lambda},t\vec{\eta}*_0 t\vec{\lambda}}(\epsilon *_0 \delta)\\ &= \big(\nu_- \ *_0\ A_h(\mu_-)\big) \ *_{(m-2)}\ \left(\mathfrak{d}^{m-1}_{s\vec{\eta},t\vec{\eta}}(\epsilon) \ *_0\ A_h\big(\mathfrak{d}^{m-1}_{s\vec{\lambda},t\vec{\lambda}}(\delta)\big)\right)\\ &= \big(\nu_- \ *_{(m-2)}\ \mathfrak{d}^{m-1}_{s\vec{\eta},t\vec{\eta}}(\epsilon)\big) \ *_0\ \left(A_h(\mu_-) \ *_{(m-2)}\ A_h\big(\mathfrak{d}^{m-1}_{s\vec{\lambda},t\vec{\lambda}}(\delta)\big)\right)\\ &= \mathfrak{d}^m_{\vec{\eta},\vec{\nu}}(\epsilon) \ *_0\ A_h\big(\mathfrak{d}^m_{\vec{\lambda},\vec{\mu}}(\delta)\big).\end{aligned}$$

Here the first equation is by definition of $\mathfrak{d}^m$. The second equation is by the induction hypothesis and definition of $(\vec{\nu} *_0 \vec{\mu})_-$. The third equation is by interchange, and the final equation is by definition of $\mathfrak{d}^m$ together with functoriality of A_h. In this case the second condition (2) from above is an immediate consequence. The induction step when m is odd is by a "dual" argument. Finally, the case where $m = 1$ is trivial. □

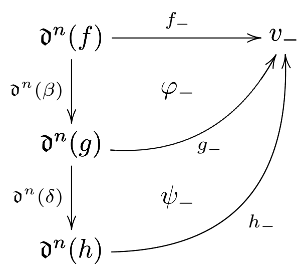
The proof of Lemma 4.4 is typical of the kind of argument involved in proving that $\int A$ is an ω-category. Indeed, we will see that an analogous argument holds for composition along n-cells for $n > 0$. It is to this which we now turn.

4.1.4. Vertical composition. Assume $n > 0$ and suppose we are given m-cells which are composable along an n-cell as indicated in the following diagram:

(36)



where $\vec{f}, \vec{g}$ and $\vec{h}$ are n-cells in $\int A$. Here, $\vec{\alpha}$ and $\vec{\beta}$ are the $(n+1)$-cells bounding $\vec{\varphi}$. I.e., $\vec{\beta} = t^{(m-n-1)}\vec{\varphi}$, et cetera. As such, when $m = n+1$ we have $\vec{\beta} = \vec{\alpha} = \vec{\varphi}$ and similarly for $\vec{\psi}$. We would like to define the composite $(\vec{\psi} *_n \vec{\varphi})$. Since the first component will be the composite $(\psi *_n \varphi)$ taken in $\mathcal{C}$, it remains only to define the second component $(\vec{\psi} *_n \vec{\varphi})_-$. The definition will alternate between those cases where $(n+1)$ is even and those where it is odd. First, assume $(n+1)$ is even. Then we obtain the following diagram in A_y:



where $\vec{x}$ and $\vec{y}$ are the 0-cells bounding all of the cells in question. To see that we have correctly identified the n-cells of A_y bounding φ_- and ψ_- note that when $m = n+1$ this is trivially the case. When $m > n+1$,

$$\begin{aligned} s^{(m-n)}(\varphi_-) &= s(\alpha_-) \\ &= f_- \end{aligned}$$

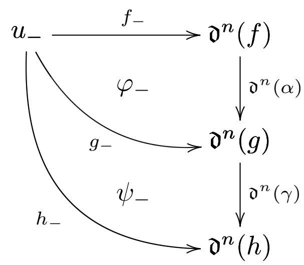
where the first equation is by Lemma 4.3 and the second is by the fact that $(n+1)$ is even. Similarly, $t^{(m-n)}\varphi_- = t\beta_-$ which, since $(n+1)$ is even, is equal to $\mathfrak{d}^{n+1}_{\vec{f},\vec{g}}(\beta)$, as required. Similar calculations show that ψ_- is as indicated in the diagram. Note also that

$$\mathfrak{d}^n_{\vec{u},\vec{v}}(\delta) \ *_{(n-1)} \ \mathfrak{d}^n_{\vec{u},\vec{v}}(\beta) = \mathfrak{d}^n(\delta *_n \beta),$$

by functoriality of $\mathfrak{d}^n_{\vec{u},\vec{v}}(-)$. These observations suggest that, when $(n+1)$ is even, we define $(\vec{\psi} *_n \vec{\varphi})_-$ to be the composite

$$\big(\psi_- \ *_{(n-1)} \ \mathfrak{d}^n_{\vec{u},\vec{v}}(\beta)\big) *_n \varphi_-.$$

Similarly, when $(n+1)$ is odd, we obtain



in A_y and we may define $(\vec{\psi} *_n \vec{\varphi})_-$ to be the analogous composite. Explicitly, given $\vec{\varphi}$ and $\vec{\psi}$ as above, we define

$$(\vec{\psi} *_n \vec{\varphi})_- := \begin{cases} \big(\psi_- \ *_{(n-1)} \ \mathfrak{d}^n_{\vec{u},\vec{v}}(\beta)\big) *_n \varphi_- & \text{if } (n+1) \text{ is even, and} \\ \\ \psi_- *_n \big(\mathfrak{d}^n_{\vec{u},\vec{v}}(\gamma) \ *_{(n-1)} \ \varphi_-\big) & \text{if } (n+1) \text{ is odd.} \end{cases}$$

where $\vec{\beta}$ and $\vec{\gamma}$ are the bounding $(n+1)$-cells as indicated in (36). The first step will be to prove that the composites occurring in the definition of composition are defined and that the resulting cells have the correct boundaries.

Lemma 4.5. *When $\vec{\varphi}$ and $\vec{\psi}$ are as indicated in (36) above and $2 < (n+1) < m$,*

$$s(\vec{\psi} *_n \vec{\varphi})_- = \begin{cases} (s\vec{\psi} *_n s\vec{\varphi})_- & \textit{if } m \textit{ is even, and} \\ \\ \mathfrak{d}^m_{(s\vec{\psi} * s\vec{\varphi}),(t\vec{\psi} * t\vec{\varphi})}(\psi *_n \vec{\varphi}) & \textit{if } m \textit{ is odd,} \end{cases}$$

and

$$t(\vec{\psi} *_n \vec{\varphi})_- = \begin{cases} (t\vec{\psi} *_n t\vec{\varphi})_- & \textit{if } m \textit{ is odd, and} \\ \\ \mathfrak{d}^m_{s\vec{\psi} *_n s\vec{\varphi}, t\vec{\psi} *_n t\vec{\varphi}}(\psi *_n \varphi) & \textit{if } m \textit{ is even,} \end{cases}$$

when $m > n+1$, and when $m = n+1$,

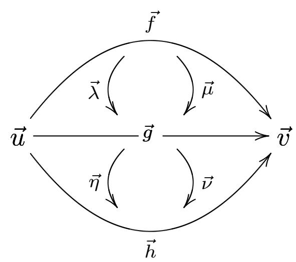
$$s(\vec{\psi} *_n \vec{\varphi})_- = s(\varphi_-), \textit{ and}$$
$$t(\vec{\psi} *_n \vec{\varphi})_- = t(\psi_-).$$

Proof. As with the proof of Lemma 4.4 it is necessary to prove also some stronger facts regarding the behavior of the weighted face functors. Namely, we prove that, where the $(m-1)$-boundaries of $\vec{\varphi}$ and $\vec{\psi}$ are as in the proof of Lemma 4.4,

(37)

$$\mathfrak{d}^m_{\vec{\eta} *_n \vec{\lambda}, \vec{\nu} *_n \vec{\mu}}(\epsilon *_n \delta) = \begin{cases} \big(\mathfrak{d}^m_{\vec{\eta},\vec{\nu}}(\epsilon) \ *_{(n-1)} \ \mathfrak{d}^n_{\vec{u},\vec{v}}(\beta)\big) \ *_n \ \mathfrak{d}^m_{\vec{\lambda},\vec{\mu}}(\delta) & \text{if } (n+1) \text{ is even, and} \\ \\ \mathfrak{d}^m_{\vec{\eta},\vec{\nu}}(\epsilon) \ *_n \ \big(\mathfrak{d}^n_{\vec{u},\vec{v}}(\gamma) \ *_{(n-1)} \ \mathfrak{d}^m_{\vec{\lambda},\vec{\mu}}(\delta)\big) & \text{if } (n+1) \text{ is odd,} \end{cases}$$

for k-cells δ and ϵ of $\mathcal{C}_m(\lambda,\mu)$ and $\mathcal{C}_m(\eta,\nu)$, respectively. Again, this is proved by induction on m simultaneously with the verification of the boundary conditions. Note that we must now consider four distinct cases. E.g., the case where $(n+1)$ is even and m is even, the case where $(n+1)$ is even and m is odd, et cetera. For the induction steps these cases are all by similar arguments to those given in the proof of Lemma 4.4. As such, we describe here only one of the base cases (the others are essentially the same). In particular, we consider the base case where $(n+1)$ is odd and m is even. I.e., $m = n+2$. In this case we have



with $\vec{\beta} = \vec{\mu}$ and $\vec{\gamma} = \vec{\eta}$. In this case, we begin by observing that,

$$\begin{aligned}\mathfrak{d}^{n+1}_{\vec{f},\vec{h}}(\epsilon *_n \delta) &= \mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon *_n \delta) \ *_{(n-1)} \ f_- \\ &= \mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon) \ *_{(n-1)} \ \mathfrak{d}^{n}_{\vec{u},\vec{v}}(\delta) \ *_{(n-1)} \ f_- \\ &= \mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon) *_{(n-1)} \mathfrak{d}^{n+1}_{\vec{f},\vec{g}}(\delta),\end{aligned}$$

where the second equation is by functoriality of $\mathfrak{d}^{n}_{\vec{u},\vec{v}}(-)$. Using this fact we obtain

$$\begin{aligned}\mathfrak{d}^{m}_{\vec{\eta}*_n\vec{\lambda},\vec{\nu}*_n\vec{\mu}}(\epsilon *_n \delta) &= \Big(\nu_- \ *_n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\nu) \ *_{(n-1)} \ \mu_-\big)\Big) \ *_n \ \mathfrak{d}^{n+1}_{\vec{f},\vec{h}}(\epsilon *_n \delta) \\ &= \Big(\nu_- \ *_n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\nu) \ *_{(n-1)} \ \mu_-\big)\Big) \ *_n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon) *_{(n-1)} \mathfrak{d}^{n+1}_{\vec{f},\vec{g}}(\delta)\big) \\ &= \nu_- \ *_n \ \Big(\big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\nu) \ *_n \ \mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon)\big) \ *_{(n-1)} \ \big(\mu_- \ *_n \ \mathfrak{d}^{n+1}(\delta)\big)\Big) \\ &= \nu_- \ _n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon) \ *_{(n-1)} \ \mathfrak{d}^{n+2}_{\vec{\lambda},\vec{\mu}}(\delta)\big),\end{aligned}$$

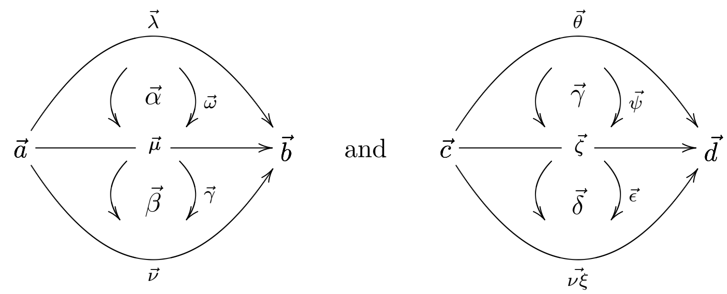
where the final equation is by the fact that $t^{(k+1)}(\epsilon) = \nu$. On the other hand, a straightforward calculation shows that

$$\mathfrak{d}^{m}_{\vec{\eta},\vec{\nu}}(\epsilon) \ *_n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\gamma) \ *_{(n-1)} \ \mathfrak{d}^{m}_{\vec{\lambda},\vec{\mu}}(\delta)\big) = \nu_- \ _n \ \big(\mathfrak{d}^{n}_{\vec{u},\vec{v}}(\epsilon) \ *_{(n-1)} \ \mathfrak{d}^{n+2}_{\vec{\lambda},\vec{\mu}}(\delta)\big),$$

as well. Using this fact, the boundary condition is immediate. □

PROPOSITION 4.6. *Given a (small) strict ω-category $\mathcal{C}$ and a functor $A : \mathcal{C} \longrightarrow \omega$-**Cat**, the Grothendieck construction $\int A$ of A is a (small) strict ω-category.*

PROOF. We have already seen in Lemma 4.2 that $\int A$ is a reflexive globular set and we have described the candidates for composition. As such, it remains to verify the unit, associativity and interchange laws. These are all routine (although somewhat lengthy) computations using the facts established in the proofs of Lemmata 4.4 and 4.5. To give some indication of the kind of calculation involved we consider the case where, for a fixed $p > 0$, $p > n$ and both $(p+1)$ and $(n+1)$ are even. Suppose we are given m-cells, with $m > p$, $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ and $\vec{\delta}$ which are bounded at dimension p as indicated in the following diagram:



I.e., $t^{m-p}(\vec{\alpha}) = \vec{\omega}$ and so forth. Observe that, under these hypotheses, *at dimension* $(p-1) \geq (n+1)$ $\vec{\alpha}$ and $\vec{\beta}$ are parallel (and similarly for $\vec{\gamma}$ and $\vec{\delta}$). Assume that the composites $(\vec{\gamma} *_n \vec{\alpha})$ and $(\vec{\delta} *_n \vec{\beta})$ are defined. By the remark just made it follows that

$$\begin{aligned} t^{m-n}\vec{\alpha} &= \vec{g} \\ &= t^{m-n}\vec{\beta} \\ &= s^{m-n}\vec{\gamma} \\ &= s^{m-n}\vec{\delta}. \end{aligned}$$

Moreover, where $\vec{\varphi}$ is $t^{m-n-1}\vec{\alpha}$ we have also $\vec{\varphi} = t^{m-n-1}\vec{\beta}$. We will prove that the interchange law

$$(\vec{\delta} *_p \vec{\gamma}) *_n (\vec{\beta} *_p \vec{\alpha}) = (\vec{\delta} *_n \vec{\beta}) *_p (\vec{\gamma} *_n \vec{\alpha})$$

holds. First, observe that $\big((\vec{\delta} *_p \vec{\gamma}) *_n (\vec{\beta} *_p \vec{\alpha})\big)_-$ is equal to

$$\Big(\big((\delta_- *_{(p-1)} \mathfrak{d}^p_{\vec{c},\vec{d}}(\psi)) \ *_p \ \gamma_-\big) \ *_{(n-1)} \ \mathfrak{d}^n(\varphi)\Big) \ *_n \ \big((\beta_- *_{(p-1)} \mathfrak{d}^p_{\vec{a},\vec{b}}(\omega)\big) \ *_p \ \alpha_-\big)$$

which, by two applications of interchange, is in turn equal to

$$\Big(\big((\delta_- *_{(p-1)} \mathfrak{d}^p(\psi)) *_{(n-1)} \mathfrak{d}^n(\varphi)\big) \ *_n \ (\beta_- *_{(p-1)} \mathfrak{d}^p(\omega))\Big) *_p \big((\gamma_- *_{(n-1)} \mathfrak{d}^n(\varphi)\big) *_n \alpha_-\big)$$

On the other hand, $\big((\vec{\delta} *_n \vec{\beta}) *_p (\vec{\gamma} *_n \vec{\alpha})\big)_-$ is equal to

$$\Big(\big((\delta_- *_{(n-1)} \mathfrak{d}^n(\varphi)) \ *_n \ \beta_-\big) \ *_{(p-1)} \ \mathfrak{d}^p(\psi *_n \omega)\Big) \ *_p \ \big((\gamma_- *_{(n-1)} \mathfrak{d}^n(\varphi)) \ *_n \ \alpha_-\big).$$

Thus, it suffices to prove that

$$\Big(\big((\delta_- *_{(p-1)} \mathfrak{d}^p(\psi)) *_{(n-1)} \mathfrak{d}^n(\varphi)\big) \ *_n \ (\beta_- *_{(p-1)} \mathfrak{d}^p(\omega))\Big)$$

is equal to

$$\Big(\big((\delta_- *_{(n-1)} \mathfrak{d}^n(\varphi)) \ *_n \ \beta_-\big) \ *_{(p-1)} \ \mathfrak{d}^p(\psi *_n \omega)\Big)$$

This in turn follows, after a further routine calculation, from the fact that, by the proof of Lemma 4.5,

$$\mathfrak{d}^p_{\vec{c} *_n \vec{a}, \vec{d} *_n \vec{b}}(\psi *_n \omega) = \big(\mathfrak{d}^p_{\vec{c},\vec{d}}(\psi) \ *_{(n-1)} \ \mathfrak{d}^n(\varphi)\big) \ *_n \ \mathfrak{d}^p_{\vec{a},\vec{b}}(\omega).$$

□

REMARK 4.7. There are several reasons for referring to this construction as "the" Grothendieck construction. First, it generalizes the usual 1-dimensional Grothendieck construction. Secondly, it should be possible to show that it is a weighted colimit of the given functor, with a suitable weight, just as in the case of the ordinary construction. Such a characterization of this construction is not required for our purposes and we therefore do not address this matter here.

We now turn to the definition of (strict) ω-groupoids.

DEFINITION 4.8. A strict ω-category $\mathcal{C}$ is a **(strict) ω-groupoid** if every $(n+1)$-cell $f : a \longrightarrow b$ possesses a *strict* $*_n$-inverse $f^{-1} : b \longrightarrow a$. I.e.,

$$(f^{-1} \ *_n \ f) = a, \text{ and} \tag{38}$$

$$(f \ *_n \ f^{-1}) = b. \tag{39}$$

This definition generalizes both the usual definition of (1-)groupoid as well as the definition of (strict) 2-groupoid occurring in the work of Moerdijk and Svensson [**65**]. It should be contrasted with the *weaker* notions of ω-groupoid, also defined in the general setting of strict ω-categories, due to Street [**77**], and Kapranov and Voevodsky [**49**]. The essential difference with the definition from [**77**] is that there the notion of inverse is weakened so that, instead of equations, it is required that there exist (systems of) higher-dimensional cells $(f^{-1} *_n f) \Longrightarrow a$, *et cetera.* In [**49**] it is further required that the higher-dimensional cells witnessing invertibility of f satisfy additional coherence conditions. Such weaker notions of ω-groupoid are also of interest for interpreting type theory, but will not be considered here.

With Definition 4.8 at hand we obtain the following Corollary to Proposition 4.6:

COROLLARY 4.9. *If $\mathcal{C}$ is a (small) strict ω-groupoid and $A : \mathcal{C} \longrightarrow \omega$-**Gpd**, then $\int A$ is a (small) strict ω-groupoid.*

PROOF. Given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$, the inverse $(\vec{f})^{-1}$ is the pair $\big(f^{-1}, A_{f^{-1}}(f_-)^{-1}\big)$. For $n > 0$, given a $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ we define

$$(\vec{\varphi})^{-1} := \big(\varphi^{-1}, \varphi_-^{-1} *_{(n-1)} \mathfrak{d}^n_{s\vec{\alpha},t\vec{\alpha}}(\varphi^{-1})\big)$$

when $(n+1)$ is even, and

$$(\vec{\varphi})^{-1} := \big(\varphi^{-1}, \mathfrak{d}^n_{s\vec{\alpha},t\vec{\alpha}}(\varphi^{-1}) *_{(n-1)} \varphi_-^{-1}\big)$$

when $(n+1)$ is odd. It is straightforward to verify that these are indeed inverses with respect to composition in $\int A$. □

4.2. The dual Grothendieck construction

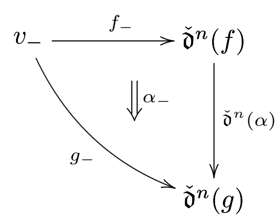
The purpose of this section is to describe the *dual Grothendieck construction* $\int^* A$ of a functor $A : \mathcal{C} \longrightarrow \omega$-**Cat** obtained by choosing the opposite orientation for the weighted faces of cells from that in $\int A$. Accordingly, we also introduce the "co-weighted face" or "dual weighted face" functors $\check{\mathfrak{d}}^n$ associated with this construction. Finally, we describe the "duality" functor $\neg : \int A \longrightarrow \int^* A$. The action of $\neg$ is, essentially, to "turn around" the triangles which constitute the cells of $\int A$ in a functorial way.

4.2.1. The dual construction. Given a functor $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Cat}$, the **dual Grothendieck construction** $\int^* A$ is the ω-category obtained by "reversing" the weighting decision made in the definition of the Grothendieck construction $\int A$ of A. I.e., $\int^* A$ has the same 0-cells as $\int A$, but 1-cells $\vec{f} : \vec{x} \longrightarrow \vec{y}$ are pairs (f, f_-) with $f : x \longrightarrow y$ an arrow in $\mathcal{C}$ and $f_- : y_- \longrightarrow A_f(x_-)$ an arrow in A_x. As with $\int A$, we define the dual weighted face of such an arrow f_- to be $A_f(x_-)$ and obtain a dual weighted face functor:

$$\mathcal{C}_1(x, y) \xrightarrow{\check{\mathfrak{d}}^1_{\vec{x},\vec{y}}} A_y$$

by setting $\check{\mathfrak{d}}^1_{\vec{x},\vec{y}}(\gamma)$ to be $x_-.\gamma$. Thus, in particular, $\check{\mathfrak{d}}^1_{\vec{x},\vec{y}} = \mathfrak{d}^1_{\vec{x},\vec{y}}$. The construction is given inductively as for $\int A$ by the following steps:

($(n+1)$ is even): A $(n+1)$-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$, with $\vec{f}, \vec{g} : \vec{v} \rightrightarrows \vec{w}$, is a pair consisting of a $(n+1)$-cell $\alpha : f \rightrightarrows g$ in $\mathcal{C}$ and a $(n+1)$-cell α_- in A_y as indicated in the following diagram:

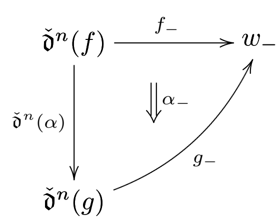


The dual weighted face functor

$$\mathcal{C}_{n+1}(f, g) \xrightarrow{\check{\mathfrak{d}}^{n+1}_{\vec{f},\vec{g}}} (A_y)_n\big(v_-, \check{\mathfrak{d}}^n(g)\big)$$

is given by defining $\check{\mathfrak{d}}^{n+1}_{\vec{f},\vec{g}}(\gamma)$ to be $\check{\mathfrak{d}}^n(\gamma) *_{(n-1)} f_-$.

($(n+1)$ is odd): On the other hand, when $(n+1)$ is odd a $(n+1)$-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$ is given by $\alpha : f \Longrightarrow g$ as above together with a $(n+1)$-cell of A_y as indicated in the following diagram:



Here the dual weighted face functor

$$\mathcal{C}_{n+1}(f, g) \xrightarrow{\check{\mathfrak{d}}^{n+1}_{\vec{f},\vec{g}}} (A_y)_n\big(\check{\mathfrak{d}}^n(f), w_-\big)$$

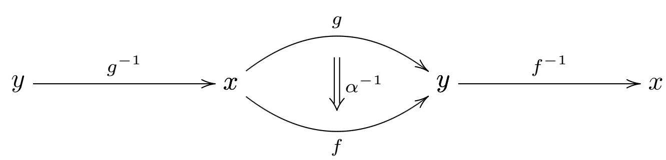
is obtained by defining $\check{\mathfrak{d}}^{n+1}_{\vec{f},\vec{g}}(\gamma)$ to be $g_- *_{(n-1)} \check{\mathfrak{d}}^n(\gamma)$.

Composition in $\int^* A$ is obtained in the obvious way and, using arguments essentially identical (modulo the evident shift in dimension) to those from Section 4.1, we obtain the following proposition:

PROPOSITION 4.10. *Given a functor* $A : \mathcal{C} \longrightarrow \omega$-**Cat** *with* $\mathcal{C}$ *an* ω*-category,* $\int^* A$ *is an* ω*-category. Moreover, if* $\mathcal{C}$ *is an* ω*-groupoid and* $A : \mathcal{C} \longrightarrow \omega$-**Gpd**, *then* $\int^* A$ *is also an* ω*-groupoid.*

We will ultimately show that, when $\mathcal{C}$ is an ω-groupoid and $A : \mathcal{C} \longrightarrow \omega$-**Gpd**, there exists a functor $\neg : \int A \longrightarrow \int^* A$ which we will employ in the construction of the identity types. However, before we can describe this functor we will require some auxiliary notions.

4.2.2. Duals and functors induced by composition. When a category $\mathcal{C}$ is an ordinary 1-dimensional groupoid, then there exists an isomorphism $\sigma : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ which is the identity on objects and sends an arrow $f : x \longrightarrow y$ to its inverse $f^{-1} : y \longrightarrow x$. Now, when $\mathcal{C}$ is an ω-groupoid there is also a "dual" functor $\sigma : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ we now consider. Recall that $\mathcal{C}^{\mathrm{op}}$ is obtained by reversing *only* 1-cells. For example, given a 2-cell $\alpha : f \longrightarrow g$ in $\mathcal{C}$, $\sigma\alpha : f^{-1} \longrightarrow g^{-1}$ is defined to be $(g^{-1} *_0 \alpha^{-1} *_0 f^{-1})$. As a diagram:



Then, where $\varphi : \alpha \longrightarrow \beta$ is a 3-cell,

$$\sigma(\varphi) := g^{-1} \;*_0\; (\beta^{-1} \;*_1 \varphi^{-1} \;*_1\; \alpha^{-1}) \;*_0\; f^{-1}.$$

At higher-dimensions the construction is the same, adding a new "inner" block obtained by composing φ^{-1} with the inverses of its boundary maps. Although we will make some minor use of this functor for arbitrary ω-groupoids, we are more concerned with a related construction which yields a similar kind of functor $\neg : \int A \longrightarrow \int^* A$. Rather than describing σ in detail, we instead focus our attention on the construction of the map $\neg$ and mention that the construction of σ is essentially the same (with obvious modifications taking into account the shift from triangles to globes).

In an ω-category $\mathcal{A}$, when $f : x \longrightarrow y$ and $g : u \longrightarrow v$ are fixed $(n+1)$-cells with $n \geq 0$, if x, y, u and v are all parallel, then there exists a functor

$$\mathcal{A}_{n+1}(y,u) \xrightarrow{(g \,*_n\, - \,*_n\, f)} \mathcal{A}_{n+1}(x,v)$$

which acts in the obvious way on cells of $\mathcal{A}_{n+1}(y,u)$.

Now, consider the case where the ω-category $\mathcal{A}$ is itself of the form $\int A$ for $A : \mathcal{C} \longrightarrow \omega$-**Gpd** with $\mathcal{C}$ an ω-groupoid. Assume we are given parallel $(n+1)$-cells $\vec{\alpha}$ and $\vec{\beta}$ of $\int A$, with $n > 0$, bounded by n-cells $\vec{f}$ and $\vec{g}$. Then, when $(n+1)$ is even, there exists a span

$$\mathfrak{d}^{n+1}(\alpha) \xleftarrow{\;\alpha_-\;} f_- \xrightarrow{\;\beta_-\;} \mathfrak{d}^{n+1}(\beta)$$

of $(n+1)$-cells in A_y. Taking inverses yields the functor

$$(A_y)_{n+1}\big(f_-, \mathfrak{d}^{n+1}(\beta)\big) \xrightarrow{(\beta_-^{-1} \,*_n\, - \,*_n\, \alpha_-^{-1})} (A_y)_{n+1}\big(\mathfrak{d}^{n+1}(\alpha), f_-\big).$$

In the same way, if $(n+1)$ is odd, we obtain the functor

$$(A_y)_{n+1}\big(\mathfrak{d}^{n+1}(\alpha), g_-\big) \xrightarrow{(\beta_-^{-1} *_n - *_n \alpha_-^{-1})} (A_y)_{n+1}\big(g_-, \mathfrak{d}^{n+1}(\beta)\big).$$

When such parallel $(n+1)$-cells $\vec{\alpha}$ and $\vec{\beta}$ of $\int A$ are fixed, we define the functor

$$(A_y)_{n+1}(s\alpha_-, t\beta_-) \xrightarrow{\rho^0_{\vec{\alpha},\vec{\beta}}} (A_y)_{n+1}(t\alpha_-, s\beta_-)$$

to be $(\beta_-^{-1} *_n - *_n \alpha_-^{-1})$. When $\vec{\varphi}$ is a fixed $(n+2)$-cell, we denote by $\rho^0_{\partial\vec{\varphi}}$ the functor $\rho^0_{s\vec{\varphi},t\vec{\varphi}}$.

4.2.3. The functors $\rho^k_{\vec{\alpha},\vec{\beta}}$. Part of the utility of the functors $\rho^0_{\vec{\alpha},\vec{\beta}}$ is that, in a suitable sense, they can be iterated. To see this, assume $n > 1$ and let $\vec{\alpha}$ and $\vec{\beta}$ be given. Then, since $n > 1$, we have

$$(A_y)_n(s^2\alpha_-, t^2\beta_-) \xrightarrow{\rho^0_{s\vec{\alpha},t\vec{\alpha}}} (A_y)_n(t^2\alpha_-, s^2\beta_-)$$

and we may define the functor

$$(A_y)_{n+1}(s\alpha_-, t\beta_-) \xrightarrow{\rho^1_{\vec{\alpha},\vec{\beta}}} \big((A_y)_n(t^2\alpha_-, s^2\beta_-)\big)^+$$

as follows:

$$\rho^1_{\vec{\alpha},\vec{\beta}} := \big(\rho^0_{\partial\vec{\alpha}}\big)^+ \circ j \circ \rho^0_{\vec{\alpha},\vec{\beta}},$$

where j is the (usually nameless) inclusion

$$(A_y)_{n+1}(t\alpha_-, s\beta_-) \longrightarrow \big((A_y)_n(s^2\alpha_-, t^2\beta_-)\big)^+$$

from Section A.3.4 of Appendix A. Explicitly, given a cell γ of $(A_y)_{n+1}(s\alpha_-, t\beta_-)$,

$$\rho^1_{\vec{\alpha},\vec{\beta}}(\gamma) = g_-^{-1} *_{(n-1)} (\beta_-^{-1} *_n \gamma *_n \alpha_-^{-1}) *_{(n-1)} f_-^{-1},$$

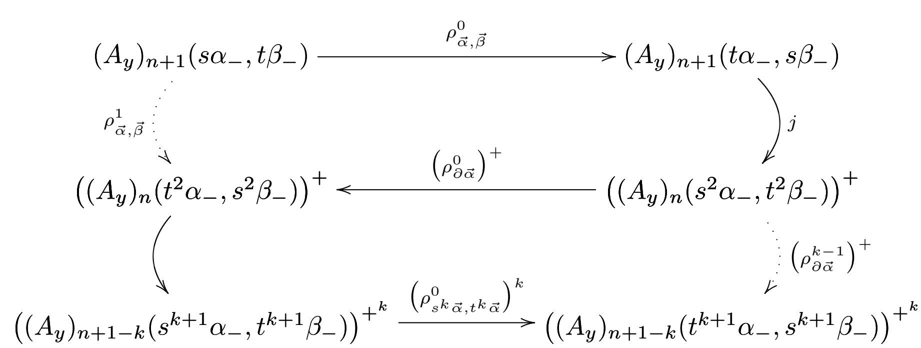
where $\vec{f} = s\vec{\alpha}$ and $\vec{g} = t\vec{\alpha}$. In general, given parallel $(n+1)$-cells $\vec{\alpha}$ and $\vec{\beta}$, if $0 < k \leq n$, then we define functors

$$(A_y)_{n+1}(s\alpha_-, t\beta_-) \xrightarrow{\rho^k_{\vec{\alpha},\vec{\beta}}} \big((A_y)_{n+1-k}(t^{k+1}\alpha_-, s^{k+1}\beta_-)\big)^{+^k}$$

by setting

$$\rho^k_{\vec{\alpha},\vec{\beta}} := \big(\rho^{k-1}_{\partial\vec{\alpha}}\big)^+ \circ j \circ \rho^0_{\vec{\alpha},\vec{\beta}}.$$

This definition can be visualized as in the following diagram:



Henceforth, when no confusion will result we omit the map i and the superscripts $(-)^+$ when dealing with these maps. This convention is justified by the fact that whenever we actually compute with these maps their action is the identity.

4.2.4. Definition of the duality functor. We now describe a functor $\neg : \int A \longrightarrow \int^* A$, called the **duality functor**, which will be required for the construction of the identity types in Section 4.3 below. Because the definition is somewhat technical, we begin by describing it in the first two dimensions where the geometry involved is more apparent. Throughout this section we assume that $A : \mathcal{C} \longrightarrow \omega$-**Gpd** with $\mathcal{C}$ a small ω-groupoid.

Given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ in $\int A$, we have in A_y the map $f_- : \mathfrak{d}^1_{\vec{x},\vec{y}}(f) \longrightarrow y_-$ and, by taking its inverse, we obtain an arrow $(f, f_-^{-1}) : \vec{x} \longrightarrow \vec{y}$ in $\int^* A$ also. I.e.,

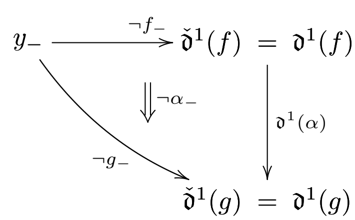
$$y_- \xrightarrow{f_-^{-1}} \check{\mathfrak{d}}^1_{\vec{x},\vec{y}}(f) \;=\; \mathfrak{d}^1_{\vec{x},\vec{y}}(f)$$

in A_y. Thus, at dimension 0 we define $\neg(\vec{x})$ to be just $\vec{x}$ and at dimension 1,

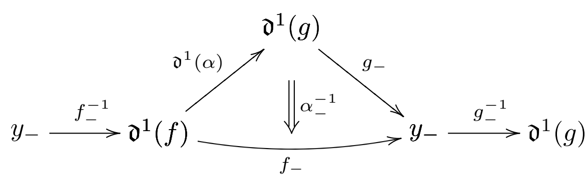
$$\neg(\vec{f}) := (f, f_-^{-1}).$$

We sometimes commit a slight abuse of notation and denote the second component of $\neg(\vec{\varphi})$ by $\neg\varphi_-$. Because the first component of $\neg(\vec{\varphi})$ is φ in all dimensions this should lead to no confusion.

Matters become more interesting when we consider a 2-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$ in $\int A$. In this case, we would like to obtain a 2-cell $\neg\alpha_-$ as indicated in the following diagram



in A_y where $\vec{f}, \vec{g} : \vec{x} \rightrightarrows \vec{y}$. Because A_y is an ω-groupoid we may form the composite



which possesses the appropriate boundary to be $\neg\alpha_-$. Indeed, we define

$$\begin{aligned}\neg\alpha_- &:= \rho^0_{\vec{f},\vec{g}}(\alpha_-^{-1}) \\ &= g_-^{-1} *_0 \alpha_-^{-1} *_0 f_-^{-1}.\end{aligned}$$

This observation, that $\neg\alpha_-$ can be defined using the functors ρ^k from above, does in fact generalize to higher dimensions where we set

$$\neg\varphi_- := \rho^{m-2}_{\partial\vec{\varphi}}(\varphi_-^{-1})$$

when $\vec{\varphi}$ is a m-cell of $\int A$ with $m \geq 2$.

LEMMA 4.11. *With this definition, $\neg : \int A \longrightarrow \int^* A$ is a map of (reflexive) globular sets.*

PROOF. We must show that $\neg\vec{\varphi}$ satisfies the co-weighted face conditions from Section 4.2.1. We have already seen that, when $\vec{\varphi}$ is a $(n+1)$-cell of $\int A$, these conditions are satisfied for $n = 0, 1$. In general, when $\vec{\varphi}$ is a $(n+1)$-cell, with $n \geq 1$, with source $\vec{f}$ and target $\vec{g}$ we will prove simultaneously by induction on n the following two facts at each stage $(n+1)$:

(1) The co-weighted face conditions are satisfied.
(2) The following equations are satisfied:

$$\rho^{n-1}_{\partial\vec{\varphi}}\big(\mathfrak{d}^{n+1}_{\vec{f},\vec{g}}(\gamma)\big) = \check{\mathfrak{d}}^{n+1}_{\neg\vec{f},\neg\vec{g}}(\gamma)$$

whenever γ is an appropriate cell.

The co-weighted face conditions are easily seen to hold in the base cases. For condition (2), assume $n = 1$ and observe that in this case

$$\begin{aligned}\rho^{0}_{\partial\vec{\alpha}}\big(\mathfrak{d}^{2}_{\vec{f},\vec{g}}(\gamma)\big) &= g_-^{-1} \ *_0\ g_- \ *_0\ \mathfrak{d}^1(\gamma) \ *_0\ f_-^{-1} \\ &= \check{\mathfrak{d}}^{2}_{\neg\vec{f},\neg\vec{g}}(\gamma).\end{aligned}$$

For the induction step of (1) when $(n+1)$ is even we note that

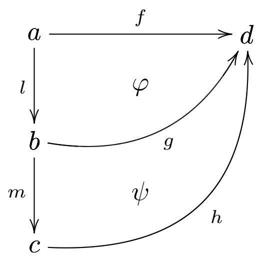
$$\begin{aligned}t(\neg\varphi_-) &= \rho^{n-1}_{\partial\vec{\varphi}}(t\varphi_-^{-1}) \\ &= \rho^{n-1}_{\partial\vec{\varphi}}(s\varphi_-) \\ &= \rho^{n-2}_{\partial\vec{\varphi}}(g_-^{-1} \ *_{(n-1)}\ f_- \ *_{(n-1)}\ f_-^{-1}) \\ &= \rho^{n-2}_{\partial\vec{\varphi}}(g_-^{-1}) \\ &= \neg(t\vec{\varphi})_-.\end{aligned}$$

On the other hand,

$$\begin{aligned}s(\neg\varphi_-) &= \rho^{n-2}_{\partial\vec{f}}\big(g_-^{-1} \ *_{(n-1)}\ \mathfrak{d}^{n+1}_{\vec{f},\vec{g}}(\varphi) \ *_{(n-1)}\ f_-^{-1}\big) \\ &= \rho^{n-2}_{\partial\vec{f}}\big(\mathfrak{d}^{n}(\varphi) \ *_{(n-1)}\ f_-^{-1}\big) \\ &= \rho^{n-2}_{\partial\vec{f}}\big(\mathfrak{d}^{n}(\varphi)\big) \ *_{(n-1)}\ \rho^{n-2}_{\partial\vec{f}}(f_-^{-1}) \\ &= \check{\mathfrak{d}}^{n}(\varphi) \ *_{(n-1)}\ \neg f_- \\ &= \check{\mathfrak{d}}^{n+1}_{\neg\vec{f},\neg\vec{g}}(\varphi)\end{aligned}$$

where the penultimate equation is by the induction hypothesis. The induction step of (2) is by a (similar and) straightforward calculation. The case where $(n+1)$ is odd is essentially "dual". Finally, observe that $\neg$ trivially preserves identities and so constitutes a homomorphism of reflexive globular sets. □

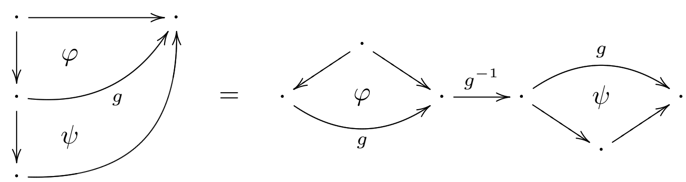
4.2.5. Composing triangles in ω-groupoids. In order to show that $\neg$ is a functor it will be convenient to have an alternative description of composition in $\int A$ and $\int^* A$ in the case where $\mathcal{C}$ is an ω-groupoid and $A : \mathcal{C} \longrightarrow \omega$-**Gpd**. In particular, we observe that, whenever $\mathcal{C}$ is an ω-groupoid, given a diagram as follows:



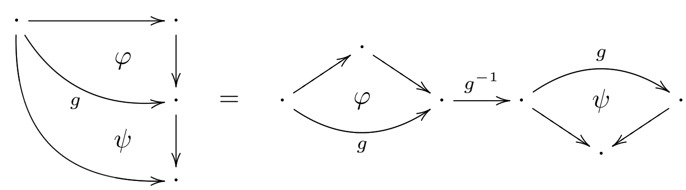
with f, g, h, $\ldots$ n-cells with $n > 0$, then

$$(\psi \ *_n \ \varphi) = (\psi \ *_{(n-1)} \ g^{-1} \ *_{(n-1)} \ \varphi).$$

As a diagram:



Similarly,



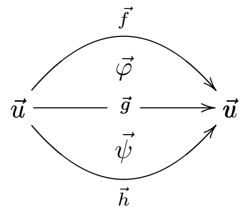
I.e., in this case also $(\psi *_n \varphi)$ is equal to $(\psi *_{(n-1)} g^{-1} *_{(n-1)} \varphi)$. Accordingly, when dealing with $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$ where $\mathcal{C}$ is itself an ω-groupoid, we obtain a description of composition terms $(\vec{\psi} *_n \vec{\varphi})_-$ which is independent of whether $(n+1)$ is even or odd. Namely,

$$(\vec{\psi} \ *_n \ \vec{\varphi})_- = \psi_- \ *_{(n-1)} \ g_-^{-1} \ *_{(n-1)} \ \varphi_-,$$

where $\vec{g}$ is the appropriate n-cell bounding $\vec{\varphi}$ and $\vec{\psi}$. I.e., $\vec{g} = t^{m-n}\vec{\varphi} = s^{m-n}\vec{\psi}$. It is this decomposition of the terms $(\vec{\psi} *_n \vec{\varphi})_-$ which allows us to prove the following result.

PROPOSITION 4.12. *The map* $\neg : \int A \longrightarrow \int^* A$ *is a functor.*

PROOF. By Lemma 4.11 $\neg$ is a homomorphism of reflexive globular sets. As such, it remains to verify that composition is preserved. Suppose we are given m-cells $\vec{\varphi}$ and $\vec{\psi}$ for which $s^{m-n}\vec{\psi} = \vec{g}$ and $t^{m-n}\vec{\varphi} = \vec{g}$. We must verify that $\neg(\vec{\psi} *_n \vec{\varphi})_-$ is equal to $\big(\neg\vec{\psi} *_n \neg\vec{\varphi}\big)_-$. The proof is divided into the cases where $n = 0$ and $n > 0$. Both cases are essentially straightforward verifications and we leave the first to the reader. For the second case, suppose $\vec{\varphi}$ and $\vec{\psi}$ are bounded by n-cells $\vec{f}, \vec{g}$ and $\vec{h}$ as indicated in the following diagram:



Then, by the observation above about composition in ω-groupoids together with the interchange law, we obtain

$$\begin{aligned}\neg(\vec{\psi} \ *_n \ \vec{\varphi})_- &= \rho^{m-2}_{\partial(\vec{\psi} *_n \vec{\varphi})}\big(\psi_-^{-1} \ *_{(n-1)} \ g_-^{-1} \ *_{(n-1)} \ \varphi_-^{-1}\big) \\ &= \rho^{n-1}_{\vec{f},\vec{h}}\big(\rho^{m-n}_{\partial\vec{\psi}}(\psi_-^{-1}) \ *_{(n-1)} \ g_-^{-1} \ *_{(n-1)} \ \rho^{m-n}_{\partial\vec{\varphi}}(\varphi_-^{-1})\big).\end{aligned}$$

Moreover, by definition of $\rho^{n-1}_{\vec{f},\vec{h}}$ this is equal to

$$\rho^{n-2}_{\partial\vec{f}}\big(\rho^{m-n+1}_{\partial\vec{\psi}}(\psi_-^{-1}) \ *_{(n-1)} \ g_- \ *_{(n-1)} \ \rho^{m-n+1}_{\partial\vec{\varphi}}(\varphi_-)\big).$$

Finally, by functoriality of $\rho^{n-2}_{\partial\vec{f}}$ and the definition of $\neg$, this is the same as

$$\neg\psi_- \ *_{(n-1)} \ (\neg g_-)^{-1} \ *_{(n-1)} \ \neg\varphi_- = \big((\neg\vec{\psi}) \ *_n \ (\neg\vec{\varphi})\big)_-.$$

□

4.3. Identity types

When $A : \mathcal{C} \longrightarrow \omega$-**Gpd** has as its domain an ω-groupoid $\mathcal{C}$, the **identity type (for A)** is a functor $I_A : \int A \circ \pi \longrightarrow \omega$-**Gpd** where $\pi : \int A \longrightarrow \mathcal{C}$ is the projection. By definition, $\int A \circ \pi$ is has as objects tuples $\vec{x} = (x, x_-, x_+)$ where x_- and x_+ are themselves objects of A_x. Similarly, n-cells $\vec{f}$ in $\int A\pi$ are tuples (f, f_-, f_+) such that both (f, f_-) and (f, f_+) are n-cells in $\int A$. I.e., we have the following:

PROPOSITION 4.13. *Given a functor $A : \mathcal{C} \longrightarrow \omega$-**Gpd**, the following diagram is a pullback*

$$\begin{array}{ccc} \int A\pi & \longrightarrow & \int A \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \int A & \xrightarrow[\pi]{} & \mathcal{C} \end{array}$$

where the nameless functors the evident projections.

With this in mind, it is straightforward to describe the action of I_A on objects. Namely, $I_A(x, x_-, x_+)$ is defined to be the ω-groupoid $A_x(x_-, x_+)$. Perhaps though, in light of the discussion of the combinatorics of the Grothendieck construction from the previous sections, matters are more complicated in higher dimensions. It is to this task which we now turn.

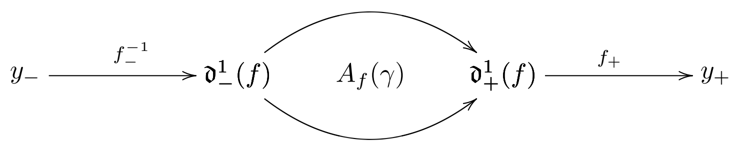
REMARK 4.14. Because, when $\int A\pi$ is involved, we are dealing with two instances of the Grothendieck construction $\int A$ it will be convenient to introduce some notation to describe the various weighted face functors. In particular, because we adopt the convention of notating cells $\vec{f}$ of $\int A\pi$ by (f, f_-, f_+) we will also notate the corresponding weighted face functors accordingly. I.e., we write $\mathfrak{d}^n_-(f)$ and

$\mathfrak{d}^n_+(f)$ for the instances of these functors corresponding to the appropriate "negative" and "positive" projections $\int A\pi \longrightarrow \int A$. When subscripts are necessary we write, e.g., $\mathfrak{d}^n_{\vec{\alpha}\vec{\beta};\xi}$ with $\xi = +, -$. We adopt also a corresponding convention for the "co-weighted face" functors.

4.3.1. Identity types in dimensions 1 and 2. Given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ in $\int A\pi$, $I_A(\vec{f})$ is the functor

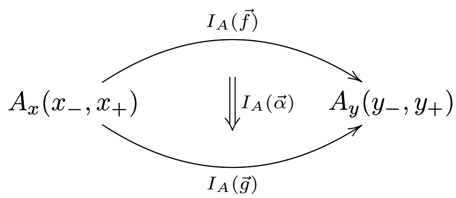
$$A_x(x_-, x_+) \longrightarrow A_y(x_-, x_+)$$

which sends any cell γ of $A_x(x_-, x_+)$ to the following composite:



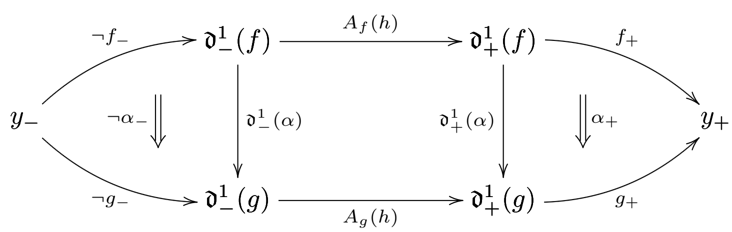
I.e., $I_A(\vec{f})(\gamma)$ is defined to be $(f_+ \ *_0 \ A_f(\gamma) \ *_0 \ f_-^{-1})$. Already at this stage we have tacitly made use of the dual functor $\neg$ since $\neg f_-$ is f_-^{-1}.

Now, given a 2-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$ we must provide a natural transformation $I_A(\vec{\alpha})$ as indicated in the following diagram:



Fixing an object $h : x_- \longrightarrow x_+$ of $A_x(x_-, x_+)$, the component $I_A(\vec{\alpha})_h$ of this transformation at h is described by the composite of the following diagram in A_y:

(40)



where the middle square commutes (on the nose) by naturality of A_α. Explicitly,

$$I_A(\vec{\alpha})_h := (f_+ \ *_0 \ A_g(h) \ *_0 \ \neg\alpha_-) \ *_1 \ (\alpha_+ \ *_0 \ A_f(h) \ *_0 \ \neg f_-).$$

With this definition in mind, we now turn to the introduction of some auxiliary functors which will allow us to describe the identity types in higher dimensions.

4.3.2. Auxiliary functors. Holding an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ of $\int A\pi$ fixed together with an object h of $A_x(x_-, x_+)$ we define functors

$$(A_y)_1\big(\mathfrak{d}^1_+(f), y_+\big) \xrightarrow{\Psi^1_{\vec{f},h}} (A_y)_1(y_-, y_+), \text{ and}$$

$$(A_y)_1\big(y_-, \mathfrak{d}^1_-(f)\big) \xrightarrow{\check{\Psi}^1_{\vec{f},h}} (A_y)_1(y_-, y_+)$$

by setting

$$\Psi^1_{\vec{f},h}(-) := (- \ *_0 \ A_f h \ *_0 \neg f_-), \text{ and}$$
$$\check{\Psi}^1_{\vec{f},h}(-) := (f_+ \ *_0 \ A_f h \ *_0 -).$$

As usual, we omit either one or both of the subscripts when no confusion will result. The first thing we observe about these functors is that

$$\Psi^1_{\vec{f}}(f_+) = \check{\Psi}^1_{\vec{f}}(\neg f_-). \tag{41}$$

The next feature which should be emphasized is that these functors interact in an important way with the usual weighted face functors. In particular, the following diagram (of ω-categories) commutes:

$$\begin{array}{ccc} \mathcal{C}_2(f,g) & \xrightarrow{\check{\mathfrak{d}}^2_{\neg\vec{f},\neg\vec{g};-}} & (A_y)_1\big(y_-, \check{\mathfrak{d}}^1_-(g)\big) \\ \Big\downarrow{\scriptstyle \mathfrak{d}^2_{\vec{f},\vec{g};+}} & & \Big\downarrow{\scriptstyle \check{\Psi}^1_{\vec{g},h}} \\ (A_y)_1\big(\mathfrak{d}^1_+(f), y_+\big) & \xrightarrow[\Psi^1_{\vec{f},h}]{} & (A_y)_1(y_-, y_+) \end{array} \tag{42}$$

To see this, we note that

$$\begin{aligned} \check{\Psi}^1_{\vec{g}}\big(\check{\mathfrak{d}}^2_-(\gamma)\big) &= g_+ \ *_0 \ A_g h \ *_0 \mathfrak{d}^1_-(\gamma) \ *_0 \ \neg f_- \\ &= g_+ \ *_0 \ \mathfrak{d}^1_+(\gamma) \ *_0 \ A_f h \ *_0 \ \neg f_- \\ &= \Psi^1_{\vec{f}}\big(\mathfrak{d}^2_+(\gamma)\big), \end{aligned}$$

where the second equation is by naturality of A_γ. We now observe that, when $\vec{\alpha}$ is as above, the component $I_A(\vec{\alpha})$ at h can be described using these functors as follows:

$$I_A(\vec{\alpha})_h = \check{\Psi}^1_{\vec{g},h}(\neg\alpha_-) \ *_1 \ \Psi^1_{\vec{f},h}(\alpha_+)$$

In particular, $I_A(\vec{\alpha})_h$ is obtained by composing

$$\Psi^1_{\vec{f}}(f_+) \xrightarrow{\Psi^1_{\vec{f}}(\alpha_+)} \Psi^1_{\vec{f}}\big(\mathfrak{d}^2_+(\alpha)\big) \ = \ \check{\Psi}^1_{\vec{g}}\big(\check{\mathfrak{d}}^2_-(\alpha)\big) \xrightarrow{\check{\Psi}^1_{\vec{g}}(\neg\alpha_-)} \check{\Psi}^1_{\vec{g}}(\neg g_-) \ = \ \Psi^1_{\vec{g}}(g_+). \tag{43}$$

As such, we have employed both (41) and (42) in order to show that the composite defining $I_A(\vec{\alpha})_h$ makes sense. We emphasize this point because it provides the first look at what will be required in higher dimensions.

At the next stage, holding a 2-cell $\vec{\alpha} : \vec{f} \longrightarrow \vec{g}$ and an arrow $h : x_- \longrightarrow x_+$ as above fixed, we define functors

$$(A_y)_2\big(f_+, \mathfrak{d}^2_{\vec{f},\vec{g};+}(\alpha)\big) \xrightarrow{\Psi^2_{\vec{\alpha},h}} (A_y)_2\big(\Psi^1_{\vec{f},h}(f_+), \Psi^1_{\vec{g},h}(g_+)\big), \text{ and}$$
$$(A_y)_2\big(\check{\mathfrak{d}}^2_{\neg\vec{f},\neg\vec{g};-}(\alpha), \neg g_-\big) \xrightarrow{\check{\Psi}^2_{\vec{\alpha},h}} (A_y)_2\big(\Psi^1_{\vec{f},h}(f_+), \Psi^1_{\vec{g},h}(g_+)\big)$$

as follows

$$\Psi^2_{\vec{\alpha},h}(-) := \check{\Psi}^1_{\vec{g},h}\big(\neg\alpha_-\big) \ *_1 \ \Psi^1_{\vec{f},h}(-), \text{ and}$$
$$\check{\Psi}^2_{\vec{\alpha},h}(-) := \check{\Psi}^1_{\vec{g},h}(-) \ *_1 \ \Psi^1_{\vec{f},h}(\alpha_+).$$

The motivation for these definitions can perhaps best be seen in consultation with (40). It follows, using the same reasoning from (43), that these functors are well-defined and possess the appropriate boundaries. An immediate consequence of the definition is that

$$\Psi^2_{\vec{\alpha},h}(\alpha_+) = \check{\Psi}^2_{\vec{\alpha},h}(\neg\alpha_-).$$

Moreover, (42) also generalizes to dimension 2 to give:

$$\begin{array}{ccc}
\mathcal{C}_3(\alpha,\beta) & \xrightarrow{\check{\mathfrak{d}}^3_{\neg\vec{\alpha},\neg\vec{\beta};-}} & (A_y)_2\big(\check{\mathfrak{d}}^2_-(\alpha),\neg g_-\big) \\
{\scriptstyle \mathfrak{d}^3_{\vec{\alpha},\vec{\beta};+}}\big\downarrow & & \big\downarrow{\scriptstyle \check{\Psi}^2_{\vec{\alpha},h}} \\
(A_y)_2\big(f_+,\mathfrak{d}^2_+(\beta)\big) & \xrightarrow[\Psi^2_{\vec{\beta},h}]{} & (A_y)_2\big(\Psi^1_{\vec{f},h}(f_+),\Psi^1_{\vec{g},h}(g_+)\big)
\end{array}$$

when $\vec{\alpha},\vec{\beta} : \vec{f} \Longrightarrow \vec{g}$ are fixed 2-cells. To see that the equation holds we reason as follows:

$$\begin{aligned}
\Psi^2_{\vec{\beta},h}\big(\mathfrak{d}^3_+(\gamma)\big) &= \check{\Psi}^1_{\vec{g},h}(\neg\beta_-) \ *_1\ \Psi^1_{\vec{f},h}\big(\mathfrak{d}^3_+(\gamma)\big) \\
&= \check{\Psi}^1_{\vec{g},h}(\neg\beta_-) \ *_1\ \Psi^1_{\vec{f},h}\big(\mathfrak{d}^2_+(\gamma)\big) \ *_1\ \Psi^1_{\vec{f},h}(\alpha_+) \\
&= \check{\Psi}^1_{\vec{g},h}(\neg\beta_-) \ *_1\ \check{\Psi}^1_{\vec{g},h}\big(\check{\mathfrak{d}}^2_-(\gamma)\big) \ *_1\ \Psi^1_{\vec{f},h}(\alpha_+) \\
&= \check{\Psi}^1_{\vec{g},h}\big(\neg\beta_- \ *_1\ \check{\mathfrak{d}}^2_-(\gamma)\big) \ *_1\ \Psi^1_{\vec{f},h}(\alpha_+) \\
&= \check{\Psi}^1_{\vec{g},h}\big(\check{\mathfrak{d}}^3_-(\gamma)\big) \ *_1\ \Psi^1_{\vec{f},h}(\alpha_+) \\
&= \check{\Psi}^2_{\vec{\alpha},h}\big(\check{\mathfrak{d}}^3_-(\gamma)\big),
\end{aligned}$$

where the third equation is by (42). We will now show that this construction can be generalized to all dimensions $(n+1)$ with $n \geq 2$. In particular we will prove that at each stage $(n+1)$, for every $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \longrightarrow \vec{\beta}$ and arrow $h : x_- \longrightarrow x_+$, there exist functors $\Psi^{n+1}_{\vec{\varphi},h}$ and $\check{\Psi}^{n+1}_{\vec{\varphi},h}$ satisfying the following conditions:

(1) When $(n+1)$ is odd,

$$(A_y)_{n+1}\big(\mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta};+}(\varphi),\beta_+\big) \xrightarrow{\Psi^{n+1}_{\vec{\varphi},h}} (A_y)_{n+1}\big(\Psi^n_{\vec{\alpha},h}(\alpha_+),\Psi^n_{\vec{\beta},h}(\beta_+)\big), \text{ and}$$

$$(A_y)_{n+1}\big(\neg\alpha_-,\check{\mathfrak{d}}^{n+1}_{\neg\vec{\alpha},\neg\vec{\beta};-}(\varphi)\big) \xrightarrow{\check{\Psi}^{n+1}_{\vec{\varphi},h}} (A_y)_{n+1}\big(\Psi^n_{\vec{\alpha},h}(\alpha_+),\Psi^n_{\vec{\beta},h}(\beta_+)\big).$$

Similarly, when $(n+1)$ is even,

$$(A_y)_{n+1}\big(\alpha_+,\mathfrak{d}^{n+1}_{\vec{\alpha},\vec{\beta};+}(\varphi)\big) \xrightarrow{\Psi^{n+1}_{\vec{\varphi},h}} (A_y)_{n+1}\big(\Psi^n_{\vec{\alpha},h}(\alpha_+),\Psi^n_{\vec{\beta},h}(\beta_+)\big), \text{ and}$$

$$(A_y)_{n+1}\big(\check{\mathfrak{d}}^{n+1}_{\neg\vec{\alpha},\neg\vec{\beta};-}(\varphi),\neg\beta_-\big) \xrightarrow{\check{\Psi}^{n+1}_{\vec{\varphi},h}} (A_y)_{n+1}\big(\Psi^n_{\vec{\alpha},h}(\alpha_+),\Psi^n_{\vec{\beta},h}(\beta_+)\big)$$

(2) When $\vec{\varphi}$ is an $(n+1)$-cell as above,

$$\Psi^{n+1}_{\vec{\varphi},h}(\varphi_+) = \check{\Psi}^{n+1}_{\vec{\varphi},h}(\neg\varphi_-). \tag{44}$$

(3) Let parallel $(n+1)$-cells $\vec{\varphi}, \vec{\psi} : \vec{\alpha} \rightrightarrows \vec{\beta}$ be given. When $(n+1)$ is odd, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_{n+2}(\varphi,\psi) & \xrightarrow{\check{\mathfrak{d}}^{n+2}_{\neg\vec{\varphi},\neg\vec{\psi};-}} & (A_y)_{n+1}\big(\neg\alpha_-, \check{\mathfrak{d}}^{n+1}_-(\psi)\big) \\ \Big\downarrow{\scriptstyle \mathfrak{d}^{n+2}_{\vec{\varphi},\vec{\psi};+}} & & \Big\downarrow{\scriptstyle \check{\Psi}^{n+1}_{\vec{\psi},h}} \\ (A_y)_{n+1}\big(\mathfrak{d}^{n+1}_+(\varphi),\beta_+\big) & \xrightarrow[\Psi^{n+1}_{\vec{\varphi},h}]{} & (A_y)_{n+1}\big(\Psi^{n+1}_{\vec{\varphi},h}(\alpha_+), \Psi^{n+1}_{\vec{\psi},h}(\beta_+)\big) \end{array} \tag{45}$$

And, when $(n+1)$ is even,

$$\begin{array}{ccc} \mathcal{C}_{n+1}(\varphi,\psi) & \xrightarrow{\check{\mathfrak{d}}^{n+2}_{\neg\vec{\varphi},\neg\vec{\psi};-}} & (A_y)_{n+1}\big(\check{\mathfrak{d}}^{n+1}_-(\varphi), \neg\beta_-\big) \\ \Big\downarrow{\scriptstyle \mathfrak{d}^{n+2}_{\vec{\varphi},\vec{\psi};+}} & & \Big\downarrow{\scriptstyle \check{\Psi}^{n+1}_{\vec{\varphi},h}} \\ (A_y)_{n+1}\big(\alpha_+, \mathfrak{d}^{n+1}_+(\psi)\big) & \xrightarrow[\Psi^{n+1}_{\vec{\psi},h}]{} & (A_y)_{n+1}\big(\Psi^{n}_{\vec{\alpha},h}(\alpha_+), \Psi^{n}_{\vec{\beta},h}(\beta_+)\big) \end{array} \tag{46}$$

commutes.

LEMMA 4.15. *The conditions described above are satisfied when, for $\vec{\varphi} : \vec{\alpha} \longrightarrow \vec{\beta}$ an $(n+1)$-cell of $\int A\pi$ and $h : x_- \longrightarrow x_+$ as above, the functors $\Psi^{n+1}_{\vec{\varphi},h}$ and $\check{\Psi}^{n+1}_{\vec{\varphi},h}$ are defined as follows:*

$$\Psi^{n+1}_{\vec{\varphi},h}(-) := \begin{cases} \check{\Psi}^{n}_{\vec{\beta},h}(\neg\varphi_-) \ *_n \ \Psi^{n}_{\vec{\alpha},h}(-) & \text{if } (n+1) \text{ is even, and} \\ \Psi^{n}_{\vec{\beta},h}(-) \ *_n \ \check{\Psi}^{n}_{\vec{\alpha},h}(\neg\varphi_-) & \text{if } (n+1) \text{ is odd;} \end{cases}$$

and

$$\check{\Psi}^{n+1}_{\vec{\varphi},h}(-) := \begin{cases} \check{\Psi}^{n}_{\vec{\beta},h}(-) \ *_n \ \Psi^{n}_{\vec{\alpha},h}(\varphi_+) & \text{if } (n+1) \text{ is even, and} \\ \Psi^{n}_{\vec{\beta},h}(\varphi_+) \ *_n \ \check{\Psi}^{n}_{\vec{\alpha},h}(-) & \text{if } (n+1) \text{ is odd.} \end{cases}$$

PROOF. We give the proof in the case where $(n+1)$ is odd as the case where it is even is essentially dual. First, to see that $\Psi^{n+1}_{\vec{\varphi}}$ is well defined and possesses the source and target as stated in condition (1) above, suppose we are given a m-cell ζ of $(A_y)_{n+1}(\mathfrak{d}^{n+1}_+\varphi, \beta_+)$. Then, ζ is a $(m+1)$-cell of $(A_y)_n(f_+, \mathfrak{d}^n_+\beta)$ where $\vec{\alpha}, \vec{\beta} : \vec{f} \rightrightarrows \vec{g}$. As such, we may apply $\Psi^n_{\vec{\beta}}$ to obtain

$$\Psi^n_{\vec{\beta}}\big(\mathfrak{d}^{n+1}_+(\varphi)\big) \ \underset{\longrightarrow}{\overset{\longrightarrow}{\Psi^n_{\vec{\beta}}(\zeta)}} \ \Psi^n_{\vec{\beta}}(\beta_+)$$

By definition of $\neg\vec{\varphi}$, we have also $\neg\varphi_- : \neg\alpha_- \longrightarrow \check{\mathfrak{d}}^{n+1}_-(\varphi)$. By the induction hypothesis,

$$\check{\Psi}^n_{\vec{\alpha}}\big(\check{\mathfrak{d}}^{n+1}_-(\varphi)\big) = \Psi^n_{\vec{\beta}}\big(\mathfrak{d}^{n+1}_+(\varphi)\big),$$

and therefore applying $\check{\Psi}^n_{\vec{\alpha}}$ to $\neg\varphi_-$ yields

$$\Psi^n_{\vec{\alpha}}(\alpha_+) \;=\; \check{\Psi}^n_{\vec{\alpha}}(\neg\alpha_-) \xrightarrow{\check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)} \Psi^n_{\vec{\beta}}\big(\mathfrak{d}^{n+1}_+(\varphi)\big).$$

Here the equation is by the induction hypothesis. As such, the composite

$$\Psi^{n+1}_{\vec{\varphi}}(\zeta) := \Psi^n_{\vec{\beta}}(\zeta) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)$$

is defined and possesses the correct boundary. A similar argument shows that $\check{\Psi}^{n+1}_{\vec{\varphi}}$ is well-defined with the appropriate boundary. Note also that, with these definitions, condition (2) from above is trivially satisfied.

Finally, to see that (3) is satisfied we note that, when $\vec{\varphi}$ and $\vec{\psi}$ are parallel $(n+1)$-cells as above and γ is a cell of $\mathcal{C}_{n+2}(\varphi,\psi)$,

$$\begin{aligned}
\Psi^{n+1}_{\vec{\varphi}}\big(\mathfrak{d}^{n+2}_+(\gamma)\big) &= \Psi^n_{\vec{\beta}}\big(\mathfrak{d}^{n+2}_+(\gamma)\big) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)\\
&= \Psi^n_{\vec{\beta}}\big(\psi_+ \;*_n\; \mathfrak{d}^{n+1}_+(\gamma)\big) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)\\
&= \Psi^n_{\vec{\beta}}(\psi_+) \;*_n\; \Psi^n_{\vec{\beta}}\big(\mathfrak{d}^{n+1}_+(\gamma)\big) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)\\
&= \Psi^n_{\vec{\beta}}(\psi_+) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}\big(\check{\mathfrak{d}}^{n+1}_-(\gamma)\big) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}(\neg\varphi_-)\\
&= \Psi^n_{\vec{\beta}}(\psi_+) \;*_n\; \check{\Psi}^n_{\vec{\alpha}}\big(\check{\mathfrak{d}}^{n+2}_-(\gamma)\big)\\
&= \check{\Psi}^{n+1}_{\vec{\psi}}\big(\check{\mathfrak{d}}^{n+2}_-(\gamma)\big),
\end{aligned}$$

where the fourth equation is by the induction hypothesis. □

4.3.3. Definition of the identity types. With the functors Ψ^n and $\check{\Psi}^n$ at our disposal it is possible to give a very efficient description of the "identity type" functor $I_A : \int A\pi \longrightarrow \omega$-**Gpd**. In particular, the official definition of I_A in all dimensions is as follows:

- **Objects:** $I_A(x, x_-, x_+)$ is given by the ω-groupoid $A_x(x_-, x_+)$.
- **1-Cells:** Given $\vec{f} : \vec{x} \longrightarrow \vec{y}$, the functor $I(\vec{f}) : I(\vec{x}) \longrightarrow I(\vec{y})$ is defined by setting

  $$I_A(\vec{f})(\gamma) := f_+ \;*_0\; A_f(\gamma) \;*_0\; \neg f_-,$$

  for any m-cell γ of $A_x(x_-, x_+)$.
- **2-Cells:** A 2-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$ is sent to the natural transformation $I_A(\vec{\alpha})$ which is defined, for an object $h : x_- \longrightarrow x_+$ of $A_x(x_-, x_+)$, as follows:

  $$\begin{aligned} I_A(\vec{\alpha})_h &:= \Psi^2_{\vec{\alpha},h}(\alpha_+)\\ &= \check{\Psi}^2_{\vec{\alpha},h}(\neg\alpha_-). \end{aligned}$$

  That $I_A(\vec{\alpha})_h$ possesses the correct domain and codomain is an immediate consequence of the results of Section 4.3.2.
- **$(n+1)$-Cells:** In general, given an $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ in $\int A\pi$ and $h : x_- \longrightarrow x_+$, we define

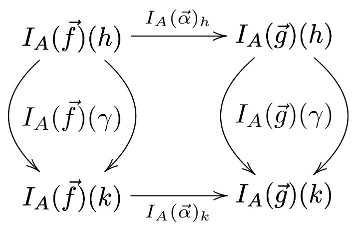
  $$\begin{aligned} I_A(\vec{\varphi})_h &:= \Psi^{(n+1)}_{\vec{\varphi},h}(\varphi_+)\\ &= \check{\Psi}^{(n+1)}_{\vec{\varphi},h}(\neg\varphi_-). \end{aligned}$$

  Again, that $I_A(\vec{\varphi})_h$ possesses the correct domain and codomain follows directly from the definition of I_A at lower dimensional cells together with the results of Section 4.3.2.

It remains only to verify that I_A is functorial. To this end, we first prove that the data given in the definition are of the appropriate kinds. E.g., that $I(\vec{\alpha})$ is a natural transformation, et cetera.

LEMMA 4.16. *As defined above, when* $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$ *is a 2-cell of* $\int A\pi$, $I_A(\vec{\alpha})$ *is an* ω*-natural transformation.*

PROOF. Explicitly, we must show that, for any m-cell γ of $A_x(x_-, x_+)$ with $m > 0$ such that $s^m(\gamma) = h$ and $t^m(\gamma) = k$, the following "schematic" diagram commutes:



in $A_y(y_-, y_+)$. I.e., we must prove that

$$I_A(\vec{\alpha})_k \;*_1\; I_A(\vec{f})(\gamma) = I_A(\vec{g})(\gamma) \;*_1\; I_A(\vec{\alpha})_h. \tag{47}$$

By definition of $\Psi^2_{\vec{\alpha},h}(\alpha_+)$ and interchange it follows that the right-hand side of (47) is equal to

$$\big(g_+ \;*_0\; A_g(\gamma) \;*_0\; \neg\alpha_-\big) \;*_1\; \Psi^1_{\vec{f},h}(\alpha_+).$$

Because A_α is itself a transformation $A_f \Longrightarrow A_g$ we obtain

$$\mathfrak{d}^1_+(\alpha) \;*_0\; A_f(\gamma) = A_g(\gamma) \;*_0\; \mathfrak{d}^1_-(\alpha).$$

Thus,

$$\begin{aligned} A_g(\gamma) \;*_0\; \neg\alpha_- &= \big(A_g(k) \;*_1\; A_g(\gamma)\big) \;*_0\; \big(\neg\alpha_- \;*_1\; \check{\mathfrak{d}}^2_-(\alpha)\big) \\ &= \big(A_g(k) \;*_0\; \neg\alpha_-\big) \;*_1\; \big(A_g(\gamma) \;*_0\; \check{\mathfrak{d}}^2_-(\alpha)\big) \\ &= \big(A_g(k) \;*_0\; \neg\alpha_-\big) \;*_1\; \big(\mathfrak{d}^1_+(\alpha) \;*_0\; A_f(\gamma) \;*_0\; f_-^{-1}\big). \end{aligned}$$

Thus, the right-hand side of (47) is equal to

$$\check{\Psi}^1_{\vec{g},k}(\neg\alpha_-) \;*_1\; \big(\mathfrak{d}^2_+(\alpha) \;*_0\; A_f(\gamma) \;*_0\; f_-^{-1}\big) \;*_1\; \Psi^1_{\vec{f},h}(\alpha_+). \tag{48}$$

Moreover, the interchange and unit laws yield

$$\big(\mathfrak{d}^2_+(\alpha) \;*_0\; A_f(\gamma) \;*_0\; f_-^{-1}\big) \;*_1\; \Psi^1_{\vec{\alpha},h}(\alpha_+) = \alpha_+ \;*_0\; A_f(\gamma) \;*_0\; f_-^{-1}.$$

Thus, the right-hand side of (47) is equal to

$$\check{\Psi}^1_{\vec{\alpha},k}(\neg\alpha_-) \;*_1\; \big(\alpha_+ \;*_0\; A_f(\gamma) \;*_0\; f_-^{-1}\big) = I_A(\vec{\alpha})_k *_1 I_A(\vec{f})(\gamma),$$

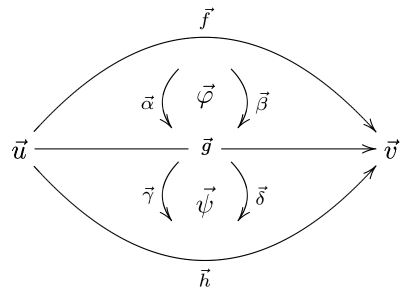
as required. □

A similar argument yields the following fact:

LEMMA 4.17. *Given parallel* n*-cells* $\vec{\alpha}$ *and* $\vec{\beta}$ *in* $\int A\pi$ *bounded by* 1*-cells* $\vec{f}, \vec{g} : \vec{x} \rightrightarrows \vec{y}$ *together with a* $(n+1)$*-cell* $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$, $I_A(\vec{\varphi})$, *as defined above, is a modification* $I_A(\vec{\alpha}) \Longrightarrow I_A(\vec{\beta})$.

PROPOSITION 4.18. *As defined,* I_A *is a functor* $\int(A \circ \pi) \longrightarrow \omega$-**Gpd**.

Proof. First we consider the case of vertical composition. Let p-cells $\vec{\varphi}$ and $\vec{\psi}$ be given, for $p \geq m \geq n+1 > 1$, which are bounded by 0-cells $\vec{x}$ and $\vec{y}$ and by n-cells $\vec{f}$, $\vec{g}$ and $\vec{h}$ as indicated in the following diagram:



Then, for any object $k : x_- \longrightarrow x_+$ of $A_x(x_-, x_+)$, we will prove by induction on m the stronger fact that when m is odd

$$\Psi^m_{s^{p-m}(\vec{\psi} *_n \vec{\varphi}),k}\big((\vec{\psi} *_n \vec{\varphi})_+\big) = \Psi^m_{s^{(p-m)}\vec{\psi},k}(\psi_+) \;\; *_n \;\; \Psi^m_{s^{(p-m)}\vec{\varphi},k}(\varphi_+), \text{ and}$$
$$\check{\Psi}^m_{t^{p-m}(\vec{\psi} *_n \vec{\varphi}),k}\big(\neg(\vec{\psi} *_n \vec{\varphi})_-\big) = \check{\Psi}^m_{t^{(p-m)}\vec{\psi},k}(\neg\psi_-) \;\; *_n \;\; \check{\Psi}^m_{t^{(p-m)}\vec{\varphi},k}(\neg\varphi_-);$$

and

$$\Psi^m_{t^{p-m}(\vec{\psi} *_n \vec{\varphi}),k}\big((\vec{\psi} *_n \vec{\varphi})_+\big) = \Psi^m_{t^{(p-m)}\vec{\psi},k}(\psi_+) \;\; *_n \;\; \Psi^m_{t^{(p-m)}\vec{\varphi},k}(\varphi_+), \text{ and}$$
$$\check{\Psi}^m_{s^{p-m}(\vec{\psi} *_n \vec{\varphi}),k}\big(\neg(\vec{\psi} *_n \vec{\varphi})_-\big) = \check{\Psi}^m_{s^{(p-m)}\vec{\psi},k}(\neg\psi_-) \;\; *_n \;\; \check{\Psi}^m_{s^{(p-m)}\vec{\varphi},k}(\neg\varphi_-);$$

when m is even.

First, assume $m = n+1$ is even. We also assume that $n+1 > 2$ since the case where $n+1 = 2$ is a straightforward calculation (using ideas essentially the same as those used here). Then

$$\Psi^{n+1}_{\vec{\delta} *_n \vec{\beta}}\big((\vec{\psi} *_n \vec{\varphi})_+\big) = \check{\Psi}^n_{\vec{h}}\big(\neg(\vec{\delta} *_n \vec{\beta})_-\big) \;\; *_n \;\; \Psi^n_{\vec{f}}\big((\vec{\psi} *_n \vec{\varphi})_+\big)$$

And this is equal, by definition of composition, to

$$\check{\Psi}^n_{\vec{h}}\big(\neg\delta_- \;\; *_n \;\; (\check{\mathfrak{d}}^n_-(\delta) *_{(n-1)} \neg\beta_-)\big) \;\; *_n \;\; \Psi^n_{\vec{f}}\big((\psi_+ *_{(n-1)} \mathfrak{d}^n_+(\beta)) \;\; *_n \;\; \varphi_+\big) \tag{49}$$

Now we will investigate in more detail each of the larger terms in this composite. First:

$$\check{\Psi}^n_{\vec{h}}\big(\neg\delta_- *_n (\check{\mathfrak{d}}^n_-(\delta) *_{(n-1)} \neg\beta_-)\big) = \check{\Psi}^n_{\vec{h}}(\neg\delta_-) *_n \check{\Psi}^n_{\vec{h}}\big(\check{\mathfrak{d}}^n_-(\delta) *_{(n-1)} \neg\beta_-)\big).$$

By definition of $\check{\Psi}^n$ and functoriality this is equal to

$$\check{\Psi}^n_{\vec{h}}(\neg\delta_-) *_n \big(\Psi^{n-1}_{\vec{v}}(h_+) *_{(n-1)} \check{\Psi}^{n-1}_{\vec{u}}(\check{\mathfrak{d}}^n_-(\delta)) *_{(n-1)} \check{\Psi}^{n-1}_{\vec{u}}(\neg\beta_-)\big),$$

which by (46) is equal to:

$$\begin{aligned} &\check{\Psi}^n_{\vec{h}}(\neg\delta_-) \;\; *_n \;\; \big(\Psi^{n-1}_{\vec{v}}(h_+) \;\; *_{(n-1)} \;\; \Psi^{n-1}_{\vec{v}}(\mathfrak{d}^n_+(\delta)) \;\; *_{(n-1)} \;\; \check{\Psi}^{n-1}_{\vec{u}}(\neg\beta_-)\big) \\ = &\check{\Psi}^n_{\vec{h}}(\neg\delta_-) \;\; *_n \;\; \big(\Psi^{n-1}_{\vec{v}}(\mathfrak{d}^{n+1}_+(\delta)) \;\; *_{(n-1)} \;\; \check{\Psi}^{n-1}_{\vec{u}}(\neg\beta_-)\big) \end{aligned}$$

Similarly, the other half of (49) is equal to

$$\big(\Psi^{n-1}_{\vec{v}}(\psi_+) \;\; *_{(n-1)} \;\; \check{\Psi}^{n-1}_{\vec{u}}(\check{\mathfrak{d}}^{n+1}_-(\beta))\big) \;\; *_n \;\; \Psi^n_{\vec{f}}(\varphi_+).$$

By these observations and a routine calculation it follows that (49) is equal to

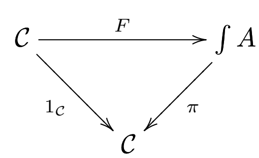
$$\check{\Psi}^n_{\vec{h}}(\neg\delta_-) \;\; *_n \;\; \big(\Psi^{n-1}_{\vec{v}}(\psi_+) \;\; *_{(n-1)} \;\; \check{\Psi}^{n-1}_{\vec{u}}(\neg\beta_-)\big) \;\; *_n \;\; \Psi^n_{\vec{f}}(\varphi_+).$$

Finally, using the unit and interchange laws this is seen to be the same as $\Psi^{n+1}_{\vec{\delta}}(\psi_+) *_n \Psi^{n+1}_{\vec{\beta}}(\varphi_+)$. The base cases where $n+1$ is odd are dual and the induction steps are trivial. Thus, I_A is a functor. $\square$

4.4. Reflexivity and elimination terms

In this section we define the functors which will interpret reflexivity and elimination terms. As in [**35**] we will interpret terms as sections of the projection map $\int A \longrightarrow \mathcal{C}$ associated to the functor A which interprets their type. We begin by summarizing some of the basic facts about such sections and the related structures resulting from the Grothendieck construction.

4.4.1. Sections of projection functors. A section F of a projection $\pi : \int A \longrightarrow \mathcal{C}$ as indicated in the following diagram:



consists exactly of the following data:

Objects: To each object x of $\mathcal{C}$ there is assigned an object x_F of A_x. I.e., $(x, x_F) = F(x)$.

1-Cells: To an arrow $f : x \longrightarrow y$ in $\mathcal{C}$ there is assigned an arrow $f_F : \mathfrak{d}^1_{F(x),F(y)}(f) \longrightarrow y_F$ of A_y.

$(n+1)$**-Cells:** When $(n+1)$ is even, there is assigned to an $(n+1)$-cell $\alpha : f \Longrightarrow g$ an $(n+1)$-cell $\alpha_F : f_F \Longrightarrow \mathfrak{d}^{n+1}_{F(f),F(g)}(\alpha)$ of A_y. When $(n+1)$ is odd, $\alpha_F : \mathfrak{d}^{n+1}_{F(f),F(g)}(\alpha) \Longrightarrow g_F$.

Note that such an assignment is made into a map of globular sets by defining $F(\varphi) := (\varphi, \varphi_F)$. These assignments are required to be functorial in the sense of preserving identities and composition. Preservation of composition amounts to the following. Given m-cells, for $m > 0$, ψ and φ in $\mathcal{C}$ such that $(\psi *_0 \varphi)$ is defined, it is required in order for the assignment $(-)_F$ to constitute a section such that

$$(\psi *_0 \varphi)_F = \psi_F *_0 A_h(\varphi_F),$$

where h is the bounding 1-cell as above. Assume that the composite $(\psi *_n \varphi)$ is defined and that $t^{m-n-1}\varphi = \beta$ and $s^{m-n-1}\psi = \gamma$. Furthermore, let u and v be the $(n-1)$-cells bounding both φ and ψ. Then it is required that

$$(\psi *_n \varphi)_F = \begin{cases} \left(\psi_F *_{(n-1)} \mathfrak{d}^n_{Fu,Fv}(\beta)\right) \ *_n \ \varphi_F & \text{if } (n+1) \text{ is even, and} \\ \\ \psi_F \ *_n \ \left(\mathfrak{d}^n_{Fu,Fv}(\gamma) *_{(n-1)} \varphi_F\right) & \text{if } (n+1) \text{ is odd.} \end{cases}$$

We also note that given any functor $\sigma : \mathcal{D} \longrightarrow \mathcal{C}$, there exists a functor $\{\sigma\}_A : \int(A \circ \sigma) \longrightarrow \int A$ such that the following diagram is a pullback in ω-**Cat**:

$$\begin{array}{ccc} \int(A\circ\sigma) & \xrightarrow{\{\sigma\}_A} & \int A \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow[\sigma]{} & \mathcal{C} \end{array}$$

Namely, $\{\sigma\}_A$ sends $\vec{x}$ in $\int A\sigma$ to $(\sigma(x), a)$ and similarly for cells in all dimensions. Consequently, there corresponds to any section F of the projection $\int A \longrightarrow \mathcal{C}$ a canonical section $F[\sigma]$ of $\int A\sigma \longrightarrow \mathcal{D}$ for which

$$F \circ \sigma = \{\sigma\}_A \circ F[\sigma].$$

Finally, note that, by taking $\mathcal{D}$ to be $\int A$ itself and σ to be π, we obtain $\int A\pi$ as the pullback of π along itself and there exists a canonical map $\delta_A : \int A \longrightarrow \int A\pi$ induced by the identity functor $1_{\int A}$.

4.4.2. Reflexivity terms. We end this section by describing briefly the "reflexivity term" associated to a functor $A : \mathcal{C} \longrightarrow \omega$-**Gpd**. By definition, the reflexivity term should be a section r_A:

$$\begin{array}{ccc} \int A & \xrightarrow{r_A} & \int(I_A \circ \delta_A) \\ & {\scriptstyle 1_{\int A}}\searrow \qquad \swarrow & \\ & \int A & \end{array} \tag{50}$$

where δ_A is as in Section 4.4.1. For objects, given an object $\vec{x}$ of $\int A$, we must provide an object $\vec{x}_r$ of $I_A(x, x_-, x_-)$. I.e., $\vec{x}_r$ should be an arrow $x_- \longrightarrow x_-$ in A_x. We define $\vec{x}_r$ to be the identity x_-. (Note that here and throughout we omit the identity maps and write x_- instead of $i(x_-)$.) Next, given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$, we need to provide an arrow

$$(\partial I)^1_{r(\vec{x}),r(\vec{y})}(\vec{f}) \xrightarrow{\vec{f}_r} \vec{y}_r \;=\; y_-$$

in A_y, where $(\partial I)^n$ denotes the weighted face functor for $\int I_A$ (and so, in this case, also $\int I_A\delta_A$). But, by definition, $(\partial I)^1(\vec{f})$ is just y_- and we therefore define $\vec{f}_r$ to be y_-. Indeed, at *every* dimension $n \geq 1$, when $\vec{\varphi}$ is a n-cell of $\int A$ bounded by objects $\vec{x}$ and $\vec{y}$, we define $\vec{\varphi}_r$ to be y_-. We claim that r is constant y_- in all dimensions $n \geq 1$. This follows directly from the following useful fact:

Lemma 4.19. *Assume $\vec{\varphi} : \vec{\alpha} \longrightarrow \vec{\beta}$ is an m-cell of $\int A$ bounded by n-cells $\vec{\alpha}$ and $\vec{\beta}$ as indicated in the following diagram:*

$$\vec{u} \;\underset{\vec{\beta}}{\overset{\vec{\alpha}}{\Downarrow \vec{\varphi}}}\; \vec{v}$$

with $1 \leq n < m$, and by objects $\vec{x}$ and $\vec{y}$. Then,

$$\check{\Psi}^n_{\vec{\beta},1_{x_-}}(\neg\varphi_-) = \Psi^n_{\vec{\alpha},1_{x_-}}\big(\rho^{m-n-2}_{\partial\vec{\varphi}}(\varphi_-^{-1})\big)$$

if $(n+1)$ is even, and

$$\check{\Psi}^n_{\vec{\alpha},1_{x_-}}(\neg\varphi_-) = \Psi^n_{\vec{\beta},1_{x_-}}\left(\rho^{m-n-2}_{\partial\vec{\varphi}}(\varphi_-^{-1})\right)$$

if $(n+1)$ is odd. Here we adopt the convention that $\rho^{-1}_{\partial\vec{\varphi}}$ is the identity and that, when regarding the cells in question as being in $\int A\pi$, $\vec{\alpha} = (\alpha, \alpha_-, \alpha_-)$.

Proof. The proof is by induction on n. For the base case $n = 1$ we observe that

$$\begin{aligned}
\check{\Psi}^1_{\vec{\beta},1_{x_-}}(\neg\varphi_-) &= \check{\Psi}^1_{\vec{\beta},1_{x_-}}\left(\rho^{m-2}_{\partial\vec{\varphi}}(\varphi_-^{-1})\right) \\
&= \left(\beta_- \ *_0\ \beta_-^{-1} \ *_0\ \rho^{m-3}_{\partial\vec{\varphi}}(\varphi_-^{-1}) \ *_0\ \alpha_-^{-1}\right) \\
&= \Psi^1_{\vec{\alpha},1_{x_-}}\left(\rho^{m-3}_{\partial\vec{\varphi}}(\varphi_-^{-1})\right).
\end{aligned}$$

The induction steps are straightforward using the same trick. □

Using Lemma 4.19 it is straightforward to prove that $(\partial I)^1_{r(\vec{x}),r(\vec{y})}(\vec{\varphi})$ is equal to y_- for any $\vec{\varphi}$ in $\int A$ bounded by $\vec{x}$ and $\vec{y}$. As such, we may define $\vec{\varphi}_r$ to be y_- in all dimensions $n \geq 1$. With this definition functoriality of r is trivial and we have proved:

Lemma 4.20. *Given $A : \mathcal{C} \longrightarrow \omega$-**Gpd** as above, the assignment r defined above induces a section r_A as indicated in (50).*

4.4.3. Setting up the construction of elimination terms. Turning now to elimination terms, suppose we are given a functor $D : \int I_A \longrightarrow \omega$-**Gpd** together with a section $d : \int A \longrightarrow \int D\{\delta_A\}_{I_A} r_A$ of the projection $\int D\{\delta_A\}_{I_A} r_A \longrightarrow \int A$. Here, recall that the composite $D\{\delta_A\}_{I_A} r_A$ is constructed by

$$\int A \xrightarrow{\ r_A\ } \int I_A \circ \delta_A \xrightarrow{\ \{\delta_A\}_{I_A}\ } \int I_A \xrightarrow{\ D\ } \omega\text{-}\mathbf{Gpd}$$

where the notation $\{\delta_A\}_{I_A}$ for the functor induced by δ_A is as described in Section 4.4.1 above.

We would like to prove that this extends to a section J of the projection $\int D \longrightarrow \int I_A$ as indicated in the following diagram:

$$\begin{array}{ccc}
\int I_A & \xrightarrow{\ J\ } & \int D \\
& {\scriptstyle 1_{\int I_A}}\searrow \quad \swarrow & \\
& \int I_A &
\end{array}$$

We begin by fixing notation.

As we are dealing with multiple cases of the Grothendieck construction it will be convenient to introduce some notation to deal with the different weighted face functions which occur. First, we denote by $\Theta^n(-)$ the weighted face functor for $\int D$. As usual, we denote by $\mathfrak{d}^n_-$ and $\mathfrak{d}^n_+$ the functors for the two projections of $\int I_A$. Finally, we denote by $\tilde{\Theta}^n$ the weighted face functor for $\int D \circ \{\delta_A\}_{I_A} \circ r_A$.

Next, we observe that there is an endofunctor $\downarrow(-) : \int I_A \longrightarrow \int I_A$ defined as the following composite:

$$\int I_A \xrightarrow{\ \pi_0\ } \int A \xrightarrow{\ r_A\ } \int I_A \circ \delta_A \longrightarrow \int I_A.$$

I.e., $\downarrow$ sends an object $\vec{x} = (x, x_-, x_+, x_\rightarrow)$ to (x, x_-, x_-, x_-) and similarly for higher-dimensional cells. Here, as throughout, we omit mention of identity arrows. I.e., writing out identities we have that $\downarrow \vec{x}$ is $(x, x_-, x_-, i(x_-))$ or $(x, x_-, x_-, 1_{x_-})$.

We will often be concerned with the situation where we consider, given $\vec{x}$ an object of $\int I_A$, the restriction (x, x_-) of $\vec{x}$ to $\int A$. Rather than write $\pi_0(\vec{x})$ every time for this object we instead denote this pair by $\breve{x}$. Similarly, $\breve{\gamma}$ denotes $\pi_0(\vec{\gamma})$ for general n-cells $\vec{\gamma}$ of $\int I_A$.

4.4.4. Naturality cells. The construction of the elimination terms is rather technical and proceeds in several stages. First, we describe "naturality cells" which exhibit what amounts to a kind of "ω-pseudonatural transformation" ε_- from $\downarrow(-)$ to the identity $1_{\int I_A}$. The construction proceeds by induction on dimension as usual.

(Dimension 0): First, in dimension 0, given $\vec{x}$ in $\int I_A$, we define an arrow $\varepsilon_{\vec{x}} : \downarrow(\vec{x}) \longrightarrow \vec{x}$ as follows:

$$\varepsilon_{\vec{x}} := (x, x_-, x_\rightarrow, x_\rightarrow).$$

Next, holding 0-cells $\vec{x}$ and $\vec{y}$ of $\int I_A$ fixed we define functors

$$(\int I_A)_1(\vec{x}, \vec{y}) \xrightarrow{\nabla^1_{\vec{x},\vec{y};\xi}} (\int I_A)_1(\downarrow \vec{x}, \vec{y})$$

for $\xi = -, +$ as follows:

$$\nabla^1_{\vec{x},\vec{y};\xi} := \begin{cases} (- \ *_0 \ \varepsilon_{\vec{x}}) & \text{if } \xi = -, \text{ and} \\ \\ \big(\varepsilon_{\vec{y}} *_0 \ \downarrow(-)\big) & \text{if } \xi = +. \end{cases}$$

As usual, we omit the subscripts $\vec{x}$ and $\vec{y}$ when these are understood.

Note that with these definitions, when $\vec{\varphi}$ is any m-cell, with $m \geq 1$,

$$\nabla^1_-(\vec{\varphi}) = (\varphi,\ \varphi_-,\ \varphi_+ *_0 A_f(x_\rightarrow),\ \varphi_\rightarrow), \text{ and} \tag{51}$$

$$\nabla^1_+(\vec{\varphi}) = (\varphi,\ \varphi_-,\ y_\rightarrow *_0 \varphi_-, y_\rightarrow), \tag{52}$$

where $s^{m-1}\vec{\varphi} = \vec{f}$ and $\vec{f} : \vec{x} \longrightarrow \vec{y}$ is as above.

Remark 4.21. Because we will sometimes want to refer to the different elements of such a pair $\nabla^1_\xi(\vec{\varphi})$ we denote by $[\nabla^1_\xi(\vec{\varphi})]_k$ the k-th component, for $k = 0, 1, 2, 3$. E.g., $[\nabla^1_+(\vec{\varphi})]_2$ is $(y_\rightarrow *_0 \varphi_-)$,.

Dimension 1: Next, we define, given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ in $\int I_A$, a 2-cell

$$\nabla^1_-(\vec{f}) \xrightarrow{\varepsilon_{\vec{f}}} \nabla^1_+(\vec{f}).$$

I.e., $\varepsilon_{\vec{f}}$ is as indicated in the following "naturality" diagram:

$$\begin{array}{ccc} \downarrow(\vec{x}) & \xrightarrow{\varepsilon_{\vec{x}}} & \vec{x} \\ {\scriptstyle \downarrow(\vec{f})}\big\downarrow & \varepsilon_{\vec{f}}\Downarrow & \big\downarrow{\scriptstyle \vec{f}} \\ \downarrow(\vec{y}) & \xrightarrow[\varepsilon_{\vec{y}}]{} & \vec{y} \end{array}$$

In particular, $\varepsilon_{\vec{f}}$ is defined to be $(f, f_-, f_\rightarrow *_0 f_-, f_\rightarrow)$. This definition is easily seen to make sense using (51) and (52). Now, holding parallel arrows $\vec{f}$ and $\vec{g}$ fixed, we define functors

$$(\int I_A)_2(\vec{f},\vec{g}) \xrightarrow{\nabla^2_\xi} (\int I_A)_2\big(\nabla^1_-(\vec{f}), \nabla^1_+(\vec{g})\big)$$

by

$$\nabla^2_\xi(\vec{\gamma}) := \begin{cases} \nabla^1_+(\vec{\gamma}) \ *_1 \ \varepsilon_{\vec{f}} & \text{if } \xi = -, \text{ and} \\ \\ \varepsilon_{\vec{g}} \ *_1 \ \nabla^1_-(\vec{\gamma}) & \text{if } \xi = +. \end{cases}$$

With these definitions it is straightforward to verify that

$$\nabla^2_-(\vec{\varphi}) = (\varphi,\ \varphi_-,\ f_\rightarrow *_0 \varphi_-,\ f_\rightarrow), \tag{53}$$

when $\vec{\varphi}$ is any cell of $(\int I_A)_2(\vec{f},\vec{g})$. In order to obtain a similar analysis of $\nabla^2_+(\vec{\varphi})$ we require a further fact about the duality functor $\neg$.

Lemma 4.22. *Given any m-cell, for $m \geq 1$, $\vec{\varphi}$ of $\int A$,*

$$\neg\varphi_- \ *_0 \ \varphi_- = \mathfrak{d}^1_{\vec{x},\vec{y}}(\varphi), \text{ and}$$
$$\varphi_- \ *_0 \ \neg\varphi_- = y_-$$

where $\vec{x}$ and $\vec{y}$ are the 0-cells bounding $\vec{\varphi}$.

Proof. This is a direct consequence of the easily proved fact that, for $0 \leq n \leq m-2$

$$\rho^{m-n-2}_{\partial\vec{\varphi}}(\varphi_-^{-1}) \ *_n \ \varphi_- = t^{m-n-1}(\vec{\varphi})^-_- \ *_n \ \big(\varphi_- \ *_{(n+1)} \ \rho^{m-n-3}_{\partial\vec{\varphi}}(\varphi_-^{-1})\big)$$

when n is even (or 0), and

$$\varphi_- \ *_n \ \rho^{m-n-2}_{\partial\vec{\varphi}}(\varphi_-^{-1}) = \big(\rho^{m-n-3}_{\partial\vec{\varphi}}(\varphi_-^{-1}) \ *_{(n+1)} \ \varphi_-\big) \ *_n \ s^{m-n-1}(\vec{\varphi})^{-1}_-,$$

when n is odd. Iteratively applying these facts and canceling inverses yields the required result. □

Using Lemma 4.22 it follows, by a (lengthy but) straightforward calculation, that, where $\vec{\beta}$

$$\nabla^2_+(\vec{\varphi}) = \Big(\varphi,\ \varphi_-,\ \big(g_\rightarrow \ *_1 \ \Psi^2_{\vec{\beta},x_\rightarrow}(\varphi_+)\big) *_0 \beta_-,\ \varphi_\rightarrow\Big) \tag{54}$$

for any m-cell $\vec{\varphi}$ of $(\int I_A)_2(\vec{f},\vec{g})$ with $\vec{\beta} = t^{m-2}\vec{\varphi}$. Using (53) and (54) we define $\varepsilon_{\vec{\alpha}}$, for $\vec{\alpha} : \vec{f} \Rrightarrow \vec{g}$ a 2-cell of $\int I_A$, as follows:

$$\varepsilon_{\vec{\alpha}} := (\alpha,\ \alpha_-,\ \alpha_\rightarrow \ *_0 \ \alpha_-,\ \alpha_\rightarrow).$$

In higher-dimensions this procedure is carried out as follows:

Dimension $(n+1)$**:** Given $\vec{\alpha}$ and $\vec{\beta}$ of dimension n together with the appropriate ∇^n_- we first observe that, using decompositions of $\nabla^n_\xi(\vec{\varphi})$ corresponding to (53) and (54), and proved by a standard calculation using

Lemma 4.22, it follows that, if, $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ is a $(n+1)$-cell, then

$$\nabla^n_-(\vec{\varphi}) = \begin{cases} (\varphi,\ \varphi_-,\ \alpha_\to *_0 \varphi_-,\ \alpha_\to) & \text{if } n \text{ is even, and} \\ \\ (\varphi,\ \varphi_-,\ (\partial I)^n(\vec{\varphi}) *_0 \varphi_-,\ \varphi_\to) & \text{if } n \text{ is odd;} \end{cases}$$

and

$$\nabla^n_+(\vec{\varphi}) = \begin{cases} (\varphi,\ \varphi_-,\ (\partial I)^n(\vec{\varphi}) *_0 \varphi_-,\ \alpha_\to) & \text{if } n \text{ is even, and} \\ \\ (\varphi,\ \varphi_-,\ \beta_\to *_0 \varphi_-,\ \beta_\to) & \text{if } n \text{ is odd.} \end{cases}$$

Thus we define

$$\nabla^n_-(\vec{\alpha}) \xrightarrow{\varepsilon_{\vec{\varphi}}} \nabla^n_+(\vec{\beta})$$

by

$$(\varphi, \varphi_-, \varphi_\to *_0 \varphi_-, \varphi_\to).$$

Now, holding $\vec{\alpha}$ and $\vec{\beta}$ fixed, we define

$$(\int I_A)_{n+1}(\vec{\alpha}, \vec{\beta}) \xrightarrow{\nabla^{n+1}_\xi} (\int I_A)_{n+1}\big(\nabla^n_-(\vec{\alpha}), \nabla^n_+(\vec{\beta})\big)$$

for $\xi = -, +$ as

$$\nabla^{n+1}_\xi(\vec{\gamma}) := \begin{cases} \nabla^n_+(\vec{\gamma})\ *_n\ \varepsilon_{\vec{\alpha}} & \text{if } \xi = -, \text{ and} \\ \\ \varepsilon_{\vec{\beta}}\ *_n\ \nabla^n_-(\vec{\gamma}) & \text{if } \xi = +. \end{cases}$$

4.4.5. Elimination terms in dimensions 0, 1. Assume we are given an object $\vec{x} = (x, x_-, x_+, x_\to)$ of $\int I_A$. We would like to provide a corresponding object, which for the sake of notational convenience we simply denote by x_J, of $D(\vec{x})$. This x_J is obtained by a kind of Yoneda style argument. Namely, we observe that, by assumption there is a term $\breve{x}_d$ in $D(\downarrow(\vec{x}))$. Applying the functor $D(\varepsilon_{\vec{x}})$ yields the required x_J in $D(\vec{x})$. I.e.,

$$x_J := D(\varepsilon_{\vec{x}})(\breve{x}_d).$$

Because it will greatly simplify matters in the later stages, we introduce a special notation for $D(\varepsilon_{\vec{x}})$ and its higher-dimensional generalizations $D(\varepsilon_{\vec{\gamma}})$. Namely, we define

$$\langle \vec{\gamma} \rangle := D(\varepsilon_{\vec{\gamma}}).$$

With this notation $x_J = \langle \vec{x} \rangle(\breve{x}_d)$.

In dimension 1, given an arrow $\vec{f} : \vec{x} \longrightarrow \vec{y}$ in $\int I_A$ we have by hypothesis the arrow $\breve{f}_d : \tilde{\Theta}^1_{\breve{x},\breve{y}}(\breve{f}) \longrightarrow \breve{y}_d$ in $D(\downarrow \vec{y})$ and, applying $\langle \vec{y} \rangle$,

$$\langle \vec{y} \rangle\big(\tilde{\Theta}^1 \breve{f}\big) \xrightarrow{\langle \vec{y} \rangle(\breve{f}_d)} y_J$$

in $D(\vec{y})$. Now,

$$\begin{aligned} \langle \vec{y} \rangle\big(\tilde{\Theta}^1 \breve{f}\big) &= \langle \vec{y} \rangle\big(D(\downarrow \vec{f})\breve{x}_d\big) \\ &= D\big(\nabla^1_+ \vec{f}\big)\breve{x}_d \end{aligned}$$

Also,

$$\begin{aligned}\Theta^1(\vec{f}) &= D(\vec{f})(x_J)\\ &= D\big(\nabla^1_- \vec{f}\big)\breve{x}_d\end{aligned}$$

and therefore we define f_J to be the composite

$$\Theta^1(\vec{f}) \xrightarrow{\langle \vec{f}\rangle_{\breve{x}_d}} D\big(\nabla^1_+ \vec{f}\big)\breve{x}_d \xrightarrow{\langle \vec{y}\rangle \breve{f}_d} y_J.$$

Again, it will be useful to introduce some additional notation to clarify the situation in higher-dimensions. First, holding fixed objects $\vec{x}$ and $\vec{y}$ of $\int I_A$, we define a functor

$$(\int I_A)_1(\vec{x},\vec{y}) \xrightarrow{\vartheta^1_{\vec{x},\vec{y}}} D(\vec{y})$$

by

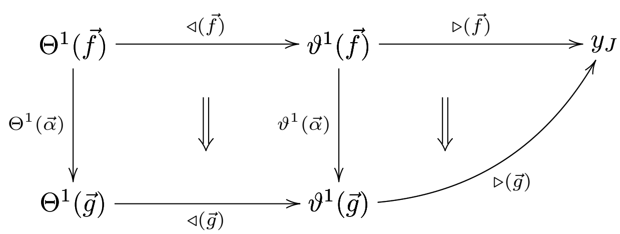
$$\vartheta^1_{\vec{x},\vec{y}}(\vec{\gamma}) := \langle \vec{y}\rangle\big(\tilde{\Theta}^1_{\vec{x},\vec{y}}(\breve{\gamma})\big).$$

The next ingredient is to define, for $\vec{f} : \vec{x} \longrightarrow \vec{y}$, arrows $\triangleleft(\vec{f}) : \Theta^1(\vec{f}) \longrightarrow \vartheta^1(\vec{f})$ and $\triangleright(\vec{f}) : \vartheta^1(\vec{f}) \longrightarrow y_J$ as follows:

$$\begin{aligned}\triangleleft(\vec{f}) &:= \langle \vec{f}\rangle_{\breve{x}_d}, \text{ and}\\ \triangleright(\vec{f}) &:= \langle \vec{y}\rangle \breve{f}_d.\end{aligned}$$

Thus, with this notation f_J is just $\triangleright(\vec{f}) *_0 \triangleleft(\vec{f})$. We will see below that, in general, γ_J will always be formed as a composite of the form $\triangleright(\vec{\gamma}) *_{k-1} \triangleleft(\vec{\gamma})$ along a $(k-1)$-cell $\vartheta^k(\vec{\gamma})$.

4.4.6. Elimination terms in dimensions 2. In dimension 2, let arrows $\vec{f},\vec{g} : \vec{x} \rightrightarrows \vec{y}$ in $\int I_A$ be given together with a 2-cell $\vec{\alpha} : \vec{f} \Longrightarrow \vec{g}$. In order to define α_J we will describe 2-cells filling both the square and triangle as indicated in the following diagram:



Defining the functor

$$(\int I_A)_2(\vec{f},\vec{g}) \xrightarrow{\vartheta^2_{\vec{f},\vec{g}}} D(\vec{y})_1\big(\Theta^1(\vec{f}), y_J\big)$$

by

$$\vartheta^2(-) := \triangleright(\vec{g}) \ *_0 \ \vartheta^1(-) \ *_0 \ \triangleleft(\vec{f}).$$

we see that our goal is precisely to provide 2-cells

$$f_J \xrightarrow{\triangleleft(\vec{\alpha})} \vartheta^2_{\vec{f},\vec{g}}(\vec{\alpha}) \xrightarrow{\triangleright(\vec{\alpha})} \Theta^2_{\vec{f},\vec{g}}(\vec{\alpha}).$$

The strategy for filling the square and triangle from above is fairly simple. For the triangle, we use $\breve{\alpha}_d$, and for the square we use the naturality cell $\varepsilon_{\vec{\alpha}}$. To begin with, we define

$$\triangleleft(\vec{\alpha}) := \langle\vec{y}\rangle(\breve{\alpha}_d) \ *_0 \ \triangleleft(\vec{f}).$$

For the square, we observe that

$$\begin{aligned}\Theta^1(\vec{\gamma}) &= D(\vec{\gamma})(x_J), \text{ and}\\ \vartheta^1(\vec{\gamma}) &= D(\nabla^1_+\vec{\gamma})(\breve{x}_d)\end{aligned}$$

where $\vec{\gamma}$ is any cell of $\int I_A(\vec{x},\vec{y})$. Thus,

$$\begin{aligned}\vartheta^1(\vec{\gamma}) \ *_0 \ \triangleleft(\vec{f}) &= D(\nabla^2_-\vec{\gamma})(\breve{x}_d), \text{ and}\\ \triangleleft(\vec{g}) \ *_0 \ \Theta^1(\vec{\gamma}) &= D(\nabla^2_+\vec{\gamma})(\breve{x}_d).\end{aligned}$$

As such, we define

$$\triangleright(\vec{\alpha}) := \triangleright(\vec{g}) \ *_0 \ \langle\vec{\alpha}\rangle_{\breve{x}_d}.$$

Thus, α_J is $(\triangleright(\vec{\alpha}) \ *_1 \ \triangleleft(\vec{\alpha}))$. It will often be convenient to omit parentheses when dealing with the arrows $\triangleright(\vec{\alpha})$ and $\triangleleft(\vec{\alpha})$. In order to avoid confusion, we adopt the convention that $\triangleright$ and $\triangleleft$ bind more tightly than composition. I.e., $\triangleleft\,\vec{\gamma} \ *_k \ \vec{\varphi}$ should be read as $\triangleleft(\vec{\gamma}) \ *_k \ \vec{\varphi}$.

Before moving on to dimension 3, we first introduce some additional machinery which is the final technical ingredient required in order to make the induction to higher dimensions possible. Namely, for $\vec{f},\vec{g} : \vec{x} \rightrightarrows \vec{y}$ parallel 1-cells, we define functors

$$D(\vec{y})_1(\vartheta^1\vec{f}, y_J) \xrightarrow{\breve{H}^1_{\vec{f},\vec{g}}} D(\vec{y})_1(\Theta^1\vec{f}, y_J) \xleftarrow{H^1_{\vec{f},\vec{g}}} D(\vec{y})_1(\Theta^1\vec{f}, \vartheta^1\vec{g})$$

as follows:

$$\begin{aligned}\breve{H}^1_{\vec{f},\vec{g}}(-) &:= (- \ *_0 \ \triangleleft\vec{f}), \text{ and}\\ H^1_{\vec{f},\vec{g}}(-) &:= (\triangleright\vec{g} \ *_0 \ -).\end{aligned}$$

With these functors at our disposal, we are in the position to make several remarks regarding their interaction with the other structures with which we are concerned. To begin with, when $\vec{\alpha}$ is a 2-cell $\vec{f} \Longrightarrow \vec{g}$,

$$\triangleright\,\vec{\alpha} = H^1_{\vec{f},\vec{g}}\big(\langle\vec{\alpha}\rangle_{\breve{x}_d}\big), \text{ and} \tag{55}$$

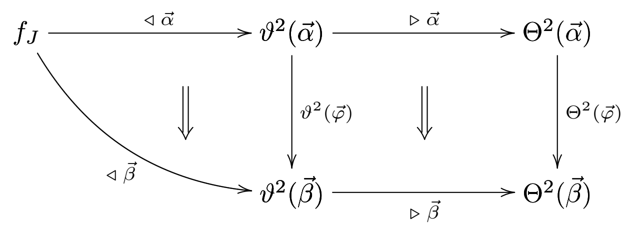
$$\triangleleft\,\vec{\alpha} = \breve{H}^1_{\vec{f},\vec{g}}\big(\langle\vec{y}\rangle(\breve{\alpha}_d)\big). \tag{56}$$

Also, these functors interact with ϑ^2 in the sense that

$$\breve{H}^1_{\vec{f},\vec{g}}\big(\triangleright\,\vec{g} \ *_0 \ \vartheta^1(\vec{\gamma})\big) \ = \ \vartheta^2_{\vec{f},\vec{g}}(\vec{\gamma}) \ = \ H^1_{\vec{f},\vec{g}}\big(\vartheta^1(\vec{\gamma}) \ *_0 \ \triangleleft\,\vec{f}\big). \tag{57}$$

In an informal sense, the problem of providing the elimination maps φ_J will be seen to always amount, as above, to filling both a triangle and a square. In each case, the tactic is essentially the same as above and the functors H^k and $\breve{H}^k$ allow us to express in the most perspicuous way the combinatorics of the situation for the squares and triangles, respectively.

4.4.7. Elimination terms in dimension 3. In dimension 3, given $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ a 3-cell of $\int I_A$, we would like to describe the 3-cells indicated in the following diagram:



(58)

With this picture in mind we begin by defining, for fixed parallel 2-cells $\vec{\alpha}, \vec{\beta} : \vec{f} \rightrightarrows \vec{g}$, functors

$$D(\vec{y})_2(f_J, \vartheta^2\vec{\beta}) \xrightarrow{\breve{H}^2_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_2(f_J, \Theta^2\vec{\beta}) \xleftarrow{H^2_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_2(\vartheta^2\vec{\alpha}, \Theta^2\vec{\beta})$$

as follows:

$$\breve{H}^2_{\vec{\alpha},\vec{\beta}}(-) := \rhd\,\vec{\beta} \ *_1 \ \breve{H}^1_{\vec{f},\vec{g}}(-), \text{ and}$$
$$H^2_{\vec{\alpha},\vec{\beta}}(-) := H^1_{\vec{f},\vec{g}}(-) \ *_1 \ \lhd\,\vec{\alpha}.$$

Next,

$$(\int I_A)_3(\vec{\alpha}, \vec{\beta}) \xrightarrow{\vartheta^3_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_2\big(f_J, \Theta^2(\vec{\beta})\big)$$

is defined by

$$\vartheta^3(-) := \rhd(\vec{\beta}) \ *_1 \ \vartheta^2(-) \ *_1 \ \lhd(\vec{\alpha})$$

Before going any further it is useful to establish several facts. First, we note that by a straightforward calculation:

(59) $$\vartheta^2_{\vec{f},\vec{g}}(\vec{\gamma}) = \breve{H}^1_{\vec{f},\vec{g}}\big(\langle\vec{y}\rangle\tilde{\Theta}^2\breve{\gamma}\big).$$

We call (59) the **triangle-law for dimension 2** and note that together with (56) it follows that the triangle from (58) may be filled with the 3-cell $\breve{H}^1_{\vec{f},\vec{g}}(\langle\vec{y}\rangle\breve{\varphi}_d)$. Accordingly, we define

$$\rhd\,\vec{\varphi} := \breve{H}^2_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{y}\rangle\breve{\varphi}_d\big).$$

Turning to the square, observe that

$$\begin{aligned}\Theta^2_{\vec{f},\vec{g}}(\vec{\gamma}) \ *_1 \ \rhd\,\vec{\alpha} &= (g_J \ *_0 \ \Theta^1\vec{\gamma}) \ *_1 \ (\rhd\,\vec{g} \ *_0 \ \langle\vec{\alpha}\rangle_{\breve{x}_d}) \\ &= (\rhd\,\vec{g} \ *_0 \ \lhd\,\vec{g} \ *_0 \ \Theta^1\vec{\gamma}) \ *_1 \ (\rhd\,\vec{g} \ *_0 \ \langle\vec{\alpha}\rangle_{\breve{x}_d}) \\ &= \rhd\,\vec{g} \ *_0 \ \big((\lhd\,\vec{g} \ *_0 \ \Theta^1\vec{\gamma}) \ *_1 \ \langle\vec{\alpha}\rangle_{\breve{x}_d}\big) \\ &= \rhd\,\vec{g} \ *_0 \ D\big(\nabla^3_-\vec{\gamma}\big)_{\breve{x}_d}.\end{aligned}$$

Consequently, we obtain the **source square-law for dimension 2**:

(60) $$\Theta^2_{\vec{f},\vec{g}}(\vec{\gamma}) \ *_1 \ \rhd\,\vec{\alpha} = H^1_{\vec{f},\vec{g}}\big(D(\nabla^3_-\vec{\gamma})_{\breve{x}_d}\big).$$

Another straightforward calculation yields the **target square-law for dimension 2**:

$$\triangleright \vec{\beta} \ *_1 \ \vartheta^2_{\vec{f},\vec{g}}(\vec{\gamma}) = H^1_{\vec{f},\vec{g}}\big(D(\nabla^3_+\vec{\gamma})_{\breve{x}_d}\big). \tag{61}$$

Thus, the filler of the square in (58) is defined to be $H^1_{\vec{f},\vec{g}}\big(\langle\vec{\varphi}\rangle_{\breve{x}_d}\big)$. Finally, we set

$$\triangleleft \vec{\varphi} := H^2_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{\varphi}\rangle_{\breve{x}_d}\big), \text{ and}$$
$$\varphi_J := \triangleright \vec{\varphi} \ *_2 \ \triangleleft \vec{\varphi}.$$

4.4.8. The construction in higher dimensions. Now, at higher-dimensions, the construction of the elimination terms is by induction on dimension. In particular, we proceed by induction on $n \geq 2$ in such a way that *at stage* $(n+1)$ — in addition to the existence of the required $(n+1)$-cells φ_J — the following conditions are satisfied:

(1) For all parallel n-cells $\vec{\alpha}, \vec{\beta} : \vec{f} \rightrightarrows \vec{g}$, there is a functor $\vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}$ parallel to $\Theta^{n+1}_{\vec{\alpha},\vec{\beta}}$. I.e., .

(2) For any $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \longrightarrow \vec{\beta}$, there exist corresponding $(n+1)$-cells $\triangleright \vec{\varphi}$ and $\triangleleft \vec{\varphi}$ such that

$$\alpha_J \xrightarrow{\triangleleft \vec{\varphi}} \vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\varphi} \xrightarrow{\triangleright \vec{\varphi}} \Theta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\varphi}$$

when $(n+1)$ is even, and

$$\Theta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\varphi} \xrightarrow{\triangleleft \vec{\varphi}} \vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\varphi} \xrightarrow{\triangleright \vec{\varphi}} \beta_J$$

when $(n+1)$ is odd.

(3) There are, for $\vec{\alpha}$ and $\vec{\beta}$ parallel n-cells, functors $H^n_{\vec{\alpha},\vec{\beta}}$ and $\breve{H}^n_{\vec{\alpha},\vec{\beta}}$ such that

$$D(\vec{y})_n\big(\vartheta^n\vec{\alpha}, g_J\big) \xrightarrow{\breve{H}^n_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_n\big(\Theta^n\vec{\alpha}, g_J\big), \text{ and}$$
$$D(\vec{y})_n\big(\Theta^n\vec{\alpha}, \vartheta^n\vec{\beta}\big) \xrightarrow{H^n_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_n\big(\Theta^n\vec{\alpha}, g_J\big)$$

if $(n+1)$ is even; and

$$D(\vec{y})_n\big(f_J, \vartheta^n\vec{\beta}\big) \xrightarrow{\breve{H}^n_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_n\big(f_J, \Theta^n\vec{\beta}\big), \text{ and}$$
$$D(\vec{y})_n\big(\vartheta^n\vec{\alpha}, \Theta^n\vec{\beta}\big) \xrightarrow{H^n_{\vec{\alpha},\vec{\beta}}} D(\vec{y})_n\big(f_J, \Theta^n\vec{\beta}\big)$$

if $(n+1)$ is odd.

(4) The following **triangle-law** is satisfied:

$$\vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\gamma} = \breve{H}^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{y}\rangle\tilde{\Theta}^{n+1}\breve{\gamma}\big),$$

when $\vec{\gamma}$ is any cell in the domain of ϑ^{n+1}.

(5) If $\vec{\varphi}, \vec{\psi} : \vec{\alpha} \rightrightarrows \vec{\beta}$ are parallel $(n+1)$-cells, the following **square-laws** are satisfied:

$$H^n_{\vec{\alpha},\vec{\beta}}\big(D(\nabla^{n+2}_-\vec{\gamma})_{\breve{x}_d}\big) = \begin{cases} \Theta^{n+1}\vec{\gamma} \ *_n \ \triangleright \vec{\varphi} & \text{if } (n+1) \text{ is even, and} \\ \vartheta^{n+1}\vec{\gamma} \ *_n \ \triangleleft \vec{\varphi} & \text{if } (n+1) \text{ is odd;} \end{cases}$$

and

$$H^n_{\vec{\alpha},\vec{\beta}}\big(D(\nabla^{n+2}_{+}\vec{\delta})_{\breve{x}_d}\big) = \begin{cases} \rhd\vec{\psi} \ *_n \ \vartheta^{n+1}\vec{\delta} & \text{if } (n+1) \text{ is even, and} \\ \\ \lhd\vec{\psi} \ *_n \ \Theta^{n+1}\vec{\delta} & \text{if } (n+1) \text{ is odd,} \end{cases}$$

for appropriate cells $\vec{\gamma}$ and $\vec{\delta}$. Note that the ∇^{n+2}_{ξ} here are defined with respect to $\vec{\varphi}$ and $\vec{\psi}$.

Assuming we have carried out the construction up to stage n, we claim that the following definitions at stage $(n+1)$ will satisfy the required conditions:

- For parallel n-cells $\vec{\alpha}$ and $\vec{\beta}$,

$$\vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}(-) := \rhd\vec{\alpha} \ *_{(n-1)} \ \vartheta^n(-) \ *_{(n-1)} \ \lhd\vec{\beta}.$$

- If $\vec{\alpha}$ and $\vec{\beta}$ are parallel n-cells $\vec{f} \rightrightarrows \vec{g}$, then we define

$$H^n_{\vec{\alpha},\vec{\beta}}(-) := \begin{cases} \rhd\vec{\beta} \ *_n \ H^{n-1}_{\vec{f},\vec{g}}(-) & \text{if } (n+1) \text{ is even, and} \\ \\ H^{n-1}_{\vec{f},\vec{g}}(-) \ *_n \ \lhd\vec{\alpha} & \text{if } (n+1) \text{ is odd;} \end{cases}$$

and

$$\breve{H}^n_{\vec{\alpha},\vec{\beta}}(-) := \begin{cases} \breve{H}^{n-1}_{\vec{f},\vec{g}}(-) \ *_n \ \lhd\vec{\alpha} & \text{if } (n+1) \text{ is even, and} \\ \\ \rhd\vec{\beta} \ *_n \ \breve{H}^{n-1}_{\vec{f},\vec{g}}(-) & \text{if } (n+1) \text{ is odd.} \end{cases}$$

- Given $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ a $(n+1)$-cell, we define

$$\lhd\vec{\varphi} := \begin{cases} \breve{H}^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{y}\rangle\breve{\varphi}_d\big) & \text{if } (n+1) \text{ is even, and} \\ \\ H^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{\varphi}\rangle_{\breve{x}_d}\big) & \text{if } (n+1) \text{ is odd;} \end{cases}$$

and

$$\rhd\vec{\varphi} := \begin{cases} H^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{\varphi}\rangle_{\breve{x}_d}\big) & \text{if } (n+1) \text{ is even, and} \\ \\ \breve{H}^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{y}\rangle\breve{\varphi}_d\big) & \text{if } (n+1) \text{ is odd.} \end{cases}$$

- In all dimensions,

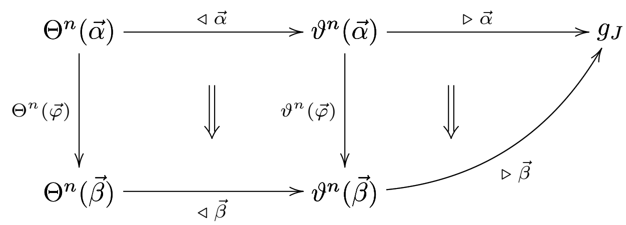
$$\alpha_J := \rhd\vec{\alpha} \ *_n \ \lhd\vec{\alpha}$$

when $\vec{\alpha}$ is a $(n+1)$-cell.

The reader can readily verify that we have already satisfied the conditions of the induction in the base case where $n = 2$. We now turn to the induction step.

Lemma 4.23. *With the definitions given above, the conditions of the construction are satisfied in all dimensions* $(n+1)$.

Proof. First, assume $(n+1)$ is even with $n > 2$ and let an $(n+1)$-cell $\vec{\varphi} : \vec{\alpha} \Longrightarrow \vec{\beta}$ in $\int I_A$ be given. Then, by the induction hypothesis and examination of the following diagram



where $\vec{\alpha}, \vec{\beta} : \vec{f} \Longrightarrow \vec{g}$, it follows that conditions (1)-(3) are satisfied with the definitions given above.

For the triangle law, we reason as follows:

$$\begin{aligned}
\breve{H}^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{y}\rangle\tilde{\Theta}^{n+1}\breve{\gamma}\big) &= \breve{H}^{n-1}_{\vec{f},\vec{g}}\big(\langle\vec{y}\rangle\breve{\beta}_d \ *_{(n-1)} \ \langle\vec{y}\rangle\tilde{\Theta}^n\breve{\gamma}\big) \ *_{(n-1)} \ \lhd\,\vec{\alpha} \\
&= \breve{H}^{n-1}_{\vec{f},\vec{g}}\big(\langle\vec{y}\rangle\breve{\beta}_d\big) \ *_{(n-1)} \ \breve{H}^{n-1}_{\vec{f},\vec{g}}\big(\langle\vec{y}\rangle\tilde{\Theta}^n\breve{\gamma}\big) \ *_{(n-1)} \ \lhd\,\vec{\alpha} \\
&= \rhd\,\vec{\beta} \ *_{(n-1)} \ \vartheta^n_{\vec{f},\vec{g}}(\vec{\gamma}) \ *_{(n-1)} \ \lhd\,\vec{\alpha} \\
&= \vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}(\vec{\gamma}),
\end{aligned}$$

where the penultimate equation is by definition of $\rhd\,\vec{\beta}$ and the induction hypothesis.

Next assume given $(n+1)$-cells $\vec{\varphi}, \vec{\psi} : \vec{\alpha} \Longrightarrow \vec{\beta}$. For the "source" square law, we have

$$\begin{aligned}
H^n_{\vec{\alpha},\vec{\beta}}\big(D(\nabla^{n+2}_{-}\vec{\gamma})_{\breve{x}_d}\big) &= H^n_{\vec{\alpha},\vec{\beta}}\big(D(\nabla^{n+1}_{+}\vec{\gamma})_{\breve{x}_d} \ *_n \ \langle\vec{\varphi}\rangle_{\breve{x}_d}\big) \\
&= \left(\rhd\vec{\beta} \ *_{(n-1)} \ H^{n-1}_{\vec{f},\vec{g}}\big(D(\nabla^{n+1}_{+}\vec{\gamma})\big)\right) \ *_n \ \rhd\,\vec{\varphi} \\
&= \big(\rhd\vec{\beta} \ *_{(n-1)} \ \lhd\,\vec{\beta} \ *_{(n-1)} \ \Theta^n\vec{\gamma}\big) \ *_n \ \rhd\,\vec{\varphi} \\
&= \big(\beta_J \ *_{(n-1)} \ \Theta^n\vec{\gamma}\big) \ *_n \ \rhd\,\vec{\varphi} \\
&= \Theta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\gamma} \ *_n \ \rhd\,\vec{\varphi},
\end{aligned}$$

where the third equation is by the induction hypothesis. For the "target" square law, we reason similarly and note that

$$\begin{aligned}
H^n_{\vec{\alpha},\vec{\beta}}\big(\langle\vec{\psi}\rangle_{\breve{x}_d} \ *_n \ D(\nabla^{n+1}_{-}\vec{\gamma})_{\breve{x}_d}\big) &= \rhd\,\vec{\psi} \ *_n \ \left(\rhd\,\vec{\beta} \ *_{(n-1)} \ H^{n-1}_{\vec{f},\vec{g}}\big(D(\nabla^{n+1}_{-}\vec{\gamma})_{\breve{x}_d}\big)\right) \\
&= \rhd\,\vec{\psi} \ *_n \ \big(\rhd\,\vec{\beta} \ *_{(n-1)} \ \vartheta^n\vec{\gamma} \ *_{(n-1)} \ \lhd\,\vec{\alpha}\big) \\
&= \rhd\,\vec{\psi} \ *_n \ \vartheta^{n+1}_{\vec{\alpha},\vec{\beta}}\vec{\gamma},
\end{aligned}$$

as required.

The induction step where $(n+1)$ is odd is essentially dual. □

Using the lemma, we now have the following fundamental result.

Proposition 4.24. *The cells of the form φ_J constitute a section $J : \int I_A \longrightarrow \int D$ of the projection map $\int D \longrightarrow \int I_A$.*

PROOF. In light of Lemma 4.23 the only thing which remains is to verify that the assignment $(-)_J$ is functorial. This however is a consequence of the functoriality of d and the construction of the "terms" φ_J using the functors H^k and $\breve{H}^k$. □

4.5. The interpretation of type theory

With the machinery from the preceding sections at our disposal it is now possible to describe explicitly an interpretation of type theory using ω-groupoids. The interpretation given generalizes directly the Hofmann-Streicher [**35**] interpretation using regular 1-dimensional groupoids. Before going into the details several remarks are in order.

First, whereas in *ibid* the entire logical framework is interpreted, we here only interpret the theory $\mathbb{T}_\omega$ as described in Chapter 1. We note, however, that we could just as well have interpreted the entire logical framework in this setting. Secondly, the interpretation we give below can be organized into a (large) comprehension category, or a category with attributes, or a category with families. In this case we believe that the model most naturally can be described as a category with attributes or a category with families [**21**]. We assume that the reader is familiar with these forms of semantics. Because the ideas behind the basic interpretation are not new, we do not go into full detail regarding the interpretation of the basic syntax. Finally, we sketch the construction of dependent products and sums before going on to the statement of the main results.

4.5.1. Contexts, types and terms. The idea of the interpretation, which should be familiar in light of the discussions in the foregoing chapters, is to regard closed types as ω-groupoids. Explicitly, contexts Γ are interpreted as small ω-groupoids. To begin with, the empty context () is interpreted as the terminal ω-groupoid:

$$[\![()]\!] := \mathbf{1}.$$

Now, given a context Γ together with its interpretation $[\![\Gamma]\!]$ as an ω-groupoid, judgements of the form $\Gamma \vdash A : \text{type}$ are interpreted as functors

$$[\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash A:\text{type}]\!]} \omega\text{-}\mathbf{Gpd}.$$

The extended context $(\Gamma, x : A)$ is then interpreted as the ω-groupoid given by applying the Grothendieck construction from Section 4.1 to the functor in question:

$$[\![\Gamma, x : A]\!] := \int [\![\Gamma \vdash A : \text{type}]\!].$$

A judgement of the form $\Gamma \vdash a : A$ is then interpreted as a section

$$\begin{array}{ccc} [\![\Gamma]\!] & \longrightarrow & \int [\![\Gamma \vdash A : \text{type}]\!] \\ & {\scriptstyle 1_{[\![\Gamma]\!]}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & [\![\Gamma]\!] & \end{array}$$

of the projection functor.

4.5.2. Substitution and weakening. Suppose we are given an ω-groupoid $\mathcal{C}$ interpreting a context Γ together with $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$, $B : \int A \longrightarrow \omega\text{-}\mathbf{Gpd}$ and $C : \int B \longrightarrow \omega\text{-}\mathbf{Gpd}$ interpreting judgements

$$\Gamma \vdash A : \mathrm{type}, \quad \Gamma, x : A \vdash B(x) : \mathrm{type}, \quad \text{and } \Gamma, x : A, y : B(x) \vdash C(x,y) : \mathrm{type},$$

respectively. Moreover, let a section a interpreting the judgement $\Gamma \vdash a : A$ be given. Then the judgement $\Gamma \vdash B[a/x]$ is interpreted as the composite functor

$$\mathcal{C} \xrightarrow{\ a\ } \int A \xrightarrow{\ B\ } \omega\text{-}\mathbf{Gpd}.$$

Similarly,

$$[\![\Gamma, y : B(a) \vdash C(a,y) : \mathrm{type}]\!] := C \circ \{a\}_B,$$

in the notation of Section 4.4.1. Finally, if c is a section of $\int C \longrightarrow \int B$ interpreting the judgement $\Gamma, x : A, y : B(x) \vdash c(x,y) : C(x,y)$ we define

$$[\![\Gamma, y : B(a) \vdash c(a,y) : C(a,y)]\!] := c[a].$$

Finally, for weakening, we note that when functors $A, B : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$ interpret the judgements $\Gamma \vdash A : \mathrm{type}$ and $\Gamma \vdash B : \mathrm{type}$, the "weakened" judgement $\Gamma, x : A \vdash B : \mathrm{type}$ is interpreted by the composite

$$\int A \xrightarrow{\ \pi\ } \mathcal{C} \xrightarrow{\ B\ } \omega\text{-}\mathbf{Gpd}.$$

4.5.3. Dependent sums. Before defining dependent products and sums, we begin by describing some basic features of the general setup. First, given a functor $A : \mathcal{C} \longrightarrow \omega\text{-}\mathbf{Gpd}$ we note that, by the basic properties of $\omega\text{-}\mathbf{Gpd}$, there exists an ω-groupoid denoted by $\Gamma(A)$ of sections of the projection $\int A \longrightarrow \mathcal{C}$. I.e., the objects of $\Gamma(A)$ are sections

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ a\ } & \int A \\ & {\scriptstyle 1_{\mathcal{C}}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathcal{C} & \end{array}$$

and arrows are 2-cells $\alpha : a \Longrightarrow b$ for which $\pi \circ \alpha = 1_{\mathcal{C}}$ and so forth at higher dimensions. Now, given a further functor $B : \int A \longrightarrow \omega\text{-}\mathbf{Gpd}$ and an object x of $\mathcal{C}$, we define a functor $\ulcorner B_x \urcorner : A_x \longrightarrow \omega\text{-}\mathbf{Gpd}$ as follows:

Objects: Given an object x_- of A_x, $\ulcorner B_x \urcorner(x_-)$ is the ω-groupoid $B(x, x_-)$.

1-Cells: Given an arrow $h : x_- \longrightarrow x_+$ in A_x, we define the functor $\ulcorner B_x \urcorner(h) : B(x, x_-) \longrightarrow B(x, x_+)$ as follows:

$$\ulcorner B_x \urcorner(h)(-) := B(x,h)(-).$$

Here we note that this is possible since $(x,h) : (x, x_-) \longrightarrow (x, x_+)$, where we have written (x,h) instead of $(1_x, h)$.

$(n+1)$**-Cells:** Given a $(n+1)$-cell $\alpha : f \Longrightarrow g$ in A_x bounded by 0-cells x_- and x_+, together with an object y of $B(x, x_-)$, the transformation $\ulcorner B_x \urcorner(\alpha)$ is defined at y by

$$\ulcorner B_x \urcorner(\alpha)_y := B(x, \alpha)_y.$$

Now, we would like to describe, in this same setting, the dependent sum $\Sigma_{A,B} : \mathcal{C} \longrightarrow \omega$-**Gpd**. Note that, when A and B are apparent, we will often omit the subscripts.

Objects: Given an object x of $\mathcal{C}$, we define

$$\Sigma_{A,B}(x) := \int \ulcorner B_x \urcorner.$$

Given $f : x \longrightarrow y$ in $\mathcal{C}$ we would like to define $\Sigma_{A,B}(f) : \int \ulcorner B_x \urcorner \longrightarrow \int \ulcorner B_y \urcorner$, which we will write, for the sake of avoiding too many parentheses, as Σ_f. An object $\vec{v}$ of $\int \ulcorner B_x \urcorner$ is a pair $(v_-, v_\sharp)$ such that v_- is an object of A_x and $v_\sharp$ is an object of $B(x, v_-)$. We adopt a similar notation in higher dimensions. We also write $\partial(x)^n$ for the weighted face functor of $\int \ulcorner B_x \urcorner$ and similarly for $\partial(y)^n$, *et cetera*. With this in mind we adopt the following definition (which we will discuss below):

1-Cells: Given $f : x \longrightarrow y$ in $\mathcal{C}$, Σ_f is defined on objects $\vec{v}$ of $\int \ulcorner B_x \urcorner$ by

$$\Sigma_f(\vec{v}) := \big(v_-.f,\ B(f, v_-.f)(v_\sharp)\big),$$

and on $(n+1)$-cells $\vec{\varphi}$ of $\int \ulcorner B_x \urcorner$ by

$$\Sigma_f(\vec{\varphi}) := \big(\varphi_-.f,\ B(f, w_-.f)(\varphi_\sharp)\big),$$

where $\vec{v}$ and $\vec{w}$ are the objects bounding $\vec{\varphi}$ at source and target position, respectively.

Note that this is a correct definition since, by functoriality of B, it is proved (simultaneously with the verification of the definitions) at each stage that

$$B(f, w_-.f)\big(\partial(x)^{n+1}_{\vec{\alpha},\vec{\beta}}(\varphi_-)\big) = \partial(y)^{n+1}_{\Sigma_f\vec{\alpha},\Sigma_f\vec{\beta}}(\varphi_-.f),$$

where $\vec{\alpha}$ and $\vec{\beta}$ are the n-cells bounding $\vec{\varphi}$.

In dimension $(n+1)$, $\Sigma_{A,B}$ is given as follows:

$(n+1)$**-Cells:** Given a $(n+1)$-cell $\varphi : \alpha \longrightarrow \beta$ in $\mathcal{C}$, $\Sigma_\varphi : \Sigma_\alpha \longrightarrow \Sigma_\beta$ has as its component at the object $\vec{v}$ of $\int \ulcorner B_x \urcorner$ the n-cell

$$(\Sigma_\varphi)_{\vec{v}} := \big(v_-.\varphi,\ B(\varphi, v_-.\varphi)(v_\sharp)\big)$$

of $\int \ulcorner B_y \urcorner$.

To see that this makes sense, let n-cells $\lambda, \mu : \alpha \longrightarrow \beta$ be given. Then it follows, by induction on n that,

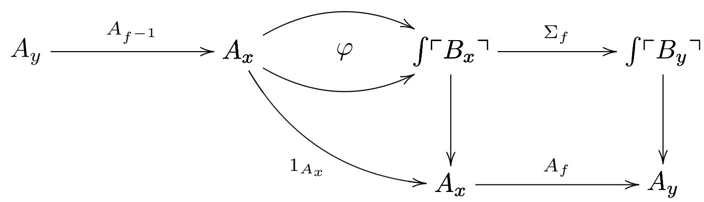
$$\partial(y)^{(n+1)}_{(\Sigma_\lambda)_{\vec{v}},(\Sigma_\mu)_{\vec{v}}}(v_-.\gamma) = \begin{cases} B(\mu,\ v_-.\gamma)_{v_\sharp} & \text{if } (n+1) \text{ is even, and} \\ \\ B(\lambda,\ v_-.\gamma)_{v_\sharp} & \text{if } (n+1) \text{ is odd,} \end{cases}$$

where γ is any m-cell with $m \geq n+1$ bounded by λ and μ (i.e., $s^{m-n}\gamma = \lambda$ and $t^{m-n}\gamma = \mu$). This is a straightforward induction using functoriality of B. This completes the description of the interpretation $\Sigma_{A,B}$ of dependent sums in this setting. With these definitions, the verification of functoriality is a routine calculation.

Next, we define the dependent product $\Pi_{A,B} : \mathcal{C} \longrightarrow \omega$-**Gpd** to be the functor which sends an object x to the ω-groupoid $\Gamma(\ulcorner B_x \urcorner)$ of sections of the projection $\int \ulcorner B_x \urcorner \longrightarrow A_x$. Explicitly, $\Pi_{A,B}$ is as follows:

Objects: $\Pi_{A,B}(x)$ is defined to be $\Gamma(\ulcorner B_x \urcorner)$.

1-Cells: Given an arrow $f : x \longrightarrow y$ in $\mathcal{C}$, $\Pi_{A,B}(f)$ is the functor sending a section a in $\Gamma(\ulcorner B_x \urcorner)$ to the section $\Sigma_f \circ a \circ A_{f^{-1}}$. In general, $\Pi_{A,B}(f)$ sends an arbitrary cell φ of $\Gamma(\ulcorner B_x \urcorner)$ to the composite $\Sigma_f \circ \varphi \circ A_{f^{-1}}$ as indicated in the following diagram:



$(n+1)$**-Cells:** In general, when $\gamma : \alpha \longrightarrow \beta$ is a $(n+1)$-cell in $\mathcal{C}$ and a is an object of $\Pi_{A,B}(x)$, the dependent product is defined by setting

$$\Pi_{A,B}(\gamma)_a := \Sigma_\gamma \circ a \circ A_{\sigma(\gamma)},$$

where $\sigma : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ is the dual functor for $\mathcal{C}$ as described in Section 4.2.2.

This completes the description of the dependent sums and products. The introduction and elimination terms are then obtained as direct generalizations of those given in [**35**] using techniques similar to those employed here.

4.5.4. Independence results. With the interpretations given we obtain the following result extending the original groupoids model from *ibid* to the setting of ω-groupoids:

THEOREM 4.25. *With the interpretation just described, we obtain a model of* $\mathbb{T}_\omega$ *using* ω-**Gpd**. *Moreover, this model is both split and coherent.*

PROOF. It is straightforward to verify that all of the required laws, including the coherence laws, are satisfied. Moreover, because substitution is interpreted functorially, the model is split. □

It is a trivial consequence of the interpretation that, for all $n \geq 1$, the model refutes UIP_n. Indeed, these are refuted already in the empty context. Thus, the following corollaries are immediate:

COROLLARY 4.26. *For all $n \geq 1$, the principle* UIP_n *is not derivable in* $\mathbb{T}_\omega$.

COROLLARY 4.27. *For $n \geq 0$, neither* TR_n *nor* OUP_n *are derivable in* $\mathbb{T}_\omega$.

Moreover, by truncating the construction of the model in ω-**Gpd** to n-groupoids, all of the theories in the hierarchy of theories are distinct.

COROLLARY 4.28. *For all $i, j \geq 0$, if $i \neq j$, then $\mathbb{T}_i \neq \mathbb{T}_j$. Moreover, when $i, j \geq 1$, $\mathbb{P}_i \neq \mathbb{P}_j$.*

CHAPTER 5

Future work

There are several directions that we regard as being fruitful for future work in the areas considered in this dissertation. In this final chapter we will survey, briefly, some of these. The various topics discussed below are presented in no particular order.

Applications to homotopy theory and computer science. Another direction in which the material of this dissertation can be further developed is with a view toward applications to homotopy theory and computer science. Specifically, we would like to better understand the type theory which provides the internal language of categories possessing weak factorization systems. As the discussion of quasi-models in Chapter 2 indicates, such models have, in some sense, suitable structure to interpret identity types. However, substitution in these models need not behave correctly. It seems likely though that such categories are genuine models of a form of intensional type theory with explicit substitution [**1**]. Developing such a calculus would be important, not only because it would open the door to more direct applications to homotopy theory, but also because calculi of explicit substitution inevitably arise in computer implementations of type theory. As such, it might be hoped that a reasonable calculus of explicit substitution corresponding to the internal language of categories with weak factorization systems might be of use to both homotopy theory and theoretic computer science (e.g., we would expect applications to the study and development of proof assistants [**9, 17**]).

Higher-dimensional intervals. It should be possible to generalize the results of Chapter 3 to the (strict) higher-dimensional setting, thereby relating the constructions of Chapter 4 to manipulations involving the co-ω-groupoid interval $\mathbf{I}^{\omega}$ in the same way that Chapter 3 serves to relate the Hofmann-Streicher model [**35**] to the structure of the groupoid interval $\mathbf{I}$. Such a task was one of the motivations for considering the model constructed in Chapter 4; for, as described at the end of Chapter 3, in order to model type theory using a (strict) higher-dimensional interval it seems to be necessary to know not just what the interval is, but also what is the appropriate notion of split fibration associated to that interval. Thus, it is desirable to show that the constructions from Chapter 4 relate in the expected way to manipulations of some kind of split fibrations of ω-groupoids and their corresponding infinite dimensional interval.

Weakening the structure. Although the model constructed using strict ω-groupoids does refute the kind of truncation principles we have considered, it nonetheless validates certain conditions that we do not expect to hold syntactically. In particular, because the groupoid laws themselves are strict, this model validates the corresponding type theoretic rules on the nose (i.e., up to definitional equality). We expect that (at least some of) these groupoid laws are not derivable in $\mathbb{T}_\omega$. Rather, they should be valid only up to the existence of propositional equality. As such, it is reasonable to expect that there exist models of $\mathbb{T}_\omega$ which are not just higher-dimensional in the sense of the model from Chapter 4, but which are also suitably *weak*. In particular, it should be possible to obtain models of type theory using also *a suitable notion of* weak ω-groupoids. We emphasize that not all notions of weak ω-groupoids may work for this purpose: some may simply be too weak. After all, it is not clear *a priori* what equations on composites are forced by $\mathbb{T}_\omega$ to hold in all dimensions. For example, already at dimension 1, the composite operation $(f \cdot g)$ on identity proofs is strictly unital on one side. Even aside from this concern, it is not clear what coherence laws the higher-dimensional forms of composition satisfy and there is no reason to think that they will be the same as those required by X, Y, or Z definitions of weak ω-groupoid. With these caveats in place, we do believe that weaker models of $\mathbb{T}_\omega$ are waiting to be discovered and that some (if not all) of the techniques developed in this dissertation can be modified to suit this weaker setting.

Truly intensional models. In his *Habilitationsschrift* [**80**], Streicher enumerates a number of properties which he regards as being indicative of true intensionality for models or type theories. First, the reflection rule cannot be valid in arbitrary contexts. Second, function extensionality cannot be valid. Finally, the reflection rule must be valid in the empty context. In [**80**] Streicher constructs, using a form of modified realizability, a model of type theory which is truly intensional in this sense. However, this model is not able to refute the principle of uniqueness of identity proofs. On the other hand, both the Hofmann-Streicher groupoid model [**35**] and our generalization to ω-groupoids refute this principle; but neither are truly intensional in the sense described above. For example, they validates function extensionality and fail to validate the reflection rule in the empty context. Ideally, it should be possible to obtain interesting models that are both fully intensional in the sense of [**80**] and also clearly exhibit the higher-dimensional structure of the identity type construction. Accordingly, one very important avenue for future research is to pursue such models. In particular, it might be possible to construct a model which is, in a suitable sense, a hybrid of the realizibility techniques of *ibid* together with approach using higher-dimensional groupoids that we have considered in this dissertation.

APPENDIX A

Categorical background

The purpose of this Appendix is to collect in one place some of the basic category theoretic background of which we make use. For basic categorical background we refer the reader to [**58**].

A.1. Internal groupoids

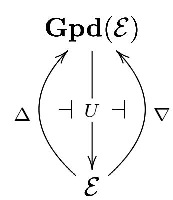
Assume $\mathcal{E}$ is a finitely bicomplete, cartesian closed category. Recall that an **internal groupoid** $\mathbb{G}$ **in** $\mathcal{E}$ consists of an internal category $\mathbb{G}$ together with a "symmetry" map $r : \mathbb{G}_1 \longrightarrow \mathbb{G}_1$ such that the following diagrams commute:

$$(62) \qquad \begin{array}{ccccc} & \swarrow^{s} & \mathbb{G}_1 & {}^{t}\searrow & \\ & & \downarrow r & & \\ \mathbb{G}_0 & \xleftarrow[t]{} & \mathbb{G}_1 & \xrightarrow[s]{} & \mathbb{G}_0, \end{array}$$

$$(63) \qquad \begin{array}{ccc} \mathbb{G}_1 & \xrightarrow{\langle 1,r\rangle} & \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbb{G}_0 & \xrightarrow[i]{} & \mathbb{G}_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{G}_1 & \xrightarrow{\langle r,1\rangle} & \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbb{G}_0 & \xrightarrow[i]{} & \mathbb{G}_1. \end{array}$$

Together, (62) and (63) state that r takes an (internal) arrow to its inverse. The category $\mathbf{Gpd}(\mathcal{E})$ of internal groupoids in $\mathcal{E}$ has as objects internal groupoids and as arrows internal functors between them.

A.1.1. Discrete and codiscrete groupoids. There is an evident forgetful functor $U : \mathbf{Gpd}(\mathcal{E}) \longrightarrow \mathcal{E}$ which sends an internal groupoid $\mathbb{G}$ to its object of objects $\mathbb{G}_0$ and similarly sends an internal functor to its object part. This forgetful functor possess both left and right adjoints



with Δ sending an object A to the **discrete** internal groupoid generated by A. I.e., $\Delta(A)_i := A$ for $i = 0, 1$ and all of the additional maps s, t, r, i and comp are defined to be the identity 1_A. On the other hand, ∇ sends A to the **codiscrete** internal groupoid generated by A. I.e.,

$$\begin{aligned}
\nabla(A)_0 &:= A \\
\nabla(A)_1 &:= A \times A \\
s &:= \pi_0 \\
t &:= \pi_1 \\
i &:= \Delta \\
r &:= \langle \pi_1, \pi_0 \rangle \\
c &:= \langle s \circ p_0, t \circ p_1 \rangle,
\end{aligned}$$

where $\Delta : A \longrightarrow A \times A$ is the diagonal and p_0, p_1 are the pullback projections $\nabla(A)_1 \times_{\nabla(A)_0} \nabla(A)_1 \rightrightarrows \nabla(A)_1$. $\mathbf{Gpd}(\mathcal{E})$ is finitely complete and cartesian closed.

A.1.2. Further basic facts. It is convenient to remember some very basic facts about internal groupoids. First, note that if $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a functor between internal groupoids, then, because inverses in internal categories are always unique, it the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{G}_1 & \xrightarrow{f_1} & \mathbb{H}_1 \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\
\mathbb{G}_1 & \xrightarrow[f_1]{} & \mathbb{H}_1.
\end{array}$$

The inverse map $r : \mathbb{G}_1 \longrightarrow \mathbb{G}_1$ induces in an obvious way a map

$$\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \xrightarrow{\bar{r}} \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1$$

which, intuitively, sends a composable pair (ϕ, ψ) to the pair (ψ^{-1}, ϕ^{-1}). This operation commutes in the obvious way with the composition map as indicated in the following diagram:

$$\begin{array}{ccc}
\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 & \xrightarrow{\bar{r}} & \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
\mathbb{G}_1 & \xrightarrow[r]{} & \mathbb{G}_1.
\end{array}$$

A.2. Simplicial sets

In this section we review some basic facts regarding the category of simplicial sets. The references for this section are [**58**] and [**26**].

A.2.1. Definition and examples. The **simplicial category $\boldsymbol{\Delta}$** has as objects all non-empty, finite ordinals $[n] := \{0, \ldots, n\}$ and monotone maps between them as arrows. In the category $\boldsymbol{\Delta}$ there is, for each $0 \leq i \leq n$, a distinguished map $d^i : [n-1] \longrightarrow [n]$ which is defined to be the injective monotone map which omits i. Similarly, there is a distinguished map $s^i : [n+1] \longrightarrow [n]$, for $0 \leq i \leq n$, defined to be the surjective monotone map which repeats i. The maps d^i are called the **coface maps** and the maps s^i are called the **codegeneracy maps**. The coface and codegeneracy maps freely generate all arrows in $\boldsymbol{\Delta}$ together with the following relations called the **cosimplicial identities**:

$$
\begin{aligned}
d^j d^i &= d^i d^{j-1} && \text{if} \quad i < j\,, \\
s^j d^i &= \begin{cases} d^i s^{j-1} & \text{if} \quad i < j\,, \\ 1 & \text{if} \quad i = j\,,\ i = j+1\,, \\ d^{i-1} s^j\,, & \text{if} \quad i > j+1\,,\ \text{and} \end{cases} \\
s^j s^i &= s^i s^{j+1} && \text{if} \quad i \leq j\,.
\end{aligned}
$$

A **simplicial set** X is a presheaf $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ on $\mathbf{\Delta}$. We write **SSet** for the category of simplicial sets, $\Delta[n]$ for the image of the ordinal $[n]$ under the Yoneda embedding. Explicitly, a simplicial set X consists of a sequence of sets X_n for $0 \leq n$ together with maps $d_i : X_n \longrightarrow X_{n-1}$ (the **face maps**) and $s_i : X_n \longrightarrow X_{n+1}$ (the **degeneracy maps**) for $0 \leq i \leq n$ subject to the following **simplicial identities**:

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i && i < j\,, \\
d_i s_j &= \begin{cases} s_{j-1} d_i\,, & i < j\,, \\ 1_{X_n}\,, & i = j\,, i = j+1\,, \\ s_j d_{i-1}\,, & i > j+1\,, \end{cases} \\
s_i s_j &= s_{j+1} s_i && i \leq j\,.
\end{aligned}
$$

The elements of X_0 are called **vertices** of X and the elements of X_n are called **n-simplices** of X.

Example A.1. The following are some of the basic examples of simplicial sets.

(1) Let an arbitrary set X_0 be given. We define a simplicial set X by setting:
  - The vertices of X are exactly the elements of X_0.
  - The elements of X_1 are binary words $(x * y)$ on the alphabet X_0, where $*$ is concatenation and where neither x nor y is the empty string. The map $s_0 : X_0 \longrightarrow X_1$ sends a vertex x to the word $(x * x)$. The map $d_0 : X_1 \longrightarrow X_0$ sends $(x * y)$ to y and d_1 sends it to x.
  - In general, X_n consists of $(n+1)$-ary words $(x_0 * \cdots * x_n)$ on the alphabet X_0, where we again disallow the empty string. In this case the maps $s_i : X_{n-1} \longrightarrow X_n$, for $0 \leq i < n$, are obtained by setting

    $$s_i(x_0 * \cdots * x_{n-1}) := (x_0 * \cdots * x_i * x_i * \cdots * x_{n-1}),$$

    and the maps $d_i : X_n \longrightarrow X_{n-1}$, for $0 \leq i \leq n$, by

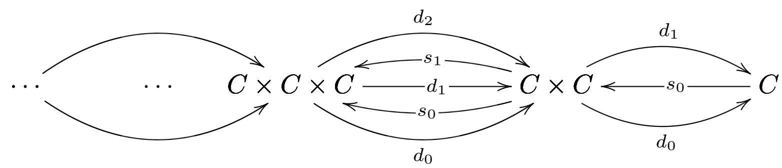
    $$d_i(x_0 * \cdots * x_n) := (x_0 * \cdots * \hat{x}_i * \cdots x_n).$$

    Here we have written $(x_0 * \cdots * \hat{x}_i * \cdots x_n)$ for the n-ary word obtained by omitting x_i. This "hat" notation for lists or words with omitted elements is quite convenient and we employ it frequently when dealing with simplicial sets.

  The reader should now verify that, using these definitions, the simplicial identities are satisfied.

(2) The foregoing example can be formulated in any category $\mathcal{C}$ with finite products by associating to any object C of $\mathcal{C}$ the simplicial object in $\mathcal{C}$

illustrated in the following diagram:



where the d_i are projections and the s_i are given by inserting the diagonal $C \longrightarrow C \times C$.

(3) Recall that a **simplicial complex** K is given by a set K_0 of *vertices* together with a set K of finite non-empty subsets of K_0 such that all of the singletons $\{v\}$, for $v \in K_0$, are in K and if $\sigma \in K$ and τ is a non-empty subset of σ, then $\tau \in K$. Elements of K are called *simplices*.

Given a simplicial complex K we define a simplicial set $\hat{K}$ as follows:

- $\hat{K}_0 := K_0$.
- $\hat{K}_1$ is the set of binary words $(x * y)$ on the alphabet K_0 such that

$$(x * y) \in \hat{K}_1 \qquad \text{iff} \qquad \{x, y\} \in K.$$

  As in the first example, the map $s_0 : \hat{K}_0 \longrightarrow \hat{K}_1$ sends a vertex x to $(x * x)$ and the maps d_i are defined similarly.
- In general, $\hat{K}_n$ is the set of words of length $n+1$ on the alphabet K_0 such that

$$(x_0 * \cdots * x_n) \in \hat{K}_n \qquad \text{iff} \qquad \{x_0, \ldots, x_n\} \in K.$$

  The face and degeneracy maps are defined exactly as in the previous example.

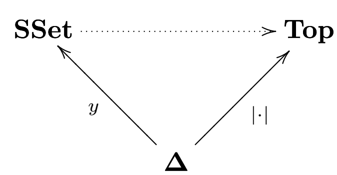
A.2.2. Geometric realization. Recall that that **geometric n-simplex** Δ_n is defined to be:

$$\Delta_n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\}.$$

I.e., Δ_n is (up to isomorphism) the convex hull in $\mathbb{R}^{n+1}$ of the standard unit vectors e_i. The construction of geometric n-simplices is functorial:

$$\mathbf{\Delta} \xrightarrow{|\cdot|} \mathbf{Top},$$

where $|[n]| := \Delta_n$ and $|\cdot|$ acts on arrows by reparameterization. There exists, by left-Kan extension, a canonical functor $\mathbf{SSet} \longrightarrow \mathbf{Top}$ extending $|\cdot|$ as indicated in the following diagram:



We write $|\cdot|$ also for the extension $\mathbf{SSet} \longrightarrow \mathbf{Top}$ and call this functor **geometric realization**. As always in this situation, $|\cdot|$ has a right-adjoint $S(-) : \mathbf{Top} \longrightarrow \mathbf{SSet}$

defined by

$$S(X)_n := \mathbf{Top}(\Delta_n, X).$$

$S(X)$ is called the **singular complex of** X. By definition of left-Kan extension, the geometric realization of a simplicial set is obtained by gluing together polygons.

A.2.3. The Nerve. By taking instead the Kan extension of the Yoneda embedding along the inclusion $\mathbf{\Delta} \longrightarrow \mathbf{Cat}$ we obtain the **fundamental category** functor $\tau_1 : \mathbf{SSet} \longrightarrow \mathbf{Cat}$ and its right-adjoint $N : \mathbf{Cat} \longrightarrow \mathbf{SSet}$ the **nerve functor**.

EXAMPLE A.2. The geometric realization of the nerve of a category $\mathcal{C}$ is sometimes called the *classifying space of* $\mathcal{C}$ and is written as $\mathbf{B}\mathcal{C}$ (in fact, some authors simply write $\mathbf{B}\mathcal{C}$ for both the nerve of $\mathcal{C}$ and the geometric realization of the nerve). This terminology and notation arise from the case where $\mathcal{C}$ is a group G. When G is an abelian group $\mathbf{B}G$ is in fact the **Eilenberg-Mac Lane space** $K(G,1)$ **of** G which has the property that

$$\pi_n\big(K(G,1)\big) = \begin{cases} G & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_n(X)$ denotes the n-th homotopy group of X.

A.3. Globular sets and strict ω-categories

Like simplicial sets, globular sets are one of the fundamental combinatorial structures employed in higher-dimensional category theory. Strict ω-categories arise as algebras for a monad on the category of globular sets and it is the aim of this section to recall the definitions of both globular sets and strict ω-categories. Our sources for much of this background material are [**77, 78, 83, 55**].

A.3.1. Globular sets. We denote by $\mathbf{\Gamma}$ the free category generated by the following graph

(64)

$$(0) \underset{t}{\overset{s}{\rightrightarrows}} (1) \underset{t}{\overset{s}{\rightrightarrows}} (2) \underset{t}{\overset{s}{\rightrightarrows}} \cdots (n) \underset{t}{\overset{s}{\rightrightarrows}} (n+1) \underset{t}{\overset{s}{\rightrightarrows}} \cdots$$

subject to the relations

$$s \circ s = t \circ s, \text{ and}$$
$$t \circ t = s \circ t.$$

A **globular set** is a presheaf $X : \mathbf{\Gamma}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ on $\mathbf{\Gamma}$. Explicitly, a globular set is given by a diagram

$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} \cdots X_n \underset{t}{\overset{s}{\leftleftarrows}} X_{n+1} \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

in **Set** such that the following **globular identities** are satisfied:

$$s \circ s = s \circ t, \text{ and}$$
$$t \circ t = t \circ s.$$

The maps s is referred to as **source maps** and t as **target maps**. We denote by **GSet** the category $[\mathbf{\Gamma}^{\mathrm{op}}, \mathbf{Set}]$ of globular sets. An arrow $f : X \longrightarrow Y$ in **GSet** is then a natural transformation. I.e., such an f consists of a morphism of graded sets which commutes with the source and target maps. By truncating the diagram (64) at a fixed n we obtain the n-truncated co-globular category $\mathbf{\Gamma}_n$ and, by taking presheaves, the category of n-trucated globular sets n-**GSet**. A 2-truncated globular set is precisely a (directed) graph. Thus, globular sets and n-globular sets are higher-dimensional graphs. By taking the left-Kan extension along the Yoneda embedding $\mathbf{\Gamma} \longrightarrow \mathbf{GSet}$ of the map $\mathbf{\Gamma} \longrightarrow \mathbf{Top}$ which sends n to the n-dimensional globe and which acts on the arrows of Γ by projecting the $(n+1)$-dimensional globe onto its hemispheres we obtain, as in the case of simplicial sets, a geometric realization functor $|\cdot| : \mathbf{GSet} \longrightarrow \mathbf{Top}$ which possesses a right-adjoint.

A.3.2. Strict ω-categories. Just as categories are graphs equipped with the additional structure given by identities and composition, so too n-categories are n-globular sets with additional structure and ω-categories are globular sets with additional structure. We here describe ω-categories explicitly and remark that n-categories are obtained by truncating the definition given below at n. Throughout we are dealing with *strict* ω-categories.

Let A be a globular set and define $A_n \times_{A_p} A_n$ to be the pullback

$$\begin{array}{ccc} A_n \times_{A_p} A_n & \longrightarrow & A_n \\ \downarrow & & \downarrow{\scriptstyle t^{(n-p)}} \\ A_n & \xrightarrow[s^{(n-p)}]{} & A_p \end{array}$$

where

$$t^k := \underbrace{t \circ t \circ \cdots \circ t}_{k\text{-times}}$$

for any $k \geq 0$. A **(strict) ω-category** consists of a globular set A together with maps $i : A_n \longrightarrow A_{n+1}$ and

$$A_n \times_{A_p} A_n \xrightarrow{*_p} A_n,$$

such that the following conditions are satisfied:

(Domain and Codomain Laws): For each $n \geq 0$,

$$s \circ i \; = \; 1_{A_n} \; = \; t \circ i.$$

And

$$\ell(g *_p f) = \begin{cases} \ell(g) *_p \ell(f) & \text{if } p < (n-1) \\ \\ = \begin{cases} s(f) & \text{if } \ell = s \\ t(g) & \text{if } \ell = t \end{cases} & \text{if } p = (n-1). \end{cases}$$

for $\ell = s, t$.

(Associativity Laws): Each operation $*_p$ is associative.

(Unit Laws): Given f in A_n,

$$i^{(n-p)}\big(t^{(n-p)}(f)\big) *_p f \; = \; f \; = \; f *_p i^{(n-p)}\big(s^{(n-p)}(f)\big).$$

(Interchange Laws): Given $q < p < n$ and f, g, h, k in A_n such that the composites $(y *_q x)$, $(k *_q h)$, $(h *_p f)$ and $(k *_p g)$ are defined,

$$(k *_p g) *_q (h *_p f) = (k *_q h) *_p (g *_q f),$$

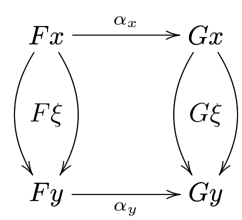
and

$$i(g) *_q i(f) = i(g *_q f).$$

The maps i are referred to as **identity maps** and the $*_p$ are compositions. Sometimes we refer to composition $*_0$ along 0-cells as **horizontal composition**. Note that if f and g are n-cells such that $(g *_p f)$ is defined, then $s^{(n-p)}(f)$, $s^{(n-p)}(g)$, $t^{(n-p)}(f)$ and $t^{(n-p)}(g)$ are all parallel. When no confusion will result we often omit mention of identity maps. Thus, e.g., if $f : x \longrightarrow y$ is a 1-cell and $\alpha : g \Longrightarrow h$ is a 2-cell with $s(g) = y$, we denote $\alpha *_0 i(f)$ by $\alpha *_0 f$.

A **(strict)** ω**-functor** $F : A \longrightarrow B$ between ω-categories is then simply a map of globular sets which preserves compositions and identities. We often refer to ω-functors simply as *functors* when it is understood that we are dealing with ω-categories. The category of small ω-categories and functors between them is denoted by ω-**Cat**. Just as **Cat** is monadic over 1-**GSet**, so too ω-**Cat** is monadic over **GSet** (cf. [**55**] for an explicit description of the monad). Indeed, ω-**Cat** is a bicomplete cartesian closed category. Henceforth we often denote ω-categories by $\mathcal{A}, \mathcal{B}, \ldots$. Clearly every ω-category is also an n-category, for $1 \leq n$ and similarly for ω-functors.

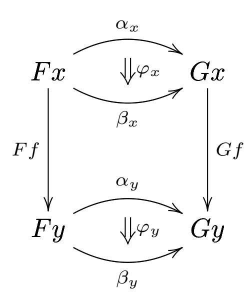
A.3.3. Higher-dimensional transformations. Given functors $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ between ω-categories, a **natural transformation** $\alpha : F \Longrightarrow G$ consists of an assignment of 1-cells $\alpha_x : Fx \longrightarrow Gx$ for objects x of $\mathcal{A}$ such that the following (somewhat schematic) diagram commutes:



for every k-cell ξ bounded by 0-cells x and y. I.e., if ξ is any k-cell, for $k \geq 1$, such that $s^k \xi = x$ and $t^k \xi = y$, then

$$\alpha_y *_0 F\xi = G\xi *_0 \alpha_x. \tag{65}$$

Passing up one dimension, suppose we are given functors F and G as above together with natural transformations α and β from F to G. Then, a **modification** or **2-transformation** $\varphi : \alpha \Longrightarrow \beta$ consists of an assignment of 2-cells $\varphi_x : \alpha_x \longrightarrow \beta_x$ of $\mathcal{B}$ parameterized by objects x of $\mathcal{A}$ subject to the condition that, for any arrow $f : x \longrightarrow y$ of $\mathcal{A}$, the following diagram commutes:



I.e.,

$$\varphi_y *_0 Ff = Gf *_0 \varphi_x \tag{66}$$

for $f : x \longrightarrow y$ an arrow of $\mathcal{A}$.

It is possible to generalize inductively to higher-dimensional transformations. In particular, assuming we have defined n-transformations, for $n \geq 2$, in such a way that the obvious boundary conditions are satisfied a $(n+1)$**-transformation** ψ from an n-transformation γ to a n-transformation δ consists of a family of n-cells $\psi_x : \gamma_x \Longrightarrow \delta_x$ in $\mathcal{B}$ parameterized by objects x of $\mathcal{A}$ such that, whenever $f : x \longrightarrow y$ is an arrow in $\mathcal{A}$, the naturality condition

$$\psi_y *_0 Ff = Gf *_0 \psi_x \tag{67}$$

is satisfied. With these definitions it is straightforward to verify that the following more general naturality conditions are also satisfied:

SCHOLIUM A.3. *If ξ is a k-cell of $\mathcal{A}$ bounded by 1-cells $f, g : x \rightrightarrows y$ and φ is a $(n+1)$-transformation bounded by functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, then*

$$\varphi_y *_0 F\xi = G\xi *_0 \varphi_x.$$

With ω-functors and these higher-dimensional transformations ω-**Cat** itself exhibits the combinatorial structure of an ω-category.

PROPOSITION A.4. *The category ω-**Cat** is itself a large ω-category with $(n+1)$-cells given by n-transformations.*

A.3.4. Dimension shift and hom-ω-categories. There exists a functor $(-)^+ : \omega\text{-}\mathbf{Cat} \longrightarrow \omega\text{-}\mathbf{Cat}$ called the **dimension shift functor** which shifts the dimension of an ω-category. Specifically, given an ω-category $\mathcal{A}$, $\mathcal{A}^+$ has as objects 1-cells of $\mathcal{A}$ and, in general, n-cells of $\mathcal{A}^+$ are $(n+1)$-cells of $\mathcal{A}$. I.e., $\mathcal{A}^+$ is the result of applying the obvious dimension shift functor on globular sets to the underlying globular set of $\mathcal{A}$. Trivially, $\mathcal{A}^+$ is an ω-category and $(-)^+$ is functorial. Similarly, given ω-category $\mathcal{A}$ and objects x and y of $\mathcal{A}$, the hom set $\mathcal{A}(x, y)$ can be made into an ω-category — which we sometimes denote by $\mathcal{A}_1(x, y)$ to emphasize the dimension — by defining 0-cells to be arrows $f : x \longrightarrow y$ and $(n+1)$-cells to be n-cells in the obvious way. Similarly, given parallel $(n+1)$-cells f, g of $\mathcal{A}$, there is an ω-category $\mathcal{A}_{n+2}(f, g)$ which has 0-cells $(n+2)$-cells $\alpha : f \Longrightarrow g$ and so forth.

Now, if $\mathcal{A}$ is an ω-category and f, g are n-cells with $n \geq 1$, then there exists a faithful inclusion functor

$$\mathcal{A}_{(n+1)}(f, g) \longrightarrow \big(\mathcal{A}_n(sf, tg)\big)^+$$

which sends a $(n+1)$-cell $\alpha : f \longrightarrow g$ to itself, and similarly for higher-dimensional cells.

APPENDIX B

Type theoretic background

B.1. The syntax of type theory

In this section we state the general rules of the basic form of type theory $\mathbb{T}_\omega$ which we consider. This is, in some sense, the simplest form of Martin-Löf type theory without natural numbers or universes. Of course, these additional features could well be considered, but, because we are primarily interested in analysing the higher-dimensional structure to which identity types give rise, this is the more basic theory. We refer the reader to [**60, 67, 80, 21, 32**] for additional details regarding the syntax of type theory.

In order to avoid superfluous repetition, some contexts are elided from the rules. Evident judgements are also omitted from the statements of rules when no confusion will result.

B.1.1. Forms of judgement. The formulation of type theory which we consider has six forms of judgement. The first two forms govern contexts. Namely,

$$\vdash \Gamma : \text{context}, \quad \text{and}$$
$$\vdash \Gamma = \Delta : \text{context},$$

which indicate that Γ is context and that Γ and Δ are *definitionally equal* as contexts. The next two forms of judgement,

$$\Gamma \vdash A : \text{type}, \quad \text{and}$$
$$\Gamma \vdash A = B : \text{type},$$

express that A is a type in context Γ and that A and B are definitionally equal types in context Γ. Finally,

$$\Gamma \vdash a : A, \quad \text{and}$$
$$\Gamma \vdash a = b : A$$

state that a is a term of type A in context Γ and that a and b are definitionally equal terms of type A in context Γ.

Although we formulate the theory with forms of judgement governing contexts, the theory can also be formulated without these rules and the system given here is conservative over the system without such forms of judgement.

B.1.2. Contexts. Contexts are finite lists of variable declarations $(x_1 : A_1, \ldots, x_n : A_n)$, for $n \geq 0$, such that $\mathrm{FV}(A_i) \subseteq \{x_1, \ldots, x_{i-1}\}$ when $1 \leq i \leq n$. Explicitly, the judgements governing context formation are axiomatized by certain rules which we now describe.

To begin with, the following rule expresses that the **empty context** () is a context:

$$\frac{}{\vdash\ () : \text{context}}\ () \text{ context}$$

The following rule allows for the extension of contexts:

(68) $$\frac{\Gamma\ \vdash\ A : \text{type}}{\vdash\ (\Gamma, x : A) : \text{context}}\ \text{Context extension}$$

where x is a fresh variable. A few remarks about this rule are in order. Here we adopt the convention of tacitly suppressing some of the hypotheses of rules when they are apparent. Thus, the true form of the rule (68) should also include as a hypothesis the judgement $\vdash\ \Gamma : \text{context}$. The next point to make about (68) is that when dealing with contexts we follow the convention of omitting unnecessary parentheses. Thus, for example, if A is a basic type, then (68) yields that $((), x : A)$ is a well-formed context and the expression $((), x : A)$ will be identified with the list $(x : A)$. Similarly, the context $\big(((), x_1 : A_1), x_2 : A_2\big)$ is identified with the list $(x_1 : A_1, x_2 : A_2)$, *et cetera.*

B.1.3. Structural rules.

$$\frac{}{() : \text{context}}\ \text{Empty context}$$

$$\frac{\Gamma\ \vdash\ A : \text{type}}{\vdash\ (\Gamma, x : A) : \text{context}}\ \text{Context extension}$$

where x is assumed to be a fresh variable in the context extension rule.

We now state the basic structural rules of type theory. First, we have the weakening rule,

$$\frac{B : \text{type}}{\Delta\ \vdash\ B : \text{type}}\ \text{Weakening}$$

The additional structural rules are as follows:

$$\frac{a : A \qquad x : A, \Delta\ \vdash\ B(x) : \text{type}}{\Delta[a/x]\ \vdash\ B[a/x] : \text{type}}\ \text{Type substitution}$$

$$\frac{a : A \qquad x : A, \Delta\ \vdash\ b(x) : B(x)}{\Delta[a/x]\ \vdash\ b[a/x] : B[a/x]}\ \text{Term substitution}$$

$$\frac{A : \text{type}}{x : A, \Delta\ \vdash\ x : A}\ \text{Variable declaration}$$

B.1.4. Rules governing definitional equality. The behavior of definitional equality of terms and types is specified by a number of rules. To begin with, the following rules stipulate that definitional equality constitutes an equivalence relation on both terms and types:

$$\frac{A : \text{type}}{A = A : \text{type}}\ \text{Type ref.} \qquad \frac{A = B : \text{type}}{B = A : \text{type}}\ \text{Type sym.}$$

$$\frac{A = B : \text{type} \qquad B = C : \text{type}}{A = C : \text{type}} \text{ Type trans.}$$

$$\frac{a : A}{a = a : A} \text{ Term ref.} \qquad \frac{a = b : A}{b = a : A} \text{ Term sym.}$$

$$\frac{a = b : A \qquad b = c : A}{a = c : A} \text{ Term trans.}$$

Additional rules ensure that definitional equality is well-behaved with respect to substitution and inhabitation.

$$\frac{a = a' : A \qquad x : A \vdash B(x) : \text{type}}{B[a/x] = B[a'/x] : \text{type}} \text{ Type congruence}$$

$$\frac{a = a' : A \qquad x : A \vdash b(x) : B(x)}{b[a/x] = b[a'/x] : B[a/x]} \text{ Term congruence}$$

$$\frac{A = B : \text{type} \qquad a : A}{a : B} \text{ Term conv.}$$

B.1.5. Formation rules. The formation rules for dependent sums and products are as follows:

$$\frac{x : A \vdash B(x) : \text{type}}{\Sigma_{x:A}.B(x) : \text{type}} \, \Sigma \text{ form.} \qquad \frac{x : A \vdash B(x) : \text{type}}{\Pi_{x:A}.B(x) : \text{type}} \, \Pi \text{ form.}$$

The (categorical) formation rule for identity types is given by

$$\frac{A : \text{type} \qquad a, b : A}{\vdash \mathrm{Id}_A(a, b) : \text{type}} \text{ Id form.}$$

B.1.6. Introduction and elimination rules for dependent products and sums. Introduction rules are as follows:

$$\frac{x : A \vdash f(x) : B(x)}{\lambda_{x:A}.f(x) : \Pi_{x:A}.B(x)} \, \Pi \text{ intro.}$$

$$\frac{x : A \vdash B(x) : \text{type} \qquad a : A \qquad b : B(a)}{\mathrm{pair}(a, b) : \Sigma_{x:A}.B(x)} \, \Sigma \text{ intro.}$$

Elimination rules are as follows:

$$\frac{f : \Pi_{x:A}.B(x) \qquad a : A}{\mathrm{app}(f, a) : B(a).} \, \Pi \text{ Elim.}$$

$$\frac{c : \Sigma_{x:A}.B(x) \qquad x : A, y : B(x) \vdash d(x, y) : C\big(\mathrm{pair}(x, y)\big)}{R^{\Sigma}_{A,B,C}(d, c) : C(c)} \, \Sigma \text{ weak elim.}$$

B.1.7. Introduction and elimination rules for identity types. The introduction, elimination and conversion rules are as follows:

$$\frac{a : A}{r_A(a) : \mathrm{Id}_A(a,a)}\ \text{Id intro.}$$

$$\frac{\begin{array}{c} x : A, y : A, z : \mathrm{Id}_A(x,y) \ \vdash \ B(x,y,z) : \text{type} \\ u : A \ \vdash \ b(u) : B\big(u,u,r_A(u)\big) \\ p : \mathrm{Id}_A(a,a') \end{array}}{J_{A,B}(b,a,a',p) : B(a,a',p)}\ \text{Id elim.}$$

B.1.8. Conversion rules.

$$\frac{\lambda_{x:A}.f(x) : \Pi_{x:A}.B(x) \qquad a : A}{\mathrm{app}\big(\lambda_{x:A}.f(x), a\big) \ = \ f(a) : B(a)}\ \Pi \text{ conversion}$$

$$\frac{a : A \qquad b : B(a) \qquad x : A, y : B(x) \ \vdash \ d(x,y) : C\big(\mathrm{pair}(x,y)\big)}{R^{\Sigma}_{A,B,C}\big(d, \mathrm{pair}(a,b)\big) \ = \ d(a,b) : C\big(\mathrm{pair}(a,b)\big)}\ \Sigma \text{ conversion}$$

$$\frac{a : A}{J_{A,B}\big(b,a,a,r_A(a)\big) \ = \ b(a) : B\big(a,a,r_A(a)\big)}\ \text{Id conversion}$$

B.1.9. Coherence rules for identity types. Finally, all of the data given by the formation, introduction and elimination rules is subject at the meta-level to cohrence ("Beck-Chevalley") conditions. We state only the coherence rules for identity types, as the corresponding rules for dependent products and sums follow the same pattern. We emphasize that these are meta-rules and should be understood as taking place in the logical framework.

$$\frac{x : C \ \vdash \ A(x) : \text{type} \quad x : C \ \vdash \ a(x), b(x) : A(x) \qquad \vdash \ c : C}{\vdash \ \big(\mathrm{Id}_{A(x)}(a(x), b(x))\big)[c/x] \ = \ \mathrm{Id}_{A[c/x])}(a[c/x], b[c/x])}\ \text{Id coherence}$$

$$\frac{x : C \ \vdash \ A(x) : \text{type} \quad x : C \ \vdash \ a(x) : A(x) \qquad \vdash \ c : C}{\vdash \ \big(r_{A(x)}(a(x))\big)[c/x] \ = \ r_{A[c/x]}(a[c/x]) : \mathrm{Id}_{A[c/x]}(a[c/x], a[c/x])}\ r \text{ coherence}$$

$$\frac{\begin{array}{c} x : C, y : A(x), z : A(x), v : \mathrm{Id}_{A(x)}(y,z) \ \vdash \ B(x,y,z,v) : \text{type} \\ x : C, u : A(x) \ \vdash \ b(x,u) : B\big(x,u,u,r_{A(x)}(u)\big) \\ x : C \ \vdash \ p(x) : \mathrm{Id}_{A(x)}(a(x), a'(x)) \qquad \vdash \ c : C \end{array}}{\Big(J_{A(x),B}\big(b(x), a(x), a'(x), p(x)\big)\Big)[c/x] \ = \ J_{A(c),B}\big(b(c), a(c), a'(c), p(c)\big)}\ J \text{ coherence}$$

B.2. Additional and derived rules

This section contains assorted additional rules which are not assumed as part of the basic theory $\mathbb{T}_\omega$ (although they may be derivable).

B.2.1. Rules for products and exponentials. For convenience we include also the rules governing products and exponentials.

$$\frac{A : \text{type} \qquad B : \text{type}}{A \times B : \text{type}} \times \text{ form.} \qquad \frac{A : \text{type} \qquad B : \text{type}}{B^A : \text{type}} \text{ Exp. form.}$$

$$\frac{a : A \qquad b : B}{\text{pair}(a,b) : A \times B} \times \text{ intro.}$$

$$\frac{d : A \times B}{\pi_1(d) : A} \times \text{ elim. 1} \qquad \frac{d : A \times B}{\pi_2(d) : B} \times \text{ elim. 2}$$

$$\frac{x : A \ \vdash \ f(x) : B}{\lambda_{x:A}.f(x) : B^A} \text{ Exp. intro.}$$

$$\frac{f : B^A \qquad a : A}{\text{app}(f,a) : B} \text{ Exp. elim.}$$

B.2.2. Strong elimination rules for dependent sums.

$$\frac{c : \Sigma_{x:A}.B(x)}{\pi_1(c) : A} \Sigma \text{ strong elim. 1} \qquad \frac{c : \Sigma_{x:A}.B(x)}{\pi_2(c) : B[\pi_1(c)/x]} \Sigma \text{ strong elim. 2}$$

B.2.3. Hypothetical rules for identity types.

$$\frac{A : \text{type}}{x : A, y : A \ \vdash \ \text{Id}_A(x,y) : \text{type}} \text{ Id form. (hypothetical)}$$

$$\frac{A : \text{type}}{x : A \ \vdash \ r_A(x) : \text{Id}_A(x,x)} \text{ Id intro. (hypothetical)}$$

$$\frac{\begin{array}{c} x : A, y : A, z : \text{Id}_A(x,y) \ \vdash \ B(x,y,z) : \text{type} \\ u : A \ \vdash \ b(u) : B[u/x][u/y][r_A(u)/z] \end{array}}{x : A, y : A, z : \text{Id}_A(x,y) \ \vdash \ J_{A,B}(b,x,y,z) : B(x,y,z)} \text{ Id elim. (hypothetical)}$$

$$\frac{x : A, y : A, z : \text{Id}_A(x,y) \ \vdash \ B(x,y,z) : \text{type}}{x : A \ \vdash \ J_{A,B}(b,x,x,r_A(x)) = b(x) : B\big(x,x,r_A(x)\big)} \text{ Id conversion (hypothetical)}$$

B.2.4. Derived rules for identity types.

$$\frac{x : A \ \vdash \ B(x) : \text{type} \qquad p : \text{Id}_A(a_1,a_2) \qquad q : B(a_1)}{\text{subst}_{A,B}(a_1,a_2,p,q) : B(a_2).} \text{ Id Sub.}$$

$$\frac{p : \text{Id}_A(a_1,a_2)}{s_A(p) : \text{Id}_A(a_2,a_1)} \text{ Id Sym.} \qquad \frac{p : \text{Id}_A(a_1,a_2) \qquad q : \text{Id}_A(a_2,a_3)}{t_A(p,q) : \text{Id}_A(a_1,a_3)} \text{ Id Trans.}$$

$$\frac{}{\text{subst}_{A,B}(a,a,r_A(a),q) \ = \ q} \text{ Id Sub. Conv.}$$

$$\frac{}{s_A(r_A(a)) \;=\; r_A(a)}\ \text{Id Sym. Conv.} \qquad \frac{}{t_A(p, r_A(b)) \;=\; p}\ \text{Id Trans. Conv.}$$

B.2.5. Truncation principles for identity types.

$$\frac{\vdash\ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n) \qquad \vdash\ p : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, b_{n+1})}{\vdash\ a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}\ \text{TR}_n$$

$$\frac{\vdash\ a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}{\vdash\ a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}\ \text{UIP}_n$$

$$\frac{\vdash\ a_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n) \qquad \vdash\ p : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, a_{n+1})}{\vdash\ p \;=\; r_{\underline{A}^n(a_1, b_1; \cdots; a_n, b_n)}(a_{n+1}) : \underline{A}^{n+1}(a_1, b_1; \cdots; a_{n+1}, a_{n+1})}\ \text{OUP}_n$$

B.2.6. Streicher's eliminator K.

$$\frac{\begin{array}{c} x : A, y : \mathrm{Id}_A(x, x)\ \vdash\ C(x, y) : \text{type} \\ x : A\ \vdash\ d(x) : C(x, r_A(x)) \\ \vdash\ p : \mathrm{Id}_A(a, a) \end{array}}{\vdash\ K_{A,D}([x : A]d(x), a, p) : C(a, p)}\ K \text{ elim.}$$

$$\frac{\vdash\ a : A}{\vdash\ K_{A,D}([x : A]d(x), a, r_A(a)) \;=\; d(a) : C(a, r_A(a))}\ K \text{ conv.}$$

In the presence of the reflection rule the K rules are equivalent to the usual J rules. For more on K see (cf. [**80**]).

B.3. Interpreting type theory in comprehension categories

Comprehension categories are significant as a semantics for type theory because they capture much of the "fibrational" information regarding type theory in a reasonably concise manner. They also provide a uniform setting for describing several of the coherence issues which arise in the categorical treatment of type theory. We refer the reader to [**38, 79, 69, 21, 34**] for more on the semantics of type theory.

Definition B.1. A **comprehension category** consists of the following data:

- A finitely complete category $\mathcal{C}$.
- A Grothendieck fibration $\mathbf{P}(-) : \mathcal{P} \longrightarrow \mathcal{C}$.
- A fibred functor $\chi : \mathcal{P} \longrightarrow \mathcal{C}^{\rightarrow}$ over $\mathcal{C}$, where the $\mathcal{C}^{\rightarrow} \longrightarrow \mathcal{C}$ is the codomain fibration.

A comprehension category is said to be **split** if $\mathbf{P}(-)$ is a split fibration.

Note that, unless it is split, a comprehension category only interprets substitution correctly up to (coherent) isomorphism (cf. [**18, 33**]). As such, unless otherwise stated, we will assume the comprehension categories with which we deal in this Appendix are split.

B.3.1. The interpretation of contexts and type judgements. When interpreting type theory using a fixed comprehension category, the category $\mathcal{C}$ is to be thought of as the category of contexts and an elements of the fibre $\mathcal{P}(\Gamma)$ over a "context" Γ as the types in context Γ. We mention that some of the notation employed below is explained in Remark 2.15 of Chapter 2.

The interpretation of contexts and type judgements in such a comprehension category is summarized as follows:

- The empty context () is interpreted as the terminal object of $\mathcal{C}$:
$$[\![()]\!] := 1.$$
- If Γ is an arbitrary context which has already received an interpretation $[\![\Gamma]\!]$, then a judgement of the form $\Gamma \vdash A : \text{type}$ is interpreted as an object of the fibre $\mathcal{P}([\![\Gamma]\!])$:
$$[\![\Gamma \vdash A : \text{type}]\!] := \text{an object of } \mathcal{P}([\![\Gamma]\!]).$$
- Given the foregoing situation, the extended context $(\Gamma, x : A)$ is interpreted as the domain of the map obtained by applying $\chi : \mathcal{P} \longrightarrow \mathcal{C}^{\to}$ to $[\![\Gamma \vdash A : \text{type}]\!]$. I.e.,
$$[\![\Gamma, x : A]\!] := [\![\Gamma]\!]_{[\![\Gamma \vdash A : \text{type}]\!]}.$$

Type identity judgements $\Gamma \vdash A = B : \text{type}$ are then interpreted as actual equalities in $\mathcal{P}([\![\Gamma]\!])$.

B.3.2. The interpretation of terms. Given interpretations of a context Γ and a judgement $\Gamma \vdash A : \text{type}$ as an object $[\![\Gamma]\!]$ of $\mathcal{C}$ and an element
$$\alpha = [\![\Gamma \vdash A : \text{type}]\!] \in \mathcal{P}([\![\Gamma]\!]),$$
respectively, a judgement of the form $\Gamma \vdash a : A$ is interpreted as a section $[\![\Gamma \vdash a : A]\!]$ of π_α as follows:
$$\begin{array}{ccc} [\![\Gamma]\!] & \xrightarrow{[\![\Gamma \vdash a : A]\!]} & [\![\Gamma]\!]_\alpha \\ & {}_{1_\Gamma}\searrow \quad \swarrow_{\pi_\alpha} & \\ & [\![\Gamma]\!], & \end{array}$$
in $\mathcal{C}$. Finally, definitional equality of terms $\Gamma \vdash a = b : A$ is interpreted as actual equality of arrows in $\mathcal{C}$.

B.3.3. The interpretation of substitution into a type. Assume given interpretations of the judgements $\Gamma, x : A \vdash B(x) : \text{type}$ and $\Gamma \vdash a : A$ as described above. Thus, there are objects α of $\mathcal{P}([\![\Gamma]\!])$ and β of $\mathcal{P}([\![\Gamma]\!]_\alpha)$ interpreting
$$\Gamma \vdash A : \text{type} \quad \text{and} \quad \Gamma, x : A \vdash B(x) : \text{type},$$
respectively, together with a section $[\![\Gamma \vdash a : A]\!]$ of π_α. We henceforth will simply abbreviate this section by a and similarly for other sections. Thus, because β is in the fibre of $\mathbf{P}(-)$ over $[\![\Gamma]\!]_\alpha$, there is cartesian lift $a_\beta : (\beta \cdot a) \longrightarrow \beta$ in $\mathcal{P}$ over the map
$$[\![\Gamma]\!] \xrightarrow{\;a\;} [\![\Gamma]\!]_\alpha.$$
We then define
$$[\![\Gamma \vdash B[a/x] : \text{type}]\!] := (\beta \cdot a).$$

Note that substitution is not functorial unless $\mathbf{P}(-)$ is a split fibration.

B.3.4. The interpretation of substitution into a term. Given judgements $\Gamma \vdash a : A$ and $\Gamma, x : A \vdash B(x) : \text{type}$, the judgement $\Gamma \vdash B(a) : \text{type}$ is interpreted as

$$[\![\Gamma \vdash B(a) : \text{type}]\!] := \big(\beta \cdot [\![\Gamma \vdash a : A]\!]\big). \tag{69}$$

where

$$\begin{aligned} \alpha &= [\![\Gamma \vdash A : \text{type}]\!] \in \mathcal{P}([\![\Gamma]\!]), \quad \text{and} \\ \beta &= [\![\Gamma, x : A \vdash B(x) : \text{type}]\!] \in \mathcal{P}([\![\Gamma]\!]_\alpha). \end{aligned}$$

Similarly, given $\Gamma, x : A \vdash b(x) : B(x)$, the judgement $\Gamma \vdash b(a) : B(a)$ is interpreted as $b[a]$ where we write a as an abbreviation for $[\![\Gamma \vdash a : A]\!]$ and b as an abbreviation for $[\![\Gamma, x : A \vdash b(x) : B(x)]\!]$.

B.3.5. The interpretation of weakening. Suppose given judgements $\Gamma \vdash A : \text{type}$ and $\Gamma \vdash B : \text{type}$ together with their interpretations $\alpha, \beta \in \mathcal{P}([\![\Gamma]\!])$. In this situation, the judgement $\Gamma, x : A \vdash B : \text{type}$ is interpreted as

$$[\![\Gamma, x : A \vdash B : \text{type}]\!] := \beta \cdot \pi_\alpha.$$

where $\pi_\alpha : [\![\Gamma]\!]_\alpha \longrightarrow [\![\Gamma]\!]$. Similarly, given a context Γ and a judgement in the empty context $\vdash A : \text{type}$ the we interpret

$$[\![\Gamma \vdash A : \text{type}]\!] := \alpha \cdot ! \in \mathcal{P}([\![\Gamma]\!]),$$

where $\alpha \in \mathcal{P}(1)$ is the interpretation of $\vdash A : \text{type}$ and ! is the canonical map $[\![\Gamma]\!] \longrightarrow 1$.

Given $\Gamma \vdash A : \text{type}$, the judgement $\Gamma, x : A \vdash A : \text{type}$ is interpreted, as above, as $\alpha \cdot \pi_\alpha \in \mathcal{P}([\![\Gamma]\!]_\alpha)$. The corresponding context $(\Gamma, x : A, y : A)$, for y a fresh variable, is then

$$[\![\Gamma, x : A, y : A]\!] := [\![\Gamma]\!]^+_\alpha.$$

It is convenient to introduce some additional notation associated with the operation of weakening. Given objects Δ and Γ of $\mathcal{C}$ together with an element $\alpha \in \mathcal{P}(\Gamma)$ and a map $\sigma : \Delta \longrightarrow \Gamma$, the induced map

$$\big(\Delta_{\alpha\cdot\sigma}\big)_{(\alpha\cdot\pi_\alpha)\cdot\sigma_\alpha} \xrightarrow{\ (\sigma_\alpha)_{\alpha\cdot\pi_\alpha}\ } \Gamma^+_\alpha$$

is abbreviated by $\sigma^\dagger$ as indicated in the following (two pullback) diagram:

$$\begin{array}{ccc}
\big(\Delta_{\alpha\cdot\sigma}\big)_{(\alpha\cdot\pi_\alpha)\cdot\sigma_\alpha} & \xrightarrow{\ \sigma^\dagger\ } & \Gamma^+_\alpha \\
{\scriptstyle \pi_{(\alpha\cdot\pi_\alpha)\cdot\sigma_\alpha}}\downarrow & & \downarrow{\scriptstyle \pi^+_\alpha} \\
\Delta_{\alpha\cdot\sigma} & \xrightarrow{\ \sigma_\alpha\ } & \Gamma_\alpha \\
{\scriptstyle \pi_{\alpha\cdot\sigma}}\downarrow & & \downarrow{\scriptstyle \pi_\alpha} \\
\Delta & \xrightarrow[\ \sigma\ ]{} & \Gamma.
\end{array}$$

B.4. The initial model of $\mathbb{T}_\omega$

We now recall the details of the initial model of $\mathbb{T}_\omega$ obtained using the category of contexts. We also prove a basic fact relating this model with the weak factorization system from [**23**]. We assume that the reader is familiar with the basic definitions and properties of context morphisms. We denote by $\mathcal{C}(\mathbb{T}_\omega)$ the **category of contexts of** $\mathbb{T}_\omega$ which has as objects contexts and as arrows context morphisms.

DEFINITION B.2. A context morphism $\Gamma \longrightarrow \Delta$ is a **dependent projection** if it is either an identity or of the form

$$(x_1 : A_1, \ldots, x_{n+1} : A_{n+1}) \xrightarrow{(x_1,\ldots,x_n)} (x_1 : A_1, \ldots, x_n : A_n)$$

for $0 \leq n$.

The set of dependent projections is denoted by $\mathfrak{D}$. The intial comprehension category modelling $\mathbb{T}_\omega$ is obtained as the Grothendieck fibration $\mathcal{C}(\mathbb{T}_\omega)_{\mathfrak{D}} \longrightarrow \mathcal{C}(\mathbb{T}_\omega)$ which is obtained as the restriction of the codomain fibration to the full subcategory of the arrow category having as objects dependent projections. That this is the initial model is well-known. (The proof of this for extensional type theory can be found in [**38**] and is easily modified to give a proof for $\mathbb{T}_\omega$. Alternatively, an equivalent result, stated in terms of categories with families, can be found in [**32**]).

DEFINITION B.3. Let a context $\Gamma = (x_0 : A_0, \ldots, x_n : A_n)$ be given. A context morphism $\Gamma \longrightarrow \Delta$ in $\mathcal{C}(\mathbb{T}_\omega)$ is said to be **generalized dependent projection** if there exists a natural number $0 \leq m \leq n$ together with an inclusion $[m] \longrightarrow [n]$ of an initial segment such that

$$\Delta = (x_{\alpha(0)} : A_{\alpha(0)}, \ldots, x_{\alpha(m)} : A_{\alpha(m)}), \quad \text{and}$$
$$f = (x_{\alpha(0)}, \ldots, x_{\alpha(m)}).$$

The collection of generalized dependent projections is denoted by $\mathfrak{P}$.

It is straightforward to verify that $\mathfrak{P}$ is the closure of the set $\mathfrak{D}$ of (ordinary) dependent projections under composition. Note that we have the following result:

SCHOLIUM B.4. *In a category $\mathcal{C}$, if $\mathfrak{A}$ is a collection of arrows of $\mathcal{C}$ and $\mathfrak{A}'$ is the closure of $\mathfrak{A}$ under composition, then*

$$^{\pitchfork}\mathfrak{A} = ^{\pitchfork}(\mathfrak{A}').$$

PROOF. Since $\mathfrak{A}$ is contained in $\mathfrak{A}'$ it follows that $^{\pitchfork}(\mathfrak{A}')$ is contained in $^{\pitchfork}\mathfrak{A}$. The converse follows a basic lifting argument using the fact that, as the closure of $\mathfrak{A}$ under composition, maps in $\mathfrak{A}'$ can be decomposed into composites of maps in $\mathfrak{A}$. □

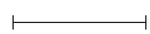
APPENDIX C

A schematic picture of the definition of strict intervals

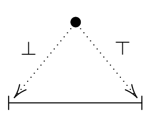
We will now give a brief presentation of the "intended picture" of cocategory objects which should help the reader understand the intuition a little better (this picture is in some sense just a way of illustrating the cocategory object in **Gpd** discussed below). To begin with, we will regard C_0 as a single point:

$$\bullet,$$

and C_1 will be regarded as the "unit interval":

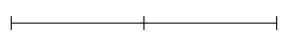


The maps $\bot$ and $\top$ then are simply points of the interval:

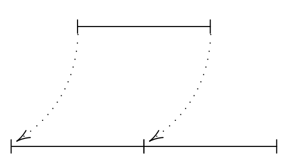


where $\bot$ is identified with the "bottom" end of the interval and $\top$ is identified with the "top" end.

C_2 is then, by definition, the result of gluing the interval to itself by identifying the top and bottom:

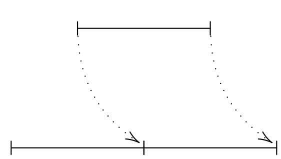


and the maps $\downarrow, \uparrow \colon C_1 \rightrightarrows C_2$ have the actions illustrated as follows:



$$\begin{array}{c} C_1 \\ \big\downarrow {\scriptstyle \downarrow} \\ C_2 \end{array}$$

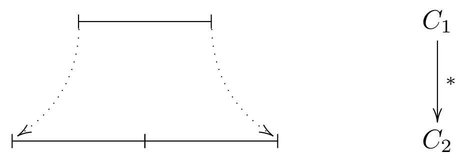
and:



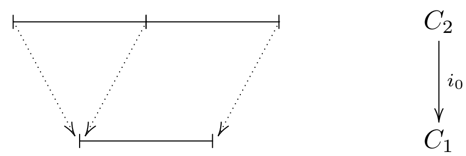
$$\begin{array}{c} C_1 \\ \big\downarrow {\scriptstyle \uparrow} \\ C_2 \end{array}$$

The point $\downarrow \circ \top = \uparrow \circ \bot$ may (in some sense) be identified with the midpoint of C_2.

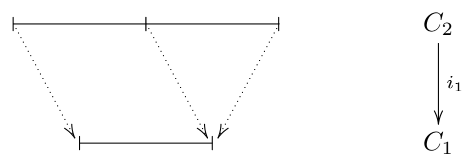
The cocomposition $* : C_1 \longrightarrow C_2$ is then the "magnification" operation:



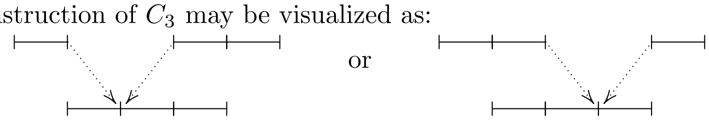
The maps $i_0, i_1 : C_2 \rightrightarrows C_1$ mentioned in the fourth axiom for cocategory objects have the action, in this case, of collapsing the initial segment to $\bot$ and collapsing the final segment to $\top$, respectively. This is illustrated as follows:



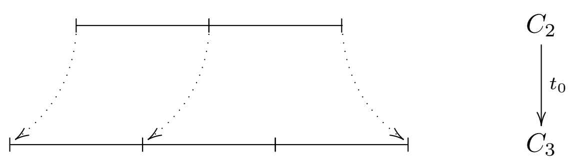
and



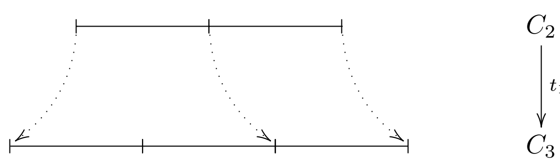
The construction of C_3 may be visualized as:



The maps $t_0, t_1 : C_2 \rightrightarrows C_3$ are then given schematically by:



and:



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