A Survey of Computations of Homotopy Groups of Spheres and Cobordisms

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Chapter 1

Introduction

Among all the interesting spaces in topology, the spheres are no doubt beautiful objects and of most consideration since antiquity. Any nontrivial observations of them are of course very important. From the categorical point of view, we should not just focus on the objects themselves, but the morphisms between them as well. For this purpose, in algebraic topology, we do want to classify the set of continuous morphisms between spheres under the equivalent relation named homotopy, which describes a continuous deformation between two continuous maps. Let S^n be the n-sphere and $\pi_k(X)$ be the set of homotopic equivalent based maps from S^k to X. For the reason that S^k is a double suspension when $k \geq 2$, the set is actually an abelian group. A natural question is, what are these abelian groups?

The importance of this question is not just aesthetical, but also from the fact that it is connected to many other areas in mathematics, such as geometric topology, algebra and algebraic geometry. For example, the groups of differential structures on spheres is somehow determined by the stable homotopy groups of spheres (By the Freudenthal suspension theorem, the group $\pi_n(S^{n+k})$ is independent of n when n is larger than $k+1$, and is called the k^{th} stable homotopy group of spheres, denoted by $\pi_k^{st}(S^0)$). Another impressive example is the theory of topological modular forms, which relates certain parts of $\pi_*^{st}(S^0)$ to the moduli stack of elliptic curves. A quick look of the known table of these groups shows that the pattern is not simple at all. Many great works has been done for the last 90 years, but there are much more we want to know and want to understand.

In Chapter 2, the theory of framed cobordism was discussed to calculate stable homotopy groups of spheres. From Section 2.1 to 2.3, we introduce basic notions of framed cobordism and discuss its relationship with stable homotopy groups of spheres. In Section 2.4 and 2.5, following Pontrjagin's method of surgery, we sketch the proof of $\pi_2^{st}(S^0) \cong \mathbb{Z}/2$. In Section 2.6, following Atiyah's methods and citing some facts of $K3$-surfaces, we work out the details of the proof of $\pi_3^{st}(S^0) \cong \mathbb{Z}/24$.

In Chapter 3, we talk about Serre's method in Section 3.1, where Serre fibration and Serre spectral sequences are discussed. In Section 3.2, we introduce the EHP-sequences and Toda brackets. Using the EHP-sequences, we calculate the 2-components of $\pi_{n+k}(S^n)$ for $k \leq 3$. A table for 2-components of $\pi_{n+k}(S^n)$ for $k \leq 24$ can be found in in appendix A, where the data come from [Tod62, MT63, Mim65, MMO74]. We also follow Toda's proof for the non-existance of Hopf invariant one problem in dimension 16 in Section 3.2.5.

In Chapter 4, we introduce the stable homotopy category in Section 4.1, which is the right category to discuss stable groups. In Section 4.2 and 4.3, using the notion of spectra, we introduce Adams spectral sequences and some basic properties of E_2-term for sphere spectrum at prime 2. To calculate the E_2-term of Adams spectral sequence, May spectral sequence was introduced in Section 4.4. Following Tangora's method, we actually work out detailed calculations to compute the E_∞-term of May spectral sequence up to stem 29 at prime 2 with illustrations. Citing some differentials of Adams spectral sequence in Section 4.5, we have the table of the first 29 stable homotopy groups of spheres at prime 2.

In Chapter 5, we talk about different kinds of cobordism theories characterized by G-structures. We introduce basic notions in Section 5.1 and convert the problem of computing cobordism ring to the one of computing the homotopy ring of Thom spectra by Pontrjagin-Thom construction in Section 5.2. In Section 5.3, we do the calculations of the unoriented cobordism ring, the oriented cobordism ring and the first three groups of spin cobordism. In Section 5.4, we compute the complex cobordism ring and introduce the Adams-Novikov spectral sequence, which is based on complex cobordism MU and is efficient in computing stable homotopy groups of spheres at large primes.

In Chapter 6, we give a short introduction to chromatic homotopy theory. In Section 6.1, we discuss basic notions of formal group laws in that Quillen discovers that complex cobordism in fact gives the universal formal group law. We talk about the Brown-Peterson spectrum BP in Section 6.2, which is a simpler version of complex cobordism when localized at a prime p. Similarly as the Hopf algebroid (MU_*, MU_*MU), (BP_*, BP_*BP) represents the moduli stack of formal groups localized at a prime p. We therefore introduce Morava K-theories as the "points" of this geometric object in Section 6.3. Using the filtration of BP given by the height of formal groups, we introduce the chromatic spectral sequence in Section 6.4. In Section 6.5, we shortly describe the geometric explanation of the chromatic filtration using Hopkins' Nilpotence and periodicity theorems. In Section 6.6, we present the result of calculation in chromatic level one and discuss its relation to J-homomorphism and localization with respect to K-theory. In Section 6.7, we present the result of calculation of homotopy groups of $K(2)$-local sphere and the resolution concerning spectra, which are homotopy fixed points of Lubin-Tate spectrum under actions of certain finite groups. We also discuss its relationship with the moduli stack of elliptic curves.

We do not claim originality of any results in this survey and we have to mention that the sources we quote here is where these results we learned from.

Table 1.1: Symbols

$\mathbb{Z}$	the ring of integers
$\mathbb{F}_p$	the field with p elements
$\mathbb{Z}_{(p)}$	the integers localised at p
$\mathbb{H}$	the quaternions
S^n	the n-dimensional sphere
$\mathbb{R}^n$	the n-dimensional euclidian space
$\mathbb{S}$	the sphere spectrum
$H\mathbb{F}_2$	the Eilenberg-MacLane spectrum of $\mathbb{F}_2$
$\pi_*(X)$	the homotopy groups of X
Σ	the suspension
$A(2)^*$	the Steenrod algebra with coefficient $\mathbb{F}_2$
$A(2)_*$	the dual Steenrod algebra with coefficient $\mathbb{F}_2$
pt	one point
MO	the spectrum of unoriented cobordism
MSO	the spectrum of oriented cobordism
MU	the spectrum of complex cobordism
MU_*	the coefficient ring of complex cobordism, $\mathbb{Z}[x_1, x_2, ...]$
MU_*MU	the MU-homology of MU, $MU_*[b_1, b_2, ...]$
BP	the Brown–Peterson spectrum
BP_*	the coefficient ring of BP, $\mathbb{Z}_{(p)}[v_1, v_2, ...]$
BP_*BP	the BP-homology of BP, $BP_*[t_1, t_2, ...]$
$K(n)$	the spectrum of the n^{th} Morava K-theory
$K(n)_*$	the coefficient ring of $K(n)$, $\mathbb{F}_p[v_n, v_n^{-1}]$

Chapter 2

The Method of Framed Cobordism

In this chapter, we take the north pole as the base point of the S^n, and identify the complement of it with $\mathbb{R}^n$.

2.1 Mapping degree and $\pi_n(S^n)$

It is easy to show $\pi_i(S^n) = 0$ for $i < n$. So the first nontrivial homotopy groups of spheres are $\pi_n(S^n)$. The computation of $\pi_n(S^n)$ is standard. Here we give an elementary argument.

To see the group is nontrivial, we define the notion of mapping degree. Assume $f : M \to N$ is a smooth map between n-dimensional compact oriented manifolds M and N. Then for a generic point p of N, $f^{-1}(p)$ is a finite set and the Jacobian of f at each of the points $x \in f^{-1}(p)$ does not vanish for some (hence any) oriented coordinate chart, and $ind(x)$ is defined to be the sign of the Jacobian.

Definition 2.1.1. *The mapping degree of f, $deg(f)$, is $\Sigma_{x\in f^{-1}(p)} ind(x)$.*

It can be shown that $deg(f)$ is independent of the choice of p, and it is homotopy invariant. So we can define the degree of any continuous map to be that of any smooth map homotopy to it.

In the case of spheres, the degree gives a homomorphism from $\pi_n(S^n)$ to $\mathbb{Z}$, since by definition of the sum of two maps, the preimage of a generic point for the sum correspond bijectively to the disjoint union of the preimages of each map. Since the mapping degree of the identity map is 1, this is an epimorphism.

In fact we have the following:

Theorem 2.1.2. *The mapping degree gives an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.*

Proof. It suffices to show it is a monomorphism, that is, if $f : S^n \to S^n$ has mapping degree 0, then f can be extended to a map $\tilde{f} : D^{n+1} \to S^n$, where D^{n+1} is the $(n+1)$-dimensional unit disk.

We can suppose f is smooth. Let p be a generic point of S^n, and $f^{-1}(p) = \{x_1, \dots, x_k, y_1, \dots, y_k\}$, with $ind(x_i) = 1$ and $ind(y_i) = -1$. By suitable choice of sub-indices, we can find smooth paths $l_1, \dots, l_k$ in D^n, which are embedded submanifolds not intersecting each other and l_i intersects the boundary transversely on x_i, y_i.

Further more, we can find local charts $\phi, \psi_1, \dots, \psi_k, \varsigma_1, \dots, \varsigma_k : D^n \to S^n$ around $p, x_1, \dots, x_k, y_1, \dots, y_k$, respectively, and tubular neighborhoods $\tau_1, \dots, \tau_k : D^n \times [0,1] \to D^{n+1}$ of $l_1, \dots, l_k$, such that:

1. $\phi, \psi_1, \dots, \psi_k$ preserve orientation and $\varsigma_1, \dots, \varsigma_k$ reverse orientation .
2. The images of $\psi_1, \dots, \psi_k, \varsigma_1, \dots, \varsigma_k$ do not intersect each other.
3. the images of $\tau_1, \dots, \tau_k$ do not intersect each other.
4. $f \circ \psi_i = \phi$, $f \circ \varsigma_i = \phi$.
5. $\tau_i|_{D^n \times \{0\}} = \psi_i$, $\tau_i|_{D^n \times \{1\}} = \varsigma_i$.

This can be achieved, for example, by giving Riemannian metrics to S^n and D^n which is euclidian near $p, x_1, \dots, x_k, y_1, \dots, y_k$, and take the exponential map.

Now we can extend f to $Im(\tau_i)$ by the formula $\tilde{f}|_{Im(\tau_i)} = pr_1 \circ \tau_i^{-1}$ where $pr_1 : D^n \times [0,1] \to D^n$ is the projection. Let M be the complement of the interior of $\cup Im(\tau_i)$, then $\tilde{f}$ is already defined on the boundary of M, and take values away from p. Since the complement of p in S^n is homeomorphic to the euclidian space, $\tilde{f}$ can be extended to the entire M, and we find the desired map. $\square$

2.2 Framed cobordism

To generalize the method in the preceding section to calculate $\pi_{n+k}(S^n)$ for $k > 0$, one has to investigate the preimage of maps from the $(n+k)$-dimensional sphere to the n-dimensional sphere, which is in general a manifold of dimension k. So the problem is to give manifolds some kind of indices generalizing those in the preceding section for points. This is done by Pontrjagin, who introduced the notion of framed cobordism to classify the preimage of maps between spheres.

Definition 2.2.1. *A framed submanifold of $\mathbb{R}^{n+k}$ of dimension k is an k-dimensional submanifold of $\mathbb{R}^{n+k}$ together with a framing (i.e. n linearly independent sections) on its normal bundle.*

Let $f : S^{n+k} \to S^n$ be a based smooth map, and $p \in S^n$ be a generic point. Then the preimage of p is a k-dimensional submanifold with normal bundle $f^*T_pS^n$, and any frame of T_pS^n pulls back to a framing of it. By deleting the base point, we get a framed submanifold of $\mathbb{R}^{n+k}$, depending on the map f, the point p, and a frame of T_pS^n.

This submanifold is of course not homotopy independent. But it is homotopy independent up to an equivalence relation called cobordism.

Definition 2.2.2. *A framed cobordism between M and N, two framed submanifolds of $\mathbb{R}^{n+k}$ of dimension k, is a submanifold L of $\mathbb{R}^{n+k} \times [0,1]$, whose boundary is contained in $\mathbb{R}^{n+k} \times \{0,1\}$, (also required to intersect transversely with $\mathbb{R}^{n+k} \times \{0,1\}$), together with a framing on its normal bundle, such that $L \cap \mathbb{R}^{n+k} \times \{0\} = M$ and $L \cap \mathbb{R}^{n+k} \times \{1\} = N$ as framed submanifolds.*

In such a case, we say M is framed cobordant to N.

Conversely, let M be a framed submanifold of $\mathbb{R}^{n+k}$ of dimension k. Then we can find a tubular neighborhood $\sigma : M \times D^n \to \mathbb{R}^{n+k}$ of M such that the framing pulls back to the one induced from some fixed frame of T_0D^n. Define a map $M \times D^n \to S^n$ by composing the projection to D^n with the map $D^n \to S^n$ which collapses the boundary of D^n to the base point. We can extend this map to S^{n+k} by sending the other points to the base point. Thus we construct from a framed submanifold a map between spheres, which is easily seen to be a right inverse to the preceding construction.

Moreover, starting from a framed cobordism, a similar construction gives a homotopy between two maps. We conclude:

Theorem 2.2.3. *The homotopy class of continuous maps from S^{n+k} to S^n, i.e. $\pi_{n+k}(S^n)$, correspond bijectively to the cobordism class of framed submanifolds of $\mathbb{R}^{n+k}$ of dimension k.*

Proof. See [Pon76]. □

Remark 2.2.4. *If we define the addition of cobordism classes to be disjoint and 'untangled' union, then this is an isomorphism of abelian groups.*

Remark 2.2.5. *The suspension homomorphism correspond to the inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$, and a general position argument shows the Freudenthal theorem for spheres that the suspension homomorphism $\Sigma : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$ is an epimorphism for $n = k+1$ and an isomorphism for $n \geq k+2$.*

Remark 2.2.6. *The smash product of homotopy groups correspond to the cartesian product of framed manifolds.*

2.3 Hopf fibration and $\pi_{n+1}(S^n)$

The first nontrivial map from a higher dimensional sphere to a lower dimensional one is given by the Hopf fibration.

Let $S^3 \subset \mathbb{C}^2$ be the unit sphere. Then the group $U(1)$ acts freely on it, and the quotient is the complex projective line $\mathbb{CP}^1 = S^2$. The quotient map defines the Hopf fibration $\eta_2 : S^3 \to S^2$.

To show this map is not homotopoy to the trivial map, look at the corresponding framed manifold, which is a circle (in the sense of euclidian geometry) with a certain framing.

Define the canonical framing of the standard embedding $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$ to be the orthonormal framing with the first vector lay in $\mathbb{R}^2$ and pointing out and the second vector to be $(0, 0, 1)$. Then any framing differs from a canonical one by a map $S^1 \to O(2)$. The orientation of S^1 can be determined by the framing so that when we move it isotopically to the standard one in $\mathbb{R}^2$ with the standard orientation, the difference to the canonical framing is in fact a map $S^1 \to SO(1)$. One verifies the homotopy class of this map depends only on the original framed circle.

Now the difference of the framing corresponding to the Hopf map from the canonical one is just a degree one map. But if a framed circle in $\mathbb{R}^3$ is null cobordant, then the framing should be extended to some disk and thus homotopic to the canonical one. So the Hopf map is not null homotopic.

The above argument can be extended to higher dimensions to show the suspension of the Hopf map is also nontrivial since this correspond to the nontrivial element of $\pi_1 SO(n)$ which is $\mathbb{Z}/2$ when $n \geq 3$.

To calculate the group $\pi_{n+1}(S^n)$, observe that any 1-dimensional framed submanifold of $\mathbb{R}^n$ is cobordant to a circle, though the case of knots and links cost some meditation. Thus the preceding argument gives the following result, which is first obtained in [Hop30] and [Fre37]:

Theorem 2.3.1. *$\pi_3(S^2) \cong \mathbb{Z}$ generated by the Hopf map and $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ generated by the suspension of the Hopf map.*

Proof. See [Pon76]. □

2.4 Surgery

For the sake of simplicity, we consider only stable groups from now on. And in the stable case, the background manifold $\mathbb{R}^n$ can be gotten rid with.

Definition 2.4.1. *A stably framed manifold $(X, N, \sigma, \varsigma)$ is a smooth manifold X, a vector bundle N over X, together with framings σ, ς of N and $TX \oplus N$ respectively.*

Another stable framing $(X, N', \sigma', \varsigma')$ of X is said to be a subframing of $(X, N, \sigma, \varsigma)$ if there is a vector bundle V over X with a framing ξ, such that $N = N' \oplus V$, $\sigma = \sigma' \oplus \xi$, and $\varsigma = \varsigma \oplus \xi$.

Two stable framings of the same manifold is said to be equivalent if they are subframings of some common stable framing. Equivalent stable framings will be identified as the same stable framing.

A homotopy between two stable framings is any homotopy between two equivalent ones.

Definition 2.4.2. *A framed cobordism between two stably framed manifolds $(X, N, \sigma, \varsigma)$ and $(X', N', \sigma', \varsigma')$, is a stably framed manifold with boundary (Y, P, ξ, η),* *such that ∂Y, the boundary of Y, is $X \cup X'$, $(X, N, \sigma, \varsigma)$ is equivalent to $(X, P|_X \oplus V, \xi|_X \oplus \zeta, \eta|_X)$, and $(X', N', \sigma', \varsigma')$ is equivalent to $(X', P|_{X'} \oplus V, \xi|_{X'} \oplus (-\zeta), \eta|_{X'})$, where V is the normal bundle of ∂Y in Y, and ζ is the inward pointing unit vector of V.*

Remark 2.4.3. *When restricting a stable framing to an orientable codimension one submanifold, we mean to add the normal vector of the submanifold to the framing of the normal bundle in the definition of a stable framing as the last factor.*

In general there are two choices of the normal vector. An orientation specifies a choice of the normal vector. When restricting to the boundary, we call the two choices inward pointing framing and outward pointing framing.

In the preceding definition, X gets the inward pointing framing, and X' gets the outward pointing framing.

Theorem 2.4.4. *The cobordism class of stably framed manifolds is isomorphic with the stable homotopy group of spheres.*

Remark 2.4.5. *When taking the product of $(X, N, \sigma, \varsigma)$ and $(X', N', \sigma', \varsigma')$, we force N to be even dimensional, or a sign $(-1)^{dim(N) \cdot dim(X')}$ must be introduced.*

To study the cobordism class of manifolds we need the notion of surgery.

In the following, all corners are to be smoothed without explicit mentioning.

Suppose Y is a cobordism between X and X'. Let $f : Y \to \mathbb{R}$ be a Morse function such that $f|_X = 0$, $f|_{X'} = 1$ and $0 < f < 1$ in the interior of Y. Such a function gives a handle decomposition of Y. Precisely, whenever a is a critical value of f, $f^{-1}[0, a + \epsilon]$ is obtained from $f^{-1}[0, a - \epsilon]$ by attaching a handle $D^k \times D^t$ where k is the index of the critical point and ϵ is a small positive real number. And $f^{-1}(a + \epsilon)$ differs from $f^{-1}(a - \epsilon)$ by cutting $D^k \times \partial D^t$ and pasting $\partial D^k \times D^t$ along $\partial D^k \times \partial D^t$. And we see X' is obtain from X by a sequence of such an operation called surgery. Conversely, if two manifolds differ from each other by a sequence of such surgeries, we can construct a cobordism by attaching handle bodies.

In the case of stably framed cobordism, we need to be careful about framings. $D^k \times D^t$ carries a canonical framing (in fact unique up to homotopy if an orientation is specified, since it is contractible), and this induces the canonical stable framing over $D^k \times \partial D^t$, $\partial D^k \times D^t$, and $\partial D^k \times \partial D^t$. Here $D^k \times \partial D^t$ gets the inward pointing framing and $\partial D^k \times D^t$ gets the outward pointing framing, while $\partial D^k \times \partial D^t$ gets the outward pointing one as the boundary of either $\partial D^k \times D^t$ or $\partial D^k \times \partial D^t$..

Definition 2.4.6. *Let $(X, N, \sigma, \varsigma)$ be a stably framed manifold of dimension $k + t - 1$. Suppose $\phi : D^k \times \partial D^t \to X$ is an embedding and τ is a homotopy from the canonical stable framing of $D^k \times \partial D^t$ to the one pulled back from X. Then a surgery along ϕ is obtained by cutting $Im(\phi)$ and gluing $\partial D^k \times D^t$ along $\partial D^k \times \partial D^t$. The framing over the remaining part of X is unaltered and over $\partial D^k \times D^t$ is the canonical one except near $\partial D^k \times \partial D^t$ it is modified according to τ so that it glues up.*

Taking care of framings in the preceding argument, we get:

Theorem 2.4.7. *Two stably framed manifolds are in the same stably framed cobordism class if and only if they differ from each other by a sequence of surgeries described above.*

2.5 Arf invariant and $\pi_{n+2}(S^n)$

The calculation of $\pi_{n+2}(S^n)$ is completed in [Whi50] and [Pon50]. Here we follow the method of Pontrjagin.

We will assume $n \geq 4$ in this section.

First we show the group is nontrivial.

Let S be an orientable surface. It is elementary that we can transform it into a sphere by surgery along some non-seperatting S^1's.

Now suppose S is stably framed. Taking the account of framings, only those surgeries along S^1's which bound stably framed disks are admissible. We know the obstruction to extend a stable framing of S^1 to the disk lies in $\pi_1(O)$ which is $\mathbb{Z}/2$. So define a function γ from immersed S^1's in S to $\mathbb{Z}/2$ by taking $\gamma(C) = 0$ if the curve C together with the stable framing restricted from S bounds a stably framed disk and $\gamma(C) = 1$ otherwise. When $C = C_1 \cup \cdots \cup C_k$ is a immersed submanifold with k components, we can extend the definition of γ by taking $\gamma(C) = \gamma(C_1) + \cdots + \gamma(C_k)$. For those C which have only double points, define a function δ by taking $\delta(C) = \gamma(C) + s(C)$ where $s(C)$ is the number of its double points mod 2.

Theorem 2.5.1. *$\delta(C)$ depends only on the mod 2 homology class of C, so that we can write $\delta(z)$ for $z \in H_1(S, \mathbb{F}_2)$ unambiguously. Moreover, δ is quadratic and*

$$\delta(z_1 + z_2) = \delta(z_1) + \delta(z_2) + J(z_1, z_2) \tag{2.5.2}$$

where J is the intersection form for $H_1(S, \mathbb{F}_2)$.

Proof. See [Pon76]. □

For quadratic forms over $\mathbb{F}_2$ satisfying equation 2.5.2 there is the Arf invariant. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a canonical basis for J in the sense that $J(a_i, b_i) = J(b_i, a_i) = 1$ for $i = 1, \ldots, g$ and any other pairings of them form 0. Then the Arf invariant of δ is defined to be $\sum_{i=1}^{g} \delta(a_i)\delta(b_i)$. This is independent of the canonical basis.

It happens that the Arf invariant of the quadratic form δ in theorem 2.5.1 is a stably framed cobordism invariant. So any stably framed surface S with $\delta(S) = 1$ would correspond to a nontrivial element of $\pi_{n+2}(S^n)$. In fact the left invariant framing on $S^1 \times S^1$ would do, which can also be described as η^2 where η is the suspension of the Hopf map.

Alternatively, this invariant can also be described as the index of the Dirac operator on S. The stable framing on S gives a spin structure on S and this determines a KO-orientation on S as described in [ABS64]. The invariant is

$p_!(1) \in KO^{-2}(pt)$ where p is the map from S to one point. This is the mod 2 demension of the solution of the Dirac equation on S. See [Ati71] for details.

Now we show in fact this Pontrjagin–Arf invariant gives an isomorphism $\pi_{n+2}(S^n) \cong \mathbb{Z}/2$. The surjectivity is already prooved. To show injectivity, suppose S is a stably framed surface with Pontrjagin–Arf invariant 0 (it can be shown that it suffices to consider the connected case). Let the genus of S be g. If $g = 0$, since $\pi_2(O) = 0$, S bounds a framed solid ball and we are done. If $g > 0$, then by the definition of Arf invariant, there exists some S^1 embedded in S which bound a framed disk and non-separating so that surgery along it reduces the genus g. We arrive at the following:

Theorem 2.5.3. *$\pi_{n+2}(S^n) \cong \mathbb{Z}/2$ generated by $\eta_n \circ \eta_{n+1}$ where $\eta_n : S^{n+1} \to S^n$ is the suspension of the Hopf map.*

Proof. See [Pon76]. □

2.6 The computation of $\pi_{n+3}(S^n)$

The calculation of $\pi_{n+3}(S^n)$ is difficult since it is hard to classify 3 dimensional manifolds. However, this difficulty can be bypassed, and the group is calculated in [Rok51].

Here we calculate the stable group, so assume $n \geq 5$.

First we show the group is nontrivial, following [AS74].

Let X be a stably framed 3 dimensional manifold. This gives a spin structure on X. The 3 dimensional spin cobordism group is trivial (this can be shown by elementary methods, see section 5.3.3), so we can find a 4 dimensional spin manifold with boundary Y such that $\partial Y = X$ as spin manifolds. The tangent bundle of Y together with the framing on X gives an element $[TY] \in KO(Y, X)$, and we can calculate its Pontrjagin class $p_1([TY]) \in H^4(Y, X; \mathbb{Z})$. By integration we get an integer $p_1(Y, X)$. Since TY has a spin structure, $p_1(Y, X)$ is in fact an even number. This number depends on the choice of Y. If Y' is another spin 4-manifold with boundary X, then consider the manifold Z obtained by gluing together Y and Y' along X. Since X has a framing, which means we can choose some connection on the stable tangent bundle of Z which is flat near X, $p_1(Z) = p_1(Y, X) - p_1(Y', X)$. The $\hat{A}$-genus of a 4 dimensional spin manifold is even (this can be shown by the Atiyah-Singer index theorem since this is the index of the Dirac operator, or calculate directly the homomorphism $p_! : KO(Z) \to KO^{-4}(pt)$ where p is the map to one point), and $\hat{A}_1 = -\frac{p_1}{24}$, so $p_1(Z)$ is divisible by 48. Thus $e(X) = -\frac{1}{2}p_1(Y, X) \mod 24$ is well defined.

In fact, $e(X)$ depends only on the stably framed cobordism class of X. Suppose $\tilde{X}$ is another framed manifold cobordant to X, and W is a stably framed manifold with $\partial W = X \cup \tilde{X}$. If $\tilde{X}$ is the boundary of the spin manifold $\tilde{Y}$, then $V = Y \cup_X W \cup_{\tilde{X}} \tilde{Y}$ is a spin manifold. One verifies $p_1(W) = p_1(Y, X) - p_1(\tilde{Y}, \tilde{X})$, so $e(X) = e(\tilde{X})$.

The Lie group $S^3 \cong Spin(3)$ with the left invariant framing has $e(S^3) = -1$, since the first obstruction for $Spin$ is $\frac{p_1}{2}$. So the map $e : \pi_{n+3}(S^n) \to \mathbb{Z}/24$ is an epimorphism.

Theorem 2.6.1. $\pi_{n+3}(S^n) \cong \mathbb{Z}/24$ *given by the* e *invariant and is generated by the element* ν *corresponding to* S^3 *with the left invariant framing.*

Proof. Only the injectivity of e remains to be proved.

Let X be a stably framed 3 dimensional manifold, and Y a spin manifold with boundary X. Let Z be the manifold obtained from Y by removing a small solid ball D^4 in Y. (Z, X) is homotopy equivalent to a CW-pair of dimension 3, and since $\pi_i(Spin) = 0$ for $i \leq 2$, the framing on X extends to one on W. So W gives a stably framed cobordism from X to S^3 with a certain framing. The framings on S^3 correspond bijectively with $\pi_3(O) \cong \mathbb{Z}$, and one verifies the addition in cobordism classes correspond to the addition in the homotopy group $\pi_3(O)$. This shows $\pi_{n+3}(S^n)$ is cyclic with generator ν.

Suppose M is a closed 4 dimensional spin manifold and N is obtained from M by removing a small solid ball D^4. Then as before there is a framing on N. This gives a null-cobordism of S^3 with the framing restricted from M. The nontriviality of this framing is the obstruction of the existence of a framing over M, and since the first obstruction for $Spin$ is $\frac{p_1}{2}$, we see this framing corresponds to $\frac{p_1}{2}$ times ν.

Now set M to be the $K3$ surface. We know that $p_1(K3) = -48$, so $24\nu = 0$. □

Chapter 3

Methods from Homotopy Theory: Sequences and Operations

The method in the last chapter of calculating $\pi_{n+k}(S^n)$ becomes more and more complicated when k grows. One might wonder if there is any systematic method in calculating the homotopy groups. In fact homotopy theory provides many methods to deal with the homotopy groups in general and these can be used to calculate the homotopy groups in a certain range. Though there is no method available at present to provide a practical way to calculate the homotopy groups to any range as one like, we understand quite a lot of the general structures of the homotopy groups by now.

The homotopical methods use as the main tool exact sequences. These exact sequences are unanimous in homotopy theory, and relates various kind of groups together, leading to a complicated network in the end. Theories and tricks are invented to unravel them so that the calculations are possible for human beings. Also, in deciding the morphisms in the sequences, certain natural transformations are needed. These are the cohomology operations and homotopy operations, which has complicated relations. In some sense, these are the origins of the seemingly random structure of the homotopy groups of spheres.

3.1 The method of J.-P Serre

The first method leading to a general scheme to calculate the homotopy groups was given by J.-P Serre. Since the homotopy group of the loop space of a topological space is just a shift in degree of the original one, by taking the n-fold loop space, the calculation of the homotopy groups reduces to the calculation of the fundamental group, which is well understood. J.-P Serre developed the method of spectral sequence so that the cohomology of the loop space can be calculated. This enabled him to prove the that the homotopy groups of spheres are finitely generated, as well as the determination of the rank of the free part. With the aid of cohomology operations such as Steenrod squares, he was able to calculate the groups $\pi_n(S^{n+k})$ for $k \leq 8$.

3.1.1 The Serre spectral sequence

The method of spectral sequence is a means to calculate the derived functors in two steps. One might ignore something in the first step, so that the derived functor, such as cohomology groups, is comparatively easy to compute. And in the second step, one computes the cohomology of the output in the first step. But in general, we lose some information in the first step. For example, applying the cohomology group functor loses those information carried by the differentials in the complex. So we do not get what we want by directly calculating the derived functor of the outcome in the first step. Instead, we must trace back the information carried by the differentials, and the result is a spectral sequence.

A spectral sequence is a sequence of graded abelian groups E_r^*, with differentials $d_r : E_r^* \to E_r^*$ of certain degrees and $d_r \circ d_r = 0$, satisfying the condition that the homology of (E_r^*, d_r) is E_{r+1}^*. The abelian groups in a spectral sequence may have more structures, such as module structures over some ring, or a second degree, etc. In some cases, we can define the limit E_∞^* of the groups E_r^* as $r \to \infty$, and the resulting groups E_∞^* will give approximations to the object we want to study. One can consult [Boa99] for a general discussion of the convergence of spectral sequences.

The Serre spectral sequence is used to calculate the homology of the total space of a fibration once that of the base and the fibre is known. It can also be used reversely to calculated the cohomology of the base or fibre from the knowledge of the other two.

Theorem 3.1.1. *Suppose $p : E \to B$ is a Serre fibration, with fibre F, base B. Let R be a commutative ring. Then there is a spectral sequence $(E_r^{p,q}, d_r)$ with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, converging to $H^*(E, R)$, and $E_2^{p,q}$ is functorially isomorphic to $H^p(B, H^q(F, R))$. Here $H^*(F, R)$ is the local system (or locally constant sheaf) defined by the cohomology of the fibre.*

Proof. See [Ser51]. $\square$

Remark 3.1.2. *The Serre spectral sequence has a multiplicative structure on each $E_r^{*,*}$ respecting degrees, and the differentials d_r are derivations (in the graded sense).*

The Serre spectral sequence can be used to prove finiteness conditions of cohomology groups.

Corollary 3.1.3. *Suppose furthermore B is 1-connected and R is a PID. Then if any two of the spaces F, B, E has the property that all cohomology groups with coefficient R are finitely generated R-modules, then the third one also has this property.*

Proof. See [Ser51]. □

This corollary show that many spaces constructed from fibrations, such as the loop spaces of simply connected finite CW-complexes, has finitely generated cohomology groups. But one should take in mind that in general this is not the case for non-simply connected spaces.

To calculate the homotopy groups of spheres, J.-P Serre computed inductively the n-fold loop spaces of the spheres using the fibration $\Omega X \to E \to X$, where E is contractible, for any space X. When the cohomology of $\Omega^n S^k$ is computed, its fundamental group is deduced from the Hurewics theorem. Then take the universal cover and apply the loop functor to compute the cohomology of the next loop space.

Theorem 3.1.4. *The groups $\pi_{n+k}(S^n)$ are all finitely generated. Their ranks are given by*

$$rank(\pi_{n+k}(S^n)) = \begin{cases} 1 & if & k = 0 \\ 1 & if & n \text{ } even \text{ } and \text{ } k = n-1 \\ 0 & otherwise & \end{cases} \tag{3.1.5}$$

Proof. See [Ser51]. □

3.1.2 Cohomology operations and Eilenberg-MacLane spaces

The cohomology operations are natural transformations between cohomology functors. They are important in determining the differentials in the Serre spectral sequence. And the structure of the algebra of all the cohomology operations is used in the Adams spectral sequence, which will be discussed in the next chapter.

The simplest cohomology operation is the Bockstein operation, which is induced by a short exact sequence of the coefficient groups.

Let p be a fixed prime. The Bockstein exact sequence $\ldots \to H^n(X,\mathbb{Z}) \xrightarrow{\times p} H^n(X,\mathbb{Z}) \to H^n(X,\mathbb{Z}/p) \xrightarrow{\beta} H^{n+1}(X,\mathbb{Z}) \to \ldots$, where β is the Bockstein operation, gives an exact couple. The resulting spectral sequence is the Bockstein spectral sequence. Let β_r to be the differential of the E_r-term in the spectral sequence. This is call the r^{th} Bockstein operation, and they give information about the elements in $H^*(X,\mathbb{Z})$ of order p^r.

The other operations are the Steenrod squares Sq^i, and the Steenrod powers $\mathcal{P}^i$. They satisfy (and characterized by) the following properties.

- For any pair of spaces $Y \subset X$,

  $Sq^i : H^q(X,Y,\mathbb{Z}/2) \to H^{q+i}(X,Y,\mathbb{Z}/2)$,

  $\beta : H^q(X,Y,\mathbb{Z}/p) \to H^{q+1}(X,Y,\mathbb{Z}/p)$ for p an odd prime,

  $\mathcal{P}^i : H^q(X,Y,\mathbb{Z}/p) \to H^{q+2i(p-1)}(X,Y,\mathbb{Z}/p)$ for p an odd prime,

- $Sq^0 = id$, $\mathcal{P}^0 = id$

- if $x \in H^q(X, Y, \mathbb{Z}/2)$ then $Sq^q x = x^2$
  if $x \in H^{2q}(X, Y, \mathbb{Z}/p)$ then $\mathcal{P}^q x = x^p$
- if $x \in H^k(X, Y, \mathbb{Z}/2), k < q$ then $Sq^q x = 0$
  if $x \in H^k(X, Y, \mathbb{Z}/p), k < 2q$ then $\mathcal{P}^q x = 0$
- Sq^1, β are the Bockstein operations induced form $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$.
- Cartan formula:

$$Sq^k(xy) = \sum_{i=0}^{k} Sq^i x \cdot Sq^{k-i} y \tag{3.1.6}$$

$$\mathcal{P}^k(xy) = \sum_{i=0}^{k} \mathcal{P}^i x \cdot \mathcal{P}^{k-i} y \tag{3.1.7}$$

- Adem relations:
  if $a < 2b$, then

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \tag{3.1.8}$$

  if $a < pb$, then

$$\mathcal{P}^a \mathcal{P}^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} \mathcal{P}^{a+b-t} \mathcal{P}^t \tag{3.1.9}$$

  if $a \leqslant pb$, then

$$\begin{aligned}\mathcal{P}^a \beta \mathcal{P}^b = \quad & \textstyle\sum_{t=0}^{[a/p]} \quad (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta \mathcal{P}^{a+b-t} \mathcal{P}^t \\ & + \sum_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} \mathcal{P}^{a+b-t} \beta \mathcal{P}^t\end{aligned}$$

- if $\delta^* : H^q(A, \mathbb{Z}/p) \to H^{q+1}(X, A, \mathbb{Z}/p)$ is the coboundary map, then $Sq^i \delta^* = \delta^* Sq^i$, $\beta \delta^* = -\delta^* \beta$, $\mathcal{P}^i \delta^* = \delta^* \mathcal{P}^i$

For the construction of these operations, see [SE62].

We can construct other cohomology operations by composition of the Steenrod squares and powers. Because of the Adem relations, only certain kind of compositions are needed.

Definition 3.1.10. *Fix a prime number p.*

- *If $p = 2$, let $I = (s_1, \ldots, s_k)$. I is admissible if $s_i \geq 2s_{i+l}$ and $s_k \geq 1$. The length, degree, and excess of I are defined by $\mathfrak{l}(I) = k$, $\mathfrak{d}(I) = \sum s_j$, and, for $I = (s, J)$, $\mathfrak{e}(I) = s - \mathfrak{d}(I)$. Define $Sq_t^I = Sq^{s_1} \ldots Sq^{s_{k-1}} Sq_t^{s_k}$. Here $Sq_t^s = Sq^s$ if $s \geq 2$, $Sq_t^1 = \beta_t$ is the Bockstein operation if $t < \infty$, and $Sq_\infty^1 = 0$. The empty sequence is admissible. Define $Sq^I = Sq_1^I$*

- *If $p > 2$, let $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k, \epsilon_{k+l})$, $\epsilon_i = 0$ or 1. I is admissible if $s_i \geq ps_{i+1} + \epsilon_{i+1}$ and $s_k \geq 1$ or if $k = 0$ (then $I = (\epsilon)$). Define $\mathfrak{l}(I) = k$, $\mathfrak{d}(I) = \sum \epsilon_i + \sum 2(p-1)s_i$, and, if $I = (\epsilon, s, J)$, $\mathfrak{e}(I) = 2s + \epsilon - \mathfrak{d}(J)$. Define $\mathcal{P}_t^I = \beta^{\epsilon_1}\mathcal{P}^{s_1} \ldots \beta^{\epsilon_k}\mathcal{P}^{s_k}\beta_t^{\epsilon_{k+1}}$. Here $\beta_t^0 = 1$ for all t, $\beta_t^1 = \beta_t$ for $t < \infty$, and $\beta_\infty^1 = 0$. Define $\mathcal{P}^I = \mathcal{P}_1^I$.*

In view of the Adem relations, the linear combinations of the form Sq^I or $\mathcal{P}^I$ exhaust all the cohomology operation obtained from the Steenrod squares and powers together with the primary Bockstein operation.

The cohomology functors are representable in the homotopy category of spaces. So by the Yonida lemma, the natural transformations between two cohomology functors correspond bijectively to the morphisms between the objects representing them.

The space representing the functor $H^i(\bullet, \Pi)$ is the Eilenberg-MacLane space $K(\Pi, i)$. So cohomology operations from $H^i(\bullet, \Pi)$ to $H^{i'}(\bullet, \Pi')$ are classified by $[K(\Pi, i), K(\Pi', i')]$, or $H^{i'}(K(\Pi, i), \Pi')$. The cohomology of Eilenberg-MacLane spaces can be calculated using the Serre spectral sequence, since we have the relation $\Omega K(\Pi, i) \cong K(\Pi, i-1)$. The main results is as follows:

- $H^i(K(G, n), \mathbb{Q}) = 0$ if G is finite and $i > 0$.

  $H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \Lambda[\iota_n]$ if k is odd.

  $H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \mathbb{Q}[\iota_n]$ if k is even.

- $H^*(K(\mathbb{Z}/2, 1), \mathbb{F}_2) = \mathbb{F}_2[\iota_1]$.

  $H^*(K(\mathbb{Z}/2^t, 1), \mathbb{F}_2) = \Lambda[\iota_1] \otimes \mathbb{F}_2[\beta_t(\iota_1)]$, for $t \geq 2$.

  $H^*(K(\mathbb{Z}/2^t, n), \mathbb{F}_2) = \mathbb{F}_2[Sq_t^I \iota_n : I$ is admissible and $\mathfrak{e}(I) < n]$, if $n \geq 2$.

  $H^*(K(\mathbb{Z}, n), \mathbb{Z}/2) = \mathbb{F}_2[Sq^I \iota_n : I$ is admissible and $\mathfrak{e}(I) < n, s_k \geq 2]$, if $n \geq 2$ and $I = (s_1, \ldots, s_k)$.

- For p an odd prime, and when $t = \infty$ the group $\mathbb{Z}/p^t$ is replaced with $\mathbb{Z}$,

  $H^*(K(\mathbb{Z}/p^t, 1), \mathbb{F}_p) = \Lambda[\iota_1] \otimes \mathbb{F}_p[\beta_t(\iota_1)]$.

  $H^*(K(\mathbb{Z}/p^t, n), \mathbb{F}_p) = \Lambda[\mathscr{T}_{n,t}] \otimes \mathbb{F}_p[\mathscr{S}_{n,t}]$, where $\mathscr{T}_{n,t}$ and $\mathscr{S}_{n,t}$ are subsets of $\{\mathcal{P}_t^I \iota_n : I$ is admissible and $\mathfrak{e}(I) < n$ or, $\mathfrak{e}(I) = n$ and $\epsilon_1 = 1\}$ of odd and even degree respectively. Here $I = (\epsilon_1, , s_1, \ldots, \epsilon_k, s_k, \epsilon_{k+l})$.

The proofs can be found in [Ser53a], [Car55] and [May70].

Remark 3.1.11. *For unequal characteristics, the groups $H^i(K(\mathbb{Z}/q^t, n), \mathbb{F}_p) = 0$ for $i > 0$ if $p \neq q$. In general, we have the notion of localization at a prime number, just as in algebraic geometry, so that we can deal with one prime at a time and ignore the other primes. See [BK72] for details of localization at a prime number.*

With the aid of cohomology operations, J.-P Serre could compute the first eight nontrivial homotopy groups for the spheres. See [Ser53c] for details.

3.1.3 The Steenrod algebra

Using the results on the cohomology of Eilenberg-MacLane spaces, we can classify all the cohomology operations for cohomology with coefficient $\mathbb{F}_p$. We are mainly interested in the stable cohomology operation, i.e. those commuting with the suspension. These operations constitute an algebra under composition, called the Steenrod algebra $A(p)^*$. And the cohomology of topological spaces are modules over $A(p)^*$.

Theorem 3.1.12. *The Steenrod algebra $A(p)^*$ is generated by the Steenrod squares if $p = 2$, the Steenrod powers and the Bockstein operation if $p > 2$, modulo the Adem relations. Moreover, it is a Hopf algebra with comultiplication $\psi(Sq^i) = \sum_{j+k=i} Sq^j \otimes Sq^k$ if $p = 2$, $\psi(\mathcal{P}^i) = \sum_{j+k=i} \mathcal{P}^j \otimes \mathcal{P}^k$ and $\psi(\beta) = 1 \otimes \beta + \beta \otimes 1$ if $p > 2$.*

Proof. See [SE62]. □

Since the Adem relations have a very complicated form, the Steenrod algebra is not easy to investigate. However, we can take the dual algebra $A(p)_*$. Since the dual algebra is a Hopf algebra with commutative multiplication, its algebra structure must have a simple form.

Theorem 3.1.13. *$A(p)_*$ is a graded commutative, noncocommutative Hopf algebra.*

- *For $p = 2$, $A(2)_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$ as an algebra where $|\xi_n| = 2^n - 1$. The coproduct $\Delta : A(2)_* \to A(2)_* \otimes A(2)_*$ is given by $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i$, where $\xi_0 = 1$.*

- *For $p > 2$, $A(p)_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots]$ as an algebra, where $|\xi_n| = 2(p^n - 1)$, and $|\tau_n| = 2p^n - 1$. The coproduct $\Delta : A(p)_* \to A(p)_* \otimes A(p)_*$ is given by $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} \otimes \xi_i$, where $\xi_0 = 1$ and $\Delta\tau_n = \tau_n \otimes 1 + \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} \otimes \tau_i$.*

Proof. See [Mil58]. □

Dually, the homology of topological spaces are comodules over the dual Steenrod algebra $A(p)_*$.

The theory of cohomology operations can be extended to the generalized cohomology theories, at least for good ones. But in general, the dual algebra would not be a Hopf algebra. Instead, it is a Hopf algebroid. See [Rav86], chapter 2 for details.

3.1.4 The Postnikov tower

We can simplify J.-P. Serre's method by using the Postnikov towers instead of applying iterated loop space functor, so that we can avoid dealing with non-simply connected spaces, and deal with stable homotopy groups directly without knowing the unstable ones.

For reasons of simplicity, we consider only simply connected spaces in this section.

Definition 3.1.14. *The Postnikov tower of a simply connected space X is a sequence*

$$\begin{tikzcd} X \arrow[d, "g_0"] & X_1 \arrow[l, "f_0"'] \arrow[d, "g_1"] & X_2 \arrow[l, "f_1"'] \arrow[d, "g_2"] & \cdots \arrow[l, "f_2"'] \\ Y_0 & Y_1 & Y_2 & \end{tikzcd} \tag{3.1.15}$$

such that Y_k is of type $K(\Pi, k+1)$ and X_i, which is the homotopy fibre of f_{i-1}, is the i-connected cover of X.

One can show that the Postnikov tower exists and is unique up to homotopy. Also, the homotopy groups of X is just the direct sum of the homotopy groups of the Y_j's.

Using the Serre spectral sequence, one can calculate the cohomology groups of the X_i inductively, and determine the spaces Y_i using the Hurewics theorem. This is particularly useful in the stable range, because in this case the Serre spectral sequence becomes an exact sequence.

Remark 3.1.16. *The Postnikov tower is a kind of resolution of spaces by the Eilenberg-MacLane spaces. In the next chapter, we will study another resolution, the Adams resolution, which makes the calculation simpler, and also gives more structures of the groups.*

Using this method, one can calculate the first nine stable homotopy groups of spheres, and to the thirteenth stem in the 2-component. The 14^{th} stem of the 2-component cannot be obtained with this method alone, essentially because the Hopf invariant one problem for $n = 16$ shows up here.

3.2 The method of Hirosi Toda

The difficulty encountered by the cohomological methods in the preceding section is overcome by H. Toda using homotopy operations, the composition and Toda brackets. He was able to compute more stems of the homotopy groups of spheres, and in particular, solved the Hopf invariant one problem for $n = 16$.

We work with only the 2-component in the section, so all the spaces involved will be implicitly assumed to be 2-local and the groups are all localized at the prime 2.

3.2.1 The EHP-sequence

The sequence used by H. Toda was the EHP-sequence of I. M. James. This sequence was primarily a long exact sequence for the 2-component, and could be extended to the other components. However, we will only discuss the 2-component in this section. so all the spaces involved will be implicitly assumed to be 2-local.

The main point in the EHP-sequence is a 2-local fibre sequence $S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$.

To prove it, let us study the spaces ΩS^{j+1} first. It turns out to be the free A_∞ space generated by S^j. Since the A_∞-operad is equivalent to the associative operad, we have a concrete model for ΩS^{j+1}. Fix a base point e_0 in S^j. Define $\mathfrak{C}S^j$ to be the quotient space obtained from $\coprod_{k\geq 1}\prod^k S^j$ by identifying $(x_1,\dots,x_{t-1},e_0,x_{t+1},\dots,x_k) \in \prod^k S^j$ with $(x_1,\dots,x_{t-1},x_{t+1},\dots,x_k) \in \prod^{k-1} S^j$, where $\prod^k S^j$ means the cartesian product of k copies of S^j.

Theorem 3.2.1. *The space $\mathfrak{C}S^j$ is homotopy equivalent to ΩS^{j+1}.*

Proof. See [May72]. □

The space $\mathfrak{C}S^j$ has a natural cell decomposition with one cell in dimension kj for each k, and a calculation with the Serre spectral sequence of the fibre sequence $\Omega S^{j+1} \to E \to S^{j+1}$, for the cohomology with coefficient $\mathbb{Q}$, shows that all the boundary maps in the complex of the cell decomposition vanish. Thus $H^{kj}(\Omega S^{j+1},\mathbb{Z}) \cong \mathbb{Z}$ and the cohomology groups of other dimensions vanish. Now in the Serre spectral sequence of $\Omega S^{j+1} \to E \to S^{j+1}$ for the coefficient $\mathbb{Z}$, the only pattern is that $d_{j+1}(x_{kj}) = x_{(k-1)j} \otimes y$ where x_{kj} is the generator of $H^{kj}(\Omega S^j,\mathbb{Z})$ and y is the generator of $H^{j+1}(S^{j+1},\mathbb{Z})$. So the multiplicative structure of $H^*(\Omega S^{j+1},\mathbb{Z})$ can be read from it. If j is odd, $x_j x_{2kj} = x_{(2k+1)j}$ and $x_{2kj}x_{2hj} = \binom{k+h}{k}x_{2(k+h)j}$. If j is even, $x_{kj}x_{hj} = \binom{k+h}{k}x_{(k+h)j}$. Modulo 2, we obtain $H^*(\Omega S^{j+1},\mathbb{F}_2) \cong \Lambda[x_{2^t j} : t \geq 0]$

Next we describe the map $\mathcal{H} : \Omega S^{j+1} \to \Omega S^{2j+1}$. Since $S^j \wedge S^j \cong S^{2j}$, we have a map $h : S^j \times S^j \to S^{2j}$. The map $\mathcal{H}$ is defined to be $\mathcal{H}(x_1,\dots,x_k) = \prod_\sigma h(x_{\sigma_1},x_{\sigma_2})$ for $(x_1,\dots,x_k) \in \mathfrak{C}S^j$, where $\sigma = (\sigma_1,\sigma_2)$ runs over all sequences with $1 \leq \sigma_1 < \sigma_2 \leq k$ and the product is the multiplication in $\mathfrak{C}S^{2j}$. One checks that this is well defined.

The map $\mathcal{H}$ maps the $2j$-cell in ΩS^{j+1} to the $2j$-cell in ΩS^{2j+1} with degree 1. So if j is odd, $\mathcal{H}^*(x'_{2kj}) = x_{2kj}$ where x_t and x'_t are the generators of $H^*(\Omega S^{j+1},\mathbb{Z})$ and $H^*(\Omega S^{2j+1},\mathbb{Z})$ respectively. If j is even, then we have $\mathcal{H}^*(x'_{2kj}) = \frac{(2k)!}{2^k k!}x_{2kj}$. Anyway, we have $\mathcal{H}^*(x'_{2kj}) = x_{2kj}$ modulo 2.

Theorem 3.2.2. *There is a 2-local fibre sequence $S^j \xrightarrow{\mathscr{E}} \Omega S^{j+1} \to \Omega S^{2j+1}$ where the map $\mathscr{E}$ is dual to the suspension.*

Proof. Let $\mathfrak{F}$ be the homotopy fibre of the map $\mathcal{H}$. Since $\mathcal{H}$ maps the j-skeleton $S^j \subset \mathfrak{C}S^j$ to the base point, the map $\mathscr{E} : S^j \to \Omega S^{j+1}$ factors through $\mathfrak{F}$. A calcualtion with the Serre spectral sequence of $\mathfrak{F} \to \Omega S^{j+1} \to \Omega S^{2j+1}$ for coefficient $\mathbb{F}_2$ shows that $H^*(\mathfrak{H}, \mathbb{F}_2) \cong \Lambda[u_j]$ with u_j the restriction of x_j, and the differential vanish. This shows the map $S^j \to \mathfrak{F}$ is a 2-local equivalence. ☐

Corollary 3.2.3. *There is a long exact sequence*

$$\ldots \to \pi_i(S^n) \xrightarrow{\Sigma} \pi_{i+1}(S^{n+1}) \xrightarrow{H} \pi_{i+1}(S^{2n+1}) \xrightarrow{\Delta} \pi_{i-1}(S^n) \to \ldots \tag{3.2.4}$$

This is the EHP-sequence. The map H is called the Hopf invariant.

3.2.2 Homotopy operations

We will discuss homotopy operations in a general setting, including those with many variables and the underlying space might vary.

So suppose $\mathfrak{F}$ to be a functor sending the sequence of pairs of topological spaces $(X_1, Y_1), \ldots, (X_n, Y_n)$ to the pair (W, Z). Then a homotopy operation would mean a natural transformation from $[X_1, Y_1] \times \cdots \times [X_n, Y_n]$ to $[W, Z]$, where $[X, Y]$ means the set of homotopy classes of maps from X to Y.

The first example of a homotopy operation is the composition of maps. These define in particular maps $\pi_m(S^n) \times \pi_n(S^k) \to \pi_m(S^k)$. These satisfy:

Let $\iota_n \in \pi_n(S^n)$ be the identity map, and α, β are elements of the homotopy groups of spheres.

$$\iota_n \circ \alpha = \alpha \circ \iota_p = \alpha \text{ for } \alpha \in \pi_p(S^n) \tag{3.2.5}$$

$$\alpha \circ (\beta_1 \pm \beta_2) = \alpha \circ \beta_1 \pm \alpha \circ \beta_2 \tag{3.2.6}$$

$$(\alpha_1 \pm \alpha_2) \circ \Sigma\beta = \alpha_1 \circ \Sigma\beta \pm \alpha_2 \circ \Sigma\beta \tag{3.2.7}$$

Note the the composition is not distributive in general.

The suspension commutes with the composition:

$$\Sigma(\alpha \circ \beta) = \Sigma\alpha \circ \Sigma\beta \tag{3.2.8}$$

The Hopf invariant behaves well with composition if one of the components is a suspension:

$$H(\alpha \circ \Sigma\beta) = H(\alpha) \circ \Sigma\beta \tag{3.2.9}$$

$$H(\Sigma\gamma \circ \alpha) = \Sigma(\gamma \wedge \gamma) \circ H(\alpha) \tag{3.2.10}$$

Remark 3.2.11. *This equation shows in particular that $H(k\alpha) = k^2 H(\alpha)$. Letting α to be the generator of $\pi_3(S^2)$, we see that the composition is not bilinear in this case.*

For the map Δ in the EHP-sequence, we have:

$$\Delta(\alpha \circ \Sigma^2 \beta) = \Delta(\alpha) \circ \beta \tag{3.2.12}$$

The proof of 3.2.9, 3.2.10 and 3.2.12 can be found in [Tod62], chapter 2.

The smash product is closely related to the composition. In the stable range, they define the same ring structure on the stable homotopy groups of spheres.

$$\alpha \wedge \iota_n = \Sigma^n \alpha \tag{3.2.13}$$

$$(\alpha_1 \circ \alpha_2) \wedge (\beta_1 \circ \beta_2) = (\alpha_1 \wedge \beta_1) \circ (\alpha_2 \circ \beta_2) \tag{3.2.14}$$

If $\alpha \in \pi_{p+k}(S^p)$ and $\beta \in \pi_{q+h}(S^q)$,

$$\alpha \wedge \beta = (-1)^{(p+k)(q+h)-pq} \beta \wedge \alpha \tag{3.2.15}$$

$$\alpha \wedge \beta = (-1)^{h(p+k)} \Sigma^q \alpha \circ \Sigma^{p+k} \beta = (-1)^{ph} \Sigma^p \beta \Sigma^{q+h} \alpha \tag{3.2.16}$$

Corollary 3.2.17. *The composition is commutative (in the graded sense) in the stable range, i.e.*

$$\Sigma^q \alpha \circ \Sigma^{p+k} \beta = (-1)^{kh} \Sigma^p \beta \circ \Sigma^{q+h} \alpha \tag{3.2.18}$$

Another important homotopy operation is the Whitehead product. To avoid complicity, suppose the spaces are simply connected.

The space $S^m \times S^n$ has a standard cell decomposition, with one $(m+n)$-cell and the $(m+n-1)$-skeleton is $S^m \vee S^n$. The attaching map gives a map $\psi : S^{m+n-1} \to S^m \vee S^n$. This map can also be described as follows. S^{m+n-1} can be viewed as the boundary of $D^m \times D^n$, so $S^{m+n-1} = (D^m \times S^n) \cup_{S^m \times S^n} (S^m \times D^n)$. The map ψ on $D^m \times S^n$ is the composition of the projection to D^m followed by the quotient map to S^m. The restriction to $S^m \times D^n$ is similar.

Let $\alpha \in \pi_m(X)$ and $\beta \in \pi_n(X)$, then their Whitehead product is defined by $[\alpha, \beta] = (\alpha, \beta) \circ \psi \in \pi_{m+n-1}(X)$ where $(\alpha, \beta) : S^m \vee S^n \to X$ restricts to α and β on S^m and S^n respectively.

The Whitehead product is bilinear, and if we set $\deg(\alpha) = m - 1$ for $\alpha \in \pi_m(X)$, then it gives a graded Lie algebra structure (if the sign is chosen suitably):

Theorem 3.2.19. *If α, β, γ are elements in $\pi_p(X), \pi_q(X), \pi_r(X)$ respectively, with $p, q, r > 1$, then*

$$\begin{aligned} & (-1)^{p(r-1)}[\alpha,[\beta,\gamma]] + (-1)^{q(p-1)}[\beta,[\gamma,\alpha]] + (-1)^{r(q-1)}[\gamma,[\alpha,\beta]] \\ = \; & (-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{qp}[[\beta,\gamma],\alpha] + (-1)^{rq}[[\gamma,\alpha],\beta] \\ = \; & 0 \end{aligned} \tag{3.2.20}$$

Proof. See [NT54]. ∎

This suggests the Whitehead product has something to do as a commutator. In fact, the Whitehead product is a commutator on the free part.

Theorem 3.2.21. *Let ϕ be the composition $\pi_{q+1}(X) \xrightarrow{\partial} \pi_q(\Omega X) \to H_q(X, \mathbb{Z})$, then*

$$\phi[\alpha, \beta] = (-1)^{deg(\alpha)}[\phi(\alpha), \phi(\beta)] \tag{3.2.22}$$

Where the bracket on the left side is the Whitehead product, and the one on the right is the commutator of the Pontrjagin product on $H_*(\Omega X, \mathbb{Z})$.

Proof. See [Sam53]. □

Remark 3.2.23. *The Whitehead product, together with the composition, generate all the primary homotopy operations on the homotopy groups of a fixed space, as proved in [Hil55]. Also, the data given by the Whitehead product completely determine the rational homotopy type of a simply connected space, see [Qui69b].*

The Whitehead product also gives the commutator of the composition product just below the stable range:

Theorem 3.2.24. *Let $\alpha \in \pi_p(S^m)$ and $\beta \in \pi_q(S^n)$. Define*

$$\theta = \Sigma^{n-1}\alpha \circ \Sigma^{p-1}\beta - (-1)^{(p-m)(q-n)}\Sigma^{m-1}\beta \circ \Sigma^{q-1}\alpha$$

Then $2\theta = 0$ and

$$\theta = [\iota, \iota] \circ \Sigma^{2m-2}H(\beta) \circ \Sigma^{q-1}H(\alpha)$$

Proof. See [Bar61]. □

It can be proved that

$$\Delta(\iota_{2m+1}) = \pm[\iota_m, \iota_m] \tag{3.2.25}$$

Since the Hopf invariant of $[\iota_m, \iota_m]$ is 2,

$$H(\Delta(\iota_{2m+1})) = \pm 2\iota_{2n-1} \tag{3.2.26}$$

Combining 3.2.12, 3.2.25 we get

$$\Delta(\Sigma^2\gamma) = \pm[\iota_m, \iota_m] \circ \gamma \tag{3.2.27}$$

So the θ in theorem 3.2.24 equals $\pm\Delta(\Sigma^{2m}H(\beta) \circ \Sigma^{q+1}H(\alpha))$.

3.2.3 The Toda bracket

The most important secondary homotopy operation is the Toda bracket. It is defined for triples (α, β, γ) whenever $\alpha \circ \beta = \beta \circ \gamma = 0$, and takes value a coset for certain subgroup.

Let $n \geq 0$, and $\alpha \in [\Sigma^n Y, Z]$, $\beta \in [X, Y]$, $\gamma \in [W, X]$ such that $\alpha \circ \Sigma^n \beta = 0$, $\beta \circ \gamma = 0$. Then there exist homotopies $\Phi : \Sigma^n X \times [0,1] \to Z$ from $\alpha \circ \Sigma^n \beta$ to 0, and $\Psi : W \times [0,1] \to Y$ from $\beta \circ \gamma$ to 0.

Define $\Theta : \Sigma^{n+1} W \to Z$ by the formula

$$\Theta(w,t) = \begin{cases} \alpha(\Sigma^n \Psi(w, 2t-1)) & \text{for} \quad \frac{1}{2} \leq t \leq 1 \\ \\ \Phi(\Sigma^n \gamma(w), 1-2t) & \text{for} \quad 0 \leq t \leq \frac{1}{2} \end{cases}$$

Then the Toda bracket $< \alpha, \beta, \gamma >_n$ is defined to be the set of Θ when Φ, Ψ run over all possible homotopies. This is a double coset in $[\Sigma^{n+1} W, Z]$ of the subgroups $[\Sigma^{n+1} X, Z] \circ \Sigma^{n+1} \gamma$ and $\alpha \circ \Sigma^n [\Sigma W, Y]$. Sometimes we will omit the subscript n.

Alternatively, we can describe the Toda bracket as follows. We call $\bar{\alpha} \in [Y \cup_\beta \mathcal{C}X, Z]$ an extension of α if $\bar{\alpha}|_Y = \alpha$, and call $\tilde{\gamma} \in [\Sigma W, Y \cup_\beta \mathcal{C}X]$ a coextension of γ if

$$\tilde{\gamma}(w,t) = \begin{cases} (\gamma(w), 1-2t) & if \quad 0 \leq t \leq \frac{1}{2} \\ \\ \in Y & if \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then $< \alpha, \beta, \gamma >_n = \{(-1)^n \bar{\alpha} \circ \Sigma^n \tilde{\gamma}\}$ where $\bar{\alpha}$ and $\tilde{\gamma}$ run over all the extensions of α and coextensions of γ respectively. This description shows that the Toda bracket in the stable range coincides with the Massey product in the triangulated category of spectra.

The Toda bracket has many good properties. Firstly, it is trilinear:

Theorem 3.2.28. *Let $\alpha, \alpha' \in [\Sigma^n Y, Z]$, $\beta, \beta' \in [X, Y]$, $\gamma, \gamma' \in [W, X]$. Then*

$$< \alpha, \Sigma^n \beta, \Sigma^n(\gamma + \gamma') >_n \quad \subset \quad < \alpha, \Sigma^n \beta, \Sigma^n \gamma >_n + < \alpha, \Sigma^n \beta, \Sigma^n \gamma' >_n$$

if $n \geq 1$ or $W = \Sigma W'$, (3.2.29)

$$< \alpha, \Sigma^n(\beta + \beta'), \Sigma^n \gamma >_n \quad = \quad < \alpha, \Sigma^n \beta, \Sigma^n \gamma >_n + < \alpha, \Sigma^n \beta', \Sigma^n \gamma >_n$$

if $n \geq 1$ or $\gamma = \Sigma \hat{\gamma}$, (3.2.30)

$$< \alpha + \alpha', \Sigma^n \beta, \Sigma^n \gamma >_n \quad \subset \quad < \alpha, \Sigma^n \beta, \Sigma^n \gamma >_n + < \alpha', \Sigma^n \beta, \Sigma^n \gamma >_n$$

if $n \geq 1$ or $\beta = \Sigma \hat{\beta}$ and $\gamma = \Sigma \hat{\gamma}$, (3.2.31)

Proof. See [Tod62], chapter 1. ∎

We also have the juggling formula:

Theorem 3.2.32. *When the Toda brackets in the following formula are defined, we have*

$$< \alpha, \Sigma^n \beta, \Sigma^n \gamma >_n \circ \Sigma^{n+1} \delta \quad \subset \quad < \alpha, \Sigma^n \beta, \Sigma^n(\gamma \circ \delta) >_n \qquad (3.2.33)$$

$$< \alpha, \Sigma^n \beta, \Sigma^n(\gamma \circ \delta) >_n \quad \subset \quad < \alpha, \Sigma^n(\beta \circ \gamma), \Sigma^n \delta >_n \qquad (3.2.34)$$

$$< \alpha \circ \Sigma^n \beta, \Sigma^n \gamma, \Sigma^n \delta >_n \quad \subset \quad < \alpha, \Sigma^n(\beta \circ \gamma), \Sigma^n \delta >_n \qquad (3.2.35)$$

$$\alpha \circ < \beta, \Sigma^n \beta, \Sigma^n \delta >_n \quad \subset \quad < \alpha \circ \beta, \Sigma^n \gamma, \Sigma^n \delta >_n \qquad (3.2.36)$$

Proof. See [Tod62], chapter 1. □

We also have

$$<< \alpha,\beta,\gamma>,\Sigma\delta,\Sigma\epsilon> + <\alpha,<\beta,\gamma,\delta>,\Sigma\epsilon> + <\alpha,\beta,<\gamma,\delta,\epsilon>> = 0 \tag{3.2.37}$$

In the stable range, there is commutation relations:

Theorem 3.2.38. *Let $\alpha\in\pi_{p+h}(S^p)$, $\beta\in\pi_{q+k}(S^q)$ and $\gamma\in\pi_{r+l}(S^r)$. Suppose $\alpha\wedge\beta=\beta\wedge\gamma=0$, then $<\Sigma^{q+r}\alpha,\Sigma^{p+h+r}\beta,\Sigma^{p+h+q+k}\gamma>$ and $(-1)^{hk+kl+lh+1}<\Sigma^{p+q}\gamma,\Sigma^{p+r+l}\beta,\Sigma^{q+k+r+l}\alpha>$ have a common element.*

Further assume $\alpha\wedge\gamma=0$, then $(-1)^{hl}<\Sigma^{q+r}\alpha,\Sigma^{p+h+r}\beta,\Sigma^{p+h+q+k}\gamma> + (-1)^{kh}<\Sigma^{p+r}\beta,\Sigma^{p+q+k}\gamma,\Sigma^{q+k+r+l}\alpha> + (-1)^{lk}<\Sigma^{p+q}\gamma,\Sigma^{q+r+l}\alpha,\Sigma^{p+h+r+l}\beta>$ contains 0.

Proof. See [Tod62], chapter 3. □

Suspension commutes with Toda bracket:

$$-\Sigma<\alpha,\Sigma^n\beta,\Sigma^n\gamma>_n \subset <\Sigma\alpha,\Sigma^{n+1}\beta,\Sigma^{n+1}\gamma>_n \tag{3.2.39}$$

The Hopf invariant of the Toda bracket can be calculated:

Theorem 3.2.40. *Suppose $n\ge 1$, then*

$$H(<\alpha,\Sigma^n\beta,\Sigma^n\gamma>_n) \subset <H(\alpha),\Sigma^n\beta,\Sigma^n\gamma>_n \tag{3.2.41}$$

Proof. See [Tod62], chapter 2. □

If all three variables are suspensions, we can be more precise:

Theorem 3.2.42. *Suppose $\Sigma(\alpha\circ\beta)=\beta\circ\gamma=0$, then*

$$H(<\Sigma\alpha,\Sigma\beta,\Sigma\gamma>_1) = -\Delta^{-1}(\alpha\circ\beta)\circ\Sigma^2\gamma \tag{3.2.43}$$

Proof. See [Tod62], chapter 2. □

As an example, we calculate the Toda bracket $<2,\eta,2>$ where η is the generator of $\pi_1(\mathbb{S})$.

Theorem 3.2.44. *The set $<2,\eta,2>$ contains a single element η^2.*

Proof. We know $\pi_2(\mathbb{S})\cong\mathbb{Z}/2$ generated by η^2. Suppose on the contrary $<2\iota_n,\eta_n,2\iota_{n+1}>$ contains 0. Let $K=S^n\cup_{\eta_n}D^{n+2}$ and $\alpha\in[K,S^n]$ be an extension of $2\iota_n$ and $\beta\in[S^{n+2},K]$ be a coextension of $2\iota_{n+1}$. Since $0\in<2\iota_n,\eta_n,2\iota_{n+1}>$, we can choose α,β so that $\alpha\circ\beta=0$. Let $L=S^n\cup_\alpha\mathcal{C}K$. Then there is a coextension $\tilde\beta\in[S^{n+3},L]$ of β. Define $M=L\cup_{\tilde\beta}\mathcal{C}S^{n+3}$. K has two cells, and we have $Sq^2(x_n)=x_{n+2}$ where x_n and x_{n+2} are the generators in $H^*(K,\mathbb{F}_2)$. L has three cells. The two lower dimensional cells is $S^n\cup_{2\iota_n}D^{n+1}$ so $Sq^1(y_n)=y_{n+1}$. When the bottom cell is collapsed we get ΣK so $Sq^2(y_{n+1})=y_{n+3}$. Here y_n,y_{n+1},y_{n+3} are the generators in $H^*(L,\mathbb{F}_2)$. Similarly, in M,

$Sq^1(z_n) = z_{n+1}$, $Sq^2(z_{n+1}) = z_{n+3}$ and $Sq^1(z_{n+3}) = z_{n+4}$ for the generators $z_n, z_{n+1}, z_{n+3}, z_{n+4}$ of $H^*(M, \mathbb{F}_2)$. But $Sq^1Sq^2Sq^1 = Sq^3Sq^1 = Sq^2Sq^2$ by the Adem relations. We have $Sq^2(z_n) = 0$ because there is no $(n+2)$-cell. But $Sq^2Sq^2(z_n) = Sq^2Sq^2Sq^1(z_n) = z_{n+4}$, contradiction. □

Remark 3.2.45. *Using the decomposition $S^{n+2} \xrightarrow{\beta} K \xrightarrow{\alpha} S^n$ of $< 2\iota_n, \eta_n, 2\iota_{n+1} >$, one can calculate directly to show that this represents the element h_1^2 in the Adams spectral sequence, so $< 2, \eta, 2 > = \eta^2$.*

In general, the Toda brackets of the form $< \alpha, \beta, \alpha >$ often have the form $\beta \circ \alpha^*$ with α^* depending only on α.

Theorem 3.2.46. *Let $\alpha \in \pi_{n+k}(S^n)$ and $h \geq 0$ satisfy*

$$(1 - (-1)^k)\Sigma^{n-h}\alpha \circ \Sigma^{n+k-h}\alpha = 0 \tag{3.2.47}$$

Assume $k \leq 2n - 2$. Then there exists $\alpha^* \in \pi_{2n+2k+1}(S^{2n})$ such that for any $\beta \in \pi_{n+t}(S^m)$ with $\beta \circ \Sigma^t\alpha = 0$,

$$\begin{aligned}\Sigma^n\beta \circ \Sigma^t\alpha^* \quad \in \quad & (-1)^{km+kt+t} < \Sigma^m\alpha, \Sigma^{n+k}\beta, \Sigma^{n+k+t}\alpha >_{n+k} \\ & +(-1)^{kn+h+t+1} < \Sigma^n\beta, \Sigma^{n+t}\alpha, (1-(-1)^k)\Sigma^{n+k+t}\alpha >_{h+t}\end{aligned} \tag{3.2.48}$$

Further if $(1 - (-1)^k)\Sigma^{n+k+1}\alpha = 0$ then

$$\Sigma^n\beta \circ \Sigma^t\alpha^* \in (-1)^{km+kt+t} < \Sigma^m\alpha, \Sigma^{n+k}\beta, \Sigma^{n+k+t}\alpha >_{n+k} \tag{3.2.49}$$

Proof. See [Tod62], chapter 3. □

In particular, from theorem 3.2.44, it follows that the α^* associated with 2ι is η. So we have:

Theorem 3.2.50. *Let $\beta \in \pi_{m+k}(S^m)$ with $k > 0$. Assume $2\beta = 0$, then*

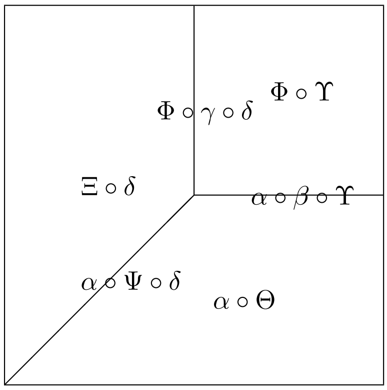
$$\Sigma\beta \circ \eta_{m+k+1} \in < 2\iota_m, \Sigma\beta, 2\iota_{m+k+1} >_1 \tag{3.2.51}$$

There is also higher Toda brackets with more than three variables. For example, let $\alpha \in [Y, Z]$, $\beta \in [X, Y]$, $\gamma \in [W, X]$, $\delta \in [V, W]$ satisfy $\alpha \circ \beta = \beta \circ \gamma = \gamma \circ \delta = 0$, $< \alpha, \beta, \delta >$ and $< \beta, \gamma, \delta >$ both contain 0. Assume further that we can choose homotopies Φ from $\alpha \circ \beta$ to 0, Ψ from $\beta \circ \gamma$ to 0, Υ from $\gamma \circ \delta$ to 0, such that the elements κ, ρ defined by them in $< \alpha, \beta, \delta >$ and $< \beta, \gamma, \delta >$ are homotopic to 0. Choose homotopies Ξ from κ to 0 and Θ from ρ to 0. Then define the bracket $< \alpha, \beta, \gamma, \delta >$ to be the class defined by the maps $\Lambda : \Sigma^2 V \to Z$, defined by

$$\Lambda(v, s, t) = \begin{cases} \Xi(\delta(v), 1-2s, t) & if \quad 0 \leq s \leq \frac{1}{2}, 0 \leq t \leq \frac{1}{2} \\ \Xi(\delta(v), 1 - \frac{s}{1-t}, t) & if \quad 0 \leq s \leq 1-t, \frac{1}{2} \leq t \leq 1 \\ \Phi(\Upsilon(v, 2s-1), 1-2t) & if \quad \frac{1}{2} \leq s \leq 1, 0 \leq t \leq \frac{1}{2} \\ \alpha(\Theta(v, s, 1 - \frac{1-t}{s})) & if \quad 1-t \leq s \leq \frac{1}{2}, \frac{1}{2} \leq t \leq 1 \\ \alpha(\Theta(v, s, 2t-1)) & if \quad \frac{1}{2} \leq s \leq 1, \frac{1}{2} \leq t \leq 1 \end{cases} \tag{3.2.52}$$

when the homotopies run over all choices.

This is sketched in the following picture:



3.2.4 Calculation with the EHP-sequence

We give some examples of calculations of the 2-component of the homotopies groups of spheres using the EHP-sequences and homotopy operations. The detailed calculation of $\pi_{n+k}(S^n)$ for $k \leq 19$ can be found in [Tod62], and more calculations can be found in [MT63, Mim65, MMO74]. Their results are summarized in appendix A.

To begin with, we first give the structure of the homotopy groups implied by the existence of the division algebras.

Theorem 3.2.53. *Let $n = 2, 4$ or 8, and $\mathfrak{h} : S^{2n-1} \to S^n$ the map with Hopf invariant one. Then*

$$\Sigma + \mathfrak{h}_* : \pi_{i-1}(S^{n-1}) \oplus \pi_i(S^{2n-1}) \to \pi_i(S^n) \tag{3.2.54}$$

is an isomorphism for all i.

Proof. The case $n = 2$ is obvious. So assume $n = 4, 8$ in the following.

Define a map $\mathfrak{j} : S^{n-1} \times \Omega S^{2n-1} \to \Omega S^n$ by the formula $\mathfrak{j}(x) = \mathscr{E}(x) \cdot \Omega\mathfrak{h}(x)$, the loop-multiplication in ΩS^n of the inclusion $\mathscr{E} : S^{n-1} \to \Omega S^n$ and the map $\Omega\mathfrak{h}$. The cohomology ring of the loop space of spheres is calculated in 3.2.1. The fact that $\mathfrak{h}$ has Hopf invariant one means it pulls back the generator of $H^{2n-2}(\Omega S^n, \mathbb{Z})$ to the generator of $H^{2n-2}(\Omega S^{2n-1}, \mathbb{Z})$. It follows that the map $\mathfrak{j}$ is a homotopy equivalence. □

Remark 3.2.55. *One can also use the long exact sequence*

$$\ldots \to \pi_i(S^{n-1}) \to \pi_i(S^{2n-1}) \to \pi_i(S^n) \xrightarrow{\partial} \pi_{i-1}(S^{n-1}) \to \ldots$$

from the fibration $S^{n-1} \to S^{2n-1} \xrightarrow{\mathfrak{h}} S^n$, and show directly that the map ∂ is a left inverse to the suspension.

Now we can give the calculation of $\pi_{n+k}(S^n)$ for $k \le 3$.

When $k = 1$, we know from the preceding theorem that $\pi_3(S^2) \cong \mathbb{Z}$ generated by the Hopf map η_2, and the Hopf invariant is an isomorphism. Next use the EHP-sequence $\pi_5(S^5) \xrightarrow{\Delta} \pi_3(S^2) \xrightarrow{\Sigma} \pi_4(S^3) \to 0$. Since $H(\Delta(\iota_5)) = 2\iota_3$ by 3.2.26, $\pi_4(S^3) \cong \mathbb{Z}/2$ generated by η_3 the suspension of η_2. Thus:

Theorem 3.2.56. *$\pi_3(S^2) \cong \mathbb{Z}$, $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \ge 4$.*

The following lemma will be useful:

Theorem 3.2.57. *Let $\alpha \in \pi_i(S^3)$ such that $2\alpha = 0$, so $< \eta_3, 2\iota_4, \Sigma\alpha >_1$ is defined. Let $\beta \in < \eta_3, 2\iota_4, \Sigma\alpha >_1$. Then $H(\beta) = \Sigma^2\alpha$ and $2\beta = \eta_3 \circ \Sigma\alpha \circ \eta_{i+1}$.*

Proof. By 3.2.43, $H(\beta) \in \Delta^{-1}(\eta_2 \circ 2\iota_3) \circ \Sigma^2\alpha = \Delta^{-1}(2\eta_2) \circ \Sigma^2\alpha = \pm\iota_5 \circ \Sigma^2\alpha = \Sigma^2\alpha$.

By 3.2.32 and 3.2.39, $2\beta \in < \eta_3, 2\iota_4, \Sigma\alpha >_1 \circ 2\iota_{i+2} = \eta_3 \circ \Sigma < 2\iota_3, \alpha, 2\iota_i > \subset \eta_3 \circ - < 2\iota_4, \Sigma\alpha, 2\iota_{i+1} >_1$, and this contains $\eta_3 \circ -(\Sigma\alpha \circ \eta_{i+1}) = \eta_3 \circ \Sigma\alpha \circ \eta_{i+1}$ by 3.2.50.

A calculation of the indeterminacy of the bracket shows in fact $2\beta = \eta_3 \circ \Sigma\alpha \circ \eta_{i+1}$. ∎

For $k = 2$, by 3.2.53, $\pi_4(S^2) \cong \mathbb{Z}/2$ generated by $\eta_2^2 = \eta_2 \circ \eta_3$. Next use the EHP-sequence

$$\pi_6(S^3) \xrightarrow{H} \pi_6(S^5) \xrightarrow{\Delta} \pi_4(S^2) \xrightarrow{\Sigma} \pi_5(S^3) \xrightarrow{H} \pi_5(S^5) \xrightarrow{\Delta} \pi_3(S^2)$$

We know $\Delta : \pi_5(S^5) \to \pi_3(S^2)$ is injective. So $\Sigma : \pi_4(S^2) \to \pi_5(S^3)$ is surjective.

Let $\nu' \in < \eta_3, 2\iota_4, \eta_4 >$, then by 3.2.57, $H(\nu') = \eta_5$ and $2\nu' = \eta_3^3$.

This shows $H : \pi_6(S^3) \to \pi_6(S^5)$ is surjective, so $\Sigma : \pi_4(S^2) \to \pi_5(S^3)$ is an isomorphism. By 3.2.53, $\pi_6(S^4) \cong \pi_5(S^3) \oplus \pi_6(S^7)$. Hence:

Theorem 3.2.58. *$\pi_{n+2}(S^n) \cong \mathbb{Z}/2$ generated by η_n^2.*

Now we calculate the case $k = 3$. As before, $\pi_5(S^2) \cong \mathbb{Z}/2$ generated by η_2^3. Next use the sequence $\pi_7(S^3) \xrightarrow{H} \pi_7(S^5) \xrightarrow{\Delta} \pi_5(S^2) \xrightarrow{\Sigma} \pi_6(S^3) \xrightarrow{H} \pi_6(S^5)$. We have $H(\nu') = \eta_5$ and $2\nu' = \Sigma\eta_2^3$. We also have $H(\nu' \circ \eta_6) = H(\nu') \circ \eta_6 = \eta_5^2$ by 3.2.9. So $H : \pi_7(S^3) \to \pi_7(S^5)$ is surjective. So $\pi_6(S^3) \cong \mathbb{Z}/4$ generated by ν'.

By 3.2.53, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ generated by ν_4 and $\Sigma\nu'$ where ν_4 is the map with Hopf invariant one.

The next step is a little elaborated. But since we have already calculated the stable group $\pi_{n+3}(S^n)$ for $n \ge 5$ in 2.6, there is a shortcut. By the EHP-sequence, $\pi_8(S^5) \cong \pi_7(S^4)/\Delta(\pi_9(S^9))$. Since $\Delta(\iota_9)$ has Hopf invariant 2 by 3.2.26, $\Delta(\iota_9) = \pm 2\nu_4 + k\Sigma\nu'$ for some k. And the only way to obtain the expected group $\mathbb{Z}/8$ is $\Delta(\iota_9) = \pm 2\nu_4 \pm \Sigma\nu'$. By suitable choice of ν', we may assume $\Delta(\iota_9) = \pm(2\nu_4 - \Sigma\nu')$ so that $2\Sigma\nu_4 = \Sigma^2\nu'$.

Theorem 3.2.59. *$\pi_5(S^2) \cong \mathbb{Z}/2$, $\pi_6(S^3) \cong \mathbb{Z}/4$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/4$, $\pi_{n+3}(S^n) \cong \mathbb{Z}/8$ for $n \ge 5$.*

3.2.5 The Hopf invariant one problem for $n = 16$

Here we show how the composition methods can be used to prove the nonexistence of a map from S^{31} to S^{16}.

First we need the result of the groups $\pi_{n+7}(S^n)$.

Theorem 3.2.60. *$\pi_{13}(S^6) \cong \mathbb{Z}/4$ generated by σ'' with $H(\sigma'') = \eta_{11}^2$. $\pi_{14}(S^7) \cong \mathbb{Z}/8$ generated by σ' with $H(\sigma') = \eta_{13}$, and $2\sigma' = \Sigma\sigma''$. $\pi_{15}(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ generated by σ_8 and $\Sigma\sigma'$, where σ_8 has Hopf invariant one. $\pi_{n+7}(S^n) \cong \mathbb{Z}/16$ generated by σ_n for $n \geq 9$. $\Delta(\iota_{17}) = \pm(2\sigma_8 - \Sigma\sigma')$ and $2\Sigma\sigma_8 = \Sigma^2\sigma'$.*

Proof. See [Tod62], chapter 5. □

We also need the result of $\pi_{n+k}(S^n)$ for $k \leq 9$, which is collected in appendix A. These calculations are also obtainable by the cohomological methods in section 3.1.

Using the sequence $\pi_{31}(S^{16}) \xrightarrow{H} \pi_{31}(S^{31}) \xrightarrow{\Delta} \pi_{29}(S^{15})$ we see that to prove the nonexistence of a map in $\pi_{31}(S^{16})$, it suffices to prove that $\Delta(\iota_{31}) \neq 0$ in $\pi_{29}(S^{15})$.

Using theorem 3.2.24, $\Delta(\iota_{31}) = \Delta(\Sigma^{16}H(\sigma_8) \circ \Sigma^{16}H(\sigma_6)) = \Sigma^7\sigma_8 \circ \Sigma^{14}\sigma_8 + \Sigma^7\sigma_8 \circ \Sigma^{14} \circ \sigma_8 = 2\sigma_{15}^2$, so it suffices to show $2\sigma_{15}^2 \neq 0$.

We begin with $\pi_{22}(S^8)$, which by theorem 3.2.53, equals $\mathbb{Z}/16\{\sigma_8^2\} \oplus \Sigma\pi_{21}(S^7)$. So $2\sigma_8^2 \neq 0$. We have to prove this element suspends to a nonzero one in $\pi_{29}(S^{15})$. So we calculate the kernel of Σ, which equals the image of Δ by the EHP-sequence.

$\Delta(\pi_{24}(S^{17}))$ is generated by $\Delta(\sigma_{17})$. By 3.2.12, $\Delta(\sigma_{17}) = \Delta(\iota_{17}) \circ \sigma_{15} = \pm(2\sigma_8 - \Sigma\sigma') \circ \sigma_{15} = \pm(2\sigma_8^2 - \Sigma(\sigma' \circ \sigma_{14}))$. We will show $\sigma' \circ \sigma_{14} \neq 0$ in $\pi_{21}(S^7)$ so $2\sigma_9^2 \neq 0$. In fact, $H(\sigma' \circ \sigma_{14}) = H(\sigma') \circ \sigma_{14} = \eta_{13} \circ \sigma_{14}$. This element is nonzero, and equals $\bar{\nu}_{13} + \epsilon_{13}$ in the table of appendix A.

Next $\Delta(\pi_{25}(S^{19}))$ is generated by $\Delta(\nu_{19}^2) = \Delta(\iota_{19}) \circ \nu_{17}^2$ using 3.2.12. $\Delta(\iota_{19})$ cannot vanish because there is no map with Hopf invariant one in $\pi_{19}(S^{10})$. In fact $\Delta(\iota_{19}) = \sigma_9 \circ \eta_{16} + \bar{\nu}_9 + \epsilon_9$, and $\Delta(\nu_{19}^2) = \bar{\nu}_9 \circ \nu_{17}^2$. We have to prove that $\bar{\nu}_7 \circ \nu_{15}^2 \neq \sigma' \circ \sigma_{14}$ in $\pi_{21}(S^7)$. This is the case since $\bar{\nu}_7 \circ \nu_{15}^2$ is a suspension so have Hopf invariant 0. Thus $2\sigma_{10}^2 \neq 0$.

The two groups $\pi_{26}(S^{21})$ and $\pi_{27}(S^{23})$ vanish so that $2\sigma_{12}^2 \neq 0$.

Now $\Delta(\pi_{28}(S^{25}))$ is generated by $\Delta(\nu_{25}) = \Delta(\iota_{25}) \circ \nu_{23}$ using 3.2.12. Since $H(\Delta(\iota_{25}) \circ \nu_{23}) = 2\nu_{23}$ using 3.2.9 and 3.2.26, $\Delta(\nu_{25})$ and $2\Delta(\nu_{25})$ are not suspensions. $\Delta(4\nu_{25}) = \Delta(\eta_{25}^3)$. By 3.2.24, $\Delta(\eta_{25}^3) = \Delta(\Sigma^{14}H(\sigma'') \circ \Sigma^{14}H(\sigma')) = \Sigma^6\sigma'' \circ \Sigma^{12}\sigma' + \Sigma^5\sigma' \circ \Sigma^{13}\sigma'' = 16\sigma_{12}^2 = 0$. So the image of Δ does not contain common element with the image of Σ in $\pi_{26}(S^{12})$. And we have $2\sigma_{13}^2 \neq 0$.

$\Delta(\pi_{29}(S^{27}))$ is generated by $\Delta(\eta_{27}^2)$. Using 3.2.24 one proves $\Delta(\eta_{27}^2) = 8\sigma_{13}^2$, so $2\sigma_{14}^2 \neq 0$.

Similarly, $\Delta(\pi_{30}(S^{29}))$ is generated by $\Delta(\eta_{29}) = 4\sigma_{14}^2$, and we have finally proved $2\sigma_{15}^2 \neq 0$.

Theorem 3.2.61. *There is no map in $\pi_{31}(S^{16})$ with Hopf invariant one.*

The general theorem that the only maps with Hopf invariant one are those induced by $\mathbb{C}$,$\mathbb{H}$,$\mathbb{O}$, is proved by using secondary cohomology operations or using K-theory, see [Ada60, AA66].

Chapter 4

The Adams Spectral Sequence

In this chapter, a basic reference is [Rav86]. To make Serre's method more clear, we use the language of spectra.

4.1 Stable homotopy theory

We first state Brown's representability theorem.

Theorem 4.1.1. *Let H be a contravariant functor H on the homotopy category of based CW complexes to the category of Abelian groups satisfies wedge axiom and MV axiom. Then H is representable by a CW complex Y, that is, there is an isomorphism between $H(\cdot)$ and $[\cdot, Y]$ for any finite CW complex.*

Proof. See [Bro62] □

As a corollary, we have the following

Theorem 4.1.2. *Any generalised cohomology theory $\widetilde{E^*}$ is represented by $\{E_i\}_{i\in\mathbb{Z}}$, such that $\widetilde{E^*}(X) = [X, E_i]$ and the maps $E_i \to \Omega E_{i+1}$ are weak equivalences.*

For example, for ordinary cohomology theories, we have natural isomorphisms $\widetilde{H}^n(X;\pi) \cong [X, K(\pi, n)]$ for any CW complex X, integer $n > 0$ and Abelian group π. To better understand relations between these E_i, we introduce the notation of prespectra.

Definition 4.1.3. *A prespectrum is a sequence of based spaces $E = \{E_i\}_{i\geq 0}$, and based maps $\sigma : \Sigma E_i \to E_{i+1}$ as structure maps. A map $f : E \to F$ of prespectra E and F is a sequence of based maps $f_i : E_i \to F_i$ commuting with structure maps. Hence we get a category of prespectra, denoted by $\mathfrak{PreSp}$.*

As shown in [May99], we can associate a generalized homology theory for a given prespectrum under certain connectness conditions. To discuss more properly with respect to the stability and generalized cohomology theories, we introduce the notation of spectra.

Definition 4.1.4. *A spectrum is a prespectrum such that the adjoints $E_i \to \Omega E_{i+1}$ to the structure maps are homeomorphisms. The category of spectra, denoted by $\mathfrak{Sp}$, is defined as a full subcategory of $\mathfrak{PreSp}$.*

Just like we can associate a sheaf to a presheaf, the inclusion functor $\mathfrak{PreSp} \to \mathfrak{Sp}$ has a left adjoint named spectrification, denoted by L, which can be shown by Freyd's adjoint functor theorem.

Definition 4.1.5. *The sphere spectrum, denoted by $\mathbb{S}$, is defined to be the spectrification of the suspension prespectrum S^i with obvious structure maps.*

Remark 4.1.6. *A similar statement of Brown's representability theorem holds for spectra.*

As discussed in [Ada74], we can properly define generalized homology, cohomology and homotopy groups of a spectrum. For instance, denote $HZ/(p)$ the mod (p) Eilenberg-Mac Lane spectrum, then $HZ/(p)^*(HZ/(p))$ is just the mod (p) Steenrod algebra A_p.

Remark 4.1.7. *If we define the fibrations (resp. weak equivalences) to be the maps $f : E \to F$ such that $f_i : E_i \to F_i$ are fibrations (resp. weak equivalences) and the cofibrations to be precisely those maps which have the LLP with respect to the acyclic fibrations, then the category $\mathfrak{Sp}$ becomes a closed model category.*

Remark 4.1.8. *If we defined suspension to be the translation functor, and distinguished triangles to be the cofibration sequences, then the homotopy category of $\mathfrak{Sp}$, which is also called the stable homotopy category, becomes a triangulated category.*

As in the category of based topological spaces, smash product is also an important operation in category $\mathfrak{Sp}$.

Theorem 4.1.9. *There is a functor $\bigwedge : \mathfrak{Sp} \times \mathfrak{Sp} \to \mathfrak{Sp}$ named smash product. Smash product is associative, commutative (up to homotopy), and has the sphere spectrum $\mathbb{S}$ as a unit, up to coherent natural equivalences.*

Proof. For detailed constructions and proof, see [Ada74]. □

4.2 Construction of the Adams spectral sequence

Definition 4.2.1. *A spectrum E is called connective, if $\pi_i(E) = 0$ for all $i < n$ for some n.*

Definition 4.2.2. *A mod (p) Adams resolution (X_s, f_s) for a spectrum X is defined to be the diagram:*

$$\begin{tikzcd} X_0 \arrow[d, "g_0"] & X_1 \arrow[l, "f_0"'] \arrow[d, "g_1"] & X_2 \arrow[l, "f_1"'] \arrow[d, "g_2"] & \cdots \arrow[l, "f_2"'] \\ Y_0 & Y_1 & Y_2 & \end{tikzcd}$$

where $X = X_0$, $Y_s = X_s \wedge HZ/(p)$,and $H^*(g_s)$ is onto.

Theorem 4.2.3. *Let X be a connective spectrum of finite type, then there is a spectral sequence*

$$(E_*^{*,*}(X), d_r : E_r^{s,t} \to E_r^{s+r,t+r-1})$$

such that

$$E_2^{s,t} = Ext_{A_p}^{s,t}(H^*(X, \mathbb{F}_p)), \mathbb{F}_p) \Rightarrow \pi_*(X) \otimes \mathbb{Z}_{(p)}$$

is conditionally convergent in the colimit sense (in the sense of [Boa99]).

Proof. We sketch the proof. As in Serre's methods, we can show that for a given spectrum X as above, there exist a mod (p) Adams resolution (X_s, f_s) such that each sequence

$$X_{s+1} \xrightarrow{f_s} X_s \xrightarrow{g_s} Y_s \longrightarrow \sum X_{s+1}$$

is a fiber sequence. Then we get a short exact sequence:

$$0 \longrightarrow H^*(\Sigma X_{s+1}, \mathbb{F}_p) \longrightarrow H^*(Y_s, \mathbb{F}_p) \xrightarrow{g_s} H^*(X_s, \mathbb{F}_p) \longrightarrow 0.$$

Since each $Y_s = X_s \wedge HZ/(p)$ is a wedge of suspensions of $HZ/(p)$ and $HZ/(p)^*(HZ/(p) \wedge X_s)$ is free as a module over the Steenrod algebra A_p, we obtain a free A_p-resolution of $H^*(X, \mathbb{F}_p)$:

$$\cdots \longrightarrow H^*(\Sigma^2 Y_2, \mathbb{F}_p) \longrightarrow H^*(\Sigma Y_1, \mathbb{F}_p) \longrightarrow H^*(Y_0, \mathbb{F}_p) \longrightarrow H^*(X, Z/(p)) \longrightarrow 0.$$

Also from the fiber sequence, we get a long exact sequence of homotopy groups, and by setting $D_1^{s,t} = \pi_{t-s}(X_s), E_1^{s,t} = \pi_{t-s}(Y_s)$, an exact couple is obtained:

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

where $i_1 = \pi_{t-s}(f_s), j_1 = \pi_{t-s}(g_s)$, and k_1 is a boundary map. This leads to a spectral sequence. Let $X_\infty = holim(X_s)$, we get an exact sequence:

$$0 \longrightarrow \varprojlim \pi_*(X_\infty) \longrightarrow \pi_*(X_\infty) \longrightarrow \varprojlim{}^1 \pi_*(X_\infty) \longrightarrow 0.$$

Define Z_s such that there is a fiber sequence

$$X_\infty \longrightarrow X_s \longrightarrow Z_s \longrightarrow \sum X_\infty.$$

Then use the octahedral axiom of the stable homotopy category as a triangulated category, this spectral sequence converge to $\pi_*(Z_0)$, that is, $\pi_*(X) \otimes \mathbb{Z}_{(p)}$. $\square$

Generally, we have the following.

Theorem 4.2.4. *Let X be a connective spectrum of finite type, then there is a spectral sequence*

$$(E_*^{*,*}(X), d_r : E_r^{s,t} \to E_r^{s+r,t+r-1})$$

such that

$$E_2^{s,t} = Ext_{A_p}^{s,t}(H^*(X, \mathbb{F}_p)), H^*(Y, \mathbb{F}_p)) \Rightarrow [Y, X \wedge \widehat{\mathbb{S}_{(p)}}]$$

is conditionally convergent in the colimit sense.

Definition 4.2.5. *A map $f : X \to Y$ has Adams filtration $\geq s$, if it can be factorized as*

$$X \xrightarrow{f_1} W_1 \xrightarrow{f_2} W_2 \longrightarrow \cdots \longrightarrow W_{s-1} \xrightarrow{f_s} Y.$$

such that $HZ/(p)_*(f_i) = 0$ for all i.

4.3 Properties of the E_2-term of the Adams spectral sequence

In this section, we state some results of the E_2-term of the Adams spectral sequence of prime 2. For primes $p > 2$, the results are similar and can be found in [Rav86].

Theorem 4.3.1. *1.* $Hom_{A_2}(\mathbb{F}_2, \sum^t \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$

2. $Ext^1_{A_2}(\mathbb{F}_2, \sum^t \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } t = 2^i \text{for some } i \\ 0 & \text{else} \end{cases}$

Proof. Compute directly. $\square$

Denote $h_i \neq 0 \in Ext^1_{A_2}(\mathbb{F}_2, \sum^{2^i} \mathbb{F}_2)$.

Theorem 4.3.2. *1. $Ext^s_{A_2}(\mathbb{F}_2, \sum^t \mathbb{F}_2) = 0$ for (s, t) such that $t - s < 0$.*

2. $Ext^s_{A_2}(\mathbb{F}_2, \sum^s \mathbb{F}_2) = \mathbb{F}_2$ generated by ${h_0}^s$.

3. $Ext^s_{A_2}(\mathbb{F}_2, \sum^t \mathbb{F}_2) = 0$ for all (s, t) such that $0 < s < t < U(s)$, where $U(s)$ is the following numerical function: $U(4s) = 12s - 1$,$U(4s + 1) = 12s + 2$,$U(4s + 2) = 12s + 4$,$U(4s + 3) = 12s + 6$.

Proof. See [Ada66b]. □

We call this theorem Adams edge theorem.Roughly speaking, we say there is a vanishing line.

Theorem 4.3.3. *$Ext^2_{A_2}(\mathbb{F}_2, \sum^* \mathbb{F}_2) = 0$ is generated by h_ih_j satisfying the relations $h_ih_j = h_jh_i$,$h_ih_{i+1} = 0$.*

Remark 4.3.4. *$\langle h_0, h_1, h_0\rangle = h_1^2$,$\langle h_1, h_0, h_1\rangle = h_0h_2$.*

Theorem 4.3.5. *$Ext^3_{A_2}(\mathbb{F}_2, \sum^* \mathbb{F}_2) = 0$ is generated by $h_ih_jh_k$ and c_i satisfying the relations $h_ih_{i+2}^2 = 0$,$h_i^2h_{i+2} = h_{i+1}^2$,$c_i = \langle h_{i+1}, h_i, h_{i+2}^2\rangle$.*

Remark 4.3.6. *$d_2h_i = h_{i-1}h_0^2$.*

Remark 4.3.7. *$h_0^4h_3 = 0$. If $x \in E_2^{s,t}$ such that h_0^4x is above the vanishing line, then we get the Massey product $\langle x, h_0^4, h_3\rangle$.*

Theorem 4.3.8. *When $(s + 4, t + 4)$ is above the vanishing line,*

$$\langle \bullet, h_0^4, h_3\rangle : Ext^s_{A_2}(\mathbb{F}_2, \sum^t \mathbb{F}_2) \to Ext^{s+4}_{A_2}(\mathbb{F}_2, \sum^{t+12} \mathbb{F}_2)$$

is an isomorphism.

Proof. See [Ada66b]. □

Remark 4.3.9. *$h_0^{2^n}h_{n+1} = 0$, for $n > 1$. Similarly, near the vanishing line, we have $\langle \bullet, h_0^{2^n}, h_{n+1}\rangle$ as an isomorphism. This is a line of slope $\frac{1}{5}$.*

4.4 The May spectral sequence

In this section, citing a differential in 24-stem, we follow Tangora's method to compute the E_∞ term of the May spectral sequence up to 29-stem at prime 2. He actually computed through dimension 70 in [Tan70a].

Theorem 4.4.1. *There exists a spectral sequence $(E_r^{u,v,t}, d_r)$, called the May spectral sequence, converging to the E_2 term of the Adams spectral sequence, where each d_r is a homomorphism of the tri-graded algebra: $d_r : E_r^{u,v,t} \to E_r^{u+r,v-r+1,t}$ as a derivation with respect to the algebra structure. The E_2 term of this spectral sequence is generated by the following generators in the range $t - s \leq 31$ and $s < 3$ in Table 4.1 below subject to the relations(at least) in Table 4.2 below.*

Proof. See [May65a, May65b, May66]. □

Remark 4.4.2. *In the tri-graded algebra $E_2^{u,v,t}$, u is the filtration degree, v is the complementary degree, t is associated with the degree in the Steenrod algebra, and homological degree $s = u + v$, which is the same as in Adams Spectral Sequences.*

Table 4.1: Generators of E_2 for $t-s \leq 31$

$t-s$	s	Name	u	v	t
0	1	h_0	0	1	1
1	1	h_1	0	1	2
3	1	h_2	0	1	4
4	2	b_{02}	-2	4	6
7	1	h_3	0	1	8
7	2	$h_0(1)$	-2	4	9
10	2	b_{12}	-2	4	12
12	2	b_{03}	-4	6	14
15	1	h_4	0	1	16
16	2	$h_1(1)$	-2	4	18
22	2	b_{22}	-2	4	24
26	2	b_{13}	-4	6	28
28	2	b_{04}	-6	8	30
31	1	h_5	0	1	32

Table 4.2: Relations in E_2 for $t-s \leq 31$

$h_i h_{i+1} = 0 \ (i \geq 0)$
$h_2 b_{02} = h_0 h_0(1)$
$h_0 b_{12} = h_2 h_0(1)$
$h_0(1)^2 = b_{02} b_{12} + h_1^2 b_{03}$
$h_3 h_0(1) = 0$
$h_1 h_1(1) = h_3 b_{12}$
$h_3 h_1(1) = h_1 b_{22}$
$b_{02} h_1(1) = h_1 h_3 b_{03}$
$h_0(1) h_1(1) = 0$
$b_{02} b_{22} = h_0^2 b_{13} + h_3^2 b_{03}$
$b_{22} h_0(1) = h_0 h_2 b_{13}$

There are some quick observations from the theorem above.

Remark 4.4.3. *We have $d_r(h_i) = 0$ for any h_i $(i \geq 0)$ and d_r $(r \geq 2)$ and $d_r = 0$ for r odd by dimension reasons.*

Remark 4.4.4. *If $x \in E_2$ and $h_i x \neq 0$, then $h_i^n x \neq 0$ for any $n > 0$, at least in the range of $t - s \leq 31$. Any nonzero element in E_2 cannot be of the form $h_i y$ and $h_{i+1} z$ at the same time.*

Remark 4.4.5. *If $d_2 x = y$ in E_2 and that $h_i x$ and $h_i y$ are nonzero, then ${h_i}^n x$ and $h_i^n y$ are nonzero, and $d_2({h_i}^n x) = {h_i}^n y$. Therefore, all of these elements disappear in E_3.*

Since there are many generators, E_2 becomes large very soon. A basic technique offered in [Tan70a] to simplify this issue is to make use of the remark above. We call $d_2 x$ is h_i-stable and these elements involved is a h_i-ladder. Then when we compute E_2 term stem by stem, we throw out all these ladders, that is, quotient these acyclic complexes, to make later stems in E_2 smaller. Doing this will not change E_3 term, however, we need to be careful in later stems of E_2 since the d_2 of certain elements will not contain only one element. Details will be discussed as follows.

0-stem: We have h_0^s, $s \geq 0$.

1-stem: We have h_1. Since h_1 is a permanent cycle, h_0^n survive.

2-stem: We have h_1^2. By Leibnitz's rule, $d_i(h_1^2) = 0$. Therefore h_1 survives.

3-stem: We have h_1^3 and $h_0^{s-1} h_2$, $s \geq 1$. These are permanent cycles by direct computation. Therefore h_1^2 survives.

4-stem: We have h_1^4 and $h_0^{s-2} b_{02}$, $s \geq 2$. h_1^4 is a permanent cycle.

Theorem 4.4.6. $d_2(b_{02}) = h_1^3 + h_0^2 h_2$.

Proof. By dimension reasons, we can assume $d_2(b_{02}) = a h_1^3 + b h_0^2 h_2$, where $a, b = 0, 1$ to be decided. By Adams edge theorem, $h_0^s h_2$ cannot survive for large s. Since $h_0^s h_2$ is a cycle, it is also a boundary. Then $h_0^s h_2$ is in the image of $d_r(h_0^{s-2} b_{02})$ for some r since this is the only possible element in position. The filtration degree u of $h_0^s h_2$ and $h_0^{s-2} b_{02}$ are 0 and -2 respectively, hence we have $r = 2$. Then we have $d_2(h_0^{s-2} b_{02}) = h_0^{s-2} d_2(b_{02}) = h_0^s h_2$, so $b = 1$. Also by Adams edge theorem and "slope" reasons, h_1^s cannot survive. Then we have $h_1^s = d_r(h_1 x)$ for some x. (Otherwise, if $h_1^s = d_r(h_0 x)$, then $0 = d_r(h_1 h_0 x) = h_1 d_r(h_0 x) = h_1^{s+1}$, which is a contradiction.) The only possibility is that $d_2(h_1^{s_3} b_{02}) = h_1^s$. Therefore $a = 1$. ∎

Remark 4.4.7. *This conclusion can also be obtained by using the result $h_1^3 = h_0^2 h_2$ of Adams in the E_2 term of Adams spectral sequence or computing directly in bar complex.*

Notice that $d_2(h_i b_{02})$ is h_i-stable, $i = 0, 1$, then we can throw these two ladders away.

5-stem: Nothing is left after throwing out these two ladders above containing h_1^5, and $h_1 b_{02}$. Therefore, there is no survivors in 4-stem and 5-stem.

6-stem: We have h_2^2. For simplicity, we don't mention those factored already and h_0-multiplies any more.

7-stem: We have h_3 and $h_0(1)$. h_3 is a permanent cycle.

Theorem 4.4.8. $d_2(h_0(1)) = h_0h_2^2$.

Proof. One way to see this is to use the relation $h_2b_{02} = h_0h_0(1)$. Then $h_0d_2(h_0(1)) = d_2(h_0h_0(1)) = d_2(h_2b_{02}) = h_0^2h_2^2$. Therefore $d_2(h_0(1)) \neq 0$. Then the only possibility is that $d_2(h_0(1)) = h_0h_2^2$. Another way is using Adams edge theorem for $h_0^{s-2}h_2^2$ and filtration degree u for differential d_r to decide the equality. □

This differential is h_0-stable. Therefore only h_2^2 survives in 6-stem.

8-stem: We have h_1h_3, $h_1h_0(1)$ and b_{02}^2. h_1h_3 is a permanent cycle. $d_2(h_1h_0(1)) = h_1d_2(h_0(1)) = h_1h_0h_2^2 = 0$. Since the filtration degree u of $h_1h_0(1)$ is -2, we get $h_1h_0(1)$ is a permanent cycle. Denote $h_1h_0(1)$ by c_0. $d_2(b_{02}) = 0$.

Theorem 4.4.9. $d_4(b_{02}^2) = h_0^4h_3$.

Proof. For large n, $h_0^nh_3$ cannot survive, then we must have $d_r(b_{02}^2) = h_0^4h_3$. Since the filtration degree u of b_{02}^2 and $h_0^4h_3$ are -4 and 0 respectively, we get $r = 4$. □

This differential is h_0-stable. Therefore only $h_0^ih_2$, $0 \leq i \leq 3$ survives in 7-stem.

9-stem: We have h_2^3, $h_1^2h_3$, $h_1^2h_0(1)$ and h_2^3. These are all permanent cycles. Therefore the survivors of 8-stem is h_1h_3 and $c_0 = h_1h_0(1)$.

10-stem: We have $h_1^3h_3$, $h_1^2b_{02}^2$, $h_1^3h_0(1)$ and b_{12}. First three elements are permanent cycles.

11-stem: We have h_3b_{02}, $b_{02}h_0(1)$, $h_1^4h_3$, $h_1^3b_{02}^2$, $h_1^4h_0(1)$ and h_1b_{12}.

12-stem: We have b_{03}, h_2^4, b_{02}^3, $h_1h_3b_{02}$, $h_1b_{02}h_0(1)$, $h_1^5h_3$, $h_1^4b_{02}^2$, $h_1^5h_0(1)$ and h_2b_{12}.

Theorem 4.4.10. $d_2(b_{12}) = h_2^3 + h_1^2h_3$, $d_2(b_{03}) = h_1b_{12} + h_3b_{02}$.

Proof. For dimension reasons, we can assume that $d_2(b_{12}) = A_1h_2^3 + A_2h_1^2h_3$ and $d_2(b_{03}) = A_3h_1b_{12} + A_4h_3b_{02}$, where $A_i = 0, 1$, $i = 1, 2, 3, 4$. To kill $h_0^nh_2^3$, only possibility is $h_0^nb_{12}$, then we get $A_2 = 1$. To kill $h_0^nh_3b_{02}$, only possibility is $h_0^nb_{03}$, then we get $A_3 = 1$. $0 = d_2d_2(b_{03}) = d_2(h_3b_{02}) + A_4d_2(h_1b_{12}) = h_1^3h_3 + A_1A_4h_1^3h_3$, then we must have $A_1 = A_4 = 1$. □

$d_2(h_ib_{12}), d_2(h_ib_{03})$ are h_i-stable for $i = 0, 1$, so we can throw four ladders away. Therefore, in 9-stem, there are survivors $h_1^2h_3 = h_2^3$, $h_1^2h_0(1) = h_1c_0$ and $h_1b_{02}^2$.

Remark 4.4.11. *We denote $P^1h_1 = h_1b_{02}^2$, and generally $P^kx = b_{02}^{2k}x$, where P is short for periodicity, for reasons discussed in the last chapter.*

In 11-stem, $d_2(b_{02}h_0(1)) = b_{02}d_2(h_0(1))+d_2(b_{02})h_0(1) = b_{02}h_0h_2^2+h_0(1)(h_1^3+h_0^2h_2)$. Since $h_2b_{02} = h_0h_0(1)$, we have $d_2(b_{02}h_0(1)) = h_1^3h_0(1) = h_1^2c_0$. $d_2(h_2b_{02}^2) = 0$. This is h_1-stable. Since the filtration degree u of $h_1^2b_{02}^2$ and $h_2b_{02}^2$ are -4 and -4 respectively, $h_2b_{02}^2$ is a permanent cycle. Therefore the only survivor in 10-stem is $h_1^2b_{02}^2 = P^1h_1^2$.

In 12-stem, $d_2(b_{02}^3) = b_{02}^2d_2(b_{02}) = h_0^2h_2b_{02}^2 + h_1^3b_{02}^2$ and it is h_i-stable for $i = 0, 1$. Then we can throw two ladders away. Therefore, in 11-stem, there are survivors $h_0^{i+1}b_{02}h_0(1) = h_0^ih_2b_{02}^2 = P^1h_0^ih_2$ for $i = 0, 1, 2$, where $P^1h_0^2h_2 = P^1h_1^3$.

13-stem: We have only h_2b_{12}. $d_2(h_2b_{12}) = h_2d_2(b_{12}) = h_2(h_1^2h_3 + h_2^3) = h_2^4$. This is h_0-stable. We can throw this ladder away. Therefore, there are no survivors in 12-stem and 13-stem.

Remark 4.4.12. *It should be mentioned that this differential is also h_2-stable. We will also throw this h_2-ladder away, such as $d_2(h_2^2b_{12}) = h_2^5$ and their related h_0-ladder together.*

For now, the above process can be shown by the Table 4.3 below. We use very short lines to denote h_0 and h_1-stable derivations, see $d_2(b_{02}) = h_1^2 + h_0^2h_2$ as an example.

Therefore, we have the E_∞ term of the May spectral sequence in the range of $t - s \leq 13$ in Table 4.4 below.

We move on.

14-stem: We have h_3^2 and $b_{02}b_{12}$. (We have the relation $h_0(1)^2 = b_{02}b_{12} + h_1^2b_{03}$, so we don't need to consider $h_0(1)^2$ since $h_1^2b_{03}$ is already thrown away in the quotient complex.) Since 13-stem is already empty, these are all permanent cycles. It should be mentioned that $b_{02}b_{12}$ is not actually a cycle, but equivalent to a cycle $h_0(1)^2$ in the quotient complex. We denote $d_0 = b_{02}b_{12}$.

15-stem: We have h_4, h_2b_{03}, $h_3b_{02}^2$, $h_0(1)b_{02}^2$ and some h_1-multiplies. For simplicity, we do not mention these h_1-multiplies explicitly from now on, since they can be viewed directly in the table. $d_2(h_2b_{03}) = 0$ $d_2(h_0(1)b_{02}^2) = h_0h_2^2b_{02}^2 = h_0^3d_0$. This is h_0-stable. All these are d_2-cycles.

Remark 4.4.13. *Since $h_0^nh_3^2$ in 14-stem cannot be killed by d_2 as discussed above, by Adams edge theorem, it must be killed by d_r, $r \geq 4$. For this purpose, we need to discuss which d_2-cycles in 15-stem can survive in E_4 term. Therefore we move to 16-stem.*

16-stem: We haveb_{02}^4, $b_{03}b_{02}$ and $h_1(1)$.

Theorem 4.4.14. $d_2(h_1(1)) = h_1h_3^2$.

Proof. For dimension reasons, we can assume $d_2(h_1(1)) = A_1h_1h_3^2 + A_2h_0^2h_4 + A_3h_2b_{03}$, where $A_1, A_2, A_3 = 0, 1$. The filtration degree u of $d_2(h_1(1))$, $h_1h_3^2$, $h_0^2h_4$ and h_2b_{03} are $-2, 0, 0$ and -4 respectively. Then we get $A_3 = 0$. By relation $h_0h_1(1) = 0$, we get $0 = d_2(h_0h_1(1)) = A_2h_0^3h_4$, hence $A_2 = 0$. By relation $h_1h_1(1) = h_3b_{12}$, we get $A_1h_1^2h_3^2 = d_2(h_1h_1(1)) = h_1^2h_3^2$, hence $A_1 = 0$. ∎

Table 4.3: Quotient complex of E_2 in the range of $t-s \leq 13$

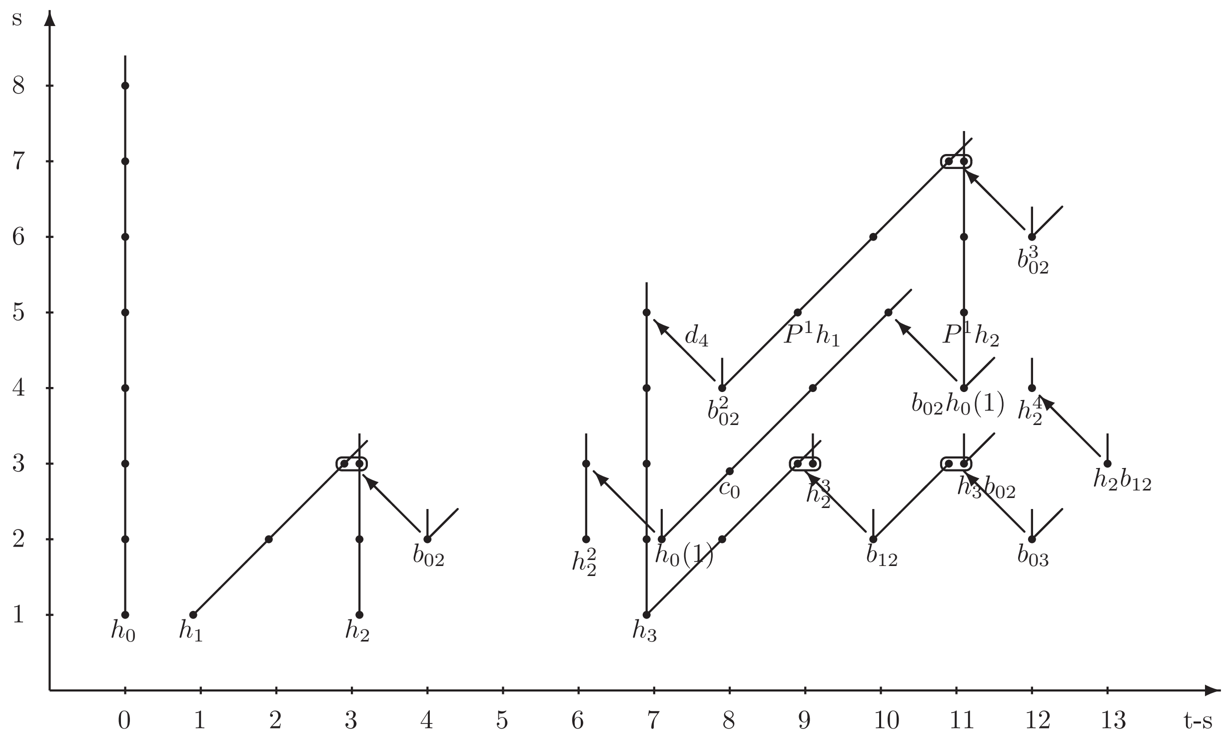
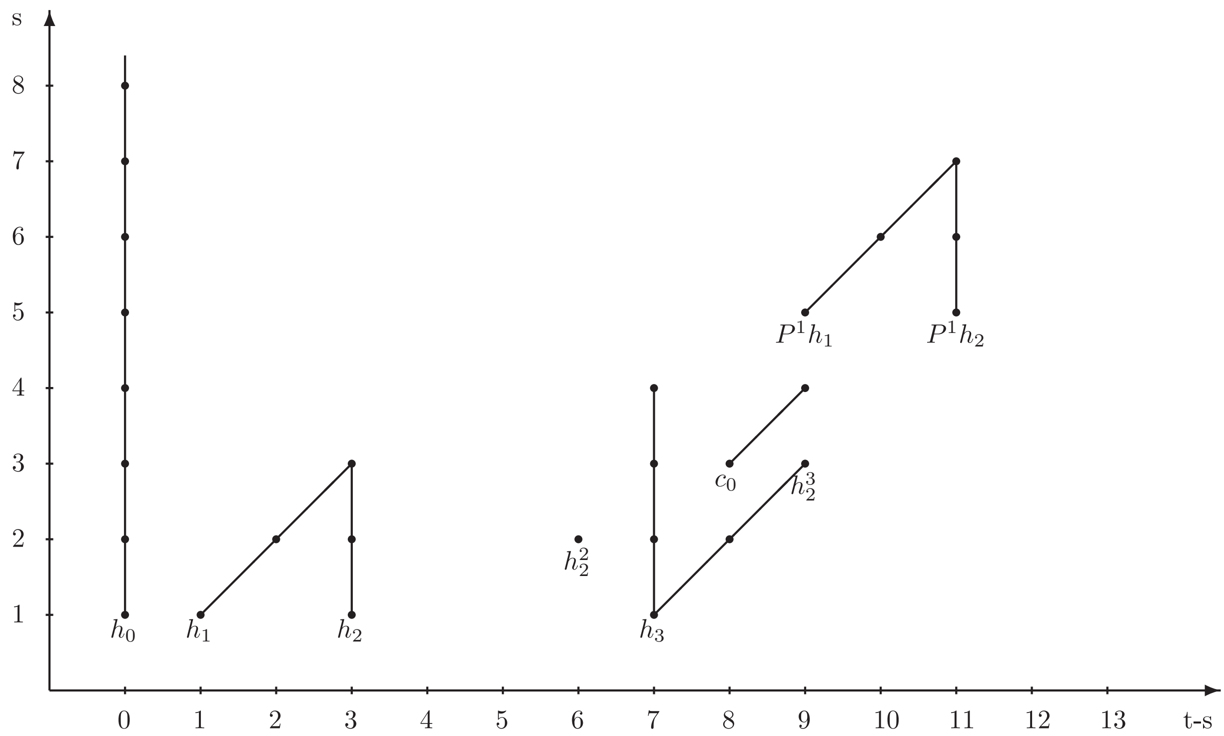


Table 4.4: E_∞ term of the May spectral sequence in the range of $t-s \leq 13$



This is h_1-stable. $d_2(b_{02}^4) = 0$. $d_2(b_{03}b_{02}) = b_{02}d_2(b_{03}) + b_{03}d_2(b_{02}) = h_1d_0 + h_3b_{02}^2 + h_0^2h_2b_{03} + h_1^3b_{03}$. Therefore, we have E_3 term of 15-stem and $h_3b_{02}^2$ remains nonzero. Then $d_4(h_3b_{02}^2) = h_3d_4(b_{02}^2) = h_0^4h_3^2$.

Theorem 4.4.15. $d_4(h_2b_{03}) = h_0^2h_3^2$.

Proof. Since $d_2(h_0^nb_{03}b_{02}) = h_0^nh_3b_{02}^2 + h_0^{n+2}h_2b_{03}$, $h_0^nh_3b_{02}^2$ and $h_0^{n+2}h_2b_{03}$ represent the same element in E_4. Therefore $d_4(h_0^3h_2b_{03}) = d_4(h_0h_3b_{02}^2) = h_0^5h_3^2$. □

Therefore, the survivors of 14-stem are h_3^2, $h_0h_3^2$ and $h_0^id_0$ for $i = 0, 1, 2$.

Theorem 4.4.16. $d_8(b_{02}^4) = h_0^8h_4$.

Proof. The only opportunity to kill $h_0^nh_4$ is b_{02}^4, then we have $d_r(b_{02}^4) = h_0^8h_4$. The filtration degree u of $b_{02}^4)$ and $h_0^8h_4$ are -8 and 0 respectively, so we get $r = 8$. □

Therefore, the survivors of 15-stem are $h_0^ih_4$ for $0 \leq i \leq 7$ and h_1d_0.

Remark 4.4.17. *Since $d_2(b_{03}b_{02}) = h_1d_0 + h_3b_{02}^2 + h_0^2h_2b_{03}$ in the quotient complex, we use $h_3b_{02}^2 + h_0^2h_2b_{03}$ to kill h_0-ladder of $b_{03}b_{02}$. Remaining elements could be viewed as h_1d_0 and one of $h_3b_{02}^2$ and $h_0^2h_2b_{03}$ equivalently, and the later is killed by d_4. Hence we get only h_1d_0 in 14-stem.*

17-stem: We have only $h_0(1)b_{12}$ since we don't mention h_1-multiplies anymore. $d_2(h_0(1)b_{12}) = h_0(1)d_2(b_{12}) + b_{12}d_2(h_0(1)) = h_2^3h_0(1) + b_{12}h_0h_2^2 = 0$. Therefore, the survivors of 16-stem are h_1h_4, $h_1^2d_0$ and $h_1h_0(1)b_{02}^2 = P^1c_0$.

Remark 4.4.18. *We denote $e_0 = h_0(1)b_{12}$.*

18-stem: We have h_2h_4, $h_2^2b_{03}$, $h_3^2b_{02}$ and $b_{02}^2b_{12}$. $d_2(h_2h_4) = 0$. $d_2(h_2^2b_{03}) = h_2^2d_2(b_{03}) = 0$. $d_2(h_3^2b_{02}) = h_3^2d_2(b_{02}) = h_1^3h_3^2 \sim 0$. $d_2(b_{02}^2b_{12}) = b_{02}^2d_2(b_{12}) = h_2^3b_{02}^2 + h_1^2h_3b_{02}^2 \sim h_0^3e_0 + h_1^3d_0$. $d_2(h_ib_{02}^2b_{12})$ is h_i-stable for $i = 0, 1$. Therefore, the survivors of 17-stem are $h_1^2h_4$, $h_0^ie_0$ for $0 \leq i \leq 3$, $h_1P^1c_0$ and $h_1b_{02}^4 = P^2h_1$, where $h_0^3e_0 = P^1h_2^3 \sim h_1^3d_0$.

Remark 4.4.19. *We denote $f_0 = h_2^2b_{03}$.*

19-stem: We have $h_2h_1(1)$, h_4b_{02}, h_3b_{03}, $h_0(1)b_{03}$, $h_3b_{02}^3$ and $h_0(1)b_{02}^3$. $d_2(h_2h_1(1)) = 0$. $d_2(h_4b_{02}) = h_4d_2(b_{02}) = h_1^3h_4 + h_0^2h_2h_4$, and $d_2(h_ih_4b_{02})$ is h_i-stable for $i = 0, 1$. $d_2(h_3b_{03}) = h_3d_2(b_{03}) = h_3^2b_{02} + h_1h_3b_{12} \sim h_3^2b_{02}$, and it is h_i-stable for $i = 0, 1$. $d_2(h_0(1)b_{03}) = h_0(1)d_2(b_{03}) + b_{03}d_2(h_0(1)) = h_1e_0 + h_0h_2^2b_{03}$, and $d_2(h_ih_0(1)b_{03})$ is h_i-stable for $i = 0, 1$. $d_2(h_3b_{02}^3) = h_3d_2(b_{02}^3) = h_1^3h_3b_{02}^2 \sim 0$. $d_2(h_0(1)b_{02}^3) = b_{02}^2d_2(h_0(1)b_{02}) = h_1^2P^1c_0$, and it is h_1-stable. Therefore, the survivors of 18-stem are $h_0^ih_2h_4$ for $0 \leq i \leq 2$, $h_0^jh_2^2b_{03}$ for $j = 0, 1$ and $P^2h_1^2$, where $h_0^2h_2h_4 \sim h_1^3h_4$ and $h_1e_0 \sim h_0f_0$.

Remark 4.4.20. *We denote $c_1 = h_2h_1(1)$.*

Remark 4.4.21. *In the quotient complex, we have $d_2(h_3b_{03}) = h_3^2b_{02}$. This is also h_3-stable. Therefore we can throw this h_3-ladder away together with related h_i-ladders for $i = 0, 1$. We do not draw this differential in Table 4.5 below.*

20-stem: We have b_{12}^2, b_{02}^5 and $b_{02}^2b_{03}$. $d_2(b_{12}^2) = 0$. $d_2(b_{02}^2b_{03}) = b_{02}^2d_2(b_{03}) = h_3b_{02}^2 + h_1b_{02}^2b_{12} \sim h_3b_{02}^2$ and this is h_i-stable for $i = 0, 1$. $d_2(b_{02}^5) = b_{02}^4d_2(b_{02}) = P^2h_1^3 + P^2h_0^2h_2$, and $d_2(h_ib_{02}^5)$ is h_i-stable for $i = 0, 1$. Therefore, the survivors of 19-stem are $c_1 = h_2h_1(1)$ and $h_0^iP^2h_2$ for $0 \leq i \leq 2$, where $h_0^2P^2h_2 \sim P^2h_1^3$.

Remark 4.4.22. *In the quotient complex, we have $d_2(b_{02}^2b_{03}) = h_3b_{02}^3$. This is also h_3-stable. Therefore we can throw this h_3-ladder away together with related h_i-ladders for $i = 0, 1$.*

21-stem: We have $h_2^2h_4$, h_3^3, $h_2^3b_{03}$ and $d_0h_0(1)$. $d_2(h_2^2h_4) = 0$. $d_2(h_3^3) = 0$. $d_2(h_2^3b_{03}) = h_2^3d_2(b_{03}) = 0$. $d_2(d_0h_0(1)) = d_0d_2(h_0(1)) = h_0h_2^2d_0 \sim h_0^3b_{12}^2$ and this is h_0-stable. Therefore, the survivors of 20-stem are $h_0^ib_{12}^2$ for $0 \leq i \leq 2$, where $h_0b_{12}^2 = h_2e_0$.

Remark 4.4.23. *We denote $g = b_{12}^2$.*

22-stem: We have b_{22}, $h_2^2h_1(1) = h_2c_1$, $h_3^2b_{02}^2$, $b_{02}^3b_{12} = P^1d_0$, $b_{12}b_{03}$ and $h_4h_0(1)$. $d_2(h_2c_1) = 0$. $d_2(h_3^2b_{02}^2) = 0$. $d_2(P^1d_0) = 0$. $d_2(b_{12}b_{03}) = b_{12}d_2(b_{03}) + b_{03}d_2(b_{12}) = h_1b_{12}^2 + h_2^3b_{03}$, where we use relations $b_{02}h_1(1) = h_1h_3b_{03}$ and $h_3b_{12} = h_1h_1(1)$. $d_2(h_ib_{12}b_{03})$ is h_i-stable for $i = 0, 1$. $d_2(h_4h_0(1)) = h_4d_2(h_0(1)) = h_0h_2^2h_4$ and it is h_0-stable. Citing the result $d_2(b_{22}) = h_2^2h_4 + h_3^3$ from Theorem 4.4.23 below, the survivors of 21-stem are $h_2^2h_4 \sim h_3^3$ and $h_1g \sim h_2f_0 = h_2^3b_{03}$.

Remark 4.4.24. *$d_2(h_ib_{22})$ is h_i-stable for $i = 0, 1$. As in the case of $d_2(h_2b_{12})$ in Remark 4.4.12, $d_2(h_2b_{22})$ is also h_2-stable, hence we can throw this h_2-ladder away together with related h_0-ladder. What's more, $d_2(h_3b_{22})$ is also h_3-stable, hence we can throw this h_3-ladder away together with related h_0, h_1-ladders.*

For now, the above process can be shown by the Table 4.5 below.

Therefore, we have the E_∞ term of the May spectral sequence in the range of $14 \leq t - s \leq 21$ in Table 4.6 below.

Theorem 4.4.25. $d_2(b_{22}) = h_2^2h_4 + h_3^3$.
$d_2(b_{13}) = h_4b_{12} + h_2b_{22}$.
$d_2(b_{04}) = h_4b_{03} + h_1b_{13}$.

Proof. By dimension reasons, the six elements are all we can expect. We need to determine the coefficients. Firstly, by resulting the calculations done above of 22-stem, the only hope to kill $h_0^nh_3^3$ is $h_0^nb_{22}$, therefore the coefficient of h_3^3 in $d_2(b_{22}$ is 1. Secondly, we assume the coefficient of h_4b_{03} is 0, and we conclude that there is a contradiction. Under this assumption, $d_2(h_0b_{04}) = 0$. Since the filtration degree u of h_0b_{04} is -6, we need some element in 29-stem with u less than -8 to kill $h_0^nb_{04}$. We only have $b_{03}b_{12}h_0(1) = b_{03}e_0$ with $u = -8$ and $b_{02}^3b_{12}h_0(1) = b_{02}^3e_0$ with $u = -10$. $d_2(h_0b_{03}e_0) = h_0e_0d_2(b_{03}) = 0$, hence cannot kill $h_0^nb_{04}$. $d_2(h_0b_{02}^3e_0) = h_0^3h_2P^1e_0$, hence d_4 cannot kill $h_0^nb_{04}$. By dimension reasons, $d_4(h_0b_{04}) = 0$. $d_6(h_0b_{04}) = 0$. Therefore, $h_0^nb_{04}$ is a permanent cycle, which contradicts to Adams edge theorem. At last, by $0 = d_2(d_2(b_{04})) = d_2(h_4b_{03} + Ah_1b_{13}) = h_1h_4b_{12} + Ah_1d_2(b_{13})$, we get $A = 1$, that is, $d_2(b_{04}) = h_4b_{03} + h_1b_{13}$ and the coefficient of h_4b_{12} in $d_2(b_{13})$ is 1. By

Table 4.5: Quotient complex of E_2 in the range of $14 \leq t-s \leq 21$

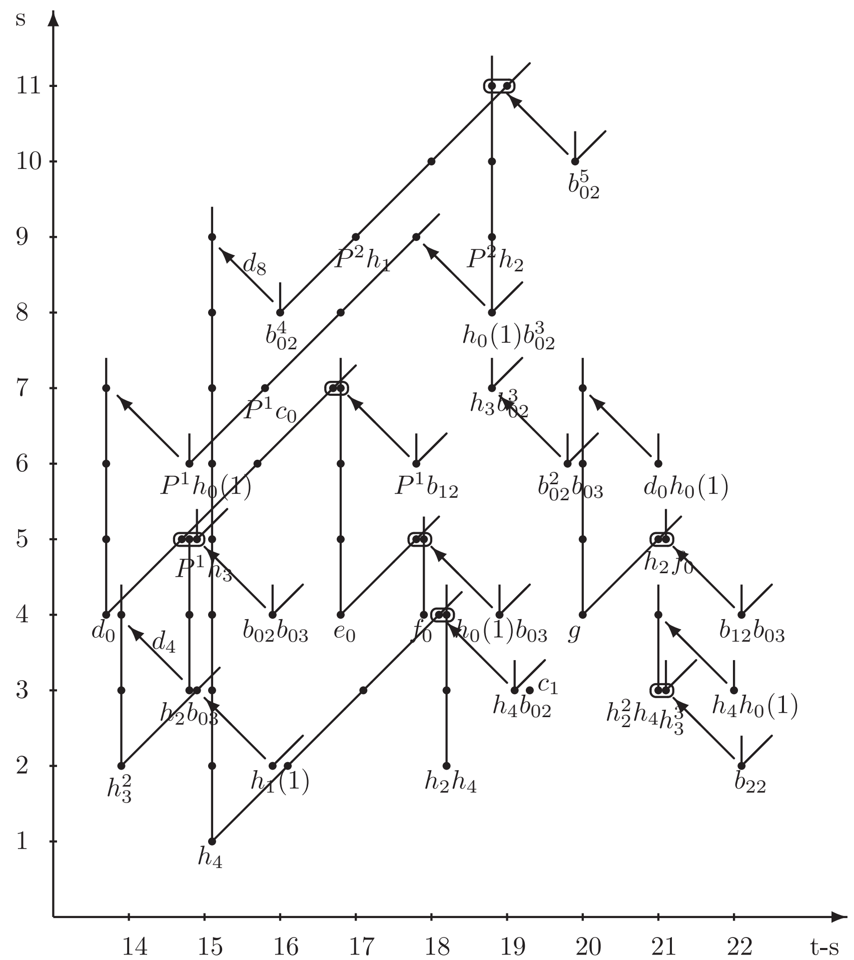
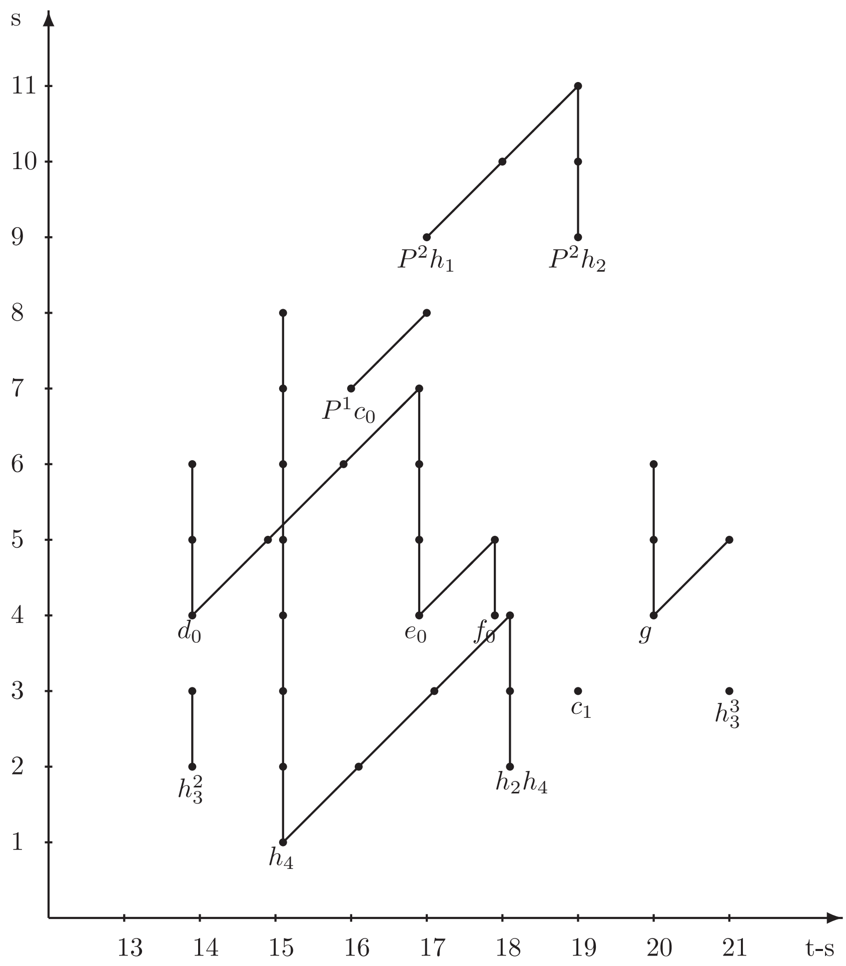


Table 4.6: E_∞ term of the May spectral sequence in the range of $14 \leq t-s \leq 21$



$0 = d_2(d_2(b_{13})) = d_2(h_4b_{12} + Bh_2b_{22}) = h_2^3h_4 + Bh_2d_2(b_{22})$, we get $B = 1$, that is, $d_2(b_{13}) = h_4b_{12} + h_2b_{22}$ and the coefficient of $h_2^2h_4$ in $d_2(b_{22})$ is 1. □

Remark 4.4.26. *$d_2(h_ib_{04})$ is h_i-stable for $i = 0, 2$. Therefore we can throw these two ladders away together with another h_0-ladder related to $d_2(h_2b_{04})$.*

23-stem: We have P^1h_4, P^2h_3, $P^2h_0(1)$, h_2g, $h_3b_{02}b_{03}$ and $h_0(1)b_{02}b_{03}$. $d_2(P^1h_4) = 0$. $d_2(P^2h_3) = 0$. $d_2(P^2h_0(1)) = b_{02}^2d_2(P^1h_0(1)) = P^1h_0^3d_0$. This is h_0-stable. $d_2(h_2g) = 0$. $d_2(h_3b_{02}b_{03}) = h_3d_2(b_{02}b_{03}) \sim P^1h_3^2$ and this is h_i-stable for $i = 0, 1$. $d_2(h_0(1)b_{02}b_{03}) \sim h_1h_0^3(1)$ and this is h_1-stable. Therefore, the survivors of 22-stem are $h_2^2h_1(1) = h_2c_1$ and $h_0^iP^1d_2$ for $0 \le i \le 2$.

Remark 4.4.27. *In the quotient complex, we have $d_2(h_3b_{02}b_{03}) = P^1h_3^2$. This is also h_3-stable. Therefore we can throw this h_3-ladder away together with related h_i-ladders for $i = 0, 1$.*

24-stem: We haveb_{02}^6, $b_{03}b_{02}^3$, $b_{02}b_{12}^2$, b_{03}^2 and $h_2^4b_{03}$. $d_2(b_{02}^6) = 0$, $d_4(b_{02}^6) = b_{02}^4d_4(b_{02}^2) = P^2h_0^4h_3$ and this is h_0-stable. $d_2(b_{03}b_{02}^3) = P^1d_2(b_{03}b_{02}) = h_1P^1d_0 + P^2h_3 + h_0^3h_0(1)b_{02}b_{03}$, and $h_id_2(b_{03}b_{02}^3)$ is h_i-stable for $i = 0, 1$. $d_2(b_{02}b_{12}^2) \sim h_0^2h_2b_{12}^2$ and this is h_0-stable. $d_2(b_{03}^2) = 0$. $d_2(h_2^4b_{03}) = 0$.

Theorem 4.4.28. *$d_4(b_{03}^2) = h_2b_{12}^2 + h_4b_{02}^2$.*

Proof. To kill $h_0^nP^1h_4$, we need an element with $u \le -8$, which could only be b_{03}^2. Therefore we get the coefficient of $h_4b_{02}^2$ is 1. For the other coefficient, we need to use calculations in the range of $42 \le t - s \le 44$, see [Tan70a]. □

Remark 4.4.29. *$d_4(h_ib_{03}^2)$ is h_i-stable for $i = 0, 1$. Actually, some $h_1^nb_{03}^2$ and $h_1^nP^1h_4$ will not survive to E_4 in that they will be killed by $h_1^mh_4b_{02}^3$ and $h_1^mb_{02}b_{03}^2$ respectively with $n = m + 3$ as discussed later.*

Therefore, the survivors of 23-stem are h_4c_0, $h_2g \sim P^1h_4$, h_0h_2g, $h_1P^1d_0$ and h_0^ii for $0 \le i \le 5$, where $i = h_0b_{02}b_{03}h_0(1) = P^1h_2b_{03}$ and $h_0^2i \sim P^2h_3 + h_1P^1d_0$.

25-stem: We have $h_2b_{12}b_{03}$, h_4b_{12}, $h_2^2c_1$ and P^1e_0. $d_2(h_2b_{12}b_{03}) = h_2d_2(b_{12}b_{03}) = h_2^4b_{03}$ and this is h_0-stable. $d_2(h_4b_{12}) \sim 0$. $d_2(h_2^2c_1) = 0$. $d_2(P^1e_0) = 0$. Therefore, the survivors of 24-stem are P^2c_0, $h_1^2P^1d_0$ and $h_1h_4c_0$.

Remark 4.4.30. *$d_2(h_2b_{12}b_{03})$ is also h_2-stable. Therefore we can throw this ladder away together with its related h_0-ladder.*

26-stem: We have b_{13}, $b_{12}h_1(1)$, $h_3^2b_{03}$, $b_{02}b_{22} = h_0^2b_{13} + h_3^2b_{03}$, $h_4b_{02}h_0(1)$, d_0b_{03}, P^2b_{12} and h_2^2g. $d_2(b_{13}) \sim h_4b_{12}$ as in Theorem 4.4.24 and it is h_i-stable for $i = 0, 1$. $d_2(b_{12}h_1(1)) = h_2^2c_1$. $d_2(h_3^2b_{03}) = h_3^3b_{02}$ has already been thrown in Remark 4.4.21. $d_2(h_4b_{02}h_0(1)) = h_1^3h_4h_0(1) = h_1^2h_4c_0$ and it is h_1-stable. $d_2(d_0b_{03}) \sim h_1P^1b_{02}$ and it is h_1-stable. $d_2(P^2b_{12}) = P^1d_2(P^1b_{12}) = h_1^3P^1d_0 + h_0^3P^1e_0$ and $d_2(h_iP^2b_{12})$ is h_i-stable for $i = 0, 1$. $d_2(h_2^2g) = 0$. Therefore, the survivors of 25-stem are $h_1P^2c_0$, P^3h_1 and $h_0^iP^1e_0$ for $0 \le i \le 3$, where $h_1^3P^1d_0 \sim h_0^3P^1e_0$.

Remark 4.4.31. *In the quotient complex, $d_2(h_2b_{13}) = h_2h_4b_{12}$ is h_2-stable and so is $d_2(b_{12}h_1(1))$. Therefore we can throw these two ladders away together with their related h_0-ladders.*

27-stem: We have $h_4b_{02}^3$, $P^1h_0(1)b_{03}$, $h_3b_{02}^5$, $h_0(1)b_{02}^5$, h_4b_{03}, $h_2b_{03}^2$ and $h_0(1)g$. $d_2(h_4b_{02}^3) = h_1^3P^1h_4 + h_0^2P^1h_2h_4 = h_1^3P^1h_4 + h_0^3h_4b_{02}h_0(1)$ and it is h_i-stable for $i = 0, 1$. We should mention that $h_1^3P^1h_4$ has not been quotient yet since it is killed by h_1-stable elements of $d_4(b_{03}^2)$. $d_2(P^1h_0(1)b_{03}) \sim h_1P^1e_0 + h_0^3d_0b_{03}$, and $d_2(h_iP^1h_0(1)b_{03})$ is h_i-stable for $i = 0, 1$. $d_2(h_3b_{02}^5) =\sim 0$. $d_2(h_0(1)b_{02}^5) = h_1^2P^2c_0$, and it is h_1-stable. $d_2(h_4b_{03}) = h_1h_4b_{12} \sim 0$. $d_2(h_0(1)g) = h_0h_2^2g$ and it is h_0-stable. $d_2(h_2b_{03}^2) = 0$. By checking filtration degree u of corresponding dimension in 28-stem, we know that $h_2b_{03}^2$ can not be killed by d_2. Then we have $d_4(h_2b_{03}^2) = h_2d_4(b_{03}^2) = P^1h_2h_4 + h_2^2g$ and it is h_0-stable. $d_2(h_0(1)g) = h_0h_2^2g$ and it is h_0-stable. Therefore, the survivors of 26-stem are $h_2^2g \sim P^1h_2h_4$, $P^3h_1^2$ and h_0^ij for $0 \le i \le 3$, where $j = h_0d_0b_{03}$, $h_0j = h_2i = P^1f_0$ and $h_0^2j \sim h_1P^1e_0$.

Remark 4.4.32. *$d_4(h_2b_{03}^2)$ is also h_2-stable. Therefore we can throw this ladder away together with its related h_0-ladder.*

28-stem: We have P^2b_{03}, b_{02}^7, b_{04}, $h_1(1)b_{03}$, $b_{02}b_{03}^2$ and $P^1g = d_0^2$. $d_2(P^2b_{03}) = h_3b_{02}^5$ and it is h_i-stable for $i = 0, 1$. $d_2(b_{02}^7) = P^3h_1^3 + h_0^2P^3h_2$ and $h_ib_{02}^7$ is h_i-stable for $i = 0, 1$. $d_2(b_{04}) \sim h_4b_{03}$ as proved in Theorem 4.4.25 and it is h_i-stable for $i = 0, 1$. $d_2(h_1(1)b_{03}) = h_1(1)d_2(b_{03}) + b_{03}d_2(h_1(1)) \sim h_1h_1(1)b_{12} = h_3g$ and it is h_1-stable. $d_2(b_{02}b_{03}^2) = b_{03}^2d_2(b_{02}) = h_1^3b_{03}^2 + h_0^2h_2b_{03}^2$ and $d_2(h_ib_{02}b_{03}^2)$ is h_i-stable for $i = 0, 1$. $d_2(P^1g) = 0$. Therefore, the survivors of 27-stem are $h_0^iP^3h_2$ for $0 \le i \le 2$ where $h_0^2P^3h_2 \sim P^3h_1^3$.

29-stem: We have $P^1d_0h_0(1)$, $h_0(1)b_{12}b_{03}$, $h_2d_0b_{03}$ and h_4d_0. $d_2(P^1d_0h_0(1)) = P^1d_2(d_0h_0(1)) = h_0^3P^1g$ and it is h_0-stable. $d_2(h_0(1)b_{12}b_{03}) = e_0d_2(b_{03}) = h_1e_0b_{12} = h_1h_0(1)g$ and it is h_1-stable. $d_2(h_2d_0b_{03}) = h_2d_2(d_0b_{03}) \sim 0$. $d_2(h_4d_0) = 0$. Therefore, the survivors of 28-stem are $h_0^iP^1g$ for $0 \le i \le 2$ where $P^1g = d_0^2$.

Remark 4.4.33. *We denote $k = h_2d_0b_{03}$.*

30-stem: We only mention that $d_2(b_{02}b_{13}) = b_{13}d_2(b_{02}) + b_{02}d_2(b_{13}) \sim h_4d_0$ and it is h_i-stable for $i = 0, 1$. $d_2(P^1b_{12}b_{03}) = P^1d_2(b_{12}b_{03}) = P^1h_1g + P^1h_2^3b_{03} \sim P^1h_1g + h_0^2k$ and $d_2(h_iP^1b_{12}b_{03})$ is h_i-stable for $i = 0, 1$. Therefore, the survivors of 29-stem are h_0^ik for $0 \le i \le 2$ where $h_0k = h_2j$ and $h_0^2k \sim h_1P^1g$.

By now, the process above can be shown by the Table 4.7 below.

Therefore, we have the E_∞ term of the May spectral sequence in the range of $22 \le t - s \le 29$ in Table 4.6 below.

Table 4.7: Quotient complex of E_2 in the range of $22 \leq t-s \leq 29$

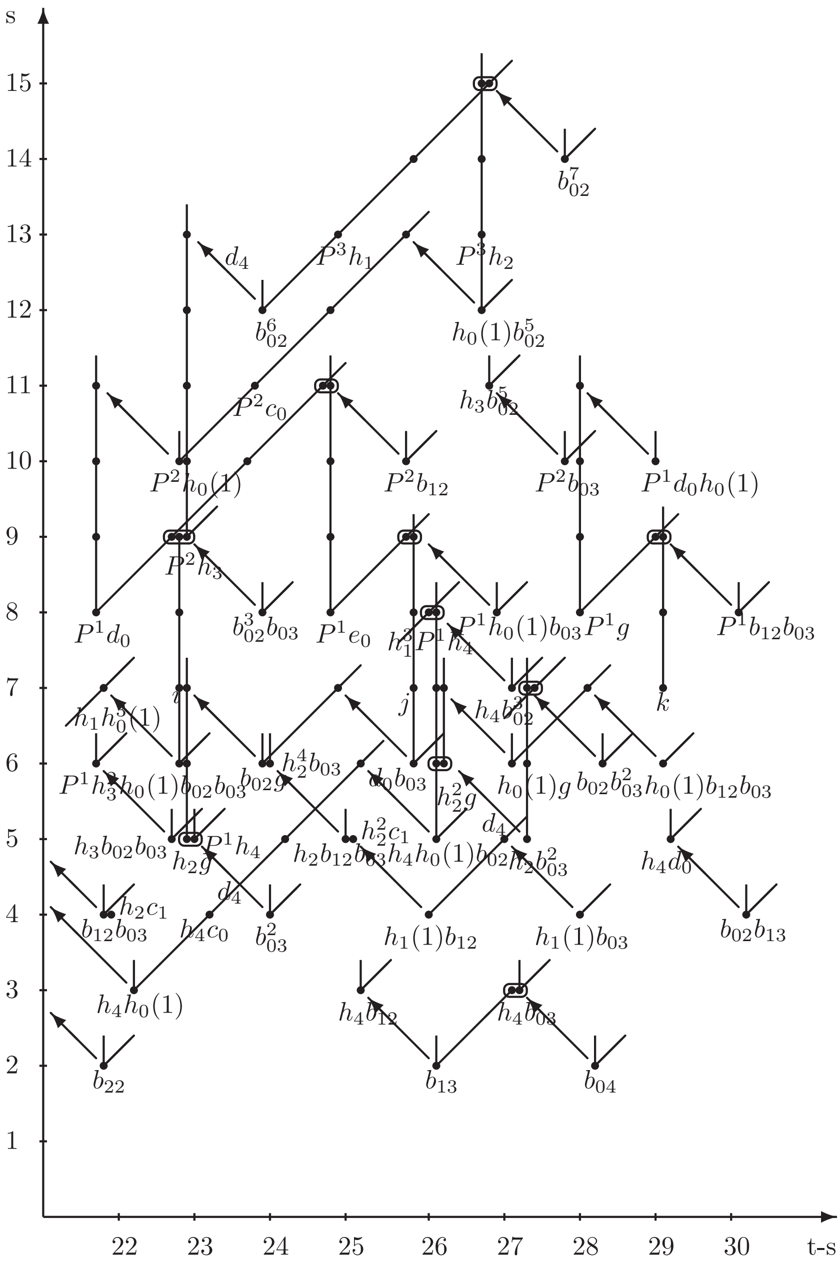
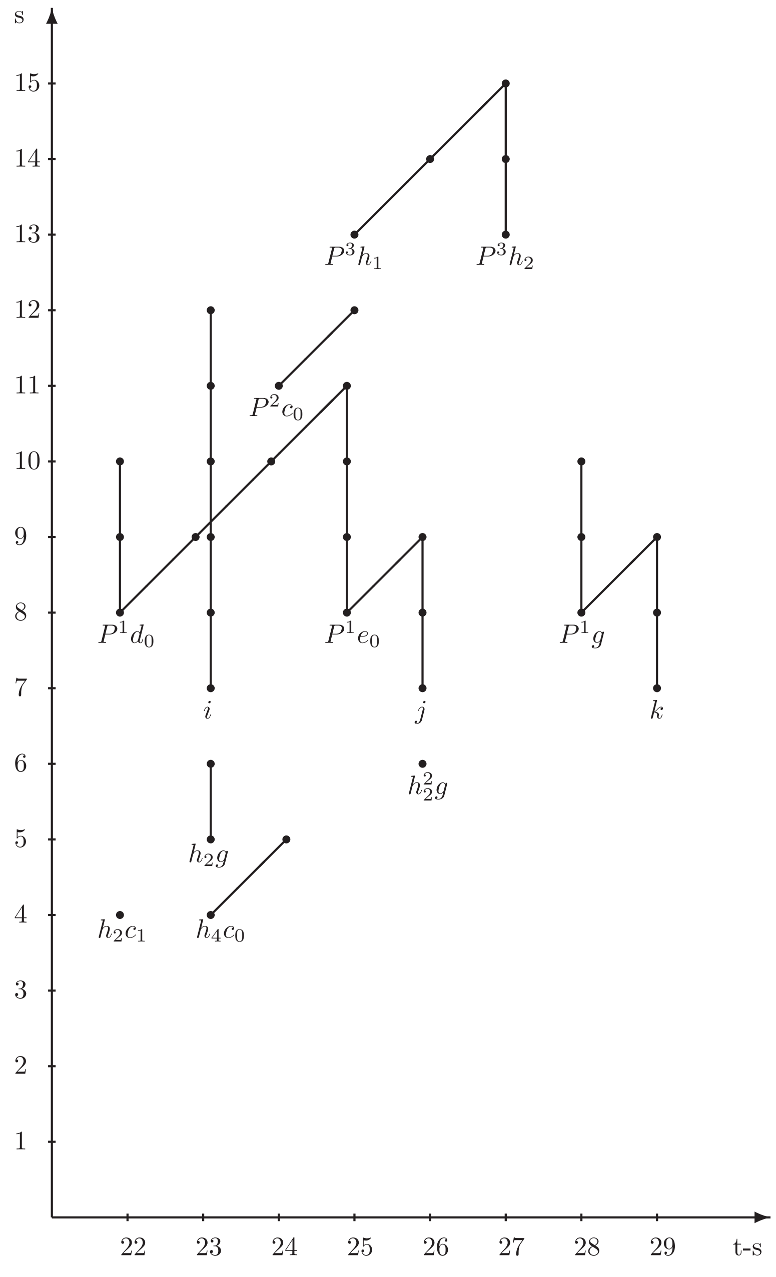


Table 4.8: E_∞ term of the May spectral sequence in the range of $22 \leq t-s \leq 29$



4.5 Differentials in the Adams spectral sequence

Theorem 4.5.1. *In the Adams spectral sequence, all differentials in the range of* $0 \leq t-s \leq 29$ *are the following:* $d_2(h_4) = h_0h_3^2$, $d_3(h_0h_4) = h_0d_0$, $d_3(h_0^2h_4) = h_0^2d_0$, $d_2(e_0) = h_1^2d_0$, $d_2(f_0) = h_0^2e_0$, $d_2(h_0f_0) = h_0^3e_0$, $d_2(i) = h_0P^1d_0$, $d_2(h_0i) = h_0^2P^1d_0$, $d_2(P^1e_0) = h_1^2P^1d_0$, $d_2(h_0^nj) = h_0^{n+1}P^1e_0$ *for* $n = 0, 1, 2$, $d_2(k) = h_0P^1g$, $d_2(h_0k) = h_0^2P^1g$ *and* $d_3(r) = h_0^2k$.

Proof. See [Rav86]. □

Remark 4.5.2. *The is a nontrivial extension at the* 23*-stem. This must be proved using some other methods. For example, one can compare it with the Adams–Novikov spectral sequence, see [Rav86] for details.*

Since there are no differentials in the range of $0 \leq t-s \leq 13$, the E_2-term and E_∞-term of Adams spectral sequence will be just the same as Table 4.4. The differentials in the above theorem can be viewed in Table 4.9 and Table 4.10 below.

Table 4.9: differentials of the Adams spectral sequence in the range of $14 \leq t-s \leq 21$

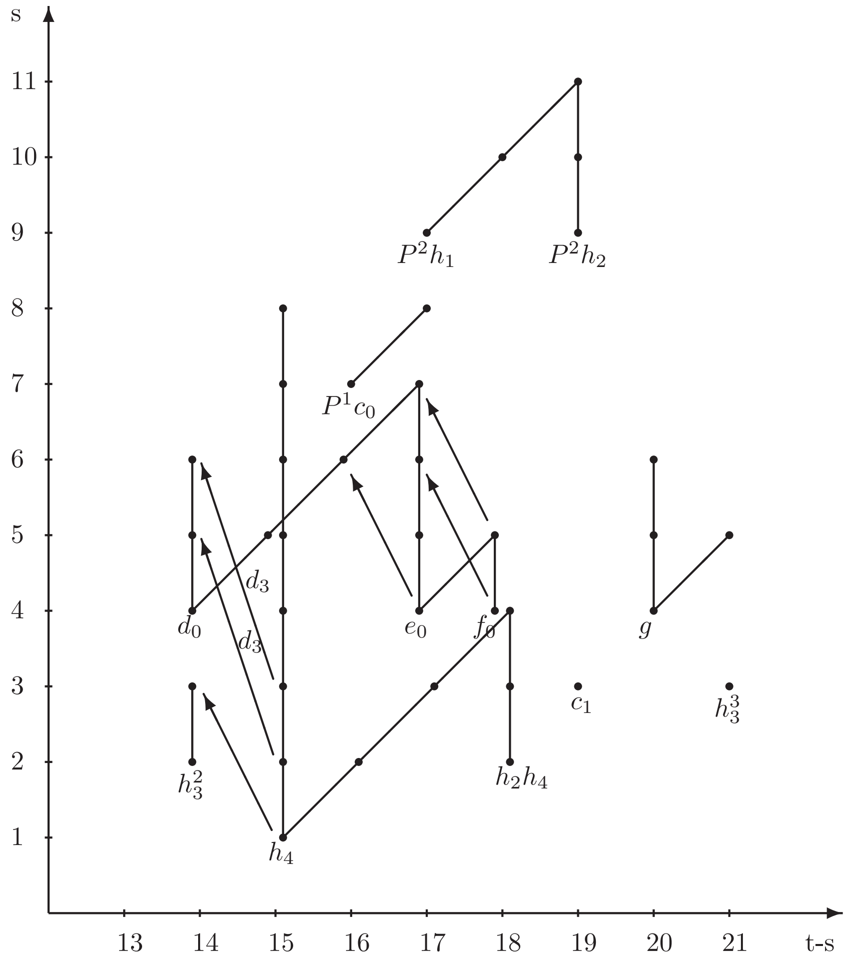
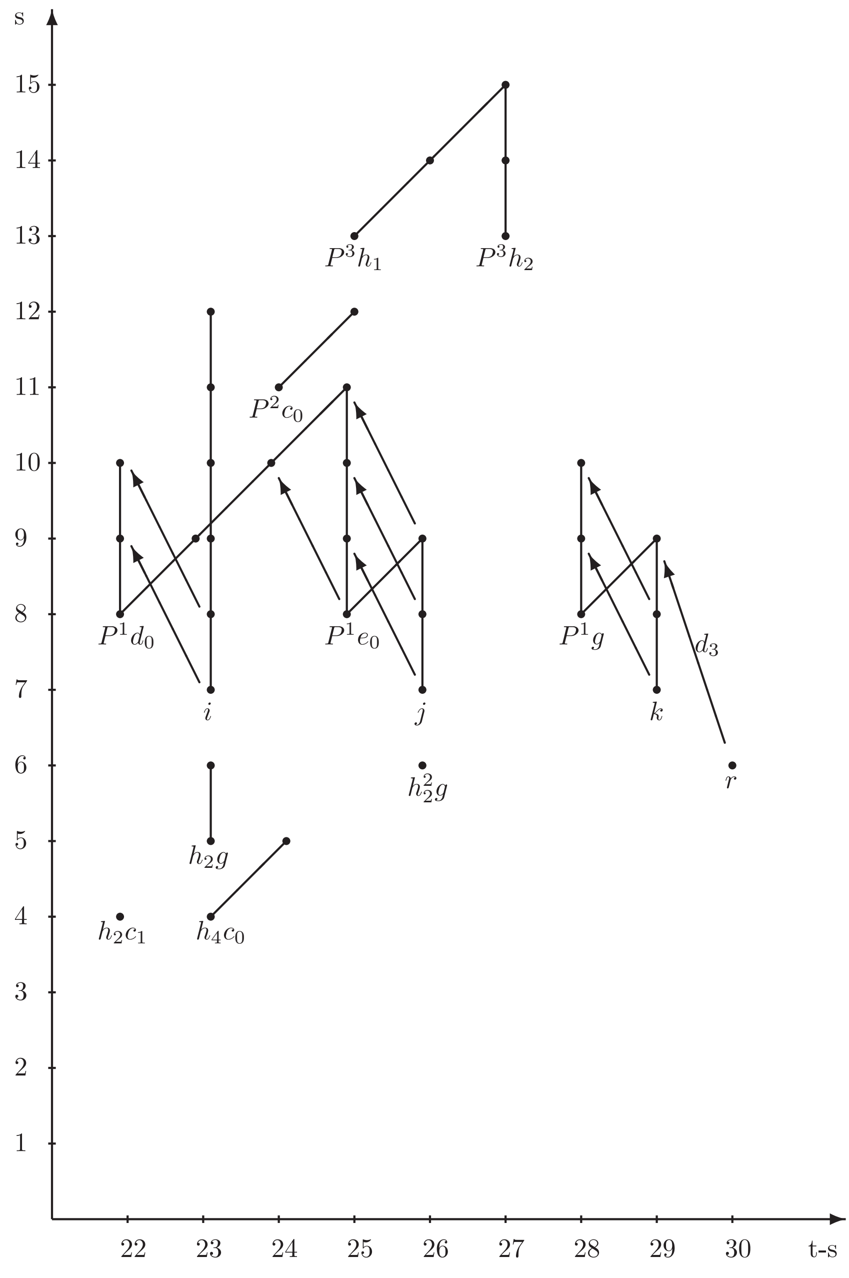


Table 4.10: differentials of the Adams spectral sequence in the range of $22 \leq t-s \leq 29$



Therefore, we have the E_∞-term of the Adams spectral sequence as in Table 4.11 and Table 4.12 below.

Table 4.11: E_∞ term of the Adams spectral sequence in the range of $14 \leq t-s \leq 21$

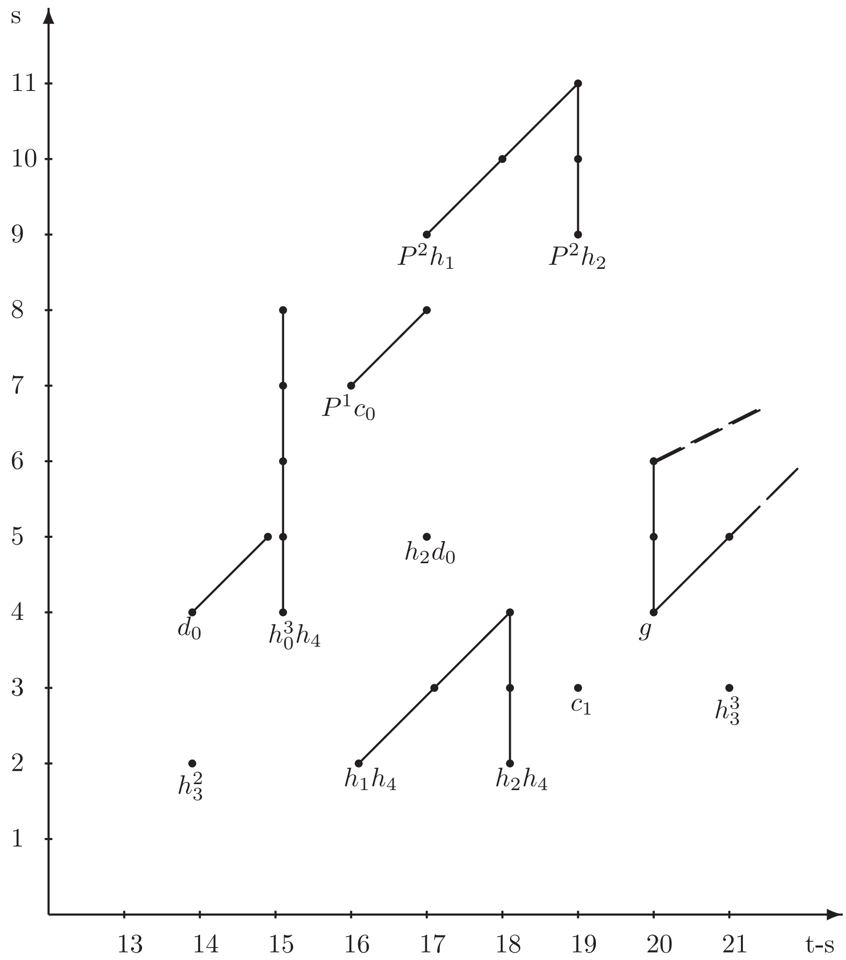
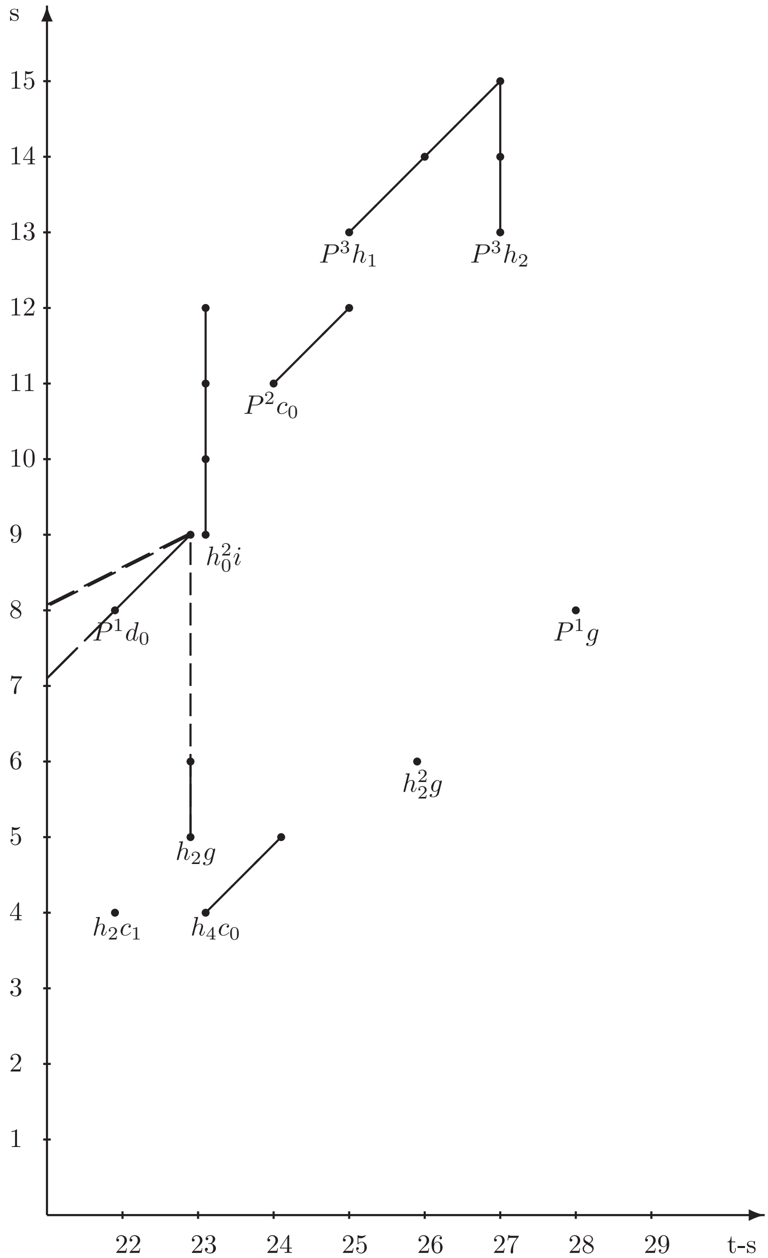


Table 4.12: E_∞ term of the Adams spectral sequence in the range of $22 \le t-s \le 29$



Therefore, we have the first 29 stable homotopy groups of spheres at prime 2 listed in Table 4.13 below.

Table 4.13: 2-component of $\pi_k(\mathbb{S})$ for $k \leq 29$

k	0	1	2	3
$\pi_k(\mathbb{S})$	$\mathbb{Z}_{(2)}$	$Z/2$	$Z/2$	$Z/8$
$\pi_{4+k}(\mathbb{S})$	0	0	$Z/2$	$Z/16$
$\pi_{8+k}(\mathbb{S})$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2 \oplus Z/2$	$Z/2$	$Z/8$
$\pi_{12+k}(\mathbb{S})$	0	0	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/32$
$\pi_{16+k}(\mathbb{S})$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$	$Z/2 \oplus Z/8$	$Z/2 \oplus Z/8$
$\pi_{20+k}(\mathbb{S})$	$Z/8$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/8 \oplus Z/16$
$\pi_{24+k}(\mathbb{S})$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2$	$Z/8$
$\pi_{28+k}(\mathbb{S})$	$Z/2$	0		

Chapter 5

Cobordism Theory

5.1 Bordism and Cobordism

The theory of cobordism is important in algebraic topology for many reasons. Primarily, it gives a classification of manifolds, which is computable by the tools of homotopy theory. This classification, though very coarse, is surprisingly useful since many things, such as the Chern numbers and Pontrjagin numbers, the signature, even things so complicated as the index of elliptic operators and the existence of metrics with positive scaler curvature, are invariant under certain kind of cobordisms. Also, as shown in chapter 2, the homotopy group of spheres is a special kind of cobordism group. Further more, the theory of complex cobordism gives deep insights into homotopy theory, which will be discussed in detail later.

Different kind of cobordism theory are characterized by a G-structure, where G is a group with a morphism $\mu : G \to O$, the infinite orthogonal group. (Precisely speaking, we should regard both G and O as Ind-objects of the category of compact Lie groups, but all we need in the following is a fibration $\nu : BG \to BO$, so we do not discuss the details of the groups here.) Roughly speaking, a G-structure of a vector bundle V is a G-principle bundle P together with a isomorphism from the associated O-bundle $P \times_G O$ to the one corresponding to the stabilization of V.

Definition 5.1.1. *Suppose $f : X \to BO$ is a map classifying a stable vector bundle U (or a virtual vector bundle of dimension 0), then a G-structure on U is a lifting of f to BG, and two liftings are regarded as the same if they are homotopic relative to ν.*

The space BO is an H-space with the addition induced by direct sum. This has a homotopy inverse, so BO is an H-group. Fix a choice of homotopy inverse $\zeta : BO \to BO$. If M is a manifold and $f : M \to BO$ classifies its stable tangent bundle, then $\zeta \circ f$ classifies its normal bundle (in fact we may choose this as the definition of the normal bundle).

Definition 5.1.2. *A manifold with a G-structure (abbreviated as G-manifold), is a compact manifold with a G-structure on its normal bundle.*

Definition 5.1.3. *A G-cobordism between compact G-manifolds X and X' is a compact G-manifold with boundary Y such that $\partial Y = X \cup X'$ as G-manifolds.*

The cobordism class of G-manifolds form a group, with the addition induced by disjoint union, and the inverse induced by changing the orientation, which means the following:

Definition 5.1.4. *Let $f : M \to BO$ be a map and $g : M \to BG$ a lifting of f. Since the fundamental group acts trivially on BO, we have a map $F : M \times [0,1] \to BO$ such that $F|_{M\times 0} = F|_{M\times 1} = f$ and $F|_{p\times[0,1]}$ represents the nontrivial element in the fundamental group for every $p \in M$. F lifts to a map $G : M \times [0,1] \to BG$ with $G|_{M\times 0} = g$. Define the map $G|_{M\times 1}$ to give the reverse orientation of g.*

If BG is an H-group with $\nu : BG \to BO$ a morphism of H-groups, then the cobordism group has a ring structure induced from cartesian product. We assume this from now on.

We can generalize the preceding definition to give a generalized homology and cohomology theory.

Let T be a space. Then $MG_*(T)$ is the cobordism group of maps from G-manifolds to T. Precisely, we call to maps $f : X \to T$ and $f' : X' \to T$ from compact G-manifolds X and X' to T cobordant if there is a compact G-manifold Y with boundary $X \cup X'$ as G-manifolds and a map from Y to T which restricts to f and f' on the boundary. The group $MG_k(T)$ is formed from the set of maps from compact k-dimensional G-manifolds to T modulo the relation generated by cobordism. This is the bordism group. It is a generalized homology theory.

Remark 5.1.5. *The boundary map in the Mayer-Vietoris sequences is defined as follows:*

Let $T = U \cup V$ with U, V open, and $\lambda : X \to T$ represent an element of $MG_*(T)$. Choose a smooth function $f : T \to \mathbb{R}$ such that the support of $|f|-1$ is contained in $\lambda^{-1}(U\cap V)$ and f takes the value -1 outside V and value 1 outside U. Further, we may suppose 0 is not a critical value of f. Then $f^{-1}(0)$ is a manifold with G-structure contained in $\lambda^{-1}(U\cap V)$, and define $\partial[\lambda] = [\lambda|_{f^{-1}(0)}]$.

To define the corresponding cohomology theory, we work in the category of compact manifolds. It can be easily generalized to noncompact manifolds to give the definition of a generalized cohomology theory with compact support. We will give the definition for a general space in the next section.

Definition 5.1.6. *Let $f : X \to Y$ be a map between manifolds. Then a G-structure on f is a G-structure on the virtual bundle $[f^*(TY)] - [TX]$.*

So a G-structure on a manifold M is the same as a G-structure on the map from M to one point. Also observe that the composition of maps with G-structure has a specified G-structure.

Let M be a manifold of dimension m. The cobordism group, $MG^k(M)$, is the cobordism classes of maps with G-structure from compact $(m-k)$-dimensional manifolds to M.

To define the pullback along maps, let $f : M \to N$ be a map, and $\nu : X \to N$ a representative of an element of $MG^k(N)$. By suitable choice, we can make f and ν intersect transversely so that $W = X \times_N M$ is a smooth manifold. Let $\mu : W \to M$ and $\sigma : W \to X$ be the projection, then we have $[\mu^* TM] - [TW] = \sigma^*([\nu^* TN] - [TX])$. Let μ have the induced G-structure and define $f^*[\nu]$ to be the cobordism class of μ. One verifies this is well defined.

Remark 5.1.7. *For manifolds with boundary, we can define the cobordism groups to be cobordism classes of maps with G-structure from manifolds with boundary which maps the boundary into boundary. If one extends the definition further to manifolds with corners, then there would be a quite satisfactory cohomology theory.*

For a vector bundle V with a G-structure, there is a MG-orientation with the Thom class given by the cobordism class of the zero section map.

If M is an m-dimensioanl G-manifold, then it is MG-oriented, so we have Poincaré duality. In fact, by composition with the map to one point, we see a map $f : X \to M$ has a G-structure if and only if X itself has a G-structure, so $MG^(m-k)(M) \cong MG_k(M)$.

Let $f : M \to N$ be a map with G-structure, then the map $f_! : MG^k(M) \to MG^{k-t}(N)$ is defined by composition with f, where t is the dimension of M minus the dimension of N.

If $\lambda : X \to M$ represents an element of $MG_*(M)$, then $p_! \circ \lambda^*$ defines the pairing of bordism and cobordism $MG_k(M) \times MG^t(M) \to MG_{k-t}(pt)$ where $p : X \to pt$ is the map to one point.

If $\nu : X \to M$ and $\mu : Y \to N$ represent element of $MG^k(M)$ and $MG^t(N)$ respectively, then $\nu \times \mu : X \times Y \to M \times N$ represents their product in $MG^{k+t}(M \times N)$. The cup product is defined by pulling this product along the diagonal map.

Summarizing, we define a pair of productive generalized homology and cohomology theory (defined for manifolds for now), and a manifold (or vector bundle) is MG-orientable if (in fact only if) it has a G-structure.

5.2 The Pontrjagin–Thom Construction

The computation of the cobordism groups are possible because we can define spectra with homotopy groups isomorphic to the cobordism groups. In the preceding section, we constructed homology theories MG_* with coefficient group the cobordism group. By the Brown representability theorem, there exist spectra representing these homology theories. It is a remarkable fact that these spectra can be constructed in concrete ways by the Pontrjagin–Thom construction, making the tools in homotopy theory available in computations.

First define the notion of Thom spectra. For a finite dimensional vector bundle, the Thom space is the space obtained from the total space of the vector bundle by adding one point at infinity. Now let V be a virtual vector bundle of dimension n over X with classifying map $\nu : X \to BO$. The space BO has a filtration $BO(0) \subset BO(1) \subset \cdots \subset BO(k) \subset \cdots \subset BO$. This induces vector bundles $V_0, V_1, \ldots, V_k, \ldots$ of dimension $0, 1, \ldots, k, \ldots$ over $\nu^{-1}(BO(0)), \nu^{-1}(BO(1)), \ldots, \nu^{-1}(BO(k)), \ldots$ classified by the restriction of ν to these spaces.

Definition 5.2.1. *The Thom spectrum of V is obtained from the prespcectrum $\{W_k\}$, where W_k is the Thom space of V_{k+n} if $k + n \geq 0$ and one point otherwise, and the map $\Sigma W_k \to W_{k+1}$ is induced from the isomorphism $V_k \oplus \mathbb{R} \cong V_{k+1}|_{\nu^{-1}(BO(k))}$.*

Definition 5.2.2. *The spectrum MG is defined to be the Thom spectrum of the virtual vector bundle of dimension 0 induced from $\nu : BG \to BO$.*

We will show that the spectrum MG represents the homology theory MG_* and the cohomology theory MG_*, justifying the notation.

Now let X be a t-dimensional G-manifold. Embed it into some euclidian space $\mathbb{R}^{n+t}$. Then the normal bundle has a G-structure. The classifying map for the normal bundle induces a map from the Thom spectrum of $t - [TX]$ to MG. The tubular neighborhood N of X in $\mathbb{R}^n$ is homeomorphic to the normal bundle, so $\mathbb{R}^{n+t}/(\mathbb{R}^{n+t} \setminus N)$ is homeomorphic to the Thom space, and we get a map $\Sigma^t\mathbb{S} \to MG$ by composing the quotient map to $\mathbb{R}^{n+t}/(\mathbb{R}^{n+t} \setminus N)$ with the map between Thom Spectra.

Conversely, given a map $\Sigma^t\mathbb{S} \to MG$, we can realize it as a map from S^{n+t} to the Thom space of the vector bundle V_n classified by $\nu^{-1}(BO(n)) \to BO(n)$ for some n and the preimage of some generic section of V_n is a G-manifold embedded in $\mathbb{R}^{n+t}$.

One shows these maps give an isomorphism between the cobordism groups and the homotopy groups of MG.

For a map λ from a G-manifold X to a space T representing some element of $MG_*(T)$, and an embedding of X in $\mathbb{R}^{n+t}$ with normal bundle N, we can define a map from M, the Thom space of N, to the space $M \wedge T_+$ where T_+ is the disjoint union of T and a base point. When $p \in N$, then map p to the point $p \wedge \lambda \circ \pi(p)$ where $\pi : N \to M$ is the projection to the base, and send the point at infinity to the base point. One verifies this is indeed continuous.

This give a homomorphism from $MG_*(T)$ to $\pi_*(MG \wedge T_+)$, which is a morphism between homology theories. Since it induces an isomorphism between coefficient groups, it is an isomorphism of homology theories. So we get the following:

Theorem 5.2.3. *The homology theory MG_* and cohomology theory MG^* are represented by the Thom spectrum MG.*

5.3 The computation of various cobordism groups

5.3.1 The unoriented cobordism ring

Setting $G = O$, we get the unoriented cobordism. The unoriented cobordism ring was first calculated in [Tho54]. After that, more and more cobordism rings are calculated by more and more elaborated methods.

The main step in calculating the unoriented cobordism ring is to show that the cohomology ring of MO is free over the Steenrod algebra, so that it is a cartesian product of Eilenberg-MacLane spectra.

Since every element of the unoriented cobordism group has order 2, we will work with the cohomology group with coefficient $\mathbb{F}_2$ only.

The cohomology of the space $BO(n)$ is a polynomial ring generated by the Stiefel-Whitney classes $\{w_1, \ldots, w_n\}$. By Thom isomorphism, the Thom space of the universal bundle, $MO(n)$, has the same cohomology as a group. The zero section map $BO(n) \to MO(n)$ induces a map between cohomology by sending the Thom class ξ to the Euler class w_n, so we can identify $\tilde{H}^*(MO)$ as the ideal in $H^*(BO(n))$ generated by w_n. The map $(\mathbb{RP}^1)^n \to BO(n)$ classifying the product of the tautological bundle, identifies the cohomology of $BO(n)$ as the subalgebra of the symmetric polynomials in $\mathbb{F}_2[t_1, \ldots, t_n]$, where t_i corresponds to the generator of the cohomology of the i^{th} component of $(\mathbb{RP}^1)^n$, And w_i corresponds to the i^{th} elementary symmetric polynomial.

Lemma 5.3.1. *When I runs over the set of admissible sequences of total degree $h \leq n$ (see 3.1.10 for the definition of admissability), the classes $Sq^I(t_1 t_2 \ldots t_n)$ in $H^*((\mathbb{RP}^1)^n)$ are linearly independent symmetric functions in t_i.*

Proof. See [Tho54]. □

So the classes $Sq^I(\xi)$ with I admissible of total degree $h \leq n$, are also independent. Letting $n \to \infty$, we see that the map $A(2)^* \to H^*(MO)$ induced by the map $MO \to H\mathbb{F}_2$ defined by the Thom class, is injective. Dually, the map $H_*(MO) \to A(2)_*$ is surjective. One also verifies the map $MO \to H\mathbb{F}_2$ preserves multiplication (at least up to homotopy). The following theorem of Milnor and Moore shows that these suffice to show the freeness of the action of the Steenrod algebra on $H^*(MO)$.

Theorem 5.3.2. *Let Σ be a commutative graded connected Hopf algebra over a field K. Let M be a K-algebra and a right Σ-comodule. Let $C = M \,\square_\Sigma\, K$ (defined by the exact sequence $0 \to M \,\square_\Sigma\, K \to M \xrightarrow{\psi - M\otimes\eta} M \otimes \Sigma$, where ψ denotes the comodule structure map of M and η the unit map of Σ). If there is a surjection $f : M \to \Sigma$ which is a homomorphism of algebras and Σ-comodules, then M is isomorphic to $C \otimes \Sigma$ simultaneously as a left C-module and as a right Σ-comodule.*

Proof. See [MM65] or [Rav86], Appendix A1. □

By comparing dimensions, we get the following:

Theorem 5.3.3. *The homology group of the spectrum MO is isomorphic to $\mathbb{F}_2[x_n : n \neq 2^i - 1] \otimes A(2)_*$ as an $A(2)_*$-comodule.*

Thus the spectrum MO is a product of Eilenberg-MacLane spectra by results in the preceding chapters.

Theorem 5.3.4. *The unoriented cobordism ring is isomorphic to $\mathbb{F}_2[x_n : n \neq 2^i - 1]$.*

5.3.2 The oriented cobordism ring

The oriented cobordism is the case $G = SO$. In this case, an SO-structure is the same as an orientation. We will also call a map with SO-structure an oriented map.

The rank of the free part of the oriented cobordism group is easy to compute, since $\pi_*(MSO) \otimes \mathbb{Q}$ is isomorphic to the rational homology.

Recall the cohomology of BSO.

- $H^*(BSO, \mathbb{Q}) = \mathbb{Q}[p_1, p_2, \dots]$, $dim(p_i) = 4i$
- $H^*(BSO, \mathbb{F}_p) = \mathbb{F}_p[p_1, p_2, \dots]$ for p odd, $dim(p_i) = 4i$
- $H^*(BSO, \mathbb{F}_2) = \mathbb{F}_2[w_2, w_3, \dots]$, $dim(w_i) = i$

The cohomology of MSO follows from Thom isomorphism.

The calculation of the multiplication of the homology ring is deduced from the multiplicative properties of the Pontrjagin numbers and Stiefel-Whitney numbers. In particular, for the rational case, we have:

Theorem 5.3.5. *$\pi_*(MSO) \cong \mathbb{Q}[x_1, x_2, \dots]$ where x_i can be chosen to be the class of $\mathbb{CP}^{2i}$.*

Proof. See [Tho54]. □

The structure of the odd component of the oriented cobordism ring can be calculated by the Adams spectral sequence. Fix an odd prime p.

Let $t_1, t_2, \dots$ be indeterminates, and set p_i to correspond to the i^{th} elementary symmetric polynomial of the t_i. Define $v_\omega = \sum t_1^{q_1} t_2^{q_2} \dots$ be a symmetric polynomial, where $\omega = (q_1, \dots, q_s)$ is a sequence of nondecreasing positive integers. Then the v_ω can be expressed by a polynomial in p_i, and we can correspond them to elements in $H^*(BSO, \mathbb{F}_p)$. These form a basis for the group. Let $V_\omega \in H^*(MSO, \mathbb{F}_p)$ be the image of v_ω under the Thom isomorphism. Call ω p-adic if at least one of the summands is equal to $\frac{p^i - 1}{2}$.

Theorem 5.3.6. *The Bockstein operation acts trivially on $H^*(MSO, \mathbb{F}_p)$. When ω runs over all non-p-adic sequences (including the empty one), and I runs over all admissible sequences, the V_ω's together with the $\mathcal{P}^I V_\omega$'s form a basis for $H^*(MSO, \mathbb{F}_p)$.*

Proof. See [Nov62]. □

This meas the cohomology group of MSO is free over the algebra generated by P^i, which is the quotient of the mod p Steenrod algebra by the two sided ideal generated by the Bockstein operation. The calculation of the Adams spectral sequence of this kind of $A(p)^*$-modules will be done in the next section. In particular, it follows that $\pi_*(MSO)$ has no p-torsion.

A complete calculation with the Adams spectral sequence gives the ring structure of the free part of the oriented cobordism ring, which was calculated in [Mil60].

Theorem 5.3.7. *The quotient ring of $\pi_*(MSO)$ by the torsion part is isomorphic to the ring of polynomials with generatord u_i of dimension $4i$ for $i \geq 1$.*

Proof. See [Nov62]. $\square$

Remark 5.3.8. *The generators $x_i = [\mathbb{CP}^{2i}]$ has the property that modulo decomposables, $x_i = (2i+1)u_i$ if $2i+1 \neq p^k$ for any p and $x_i = \frac{2i+1}{p}u_i$ if $2i+1 = p^k$ for some p.*

The computation of the 2-component is more difficult. It is calculated in [Wal60]. It turns out that after localization at the prime 2, the spectrum MSO becomes a product of Eilenberg-MacLane spectra, so in principle we can try to prove that the cohomology of MSO is a sum of the $A(2)^*$-modules $A(2)^*$ and $A(2)^*/A(2)^*(Sq^1)$. This needs some work in algebra, see [Pen82a].

In the rest of this chapter, all coefficients of cohomology will be $\mathbb{F}_2$.

To avoid the complication of algebra, one starts with the exact sequence $\pi_*(MSO) \xrightarrow{\times 2} \pi_*(MSO) \to \pi_*(MO)$ due to V. A. Rokhlin. Here we give the argument in [Ati61a].

First observe $MO^{-k}(pt) = MSO^{2N-k}(\mathbb{RP}^{2N})$ for some $N \gg k$. For any manifold M, the first Stiefel-Whitney class of its tangent bundle is classified by a map $M \to \mathbb{RP}^\infty$, and this map descends to some map $\phi : M \to \mathbb{RP}^{2N}$, unique up to homotopy. Since the first Stiefel-Whitney class of $\mathbb{RP}^{2N}$ is the nontrivial element in $H^1(\mathbb{RP}^{2N}, \mathbb{F}_2)$, the map ϕ is always orientable, and the orientation is irrelevant since the action of $\pi_1(\mathbb{RP}^{2N})$ reverses the orientation. Reversely, any oriented map $M \to \mathbb{RP}^{2N}$ from a k-dimensional manifold necessarily preserve the first Stiefel-Whitney class.

Define $\mathscr{W}_k = MSO^{2-k}(\mathbb{RP}^2, \mathbb{RP}^0)$. Since $\mathbb{RP}^{2N}/\mathbb{RP}^{2N-2} \cong \mathbb{RP}^2/\mathbb{RP}^0$, $\mathscr{W}_k \cong MSO^{2N-k}(\mathbb{RP}^{2N}, \mathbb{RP}^{2N-2})$.

From the triad $(\mathbb{RP}^2, \mathbb{RP}^1, \mathbb{RP}^0)$, observing $\mathbb{RP}^2/\mathbb{RP}^1 \cong S^2$ and $\mathbb{RP}^1/\mathbb{RP}^0 \cong S^1$, we get an exact sequence

$$\ldots \to \mathscr{W}_{k+1} \xrightarrow{\partial} MSO^{-k}(pt) \xrightarrow{\psi} MSO^{-k}(pt) \xrightarrow{\iota} \mathscr{W}_k \to \ldots \tag{5.3.9}$$

The map ψ is induced by the map which attaches the 2-cell of $\mathbb{RP}^2$ to its 1-skeleton, so ψ is multiplication by 2.

From the triad $(\mathbb{RP}^{2N}, \mathbb{RP}^{2N-2}, \emptyset)$, we get the exact sequence

$$\ldots \to \mathscr{W}_k \to MO^{-k}(pt) \xrightarrow{\xi} MO^{-(k-2)}(pt) \to \ldots$$

The map ξ is induced by taking the preimage of $\mathbb{RP}^{2N-2} \subset \mathbb{RP}^{2N}$ for the map classifying w_1.

The most technical step is the following construction. Let $f : M \to \mathbb{RP}^{2N-2}$ be the map classifying w_1. Denote by $\mathscr{N}$ the normal bundle of $\mathbb{RP}^{2N-2}$ in $\mathbb{RP}^{2N}$. Define Q to be the projective bundle of $f^*(\mathscr{N}) \oplus \mathbb{R}$, which is a fibre bundle over M with fibre the projective plane associated with the affine plane $f^*(\mathscr{N})|_x$ over any point $x \in M$. Fix a projective line $\mathscr{L} \subset \mathbb{RP}^{2N}$ which do not intersect $\mathbb{RP}^{2N-2}$. For any point $y \in \mathbb{RP}^{2N-2}$, the projective plane passing through y and $\mathscr{L}$ can be identified with the projective plane associated with $\mathscr{N}_y$, with $\mathscr{L}$ the line at infinity. We have a map from W to $\mathbb{RP}^{2N}$ mapping the fibre at x to the projective plane passing through $f(x)$ and $\mathscr{L}$ by the above identification.

One sees the preimage of $\mathbb{RP}^{2N-2}$ in W is just M, so the above construction gives a right inverse to $\xi : MO^{-(k-2)}(pt) \to MO^{-k}(pt)$, so the long exact sequence splits and we get a split exact sequence $0 \to \mathscr{W}_k \to MO^{-k}(pt) \xrightarrow{\xi} MO^{-(k-2)}(pt) \to 0$ identifying $\mathscr{W}_k$ with a direct summand of $\pi_k(MO)$, and Rokhlin's exact sequence is proved in view of the sequence 5.3.9.

Remark 5.3.10. *One may want to have a explicit construction, for any oriented manifold X bounding a not necessarily oriented manifold Z, of a oriented manifold whose boundary is $X \cup 2Y$ for some Y. In the following, transversality is assumed whenever necessary. Let $f : (Z, X) \to (\mathbb{RP}^{2N}, pt)$ be the map classifying $w_1(Z, X)$. Denote by M the inverse image of $\mathbb{RP}^{2N-2}$. Apply the above construction to M, we obtain a manifold W and a map $\nu : W \to \mathbb{RP}^{2N}$ such that $\nu^{-1}(\mathbb{RP}^{2N-1}) = M$. Let T be a sufficiently small tubular neighborhood of $\mathbb{RP}^{2N-2}$. Cut out $f^{-1}(T)$ from Z and paste $\nu^{-1}(\mathbb{RP}^{2N} \setminus T)$ in (This can be done whenever T is sufficiently small since then $f^{-1}(T)$ is the disk bundle of $f^*(\mathscr{N})$), then we get a manifold U with boundary X and the classifying map for $w_1(U, X)$ lies in $\mathbb{RP}^{2N} \setminus \mathbb{RP}^{2N-1}$. Since $\mathbb{RP}^{2N} \setminus \mathbb{RP}^{2N-2}$ is homotopic to $\mathbb{RP}^1$, we have a map $g : (U, X) \to (\mathbb{RP}^1, pt)$ classifying $w_1(U, X)$. Let $p \in \mathbb{RP}^1$ be a generic point. Then $g^{-1}(p)$ is an orientable manifold with trivial normal bundle in U. Define V to be the manifold obtained by cutting U along $g^{-1}(p)$. Then V is orientable and has boundary $X \cup 2g^{-1}(p)$. The induced orientation on X may not be the original one, but this is irrelevant modulo 2.*

Remark 5.3.11. *The group $\mathscr{W}_k = MSO^{2-k}(\mathbb{RP}^2, \mathbb{RP}^0)$ can be identified with $MSO_c^{2-k}(\mathbb{RP}^2 \setminus \mathbb{RP}^0)$, the cohomology with compact support. So it is represented by manifolds whose classifying map for w_1 lies in $\mathbb{RP}^2 \setminus \mathbb{RP}^0$ which is homotopic to $\mathbb{RP}^1$. Hence they are manifolds with w_1 coming from cohomology with integral coefficients. The map $\mathscr{W}_k \xrightarrow{\partial} \pi_{k-1}(MSO)$ can be described as follows. Let M be a manifold with w_1 integral, so it is classified by a map $M \to S^1$ and the preimage of a generic point is the desired orientable manifold.*

Next we need the explicit basis for $\pi_*(MO)$ introduced in [Dol56].

First define an action of $\mathbb{Z}/2$ on $S^m \times \mathbb{CP}^n$ by the formula $(x, y) \mapsto (-x, \bar{y})$. This is a free action and define $\mathscr{P}_{m,n}$ to be the quotient. This can also be regarded as the quotient of $S^m \times S^{2n+1}$ by the action of $O(2)$ defined by

$$g(x,z) = \left(\det(g)x, \begin{pmatrix} g & & & \\ & g & & \\ & & \dots & \\ & & & g \end{pmatrix} z \right)$$

We have a fibration $\mathbb{CP}^n \to \mathscr{P}_{m,n} \to \mathbb{RP}^m$ induced by the projection to S^m. When $m, n \to \infty$, the space $\mathscr{P}_{m,n}$ approximates $BO(2)$, which has cohomology ring $\mathbb{F}_2[w_1, w_2]$. By the naturality of the Serre spectral sequence, we see all the differentials in the Serre spectral sequence of the fibration $\mathbb{CP}^n \to \mathscr{P}_{m,n} \to \mathbb{RP}^m$ vanishes, so $H^*(\mathscr{P}_{m,n})$ is isomorphic to $\mathbb{F}_2[c,d]/(c^{m+1}, d^{n+1})$ as a module over $H^*(\mathbb{RP}^m)$ where c, d correspond to the generator of $H^*(\mathbb{RP}^m)$ and $H^*(\mathbb{CP}^n)$ respectively. Moreover, since the $\mathbb{Z}/2$ action preserve any hyperplane in $\mathbb{CP}^n$, the class of the quotient of $S^m \times \mathscr{H}$ for any hyperplane $\mathscr{H}$ is the Poincaré dual of d. Since the intersection of $n+1$ hyperplanes is empty, we see $d^{n+1} = 0$, so $H^*(\mathscr{P}_{m,n}) \cong \mathbb{F}_2[c,d]/(c^{m+1}, d^{n+1})$ as a ring.

The Steenrod squares on $\mathscr{P}_{m,n}$ is the pull back of that on $BO(2)$, so by Wu formula, $Sq^1(d) = cd$ and others are determined by the axioms and Cartan formula. Once the Steenrod squares are calculated, the Stiefel-Whitney classes of $\mathscr{P}_{m,n}$ follows by a calculation using Wu's theorem.

Theorem 5.3.12. *The Stiefel-Whitney classes of $\mathscr{P}_{m,n}$ is given by*

$$W(\mathscr{P}_{m,n}) = (1+c)^m(1+c+d)^{n+1}$$

Proof. See [Dol56]. □

Remark 5.3.13. *Alternatively, we can argue as follows. The manifold $\mathscr{P}_{m,n}$, as a bundle over $\mathbb{RP}^m$, has tangent bundle the direct sum of the vertical part and the horizontal part. The horizontal part is simply the pull back of the tangent bundle of $\mathbb{RP}^m$ so has Stiefel-Whitney class $(1+c)^{m+1}$. The vertical part can be described as in the case of $\mathbb{CP}^n$. The vertical part for the tangent bundle of $S^m \times \mathbb{CP}^n$ is just $(n+1)\gamma - \mathbb{C}$ where γ is the tautological bundle. This descends to the quotient. γ turns out to correspond to the pull back of the universal bundle over $BO(2)$, so has Stiefel-Whitney class $(1+c+d)$, and the trivial bundle $\mathbb{C}$, since the action is conjugation, goes to the determinant bundle of that pulled back from $BO(2)$ so has Stiefel-Whitney class $1+c$. Thus $w(\mathscr{P}_{m,n}) = (1+c)^m(1+c+d)^{n+1}$.*

By a calculation with Stiefel-Whitney numbers, we obtain a set of generators for $\pi_*(MO)$.

Theorem 5.3.14. *For any integer i not of the form $2^k - 1$, define numbers s, r by the formula $i + 1 = 2^r(2s+1)$. Define $\mathscr{P}_i = \mathbb{RP}^i$ if i is even, and $\mathscr{P}_i = \mathscr{P}_{2^r-1, s2^r}$ if i is odd. Then $\pi_*(MO) = \mathbb{F}_2[[\mathscr{P}_2], [\mathscr{P}_4], [\mathscr{P}_5], \dots]$.*

Proof. See [Dol56]. □

The manifolds $\mathscr{P}_{2i+1}$ are all orientable. Also the action of $\mathbb{Z}/2$ on $S^m \times \mathbb{CP}^n$ defined by $(x, y) \mapsto (\tilde{x}, y)$ where $\tilde{x} = (x_0, \ldots, x_{m-1}, -x_m)$ if $x = (x_0, \ldots, x_{m-1}, x_m)$, descends to an action on $\mathscr{P}_{m,n}$. This action reverses the orientation of $\mathscr{P}_{2i+1}$. Thus $\mathscr{P}_{2i+1}$ defines a class in $\pi_*(MSO)$ of order 2.

Next we will investigate the structure of $\mathscr{W}_k$. It is easy to see the product of two manifolds with integral w_1 still have w_1 integral, so $\sum_k \mathscr{W}_k$ is a subring of $\pi_*(MO)$, though this one is not induced from $MSO^*(\mathbb{RP}^2)$.

We can show that in $\pi_*(MO)$, $[\mathbb{CP}^k] = [\mathbb{RP}^k]^2$ by comparing Stiefel-Whitney numbers. So the set $\{[\mathscr{P}_{2i-1}], [\mathscr{C}_{4j}]\}$ where $\mathscr{C}_{4i} = \mathbb{CP}^{2i}$ generates a polynomial subalgebra of $\pi_*(MO)$ in the image of the map $\pi_*(MSO) \to \pi_*(MO)$.

If X represents an element in $\pi_*(MSO)$ of order 2, then their is a cobordism Z from $X \cup X$ to the empty set. Pasting together the two copies of X we get a manifold W with $\partial(W) = X$ where ∂ is defined in 5.3.9. Applying this construction to the $\mathscr{P}_{2i-1}$ we get manifolds $\mathscr{Q}_{2i}$ in $\mathscr{W}_*$ with $\partial(\mathscr{Q}_{2i}) = \mathscr{P}_{2i-1}$.

The manifolds $\{[\mathscr{Q}_{2k}^{\epsilon} \mathscr{P}_{2i+1}^{s} \mathscr{C}_{4j}^{t}]\}$ with $\epsilon \in \{0, 1\}$, s, t nonnegative integers and k, i, j positive integers such that $k \geq 2$, $k, i \neq 2^r$, are linearly independent in $\mathscr{W}_*$ because we have shown the monomials in $\{[\mathscr{P}_{2i-1}], [\mathscr{C}_{4j}]\}$ are linearly independent and $[\mathscr{Q}_{2k}] = [\mathbb{RP}^{2k}] \mod$ decomposables as shown in [Wal60]. (In fact $H^*(\mathscr{Q}_{m,n}) = \mathbb{F}_2[c, d, x]/(c^m(c + x), d^{n+1}, x^2)$, where c, d comes from $\mathscr{P}_{m,n}$ and x comes from S^1. $w(\mathscr{Q}_{m,n}) = (1 + x + c)(1 + c)^{m-1}(1 + c + d)^{n+1}$. Here $m = 2^r - 1, n = s2^r$.) By comparing dimensions, we can see these are indeed a basis for $\mathscr{W}_k$.

Now we can determine the additive structure of 2-component of $\pi_*(MSO)$. The sequence 5.3.9 is a kind of Bockstein sequence. We have the Bockstein operation ∂_1 defined by $\partial_1 = \iota \circ \partial$.

One can also show that the map ∂_1 is a derivation. In fact, if $f : X \to S^1$ and $g : Y \to S^1$ classify w_1 of X and Y, then $\mu \circ (f \times g) : X \times Y \to S^1$ classifies $w_1(X \times Y)$ where $\mu : S^1 \times S^1 \to S^1$ is the group multiplication. $\partial(X \times Y)$ is $(\mu \circ (f \times g))^{-1}(pt) = (f \times g)^{-1}(\mu^{-1}(pt))$. One show by direct construction that $\mu^{-1}(pt)$ is cobordant in $S^1 \times S^1$ to $\{pt\} \times S^1 + S^1 \times \{pt\}$, so $(f \times g)^{-1}(\mu^{-1}(pt))$ is cobordant to $f^{-1}(pt) \times Y + X \times g^{-1}(pt) = \partial(X) \times Y + X \times \partial(Y)$.

Give $\mathscr{W}_*$ a decreasing filtration by letting $\mathscr{Q}_{2k}^{\epsilon} \mathscr{P}_{2i+1}^{s} \mathscr{C}_{4j}^{t}$ have filtration $\epsilon + s + 4t$. Then $[\mathscr{Q}_{2k}]^2$ have filtration 4. The spectral sequence in calculating the homology of ∂_1 with this filtration has E_1-term the tensor product of the polynomial algebras $\mathbb{F}_2[\mathscr{C}_{4j}]$ and $\mathbb{F}_2[\mathscr{P}_{2k-1}] \otimes \Lambda[\mathscr{Q}_{2k}]$ where the differential on the former is trivial, and on the latter is given by $\tilde{\partial}_1(\mathscr{Q}_{2k}) = \mathscr{P}_{2k-1}$ so has homology $\mathbb{F}_2$ concentrated in degree 0. The E_2-term is thus the $\mathbb{F}_2[\mathscr{C}_4, \mathscr{C}_8, \ldots]$ which consists of permanent cycles. So the homology of ∂_1 is $\mathbb{F}_2[\mathscr{C}_4, \mathscr{C}_8, \ldots]$. Because the homology of ∂_1 has generators corresponding to generators of the free part, the higher Bockstein operations vanish. So the torsion part of $\pi_*(MSO)$ have order 2, and $\{\mathscr{C}_4, \mathscr{C}_8, \ldots\}$ generates the free part of the 2-component. (Indeed this is used in the proof of theorem 5.3.7 if one do not want to calculate the mod 2 Adams spectral sequence.)

Corollary 5.3.15. *Two manifolds are oriented cobordant if and only if thay have the same Stiefel-Whitney numbers and Pontrjagin numbers.*

Proof. See [Wal60]. □

Remark 5.3.16. *One can show that the* 2*-localization of* MSO *is a product of Eilenberg-MacLane spaces. This is not true for the localization of* MSO *at an odd prime.*

The ring structure of $\pi_*(MSO)$ can be deduced from these. Define h_{4k} to be the generators for the free part, and $g_\omega = \partial(\mathcal{Q}_{2a_1}\mathcal{Q}_{2a_2}\dots\mathcal{Q}_{2a_r})$ where ω is the unordered sequence $(a_1, a_2, \dots, a_r)$ with a_i pairwise unequal and not a power of 2. One can show these are indecomposable and any torsion element of $\pi_*(MSO)$ can be put into the form $\Sigma(\prod h_{4a})(\prod g_b)g_\omega$ where b denoted the sequence (b).

Theorem 5.3.17. *The ring* $\pi_*(MSO)$ *is generated by* $\{h_{4k}, g_\omega\}$*, with relation* $2g_\omega = 0$*,* $\sum_i g_{\omega_i}g_{a_i} = 0$ *where* $\omega = (a_1, \dots, a_r)$ *and* $\omega_i = (a_1, \dots, \hat{a}_i, \dots)$*, and relations of the type* $g_\phi g_\psi = \Sigma(\prod h_{4a})(\prod g_b)g_\omega$*.*

Proof. See [Wal60]. □

5.3.3 Spin cobordism

The spin cobordism is defined by taking $G = Spin$. This is considerably subtler than the unoriented and oriented cobordism.

Unlike the cases before, there are non-nullcobordant spin manifolds with all characteristic numbers zero. For example, the circle with the nontrivial spin structure, or the torus with the spin structure obtained from the left invariant framing.

Fortunately, we do have characteristic classes in KO-theory which detect them. The most important is the Atiyah-Bott-Shapiro construction. This can be described in many ways.

Firstly, for any n-dimensional spin manifold, there is the Dirac operator on it. Its index gives an element in $\pi_n(KO)$, which can be shown to be cobordism invariant.

Secondly, by the Atiyah-Singer index theorem, the index of the Dirac operation can be computed using KO-theory. As shown in [ABS64], the spin representation gives a specified KO-orientation of $Spin$-bundles, i.e. a functorial KO-Thom class for $Spin$-bundles, so a spin manifold is KO-oriented, and the index of the Dirac operator is simply $p_!(1) \in KO^{-n}(pt)$ where p is the map to one point and 1 is the KO-class represented by the trivial bundle of dimension 1. This construction can be extended to a morphism between generalized cohomology theories. An element of $MSpin^*(M)$ is represented by a map $f : X \to M$ with a $Spin$-structure of $[f^*(TM)] - [TX]$, so we have the map $f_!$ and $F_!(1) \in KO^*(M)$ defines a morphism from $MSpin^*$ theory to KO theory.

Thirdly, on the spectrum level, the universal bundle over $BSpin$ is KO-oriented and the KO-Thom class of $MSpin$ gives a map $\nu : MSpin \to KO$ which is the morphism constructed before.

This map can be used to calculate the spin cobordism groups in low dimensions. Since $MSpin$ is 0-connected, the map ν lifts to a map $\mu : MSpin \to bo$ where bo is the 0-connected cover of KO.

The first few spaces of the spectrum bo (i.e. the infinite loop spaces $\Omega^{\infty-n}(bo)$) can be described quite explicitly: the 0th space is $\mathbb{Z}\times BO$ of course. The 1st one is U/SO and the 2nd Sp/U. The 3rd is Sp, 4th BSp, 5th SU/SP. The 6th space is $Spin/SU$ induced from the lift of the limit of the maps $SU(n) \to SO(2n)$. The 7th space is less classic, the 3-connected cover of $Spin$. This object, *a priori* an E_∞-space, has now acquired a name, the $String$ group, for reasons apparent from the name. And the 8th space is just $BString$.

The infinite loop spaces of $MSpin$ cannot be so easily described, but it comes form a prespectrum with the n-th space $MSpin(n)$. So the map μ can be approximated by the maps $MSpin(n) \to \Omega^{\infty-n}(bo)$. These map can be studied with little effort, at least up to the 4th term.

We know $Spin(4) \cong Sp(1) \times Sp(1)$. Let G_1, G_2, G_3 be three copies of the group $Sp(1)$ (identified with the unit ball in $\mathbb{H}$), and H_{12}, H_{13}, H_{23} be three copies of $\mathbb{H}$, such that $G_i \times G_j$ acts on H_{ij} by the formula $(s,t)h = sht^{-1}$, and the remaining group acts trivially. $BSpin(4)$ is identified with $BG_1 \times BG_2$. Then the action of $G_1\times G_2$ on H_{12}, H_{13} defines two vector bundles M_1, M_2 over $BSpin(4)$, and the action of G_3 makes them $\mathbb{H}$-vector bundles. The action of $G_1\times G_2$ on H_{12} defines a vector bundle V over $BSpin(4)$ which is the same as the one pulled back from $BSO(4)$. The $G_1\times G_3$-equivariant map $H_{12}\times_{G_2} H_{23} \to H_{13}$ defines a map $\gamma : V\times_{BSpin(4)} M_2 \to M_1$ preserving the $\mathbb{H}$-structure. The Atiyah-Bott-Shapiro construction in this case is a sequence $0 \to p^*(M_2) \xrightarrow{\mu} p^*(M_1) \to 0$ of $\mathbb{H}$-vector bundles over V which is exact outside the 0-section, where $p : V \to BSpin(4)$ is the projection and μ is the map $\gamma(x)$ at a point $x \in V$. This gives a class $[\nu] \in \widetilde{KSp}(MSpin(4)) = \widetilde{KO}^4(MSpin(4))$ which is the KO-Thom class. So the map $MSpin(4) \to BSp$ is given by this virtual bundle. We know $H^*(BSp, \mathbb{Z})$ is a polynomial ring $\mathbb{Z}[c_2, c_4, \dots]$, and if $\alpha = c_2(M_1), \beta = c_2(M_2)$, then $H^*(BSpin(4), \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]$. The reduced cohomology of $MSpin(4)$ can be identified with the ideal in $H^*(BSpin(4), \mathbb{Z})$ generated by the Euler class of V, which is $\alpha - \beta$. The pull back along the 0-section of the KO-Thom class $[\nu]$ is the virtual bundle $[M_2] - [M_1]$ so its total Chern class is $c([M_2] - [M_1]) = \frac{1+\beta}{1+\alpha}$, in particular, $c_1([M_2] - [M_1]) = \beta - \alpha$ and $c_2([M_2] - [M_1]) = \alpha^2 - \alpha\beta$, and from this we see the map $MSpin(4) \to BSp$ induces an isomorphism of cohomology groups up to dimension 8. Since these groups have no torsions, the map of homology is also isomorphism up to dimension 8. From this follows $\pi_i(MSpin(4)) \cong \pi_i(BSp)$ for $i = 1, 2, 3$. Since $\pi_i(MSpin(4)) \to \pi_i(MSpin)$ is an isomorphism for $i = 1, 2$, epimorphism for $i = 3$, we get:

Theorem 5.3.18. *$\pi_0(MSpin) \cong \mathbb{Z}$, $\pi_1(MSpin) \cong \mathbb{Z}/2$, $\pi_2(MSpin) \cong \mathbb{Z}/2$ and $\pi_3(MSpin) \cong 0$.*

In fact one can prove that the Atiyah-Bott-Shapiro map $MSpin \to bo$ induce an isomorphism on homotopy groups up to dimension 7, by either calculating further with the above method, or apply the calculation of $H^*(bo)$ in [Sto63].

The compete calculation of the spin cobordism group is done in [ABP67]. To state their results, one needs to define more KO-valued characteristic classes.

Let ξ be a finite dimensional vector bundle. Define $\lambda_i(\xi)$ to be its i^{th} exterior power. Let t be an indeterminate and $\lambda_t(\xi) = \sum_{i\geq 0}\lambda_i(\xi)$. Let $s = \frac{t}{(1+t)^2}$ as a formal power series. Define $\varpi_s(\xi) = \frac{\lambda_t(\xi)}{(1+t)^n}$ where n is the dimension of ξ. One verifies $\varpi_s(\mathfrak{n}) = 1$ for $\mathfrak{n}$ a trivial bundle. ϖ_s has constant term 1 and satisfies $\varpi_s(\xi\oplus\eta) = \varpi_s(\xi)\varpi_s(\eta)$ so ϖ_s extends to a natural transformation $KO(X)\to KO(X)[[s]]$ for X finite CW-complexes. This can be further extended to give a class $\varpi_s \in KO(BSO)[[s]]$ which can be regarded as $\varpi_s(\nu)$ where ν is the universal virtual bundle over BSO (of any dimension). Let $\varpi_s = \sum_{i\geq 0}\varpi_i s^i$. Define $\varpi_J = \varpi_{j_1}\varpi_{j_2}\ldots\varpi_{j_k}$ where $J = (j_1, j_2, \ldots, j_k)$ is a sequence of integers with $j_i > 1$. Define $n(J) = j_1 + j_2 + \cdots + j_k$. The ϖ_J's pull back to classes in $KO(BSpin)$ and will be denoted by the same symbols.

Define $BO\langle n\rangle$ to be the n-connected cover of KO.

Theorem 5.3.19. *The map $\varpi_J : BSpin \to KO$ can be lifted to a map $\tilde{\varpi}_J : BSpin \to BO\langle l(J)\rangle$ where $l(J) = 4n(J)$ if $n(J)$ is even and $l(J) = 4n(J) - 2$ if $n(J)$ is odd.*

Proof. See [ABP67]. □

The Atiyah-Bott-Shapiro map $MSpin \to bo$ gives a bo-orientation, and since $BO\langle k\rangle$ are bo-module spectra, we have Thom isomorphisms. Denote by $f_J \in BO\langle l(J)\rangle^*(MSpin)$ to be the image of $\tilde{\varpi}_J$ under the Thom isomorphism.

Theorem 5.3.20. *There exist elements $\{z_j\} \subset H^{s_j}(MSpin, \mathbb{F}_2)$ for certain s_j such that the map $F : MSpin \to \prod BO\langle l(J)\rangle \times \prod \Sigma^{s_j} H\mathbb{F}_2$ given by $F = \prod f_J \times \prod z_j$ induces an isomorphism on cohomology with coefficient $\mathbb{F}_2$.*

Proof. See [ABP67]. □

Remark 5.3.21. *The cohomology of $MSpin$ can be calculated from that of $BSpin$ by Thom isomorphism, and $H^*(BSpin, \mathbb{F}_2) = H^*(BSO, \mathbb{F}_2)/I \cong \mathbb{F}_2[y_i : i \neq 1, 2^h{+}1]$, where I is the ideal generated by the regular sequence w_2,Sq^1w_2,$Sq^2Sq^1w_2$, $\ldots$,$Sq^{2^h}Sq^{2^{h-1}}\ldots Sq^1w_2$,$\ldots$. This can be deduced from the Serre spectral sequence for $B\mathbb{Z}/2 \to BSpin \to BSO$. (See [Qui71b] for the calculation of $H^*(BSpin(n))$.)*

The cohomology of $BO\langle n\rangle$ is calculated in [Sto63] as follows: $H^*(BO\langle n\rangle) \cong A(2)^*/A(2)^*(Sq^1, Sq^2)$ if $n \equiv 0 \mod 8$ and $H^*(BO\langle n\rangle) \cong A(2)^*/A(2)^*(Sq^3)$ if $n \equiv 2 \mod 8$. Here $A(2)^*(Sq^1, Sq^2)$ is the left ideal generated by Sq^1, Sq^2 and similar for $A(2)^*(Sq^3)$.

When the cohomology groups are calculated, the dimensions of the z_j can be deduced by dimension counting,

Remark 5.3.22. *The spin cobordism group has no odd torsion as the oriented case, see [Mil60]. So the above theorem determines the spin cobordism group.*

Corollary 5.3.23. *Let M be a spin manifold. Then it bounds a spin manifold if and only if $\varpi_J(M) = 0$ for all J and all Stiefel-Whitney numbers of M vanish.*

5.4 Complex cobordism and the Adams–Novikov spectral sequence

The complex cobordism theory correspond to the case $G = U$. This was first studied in [Mil60] and [Nov60]. It is important because it is not only a generalization of the ordinary cobordism theory, but also gives a powerful tool in investigating the homotopy groups, and reveals fine structures in homotoy theory, which will be discussed in detail in the next chapter.

The computation of the complex cobordism group can be achieved by the Adams spectral sequence.

Fix a prime p, and let all homology groups in the following calculations have coefficients $\mathbb{F}_p$.

The cohomology of $BU(n)$ is a polynomial algebra $\mathbb{F}_p[c_1, \dots, c_n]$. And as in the case of unoriented cobordism, $MU(n)$, the Thom space of the universal bundle, has cohomology the ideal in $H^*(BU(n))$ generated by c_n. The map $(\mathbb{CP}^1)^n \to BU(n)$ classifying the product of the tautological bundle identifies $H^*(BU(n))$ with the subalgebra of symmetric polynomials in $\mathbb{F}_p[t_1, \dots, t_n]$ as before. The Bockstein operation (Sq^1 if $p = 2$) acts trivially on these spaces for dimensional reasons. Consider the algebra P^* which is the quotient of the Steenrod algebra by the ideal generate by the Bockstein operation. Denote by $P^i = Sq^{2i}$ if $p = 2$.

Lemma 5.4.1. *When I runs over the set of admissible sequences of total degree $h \leq 2n$ (see 3.1.10 for the definition of admissibility), the classes $\mathcal{P}^I(t_1 t_2 \dots t_n)$ in $H^*((\mathbb{CP}^1)^n)$ are linearly independent symmetric functions in t_i.*

So by letting $n \to \infty$, the classes $P^I(\eta)$, where η is the Thom class in $H^*(MU)$ and I runs over all admissible sequences are linearly independent.

Dually, the homology ring of the ring spectrum MU is a comodule over P_*, the subalgebra of $A(p)_*$ generated by $\xi_1, \xi_2, \dots$ for $p > 2$ and $\xi_1^2, \xi_2^2, \dots$ for $p = 2$. And we have a epimorphism $H_*(MU) \to P_*$. Applying theorem 5.3.2, and counting dimensions, we obtain:

Theorem 5.4.2. *$H_*(MU) = \mathbb{F}_p[u_i : i \neq p^k - 1] \otimes P_*$ as an $A(p)_*$-comodule, where $dim(u_i) = 2i$.*

In calculating Ext groups, we can simplify the ring whenever it is free over something. In this case, we have:

Theorem 5.4.3. *$Ext_{A(p)_*}(H_*(MU)) = Ext_{A(p)_* \otimes \mathbb{F}_p}(C)$, where $C = \mathbb{F}_p[u_i : i \neq p^k - 1]$ has trivial coaction.*

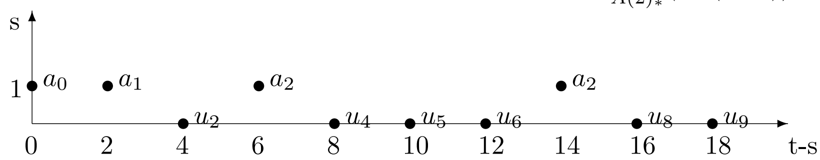
Proof. See [Rav86], chapter 3. $\square$

The algebra $A(p)_* \otimes \mathbb{F}_p$ is a primitively generated exterior algebra, so its Ext group is a polynomial algebra by direct calculation in the cobar complex.

Theorem 5.4.4. *$Ext_{A(p)_*}(H_*(MU)) = \mathbb{F}_p[u_i : i \neq p^k - 1] \otimes \mathbb{F}_p[a_0, a_1, \dots]$, where $a_i \in Ext^{1, 2p^i - 1}$.*

Proof. See [Rav86], chapter 3. □

The following picture shows the generators of $Ext^{s,t}_{A(2)_*}(H_*(MU))$ for $p = 2$.



All the differentials in the Adams spectral sequence for MU vanish because of dimensional reasons. Since $\pi_0(MU) \cong \mathbb{Z}$, the multiplication of a_0 correspond to multiplication by p in $\pi_*(MU)$. So the complex cobordism ring is:

Theorem 5.4.5. $\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, \ldots]$ *with* $dim(x_i) = 2i$.

Remark 5.4.6. *The preceding calculation gives the Hurewicz map* $h : \pi_*(MU) \to H_*(MU, \mathbb{Z})$. *We have* $H_*(MU, \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \ldots]$ *and modulo decomposables*

$$h(x_i) = \begin{cases} -pb_i & i = p^k - 1 \text{ for some } p \\ -b_i & \text{otherwise} \end{cases}$$

Unlike the case of oriented, unoriented and spin cobordism, (at the prime 2), complex cobordism do not split into a product of simple pieces as ordinary cohomology or KO-theory. As a new cohomology theory, it can give information unknown before. In fact, ordinary cohomology theory and complex K-theory are module spectra over MU, so in principle, complex cobordism always gives more information.

S. P. Novikov first observed in [Nov67] that in the Adams spectral sequence, the role of the ordinary cohomology theory can be replaced by complex cobordism theory. The resulting spectral sequence is now known as the Adams–Novikov spectral sequence.

Theorem 5.4.7. *Let* X *be a connective spectrum. There is a spectral sequence converging to* $\pi_*(X)$ *with* E_2*-term* $Ext_{A^U}(MU^*(X), MU^*(\mathbb{S}))$ *where* $A^U = MU^*(MU)$ *is the ring of operations in complex cobordism theory, or dually, with* E_2*-term* $Ext_{MU_*MU}(MU_*, MU_*(X))$ *calculated in the category of comodules over the Hopf algebroid* (MU, MU_*MU).

Proof. See [Nov67] or [Rav86], chapter 2. □

To use this spectral sequence to calculated the homotopy groups, one needs to know the structure of the Hopf algebroid (MU, MU_*MU). This is most conveniently described in terms of formal groups, which will be given in section 6.1.

In calculating stable homotopy group of spheres, for the 2-component, the Adams–Novikov spectral sequence gives information complementary to that in the classic Adams spectral sequence, and for odd primary components, the Adams–Novikov spectral sequence always gives more information. Also, the Adams–Novikov spectral sequence shows a more regular pattern, which will be explained in the next chapter.

Chapter 6

Chromatic Homotopy Theory

6.1 Formal groups

The connection between complex cobordism theory, first appeared in the sixties, has deeply influenced homotopy theory, especially after D. Quillen's discovery that complex cobordism in fact gives the universal formal group law. Since then, the theory of formal groups, well studied by algebraic geometers, was applied in homotopy theory leading to deep insight into the structure of homotopy theory. Also the study of homotopy theory reveals new structures of classical objects in algebraic geometry.

Definition 6.1.1. *Let R be a ring. A (commutative 1-dimensional) formal group law over R is a formal power series $f(x,y) = x \boxplus y = x+y+\sum_{i,j\geq 1} a_{i,j}x^iy^j \in R[[x,y]]$ satisfying the following:*

1. $x \boxplus y = y \boxplus x$
2. $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$

In other words, a formal group law is a group object in the category of formal schemes over R, together with an isomorphism from its underlying formal scheme to $Spf(R[[x]])$.

Remark 6.1.2. *Since $x \boxplus y = x + y \mod (x,y)^2$, the existence of an inverse is automatic.*

Definition 6.1.3. *Let $\gamma : R \to S$ be a morphism of rings, and $f(x,y) = x+y+\sum_{i,j\geq 1} a_{i,j}x^iy^j \in R[[x,y]]$ a formal group law over R. Then the pullback of f along γ is defined to be the formal group law $\gamma^*(f) = x+y+\sum_{i,j\geq 1} \gamma(a_{i,j})x^iy^j$ over S.*

Definition 6.1.4. *Let f, f' be two formal group laws over R. A morphism between f and f' is a formal power series $g(x) \in R[[x]]$ such that $f'(g(x), g(y)) = g(f(x, y))$.*

Definition 6.1.5. *A strict isomorphism between formal group laws is a homomorphism g of the form $x + \sum_{i \geq 2} e_i x^i$.*

Definition 6.1.6. *A logarithm of a formal group law is a formal power series $log(x) = x + \sum_{i \geq 2} b_i x^i$ such that $log(x \boxplus y) = log(x) + log(y)$, i.e. an strict isomorphism to the additive one*

Remark 6.1.7. *Over $R \otimes \mathbb{Q}$, the logarithm exists and unique. In the following logarithms are always understood to be over $R \otimes \mathbb{Q}$.*

In homotopy theory, a formal group law is associated with any complex oriented cohomology theory.

Definition 6.1.8. *A complex orientation of a multiplicative generalized cohomology theory E^* is a class $x \in E^2(\mathbb{CP}^\infty)$ such that the restriction of x to $\mathbb{CP}^1 = S^2$ equals the suspension of the unit $1 \in E^*(pt)$.*

A complex oriented cohomology theory is a multiplicative cohomology theory together with a complex orientation.

Remark 6.1.9. *One can show that in this case, any complex vector bundle has a specified E^*-orientation, i.e. a functorial Thom class in E^* theory for every complex vector bundle. Moreover, if $W = U \oplus V$, then the Thom class of W is the pull back along the diagonal of the exterior product of those of U and V. This is equivalent to a multiplicative homomorphism from MU^* to E^*.*

Reversely, given such an E^*-orientation of complex vector bundles, a complex orientation of E^* is obtained by taking the Thom class of the tautological bundle over $\mathbb{CP}^\infty$, since in this case the Thom space is homotopic to $\mathbb{CP}^\infty$. One verifies this gives a bijection between complex orientations of E^* and multiplicative homomorphisms from MU^* to E^*.

Given a complex oriented cohomology theory E^* with orientation x, a calculation using Atiyah-Hirzebruch spectral sequence shows $E^*(\mathbb{CP}^\infty) \cong E^*(pt)[[x]]$ and $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*(pt)[[x, y]]$. The tensor product of complex line bundles induce a map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, and this gives a map $m^* : E^*(pt)[[x]] \to E^*(pt)[[x, y]]$. Since the tensor product is commutative and associative, m^* makes $Spf(E^*(pt)[[x]])$ into a commutative group object. So there is a formal group law over $E^*(pt)$.

In the special case $E^* = MU^*$, the identity map on MU gives MU^* a canonical complex orientation, and for any complex oriented cohomology theory E^*, the homomorphism from MU^* to E^* preserves complex orientation, so the formal group law over $E^*(pt)$ is the pull back of the canonical one over $MU^*(pt)$. This shows the formal group law over $MU^*(pt)$ is universal among formal group laws coming from complex oriented cohomology theories.

In fact, it is the universal formal group law.

Definition 6.1.10. *A formal group law F over some ring L is called universal if for ever formal group law over a ring R, there exists one and only one morphism of rings $\gamma : L \to R$ such that the formal group law over R is the pullback of F along γ.*

One sees that a universal formal group law exists and unique up to isomorphism, and the ring L can be easily described by generators and relations. Its structure was determined by M. Lazard, so the universal ring of formal group laws is called the Lazard ring.

Theorem 6.1.11. *The universal ring of formal group laws is isomorphic to $\mathbb{Z}[x_1, x_2 \dots]$. Moreover, if its logarithm over $\mathbb{Q}[x_1, x_2, \dots]$ is $log(x) = x + \sum_{i\geq 2} m_{i-1}x^i$, then the generators can be chosen so that:*

$$m_i = \begin{cases} -\frac{x_i}{p} & i = p^k - 1 \text{ for some } p \\ -x_i & \text{otherwise} \end{cases}$$

Proof. See [Laz55]. □

The definition of the universal property of the Lazard ring L gives a morphism $L \to MU_*$. This morphism can be computed explicitly once we have the formula for the formal group law over MU_*.

This can be described quite explicitly. The pullback of $x \in MU^*(\mathbb{CP}^\infty)$, the complex orientation of MU^*, to $\mathbb{CP}^n$ is the class of a hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. The map classifying the tensor product is the limit of the Segre embedding $\mathbb{CP}^r \times \mathbb{CP}^t \to \mathbb{CP}^N$ defined by $m : (a_0, \dots, a_r) \times (b_0, \dots, b_s) \to (\dots, a_i b_j, \dots)$ where $N = rs + r + s$. So the pullback of x along this map is a hypersurface in $\mathbb{CP}^r \times \mathbb{CP}^t$ of bidegree $(1, 1)$. Denote the hyperplane class in $MU^*(\mathbb{CP}^s)$ and $MU^*(\mathbb{CP}^t)$ by X and Y respectively. We want to express $m^*(x)$ in terms of X and Y. Let $m^*(x) = \sum a_{i,j} X^i Y^j$. It is computed that $p_!(X^iY^j) = [\mathbb{CP}^{s-i} \times \mathbb{CP}^{t-j}] \in MU^*(pt)$ where p is the map to one point. Also $p_!(X^iY^jm^*(x)) = [H_{s-i,t-j}] \in MU^*(pt)$ where $H_{i,j}$ is the hypersurface in $\mathbb{CP}^i \times \mathbb{CP}^j$ of bidegree $(1, 1)$. So we have the equation $[H_{s-p,t-q}] = \sum[\mathbb{CP}^{s-i-p} \times \mathbb{CP}^{s-j-q}]a_{i,j}$. Letting $s, t \to \infty$, we obtain the formula for the formal group law over MU_*.

Theorem 6.1.12. *The formal group law $F(x, y)$ over MU_* is defined by the formula*

$$F(x, y) = \frac{x + y + \sum_{i,j\geq 1}[H_{i,j}]x^iy^j}{(1 + \sum_{i\geq 1}[\mathbb{CP}^i]x^i)(1 + \sum_{j\geq 1}[\mathbb{CP}^j]y^j)}$$

Proof. See [BMN71]. □

Corollary 6.1.13. *The logarithm of $F(x, y)$ is $log_{MU}(x) = \sum_{n\geq 0} \frac{[\mathbb{CP}^n]}{n+1} x^{n+1}$.*

This gives the morphism $L \to MU_*$, from which we deduce the theorem of D. Quillen:

Theorem 6.1.14. *The homomorphism from the Lazard ring to MU_* is an isomorphism.*

Proof. See [Qui69a]. □

This theorem shows a close connection between the theory of formal groups and the theory of complex cobordism. The structure of complex cobordism can be studied more easily with the aid of the theory of formal groups.

We can describe the structure of the Hopf algebroid (MU_*, MU_*MU) in terms of formal groups as follows:

First define a contravariant functor $\mathfrak{M}^s_{FG}$ from the category of affine schemes to grouproids. Let R be a ring, then $\mathfrak{M}^s_{FG}(Spec(R))$ is the groupoid with objects formal group laws over R and morphisms the strict isomorphism between formal group laws.

Remark 6.1.15. *The functor $\mathfrak{M}^s_{FG}$ is similar to the moduli stack of formal groups except that in the latter case we take all isomorphism as morphisms in the groupoid associated with $Spec(R)$.*

It is readily seen that the functor is represented by the Hopf algebroid (L, LB) where L is the Lazard ring and $LB \cong L \otimes \mathbb{Z}[b_1, b_2, \dots]$, since a strict isomorphism is completely determined by a formal group law and a formal power series of the form $g(x) = x + b_1x^2 + b_2x^3 + \dots$. The structure of the Hopf algebroid can be calculated from the above description:

Recall that the generators of $L \cong \mathbb{Z}[x_1, x_2, \dots]$ can be chosen so that if we set

$$m_i = \begin{cases} -\frac{x_i}{p} & i = p^k - 1 \text{ for some p} \\ -x_i & \text{otherwise} \end{cases}$$

then $log(x) = x + \sum_{i \geq 1} m_i x^{i+1}$.

Counit $\epsilon(b_i) = 0$.

Left unit η_L is the inclusion $L \to L[b_1, b_2, \dots]$.

Right unit $\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i (\sum_{j \geq 0} c(b_j))^{i+1}$, where $m_0 = b_0 = 1$ and c is the conjugation.

Conjugation $\sum_{i \geq 0} c(b_i)(\sum_{j \geq 0} b_j)^{i+1} = 1$ and $c(m_i) = \eta_R(m_i)$.

Coproduct $\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} (\sum_{i \geq 0} b_i)^{j+1} \otimes b_j$.

See [Rav86], appendix A2 for details.

Now suppose E^* is a multiplicative generalized cohomology theory and x, y two complex orientations of E^*. They give two formal group laws F^x, F^y. Any cohomology class in $E^*(\mathbb{CP}^\infty)$ can be expressed in a formal power series in x. Since the restriction of x and y to $\mathbb{CP}^1$ equal to the suspension of the unit, we have $y = x + \sum_{i \geq 1} b_i x^i$. This gives a strict isomorphism between F^x and F^y.

Over $MU \wedge MU$, there are two complex orientations induced from the morphism $MU = MU \wedge \mathbb{S} \xrightarrow{id \wedge \eta} MU \wedge MU$ and $MU = \mathbb{S} \wedge MU \xrightarrow{\eta \wedge id} MU \wedge MU$ where $\eta : \mathbb{S} \to MU$ is the unit map for the ring spectrum MU. And there is a strict isomorphism between the two formal group laws F^l and F^r. So there is a morphism of rings $LB \to MU \wedge MU$.

Since the left and right unit of the Hopf algebroid (MU_*, MU_*MU) is induced from the maps $id \wedge \eta$ and $\eta \wedge id$, the map $(L, LB) \to (MU, MU_*MU)$ preserves left and right unit. Moreover, the coproduct of (MU, MU_*MU) is induced from the map $MU \wedge MU = MU \wedge \mathbb{S} \wedge MU \xrightarrow{id\wedge\eta\wedge id} MU \wedge MU \wedge MU$. There are three complex orientations of $MU \wedge MU \wedge MU$ corresponding to the three factors, and there are three formal group laws F^l, F^m, F^r. The isomorphism constructed above is functorial so the composition of the morphisms from F^l to F^m and from F^m to F^r is just that from F^l to F^r. This means the map $(L, LB) \to (MU_*, MU_*MU)$ also preserves coproduct. In fact, it is a morphism between Hopf algebroids. By direct calculation we conclude:

Theorem 6.1.16. *There is a canonical of Hopf algebroids* $(L, LB) \cong (MU_*, MU_*MU)$.

Proof. See [Rav86], chapter 4. □

6.2 The Brown-Peterson spectrum

When localized at a prime p, the theory of formal groups can be simplified so that one can concentrate on a particular type of formal group laws. Similarly, the complex cobordism theory, after localization at a prime p, can be simplified to the Brown-Peterson cohomology theory.

First we describe Cartier's theory of typical formal group laws.

Definition 6.2.1. *Let F be a formal group law over a ring R. Call a power series $f \in R[[x]]$ without constant term a curve in the formal group defined by the law. The set of curves forms an abelian group with addition $(f \boxplus g)(x) = F(f(x), g(X))$.*

Define the operators:

$$([r]f)(x) = f(rx) \text{ where } r \in R \tag{6.2.2}$$
$$(V_n f)(x) = f(x^n) \text{ for } n \geq 1 \tag{6.2.3}$$
$$(F_n f)(x) = \sum_{i=1}^{n}{}^F f(\zeta_i x^{\frac{1}{n}}) \text{ for } n \geq 1 \tag{6.2.4}$$

Here $\sum^F a_i = a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n$, *and ζ_i are the nth roots of unity.*

Now fix a prime p.

Definition 6.2.5. *Let R be an algebra over $\mathbb{Z}_{(p)}$. Then a curve f is said to be typical if $F_q f = 0$ for any prime $q \neq p$.*

Remark 6.2.6. *If R is torsion-free then this is equivalent to the series $l(f(X))$ over $R \otimes \mathbb{Q}$ having only terms of degree a power of p, where $l(x)$ is the logarithm of the formal groups law F.*

Definition 6.2.7. *The group law F is said to be a typical formal group law, if the curve $x \in R[[x]]$ is typical.*

Any formal groups law can be transformed into a typical one by a canonical change of coordinates.

Theorem 6.2.8. *Let c_F be the curve*

$$c_F^{-1} = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n x \tag{6.2.9}$$

where $\mu(n)$ is the Möbius function. Then the formal group law $(c_F^* F)(x, y) = c_F(F(c_F^{-1}(x), c_F^{-1}(y)))$ is typical.

Proof. See [Car67]. □

It is not hard to see that there is a universal typical formal group law, over a $\mathbb{Z}_{(p)}$ algebra V. we have a ring homomorphism $\pi_*(MU) \to V$ by the universal property of $\pi_*(MU)$, and this extends to $\pi_*(MU) \otimes \mathbb{Z}_{(p)} \to V$ since V is a $\mathbb{Z}_{(p)}$ algebra.

Applying the above theorem to the universal formal group law over $\pi_*(MU)$, we get a typical formal group law over $\pi_*(MU) \otimes \mathbb{Z}_{(p)}$. This induces a ring homomorphism $V \to \pi_*(MU) \otimes \mathbb{Z}_{(p)}$.

One checks that these make V a direct summand of $\pi_*(MU) \otimes \mathbb{Z}_{(p)}$.

All the above works topologically. These maps lift to an idempotent natural transformation from $MU_{(p)}$ to itself, and defines a generalized homology theory, the Brown-Peterson homology, abbreviated by BP, whose coefficient ring is just V, the universal ring of typical formal group laws.

Theorem 6.2.10. *There is a ring spectrum BP for each prime number p, such that:*

1. *$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, ...]$ with v_i in dimension $2(p^i - 1)$.*
2. *BP is a direct summand of $MU_{(p)}$.*
3. *$MU_{(p)}$ is a direct sum of suspensions of BP's.*
4. *$BP_*BP \cong BP_*[t_1, t_2, ...]$.*

Proof. See [BP66] or [Qui69a]. □

Remark 6.2.11. *We can also construct BP by using the Landweber exact theorem. Or killing the regular ideal $(x_i : i \neq p^k - 1)$ in $\pi_*(MU)$, see [EKMM97].*

Remark 6.2.12. *The higher multiplicative structure of BP is complicated. It is not known by the authors of this report whether BP has an E_∞ structure. But it is proved that there is no map $BP \to MU$ of commutative $\mathbb{S}$-algebras, see [HKM01].*

Remark 6.2.13. *As in the case of MU, the Hopf algebroid (BP_*, BP_*BP) represents the functor from the category of affine schemes over $\mathbb{Z}_{(p)}$ to the category of groupoids, which sends Spec(R) to the groupoid with objects typical formal group laws over R and morphisms the strict isomorphisms.*

Remark 6.2.14. *One can show that all the prime ideals of BP_* which is also a sub BP_*BP-comodule are of the form $(p, v_1, \ldots, v_n)$ or $(p, v_1, v_2, \ldots)$. This enable one to prove the Landweber exact theorem. See [Rav86] for details.*

6.3 The Morava K-theories

To summarize the preceding two sections, we can say that the Hopf algebroid (MU_*, MU_*MU) represents the moduli stack of formal groups, and (BP_*, BP_*BP) represents the moduli stack of formal groups localized at a prime p. As these geometric objects can be studied by investigating their "points", the complex cobordism theory and the Brown-Peterson theory can be studied using the more simple Morava K-theories.

When we study formal groups over a finite field K of characteristic p, the formal groups can be classified by their height. Roughly speaking, the height is the first k for which the image of $v_k \in BP_*$ is nonzero under the map $BP_* \to K$ classifying some typical coordinate. A example of a formal group with height n is the formal group law over $\mathbb{F}_p$ classified by the map $BP_* \to \mathbb{F}_p$ sending v_n to 1 and other v_i's to 0.

The Morava K-theory corresponding to the formal group law over $\mathbb{F}_p$ with height n is constructed from BP, with coordinate ring $\mathbb{F}_p[v_n, v_n^{-1}]$. This is achieved by taking quotients.

In general, Let $\mathfrak{R}$ be a ring spectrum, and $\mathfrak{M}$ a module spectrum over it. Let $a \in \pi_*(\mathfrak{R})$ such that the map of multiplication by a in $\pi_*(\mathfrak{M})$ is a monomorphism. There is a map $\hat{a} : \mathfrak{M} \to \mathfrak{M}$ defined by the composition $\mathfrak{M} \cong \mathbb{S} \wedge \mathfrak{M} \xrightarrow{a \wedge id} \mathfrak{R} \wedge \mathfrak{M} \to \mathfrak{M}$. Construct the distinguished triangle $\ldots \to \mathfrak{M} \xrightarrow{\hat{a}} \mathfrak{M} \to \mathfrak{N} \to \Sigma\mathfrak{M} \to \ldots$. Using the long exact sequence one verifies that $\pi_*(\mathfrak{N}) \cong \pi_*(\mathfrak{M})/a\pi_*(\mathfrak{M})$. Applying this iteratively, one can take the quotient by a regular ideal.

So when we take the quotient of BP by the regular ideal $(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots)$, and invert v_n, we get the spectrum $K(n)$ of the Morava K-theory. See [EKMM97] for details of this construction.

Remark 6.3.1. *The Morava K-theory can also be constructed using other methods, such as the Sullivan-Baas construction using cobordism of singular manifolds.*

The Morava K-theories has many good properties. They are much simpler than the Brown-Peterson cohomology, and also have Künneth formulae. See [Wür91] for a more detailed survey of Morava K-theories.

6.4 The chromatic spectral sequence

Using the filtration of BP given by the height of formal groups, we can construct the chromatic spectral sequence introduced in [MRW77], converging to the Adams–Novikov spectral sequence, and revealing many structures of the latter spectral sequence.

To begin with, we know a formal group law of height n is something like taking the quotient of BP_* by $(p, v_1, \ldots, v_{n-1})$, and inverting v_n.

Definition 6.4.1. *Define BP_*BP-comodules inductively as follows:*

$$N_n^0 = BP_*/(p, v_1, \ldots, v_{n-1})$$

Assuming N_n^s is already defined, define $M_n^s = v_{n+s}^{-1} N_n^s$. Let $N_n^{s+1} = M_n^s / N_n^s$.

We can write the definition simply as:

$$N_n^s = BP_*/(p, v_1, \ldots, v_{n-1}, v_n^\infty, \ldots, v_{n+s-1}^\infty) \tag{6.4.2}$$

$$M_n^s = v_{n+s}^{-1} BP_*/(p, v_1, \ldots, v_{n-1}, v_n^\infty, \ldots, v_{n+s-1}^\infty) \tag{6.4.3}$$

Patching up the short exact sequences $0 \to N_n^s \to M_n^s \to N_n^{s+1} \to 0$ we get resolutions:

$$BP_*/(p, \ldots, v_{n-1}) \to M_n^0 \to M_n^1 \to \ldots \tag{6.4.4}$$

This resolution gives the chromatic spectral sequence, with E_1-term $Ext_{BP_*BP}^{*,*}(M_n^s)$, converging to $Ext_{BP_*BP}(BP_*/(p, \ldots, v_{n-1}))$.

Remark 6.4.5. *The exact sequence $0 \to N_n^s \to M_n^s \to N_n^{s+1} \to 0$ can also be used to calculate the Ext groups of M_n^s's from the Ext groups of $K(n)_*$'s, which is much easier to deal with.*

6.5 Nilpotence and periodicity

To describe the geometric explanation of chromatic filtration, we should first state two theorems of Hopkins and his collaborators.

Theorem 6.5.1. *Let X be a finite spectrum. Then $f : \Sigma^q X \to X$ is nilpotent under composition if and only if $MU_*(f) : MU_*(X) \to MU_*(X)$ is nilpotent.*

Proof. See [DHS88]. $\square$

This striking result fills the gap between homotopy and homology under the meaning of nilpotence and when $X = \mathbb{S}$, this theorem recovers the Nishida nilpotence theorem, which states that every element is nilpotent in $\pi_*^{st}(\mathbb{S})$. Based on this nilpotence theorem, they give another periodicity theorem.

Theorem 6.5.2. *Let X be a p-torsion finite spectrum and $f : \Sigma^q X \to X$. f is called a "v_n-map" if the induced map of Morava K-theory $K(i)_*(f)$:*

$K(i)_*(X) \to K(i)_*(X)$ is an isomorphism for $i = n$ and nilpotent for $i \neq n$. Then

(i)X has a v_n-map f.(We call X have type n.)

(ii)f is unique after iterating sufficient times and under homotopy equivalence and f can be chosen such that $K(i)_*(f)$ is multiplication by a unit in $K(i)_*(pt)$ for $i = n$ or zero for $i \neq n$.

Proof. See [DHS88]. □

If we say complex bordism is the right theory to detect nilpotence, then Morava K-theory is the right one to detect periodicity. A significant application is as follows.

Let p be an odd prime and $V(0) = S^n \cup_p e^{n+1}$, called Moore space, the cofiber of the degree p map between S^n with $n \geq 3$, then $V(0)$ has type 1. It is shown by Adams and Toda that there is a v_1-map $\Sigma^q V(0) \to V(0)$, where $q = 2(p-1)$. Denote the cofiber of this map by $V(1)$, then $V(1)$ has type 2 by the long exact sequence of Morava K-theory. It is showed by Smith and Toda that when $p \geq 5$, there is a v_2-map of $V(1)$. $V(2)$ is defined similarly and is showed to have a v_3-map for $p \geq 7$. All these can be recovered by the periodicity theorem. The importance of these maps is that associated to these maps, we can have some periodic families in $\pi_*^{st}(\mathbb{S})$. For example, associated to $V(0)$, the composition of these obvious maps

$$S^{n+qi} \to \Sigma^{iq} V(0) \to \cdots \to \Sigma^q V(0) \to V(0) \to S^{n+1}$$

gives a map $S^{n+qi} \to S^{n+1}$, which represents an nontrivial element in $\pi_{qi-1}^{st}(\mathbb{S}) \otimes \mathbb{Z}/(p)$. The same construction is valid also for $V(1)$ and $V(2)$. In general, any periodic map should lead to a nontrivial element in $\pi_{qi-1}^{st}(\mathbb{S}) \otimes \mathbb{Z}/(p)$ under some constructions a little complex than above, and this gives the geometric aspect of the chromatic filtration.

6.6 J-homomorphism and the K(1)-local sphere

The theory for phenomenon of low chromatic filtration is more or less well understood now. For chromatic filtration one, it corresponds to K-theory, which gives a v_1-periodic theory, and detects the image of the J-homomorphism. The theory of topological modular forms correspond to phenomenon of chromatic filtration two, and can be used to study the $K(2)$-local sphere.

The J-homomorphism is a homomorphism from $\pi_*(O)$ to $\pi_*(\mathbb{S})$. The study of the J-homomorphism was classic in algebraic topology. In [Ada66a], it was shown that the image of the J-homomorphism was detected by K-theory. Together with the Adams conjecture proved in [Qui71a, Sul74], the order of the J-homomorphism could be determined. This part of homotopy groups correspond to the chromatic filtration one part of the Adams–Novikov spectral sequence, since they are detected by K-theory.

First we define the J-homomorphism. For any map $l : S^n \to BO$, there correspond to a virtual vector bundle T over S^n of dimension 0. Let $\mathfrak{M}$ be its

Thom space. S^n has two cells e^0, e^n. The Thom space of $T|e^0$ is simply $\mathbb{S}$, and the Thom space of $T|e^n$ is $\Sigma^\infty e^n$, because they are trivial bundles. This gives a cell decomposition of $\mathfrak{M}$ into $\mathbb{S} \cup_f C\Sigma^{n-1}\mathbb{S}$ where $C\Sigma^{n-1}\mathbb{S}$ is the cone of $\Sigma^{n-1}\mathbb{S}$ and f is the attaching map $\Sigma^{n-1}\mathbb{S} \to \mathbb{S}$. The J-homomorphism is defined by $J(l) = f$. Analogously, one can also define the complex J-homomorphism using U instead of O.

Remark 6.6.1. *We can also define the J-homomorphism as follows. Since the normal bundle of S^n is trivial, any map $l : S^n \to O$ gives a stable framing for S^n. Then $J(l)$ is defined to be the cobordism class of S^n with this framing.*

As shown in [Ada66a], the image of J-homomorphism is detected by K-theory by investigating the extension of K_*K comodules defined by the cofibre sequence $\mathbb{S} \to \mathfrak{M} \to \Sigma^n\mathbb{S}$. Essentially, this is the part corresponding to the term $Ext_{BP_*BP}(M_0^1)$ in the chromatic spectral sequence.

$Ext_{BP_*BP}(M_0^1)$ can be computed directly, and the relevant differentials in the chromatic spectral sequence can also be determined. To simplify the case, we discuss the odd primary components. The case for $p = 2$ can be found in the references.

Theorem 6.6.2. *For p an odd prime,*

$$Ext^{s,t}_{BP_*BP}(M_0^1) = \begin{cases} \mathbb{Q}/\mathbb{Z}_{(p)} & if \qquad (s,t) = (0,0) \\ \mathbb{Z}/p^{i+1} & if \qquad (s,t) = (0, rp^i q) \\ \mathbb{Q}/\mathbb{Z}_{(p)} & if \qquad (s,t) = (1,0) \\ 0 & otherwise \end{cases}$$

Where $q = 2p - 2$ and r is some integer not divisible by p.

Proof. See [Rav86], chapter 5. □

Using this, the groups $Ext^1(BP_*)$ can be computed:

Theorem 6.6.3. *$Ext^1(BP_*)$ is generated by the groups $Ext^{0,t}(M_0^1)$ for $t > 0$.*

Proof. See [Rav86], chapter 5. □

These elements give invariants for elements of the homotopy groups of spheres of filtration one in the Adams-Novikov resolution. These are essentially the e-invariant in [Ada66a], so we call them also e-invariant here. We can compute these invariants for the elements $J(l)$.

In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[dd] & T \arrow[r] \arrow[d, "\lambda"] & \Sigma^n\mathbb{S} \arrow[d] \\
 & MU \arrow[d] & \bar{BP} \arrow[d, "\mu"] \\
\mathbb{S} \arrow[r] & BP \arrow[r] \arrow[ur] & BP \wedge \bar{BP}
\end{tikzcd}$$

Here λ is induced from $l : S^n \to BO$ and μ is induced from the unit $\mathbb{S} \to BP$. The bottom line is the first stems in the canonical Adams-Novikov resolution of $\mathbb{S}$.

Taking the BP homology, the bottom line is the bar complex, so the composite $\Sigma^n\mathbb{S} \to \bar{BP} \to BP \wedge \bar{BP}$ gives the above mentioned e-invariant. Calculation shows that for p an odd prime, the image of J-homomorphism can take arbitrary values of the e-invariant, so that these elements in the E_2-term of Adams–Novikov spectral sequenceare permanent cycles. Moreover, the results in [Ada66a] show that there are no group extension for these elements.

Theorem 6.6.4. *When localized at an odd prime p, the elements in $Ext^1(BP_*)$ are permanent cycles. They detect the image of J-homomorphism, and have the order indicated in the Adams–Novikov spectral sequence.*

Proof. See [Rav86], chapter 5. □

These can also be studied using the notion of localization. The general notion of localization with respect to a spectrum can be found in [Bou75]. The point is that when we want to study the information given by a particular type of generalized cohomology theory E, we invert all the maps which induce an isomorphism on E-homology. This can be achieved using model category techniques, and we get a functor L_E, localization with respect to E, which assigns every spectrum an E-local spectrum whose E-homology is isomorphic to the original one, and the isomorphism is induced by a natural transformation $Id \to L_E$. Here a spectrum X being E-local means that for any spectrum Y with $E_*(Y) = 0$ we have $[Y, X] = 0$.

When studying the homotopy groups, we can use the localization functor $L_{K(n)}$, localization with respect to Morava K-theory, to concentrate on phenomenon of chromatic filtration n.

Define $L_n = L_{H\mathbb{Q}\vee K(1)\vee\cdots\vee K(n)}$, then it is proved that $L_n\mathbb{S}$ approximates $\mathbb{S}$ as $n \to \infty$. So the study of $L_n\mathbb{S}$ for small n is the first step towards a systematic understanding of the homotopy groups of spheres.

When $n = 1$, L_1 is the same as the localization with respect to K-theory. The homotopy groups of $L_1\mathbb{S}$ is computed and they are essentially the image of J-homomorphism.

Theorem 6.6.5. *For p an odd prime,*

$$\pi_i(L_1\mathbb{S}) = \begin{cases} \mathbb{Z}_{(p)} & if \quad i = 0 \\ \mathbb{Q}/\mathbb{Z}_{(p)} & if \quad i = -2 \\ \mathbb{Z}/p^{j+1} & if \quad j = rp^j q - 1 \\ 0 & otherwise \end{cases}$$

Here $q = 2p - 2$ and r is some integer not divisible by p.

Moreover, the positive dimensional part of $\pi_*(L_1\mathbb{S})$ is isomorphic to the subgroup of $\pi_*(\mathbb{S})$ detected by $Ext^1_{BP_*BP}(BP_*)$ in the Adams–Novikov spectral sequence.

Proof. See [Rav84]. □

There is another description of the $K(1)$-local sphere.

Theorem 6.6.6. *There is a fibre sequence*

$$L_{K(1)}\mathbb{S} \to KO_p \xrightarrow{\psi^N - 1} KO_p$$

Where ψ^N is the Adams operation with N a topological generator of $\mathbb{Z}_p^\times$.

Proof. See [Rav84]. □

Remark 6.6.7. *In fact, K-theory suffices at odd prime, but we must use KO-theory at the prime* 2.

6.7 Topological modular forms and the K(2)-local sphere

The theory for phenomenon of chromatic filtration two is considerably more complex. There are extensive computations of the $Ext(M^2)$-term in the chromatic spectral sequence, and computations of the $K(2)$-local sphere. To understand these results, one uses the topological modular forms, which is a ring spectrum, generalizing the ring of the classical modular form. In particular, there is a finite resolution of the $K(2)$-local sphere by topological modular forms, so the structure of the chromatic filtration two part of the homotopy groups of spheres if understood by this method.

For the sake of simplicity, we work with the prime 3 in the rest of this section, so all thing are assumed to be 3-local.

The homotopy groups of the $K(2)$-local sphere is as follows:

Theorem 6.7.1.

$$\pi_*(L_{K(2)}\mathbb{S}) = (\mathbb{Z}_3 \oplus \mathfrak{A}_+) \otimes \Lambda(\zeta_2) \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_2$$

Where $\mathbb{Z}_3$ is the ring of 3-adic numbers and

$$\mathfrak{A}_+ = \mathbb{Z}_3\{\alpha_{3^i s/i+1} : \alpha_{3^i s/i+1} \in \mathfrak{A}, s > 0\}$$

with

$$\mathfrak{A} = \sum_{i\geq 0} \mathbb{Z}/3^{i+1} < \alpha_{3^i s/i+1} : s \in \mathbb{Z} \textit{ not divisible by } 3 >$$

and

$$\mathfrak{G}_1 = \mathfrak{B} \oplus \mathfrak{C} \oplus \mathfrak{C}\mathfrak{I} \oplus \mathfrak{B}^* \oplus (\mathfrak{B}_1 \oplus \mathfrak{C})\zeta_2$$

$$\mathfrak{G}_2 = \hat{\mathfrak{G}} \oplus \hat{\mathfrak{G}}^* \oplus \hat{\mathfrak{G}\mathfrak{Z}} \oplus \hat{\mathfrak{G}\mathfrak{Z}}^*$$

with

$$
\begin{aligned}
\mathfrak{B} &= \mathbb{Z}_3\{\beta_{3^n s/3^i m, i+1} : n \geq 0, s \in \mathbb{Z} \textit{ not divisible by } 3, i \geq 0; \\
&\quad 1 \leq m < 4 \cdot 3^{n-2i-1} \textit{ with } m \textit{ not divisible by } 3 \textit{ or } m \geq 4 \cdot 3^{n-2i-2}\} \\
\mathfrak{B}_1 &= \mathbb{Z}_3\{\beta_{3^n s/3^i m, i+1} : \beta_{3^n s/3^i m, i+1} \in \mathfrak{B}, (s+1) \textit{ not divisible by } 3, m \textit{ divisible by } 3, \\
&\quad \textit{or } s = 3^{k+2}t - 1 \textit{ with } k \geq 0, t \textit{ not divisible by } 3 \textit{ and } i \neq k+1\} \\
\mathfrak{C} &= \mathfrak{C}_1 \oplus \mathfrak{C}_2 \\
\mathfrak{C}_1 &= \mathbb{Z}_3\{\widetilde{\alpha_1\beta}_{3^n(3t+1)/3^i m+1, i} : 0 < i \leq n, 2 \cdot 3^{n-i} < 3^i m \leq 2 \cdot 3^{n-i+1}\} \\
\mathfrak{C}_2 &= \mathbb{Z}_3\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^i m+1, i} : 0 < i \leq n, 2 \cdot 3^{n-i-1} < 3^i m - 8 \cdot 3^n \leq 2 \cdot 3^{n-i}\} \\
\mathfrak{C}\mathfrak{I} &= \mathfrak{C}\mathfrak{I}_0 \oplus \mathfrak{C}\mathfrak{I}_1 \oplus \mathfrak{C}\mathfrak{I}_2 \oplus \mathfrak{C}\mathfrak{I}_3 \\
\mathfrak{C}\mathfrak{I}_0 &= \mathbb{Z}_3\{c_{3^n s} : n \geq 0, s = 3t+1 \textit{ or } s = 9t-1, t \in \mathbb{Z}\} \\
\mathfrak{C}\mathfrak{I}_1 &= \mathbb{Z}_3\{\widetilde{\alpha_1\beta}_{3^n(3t+1)/3^i m+1, i+1} : 0 \leq i \leq n, 2 \cdot 3^{n-i-1} < 3^i m \leq 2 \cdot 3^{n-i}, m \textit{ not divisible by } 3\} \\
\mathfrak{C}\mathfrak{I}_2 &= \mathbb{Z}_3\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^i m+1, i+1} : m \textit{ not divisible by } 3 \textit{ and } m \leq 8 \cdot 3^{n-i}, \\
&\quad 0 \leq i < n, i < k+1 \textit{ when } t \textit{ is divisible by } 3^k\} \\
\mathfrak{C}\mathfrak{I}_3 &= \mathbb{Z}_3\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^n m+1, i} : i = \begin{cases} n+1 & \textit{if} \quad m = 1,3,4,6,7 \\ n+2 & \textit{if} \quad m = 2,8 \\ n+3 & \textit{if} \quad m = 5, n < k+1 \textit{ when } t \textit{ is divisible by } 3^k \end{cases} \} \\
\mathfrak{B}^* &= \mathbb{Z}_3\{\beta(n)^*_{3^{n+l}s/3^i m, i+1} : s \in \mathbb{Z} \textit{ not divisible by } 3, i \geq 0, n \geq 0, 0 < 3^i m \leq 4 \cdot 3^n, \\
&\quad l > i \textit{ and } l > i+1 \textit{ when } (s+1) \textit{ is divisible by } 3\}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\mathfrak{G}} &= \sum_{t \in \mathbb{Z}} (B_5\{\beta_{9t+1}\} \oplus B_4\{\beta_{9t+1}\beta_{6/3}\} \oplus B_3\{\overline{\beta_{9t+7}\alpha_1}\} \oplus B_2\{\beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta'_1], [\beta_{9t+5}\beta'_1]\}) \\
&\quad \oplus B_1\{[\beta_{9t-1/2}\beta'_1]\} \\
\hat{\mathfrak{G}}^* &= \sum_{t \in \mathbb{Z}} (B_5\{\chi^1_{9t+1}\} \oplus B_4\{\chi^0_{9t+3}\} \oplus B_2\{\beta(0)^*_{9t+1}, \beta_{6/3}\beta(0)^*_{9t+1}, \beta_{6/3}\beta(0)^*_{9t+4}\} \\
&\quad \oplus \sum_{n \geq 1} (B_3\{\beta(0)^*_{3^{n+2}t+9u+3} : u \in \mathbb{Z} \setminus I(n)\} \oplus B_2\{\beta(0)^*_{3^{n+2}t+9u+3} : u \in I(n)\})) \\
\hat{\mathfrak{G}\mathfrak{Z}} &= \sum_{t \in \mathbb{Z}} (B_5\{\zeta\beta_{9t+1}\} \oplus B_3\{\zeta\beta_{9t+1}\beta_{6/3}\} \oplus B_2\{\overline{\zeta\beta_{9t+7}\alpha_1}, \zeta\beta_{9t+1}\alpha_1, \zeta[\beta_{9t+2}\beta'_1], \zeta[\beta_{9t+5}\beta'_1]\}) \\
\hat{\mathfrak{G}\mathfrak{Z}}^* &= \sum_{t \in \mathbb{Z}} (B_5\{\zeta_2\chi^1_{9t+7}\} \oplus B_4\{\zeta_2\chi^0_{9t+3}\} \oplus B_2\{\zeta_2\beta(0)^*_{9t+1}\} \oplus B_1\{\zeta_2\beta_{6/3}\beta(0)^*_{9t+1}, \zeta_2\beta_{6/3}\beta(0)^*_{9t+4}\} \\
&\quad \oplus \sum_{n \geq 1} (B_3\{\zeta_2\beta(0)^*_{3^{n+2}t+9u+3} : u \in \mathbb{Z} \setminus I(n)\} \oplus B_2\{\zeta_2\beta(0)^*_{3^{n+2}t+9u+3} : u \in I(n)\}))
\end{aligned}
$$

Here

$$B_k = \mathbb{Z}/3[\beta_1]/(\beta_1^k)$$

$$I(n) = \{x \in \mathbb{Z} : z = \frac{3^{n-1}-1}{2} \ or\ x = 5 \cdot 3^{n-2} + \frac{3^{n-2}-1}{2}\}$$

The degree and order of the generators are as follows:

generator	degree	order	generator	degree	order
$\alpha_{a/b}$	$4a-1$	3^b	β'_1	11	
$\widetilde{\beta_{a/b,c}}$	$16a-4b-2$	3^c	c_a	$16a-7$	3^{n+1} *if* $a = 3^n s$
$\widetilde{\alpha_1\beta_{a/b,c}}$	$16a-4b+1$	3^c	$\beta(a)^*_{b/c,d}$	$16b - 8 \cdot 3^a - 4c - 4$	3^d
χ^0_a	$16a+7$	3	χ^1_a	$16a+15$	3
ζ_2	-1				

In the above table s is not divisible by 3.

The following abbreviation is adopted: $\alpha_a = \alpha_{a/1}$, $\beta_{a/b} = \beta_{a/b,1}$, $\beta_a = \beta_{a/1}$, $\widetilde{\alpha_1\beta_{a/b}} = \widetilde{\alpha_1\beta_{a/b,1}}$, $\widetilde{\alpha_1\beta_a} = \widetilde{\alpha_1\beta_{a/1}}$, $\beta(a)^*_{b/c} = \beta(a)^*_{b/c,1}$, $\beta(a)^*_b = \beta(a)^*_{b/1}$.

Proof. See [SW02a]. □

To understand the structure of this complicated groups, one can take a similar procedure as in the end of last section and try to find a finite resolution of the $K(2)$-sphere by some spectra with more regular behavior.

To do this, we need the construction of the $K(n)$-local spheres as a homotopy fixed point spectrum.

First we introduce the ring spectrum E_n. The formal groups over a finite field are completely classified by their height. And the theory of deformation of a formal groups of height n is given by the Lubin-Tate theory, which asserts that the deformation can be classified by the ring $(E_n)_* = W(\mathbb{F}_{p^n})[[v_1, \ldots, v_{n-1}]][v_n, v_n^{-1}]$, where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors with coefficients in $\mathbb{F}_{p^n}$. There is a formal group law over $(E_n)_*$ which is the universal deformation of the height n formal group, so there is a map $MU_* \to (E_n)_*$. The Landweber exact theorem gives a spectrum E_n with coefficient $(E_n)_*$.

The action of the automorphism group of the height n formal group, the Morava stabilizer group $\mathcal{S}_n$, lifts to an action on the universal deformation. There is a deep theorem that the spectrum E_n is an E_∞ spectrum in a essentially unique way, and the action lifts to an action of $\mathcal{G}_n = \mathcal{S}_n \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ on E_n. See [GH04] for details.

Theorem 6.7.2. *The $K(n)$-local sphere is the homotopy fixed point spectrum of the action of $\mathcal{G}_n$ on E_n.*

Proof. See [DH04]. □

The spectra E_n are analogues of complex K-theory for higher chromatic filtration. The main features of 6.6.6 is a resolution of the $K(1)$-local sphere by spectrum which are homotopy fixed points of finite subgroups of $\mathcal{G}_1$. Since $\mathcal{G}_1$ have a maximal elementary abelien subgroup $\mathbb{Z}/2$, the Krull dimension of the ring of its cohomology with coefficient $\mathbb{F}_2$ is one, so we need to use fixed point spectrum of the action of $\mathbb{Z}/2$ which turns out to be KO when working with the prime 2.

At the chromatic filtration 2, there is a similar situation. For example, at the prime 3, there is also a finite resolution of the $K(2)$-local sphere.

We need some finite subgroups of the Morava stabilizer group. This can be done with the help of elliptic curves, since supersingular elliptic curves are natural sources of formal groups of height two. There is only one supersingular elliptic curves at the prime 3, which can be represented by the equation over $\mathbb{F}_3$

$$y^2 = x^3 - x \tag{6.7.3}$$

It gives a height two formal group, and the automorphism group of this elliptic curve gives a finite subgroup of $\mathcal{S}_2$. It turns out the automorphism group of this elliptic curve consists of pairs (s,t) with $t \in \mathbb{F}_3$ and $s \in \mathbb{F}_9{}^\times$ satisfying $s^4 = 1$, and the action being

$$\begin{cases} x & \mapsto & s^2x+t \\ y & \mapsto & sy \end{cases}$$

This is a finite group G_{12} of order 12. And we have a subgroup $G_{24} = G_{12} \rtimes Gal(\mathbb{F}_9/\mathbb{F}_3) \subset \mathcal{G}_2$. Denote the homotopy fixed point of this group acting on E_2 by $E_2^{G_{24}}$.

We also need another subgroup of $\mathcal{G}_2$. Let $\omega \in \mathbb{F}_9{}^\times$ be a generator of this cyclic group. dilation by ω, though not an automorphism of the elliptic curve, is nonetheless an automorphism of the height two formal group. And taking semi-product with the Galois group $Gal(\mathbb{F}_9/\mathbb{F}_3)$ gives a subgroup $SD_{16} \subset \mathcal{G}_2$ of order 16. Denote the homotopy fixed point of this group acting on E_2 by $E_2^{SD_{16}}$.

Theorem 6.7.4. *There is a resolution of the $K(2)$-local sphere*

$$L_{K(2)}\mathbb{S} \to E_2^{G_{24}} \to E_2^{G_{24}} \vee \Sigma^8 E_2^{SD_{16}} \to \Sigma^8 E_2^{SD_{16}} \vee \Sigma^{40} E_2^{SD_{16}} \to \Sigma^{40} E_2^{SD_{16}} \vee \Sigma^{48} E_2^{G_{24}} \to \Sigma^{48} E_2^{G_{24}}$$

meaning there is a finite resolution

$$\begin{tikzcd}
L_{K(2)}\mathbb{S} \arrow[r] & X_3 \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & E_2^{G_{24}} \\
\Sigma^{44} E_2^{G_{24}} \arrow[u] & \Sigma^{45} E_2^{G_{24}} \vee \Sigma^{37} E_2^{SD_{16}} \arrow[u] & \Sigma^{6} E_2^{SD_{16}} \vee \Sigma^{38} E_2^{SD_{16}} \arrow[u] & \Sigma^{7} E_2^{SD_{16}} \vee \Sigma^{-1} E_2^{G_{24}} \arrow[u] &
\end{tikzcd}$$

Proof. See [GHMR05]. □

Since this resolution is finite, the main structure of $\pi_*(L_{K(2)}\mathbb{S})$ is controlled by $\pi_*(E_2^{G_{24}})$ and $\pi_*(E_2^{SD_{16}})$.

This is the local point of view. We can also take a global point of view. Instead of taking a single supersingular elliptic curve, we consider all the elliptic curves, which are arranged as the moduli stack of elliptic curves, $\mathfrak{M}_{ell}$. There is a sheaf ω over $\mathfrak{M}_{ell}$, with the fibre over any point of $\mathfrak{M}_{ell}$ being the set of invariant differentials over the elliptic curve represented by that point. Let $\Omega^{\otimes}$ be the sheaf of rings $\oplus_{n\in\mathbb{Z}}\,\omega^{\otimes n}$.

As shown in [HM99], the sheaf of rings $\Omega^{\otimes}$ lifts to a sheaf of A_∞-ring spectra (it seems that it is well-known to be a sheaf of E_∞-ring spectra, though no reference of this fact is available to the authors of this report). This means roughly that any kind of elliptic curve is associated to a ring spectrum, which in the old day called elliptic cohomology. The spectrum $E_2^{G_{24}}$ is the ring spectrum of sections over the formal neighborhood of the point in $\mathfrak{M}_{ell}$ corresponding to the supersingular curve 6.7.3.

We can also take the global section of this sheaf, and obtain the ring spectrum of topological modular forms, TMF. And we can define tmf to be the (-1)-connected cover of TMF. This spectrum is important in many applications. For example, there is a string orientation $MString \to tmf$ where $MString$ means cobordism theory of manifolds with $String$-structure (See section 5.3.3 for the definition of $String$). This is quite alike the Atiyah-Bott-Shapiro orientation $MSpin \to KO$, so in a sense tmf is a chromatic two analogue of KO-theory, which is also indicated by the resolution 6.7.4. See [AHR] for details of this orientation.

Appendix A

Table of the Homotopy Groups of Spheres

The following is a table of the 2-component of the homotopy groups of spheres. The data come from [Tod62, MT63, Mim65, MMO74]. Further results can be found in [Oda77a, Oda77b]. In the table, the numbers $2, 4, \ldots, \infty$ correspond to a direct summand $\mathbb{Z}/2, \mathbb{Z}/4, \ldots, \mathbb{Z}_{(2)}$ respectively, and the corresponding generators are listed on the right. Only the first stem in the stable range is listed. The generators with the same Greek letter and different subscript are related by suspensions, and if $\alpha \in \pi_{n+k}(S^n)$, the symbol α^m means $\alpha \circ \Sigma^k \alpha \circ \cdots \circ \Sigma^{(m-1)k} \alpha$.

The meaning of the generators are as follows:

- η_2 is the map with Hopf invariant one coming from $\mathbb{C}$
- $\nu' \in < \eta_3, 2\iota_4, \eta_4 >_1$
  ν_4 is the map with Hopf invariant one coming from $\mathbb{H}$
- $\sigma''' = < \nu_5, 8\iota_8, \nu_8 >$
  σ'' satisfies $2\sigma'' = \Sigma\sigma'''$
  σ' satisfies $2\sigma' = \Sigma\sigma''$
  σ_8 is the map with Hopf invariant one coming from the octonions
- $\varepsilon_3 = < \eta_3, \Sigma\nu', \nu_7 >_1$
- $\bar{\nu}_6 \in < \nu_6, \eta_9, \nu_{10} >$
- $\mu_3 \in < \eta_3, \Sigma\nu', 8\iota_7, \nu_7 >_1$
- $\varepsilon' = < \nu', 2\nu_6, \nu_9 >_3$
- $\mu' \in < \eta_3, 2\iota_4, \mu_4 >_1$
- $\zeta_5 \in < \nu_5, 8\iota_8, \Sigma\sigma' >_1$
- $\theta' \in < \sigma_{11}, 2\nu_{18}, \eta_{21} >_1$
  $\theta \in < \sigma_{12}, \nu_{19}, \eta_{22} >_1$
- $\kappa_7 \in < \nu_7, \eta_{10}, 2\iota_{11}, \varepsilon_{11} >_1$
- $\bar{\varepsilon}_3 \in < \varepsilon_3, 2\iota_{11}, \nu_{11}^2 >_6$
- $\rho^{iv} \in < \sigma''', 2\iota_{12}, 8\sigma_{12} >_1$
  $\rho''' \in < \sigma'', 4\iota_{13}, 4\sigma_{13} >_1$
  $\rho'' \in < \sigma', 8\iota_{14}, 2\sigma_{14} >_1$
  $\rho' \in < \sigma_9, 16\iota_{16}, \sigma_{16} >_1$
  ρ_{13} satisfies $2\rho_{13} = \Sigma^4\rho'$ and $\Sigma^\infty\rho_{13} \in < \sigma, 2\sigma, 8\iota >$
- $\zeta' \in < \sigma'', \varepsilon_{13}, 2\iota_{21} >_1$
- ω_{14} satisfies $H(\omega_{14}) = \nu_{27}$
- $\eta^{*\prime} \in < \sigma_{15}, 4\sigma_{22}, \eta_{29} >_1$
  $\eta^*_{16} \in < \sigma_{16}, 2\sigma_{23}, \eta_{30} >_1$

- $\bar{\varepsilon}'$ satisfies $2\bar{\varepsilon}' = \eta_3^2 \circ \bar{\varepsilon}_5$ and $\Sigma\bar{\varepsilon}' = \Sigma\nu' \circ \kappa_7$
- $\bar{\mu}_3 \in < \mu_3, 2\iota_{12}, 8\sigma_{12} >_1$
- ε_{12}^* satisfies $H(\varepsilon_{12}^*) = \nu_{23}^2$ and $\Sigma^2\varepsilon_{12}^* = \omega_{14} \circ \eta_{30}$
- $\nu_{16}^* \in < \sigma_{16}, 2\sigma_{23}, \nu_{30} >_1$
  λ satisfies $\Sigma^3\lambda - 2\nu_{16}^* = \pm\Delta(\nu_{33})$ and $H(\lambda) = \nu_{25}^2$
  λ' satisfies $\Sigma^2\lambda' = 2\lambda$ and $H(\lambda') \equiv \varepsilon_{21} \mod \eta_{21} \circ \sigma_{22}$
  λ'' satisfies $\Sigma\lambda'' = 2\lambda'$ and $H(\lambda'') \equiv \eta_{19} \circ \varepsilon_{20} \mod \eta_{19}^2 \circ \sigma_{21}$
- $\xi_{12} \in < \sigma_{12}, \nu_{19}, \sigma_{22} >_1$
  ξ' satisfies $\Sigma\xi' - 2\xi_{12} = \pm\Delta(\sigma_{25})$ and $H(\xi') = \bar{\nu}_{21} + \varepsilon_{21} = \eta_{21} \circ \sigma_{22}$
  ξ'' satisfies $\Sigma\xi'' = 2\xi'$ and $H(\xi'') = \nu_{19}^3 + \eta_{19} \circ \varepsilon_{20} = \eta_{19}^2 \circ \sigma_{21}$
- $\bar{\mu}' \in < \mu', 4\iota_{14}, 4\sigma_{14} >_1$
- $\breve{\zeta}_5 \in < \zeta_5, 8\iota_{16}, 2\sigma_{16} >_1$
- $\bar{\sigma}_6 \in < \nu_6, \varepsilon_9 + \bar{\nu}_9, \sigma_{17} >_1$
- ω' satisfies $\Sigma^2\omega' = 2\omega_{14} \circ \nu_{30}$ and $H(\omega') \equiv \varepsilon_{23} \mod \eta_{23} \circ \sigma_{24}$
- $\bar{\kappa}' \in < \nu_6, \eta_9, \eta_{10} \circ \kappa_{11} >$
- $\bar{\kappa}_9 \in < \nu_9, \eta_{12}, 2\iota_{13}, \kappa_{13} >$
- β' satisfies $2\beta' = \pm\Delta(\zeta_{21})$ and $H(\beta') = \zeta_{19}$
  β'' satisfies $\Sigma\beta'' = \Delta(\mu_{25})$ and $H(\beta'') = \eta_{21} \circ \mu_{22}$
  β''' satisfies $\Sigma^2\beta''' = 8\Delta(\sigma_{29})$ and $H(\beta''') = \mu_{23}$
- $\bar{\beta}$ satisfies $\Sigma\bar{\beta} = \Delta(\eta_{41})$ and $H(\bar{\beta}) = \eta_{37}^2$
  $\bar{\bar{\beta}}$ satisfies $\Sigma\bar{\bar{\beta}} = \Delta(\iota_{43})$ and $H(\bar{\bar{\beta}}) = \eta_{39}$
- $\alpha \in < \nu_5^2, 2\iota_{11}, \kappa_{11} >$
- $\sigma^{*'''} \in < \sigma_{12}, \nu_{19}, \zeta_{22} >_1$
  $\sigma^{*''} \in < \sigma_{14}, 8\sigma_{21}, \sigma_{28} >_1$
  $\sigma^{*'} \in < \sigma_{15}, 4\sigma_{22}, \sigma_{29} >_1$
  $\sigma^* \in < \sigma_{16}, 2\sigma_{23}, \sigma_{30} >_1$
- $\bar{\alpha} \in < \varepsilon_3, 2\iota_{11}, \kappa_{11} >_1$
- $\bar{\rho}''' \in < \rho^{iv}, 2\iota_{20}, 8\sigma_{20} >_1$
  $\bar{\rho}'' \in < -\rho''', 4\iota_{21}, 4\sigma_{21} >_1$
  $\bar{\rho}' \in < \rho'', 8\iota_{22}, 2\sigma_{22} >_1$
  $\bar{\rho}_9 \in < \rho', 16\iota_{24}, \sigma_{24} >_1$
- $\phi_5 \in < \nu_5, 2\nu_8, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19} >_1$ with $2\phi_5 = 0$
- $\psi_{10} \in < \sigma_{10}, \bar{\nu}_{17} + \varepsilon_{17}, \sigma_{25} >_4$
- $\bar{\varepsilon}^{*'}$ satisfies $\Sigma\bar{\varepsilon}^{*'} = \Delta(\varepsilon_{33})$ and $H(\bar{\varepsilon}^{*'}) = \eta_{29} \circ \varepsilon_{30}$
  $\bar{\varepsilon}_{16}^* \in < \sigma_{16}^2, 2\iota_{30}, \varepsilon_{30} >_1$
- $\bar{\nu}^{*'}$ satisfies $\Sigma\bar{\nu}^{*'} = \Delta(\bar{\nu}_{33})$ and $H(\bar{\nu}^{*'}) = \nu_{29}^3$
  $\bar{\nu}_{16}^* = \bar{\varepsilon}_{16}^* + \eta_{16}^* \circ \sigma_{32}$
- $\delta_3 \in < \varepsilon_3, \varepsilon_{11} + \bar{\nu}_{11}, \sigma_{19} >_1$
- $\bar{\sigma}_6' \in < \bar{\nu}_6, \varepsilon_{14} + \bar{\nu}_{14}, \sigma_{22} >_1$
- $\breve{\zeta}_6' \in < \zeta', 8\iota_{22}, 2\sigma_{22} >_1$
- $\tilde{\varepsilon}_{10} \in < \sigma_{10}, \bar{\varepsilon}_{17}, \eta_{32} >_5$
- ζ^* satisfies $\Sigma^2\zeta^* = \Delta(\eta_{33} \circ \varepsilon_{34})$ and $H(\zeta^*) \equiv \zeta_{27} \mod 2\pi_{38}(S^{27})$
- $\mu^{*'}$ satisfies $\Sigma\mu^{*'} = \Delta(\mu_{33})$ and $H(\mu^{*'}) = \eta_{29} \circ \mu_{30}$
  $\mu_{16}^* \in < \sigma_{16}^2, 2\iota_{30}, \mu_{30} >_1$
- $\tilde{\eta}$ satisfies $\Sigma\tilde{\eta} = \Delta(\iota_{51})$ and $H(\tilde{\eta}) = \eta_{47}$
  $\tilde{\eta}'$ satisfies $\Sigma\tilde{\eta}' = \Delta(\eta_{49}) \pm 2\tilde{\eta}$ and $H(\tilde{\eta}') = \eta_{45}^2$

Table A.1: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=1$		$k=2$		$k=3$		$k=4$		$k=5$		$k=6$	
n=2	∞	η_2	2	η_2^2	2	η_2^3	4	$\eta_2 \circ \nu'$	2	$\eta_2 \circ \nu' \circ \eta_6$	2	$\eta_2 \circ \nu' \circ \eta_6^2$
n=3	2	η_3	2	η_3^2	4	ν'	2	$\nu' \circ \eta_6$	2	$\nu' \circ \eta_6^2$	0	
n=4			2	η_4^2	∞	ν_4	2	$\nu_4 \circ \eta_7$	2	$\nu_4 \circ \eta_7^2$	8	ν_4^2
					4	$\Sigma\nu'$	2	$\Sigma\nu' \circ \eta_7$	2	$\Sigma\nu' \circ \eta_7^2$		
n=5					8	ν_5	2	$\nu_5 \circ \eta_8$	2	$\nu_5 \circ \eta_8^2$	2	ν_5^2
n=6							0		∞	$\Delta(\iota_{13})$	2	ν_6^2
n=7									0		2	ν_7^2
n=8											2	ν_8^2

Table A.2: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=7$		$k=8$		$k=9$		$k=10$		$k=11$		$k=12$	
n=2	0		0		2	$\eta_2 \circ \varepsilon_3$	2 2	$\eta_2^2 \circ \varepsilon_4$ $\eta_2 \circ \mu_3$	4 2	$\eta_2 \circ \varepsilon'$ $\eta_2^2 \circ \mu_4$	4 2 2	$\eta_2 \circ \mu'$ $\eta_2 \circ \nu' \circ \bar{\nu}_6$ $\eta_2 \circ \nu' \circ \varepsilon_6$
n=3	0		2	ε_3	2 2	μ_3 $\eta_3 \circ \varepsilon_4$	4 2	ε' $\eta_3 \circ \mu_4$	4 2 2	μ' $\varepsilon_3 \circ \nu_{11}$ $\nu' \circ \varepsilon_6$	2 2	$\nu' \circ \mu_6$ $\nu' \circ \eta_6 \circ \varepsilon_7$
n=4	0		2	ε_4	2 2 2	ν_4^3 μ_4 $\eta_4 \circ \varepsilon_5$	8 4 2	$\nu_4 \circ \sigma'$ $\Sigma\varepsilon'$ $\eta_4 \circ \eta_5$	2 2 2 4 2 2	$\nu_4 \circ \sigma' \circ \eta_{14}$ $\nu_4 \circ \bar{\nu}_7$ $\nu_4 \circ \varepsilon_7$ $\Sigma\mu'$ $\varepsilon_4 \circ \nu_{12}$ $\Sigma\nu' \circ \varepsilon_7$	2 2 2 2 2 2	$\nu_4 \circ \sigma' \circ \eta_{14}^2$ ν_4^4 $\nu_4 \circ \mu_7$ $\nu_4 \circ \eta_7 \circ \varepsilon_8$ $\Sigma\nu' \circ \mu_7$ $\Sigma\nu' \circ \eta_7 \circ \varepsilon_8$
n=5	2	σ'''	2	ε_5	2 2 2	ν_5^3 μ_5 $\eta_5 \circ \varepsilon_6$	8 2	$\nu_5 \circ \sigma_8$ $\eta_5 \circ \mu_6$	8 2 2	ζ_5 $\nu_5 \circ \bar{\nu}_8$ $\nu_5 \circ \varepsilon_8$	2 2 2	ν_5^4 $\nu_5 \circ \mu_8$ $\nu_5 \circ \eta_8 \circ \varepsilon_9$
n=6	4	σ''	8 2	$\bar{\nu}_6$ ε_6	2 2 2	ν_6^3 μ_6 $\eta_6 \circ \varepsilon_7$	8 2	$\nu_6 \circ \sigma_9$ $\eta_6 \circ \mu_7$	8 4	ζ_6 $\bar{\nu}_6 \circ \nu_{14}$	16	$\Delta(\sigma_{13})$
n=7	8	σ'	2 2 2	$\sigma' \circ \eta_{14}$ $\bar{\nu}_7$ ε_7	2 2 2 2	$\sigma' \circ \eta_{14}^2$ ν_7^3 μ_7 $\eta_7 \circ \varepsilon_8$	8 2	$\nu_7 \circ \sigma_{10}$ $\eta_7 \circ \mu_8$	8 2	ζ_7 $\bar{\nu}_7 \circ \nu_{15}$	0	
n=8	∞ 8	σ_8 $\Sigma\sigma'$	2 2 2 2	$\sigma_8 \circ \eta_{15}$ $\Sigma\sigma' \circ \eta_{15}$ $\bar{\nu}_8$ ε_8	2 2 2 2 2	$\sigma_8 \circ \eta_{15}^2$ $\Sigma\sigma' \circ \eta_{15}^2$ ν_8^3 μ_8 $\eta_8 \circ \varepsilon_9$	8 8 2	$\sigma_8 \circ \nu_{15}$ $\nu_8 \circ \sigma_{11}$ $\eta_8 \circ \mu_9$	8 2	ζ_8 $\bar{\nu}_8 \circ \nu_{16}$	0	
n=9	16	σ_9	2 2 2	$\sigma_9 \circ \eta_{16}$ $\bar{\nu}_9$ ε_9	2 2 2 2	$\sigma_9 \circ \eta_{16}^2$ ν_9^3 μ_9 $\eta_9 \circ \varepsilon_{10}$	8 2	$\sigma_9 \circ \nu_{16}$ $\eta_9 \circ \mu_{10}$	8 2	ζ_9 $\bar{\nu}_9 \circ \nu_{17}$	0	
n=10			2 2	$\bar{\nu}_{10}$ ε_{10}	∞ 2 2 2	$\Delta(\iota_{21})$ ν_{10}^3 μ_{10} $\eta_{10} \circ \varepsilon_{11}$	4 2	$\sigma_{10} \circ \nu_{17}$ $\eta_{10} \circ \mu_{11}$	8	ζ_{10}	4	$\Delta(\nu_{21})$
n=11					2 2 2	ν_{11}^3 μ_{11} $\eta_{11} \circ \varepsilon_{12}$	2 2	$\sigma_{11} \circ \nu_{18}$ $\eta_{11} \circ \mu_{12}$	8	ζ_{11}	2	θ'
n=12							2	$\eta_{12} \circ \mu_{13}$	∞ 8	$\Delta(\iota_{25})$ ζ_{12}	2 2	θ $\Sigma\theta'$
n=13									8	ζ_{13}	2	$\Sigma\theta$
n=14											0	

Table A.3: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=13$		$k=14$		$k=15$		$k=16$	
n=2	2 2	$\eta_2\circ\nu'\circ\mu_6$ $\eta_2\circ\nu'\circ\eta_6\circ\varepsilon_7$	2	$\eta_2\circ\nu'\circ\eta_6\circ\mu_7$	2	$\eta_2\circ\varepsilon_3\circ\nu_{11}^2$	2	$\eta_2\circ\bar{\varepsilon}_3$
n=3	2	$\nu'\circ\eta_6\circ\mu_7$	2	$\varepsilon_3\circ\nu_{11}^2$	2	$\bar{\varepsilon}_3$	2 2	$\mu_3\circ\sigma_{12}$ $\eta_3\circ\bar{\varepsilon}_4$
n=4	8 2 2	$\nu_4^2\circ\sigma_{10}$ $\nu_4\circ\eta_7\circ\mu_8$ $\Sigma\nu'\circ\eta_7\circ\mu_8$	2 8 2	$\varepsilon_4\circ\nu_{12}^2$ $\nu_4\circ\zeta_7$ $\nu_4\circ\bar{\nu}_7\circ\nu_{15}$	2	$\bar{\varepsilon}_4$	2 2 2	$\nu_4^2\circ\sigma_{10}\circ\nu_{17}$ $\mu_4\circ\sigma_{13}$ $\eta_4\circ\bar{\varepsilon}_5$
n=5	2 2	$\nu_5\circ\sigma_8\circ\nu_{15}$ $\nu_5\circ\eta_8\circ\mu_9$	2 2	$\nu_5\circ\zeta_8$ $\nu_5\circ\bar{\nu}_8\circ\nu_{16}$	2 2	ρ^{iv} $\bar{\varepsilon}_5$	2 2	$\mu_5\circ\sigma_{14}$ $\eta_5\circ\bar{\varepsilon}_6$
n=6	2	$\nu_6\circ\sigma_9\circ\nu_{16}$	4 2	$\sigma''\circ\sigma_{13}$ $\bar{\nu}_6\circ\nu_{14}^2$	4 2	ρ''' $\bar{\varepsilon}_6$	8 2 2	ζ' $\mu_6\circ\sigma_{15}$ $\eta_6\circ\bar{\varepsilon}_7$
n=7	2	$\nu_7\circ\sigma_{10}\circ\nu_{17}$	8 4	$\sigma'\circ\sigma_{14}$ κ_7	8 2 2 2	ρ'' $\sigma'\circ\bar{\nu}_{14}$ $\sigma'\circ\varepsilon_{14}$ $\bar{\varepsilon}_7$	2 2 2 2	$\sigma'\circ\mu_{14}$ $\Sigma\zeta'$ $\mu_7\circ\sigma_{16}$ $\eta_7\circ\bar{\varepsilon}_8$
n=8	2 2	$\sigma_8\circ\nu_{15}^2$ $\nu_8\circ\sigma_{11}\circ\nu_{18}$	16 8 4	σ_8^2 $\Sigma\sigma'\circ\sigma_{15}$ κ_8	2 2 8 2 2 2	$\sigma_8\circ\bar{\nu}_{15}$ $\sigma_8\circ\varepsilon_{15}$ $\Sigma\rho''$ $\Sigma\sigma'\circ\bar{\nu}_{15}$ $\Sigma\sigma'\circ\varepsilon_{15}$ $\bar{\varepsilon}_8$	2 2 2 2 2 2 2	$\sigma_8\circ\nu_{15}^3$ $\sigma_8\circ\mu_{15}$ $\sigma_8\circ\eta_{15}\circ\varepsilon_{16}$ $\Sigma\sigma'\circ\mu_{15}$ $\Sigma^2\zeta'$ $\mu_8\circ\sigma_{17}$ $\eta_8\circ\bar{\varepsilon}_9$
n=9	2	$\sigma_9\circ\nu_{16}^2$	16 4	σ_9^2 κ_9	16 2 2 2	ρ' $\sigma_9\circ\bar{\nu}_{16}$ $\sigma_9\circ\varepsilon_{16}$ $\bar{\varepsilon}_9$	2 2 2 2	$\sigma_9\circ\nu_{16}^3$ $\sigma_9\circ\mu_{16}$ $\sigma_9\circ\eta_{16}\circ\varepsilon_{17}$ $\mu_9\circ\sigma_{18}$
n=10	2	$\sigma_{10}\circ\nu_{17}^2$	16 2	σ_{10}^2 κ_{10}	16 2 2	$\Sigma\rho'$ $\sigma_{10}\circ\bar{\nu}_{17}$ $\bar{\varepsilon}_{10}$	16 2	$\Delta(\sigma_{21})$ $\sigma_{10}\circ\mu_{17}$
n=11	2 2	$\theta'\circ\eta_{23}$ $\sigma_{11}\circ\nu_{18}^2$	16 2	σ_{11}^2 κ_{11}	16 2	$\Sigma^2\rho'$ $\bar{\varepsilon}_{11}$	2	$\sigma_{11}\circ\mu_{18}$
n=12	2 2	$\theta\circ\eta_{24}$ $\Sigma\theta'\circ\eta_{24}$	16 2 4	σ_{12}^2 κ_{12} $\Delta(\nu_{25})$	16 2	$\Sigma^3\rho'$ $\bar{\varepsilon}_{12}$	2	$\sigma_{12}\circ\mu_{19}$
n=13	2	$\Sigma\theta\circ\eta_{25}$	16 2	σ_{13}^2 κ_{13}	32 2	ρ_{13} $\bar{\varepsilon}_{13}$	2	$\sigma_{13}\circ\mu_{20}$
n=14	∞	$\Delta(\iota_{29})$	8 2	σ_{14}^2 κ_{14}	32 2	ρ_{14} $\bar{\varepsilon}_{14}$	8 2	ω_{14} $\sigma_{14}\circ\mu_{21}$
n=15	0		4 2	σ_{15}^2 κ_{15}	32 2	ρ_{15} $\bar{\varepsilon}_{15}$	2 2 2	$\eta^{*\prime}$ ω_{15} $\sigma_{15}\circ\mu_{22}$
n=16			2 2	σ_{16}^2 κ_{16}	32 2 ∞	ρ_{16} $\bar{\varepsilon}_{16}$ $\Delta(\iota_{33})$	2 2 2 2	η_{16}^* $\Sigma\eta^{*\prime}$ ω_{16} $\sigma_{16}\circ\mu_{23}$
n=17					32 2	ρ_{17} $\bar{\varepsilon}_{17}$	2 2 2	η_{17}^* ω_{17} $\sigma_{17}\circ\mu_{24}$
n=18							2 2	ω_{18} $\sigma_{18}\circ\mu_{25}$

Table A.4: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=17$		$k=18$		$k=19$		$k=20$	
n=2	2	$\eta_2\circ\mu_3\circ\sigma_{12}$	4	$\eta_2\circ\bar{\varepsilon}'$	4	$\eta_2\circ\mu'\circ\sigma_{14}$	4	$\eta_2\circ\bar{\mu}'$
	2	$\eta_2^2\circ\bar{\varepsilon}_4$	2	$\eta_2\circ\bar{\mu}_3$	2	$\eta_2\circ\nu'\circ\bar{\varepsilon}_6$	2	$\eta_2\circ\nu'\circ\mu_6\circ\sigma_{15}$
			2	$\eta_2^2\circ\mu_4\circ\sigma_{13}$	2	$\eta_2^2\circ\bar{\mu}_4$		
n=3	4	$\bar{\varepsilon}'$	4	$\mu'\circ\sigma_{14}$	4	$\bar{\mu}'$	2	$\nu'\circ\bar{\mu}_6$
	2	$\bar{\mu}_3$	2	$\nu'\circ\bar{\varepsilon}_6$	2	$\nu'\circ\mu_6\circ\sigma_{15}$	2	$\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}$
	2	$\eta_3\circ\mu_4\circ\sigma_{13}$	2	$\eta_3\circ\bar{\mu}_4$				
n=4	8	$\nu_4\circ\sigma'\circ\sigma_{14}$	8	$\nu_4\circ\rho''$	2	$\nu_4\circ\sigma'\circ\mu_{14}$	2	$\nu_4\circ\sigma'\circ\eta_{14}\circ\mu_{15}$
	4	$\nu_4\circ\kappa_7$	2	$\nu_4\circ\sigma'\circ\bar{\nu}_{14}$	2	$\nu_4\circ\Sigma\zeta'$	2	$\nu_4^2\circ\kappa_{10}$
	4	$\Sigma\bar{\varepsilon}'$	2	$\nu_4\circ\sigma'\circ\varepsilon_{14}$	2	$\nu_4\circ\mu_7\circ\sigma_{16}$	2	$\nu_4\circ\bar{\mu}_7$
	2	$\bar{\mu}_4$	2	$\nu_4\circ\bar{\varepsilon}_7$	2	$\nu_4\circ\eta_7\circ\bar{\varepsilon}_8$	2	$\nu_4\circ\eta_7\circ\mu_8\circ\sigma_{17}$
	2	$\eta_4\circ\mu_5\circ\sigma_{14}$	4	$\Sigma\mu'\circ\sigma_{15}$	4	$\Sigma\bar{\mu}'$	2	$\Sigma\nu'\circ\bar{\mu}_7$
			2	$\Sigma\nu'\circ\bar{\varepsilon}_7$	2	$\Sigma\nu'\circ\mu_7\circ\sigma_{16}$	2	$\Sigma\nu'\circ\eta_7\circ\mu_8\circ\sigma_{17}$
			2	$\eta_4\circ\bar{\mu}_5$				
n=5	4	$\nu_5\circ\kappa_8$	8	$\zeta_5\circ\sigma_{16}$	8	$\breve{\zeta}_5$	2	$\nu_5^2\circ\kappa_{11}$
	2	$\bar{\mu}_5$	2	$\nu_5\circ\bar{\varepsilon}_8$	2	$\nu_5\circ\mu_8\circ\sigma_{17}$	2	$\nu_5\circ\bar{\mu}_8$
	2	$\eta_5\circ\mu_6\circ\sigma_{15}$	2	$\eta_5\circ\bar{\mu}_6$			2	$\nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18}$
n=6	2	$\Delta(\Sigma\theta)$	2	$\Delta(\Sigma\theta)\circ\eta_{23}$	8	$\breve{\zeta}_6$	32	$\Delta(\rho_{13})$
	2	$\nu_6\circ\kappa_9$	8	$\zeta_6\circ\sigma_{17}$	32	$\bar{\sigma}_6$	4	$\bar{\kappa}'$
	2	$\bar{\mu}_6$	2	$\eta_6\circ\bar{\mu}_7$				
	2	$\eta_6\circ\mu_7\circ\sigma_{16}$						
n=7	2	$\sigma'\circ\eta_{14}\circ\mu_{15}$	8	$\zeta_7\circ\sigma_{18}$	8	$\breve{\zeta}_7$	8	$\bar{\kappa}_7$
	2	$\nu_7\circ\kappa_{10}$	2	$\eta_7\circ\bar{\mu}_8$	2	$\bar{\sigma}_7$		
	2	$\bar{\mu}_7$						
	2	$\eta_7\circ\mu_8\circ\sigma_{17}$						
n=8	2	$\sigma_8\circ\eta_{15}\circ\mu_{16}$	8	$\sigma_8\circ\zeta_{15}$	8	$\breve{\zeta}_8$	8	$\bar{\kappa}_8$
	2	$\Sigma\sigma'\circ\eta_{15}\circ\mu_{16}$	8	$\zeta_8\circ\sigma_{19}$	2	$\bar{\sigma}_8$		
	2	$\nu_8\circ\kappa_{11}$	2	$\eta_8\circ\bar{\mu}_9$				
	2	$\bar{\mu}_8$						
	2	$\eta_8\circ\mu_9\circ\sigma_{18}$						
n=9	2	$\sigma_9\circ\eta_{16}\circ\mu_{17}$	8	$\sigma_9\circ\zeta_{16}$	8	$\breve{\zeta}_9$	8	$\bar{\kappa}_9$
	2	$\nu_9\circ\kappa_{12}$	2	$\eta_9\circ\bar{\mu}_{10}$	2	$\bar{\sigma}_9$		
	2	$\bar{\mu}_9$						
	2	$\eta_9\circ\mu_{10}\circ\sigma_{19}$						
n=10	2	$\sigma_{10}\circ\eta_{17}\circ\mu_{18}$	8	λ'', ξ''	8	$\breve{\zeta}_{10}$	8	$\bar{\kappa}_{10}$
	2	$\nu_{10}\circ\kappa_{13}$	2		2	$\bar{\sigma}_{10}$	8	β'
	2	$\bar{\mu}_{10}$	2	$\eta_{10}\circ\bar{\mu}_{11}$				
n=11	2	$\sigma_{11}\circ\eta_{18}\circ\mu_{19}$	8	λ', ξ'	2	$\lambda'\circ\eta_{29}$	8	$\bar{\kappa}_{11}$
	2	$\nu_{11}\circ\kappa_{14}$	4		2	$\xi'\circ\eta_{29}$	2	$\theta'\circ\varepsilon_{23}$
	2	$\bar{\mu}_{11}$	2	$\eta_{11}\circ\bar{\mu}_{12}$	8	$\breve{\zeta}_{11}$	2	β''
					2	$\bar{\sigma}_{11}$		

Table A.5: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=17$		$k=18$		$k=19$		$k=20$	
n=12	2	ε^*_{12}	32	ξ_{12}	2	ω'	8	$\bar{\kappa}_{12}$
	2	$\sigma_{12}\circ\eta_{19}\circ\mu_{20}$	4	$\Sigma\lambda'$, $\Sigma\xi'$	2	$\xi_{12}\circ\eta_{30}$	2	$\Sigma\theta'\circ\varepsilon_{24}$
	2	$\nu_{12}\circ\kappa_{15}$	4		2	$\Sigma\lambda'\circ\eta_{30}$	2	$\Delta(\mu_{25})$
	2	$\bar{\mu}_{12}$	2	$\eta_{12}\circ\bar{\mu}_{13}$	2	$\Sigma\xi'\circ\eta_{30}$	2	$\theta\circ\varepsilon_{24}$
					8	$\breve{\zeta}_{12}$	2	$\theta\circ\bar{\nu}_{24}$
					2	$\bar{\sigma}_{12}$	2	β'''
n=13	2	ε^*_{13}	8	ξ_{13}	2	$\Sigma\omega'$	8	$\bar{\kappa}_{13}$
	2	$\sigma_{13}\circ\eta_{20}\circ\mu_{21}$	8	λ	2	$\xi_{13}\circ\eta_{31}$	2	$\Sigma\theta\circ\varepsilon_{25}$
	2	$\nu_{13}\circ\kappa_{16}$	2	$\eta_{13}\circ\bar{\mu}_{14}$	8	$\breve{\zeta}_{13}$	2	$\Sigma\theta\circ\bar{\nu}_{25}$
	2	$\bar{\mu}_{13}$			2	$\bar{\sigma}_{13}$	2	$\Sigma\beta'''$
n=14	2	ε^*_{14}	8	ξ_{14}	4	$\omega_{14}\circ\nu_{30}$	16	$\Delta(\sigma_{29})$
	2	$\sigma_{14}\circ\eta_{21}\circ\mu_{22}$	8	$\Sigma\lambda$	8	$\breve{\zeta}_{14}$	8	$\bar{\kappa}_{14}$
	2	$\nu_{14}\circ\kappa_{17}$	2	$\eta_{14}\circ\bar{\mu}_{15}$	2	$\bar{\sigma}_{14}$		
	2	$\bar{\mu}_{14}$						
n=15	2	$\eta^{*\prime}\circ\eta_{31}$	8	ξ_{15}	2	$\omega_{15}\circ\nu_{31}$	8	$\bar{\kappa}_{15}$
	2	ε^*_{15}	8	$\Sigma^2\lambda$	8	$\breve{\zeta}_{15}$		
	2	$\sigma_{15}\circ\eta_{22}\circ\mu_{23}$	2	$\eta_{15}\circ\bar{\mu}_{16}$	2	$\bar{\sigma}_{15}$		
	2	$\nu_{15}\circ\kappa_{18}$						
	2	$\bar{\mu}_{15}$						
n=16	2	$\eta^*_{16}\circ\eta_{32}$	8	ν^*_{16}	2	$\omega_{16}\circ\nu_{32}$	8	$\bar{\kappa}_{16}$
	2	$\Sigma\eta^{*\prime}\circ\eta_{32}$	8	ξ_{16}	8	$\breve{\zeta}_{16}$		
	2	ε^*_{16}	8	$\Sigma^3\lambda$	2	$\bar{\sigma}_{16}$		
	2	$\sigma_{16}\circ\eta_{23}\circ\mu_{24}$	2	$\eta_{16}\circ\bar{\mu}_{17}$				
	2	$\nu_{16}\circ\kappa_{19}$						
	2	$\bar{\mu}_{16}$						
n=17	2	$\eta^*_{17}\circ\eta_{33}$	8	ν^*_{17}	2	$\omega_{17}\circ\nu_{33}$	8	$\bar{\kappa}_{17}$
	2	ε^*_{17}	8	ξ_{17}	8	$\breve{\zeta}_{17}$		
	2	$\sigma_{17}\circ\eta_{24}\circ\mu_{25}$	2	$\eta_{17}\circ\bar{\mu}_{18}$	2	$\bar{\sigma}_{17}$		
	2	$\nu_{17}\circ\kappa_{20}$						
	2	$\bar{\mu}_{17}$						
n=18	∞	$\Delta(\iota_{37})$	8	ν^*_{18}	8	$\breve{\zeta}_{18}$	8	$\bar{\kappa}_{18}$
	2	ε^*_{18}	4	$\nu^*_{18}+\xi_{18}$	2	$\bar{\sigma}_{18}$	4	$\Delta(\nu_{37})$
	2	$\sigma_{18}\circ\eta_{25}\circ\mu_{26}$	2	$\eta_{18}\circ\bar{\mu}_{19}$				
	2	$\nu_{18}\circ\kappa_{21}$						
	2	$\bar{\mu}_{18}$						
n=19	2	ε^*_{19}	8	ν^*_{19}	8	$\breve{\zeta}_{19}$	8	$\bar{\kappa}_{19}$
	2	$\sigma_{19}\circ\eta_{26}\circ\mu_{27}$	2	$\nu^*_{19}+\xi_{19}$	2	$\bar{\sigma}_{19}$	2	$\bar{\beta}$
	2	$\nu_{19}\circ\kappa_{22}$	2	$\eta_{19}\circ\bar{\mu}_{20}$				
	2	$\bar{\mu}_{19}$						
n=20			8	ν^*_{20}	∞	$\Delta(\iota_{41})$	8	$\bar{\kappa}_{20}$
			2	$\eta_{20}\circ\bar{\mu}_{21}$	8	$\breve{\zeta}_{20}$	2	$\Delta(\eta_{41})$
					2	$\bar{\sigma}_{20}$	2	$\bar{\bar{\beta}}$
n=21					8	$\breve{\zeta}_{21}$	8	$\bar{\kappa}_{21}$
					2	$\bar{\sigma}_{21}$	2	$\Delta(\iota_{43})$
n=22							8	$\bar{\kappa}_{22}$

Table A.6: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=21$		$k=22$		$k=23$		$k=24$	
n=2	2 2	$\eta_2\circ\nu'\circ\bar{\mu}_6$ $\eta_2\circ\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}$	2	$\eta_2\circ\nu'\circ\eta_6\circ\bar{\mu}_7$	2	$\eta_2\circ\varepsilon_3\circ\kappa_{11}$	4	$\eta_2\circ\bar{\alpha}$
n=3	2	$\nu'\circ\eta_6\circ\bar{\mu}_7$	2	$\varepsilon_3\circ\kappa_{11}$	4	$\bar{\alpha}$	2 2 2	δ_3 $\bar{\mu}_3\circ\sigma_{20}$ $\varepsilon'\circ\kappa_{13}$
n=4	8 2 2	$\nu_4\circ\zeta_7\circ\sigma_{18}$ $\nu_4\circ\eta_7\circ\bar{\mu}_8$ $\Sigma\nu'\circ\eta_7\circ\bar{\mu}_8$	8 2 2	$\nu_4\circ\breve{\zeta}_7$ $\nu_4\circ\bar{\sigma}_7$ $\varepsilon_4\circ\kappa_{12}$	8 4	$\nu_4\circ\bar{\kappa}_7$ $\Sigma\bar{\alpha}$	2 2 2 2 2	$\nu_4\circ\eta_7\circ\bar{\kappa}_8$ $\nu_4\circ\sigma'\circ\kappa_{14}$ δ_4 $\bar{\mu}_4\circ\sigma_{21}$ $\Sigma\varepsilon'\circ\kappa_{14}$
n=5	2 2	α $\nu_5\circ\eta_8\circ\bar{\mu}_9$	2 2 2	$\nu_5\circ\breve{\zeta}_8$ $\nu_5\circ\bar{\sigma}_8$ $\varepsilon_5\circ\kappa_{13}$	8 2 2	$\nu_5\circ\bar{\kappa}_8$ $\bar{\rho}'''$ ϕ_5	2 2 2 2	$\nu_5\circ\eta_8\circ\bar{\kappa}_9$ $\nu_5\circ\sigma_8\circ\kappa_{15}$ δ_5 $\bar{\mu}_5\circ\sigma_{22}$
n=6	2	$\eta_6\circ\bar{\kappa}_7$	4 2 2 2	$\rho'''\circ\sigma_{21}$ $\bar{\nu}_6\circ\kappa_{14}$ $\nu_6\circ\bar{\sigma}_9$ $\varepsilon_6\circ\kappa_{14}$	8 4 2 8 4	$\nu_6\circ\bar{\kappa}_8$ $\bar{\rho}''$ ϕ_6 $\Delta(\lambda)$, $\Delta(\xi)$	2 2 2 8 2 2 2	$\nu_6\circ\sigma_9\circ\kappa_{16}$ δ_6 $\bar{\mu}_6\circ\sigma_{23}$ $\breve{\zeta}'_6$ $\bar{\sigma}'_6$ $\Delta(\lambda\circ\eta_{31})$ $\Delta(\xi_{13}\circ\eta_{31})$
n=7	2 2	$\eta_7\circ\bar{\kappa}_8$ $\sigma'\circ\kappa_{14}$	8 2 2 2	$\sigma'\circ\rho_{14}$ $\bar{\nu}_7\circ\kappa_{15}$ $\nu_7\circ\bar{\sigma}_{10}$ $\varepsilon_7\circ\kappa_{15}$	8 8 2 2 2 2	$\nu_7\circ\bar{\kappa}_{10}$ $\bar{\rho}'$ ϕ_7 $\bar{\kappa}_7\circ\nu_{27}-\nu_7\circ\bar{\kappa}_{10}$ $\sigma'\circ\sigma_{14}\circ\mu_{21}$ $\sigma'\circ\omega_{14}$	2 2 2 2 2 2 2	$\nu_7\circ\sigma_{10}\circ\kappa_{17}$ δ_7 $\bar{\mu}_7\circ\sigma_{24}$ $\breve{\zeta}'_7$ $\bar{\sigma}'_7$ $\sigma'\circ\bar{\mu}_{14}$ $\sigma'\circ\omega_{14}\circ\eta_{30}$
n=8	4 2 2 2	σ_8^3 $\sigma_8\circ\kappa_{15}$ $\eta_8\circ\bar{\kappa}_9$ $\Sigma\sigma'\circ\kappa_{15}$	32 2 8 2 2 2	$\sigma_8\circ\rho_{15}$ $\sigma_8\circ\bar{\varepsilon}_{15}$ $\Sigma\sigma'\circ\rho_{15}$ $\bar{\nu}_8\circ\kappa_{16}$ $\nu_8\circ\bar{\sigma}_{11}$ $\varepsilon_8\circ\kappa_{16}$	2 2 2 8 8 2 2 2 2	$\sigma_8^2\circ\mu_{22}$ $\sigma_8\circ\omega_{15}$ $\sigma_8\circ\eta^{*\prime}$ $\nu_8\circ\bar{\kappa}_{11}$ $\Sigma\bar{\rho}'$ ϕ_8 $\bar{\kappa}_8\circ\nu_{28}-\nu_8\circ\bar{\kappa}_{11}$ $\Sigma\sigma'\circ\sigma_{15}\circ\mu_{22}$ $\Sigma\sigma'\circ\omega_{15}$	2 2 2 2 2 2 2 2 2 2 2 2	$\sigma_8\circ\bar{\mu}_{15}$ $\sigma_8^2\circ\eta_{22}\circ\mu_{23}$ $\sigma_8\circ\nu_{15}\circ\kappa_{18}$ $\sigma_8\circ\varepsilon^*_{15}$ $\sigma_8\circ\eta^{*\prime}\circ\eta_{31}$ $\nu_8\circ\sigma_{11}\circ\kappa_{18}$ δ_8 $\bar{\mu}_8\circ\sigma_{25}$ $\breve{\zeta}'_8$ $\bar{\sigma}'_8$ $\Sigma\sigma'\circ\bar{\mu}_{15}$ $\Sigma\sigma'\circ\omega_{15}\circ\eta_{31}$
n=9	2 2 2	$\eta_9\circ\bar{\kappa}_{10}$ $\sigma_9\circ\kappa_{16}$ σ_9^3	16 2 2 2	$\sigma_9\circ\rho_{16}$ $\sigma_9\circ\bar{\varepsilon}_{16}$ $\nu_9\circ\bar{\sigma}_{12}$ $\varepsilon_9\circ\kappa_{17}$	2 2 16 8 2 2	$\sigma_9^2\circ\mu_{23}$ $\sigma_9\circ\omega_{16}$ $\bar{\rho}_9$ $\nu_9\circ\bar{\kappa}_{12}$ ϕ_9 $\bar{\kappa}_9\circ\nu_{29}-\nu_9\circ\bar{\kappa}_{12}$	2 2 2 2 2 2 2	$\sigma_9\circ\bar{\mu}_{16}$ $\sigma_9^2\circ\eta_{23}\circ\mu_{24}$ $\sigma_9\circ\nu_{16}\circ\kappa_{19}$ $\sigma_9\circ\varepsilon^*_{16}$ δ_9 $\bar{\mu}_9\circ\sigma_{26}$ $\bar{\sigma}'_9$

Table A.7: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=21$		$k=22$		$k=23$		$k=24$	
n=10	2	$\eta_{10}\circ\bar{\kappa}_{11}$	16	$\sigma_{10}\circ\rho_{17}$	16	$\bar{\rho}_{10}$	2	δ_{10}
	2	$\sigma_{10}\circ\kappa_{17}$	2	$\nu_{10}\circ\bar{\sigma}_{13}$	8	$\nu_{10}\circ\bar{\kappa}_{13}$	2	$\bar{\mu}_{10}\circ\sigma_{27}$
	2	σ_{10}^3	2	$\varepsilon_{10}\circ\kappa_{18}$	2	ϕ_{10}	2	$\bar{\sigma}'_{10}$
					2	$\bar{\kappa}_{10}\circ\nu_{30}-\nu_{10}\circ\bar{\kappa}_{13}$	4	$\tilde{\varepsilon}_{10}$
					4	ψ_{10}	32	$\Delta(\rho_{21})$
n=11	2	$\eta_{11}\circ\bar{\kappa}_{12}$	16	$\sigma_{11}\circ\rho_{18}$	16	$\bar{\rho}_{11}$	2	δ_{11}
	2	$\sigma_{11}\circ\kappa_{18}$	2	$\nu_{11}\circ\bar{\sigma}_{14}$	8	$\nu_{11}\circ\bar{\kappa}_{14}$	2	$\bar{\mu}_{11}\circ\sigma_{28}$
	2	σ_{11}^3	2	$\varepsilon_{11}\circ\kappa_{19}$	2	ϕ_{11}	2	$\bar{\sigma}'_{11}$
	2	$\theta'\circ\mu_{23}$			2	ψ_{11}	4	$\tilde{\varepsilon}_{11}$
n=12	2	$\eta_{12}\circ\bar{\kappa}_{13}$	32	$\sigma^{*'''}$	16	$\bar{\rho}_{12}$	2	δ_{12}
	2	$\sigma_{12}\circ\kappa_{19}$	4	$\sigma_{12}\circ\rho_{19}\pm 2\sigma^{*'''}$	8	$\nu_{12}\circ\bar{\kappa}_{15}$	2	$\bar{\mu}_{12}\circ\sigma_{29}$
	2	σ_{12}^3	2	$\nu_{12}\circ\bar{\sigma}_{15}$	2	ϕ_{12}	2	$\bar{\sigma}'_{12}$
	2	$\Sigma\theta'\circ\mu_{24}$	2	$\varepsilon_{12}\circ\kappa_{20}$	2	ψ_{12}	2	$\tilde{\varepsilon}_{12}$
	2	$\theta\circ\mu_{24}$						
n=13	2	$\eta_{13}\circ\bar{\kappa}_{14}$	16	$\rho_{13}\circ\sigma_{28}$	16	$\bar{\rho}_{13}$	2	δ_{13}
	2	σ_{13}^3	2	$\nu_{13}\circ\bar{\sigma}_{16}$	8	$\nu_{13}\circ\bar{\kappa}_{16}$	2	$\bar{\mu}_{13}\circ\sigma_{30}$
	2	$\Sigma\theta\circ\mu_{25}$	2	$\varepsilon_{13}\circ\kappa_{21}$	2	ϕ_{13}	2	$\bar{\sigma}'_{13}$
	4	$\lambda\circ\nu_{31}$			2	ψ_{13}	2	$\tilde{\varepsilon}_{13}$
n=14	2	$\eta_{14}\circ\bar{\kappa}_{15}$	16	$\sigma^{*''}$	16	$\bar{\rho}_{14}$	2	δ_{14}
	2	σ_{14}^3	2	$\omega_{14}\circ\nu_{30}^2$	8	$\nu_{14}\circ\bar{\kappa}_{17}$	2	$\bar{\mu}_{14}\circ\sigma_{31}$
	4	$\Sigma\lambda\circ\nu_{32}$	2	$\nu_{14}\circ\bar{\sigma}_{17}$	2	ϕ_{14}	2	$\bar{\sigma}'_{14}$
			2	$\varepsilon_{14}\circ\kappa_{22}$	2	ψ_{14}	2	$\tilde{\varepsilon}_{14}$
							8	ζ^*
n=15	2	$\eta\circ\bar{\kappa}_{16}$	16	$\sigma^{*'}$	16	$\bar{\rho}_{15}$	2	δ_{15}
	2	σ_{15}^3	2	$\omega_{15}\circ\nu_{31}^2$	8	$\nu_{15}\circ\bar{\kappa}_{18}$	2	$\bar{\mu}_{15}\circ\sigma_{32}$
	2	$\Sigma^2\lambda\circ\nu_{33}$	2	$\nu_{15}\circ\bar{\sigma}_{18}$	2	ϕ_{15}	2	$\bar{\sigma}'_{15}$
			2	$\varepsilon_{15}\circ\kappa_{23}$	2	ψ_{15}	2	$\tilde{\varepsilon}_{15}$
					2	$\bar{\varepsilon}^{*'}$	2	$\Sigma\zeta^*$
					2	$\bar{\nu}^{*'}$	2	$\mu^{*'}$
n=16	2	$\eta_{16}\circ\bar{\kappa}_{17}$	16	σ_{16}^*	16	$\bar{\rho}_{16}$	2	δ_{16}
	2	σ_{16}^3	16	$\Sigma\sigma^{*'}$	8	$\nu_{16}\circ\bar{\kappa}_{19}$	2	$\bar{\mu}_{16}\circ\sigma_{33}$
	2	$\Sigma^3\lambda\circ\nu_{34}$	2	$\omega_{16}\circ\nu_{32}^2$	2	ϕ_{16}	2	$\bar{\sigma}'_{16}$
	2	$\nu_{16}^*\circ\nu_{34}$	2	$\nu_{16}\circ\bar{\sigma}_{19}$	2	ψ_{16}	2	$\tilde{\varepsilon}_{16}$
			2	$\varepsilon_{16}\circ\kappa_{24}$	2	$\Sigma\bar{\varepsilon}^{*'}$	2	$\Sigma^2\zeta^*$
					2	$\Sigma\bar{\nu}^{*'}$	2	$\Sigma\mu^{*'}$
					2	ε_{16}^*	2	μ_{16}^*
					2	$\bar{\nu}_{16}^*$	2	$\eta_{16}^*\circ\varepsilon_{32}$
							2	$\eta_{16}^*\circ\bar{\nu}_{32}$
n=17	2	$\eta_{17}\circ\bar{\kappa}_{18}$	16	σ_{17}^*	16	$\bar{\rho}_{17}$	2	δ_{17}
	2	σ_{17}^3	2	$\omega_{17}\circ\nu_{33}^2$	8	$\nu_{17}\circ\bar{\kappa}_{20}$	2	$\bar{\mu}_{17}\circ\sigma_{34}$
	2	$\nu_{17}^*\circ\nu_{35}$	2	$\nu_{17}\circ\bar{\sigma}_{20}$	2	ϕ_{17}	2	$\bar{\sigma}'_{17}$
			2	$\varepsilon_{17}\circ\kappa_{25}$	2	ψ_{17}	2	μ_{17}^*
					2	$\bar{\varepsilon}_{17}^*$	2	$\eta_{17}^*\circ\varepsilon_{33}$
					2	$\bar{\nu}_{17}^*$	2	$\eta_{17}^*\circ\bar{\nu}_{33}$
n=18	2	$\eta_{18}\circ\bar{\kappa}_{19}$	16	σ_{18}^*	16	$\bar{\rho}_{18}$	2	δ_{18}
	2	σ_{18}^3	2	$\nu_{18}\circ\bar{\sigma}_{21}$	8	$\nu_{18}\circ\bar{\kappa}_{21}$	2	$\bar{\mu}_{18}\circ\sigma_{35}$
	2	$\nu_{18}^*\circ\nu_{36}$	2	$\varepsilon_{18}\circ\kappa_{26}$	2	ϕ_{18}	2	$\bar{\sigma}'_{18}$
					2	ψ_{18}	16	$\Delta(\sigma_{37})$
					2	$\bar{\varepsilon}_{18}^*$		

Table A.8: 2-component of $\pi_{n+k}(S^n)$ with generators

	$k=21$		$k=22$		$k=23$		$k=24$	
n=19	2	$\eta_{19}\circ\bar{\kappa}_{20}$	16	σ^*_{19}	16	$\bar{\rho}_{19}$	2	δ_{19}
	2	σ^3_{19}	2	$\nu_{19}\circ\bar{\sigma}_{22}$	8	$\nu_{19}\circ\bar{\kappa}_{22}$	2	$\bar{\mu}_{19}\circ\sigma_{36}$
	2	$\nu^*_{19}\circ\nu_{37}$	2	$\varepsilon_{19}\circ\kappa_{27}$	2	ϕ_{19}	2	$\bar{\sigma}'_{19}$
	2	$\bar{\beta}\circ\eta_{39}$			2	ψ_{19}		
n=20	2	$\eta_{20}\circ\bar{\kappa}_{21}$	16	σ^*_{20}	16	$\bar{\rho}_{20}$	2	δ_{20}
	2	σ^3_{20}	2	$\nu_{20}\circ\bar{\sigma}_{23}$	8	$\nu_{20}\circ\bar{\kappa}_{23}$	2	$\bar{\mu}_{20}\circ\sigma_{37}$
	2	$\Sigma\bar{\beta}\circ\eta_{40}$	2	$\varepsilon_{20}\circ\kappa_{28}$	2	ϕ_{20}		
	2	$\bar{\bar{\beta}}\circ\eta_{40}$	4	$\Delta(\nu_{41})+2\sigma^*_{20}$	2	ψ_{20}		
n=21	2	$\eta_{21}\circ\bar{\kappa}_{22}$	8	σ^*_{21}	16	$\bar{\rho}_{21}$	2	δ_{21}
	2	σ^3_{21}	2	$\nu_{21}\circ\bar{\sigma}_{24}$	8	$\nu_{21}\circ\bar{\kappa}_{24}$	2	$\bar{\mu}_{21}\circ\sigma_{38}$
	2	$\Sigma\bar{\bar{\beta}}\circ\eta_{41}$	2	$\varepsilon_{21}\circ\kappa_{29}$	2	ϕ_{21}		
					2	ψ_{21}		
n=22	∞	$\Delta(\iota_{45})$	4	σ^*_{22}	16	$\bar{\rho}_{22}$	2	δ_{22}
	2	$\eta_{22}\circ\bar{\kappa}_{23}$	2	$\nu_{22}\circ\bar{\sigma}_{25}$	8	$\nu_{22}\circ\bar{\kappa}_{25}$	2	$\bar{\mu}_{22}\circ\sigma_{39}$
	2	σ^3_{22}	2	$\varepsilon_{22}\circ\kappa_{30}$	2	ϕ_{22}	4	$\Delta(\nu_{45})$
n=23	2	$\eta_{23}\circ\bar{\kappa}_{24}$	2	σ^*_{23}	16	$\bar{\rho}_{23}$	2	δ_{23}
	2	σ^3_{23}	2	$\nu_{23}\circ\bar{\sigma}_{26}$	8	$\nu_{23}\circ\bar{\kappa}_{26}$	2	$\bar{\mu}_{23}\circ\sigma_{40}$
			2	$\varepsilon_{23}\circ\kappa_{31}$	2	ϕ_{23}	2	$\tilde{\eta}'$
n=24			2	$\nu_{24}\circ\bar{\sigma}_{27}$	∞	$\Delta(\iota_{49})$	2	δ_{24}
			2	$\varepsilon_{24}\circ\kappa_{32}$	16	$\bar{\rho}_{24}$	2	$\bar{\mu}_{24}\circ\sigma_{41}$
					8	$\nu_{24}\circ\bar{\kappa}_{27}$	2	$\Sigma\tilde{\eta}'$
					2	ϕ_{24}	2	$\tilde{\eta}$
n=25					16	$\bar{\rho}_{25}$	2	δ_{25}
					8	$\nu_{25}\circ\bar{\kappa}_{28}$	2	$\bar{\mu}_{25}\circ\sigma_{42}$
					2	ϕ_{25}	2	$\Sigma\tilde{\eta}$
n=26							2	δ_{26}
							2	$\bar{\mu}_{26}\circ\sigma_{43}$

The generators have the following relations (though we do not list all the relations):

$$\eta_3^3 = 2\nu'$$
$$\Sigma^2\nu' = 2\nu_5$$
$$\eta_3 \circ \Sigma\nu' = 0$$
$$\nu_6 \circ \eta_9 = 0$$

$$\Sigma\sigma''' = 2\sigma''$$
$$\Sigma\sigma'' = 2\sigma'$$
$$\Sigma^2\sigma' = 2\Sigma\sigma_8$$
$$\sigma_{10} \circ \eta_{17} = \bar{\nu}_{10} + \varepsilon_{10}$$

$$\eta_3^2 \circ \varepsilon_5 = 2\varepsilon'$$
$$\Sigma^2\varepsilon' = \pm 2\nu_5 \circ \sigma_8$$
$$\nu_5 \circ \Sigma\sigma' = 2\nu_5 \circ \sigma_8$$
$$\sigma' \circ \nu_{14} =?(2k+1)\nu_7 \circ \sigma_{10}$$
$$\nu_7 \circ \sigma_{10} =?(2k+1)\sigma' \circ \nu_{14}$$
$$\nu_9 \circ \sigma_{12} =?(4k+2)\sigma_9 \circ \nu_{16}$$
$$\sigma_{12} \circ \nu_{19} = 0$$

$$\varepsilon_3 \circ \nu_{11} = \nu' \circ \bar{\nu}_6$$
$$\varepsilon' \circ \eta_{13} = \nu' \circ \varepsilon_6$$
$$\eta_3 \circ \Sigma\varepsilon' =?k\eta_3^2 \circ \mu_5$$
$$\eta_3^2 \circ \mu_5 = 2\mu'$$
$$\nu_5 \circ \Sigma\sigma' \circ \eta_{15} =?4k\zeta_5$$
$$\varepsilon_5 \circ \nu_{13} = 0$$
$$\Sigma^2\mu' = \pm 2\zeta_5$$
$$\nu_7 \circ \varepsilon_{10} = 2\bar{\nu}_6 \circ \nu_{14}$$
$$\nu_6 \circ \bar{\nu}_9 = 2\bar{\nu}_6 \circ \nu_{14}$$
$$\bar{\nu}_{10} \circ \nu_{18} = 0$$

$$\eta_3 \circ \Sigma\mu' = 0$$

$$\nu_5 \circ \Sigma\sigma' \circ \eta_{15}^2 = 0$$
$$\nu_6^4 = 0$$
$$\nu_6 \circ \mu_7 = 8\Delta(\sigma_{13})$$
$$\Sigma^2\theta = 0$$

$$\nu_5^2 \circ \sigma_{11} = 0$$
$$\nu_6 \circ \zeta_9 = 2\sigma'' \circ \sigma_{13}$$
$$\bar{\nu}_7 \circ \nu_{15}^2 = 2\kappa_7 + ?4k\sigma' \circ \sigma_{14}$$

$$\Sigma\rho^{iv} = 2\rho'''$$
$$\Sigma\rho''' = 2\rho''$$
$$\Sigma^2\rho'' = 2\rho'$$
$$\Sigma^4\rho' = 2\rho_{13}$$
$$\sigma_{10} \circ \bar{\nu}_{17} = \sigma_{10} \circ \varepsilon_{17}$$
$$\sigma_{11} \circ \bar{\nu}_{18} = 0$$

$$\nu_5^2 \circ \sigma_{11} \circ \nu_{18} = 0$$
$$\Sigma\zeta' = \sigma' \circ \eta_{14} \circ \varepsilon_{15}$$
$$\nu_5 \circ \sigma_8 \circ \nu_{15}^2 = \eta_5 \circ \bar{\varepsilon}_6$$
$$\varepsilon_3^2 = \varepsilon_3 \circ \bar{\nu}_{11} = \eta_3 \circ \bar{\varepsilon}_4 = \bar{\varepsilon}_3 \circ \eta_{18}$$
$$\sigma_{10} \circ \nu_{17}^3 =?k\mu_{10} \circ \sigma_{19}$$
$$\sigma_{10} \circ \eta_{16} \circ \varepsilon_{17} =?k\mu_{10} \circ \sigma_{19}$$
$$\mu_{10} \circ \sigma_{19} = \sigma_{10} \circ \mu_{17} + \Delta(8\sigma_{21})$$
$$\Sigma^2\eta^* =? \in \Sigma^3\pi_{30}(S^{14})$$
$$\Sigma\eta_{17}^* =? \in \Sigma^3\pi_{31}(S^{15})$$

$$\eta_3^2 \circ \bar{\varepsilon}_5 = 2\bar{\varepsilon}'$$
$$\Sigma\bar{\varepsilon}' = \Sigma\nu' \circ \kappa_7$$
$$\nu_5 \circ \Sigma\sigma' \circ \sigma_{15} =?(2k+1)\Sigma^2(\varepsilon' \circ \sigma_{13}) =?$$
$$\eta_{10} \circ \mu_{11} \circ \sigma_{20} = \sigma_{10} \circ \eta_{17} \circ \mu_{18}$$

$$\eta_3 \circ \Sigma\bar{\varepsilon}' = 0$$
$$\nu_5 \circ \Sigma\rho'' =?$$
$$\nu_6 \circ \bar{\varepsilon}_9 = 0$$
$$\zeta_9 \circ \sigma_{20} =?(4k+2)\sigma_9 \circ \zeta_{16}$$
$$\sigma_{10} \circ \zeta_{17} =? \in 2\pi_{28}(S^{10}) \setminus 4\pi_{28}(S^{10})$$
$$\Sigma\lambda'' = 2\lambda'$$
$$\Sigma\xi'' = 2\xi'$$
$$\Sigma^2\lambda' = 2\lambda$$
$$\Sigma^2\xi' = 2\xi_{12}$$
$$\sigma_{12} \circ \zeta_{19} = 8\Delta(\sigma_{25}) = 16\zeta_{12}$$
$$\Sigma^4\lambda = 2\nu^*_{17}$$
$$\xi_{20} = -\nu^*_{20}$$

$$\eta_3^2 \circ \bar{\mu}_5 = 2\bar{\mu}'$$
$$\nu_5 \circ \Sigma\sigma' \circ \mu_{15} = \Sigma^2(\varepsilon' \circ \mu_{13}) =?$$
$$\nu_5 \circ \Sigma^2\zeta' = \Sigma^2(\varepsilon' \circ \eta_{13} \circ \varepsilon_{14}) =?$$

$$\nu_5 \circ \eta_8 \circ \bar{\varepsilon}_9 = \Sigma^2(\varepsilon' \circ \nu_{12}^3$$
$$\Sigma^2\bar{\mu}' = 2\breve{\zeta}_5$$
$$\nu_6 \circ \mu_9 \circ \sigma_{18} = 16\bar{\sigma}_6$$
$$\Sigma^2\lambda' \circ \eta_{31} = 0$$
$$\Sigma^2\xi' \circ \eta_{31} = 0$$
$$\xi_{14} \circ \eta_{32} = 0$$
$$\Sigma^2\omega' = 2\omega_{14} \circ \nu_{30}$$
$$\omega_{18} \circ \nu_{34} = 0$$

$$\eta_3 \circ \Sigma\bar{\mu}' = 0$$
$$\nu_6^2 \circ \kappa_{12} = 2\bar{\kappa}'$$
$$\Sigma\bar{\kappa}' = \pm 2\bar{\kappa}_7$$
$$\Sigma\beta' = \theta' \circ \varepsilon_{23}$$

$$2\bar{\alpha} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12}$$
$$4\breve{\zeta}_9 = \eta_9^2 \circ \bar{\mu}_{11}$$
$$2\tilde{\varepsilon}_{10} = \sigma_{10} \circ \nu_{17} \circ \kappa_{20}$$

The Hopf invariants of the generators are:

$$H(\eta_2) = \iota_3$$
$$H(\nu') = \eta_5$$
$$H(\nu_4) = \iota_7$$

$$H(\sigma''') = \eta_9^3$$
$$H(\sigma'') = \eta_{11}^2$$
$$H(\sigma') = \eta_{13}$$
$$H(\sigma_8) = \iota_{15}$$

$$H(\varepsilon_3) = \nu_5^2$$
$$H(\bar{\nu}_6) \equiv \nu_{11} \mod 2\nu_{11}$$
$$H(\mu_3) = \sigma'''$$

$$H(\varepsilon') = \varepsilon_5$$
$$H(\mu') = \mu_5$$
$$H(\zeta_5) = 8\sigma_9$$
$$H(\theta') = \eta_{21}^2$$
$$H(\theta) = \eta_{23}$$

$$H(\kappa_7) =?$$
$$H(\bar{\varepsilon}_3) \equiv \nu_5 \circ \sigma_8 \circ \nu_{15} \mod \nu_5 \circ \eta_8 \circ \mu_9$$
$$H(\rho^{iv}) = \eta_9^2 \circ \mu_{11} = 4\zeta_9$$
$$H(\rho''') = \eta_{11} \circ \mu_{12}$$
$$H(\rho'') \equiv \mu_{13} \mod \{\nu_{13}^3\} + \{\eta_{13} \circ \varepsilon_{14}\}$$
$$H(\rho') = 8\sigma_{17}$$
$$H(\rho_{13}) = \eta_{25}^3 = 4\nu_{25}$$

$$H(\zeta') \equiv \zeta_{11} \mod 2\zeta_{11}$$
$$H(\omega_{14}) = \nu_{27}$$
$$H(\eta^{*\prime}) = \eta_{29}^2$$

$$H(\eta_{16}^*) = \eta_{31}$$

$$H(\bar{\varepsilon}') =?$$
$$H(\bar{\mu}_3) = \rho^{iv}$$
$$H(\varepsilon_{12}^*) = \nu_{23}^2$$

$$H(\lambda'') \equiv \eta_{19} \circ \varepsilon_{20} \mod \eta_{19}^2 \circ \sigma_{21}$$
$$H(\xi'') = \nu_{19}^3 + \eta_{19} \circ \varepsilon_{20} = \eta_{19}^2 \circ \sigma_{21}$$
$$H(\lambda') \equiv \varepsilon_{21} \mod \eta_{21} \circ \varepsilon_{22}$$
$$H(\xi') = \bar{\nu}_{21} + \varepsilon_{21} = \eta_{21} \circ \sigma_{22}$$
$$H(\xi_{12}) \equiv \sigma_{23} \mod 2\sigma_{23}$$
$$H(\lambda) = \nu_{25}^2$$
$$H(\nu_{16}^*) \equiv \nu_{31} \mod 2\nu_{31}$$

$$H(\bar{\mu}') \equiv \bar{\mu}_5 \mod \Sigma^3\pi_{19}(S^2)$$
$$H(\breve{\zeta}_5) = 8\rho'$$
$$H(\bar{\sigma}_6) \equiv \sigma_{11}^2 \mod 2\sigma_{11}^2$$
$$H(\omega') \equiv \varepsilon_{23} \mod \eta_{23} \circ \sigma_{24}$$

$$H(\bar{\kappa}') =?$$
$$H(\bar{\kappa}_7) =?$$
$$H(\beta') = \zeta_{19}$$
$$H(\beta'') = \eta_{21} \circ \mu_{22}$$
$$H(\beta''') = \mu_{23}$$
$$H(\bar{\beta}) = \eta_{37}^2$$
$$H(\bar{\bar{\beta}}) = \eta_{39}$$

$$H(\alpha) = \nu_9 \circ \kappa_{12}$$
$$H(\sigma^{*\prime\prime\prime}) \equiv \zeta_{23} \mod 2\zeta_{23}$$
$$H(\sigma^{*\prime\prime}) = \nu_{27}^3 + \eta_{27} \circ \varepsilon_{28}$$

$$H(\sigma^{*\prime}) = \bar{\nu}_{29} + \varepsilon_{29}$$
$$H(\sigma^*_{16}) \equiv \sigma_{31} \mod 2\sigma_{31}$$

$$H(\bar{\alpha}) = \alpha$$
$$H(\bar{\rho}''') = 4\breve{\zeta}_9$$
$$H(\bar{\rho}'') = \eta_{11} \circ \bar{\mu}_{12}$$
$$H(\bar{\rho}') \equiv \bar{\mu}_{13} \mod \sigma_{13} \circ \eta_{20} \circ \mu_{21}$$
$$H(\bar{\rho}_9) = 16\rho_{17}$$
$$H(\phi_5) = \bar{\sigma}_9$$
$$H(\psi_{10}) = \sigma^2_{19}$$
$$H(\bar{\varepsilon}^{*\prime}) = \eta_{29} \circ \varepsilon_{30}$$
$$H(\bar{\nu}^{*\prime}) = \nu^3_{29}$$
$$H(\bar{\varepsilon}^*_{16}) = \varepsilon_{31}$$

$$H(\bar{\nu}^*_{16}) = \bar{\nu}_{31}$$

$$H(\delta_3) \equiv \nu_5 \circ \bar{\sigma}_8 \mod \nu_5 \circ \breve{\zeta}_8$$
$$H(\bar{\sigma}'_6) \equiv \bar{\sigma}_{11} \mod \xi' \circ \eta_{29}$$
$$H(\breve{\zeta}'_6) \equiv \breve{\zeta}_{11} \mod 2\breve{\zeta}_{11}$$
$$H(\tilde{\varepsilon}_{10}) = \bar{\varepsilon}_{19}$$
$$H(\zeta^*) \equiv \zeta_{27} \mod 2\pi_{38}(S^{27})$$
$$H(\mu^{*\prime}) = \eta_{29} \circ \mu_{30}$$
$$H(\mu^*_{16}) = \mu_{31}$$
$$H(\tilde{\eta}') = \eta^2_{45}$$
$$H(\tilde{\eta}) = \eta_{47}$$

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