CATEGORICAL HOMOTOPY THEORY

Garth Warner

Department of Mathematics

University of Washington

Some say follow the money; I say follow the arrows.

ABSTRACT

This book is an account of certain developments in categorical homotopy theory that have taken place since the year 2000. Some aspects have been given the complete treatment (i.e., proofs in all detail), while others are merely surveyed. Therefore a lot of ground is covered in a relatively compact manner, thus giving the reader a feel for the "big picture" without getting bogged down in the "nitty-gritty".

REFERENCES

1. Cisinski, Denis-Charles, *Les Préfaisceaux Comme Modèles Des Types D'Homotopie*, Astérisque 308 (2006).

2. Dwyer, William et al., *Homotopy Limit Functors on Model Categories and Homotopical Categories*, A.M.S. Mathematical Surveys and Monographs 113 (2004).

3. Hirschhorn, Philip, *Model Categories and Their Localizations*, A.M.S. Mathematical Surveys and Monographs 99 (2003).

4. Joyal, André, *Notes On Quasi-Categories*, 2008.

5. Lurie, Jacob, *Higher Topos Theory*, Annals of Mathematics Studies 170 (2009).

6. Maltsiniotis, Georges, *La Théorie de L'Homotopie de Grothendieck*, Astérisque 301 (2005).

7. Warner, Garth, *Topics in Topology and Homotopy Theory*, Available at: www.math.washington.edu/~ warner/

CONTENTS

PRELIMINARIES

MATTERS SIMPLICIAL

TOPICS

MATTERS SIMPLICIAL

DEFINITIONS AND NOTATION

$\underline{\Delta}$ is the category whose objects are the ordered sets $[n] \equiv \{0,1,\ldots,n\}$ $(n \geq 0)$ and whose morphisms are the order preserving maps. In $\underline{\Delta}$, every morphism can be written as an epimorphism followed by a monomorphism and a morphism is a monomorphism (epimorphism) iff it is injective (surjective). The face operators are the monomorphisms $\delta_i^n:[n-1] \to [n]$ $(n > 0,\ 0 \leq i \leq n)$ defined by omitting the value i. The degeneracy operators are the epimorphisms $\sigma_i^n:[n+1] \to [n]$ $(n \geq 0,\ 0 \leq i \leq n)$ defined by repeating the value i. Suppressing superscripts, if $\alpha \in \mathrm{Mor}([m],[n])$ is not the identity, then α has a unique factorization

$$\alpha = (\delta_{i_1} \circ \cdots \circ \delta_{i_p}) \circ (\sigma_{j_1} \circ \cdots \circ \sigma_{j_q}),$$

where $n \geq i_1 > \cdots > i_p \geq 0$, $0 \leq j_1 < \cdots < j_q < m$, and $m + p = n + q$. Each $\alpha \in \mathrm{Mor}([m],[n])$ determines a linear transformation $R^{m+1} \to R^{n+1}$ which restricts to a map $\Delta^\alpha:\Delta^m \to \Delta^n$. Thus there is a functor $\Delta^?:\underline{\Delta} \to \underline{\mathrm{TOP}}$ that sends $[n]$ to Δ^n and α to Δ^α. Since the objects of $\underline{\Delta}$ are themselves small categories, there is also an inclusion $\iota:\underline{\Delta} \to \underline{\mathrm{CAT}}$.

Given a category $\underline{C}$, write $\underline{\mathrm{SIC}}$ for the functor category $[\underline{\Delta}^{\mathrm{OP}},\underline{C}]$ and $\underline{\mathrm{COSIC}}$ for the functor category $[\underline{\Delta},\underline{C}]$ -- then by definition, a simplicial object in $\underline{C}$ is an object in $\underline{\mathrm{SIC}}$ and a cosimplicial object in $\underline{C}$ is an object in $\underline{\mathrm{COSIC}}$.

EXAMPLE The Yoneda embedding

$$Y_{\underline{\Delta}} \in \mathrm{Ob}[\underline{\Delta},\hat{\underline{\Delta}}],$$

so $Y_{\underline{\Delta}}$ is a cosimplicial object in $\hat{\underline{\Delta}}$.

SIMPLICIAL SETS

Specialize to $\underline{C} = \underline{SET}$ -- then an object in $\underline{SISET}$ is called a simplicial set and a morphism in $\underline{SISET}$ is called a simplicial map. Given a simplicial set X, put $X_n = X([n])$, so for $\alpha:[m] \to [n]$, $X\alpha:X_n \to X_m$. If $\begin{cases} d_i = X\delta_i \\ s_i = X\sigma_i \end{cases}$, then d_i and s_i are connected by the simplicial identities:

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i & (i < j) \\ s_i \circ s_j = s_{j+1} \circ s_i & (i \leq j) \end{cases}, \quad d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & (i < j) \\ \text{id} & (i = j \text{ or } i = j + 1). \\ s_j \circ d_{i-1} & (i > j + 1) \end{cases}$$

The simplicial standard n-simplex is the simplicial set $\Delta[n] = \text{Mor}(\text{--},[n])$, so for $\alpha:[m] \to [n]$, $\Delta[\alpha]:\Delta[m] \to \Delta[n]$. Owing to the Yoneda lemma, if X is a simplicial set and if $x \in X_n$, then there exists one and only one simplicial map $\Delta_x:\Delta[n] \to X$ that takes $\text{id}_{[n]}$ to x.

THEOREM $\underline{SISET}$ is complete and cocomplete, wellpowered and cowellpowered.

[Note: $\underline{SISET}$ admits an involution $X \to X^{OP}$, where $d_i^{OP} = d_{n-i}$, $s_i^{OP} = s_{n-i}$.]

Let X be a simplicial set -- then one writes $x \in X$ when one means $x \in \underset{n}{\cup} X_n$. With this understanding, an $x \in X$ is said to be degenerate if there exists an epimorphism $\alpha \neq \text{id}$ and a $y \in X$ such that $x = (X\alpha)y$; otherwise, $x \in X$ is said to be nondegenerate. The elements of X_0 (= the vertexes of X) are nondegenerate. Every $x \in X$ admits a unique representation $x = (X\alpha)y$, where α is an epimorphism and y is nondegenerate. The nondegenerate elements in $\Delta[n]$ are the monomorphisms $\alpha:[m] \to [n]$ $(m \le n)$.

A simplicial subset of a simplicial set X is a simplicial set Y such that Y is a subfunctor of X, i.e., $Y_n \subset X_n$ for all n and the inclusion $Y \to X$ is a simplicial map.

SKELETONS

The n-skeleton of a simplicial set X is the simplicial subset $X^{(n)}$ $(n \ge 0)$ of X defined by stipulating that $X_p^{(n)}$ is the set of all $x \in X_p$ for which there exists an epimorphism $\alpha:[p] \to [q]$ $(q \le n)$ and a $y \in X_q$ such that $x = (X\alpha)y$. Therefore $X_p^{(n)} = X_p$ $(p \le n)$; furthermore, $X^{(0)} \subset X^{(1)} \subset \cdots$ and $X = \operatorname{colim} X^{(n)}$. A proper simplicial subset of $\Delta[n]$ is contained in $\Delta[n]^{(n-1)}$, the frontier $\dot{\Delta}[n]$ of $\Delta[n]$. Of course, $X^{(0)}$ is isomorphic to $X_0 \cdot \Delta[0]$. In general, let $X_n^{\#}$ be the set of nondegenerate elements of X_n. Fix a collection $\{\Delta[n]_x : x \in X_n^{\#}\}$ of simplicial standard n-simplexes indexed by $X_n^{\#}$ — then the simplicial maps $\Delta_x:\Delta[n] \to X$ $(x \in X_n^{\#})$ determine an arrow $X_n^{\#} \cdot \Delta[n] \to X^{(n)}$ and the commutative diagram

$$\begin{array}{ccc} X_n^{\#} \cdot \dot{\Delta}[n] & \longrightarrow & X^{(n-1)} \\ \downarrow & & \downarrow \\ X_n^{\#} \cdot \Delta[n] & \longrightarrow & X^{(n)} \end{array}$$

is a pushout square. Note too that $\dot{\Delta}[n]$ is a coequalizer: Consider the diagram

$$\coprod_{0 \le i < j \le n} \Delta[n-2]_{i,j} \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{0 \le i \le n} \Delta[n-1]_i,$$

where u is defined by the $\Delta[\delta_{j-1}^{n-1}]$ and v is defined by the $\Delta[\delta_i^{n-1}]$ -- then the $\Delta[\delta_i^n]$ define a simplicial map $f: \coprod_{0 \le i \le n} \Delta[n-1]_i \to \Delta[n]$ that induces an isomorphism $\text{coeq}(u,v) \to \dot{\Delta}[n]$.

REMARK Call $\underline{\Delta}_n$ the full subcategory of $\underline{\Delta}$ whose objects are the $[m]$ $(m \le n)$. Given a category $\underline{C}$, denote by $\underline{\mathrm{SIC}}_n$ the functor category $[\underline{\Delta}_n^{\mathrm{OP}}, \underline{C}]$. The objects of $\underline{\mathrm{SIC}}_n$ are the "n-truncated simplicial objects" in $\underline{C}$. Employing the notation of Kan extensions, take for K the inclusion $\underline{\Delta}_n^{\mathrm{OP}} \to \underline{\Delta}^{\mathrm{OP}}$ and write $\mathrm{tr}^{(n)}$ in place of K^*, so $\mathrm{tr}^{(n)}: \underline{\mathrm{SIC}} \to \underline{\mathrm{SIC}}_n$. If $\underline{C}$ is complete and cocomplete, then $\mathrm{tr}^{(n)}$ has a left adjoint $\mathrm{sk}^{(n)}: \underline{\mathrm{SIC}}_n \to \underline{\mathrm{SIC}}$, where $\forall$ X in $\underline{\mathrm{SIC}}_n$,

$$(\mathrm{sk}^{(n)} X)_m = \operatorname*{colim}_{\substack{[m] \to [k] \\ k \le n}} X_k,$$

and a right adjoint $\mathrm{cosk}^{(n)}: \underline{\mathrm{SIC}}_n \to \underline{\mathrm{SIC}}$, where $\forall$ X in $\underline{\mathrm{SIC}}_n$,

$$(\mathrm{cosk}^{(n)} X)_m = \lim_{\substack{[k] \to [m] \\ k \le n}} X_k.$$

[Note: The colimit and limit are taken over a comma category.]

EXAMPLE Let $\underline{C} = \underline{SET}$ -- then for any simplicial set X,

$$\mathrm{sk}^{(n)}(\mathrm{tr}^{(n)}X) \approx X^{(n)}.$$

GEOMETRIC REALIZATION

The realization functor $\Gamma_{\Delta^?}$ is a functor $\underline{SISET} \to \underline{TOP}$ such that $\Gamma_{\Delta^?} \circ Y_{\underline{\Delta}} = \Delta^?$.
It assigns to a simplicial set X a topological space

$$|X| = \int^{[n]} X_n \cdot \Delta^n,$$

the <u>geometric realization</u> of X, and to a simplicial map $f:X \to Y$ a continuous function $|f|:|X| \to |Y|$, the <u>geometric realization</u> of f.

In particular: $|\Delta[n]| = \Delta^n$ and $|\Delta[\alpha]| = \Delta^\alpha$.

EXAMPLE The pushout square

$$\begin{array}{ccc} \dot{\Delta}[n] & \longrightarrow & \Delta[0] \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & S[n] \end{array}$$

defines the <u>simplicial n-sphere</u> $S[n]$. Its geometric realization is homeomorphic to S^n.

A simplicial map $f:X \to Y$ is injective (surjective) iff its geometric realization $|f|:|X| \to |Y|$ is injective (surjective). Being a left adjoint, the functor $|\ |:\underline{SISET} \to \underline{TOP}$ preserves colimits.

THEOREM Let X be a simplicial set — then $|X|$ is a CW complex with CW structure $\{|X^{(n)}|\}$.

PROOF $|X^{(0)}|$ is discrete and the commutative diagram

$$\begin{tikzcd} X_n^\# \cdot \dot{\Delta}[n] \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ X_n^\# \cdot \Delta[n] \arrow[r] & X^{(n)} \end{tikzcd}$$

is a pushout square in SISET. Since the geometric realization functor is a left adjoint, it preserves colimits. Therefore the commutative diagram

$$\begin{tikzcd} X_n^\# \cdot \dot{\Delta}^n \arrow[r] \arrow[d] & |X^{(n-1)}| \arrow[d] \\ X_n^\# \cdot \Delta^n \arrow[r] & |X^{(n)}| \end{tikzcd}$$

is a pushout square in TOP, which means that $|X^{(n)}|$ is obtained from $|X^{(n-1)}|$ by attaching n-cells $(n > 0)$. Moreover, $X = \operatorname{colim} X^{(n)} \Rightarrow |X| = \operatorname{colim} |X^{(n)}|$, so $|X|$ has the final topology determined by the inclusions $|X^{(n)}| \to |X|$. Denoting now by G the identity component of the homeomorphism group of $[0,1]$, there is a left action $G \times |X| \to |X|$ and the orbits of G are the cells of $|X|$.

[Note: If Y is a simplicial subset of X, then $|Y|$ is a subcomplex of $|X|$, thus the inclusion $|Y| \to |X|$ is a closed cofibration.]

Therefore "geometric realization" can be viewed as a functor SISET $\to$ CGH.

REMARK A colimit in CGH is calculated by taking the maximal Hausdorff quotient of the colimit calculated in TOP.

THEOREM The functor $|\ |$:SISET $\to$ CGH preserves finite limits.

N.B. $|\ |$:SISET $\to$ CGH does not preserve arbitrary limits. E.g.: The arrow

$|\Delta[1]^{\omega}| \rightarrow |\Delta[1]|^{\omega}$ is not a homeomorphism (ω the first infinite ordinal).

SINGULAR SETS

The singular functor $S_{\Delta ?}$ is a functor TOP $\rightarrow$ SISET that assigns to a topological space X a simplicial set sin X, the singular set of X: $\sin X([n]) = \sin_n X = C(\Delta^n, X)$. $|\ |$ is a left adjoint for sin.

REMARK There is a functor T from SIAB to the category of chain complexes of abelian groups: Take an X and let TX be $X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} X_2 \xleftarrow{\partial} \cdots$, where $\partial = \sum_0^n (-1)^i d_i$ $(d_i : X_n \rightarrow X_{n-1})$. That $\partial \circ \partial = 0$ is implied by the simplicial identities. One can then apply the homology functor H_* and end up in the category of graded abelian groups. On the other hand, the forgetful functor AB $\rightarrow$ SET has a left adjoint F_{AB} that sends a set X to the free abelian group $F_{AB}X$ on X. Extend it to a functor F_{AB}:SISET $\rightarrow$ SIAB. In this terminology, the singular homology $H_*(X)$ of a topological space X is $H_*(TF_{AB}(\sin X))$.

THEOREM Let X be a topological space — then the arrow of adjunction $|\sin X| \rightarrow X$ is a weak homotopy equivalence.

REMARK The class of CW spaces is precisely the class of topological spaces for which the arrow of adjunction $|\sin X| \rightarrow X$ is a homotopy equivalence.

THEOREM Let X be a simplicial set — then the geometric realization of the arrow of adjunction $X \rightarrow \sin|X|$ is a homotopy equivalence.

CATEGORICAL REALIZATION

The realization functor Γ_ι is a functor $\underline{\text{SISET}} \rightarrow \underline{\text{CAT}}$ such that $\Gamma_\iota \circ Y_{\underline{\Delta}} = \iota$. It assigns to a simplicial set X a small category

$$\text{cat } X = \int^{[n]} X_n \cdot [n]$$

called the <u>categorical realization</u> of X. In particular, cat $\Delta[n] = [n]$. In general, cat X can be represented as a quotient category $CX/\sim$. Here, CX is the category whose objects are the elements of X_0 and whose morphisms are the finite sequences $(x_1,\ldots,x_n)$ of elements of X_1 such that $d_0x_i = d_1x_{i+1}$. Composition is concatenation and the empty sequences are the identities. The relations are $s_0x = \text{id}_x$ $(x \in X_0)$ and $(d_0x) \circ (d_2x) = d_1x$ $(x \in X_2)$.

REMARK The functor cat:$\underline{\text{SISET}} \rightarrow \underline{\text{CAT}}$ preserves finite products but does not preserve finite limits.

NERVES

The singular functor S_ι is a functor $\underline{\text{CAT}} \rightarrow \underline{\text{SISET}}$ that assigns to a small category $\underline{C}$ a simplicial set ner $\underline{C}$, the <u>nerve</u> of $\underline{C}$: ner $\underline{C}([n])$ $(= \text{ner}_n\ \underline{C}) = \text{Mor}([n],\underline{C})$, thus $\text{ner}_0\ \underline{C} = \text{Ob}\ \underline{C}$ and $\text{ner}_1\ \underline{C} = \text{Mor}\ \underline{C}$. cat is a left adjoint for ner. Since ner is full and faithful, the arrow of adjunction cat $\circ$ ner $\rightarrow \text{id}_{\underline{\text{CAT}}}$ is a natural isomorphism.

EXAMPLE Viewing [n] as a small category, the definitions imply that ner[n] = $\Delta[n]$.

N.B. We have

$$\text{ner } \underline{C}^{OP} = (\text{ner } \underline{C})^{OP}.$$

Let $\underline{C}$ be a small category — then its classifying space $B\underline{C}$ is the geometric realization of its nerve:

$$B\underline{C} \equiv |\text{ner } \underline{C}|.$$

LEMMA If $\underline{C}$ is a small category, then

$$B\underline{C} \approx B\underline{C}^{OP}.$$

[This identification is canonical but, in general, is not realized by a functor from $\underline{C}$ to $\underline{C}^{OP}$.]

LEMMA If $\underline{C}$ and $\underline{D}$ are small categories, then in $\underline{CGH}$,

$$B(\underline{C} \times \underline{D}) \approx B\underline{C} \times_k B\underline{D}.$$

[In fact,

$$\text{ner}(\underline{C} \times \underline{D}) \approx \text{ner } \underline{C} \times \text{ner } \underline{D}.]$$

SIMPLEX CATEGORIES

Let X be a simplicial set — then X is a cofunctor $\underline{\Delta} \to \underline{SET}$, thus one can form the Grothendieck construction $\text{gro}_{\underline{\Delta}}$ X on X. So the objects of $\text{gro}_{\underline{\Delta}}$ X are the $([n],x)$ $(x \in X_n)$ and the morphisms $([n],x) \to ([m],y)$ are the $\alpha:[n] \to [m]$ such that $(X\alpha)y = x$. One calls $\text{gro}_{\underline{\Delta}}$ X the simplex category of X. It is isomorphic to the comma category

$$|Y_{\underline{\Delta}},K_X| : \quad \begin{array}{ccc} \Delta[n] & \longrightarrow & \Delta[m] \\ \downarrow & & \downarrow \\ X & = & X. \end{array}$$

N.B. The association $X \to \text{gro}_{\underline{\Delta}} X$ defines a functor

$$\text{gro}_{\underline{\Delta}}:\underline{SISET} \to \underline{CAT}.$$

• In $\underline{\mathrm{SISET}}$, a <u>simplicial weak equivalence</u> is a simplicial map $f:X \to Y$ such that $|f|:|X| \to |Y|$ is a homotopy equivalence.

• In $\underline{\mathrm{CAT}}$, a <u>simplicial weak equivalence</u> is a functor $F:\underline{C} \to \underline{D}$ such that $|\mathrm{ner}\, F|:B\underline{C} \to B\underline{D}$ is a homotopy equivalence.

LEMMA There are natural simplicial weak equivalences

$$\begin{cases} \mathrm{ner}(\mathrm{gro}_{\underline{\Delta}}\, X) \to X \\ \mathrm{gro}_{\underline{\Delta}}(\mathrm{ner}\, \underline{C}) \to \underline{C}. \end{cases}$$

[For instance, the first arrow is the rule $\mathrm{ner}_p(\mathrm{gro}_{\underline{\Delta}}\, X) \to X_p$ that sends

$$([n_0],x_0) \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{p-1}} ([n_p],x_p) \text{ to } (X\alpha)x_p,$$

where $\alpha:[p] \to [n_p]$ is defined by $\alpha(i) = \alpha_{p-1} \circ \cdots \circ \alpha_i(n_i)$ $(0 \le i \le p)$ $(\alpha(p) = n_p)$.]

EXAMPLE Put

$$\underline{\Delta}[n] = \mathrm{gro}_{\underline{\Delta}}\, \Delta[n].$$

Then there is a natural simplicial weak equivalence

$$\mathrm{ner}\, \underline{\Delta}[n] \to \Delta[n].$$

If X and Y are simplicial sets and if $f:X \to Y$ is a simplicial map, then there is a commutative diagram

$$\begin{array}{ccc} |\mathrm{ner}(\mathrm{gro}_{\underline{\Delta}}\, X)| & \longrightarrow & |X| \\ \downarrow & & \downarrow |f| \\ |\mathrm{ner}(\mathrm{gro}_{\underline{\Delta}}\, Y)| & \longrightarrow & |Y|, \end{array}$$

from which it follows that f is a simplicial weak equivalence iff $\text{gro}_{\underline{\Delta}}$ f is a simplicial weak equivalence.

EXPONENTIAL OBJECTS

$\underline{\text{CAT}}$ is cartesian closed:

$$\text{Mor}(\underline{C} \times \underline{D},\underline{E}) \approx \text{Mor}(\underline{C},\underline{E}^{\underline{D}}),$$

where

$$\underline{E}^{\underline{D}} = [\underline{D},\underline{E}].$$

$\underline{\text{SISET}}$ is cartesian closed:

$$\text{Nat}(X \times Y,Z) \approx \text{Nat}(X,Z^Y),$$

where

$$Z^Y([n]) = \text{Nat}(Y \times \Delta[n],Z).$$

EXAMPLE Let $\emptyset = \dot{\Delta}[0]$ and $* = \Delta[0]$ -- then the four exponential objects associated with $\emptyset$ and $*$ are $\emptyset^{\emptyset} = *$, $*^{\emptyset} = *$, $\emptyset^{*} = \emptyset$, $*^{*} = *$.

LEMMA The functor

$$\text{ner}:\underline{\text{CAT}} \rightarrow \underline{\text{SISET}}$$

preserves exponential objects.

PROOF $\forall$ $[n] \in \underline{\Delta}$,

$$\begin{aligned} \text{ner}_n([\underline{C},\underline{D}]) &= \text{Mor}([n],[\underline{C},\underline{D}]) \\ &\approx \text{Mor}([n] \times \underline{C},\underline{D}) \\ &\approx \text{Mor}(\text{ner}([n] \times \underline{C}),\text{ner } \underline{D}) \\ &\approx \text{Mor}(\text{ner}[n] \times \text{ner } \underline{C},\text{ner } \underline{D}) \end{aligned}$$

$$\approx \mathrm{Mor}(\mathrm{ner}\ \underline{C} \times \mathrm{ner}[n], \mathrm{ner}\ \underline{D})$$

$$\approx \mathrm{Mor}(\mathrm{ner}\ \underline{C} \times \Delta[n], \mathrm{ner}\ \underline{D})$$

$$= (\mathrm{ner}\ \underline{D})^{\mathrm{ner}\ \underline{C}}([n]).$$

Therefore

$$\mathrm{ner}([\underline{C},\underline{D}]) \approx (\mathrm{ner}\ \underline{D})^{\mathrm{ner}\ \underline{C}}.$$

REMARK Given a small category $\underline{C}$ and a simplicial set X, the map

$$(\mathrm{ner}\ \underline{C})^{\mathrm{ner}(\mathrm{cat}\ X)} \longrightarrow (\mathrm{ner}\ \underline{C})^{X}$$

induced by the arrow $X \to \mathrm{ner}(\mathrm{cat}\ X)$ is an isomorphism.

NOTATION Given simplicial sets X and Y, write map(X,Y) in place of Y^X.

[Note: The elements of $\mathrm{map}(X,Y)_0 \approx \mathrm{Nat}(X,Y)$ are the simplicial maps $X \to Y$.]

SEMISIMPLICIAL SETS

Let $M_{\underline{\Delta}}$ be the set of monomorphisms in Mor $\underline{\Delta}$; let $E_{\underline{\Delta}}$ be the set of epimorphisms in Mor $\underline{\Delta}$ -- then every $\alpha \in \mathrm{Mor}\ \underline{\Delta}$ can be written uniquely in the form $\alpha = \alpha^{\#} \circ \alpha^{\flat}$, where $\alpha^{\#} \in M_{\underline{\Delta}}$ and $\alpha^{\flat} \in E_{\underline{\Delta}}$.

$\underline{\Delta}_M$ is the category with Ob $\underline{\Delta}_M$ = Ob $\underline{\Delta}$ and Mor $\underline{\Delta}_M = M_{\underline{\Delta}}$, $\iota_M:\underline{\Delta}_M \to \underline{\Delta}$ being the inclusion.

Write $\underline{\mathrm{SSISET}}$ for the functor category $[\underline{\Delta}_M^{\mathrm{OP}}, \underline{\mathrm{SET}}]$ -- then an object in $\underline{\mathrm{SSISET}}$

is called a <u>semisimplicial set</u> and a morphism in <u>SSISET</u> is called a <u>semisimplicial map</u>. There is a commutative diagram

$$\begin{tikzcd} \underline{\Delta}_M \arrow[r, "Y_{\underline{\Delta}} \circ \iota_M"] \arrow[d, "Y_{\underline{\Delta}_M}"'] & \hat{\underline{\Delta}} \\ \hat{\underline{\Delta}}_M \arrow[r, equal] & \hat{\underline{\Delta}}_M \arrow[u, "\Gamma_{Y_{\underline{\Delta}} \circ \iota_M}"'] \end{tikzcd},$$

where $\Gamma_{Y_{\underline{\Delta}} \circ \iota_M}$ is the realization functor corresponding to $Y_{\underline{\Delta}} \circ \iota_M$. It assigns to a semisimplicial set X a simplicial set PX, the <u>prolongment</u> of X. Explicitly, the elements of $(PX)_n$ are all pairs (x,ρ) with $x \in X_p$ and $\rho:[n] \to [p]$ an epimorphism, thus $(PX\alpha)(x,\rho) = ((X(\rho \circ \alpha)^{\#})x, (\rho \circ \alpha)^{b})$ if the codomain of α is $[n]$. And P assigns to a semisimplicial map $f:X \to Y$ the simplicial map $Pf:\begin{cases} PX \to PY \\ (x,\rho) \to (f(x),\rho) \end{cases}$.

The prolongment functor is a left adjoint for the forgetful functor $U:\hat{\underline{\Delta}} \to \hat{\underline{\Delta}}_M$ (the singular functor in this setup).

Put

$$|\ |_M = |\ | \circ P.$$

Then $(|\ |_M, U \circ \sin)$ is an adjoint pair and $|\ |_M$ is the realization functor determined by the composite $\Delta^? \circ \iota_M$, i.e.,

$$|\ |_M = \Gamma_{\Delta^? \circ \iota_M}.$$

THEOREM For any simplicial set X, the arrow $|UX|_M \to |X|$ is a homotopy equivalence.

SUBDIVISION

Given n, let $\bar{\Delta}[n]$ be the simplicial set defined by the following conditions.

(Ob) $\bar{\Delta}[n]$ assigns to an object [p] the set $\bar{\Delta}[n]_p$ of all finite sequences $\mu = (\mu_0,\ldots,\mu_p)$ of monomorphisms in $\underline{\Delta}$ having codomain [n] such that $\forall\ i,j$ $(0 \le i \le j \le p)$ there is a monomorphism μ_{ij} with $\mu_i = \mu_j \circ \mu_{ij}$.

(Mor) $\bar{\Delta}[n]$ assigns to a morphism $\alpha:[q] \to [p]$ the map $\bar{\Delta}[n]_p \to \bar{\Delta}[n]_q$ taking μ to $\mu \circ \alpha$, i.e., $(\mu_0,\ldots,\mu_p) \to (\mu_{\alpha(0)},\ldots,\mu_{\alpha(q)})$.

Call $\bar{\Delta}$ the functor $\underline{\Delta} \to \hat{\underline{\Delta}}$ that sends [n] to $\bar{\Delta}[n]$ and $\alpha:[m] \to [n]$ to $\bar{\Delta}[\alpha]:\bar{\Delta}[m] \to \bar{\Delta}[n]$, where $\bar{\Delta}[\alpha]\nu = ((\alpha \circ \nu_0)^{\#},\ldots,(\alpha \circ \nu_p)^{\#})$. The associated realization functor $\Gamma_{\bar{\Delta}}$ is a functor $\underline{\text{SISET}} \to \underline{\text{SISET}}$ such that $\Gamma_{\bar{\Delta}} \circ Y_{\underline{\Delta}} = \bar{\Delta}$. It assigns to a simplicial set X a simplicial set

$$\mathrm{Sd}\ X = \int^{[n]} X_n \cdot \bar{\Delta}[n],$$

the <u>subdivision</u> of X, and to a simplicial map $f:X \to Y$ a simplicial map $\mathrm{Sd}\ f:\mathrm{Sd}\ X \to \mathrm{Sd}\ Y$, the <u>subdivision</u> of f. In particular, $\mathrm{Sd}\ \Delta[n] = \bar{\Delta}[n]$ and $\mathrm{Sd}\ \Delta[\alpha] = \bar{\Delta}[\alpha]$. On the other hand, the realization functor $\Gamma_{Y_{\underline{\Delta}}}$ associated with the Yoneda embedding $Y_{\underline{\Delta}}$ is naturally isomorphic to the identity functor id on $\underline{\text{SISET}}$:

$$X = \int^{[n]} X_n \cdot \Delta[n].$$

If $d_n:\bar{\Delta}[n] \to \Delta[n]$ is the simplicial map that sends $\mu = (\mu_0,\ldots,\mu_p) \in \bar{\Delta}[n]_p$ to

$d_n\mu \in \Delta[n]_p : d_n\mu(i) = \mu_i(m_i)\ (\mu_i : [m_i] \to [n])$, then the d_n determine a natural transformation $d : \overline{\Delta} \to Y_{\underline{\Delta}}$, which, by functoriality, leads to a natural transformation $d : \Gamma_{\overline{\Delta}} \to \Gamma_{Y_{\underline{\Delta}}}$. Thus, $\forall$ X,Y and $\forall$ $f : X \to Y$, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Sd}\ X \arrow[r, "d_X"] \arrow[d, "\mathrm{Sd}\ f"'] & X \arrow[d, "f."] \\ \mathrm{Sd}\ Y \arrow[r, "d_Y"'] & Y \end{tikzcd}$$

THEOREM For any simplicial set X, the arrow $|d_X| : |\mathrm{Sd}\ X| \to |X|$ is a homotopy equivalence.

REMARK It can be shown that for any simplicial set X, there is a homeomorphism $h_X : |\mathrm{Sd}\ X| \to |X|$.

[Note: h_X is not natural but is homotopic to $|d_X|$ which is natural.]

EXAMPLE Let X be a simplicial set -- then $|X|$ is homeomorphic to $B(\mathrm{cat}\ \mathrm{Sd}^2\ X)$. Therefore the geometric realization of a simplicial set is homeomorphic to the classifying space of a small category.

[Note: The homeomorphism is not natural.]

EXTENSION

Sd is the realization functor $\Gamma_{\overline{\Delta}}$. The associated singular functor $S_{\overline{\Delta}}$ is denoted by Ex and referred to as <u>extension</u>. Since (Sd,Ex) is an adjoint pair, there is a bijective map $\Xi_{X,Y}:\text{Nat}(\text{Sd } X,Y) \to \text{Nat}(X,\text{Ex } Y)$ which is functorial in X and Y. Put $e_X = \Xi_{X,X}(d_X)$ — then $e_X:X \to \text{Ex } X$ is the simplicial map given by $e_X(x) = \Delta_x \circ d_n$ $(x \in X_n)$, hence e_X is injective.

THEOREM For any simplicial set X, the arrow $|e_X|:|X| \to |\text{Ex } X|$ is a homotopy equivalence.

Denote by Ex^∞ the colimit of $\text{id} \to \text{Ex} \to \text{Ex}^2 \to \cdots$ — then Ex^∞ is a functor <u>SISET</u> $\to$ <u>SISET</u> and for any simplicial set X, there is an arrow $e_X^\infty:X \to \text{Ex}^\infty X$, the geometric realization of which is a homotopy equivalence.

COFIBRATIONS

A simplicial map $f:X \to Y$ is said to be a <u>cofibration</u> if its geometric realization $|f|:|X| \to |Y|$ is a cofibration.

LEMMA The cofibrations in <u>SISET</u> are the injective simplicial maps or still, the monomorphisms.

A cofibration is said to be <u>acyclic</u> if it is a simplicial weak equivalence.

EXAMPLE Let X be a simplicial set — then the arrow of adjunction $X \to \sin|X|$ is an acyclic cofibration.

EXAMPLE Let X be a simplicial set — then $e_X:X \to \text{Ex } X$ is an acyclic cofibration, as is $e_X^\infty:X \to \text{Ex}^\infty X$.

LEMMA Suppose that $f:X \to Y$ is an acyclic cofibration — then Sd f is an acyclic cofibration.

PROOF Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Sd}\, X \arrow[r, "\mathrm{Sd}\, f"] \arrow[d, "d_X"'] & \mathrm{Sd}\, Y \arrow[d, "d_Y"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Since Sd preserves injections, Sd f is a cofibration. But d_X and d_Y are simplicial weak equivalences.

Given $n \geq 1$, the <u>k^{th}-horn</u> $\Lambda[k,n]$ of $\Delta[n]$ $(0 \leq k \leq n)$ is the simplicial subset of $\Delta[n]$ defined by the condition that $\Lambda[k,n]_m$ is the set of $\alpha:[m] \to [n]$ whose image does not contain the set $[n] - \{k\}$.

<u>N.B.</u> $|\Lambda[k,n]| = \Lambda^{k,n}$ is the subset of $|\Lambda[n]| = \Delta^n$ consisting of those $(t_0,\ldots,t_n):t_i = 0$ $(\exists\ i \neq k)$, thus $\Lambda^{k,n}$ is a strong deformation retract of Δ^n.

LEMMA The inclusions $\Lambda[k,n] \to \Delta[n]$ $(0 \leq k \leq n,\ n \geq 1)$ are acyclic cofibrations.

KAN FIBRATIONS

Let $p:X \to B$ be a simplicial map -- then p is said to be a <u>Kan fibration</u> if it has the RLP w.r.t. the inclusions $\Lambda[k,n] \to \Delta[n]$ $(0 \leq k \leq n,\ n \geq 1)$.

EXAMPLE Let $\left[\begin{array}{l} X \\ Y \end{array}\right.$ be topological spaces, $f:X \to Y$ a continuous function -- then f is a Serre fibration iff $\sin f:\sin X \to \sin Y$ is a Kan fibration.

LEMMA Let $p:X \to B$ be a Kan fibration -- then $\mathrm{Ex}\, p:\mathrm{Ex}\, X \to \mathrm{Ex}\, B$ is a Kan fibration.

A simplicial set X is said to be a <u>Kan complex</u> if the arrow $X \to *$ is a Kan fibration. The Kan complexes are therefore those X such that every simplicial map $f:\Lambda[k,n] \to X$ can be extended to a simplicial map $F:\Delta[n] \to X$ $(0 \le k \le n,\ n \ge 1)$.

<u>N.B.</u> $\Delta[n]$ $(n \ge 1)$ is not a Kan complex.

EXAMPLE Let X be a topological space — then $\sin X$ is a Kan complex.

EXAMPLE Let $\underline{C}$ be a small category — then ner $\underline{C}$ is a Kan complex iff $\underline{C}$ is a groupoid.

EXAMPLE Let X be a simplicial set — then $\mathrm{Ex}^{\infty} X$ is a Kan complex.

LEMMA Suppose that $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a Kan fibration -- then the arrow $\mathrm{map}(K,X) \to \mathrm{map}(L,X) \times_{\mathrm{map}(L,B)} \mathrm{map}(K,B)$ is a Kan fibration.

[Pass from

$$\begin{tikzcd}
\Lambda[k,n] \arrow[r] \arrow[d] & \mathrm{map}(K,X) \arrow[d] \\
\Delta[n] \arrow[r] & \mathrm{map}(L,X) \times_{\mathrm{map}(L,B)} \mathrm{map}(K,B)
\end{tikzcd},$$

to

$$\begin{tikzcd}
\Lambda[k,n] \times K \cup \Delta[n] \times L \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\
\Delta[n] \times K \arrow[r] & B.
\end{tikzcd}]$$

So, as a special case, if Y is a Kan complex, then so is $\mathrm{map}(X,Y)$ $\forall$ X.

COMPONENTS

Let $\langle 2n\rangle$ be the category whose objects are the integers in the interval $[0,2n]$ and whose morphisms, apart from identities, are depicted by

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longleftarrow \cdots \longrightarrow \underset{2n-1}{\bullet} \longleftarrow \underset{2n}{\bullet}\ .$$

Put $I_{2n} = \mathrm{ner}\langle 2n\rangle$: $|I_{2n}|$ is homeomorphic to $[0,2n]$. Given a simplicial set X, a <u>path</u> in X is a simplicial map $\sigma: I_{2n} \to X$. One says that σ <u>begins</u> at $\sigma(0)$ and <u>ends</u> at $\sigma(2n)$. Write $\pi_0(X)$ for the quotient of X_0 with respect to the equivalence relation obtained by declaring that $x' \sim x''$ iff there exists a path in X which begins at x' and ends at x'' -- then the assignment $X \to \pi_0(X)$ defines a functor $\pi_0:\underline{\mathrm{SISET}} \to \underline{\mathrm{SET}}$ which preserves finite products and is a left adjoint for the functor $\mathrm{si}:\underline{\mathrm{SET}} \to \underline{\mathrm{SISET}}$ that sends X to $\mathrm{si}\ X$, the <u>constant simplicial set</u> on X, i.e., $\mathrm{si}\ X([n]) = X$ & $\begin{cases} d_i = \mathrm{id}_X \\ s_i = \mathrm{id}_X \end{cases}$ $(\forall\ n)$.

[Note: The geometric realization of $\mathrm{si}\ X$ is X equipped with the discrete topology.]

Given a simplicial set X, the decomposition of X_0 into equivalence classes determines a partition of X into simplicial subsets X_i. The X_i are called the <u>components</u> of X and X is <u>connected</u> if it has exactly one component.

[Note: $X = \coprod_i X_i \Rightarrow |X| = \coprod_i |X_i|$, $|X_i|$ running through the components of $|X|$, so $\pi_0(X) \longleftrightarrow \pi_0(|X|)$.]

EXAMPLE A small category $\underline{C}$ is connected iff its nerve ner $\underline{C}$ is connected or, equivalently, iff its classifying space $B\underline{C}$ is connected (= path connected).

LEMMA The components of a Kan complex are Kan.

RAPPEL Let K and L be CW complexes -- then a continuous function $f:K \to L$ is a homotopy equivalence iff for every CW complex Z, the arrow

$$\pi_0 \mathrm{map}(L,Z) \to \pi_0 \mathrm{map}(K,Z)$$

is bijective.

[Note: We have

$$\begin{tikzcd}
\pi_0 \mathrm{map}(L,Z) \arrow[r] & \pi_0 \mathrm{map}(K,Z) \\
\pi_0 \mathrm{map}(L,|\sin Z|) \arrow[r] \arrow[u, "\approx"] & \pi_0 \mathrm{map}(K,|\sin Z|). \arrow[u, "\approx"']
\end{tikzcd}$$

Therefore the top horizontal arrow is a bijection iff the bottom horizontal arrow is a bijection.]

LEMMA Let $\left[\begin{array}{l} X \\ Y \end{array}\right.$ be simplicial sets. Assume: Y is a Kan complex -- then there is a weak homotopy equivalence

$$|\mathrm{map}(X,Y)| \to \mathrm{map}(|X|,|Y|).$$

PROOF The assumption that Y is a Kan complex implies that the arrow $|\mathrm{map}(X,Y)| \to |\mathrm{map}(X,\sin Y)|$ is a homotopy equivalence. But $\mathrm{map}(X,\sin |Y|) \approx \sin \mathrm{map}(|X|,|Y|)$ and the arrow of adjunction

$$|\sin \mathrm{map}(|X|,|Y|)| \to \mathrm{map}(|X|,|Y|)$$

is a weak homotopy equivalence.

[Note: Here $\mathrm{map}(|X|,|Y|) = kC(|X|,|Y|)$ (compact open topology).]

CRITERION A simplicial map $f:X_1 \to X_2$ is a simplicial weak equivalence iff for every Kan complex Y, the arrow

$$\pi_0 \mathrm{map}(X_2,Y) \to \pi_0 \mathrm{map}(X_1,Y)$$

is bijective.

[The arrow $|f|:|X_1| \to |X_2|$ is a homotopy equivalence iff for every CW complex Z, the arrow

$$\pi_0 \mathrm{map}(|X_2|,|\sin Z|) \to \pi_0 \mathrm{map}(|X_1|,|\sin Z|)$$

is bijective. On the other hand,

$$\begin{cases} \pi_0 \mathrm{map}(X_1,\sin Z) \approx \pi_0 |\mathrm{map}(X_1,\sin Z)| \\ \pi_0 \mathrm{map}(X_2,\sin Z) \approx \pi_0 |\mathrm{map}(X_2,\sin Z)| \end{cases}$$

and since sin Z is a Kan complex,

$$\begin{cases} \pi_0 |\mathrm{map}(X_1,\sin Z)| \approx \pi_0 \mathrm{map}(|X_1|,|\sin Z|) \\ \pi_0 |\mathrm{map}(X_2,\sin Z)| \approx \pi_0 \mathrm{map}(|X_2|,|\sin Z|).] \end{cases}$$

CATEGORICAL WEAK EQUIVALENCES

A <u>weak Kan complex</u> is a simplicial set X such that every simplicial map $f:\Lambda[k,n] \to X$ can be extended to a simplicial map $F:\Delta[n] \to X$ $(0 < k < n,\ n > 1)$.

[Note: Every Kan complex is a weak Kan complex.]

<u>N.B.</u> If Y is a weak Kan complex, then so is $\mathrm{map}(X,Y)$ $\forall$ X.

EXAMPLE Let $\underline{C}$ be a small category -- then ner $\underline{C}$ is a weak Kan complex.

LEMMA Suppose that X is a weak Kan complex -- then X is a Kan complex iff cat X is a groupoid.

Denote by

$$c_0:\underline{\mathrm{SISET}} \to \underline{\mathrm{SET}}$$

the functor that sends X to the set of isomorphism classes of objects of cat X.

LEMMA c_0 preserves finite products.

PROOF cat and π_0 preserve finite products. This said, observe that c_0 is the composite

$$\underline{\mathrm{SISET}} \xrightarrow{\text{cat}} \underline{\mathrm{CAT}} \xrightarrow{\text{iso}} \underline{\mathrm{GRD}} \xrightarrow{\iota} \underline{\mathrm{CAT}} \xrightarrow{\text{ner}} \underline{\mathrm{SISET}} \xrightarrow{\pi_0} \underline{\mathrm{SET}}.$$

LEMMA If X is a Kan complex, then

$$c_0 X = \pi_0 X.$$

<u>N.B.</u> It therefore follows that if Y is a Kan complex, then $\forall$ X

$$c_0 \mathrm{map}(X,Y) = \pi_0 \mathrm{map}(X,Y).$$

DEFINITION A simplicial map $f:X_1 \to X_2$ is a <u>categorical weak equivalence</u> if for every weak Kan complex Y, the arrow

$$c_0 \mathrm{map}(X_2,Y) \to c_0 \mathrm{map}(X_1,Y)$$

is bijective.

EXAMPLE The inclusion $\Lambda[k,n] \to \Delta[n]$ $(0 < k < n,\ n > 1)$ is a categorical weak equivalence.

LEMMA The functor cat:$\underline{\mathrm{SISET}} \to \underline{\mathrm{CAT}}$ sends a categorical weak equivalence to a categorical equivalence.

THEOREM Suppose that $f:X_2 \to X_1$ is a categorical weak equivalence -- then $f:X_2 \to X_1$ is a simplicial weak equivalence.

PROOF For every Kan complex Y, the arrow

$$c_0\mathrm{map}(X_2,Y) \to c_0\mathrm{map}(X_1,Y)$$

is bijective. But

$$\begin{cases} c_0\mathrm{map}(X_2,Y) = \pi_0\mathrm{map}(X_2,Y) \\ c_0\mathrm{map}(X_1,Y) = \pi_0\mathrm{map}(X_1,Y), \end{cases}$$

from which the assertion.

POINTED SIMPLICIAL SETS

A <u>simplicial pair</u> is a pair (X,A), where X is a simplicial set and $A \subset X$ is a simplicial subset. Example: Fix $x_0 \in X_0$ and, in an abuse of notation, let x_0 be the simplicial subset of X generated by x_0 so that $(x_0)_n = \{s_{n-1} \cdots s_0 x_0\}$ $(n \geq 1)$ -- then (X,x_0) is a simplicial pair.

A <u>pointed simplicial set</u> is a simplicial pair (X,x_0). A <u>pointed simplicial map</u> is a base point preserving simplicial map $f:X \to Y$, i.e., a simplicial map $f:X \to Y$ for which the diagram

$$\begin{array}{ccc} \Delta[0] & = & \Delta[0] \\ {\scriptstyle \Delta_{x_0}}\downarrow & & \downarrow{\scriptstyle \Delta_{y_0}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes or, in brief, $f(x_0) = y_0$.

<u>SISET$_*$</u> is the category whose objects are the pointed simplicial sets and whose morphisms are the pointed simplicial maps. Thus $\underline{\text{SISET}}_* = [\underline{\Delta}^{\text{OP}}, \underline{\text{SET}}_*]$ and the forgetful functor $\underline{\text{SISET}}_* \to \underline{\text{SISET}}$ has a left adjoint that sends a simplicial set X to the pointed simplicial set $X_+ = X \coprod *$.

[Note: The vertex inclusion $e_0:\Delta[0] \to \Delta[1]$ defines the base point of $\Delta[1]$, hence of $\dot{\Delta}[1]$.]

$\Delta[0]$ is a zero object in $\underline{\text{SISET}}_*$ and $\underline{\text{SISET}}_*$ has the obvious products and coproducts. In addition, the pushout square

$$\begin{array}{ccc} X \vee Y & \longrightarrow & \Delta[0] \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & X \# Y \end{array}$$

defines the <u>smash product</u> $X \# Y$. Therefore $\underline{\text{SISET}}_*$ is a closed category if $X \otimes Y = X \# Y$ and $e = \dot{\Delta}[1]$. Here, the internal hom functor sends (X,Y) to $\text{map}_*(X,Y)$, the simplicial subset of $\text{map}(X,Y)$ whose elements in degree n are the $f:X \times \Delta[n] \to Y$ with $f(x_0 \times \Delta[n]) = y_0$, i.e., the pointed simplicial maps $X \# \Delta[n]_+ \to Y$, the zero morphism 0_{XY} being the base point.

SIMPLICIAL HOMOTOPY

Given simplicial sets X and Y, simplicial maps $f,g \in \text{Nat}(X,Y)$ are said to be <u>simplicially homotopic</u> ($f \underset{s}{\simeq} g$) provided that there exists a simplicial map $H:X \times \Delta[1] \to Y$ such that if

$$\begin{cases} H \circ i_0 : X \approx X \times \Delta[0] \xrightarrow{\mathrm{id}_X \times e_0} X \times \Delta[1] \xrightarrow{H} Y \\ H \circ i_1 : X \approx X \times \Delta[0] \xrightarrow[\mathrm{id}_X \times e_1]{} X \times \Delta[1] \xrightarrow[H]{} Y, \end{cases}$$

then $\begin{cases} H \circ i_0 = f \\ H \circ i_1 = g \end{cases}$, where $\begin{cases} e_0 : \Delta[0] \to \Delta[1] \\ e_1 : \Delta[0] \to \Delta[1] \end{cases}$ are the vertex inclusions per $\begin{cases} 0 \\ 1 \end{cases}$.

The relation $\underset{s}{\simeq}$ is reflexive but it needn't be symmetric or transitive.

[Note: Elements of $\mathrm{map}(X,Y)_1$ correspond to simplicial homotopies $H : X \times \Delta[1] \to Y$.]

EXAMPLE Take $X = Y = \Delta[n]$ $(n > 0)$. Let $C_0 : \Delta[n] \to \Delta[n]$ be the projection of $\Delta[n]$ onto the 0^{th} vertex, i.e., send $(\alpha_0,\ldots,\alpha_p) \in \Delta[n]_p$ to $(0,\ldots,0) \in \Delta[n]_p$. Claim: $C_0 \underset{s}{\simeq} \mathrm{id}_{\Delta[n]}$. To see this, consider the simplicial map $H : \Delta[n] \times \Delta[1] \to \Delta[n]$ defined by $H((\alpha_0,\ldots,\alpha_p),\ (0,\ldots,0,1,\ldots,1)) = (0,\ldots,0,\alpha_{i+1},\ldots,\alpha_p)$ so that $H((\alpha_0,\ldots,\alpha_p),\ (0,\ldots,0)) = (0,\ldots,0)$, $H((\alpha_0,\ldots,\alpha_p),\ (1,\ldots,1)) = (\alpha_0,\ldots,\alpha_p)$ -- then H is a simplicial homotopy between C_0 and $\mathrm{id}_{\Delta[n]}$. On the other hand, there is no simplicial homotopy H between $\mathrm{id}_{\Delta[n]}$ and C_0. For suppose that $H((1,1),(0,1)) = (\mu,\nu) \in \Delta[n]_1$. Apply d_1 & d_0 to get $\mu = 1$ & $\nu = 0$, an impossibility.

LEMMA Suppose that $\begin{cases} \underline{C} \\ \underline{D} \end{cases}$ are small categories. Let $F,G : \underline{C} \to \underline{D}$ be functors, $\Xi : F \to G$ a natural transformation -- then Ξ induces a functor $\Xi_H : \underline{C} \times [1] \to \underline{D}$ given on objects by

$$\Xi_H(X,0) = FX, \Xi_H(Y,1) = GY$$

and on morphisms by

$$\Xi_H(X \xrightarrow{f} Y, 0 \longrightarrow 0) = FX \xrightarrow{Ff} FY, \Xi_H(X \xrightarrow{g} Y, 1 \longrightarrow 1) = GX \xrightarrow{Gg} GY$$

$$\Xi_H(X \xrightarrow{h} Y, 0 \longrightarrow 1) = FX \xrightarrow{\Xi_Y \circ Fh} GY$$

or still,

$$\Xi_H(X \xrightarrow{h} Y, 0 \longrightarrow 1) = FX \xrightarrow{Gh \circ \Xi_X} GY.$$

Therefore

$$\text{ner } \Xi_H : \text{ner } \underline{C} \times \Delta[1] \to \text{ner } \underline{D}$$

is a simplicial homotopy between ner F and ner G.

Suppose that $\left[\begin{array}{l} F:\underline{C} \to \underline{D} \\ G:\underline{D} \to \underline{C} \end{array}\right.$ are an adjoint pair with arrows of adjunction

$\left[\begin{array}{l} \mu \in \text{Nat}(\text{id}_{\underline{C}}, G \circ F) \\ \\ \nu \in \text{Nat}(F \circ G, \text{id}_{\underline{D}}) \end{array}\right.$ — then

$$\left[\begin{array}{l} \text{id}_{\text{ner } \underline{C}} \underset{s}{\simeq} \text{ner } G \circ \text{ner } F \\ \\ \text{ner } F \circ \text{ner } G \underset{s}{\simeq} \text{id}_{\text{ner } \underline{D}} \end{array}\right.$$

or still, in the topological category,

$$\left[\begin{array}{l} \mathrm{id}_{B\underline{C}} \simeq |\mathrm{ner}\ G| \circ |\mathrm{ner}\ F| \\ \\ |\mathrm{ner}\ F| \circ |\mathrm{ner}\ G| \simeq \mathrm{id}_{B\underline{D}}. \end{array}\right.$$

I.e.: $\left[\begin{array}{l} B\underline{C} \\ B\underline{D} \end{array}\right.$ have the same homotopy type.

CONTRACTIBLE CLASSIFYING SPACES

DEFINITION A topological space X is <u>contractible</u> if the identity map of X is homotopic to some constant map of X to itself.

FACT A topological space is contractible iff it has the homotopy type of a one point space.

FACT Two contractible spaces have the same homotopy type.

FACT Any continuous map between contractible spaces is a homotopy equivalence.

A small category $\underline{C}$ is <u>contractible</u> if its classifying space $B\underline{C}$ is contractible.

EXAMPLE $\underline{1}$ is contractible ($B\underline{1}$ is a one point space).

LEMMA $\underline{C}$ is contractible iff the arrow $\underline{C} \to \underline{1}$ is a simplicial weak equivalence.

<u>N.B.</u> The arrow $\underline{C} \to \underline{1}$ is an equivalence of categories iff $\underline{C} \neq \underline{0}$ and every object is a final object.

LEMMA If $\underline{C}$ has a final object, then C is contractible.

[For then the functor $\underline{C} \to \underline{1}$ has the obvious right adjoint $\underline{1} \to \underline{C}$, thus $B\underline{C}$ and $B\underline{1}$ have the same homotopy type.]

[Note: If $\underline{C}$ has an initial object, then $\underline{C}$ is contractible. Proof: $\underline{C}^{OP}$ has a final object and $B\underline{C} \approx B\underline{C}^{OP}$.]

EXAMPLE $\underline{\Delta}$ is contractible ([0] is a final object).

REMARK If the functor $\underline{C} \to \underline{1}$ is an equivalence of categories, then $\underline{C}$ is contractible.

Suppose that $\underline{I}$ is a filtered category and let $\Delta:\underline{I} \to \underline{CAT}$ be a functor -- then since filtered colimits commute with finite limits in $\underline{SET}$, we have

$$\text{ner colim } \Delta \approx \text{colim ner } \Delta_i.$$

Assume now that $\forall$ morphism $i \xrightarrow{\delta} j$ in $\underline{I}$, the induced functor $\Delta\delta:\Delta_i \to \Delta_j$ is a simplicial weak equivalence -- then $\forall$ i, the functor $\Delta_i \to \text{colim } \Delta$ is a simplicial weak equivalence.

LEMMA Every filtered category $\underline{I}$ is contractible.

PROOF Define a functor $\Delta:\underline{I} \to \underline{CAT}$ by sending i to $\underline{I}/i$ — then $\underline{I} \approx \text{colim } \Delta$. But $\forall$ i, $\underline{I}/i$ has a final object, hence is contractible.

Let $\underline{C}$ be a small category, let $X \in \text{Ob } \underline{C}$, and let $F:\underline{C} \to \underline{C}$ be a functor.

LEMMA If there is a natural transformation from $\text{id}_{\underline{C}}$ to F and if there is a natural transformation from the constant functor $\underline{C} \to \underline{C}$ at X to F, then $B\underline{C}$ is contractible.

To illustrate this point, given a small category $\underline{I}$, let $\underline{\Delta}/\underline{I}$ be the category whose objects are the pairs (m,u), where $m \geq 0$ is an integer and $u:[m] \rightarrow \underline{I}$ is a functor, a morphism $(m,u) \rightarrow (n,v)$ being a morphism $f:[m] \rightarrow [n]$ of $\underline{\Delta}$ such that the diagram

$$\begin{tikzcd} {[m]} \arrow[r, "f"] \arrow[d, "u"'] & {[n]} \arrow[d, "v"] \\ \underline{I} \arrow[r, equal] & \underline{I} \end{tikzcd}$$

commutes.

FACT If $\underline{I}$ has a final object i_0, then $\underline{\Delta}/\underline{I}$ is contractible.

[Define a functor $F:\underline{\Delta}/\underline{I} \rightarrow \underline{\Delta}/\underline{I}$ as follows.

- On objects,

$$F(m,u) = (m + 1,u_+),$$

where

$$u_+(k) = \begin{cases} u(k) \text{ if } k \leq m \\ \\ i_0 \text{ if } k = m + 1. \end{cases}$$

- On morphisms,

$$Ff(k) = \begin{cases} f(k) \text{ if } k \leq m \\ \\ n + 1 \text{ if } k = m + 1. \end{cases}$$

Let $K_0:\underline{\Delta}/\underline{I} \rightarrow \underline{\Delta}/\underline{I}$ be the constant functor at $(0,K_{i_0})$ -- then $\exists$

$$\begin{cases} \alpha \in \text{Nat}(\text{id}_{\underline{\Delta}/\underline{I}},F) \\ \\ \beta \in \text{Nat}(K_0,F). \end{cases}$$

$\underline{\alpha}$: The inclusion $[m] \to [m+1]$ $(k \to k)$ induces a natural transformation $\mathrm{id}_{\underline{\Delta}/\underline{I}} \to F$. In fact,

$$\mathrm{id}_{\underline{\Delta}/\underline{I}}(m,u) \xrightarrow{\alpha_{(m,u)}} F(m,u)$$

is a morphism since the diagram

$$\begin{tikzcd}
{[m]} \arrow[r] \arrow[d, "u"'] & {[m+1]} \arrow[d, "u_+"] \\
\underline{I} \arrow[r, equal] & \underline{I}
\end{tikzcd}$$

commutes $(u(k) = u_+(k)$ if $k \le m)$.

$\underline{\beta}$: The inclusion $[0] \to [m+1]$ $(0 \to m+1)$ induces a natural transformation $K_0 \to F$. In fact,

$$K_0(m,u) \xrightarrow{\beta_{(m,u)}} F(m,u)$$

is a morphism since the diagram

$$\begin{tikzcd}
{[0]} \arrow[r] \arrow[d, "K_{i_0}"'] & {[m+1]} \arrow[d, "u_+"] \\
\underline{I} \arrow[r, equal] & \underline{I}
\end{tikzcd}$$

commutes $(K_{i_0}(0) = i_0 = u_+(m+1))$.

CHAPTER 0: MODEL CATEGORIES

CHAPTER 0: MODEL CATEGORIES

0.1 *ELEMENTS*

It is presupposed that the reader is familiar with the theory in so far as it is presented in TTHT. So in this section we shall simply establish notation and recall some standard facts.

0.1.1 DEFINITION Let $i:A \to Y$, $p:X \to B$ be morphisms in a category $\underline{C}$ -- then i is said to have the <u>left lifting property with respect to p</u> (LLP w.r.t. p) and p is said to have the <u>right lifting property with respect to i</u> (RLP w.r.t. i) if for all $u:A \to X$, $v:Y \to B$ such that $p \circ u = v \circ i$, there is a $w:Y \to X$ such that $w \circ i = u$, $p \circ w = v$, i.e., the commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, "v"'] & B \end{tikzcd}$$

admits a filler $w:Y \to X$.

0.1.2 EXAMPLE Take $\underline{C} = \underline{TOP}$ -- then $i:A \to Y$ is a cofibration iff $\forall$ X, i has the LLP w.r.t. $p_0:PX \to X$ and $p:X \to B$ is a Hurewicz fibration iff $\forall$ Y, p has the RLP w.r.t. $i_0:Y \to IY$.

[Note: As usual,

$$\begin{cases} PX = C([0,1],X) \\ IY = Y \times [0,1].] \end{cases}$$

Consider a category $\underline{C}$ equipped with three composition closed classes of morphisms termed weak equivalences (denoted $\xrightarrow{\sim}$), cofibrations (denoted $\rightarrowtail$), and fibrations (denoted $\twoheadrightarrow$), each containing the isomorphisms of $\underline{C}$. Agreeing to call a morphism which is both a weak equivalence and a cofibration (fibration) an acyclic cofibration (fibration), $\underline{C}$ is said to be a model category provided that the following axioms are satisfied.

(MC - 1) $\underline{C}$ is finitely complete and finitely cocomplete.

(MC - 2) Given composable morphisms f,g, if any two of $f,g,g \circ f$ are weak equivalences, so is the third.

(MC - 3) Every retract of a weak equivalence, cofibration, or fibration is again a weak equivalence, cofibration, or fibration.

(MC - 4) Every cofibration has the LLP w.r.t. every acyclic fibration and every fibration has the RLP w.r.t. every acyclic cofibration.

(MC - 5) Every morphism can be written as the composite of a cofibration and an acyclic fibration and the composite of an acyclic cofibration and a fibration.

0.1.3 NOTATION

$$\begin{aligned} \mathcal{W} &= \text{class of weak equivalences} \\ \mathrm{cof} &= \text{class of cofibrations} \\ \mathrm{fib} &= \text{class of fibrations.} \end{aligned}$$

N.B. The term model structure on a finitely complete and finitely cocomplete category $\underline{C}$ refers to the specification of $\mathcal{W}$, cof, fib subject to the assumptions above.

0.1.4 REMARK A weak equivalence $w:X \to Y$ which is a cofibration and a fibration is an isomorphism. Proof: The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w} \\ Y & \xrightarrow[\text{id}_Y]{} & Y \end{array}$$

admits a filler $Y \to X$.

0.1.5 EXAMPLE Every finitely complete and finitely cocomplete category $\underline{C}$ admits a model structure in which the weak equivalences are the isomorphisms and

$$\begin{cases} \text{cof} = \text{Mor } \underline{C} \\ \text{fib} = \text{Mor } \underline{C}. \end{cases}$$

A model category $\underline{C}$ has an initial object (denoted $\emptyset$) and a final object (denoted $*$). An object X in $\underline{C}$ is said to be <u>cofibrant</u> if $\emptyset \to X$ is a cofibration and <u>fibrant</u> if $X \to *$ is a fibration.

0.1.6 LEMMA Suppose that $\underline{C}$ is a model category. Let $X \in \text{Ob } \underline{C}$ -- then X is cofibrant iff every acyclic fibration $Y \to X$ has a right inverse and X is fibrant iff every acyclic cofibration $X \to Y$ has a left inverse.

0.1.7 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}$ -- then $\underline{\text{TOP}}$ is a model category if weak equivalence = homotopy equivalence, cofibration = closed cofibration, fibration = Hurewicz fibration. All objects are cofibrant and fibrant.

[Note: We shall refer to this model structure on $\underline{\text{TOP}}$ as the <u>Strøm structure</u>.]

Addendum: $\underline{\text{CG}}$ has a Strøm structure if weak equivalence = homotopy equivalence, cofibration = closed cofibration, fibration = $\underline{\text{CG}}$ fibration.

Given a model category $\underline{C}$, $\underline{C}^{\text{OP}}$ acquires the structure of a model category by stipulating that f^{OP} is a weak equivalence in $\underline{C}^{OP}$ iff f is a weak equivalence in $\underline{C}$, that f^{OP} is a cofibration in $\underline{C}^{OP}$ iff f is a fibration in $\underline{C}$, and that f^{OP} is a fibration in $\underline{C}^{OP}$ iff f is a cofibration in $\underline{C}$.

Given a model category $\underline{C}$ and objects A,B in $\underline{C}$, the categories $A\backslash\underline{C}$, $\underline{C}/B$ are again model categories, a morphism in either case being declared a weak equivalence, cofibration, or fibration if it is such when viewed in $\underline{C}$ alone.

0.1.8 EXAMPLE Take $\underline{C} = \underline{\underline{TOP}}$ (Strøm Structure) -- then an object (X,x_0) in $\underline{\underline{TOP}}_*$ ($\equiv *\backslash\underline{\underline{TOP}}$) is cofibrant iff $* \to (X,x_0)$ is a closed cofibration (in $\underline{\underline{TOP}}$), i.e., iff (X,x_0) is wellpointed with $\{x_0\} \subset X$ closed.

0.1.9 THEOREM Let $\underline{C}$ be a model category.

(1) The cofibrations in $\underline{C}$ are the morphisms that have the LLP w.r.t. acyclic fibrations.

(2) The acyclic cofibrations in $\underline{C}$ are the morphisms that have the LLP w.r.t. fibrations.

(3) The fibrations in $\underline{C}$ are the morphisms that have the RLP w.r.t. acyclic cofibrations.

(4) The acyclic fibrations in $\underline{C}$ are the morphisms that have the RLP w.r.t. cofibrations.

0.1.10 NOTATION Let $\underline{C}$ be a category and let $\mathcal{C} \subset \text{Mor }\underline{C}$ be a class of morphisms.

- Write LLP($\mathcal{C}$) for the class of morphisms having the left lifting property w.r.t. the elements of $\mathcal{C}$.

- Write RLP($\mathcal{C}$) for the class of morphisms having the right lifting property w.r.t. the elements of $\mathcal{C}$.

0.1.9 THEOREM (bis) Let $\underline{C}$ be a model category -- then

$$\text{cof} = \text{LLP}(\mathcal{W} \cap \text{fib}), \quad \mathcal{W} \cap \text{cof} = \text{LLP}(\text{fib}),$$

$$\text{fib} = \text{RLP}(\mathcal{W} \cap \text{cof}), \quad \mathcal{W} \cap \text{fib} = \text{RLP}(\text{cof}).$$

0.1.11 SCHOLIUM In a model category $\underline{C}$, any two of the classes of weak equivalences, cofibrations, and fibrations determines the third.

[Note: Suppose that

$$\left[\begin{array}{l} \mathcal{W}_1, \text{cof}_1, \text{fib}_1 \\ \\ \mathcal{W}_2, \text{cof}_2, \text{fib}_2 \end{array}\right.$$

are two model structures on $\underline{C}$ and let $\left[\begin{array}{l} F_1 \\ F_2 \end{array}\right.$ denote their classes of fibrant objects -- then

$$\text{cof}_1 = \text{cof}_2 \ \& \ F_1 = F_2 \Rightarrow \mathcal{W}_1 = \mathcal{W}_2 \ \& \ \text{fib}_1 = \text{fib}_2.$$

And

$$\left[\begin{array}{l} \text{cof}_1 = \text{cof}_2 \ \& \ F_2 \subset F_1 \Rightarrow \mathcal{W}_1 \subset \mathcal{W}_2 \\ \\ \text{cof}_1 = \text{cof}_2 \ \& \ \mathcal{W}_1 \subset \mathcal{W}_2 \Rightarrow F_2 \subset F_1.] \end{array}\right.$$

In a model category $\underline{C}$, the classes of cofibrations and fibrations possess a number of "closure" properties.

(Coproducts) If $\forall$ i, $f_i:X_i \to Y_i$ is a cofibration (acyclic cofibration), then $\coprod\limits_i f_i:\coprod\limits_i X_i \to \coprod\limits_i Y_i$ is a cofibration (acyclic cofibration).

(Products) If $\forall$ i, $f_i:X_i \to Y_i$ is a fibration (acyclic fibration), then $\prod\limits_i f_i:\prod\limits_i X_i \to \prod\limits_i Y_i$ is a fibration (acyclic fibration).

(Pushouts) Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

Assume: f is a cofibration (acyclic cofibration) -- then η is a cofibration (acyclic cofibration).

(Pullbacks) Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, define P by the pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Assume: g is a fibration (acyclic fibration) -- then ξ is a fibration (acyclic fibration).

(Sequential Colimits) If $\forall$ n, $f_n:X_n \rightarrow X_{n+1}$ is a cofibration (acyclic cofibration), then $\forall$ n, $i_n:X_n \rightarrow \operatorname{colim} X_n$ is a cofibration (acyclic cofibration).

(Sequential Limits) If $\forall$ n, $f_n:X_{n+1} \rightarrow X_n$ is a fibration (acyclic fibration), then $\forall$ n, $p_n:\lim X_n \rightarrow X_n$ is a fibration (acyclic fibration).

[Note: It is assumed that the relevant coproducts, products, sequential colimits, and sequential limits exist.]

0.1.12 EXAMPLE (Pushouts) Fix a model category $\underline{C}$. Let $\underline{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet\, 2$ -- then the functor category $[\underline{I},\underline{C}]$ is again a model category. Thus an object of $[\underline{I},\underline{C}]$ is a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$ and a morphism Ξ of 2-sources is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y'. \end{tikzcd}$$

Stipulate that Ξ is a weak equivalence or a fibration if this is the case of each of its vertical constituents. Define now P_L, P_R by the pushout squares

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[d] \\ P_L & Z' \arrow[l] \end{tikzcd}, \quad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ Z' \arrow[r] & P_R \end{tikzcd},$$

let $\rho_L:P_L \to X'$, $\rho_R:P_R \to Y'$ be the induced morphisms, and call Ξ a cofibration provided that $Z \to Z'$, ρ_L, and ρ_R are cofibrations. With these choices, $[\underline{I},\underline{C}]$ is a model category. The fibrant objects $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $[\underline{I},\underline{C}]$ are those for which X, Y, and Z are fibrant. The cofibrant objects $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $[\underline{I},\underline{C}]$ are those for which Z is cofibrant and $\left[\begin{array}{l} f:Z \to X \\ g:Z \to Y \end{array}\right.$ are cofibrations.

[Note: The story for pullbacks is analogous.]

0.1.13 EXAMPLE Fix a model category $\underline{C}$ -- then $\underline{\mathrm{FIL}}(\underline{C})$ is again a model category. Thus let $\phi:(\underline{X},\underline{f}) \to (\underline{Y},\underline{g})$ be a morphism in $\underline{\mathrm{FIL}}(\underline{C})$. Stipulate that ϕ is a weak equivalence or a fibration if this is the case of each ϕ_n. Define now P_{n+1} by the pushout square

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & X_{n+1} \arrow[d] \\ Y_n \arrow[r, "g_n"'] & P_{n+1}, \end{tikzcd}$$

let $\rho_{n+1}:P_{n+1} \to Y_{n+1}$ be the induced morphism, and call ϕ a cofibration provided that ϕ_0 and all the ρ_{n+1} are cofibrations (each ϕ_n $(n > 0)$ is then a cofibration as well). With these choices, $\underline{\mathrm{FIL}}(\underline{C})$ is a model category. The fibrant objects $(\underline{X},\underline{f})$ in $\underline{\mathrm{FIL}}(\underline{C})$ are those for which X_n is fibrant $\forall$ n. The cofibrant objects $(\underline{X},\underline{f})$ in $\underline{\mathrm{FIL}}(\underline{C})$ are those for which X_0 is cofibrant and $\forall$ n, $f_n:X_n \to X_{n+1}$ is a cofibration.

[Note: The story for $\underline{\mathrm{TOW}}(\underline{C})$ is analogous.]

0.1.14 DEFINITION Given a model category $\underline{C}$, objects X' and X'' are said to be <u>weakly equivalent</u> if there exists a path beginning at X' and ending at X'': $X' = X_0 \to X_1 \leftarrow \cdots \to X_{2n-1} \leftarrow X_{2n} = X''$, where all the arrows are weak equivalences.

0.1.15 EXAMPLE Take $\underline{C} = \underline{\mathrm{TOP}}$ (Strøm Structure) -- then X' and X'' are weakly equivalent iff they have the same homotopy type.

0.1.16 COMPOSITION LEMMA Consider the commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

in a category $\underline{C}$. Suppose that both the squares are pushouts -- then the rectangle is a pushout. Conversely, if the rectangle and the first square are pushouts, then the second square is a pushout.

0.1.17 APPLICATION Consider the commutative cube

$$\begin{tikzcd}
& \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \bullet \arrow[ur] \arrow[dd] & \\
& \bullet \arrow[rr] & & \bullet \\
\bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] &
\end{tikzcd}$$

in a category $\underline{C}$. Suppose that the top and the left and right hand sides are pushouts -- the the bottom is a pushout.

0.1.18 LEMMA Let $\underline{C}$ be a model category. Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\
X \arrow[r, "\xi"'] & P.
\end{tikzcd}$$

Assume: f is a cofibration and g is a weak equivalence -- then ξ is a weak equivalence provided that Z & Y are cofibrant.

[Note: There is a parallel statement for fibrations and pullbacks.]

0.1.19 EXAMPLE Working in $\underline{\text{TOP}}$ (Strøm Structure), suppose that $A \rightarrow X$ is a closed cofibration. Let $f:X \rightarrow Y$ be a homotopy equivalence -- then the arrow $X \rightarrow X \sqcup_f Y$ is a homotopy equivalence.

0.1.20 LEMMA Let $\underline{C}$ be a model category. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y', \end{tikzcd}$$

where $\begin{cases} f \\ f' \end{cases}$ are cofibrations and the vertical arrows are weak equivalences -- then the induced morphism $P \to P'$ of pushouts is a weak equivalence provided that Z & Y and Z' & Y' are cofibrant.

[Note: There is a parallel statement for fibrations and pullbacks.]

0.1.21 EXAMPLE Working in $\underline{\text{TOP}}$ (Strøm Structure), suppose that $\begin{cases} A \to X \\ A' \to X' \end{cases}$ are closed cofibrations. Let $\begin{cases} f:A \to Y \\ f':A' \to Y' \end{cases}$ be continuous functions. Assume that the diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes and that the vertical arrows are homotopy equivalences -- then the induced map $X \sqcup_f Y \to X' \sqcup_{f'} Y'$ is a homotopy equivalence.

0.1.22 DEFINITION Let $\underline{C}$ be a model category.

- $\underline{C}$ is said to be left proper if the following condition is satisfied.

Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P. \end{tikzcd}$$

Assume: f is a cofibration and g is a weak equivalence -- then ξ is a weak equivalence.

- $\underline{C}$ is said to be <u>right proper</u> if the following condition is satisfied.

Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, define P by the pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

Assume: g is a fibration and f is a weak equivalence -- then η is a weak equivalence.

<u>N.B.</u> $\underline{C}$ is <u>proper</u> if it is both left and right proper.

0.1.23 LEMMA If all the objects of $\underline{C}$ are cofibrant, then $\underline{C}$ is left proper (cf. 0.1.18) and if all the objects of $\underline{C}$ are fibrant, then $\underline{C}$ is right proper (cf. 0.1.18).

0.1.24 EXAMPLE The Strøm structure on <u>TOP</u> is proper (all objects are cofibrant and fibrant).

0.1.25 NOTATION Given a model category $\underline{C}$, write <u>HC</u> in place of $W^{-1}\underline{C}$ and call it the <u>homotopy category</u> of $\underline{C}$ (cf. 2.3.6).

[Note: $\mathcal{W}$ is necessarily saturated, i.e., $\mathcal{W} = \bar{\mathcal{W}}$ (cf. 2.3.20).]

0.1.26 EXAMPLE Take $\underline{C} = \underline{TOP}$ (Strøm Structure) — then $\underline{HTOP}$ "is" $\underline{HTOP}$.

0.1.27 THEOREM Suppose that $\underline{C}$ is a model category — then $\underline{HC}$ is a category (and not just a metacategory) (cf. 2.4.4).

0.1.28 EXAMPLE Consider the arrow category $\underline{C}(\rightarrow)$ of a model category $\underline{C}$ — then $\underline{C}(\rightarrow)$ can be equipped with two distinct model category structures both having the same class of weak equivalences, hence the same homotopy category. Thus let $(\phi,\psi):(X,f,Y) \rightarrow (X',f',Y')$ be a morphism in $\underline{C}(\rightarrow)$, so

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\psi]{} & Y' \end{array}$$

commutes. In the first structure, call (ϕ,ψ) a weak equivalence if ϕ & ψ are weak equivalences, a cofibration if ϕ and $X' \coprod_X Y \rightarrow Y'$ are cofibrations, a fibration if ϕ & ψ are fibrations and, in the second structure, call (ϕ,ψ) a weak equivalence if ϕ & ψ are weak equivalences, a cofibration if ϕ & ψ are cofibrations, a fibration if ψ and $X \rightarrow X' \times_{Y'} Y$ are fibrations.

[Note:

$$\underline{C} \text{ proper} \Rightarrow \underline{C}(\rightarrow) \text{ proper.}]$$

0.1.29 LEMMA If S is a set and if

$$\mathcal{W}_S,\ \mathrm{cof}_S,\ \mathrm{fib}_S$$

is a model structure on a category $\underline{C}_s$ $(s \in S)$, then

$$W = \prod_s W_s, \ \text{cof} = \prod_s \text{cof}_s, \ \text{fib} = \prod_s \text{fib}_s$$

is a model structure on $\underline{C} = \prod_s \underline{C}_s$ and the canonical arrow

$$\underline{HC} \to \prod_s \underline{HC}_s$$

is an equivalence of categories.

0.2 TOP: *QUILLEN STRUCTURE*

Take $\underline{C} = \underline{TOP}$ -- then $\underline{TOP}$ is a model category if weak equivalence = weak homotopy equivalence, cofibration = retract of a "countable composition" $X \to Y$, where $X = X_0 \to X_1 \to \cdots$, $Y = \operatorname{colim} X_k$, and $\forall$ k, the arrow $X_k \to X_{k+1}$ is defined by the pushout square

$$\begin{array}{ccc} \coprod\limits_{n\geq 0} \coprod S^{n-1} & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ \coprod\limits_{n\geq 0} \coprod D^n & \longrightarrow & X_{k+1}, \end{array}$$

fibration = Serre fibration. Every CW complex is cofibrant (and every object is weakly equivalent to a CW complex). Every cofibrant object is a compactly generated Hausdorff CW space (the quotient $[0,1]/[0,1[$ is compactly generated (and contractible) but not Hausdorff, hence not cofibrant). Every object is fibrant.

N.B. If (K,L) is a relative CW complex, then the inclusion $L \to K$ is a cofibration in the Quillen structure. Every cofibration in the Quillen structure is a closed cofibration, thus is a cofibration in the Strøm structure. And the Quillen structure is proper (even though not every object is cofibrant).

Addendum: CG, Δ-CG, and CGH each has a Quillen structure (definitions per those for TOP) which, moreover, is proper.

0.3 SISET: *KAN STRUCTURE*

Take $\underline{C}$ = SISET -- then SISET is a model category if weak equivalence = simplicial weak equivalence, cofibration = injective simplicial map, fibration = Kan fibration. Every object is cofibrant and the fibrant objects are the Kan complexes.

[Note: It is a corollary that $\underline{\text{SISET}}_*$ = $\Delta[0]\backslash$SISET is a model category.]

N.B. Recall that a simplicial map $f:X \to Y$ is a simplicial weak equivalence if $|f|:|X| \to |Y|$ is a homotopy equivalence.

0.3.1 LEMMA The Kan structure is proper.

PROOF Since all objects are cofibrant, half of this is automatic (cf. 0.1.23). This said, consider a pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in SISET. Assume: g is a Kan fibration and f is a weak equivalence -- then η is a weak equivalence. In fact,

$$\begin{tikzcd} {|P|} \arrow[r, "|\eta|"] \arrow[d, "|\xi|"'] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$

is a pullback square in $\underline{\text{CGH}}$, $|g|$ is a Serre fibration, and $|f|$ is a weak homotopy equivalence. Therefore $|\eta|$ is a weak homotopy equivalence.

0.3.2 REMARK Let fib_n stand for the class of f such that $\text{Ex}^n(f)$ is a Kan fibration ($n \geq 0$, $\text{Ex}^0(f) = f$) — then the containment

$$\text{fib}_n \subset \text{fib}_{n+1}$$

is strict and there is a model structure $\mathcal{W}_n$, cof_n, fib_n on $\underline{\text{SISET}}$ whose weak equivalences are those of the Kan structure (i.e., $\forall\, n$, $\mathcal{W}_n = \mathcal{W}_0$) and whose fibrations are the elements of fib_n. Bottom line: $\underline{\text{SISET}}$ can be equipped with a countable collection of distinct model structures all having the same homotopy category.

[Note: The containment

$$\text{cof}_{n+1} \subset \text{cof}_n$$

is strict, thus for $n > 0$, not every object is cofibrant. On the other hand, objects which are not fibrant in the Kan structure can become fibrant in structure "n" ($n > 0$), e.g., the $\Delta[m]$ ($m \geq 1$).]

0.4 $\underline{\text{SISET}}$: *JOYAL STRUCTURE*

Take $\underline{C} = \underline{\text{SISET}}$ — then $\underline{\text{SISET}}$ is a model category if weak equivalence = categorical weak equivalence, cofibration = injective simplicial map, fibration = all simplicial maps which have the RLP w.r.t. those cofibrations that are categorical weak equivalences. Every object is cofibrant and the fibrant objects are the weak Kan complexes.

<u>N.B.</u> Every weak equivalence per the Joyal structure is a weak equivalence per the Kan structure:

"categorical weak equivalence" => "simplicial weak equivalence".

0.4.1 REMARK The Joyal structure is left proper. However, it is not right proper.

0.5 SISET:*HG-STRUCTURE*

Take $\underline{C}$ = SISET and fix a nontrivial abelian group G -- then SISET is a model category if weak equivalence = HG-equivalence, cofibration = HG-cofibration, fibration = HG-fibration. Every object is cofibrant and the fibrant objects are the HG-local objects, i.e., those X such that $X \to *$ is an HG-fibration.

0.5.1 RAPPEL Let $f:X \to Y$ be a simplicial map -- then f is said to be an HG-equivalence if $\forall\ n \geq 0$, $|f|_*:H_n(|X|;G) \to H_n(|Y|;G)$ is an isomorphism. Agreeing that an HG-cofibration is an injective simplicial map, an HG-fibration is a simplicial map which has the RLP w.r.t. all HG-cofibrations that are HG-equivalences.

N.B. Every HG-fibration is a Kan fibration, hence every HG-local object is a Kan complex.

0.5.2 REMARK The HG-structure is left proper (but it need not be right proper (e.g., when $G = \mathbb{Q}$)).

0.6 SISET:ρ-*STRUCTURE*

Take $\underline{C}$ = SISET and fix an inclusion $\rho:A \to B$ of simplicial sets -- then SISET is a model category if weak equivalence = ρ-equivalence, cofibration = ρ-cofibration, fibration = ρ-fibration. Every object is cofibrant and the fibrant objects are the ρ-local objects.

0.6.1 RAPPEL Working within the Kan structure, a Kan complex Z is said to be <u>ρ-local</u> if $\rho^*:\mathrm{map}(B,Z) \to \mathrm{map}(A,Z)$ is a weak equivalence. Moreover, there is a functor $L_\rho:\underline{\mathrm{SISET}} \to \underline{\mathrm{SISET}}$ and a natural transformation $\mathrm{id} \to L_\rho$, where $\forall\ X$, $L_\rho X$ is ρ-local and $\ell_\rho:X \to L_\rho X$ is a cofibration such that for all ρ-local Z, the arrow $\mathrm{map}(L_\rho X,Z) \to \mathrm{map}(X,Z)$ is a weak equivalence.

0.6.2 RAPPEL Let $f:X \to Y$ be a simplicial map -- then f is said to be a <u>ρ-equivalence</u> if $L_\rho f:L_\rho X \to L_\rho Y$ is a weak equivalence. Agreeing that a <u>ρ-cofibration</u> is an injective simplicial map, a <u>ρ-fibration</u> is a simplicial map which has the RLP w.r.t. all ρ-cofibrations that are ρ-equivalences.

<u>N.B.</u> Every ρ-fibration is a Kan fibration.

0.7 <u>SIGR</u>: *FORGETFUL STRUCTURE*

The free group functor $F_{gr}:\underline{\mathrm{SET}} \to \underline{\mathrm{GR}}$ extends to a functor $F_{gr}:\underline{\mathrm{SISET}} \to \underline{\mathrm{SIGR}}$ which is left adjoint to the forgetful functor $U:\underline{\mathrm{SIGR}} \to \underline{\mathrm{SISET}}$. Call a morphism $f:G \to K$ of simplicial groups a weak equivalence if Uf is a weak equivalence, a fibration if Uf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations — then with these choices, <u>SIGR</u> is a model category.

[Note: Every object in <u>SIGR</u> is fibrant but not every object in <u>SIGR</u> is cofibrant. Definition: A simplicial group G is said to be <u>free</u> if $\forall\ n$, G_n is a free group with a specified basis B_n such that $s_i B_n \subset B_{n+1}$ $(0 \le i \le n)$. Every free simplicial group is cofibrant and every cofibrant simplicial group is the retract of a free simplicial group.]

0.8 $\underline{\mathrm{SISET}}_G$: *FORGETFUL STRUCTURE*

Fix a nontrivial group G. Denote by $\underline{G}$ the groupoid having a single object $*$ with $\mathrm{Mor}(*,*) = G$ -- then the category $\underline{\mathrm{SET}}_G$ of right G-sets is the functor category $[\underline{G}^{\mathrm{OP}}, \underline{\mathrm{SET}}]$ and the category of simplicial right G-sets $\underline{\mathrm{SISET}}_G$ is the functor category

$$[\underline{\Delta}^{\mathrm{OP}}, [\underline{G}^{\mathrm{OP}}, \underline{\mathrm{SET}}]] \approx [(\underline{\Delta} \times \underline{G})^{\mathrm{OP}}, \underline{\mathrm{SET}}].$$

So, if X is a simplicial right G-set, then $\forall\, n$, X_n is a right G-set and the actions are compatible with the simplicial structure maps. This said, let

$$U: \underline{\mathrm{SISET}}_G \longrightarrow \underline{\mathrm{SISET}}$$

be the forgetful functor and call a morphism $f: X \to Y$ of simplicial right G-sets a weak equivalence if Uf is a weak equivalence, a fibration if Uf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations -- then with these choices, $\underline{\mathrm{SISET}}_G$ is a model category.

[Note: Every object in $\underline{\mathrm{SISET}}_G$ is fibrant, the cofibrant objects being those X such that $\forall\, n$, X_n is a free G-set.]

0.8.1 REMARK U has a left adjoint F_G which sends X to $X \times \mathrm{si}\, G$.

0.9 $\underline{\mathrm{CXA}}$: *CANONICAL STRUCTURE*

Let $\underline{A}$ be an abelian category. Write $\underline{\mathrm{CXA}}$ for the abelian category of chain complexes over $\underline{A}$. Given a morphism $f: X \to Y$ in $\underline{\mathrm{CXA}}$, call f a weak equivalence if f is a chain homotopy equivalence, a cofibration if $\forall\, n$, $f_n: X_n \to Y_n$ has a left inverse, and a fibration if $\forall$ n, $f_n:X_n \to Y_n$ has a right inverse -- then with these choices, $\underline{CXA}$ is a model category.

0.10 $\underline{CXA}_{\geq 0}$: *STANDARD STRUCTURE*

Let $\underline{A}$ be an abelian category with enough projectives. Write $\underline{CXA}_{\geq 0}$ for the full subcategory of $\underline{CXA}$ whose objects have the property that $X_n = 0$ if $n < 0$. Given a morphism $f:X \to Y$ in $\underline{CXA}_{\geq 0}$, call f a weak equivalence if f is a homology equivalence, a cofibration if $\forall$ n, $f_n:X_n \to Y_n$ is a monomorphism with a projective cokernel, and a fibration if $\forall$ $n > 0$, $f_n:X_n \to Y_n$ is an epimorphism -- then with these choices, $\underline{CXA}_{\geq 0}$ is a proper model category. Every object is fibrant and the cofibrant objects are those X such that $\forall$ n, X_n is projective.

0.11 $\underline{CXA}$: *BEKE STRUCTURE*

Let $\underline{A}$ be a Grothendieck category with a separator -- then $\underline{A}$ is presentable, as is $\underline{CXA}$. Given a morphism $f:X \to Y$ in $\underline{CXA}$, call f a weak equivalence if f is a homology equivalence, a cofibration if f is a monomorphism, and a fibration if f has the RLP w.r.t. those cofibrations that are homology equivalences -- then with these choices, CXA is a proper model category. Every fibration is an epimorphism (but not conversely).

0.12 $\underline{CAT}$: *INTERNAL STRUCTURE*

Take $\underline{C} = \underline{CAT}$, let weak equivalence = equivalence, stipulate that a functor

$F:\underline{C} \to \underline{D}$ is a cofibration if the map

$$\begin{cases} \text{Ob } \underline{C} \longrightarrow \text{Ob } \underline{D} \\ X \longrightarrow FX \end{cases}$$

is injective and a fibration if $\forall\ X \in \text{Ob } \underline{C}$ and $\forall$ isomorphism $\psi:FX \to Y$ in $\underline{D}$, $\exists$ an isomorphism $\phi:X \to X'$ in $\underline{C}$ such that $F\phi = \psi$ -- then $\underline{CAT}$ is a model category in which all objects are cofibrant and fibrant.

[Note: These definitions restrict to give a model structure on $\underline{GRD}$.]

0.13 $\underline{CAT}$:*EXTERNAL STRUCTURE*

Take $\underline{C} = \underline{CAT}$, call a functor $F:\underline{C} \to \underline{D}$ a weak equivalence if $|\text{ner } F|:B\underline{C} \to B\underline{D}$ is a homotopy equivalence, a fibration if $\text{Ex}^2 \circ \text{ner } F$ is a Kan fibration, and a cofibration if F has the LLP w.r.t. all fibrations that are weak equivalences -- then $\underline{CAT}$ is a proper model category (but not all objects are cofibrant nor are all objects fibrant).

[Note: These definitions restrict to give a model structure on $\underline{GRD}$.]

0.14 $\underline{CAT}$:*MORITA STRUCTURE*

Take $\underline{C} = \underline{CAT}$, let the weak equivalences be those fully faithful functors $F:\underline{C} \to \underline{D}$ such that every object in $\underline{D}$ is the retract of an object in the image of F, let the cofibrations be the $F:\underline{C} \to \underline{D}$ which are injective on objects, and let the fibrations be the $F:\underline{C} \to \underline{D}$ which have the RLP w.r.t. acyclic cofibrations -- then $\underline{CAT}$ is a left proper model category (but $\underline{CAT}$ is not right proper). Every object is cofibrant and the fibrant objects are the small categories with the property that every idempotent splits.

0.15 $\underline{\text{EQU}}$: *LARUSSON STRUCTURE*

Let $\underline{\text{EQU}}$ be the category whose objects are the pairs $(X, \sim_X)$, where X is a set and $\sim_X$ is an equivalence relation on X, and whose morphisms are the maps $f:(X,\sim_X) \to (Y,\sim_Y)$, where f is a morphism in $\underline{\text{SET}}$ that sends equivalent elements in X to equivalent elements in Y. Call f a weak equivalence if f induces a bijection $X/\sim_X \to Y/\sim_Y$, a cofibration if f is injective, and a fibration if f maps each equivalence class in X onto an equivalence class in Y -- then $\underline{\text{EQU}}$ is a model category. Every object is cofibrant and fibrant.

0.16 *EXAMPLE*: $[\underline{I}, \underline{\text{SISET}}]$

Fix a small category $\underline{I}$ -- then the functor category $[\underline{I}, \underline{\text{SISET}}]$ admits two proper model category structures. However, the weak equivalences in either structure are the same, so both give rise to the same homotopy category $\underline{H}[\underline{I}, \underline{\text{SISET}}]$.

(L) Given functors $F,G:\underline{I} \to \underline{\text{SISET}}$, call $\Xi \in \text{Nat}(F,G)$ a weak equivalence if $\forall\ i$, $\Xi_i:Fi \to Gi$ is a simplicial weak equivalence, a fibration if $\forall\ i$, $\Xi_i:Fi \to Gi$ is a Kan fibration, a cofibration if Ξ has the LLP w.r.t. acyclic fibrations.

(R) Given functors $F,G:\underline{I} \to \underline{\text{SISET}}$, call $\Xi \in \text{Nat}(F,G)$ a weak equivalence if $\forall\ i$, $\Xi_i:Fi \to Gi$ is a simplicial weak equivalence, a cofibration if $\forall\ i$: $\Xi_i:Fi \to Gi$ is an injective simplicial map, a fibration if Ξ has the RLP w.r.t. acyclic cofibrations.

[Note: When $\underline{I}$ is discrete, structure L = structure R (all data is levelwise).]

Since the arguments are dual, it will be enough to outline the proof in the case of structure L.

0.16.1 NOTATION Let $f:X \to Y$ be a simplicial map -- then f admits a functorial factorization $X \xrightarrow{i_f} L_f \xrightarrow{\pi_f} Y$, where i_f is a cofibration and π_f is an acyclic Kan fibration, and a functorial factorization $X \xrightarrow{\iota_f} R_f \xrightarrow{p_f} Y$, where ι_f is an acyclic cofibration and p_f is a Kan fibration.

N.B. These factorizations extend levelwise to factorizations of $\Xi:F \to G$, viz. $F \xrightarrow{i_\Xi} L_\Xi \xrightarrow{\pi_\Xi} G$ and $F \xrightarrow{\iota_\Xi} R_\Xi \xrightarrow{p_\Xi} G$.

Write $\underline{I}_{dis}$ for the discrete category underlying $\underline{I}$ -- then the forgetful functor $U:[\underline{I},\underline{SISET}] \to [\underline{I}_{dis},\underline{SISET}]$ has a left adjoint that sends X to fr X, where

$$\mathrm{fr}\,X_j = \coprod_{i \in \mathrm{Ob}\,\underline{I}} \mathrm{Mor}(i,j) \cdot Xi.$$

0.16.2 LEMMA Fix an F in $[\underline{I},\underline{SISET}]$. Suppose that $\Phi:UF \to X$ is a cofibration in $[\underline{I}_{dis},\underline{SISET}]$ and

$$\begin{tikzcd}
\mathrm{fr}\,UF \arrow[r, "\mathrm{fr}\,\Phi"] \arrow[d, "\nu_F"'] & \mathrm{fr}\,X \arrow[d, "u"] \\
F \arrow[r] & G
\end{tikzcd}$$

is a pushout square in $[\underline{I},\underline{SISET}]$ — then the composite

$$Uu \circ \mu_X : X \xrightarrow{\mu_X} U\mathrm{fr}\,X \xrightarrow{Uu} UG$$

is a cofibration in $[\underline{I}_{dis},\underline{SISET}]$.

[The commutative diagram

$$\begin{tikzcd}
 & Xj \arrow[r, equal] \arrow[d] & Xj \arrow[d, "(\mu_X)_j"] \\
\left(\coprod\limits_{\substack{\delta \\ i \to j \\ \delta \neq \mathrm{id}_j}} Fi\right) \coprod Fj \arrow[r] \arrow[d] & \left(\coprod\limits_{\substack{\delta \\ i \to j \\ \delta \neq \mathrm{id}_j}} Fi\right) \coprod Xj \arrow[r] \arrow[d] & \left(\coprod\limits_{\substack{\delta \\ i \to j \\ \delta \neq \mathrm{id}_j}} Xi\right) \coprod Xj \arrow[d, "u_j"] \\
Fj \arrow[r, "\Phi_j"'] & Xj \arrow[r, "u_j \circ (\mu_X)_j"'] & Gj
\end{tikzcd}$$

tells the tale. Indeed, the middle row is a factorization of $(\mathrm{fr}\ \Phi)_j$ (suppression of "U"), the bottom square on the right is a pushout, and a coproduct of cofibrations is a cofibration.]

[Note: As usual, $\left[\begin{array}{l}\mu \\ \nu\end{array}\right.$ are the ambient arrows of adjunction.]

Consider any $\Xi : F \to G$. Claim: Ξ can be written as the composite of a cofibration and an acyclic fibration. Thus define F_1 by the pushout square

$$\begin{tikzcd}
\mathrm{fr}\ UF \arrow[r, "\mathrm{fr}\ Ui_{\Xi}"] \arrow[d, "\nu_F"'] & \mathrm{fr}\ UL_{\Xi} \arrow[d] \\
F \arrow[r] & F_1.
\end{tikzcd}$$

Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{fr}\ UF \arrow[r, "\mathrm{fr}\ Ui_\Xi"] \arrow[d, "\nu_F"'] & \mathrm{fr}\ UL_\Xi \arrow[r, "\mathrm{fr}\ U\pi_\Xi"] \arrow[d] & \mathrm{fr}\ UG \arrow[d, "\nu_G"] \\
F \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & G \arrow[d] \\
L_\Xi \arrow[r, equal] & L_\Xi \arrow[r, equal] & L_\Xi
\end{tikzcd}$$

in which $\mathrm{fr}\ UL_\Xi \to F_1 \to L_\Xi$ is ν_{L_Ξ}. Putting $F_0 = F$ (and $\Xi_0 = \Xi$), iterate the construction to obtain a sequence $F = F_0 \to F_1 \to \cdots \to F_\omega$ of objects in $[\underline{I},\underline{\mathrm{SISET}}]$, taking $F_\omega = \mathrm{colim}\ F_n$. This leads to a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "i_\omega"] \arrow[d, "\Xi"'] & F \arrow[d, "\Xi_\omega"] \\
G \arrow[r, equal] & G
\end{tikzcd} .$$

Here, i_ω is a cofibration (since the $F_n \to F_{n+1}$ are). Moreover, i_ω is a weak equivalence whenever Ξ is a weak equivalence and in that situation, i_ω has the LLP w.r.t. all fibrations. To see that Ξ_ω is an acyclic fibration, look at the interpolation

$$\begin{tikzcd}
UF_0 \arrow[r] \arrow[d] & UL_{\Xi_0} \arrow[r] \arrow[d] & UF_1 \arrow[r] \arrow[d] & UL_{\Xi_1} \arrow[r] \arrow[d] & \cdots \\
UG \arrow[r, equal] & UG \arrow[r, equal] & UG \arrow[r, equal] & UG \arrow[r, equal] & \cdots
\end{tikzcd}$$

in $[\underline{I}_{dis},\underline{\mathrm{SISET}}]$. Thanks to the lemma, the horizontal arrows in the top row are cofibrations. On the other hand, the arrows $UL_{\Xi_n} \to UG$ are acyclic fibrations. But then $U\Xi_\omega$ is an acyclic fibration per $[\underline{I}_{dis},\underline{SISET}]$, i.e., Ξ_ω is an acyclic fibration per $[\underline{I},\underline{SISET}]$. Hence the claim.

To finish the verification of MC - 5, one has to establish that Ξ can be written as the composite of an acyclic cofibration and a fibration. This, however, is immediate: Apply the claim to ι_Ξ. MC - 4 is equally clear. For if Ξ is a cofibration, then Ξ is a retract of i_ω, so if Ξ is an acyclic cofibration, then Ξ has the LLP w.r.t. all fibrations. Propriety is obvious.

N.B. In all of the above, it is understood that

$$[\underline{I}_{dis},\underline{SISET}] \approx \prod_{\text{Ob } \underline{I}} \underline{SISET}$$

carries the product structure of 0.1.29, where $\underline{SISET}$ itself is taken in its Kan structure.

0.16.3 EXAMPLE A functor $F:\underline{I} \to \underline{SISET}$ is said to be free if $\exists$ functors $B_n:\underline{I}_{dis} \to \underline{SET}$ $(n \geq 0)$ such that $\forall\ j \in \text{Ob } I: B_n j \subset (Fj)_n$ & $s_i B_n j \subset B_{n+1} j$ $(0 \leq i \leq n)$, with fr $B_n \approx F_n$ $(F_n j = (Fj)_n)$. Every free functor is cofibrant in structure L and every cofibrant functor in structure L is the retract of a free functor. Example: ner$(\underline{I}/—)$ is a free functor, hence is cofibrant in structure L.

0.17 *EXAMPLE*: $[\underline{I},\underline{C}]$

Consider the functor category $[\underline{I},\underline{C}]$, where $(I,\leq)$ is a finite nonempty directed set of cardinality ≥ 2 and $\underline{C}$ is a model category. Stipulate that a morphism $\Xi \in \text{Nat}(F,G)$ is a weak equivalence or a fibration if this is true levelwise, i.e., if $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i : Fi \to Gi$ is a weak equivalence or fibration. As for the cofibrations, given $i \in \text{Ob}\ \underline{I}$, let $\underline{I}_i$ be the subcategory of $\underline{I}$ whose elements are the $j \in I$ such that $j < i$ — then there is a commutative diagram

$$\begin{array}{ccc}
\text{colim}_{\underline{I}_i}\ Fj & \longrightarrow & \text{colim}_{\underline{I}_i}\ Gj \\
\downarrow & & \downarrow \\
Fi & \xrightarrow[\Xi_i]{} & Gi
\end{array}$$

and one deems Ξ a cofibration if $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow

$$Fi \underset{\text{colim}_{\underline{I}_i}\ Fj}{\sqcup} \text{colim}_{\underline{I}_i}\ Gj \longrightarrow Gi$$

is a cofibration. Using induction on the cardinality of I, it thus follows that with these choices, $[\underline{I},\underline{C}]$ is a model category.

0.18 *WEAK FACTORIZATION SYSTEMS*

Let $\underline{C}$ be a category.

0.18.1 DEFINITION A <u>weak factorization system</u> (w.f.s.) on $\underline{C}$ is a pair (L,R), where

$$\begin{cases} L \subset \text{Mor}\ \underline{C} \\ R \subset \text{Mor}\ \underline{C} \end{cases}$$

are classes of maps such that

$$\begin{cases} L = \text{LLP}(R) \\ R = \text{RLP}(L) \end{cases}$$

and every $f \in \operatorname{Mor} \underline{C}$ admits a factorization $f = \rho \circ \lambda$ with $\lambda \in L$, $\rho \in R$.

0.18.2 EXAMPLE Suppose that $\underline{C}$ is a model category — then the pairs

$$\begin{cases} (\text{cof}, \mathcal{W} \cap \text{fib}) \\ (\mathcal{W} \cap \text{cof}, \text{fib}) \end{cases}$$

are w.f.s. on $\underline{C}$ (cf. 0.1.9 (bis)).

0.18.3 LEMMA Let (L,R) be a w.f.s. on $\underline{C}$ -- then L and R are closed under the formation of retracts and each contains the isomorphisms of $\underline{C}$.

[Note: The intersection $L \cap R$ is the class of isomorphisms of $\underline{C}$. Proof: Let $f \in L \cap R$, say $f:X \to Y$, and consider the lifting problem

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd} .]$$

0.18.4 EXAMPLE Let $\underline{C}$ be a finitely complete and finitely cocomplete category -- then every w.f.s. (L,R) on $\underline{C}$ gives rise to a model structure on $\underline{C}$, viz. the triple

$$(\operatorname{Mor} \underline{C}, L, R).$$

E.g.: Take $\underline{C} = \underline{\text{SET}}$ and let L = the monomorphisms, R = the epimorphisms.

0.18.5 DEFINITION Let $\underline{C}$ be a cocomplete category. Fix a class $C \subset \operatorname{Mor} \underline{C}$.

- C is closed under the formation of pushouts if for every pushout square in $\underline{C}$

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow[\xi]{} & P \end{array} \quad : f \in C \Rightarrow \eta \in C.$$

• C is closed under the formation of transfinite compositions if for every wellordered set I with initial element 0 and for every functor $\Delta:\underline{I} \to \underline{C}$ such that $\forall\ i > 0$, the arrow

$$\mathrm{colim}_{j<i}\ \Delta_j \to \Delta_i$$

is an element of C, the arrow

$$\Delta_0 \to \mathrm{colim}_{\underline{I}}\Delta$$

is an element of C.

0.18.6 DEFINITION Let $\underline{C}$ be a cocomplete category. Suppose that $C \subset \mathrm{Mor}\ \underline{C}$ is closed under composition and contains the isomorphisms of $\underline{C}$ — then C is stable if it is closed under the formation of pushouts and transfinite compositions.

0.18.7 LEMMA Let $\underline{C}$ be a cocomplete category -- then every stable class $C \subset \mathrm{Mor}\ \underline{C}$ is closed under the formation of coproducts (taken in $\underline{C}(\to)$).

0.18.8 DEFINITION Let $\underline{C}$ be a cocomplete category — then a class $C \subset \mathrm{Mor}\ \underline{C}$ is retract stable if it is stable and closed under the formation of retracts.

0.18.9 EXAMPLE Let $\underline{C}$ be a small category — then the class $M \subset \mathrm{Mor}\ \hat{\underline{C}}$ of monomorphisms is retract stable.

[Note: The pair $(M,\mathrm{RLP}(M))$ is a w.f.s. on $\hat{\underline{C}}$.]

0.18.10 THEOREM Suppose that $\underline{C}$ is a cocomplete category — then for any class $C \subset \mathrm{Mor}\ \underline{C}$, $\mathrm{LLP}(C)$ is retract stable.

In particular: If $\underline{C}$ is cocomplete and if (L,R) is a w.f.s. system on $\underline{C}$, then L is retract stable.

Let $\underline{C}$ and $\underline{C}'$ be categories.

0.18.11 LEMMA Suppose that

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

are an adjoint pair. Let $\begin{cases} f \in \mathrm{Mor}\ \underline{C} \\ f' \in \mathrm{Mor}\ \underline{C}' \end{cases}$ — then Ff has the LLP w.r.t. f' iff f has the LLP w.r.t. $F'f'$.

PROOF There is a one-to-one correspondence between the commutative squares

$$\begin{tikzcd} FX \arrow[r] \arrow[d, "Ff"'] & X' \arrow[d, "f'"] \\ FY \arrow[r] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & F'X' \arrow[d, "F'f'"] \\ Y \arrow[r] & F'Y' \end{tikzcd}$$

and their fillers.

0.18.12 LEMMA Suppose that

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

are an adjoint pair. Let

$$\left[\begin{array}{l} (L,R) \text{ be a w.f.s. on } \underline{C} \\ (L',R') \text{ be a w.f.s. on } \underline{C}'. \end{array}\right.$$

Then

$$FL \subset L' \iff F'R' \subset R.$$

Suppose that $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$ are categories and

$$\left[\begin{array}{l} \underline{C} \text{ admits pushouts} \\ \underline{D} \text{ admits pullbacks.} \end{array}\right.$$

• Let $F_1,F_2:\underline{C} \to \underline{D}$ be functors and let $\alpha \in \text{Nat}(F_1,F_2)$. Given $f \in \text{Mor } \underline{C}$, there is a commutative diagram

$$\begin{tikzcd} F_1A \arrow[r, "\alpha_A"] \arrow[d, "F_1f"'] & F_2A \arrow[d, "F_2f"] \\ F_1B \arrow[r, "\alpha_B"'] & F_2B \end{tikzcd}$$

and a canonical arrow

$$\alpha_\bullet f:F_1B \coprod_{F_1A} F_2A \longrightarrow F_2B,$$

defining thereby a functor

$$\alpha_\bullet:\underline{C}(\to) \to \underline{D}(\to).$$

• Let $G_1, G_2 : \underline{D} \to \underline{C}$ be functors and let $\beta \in \mathrm{Nat}(G_2, G_1)$. Given $g \in \mathrm{Mor}\, \underline{D}$, there is a commutative diagram

$$\begin{tikzcd} G_2X \arrow[r, "\beta_X"] \arrow[d, "G_2g"'] & G_1X \arrow[d, "G_1g"] \\ G_2Y \arrow[r, "\beta_Y"'] & G_1Y \end{tikzcd}$$

and a canonical arrow

$$\beta^{\bullet} f : G_2X \longrightarrow G_2Y \times_{G_1Y} G_1X,$$

defining thereby a functor

$$\beta^{\bullet} : \underline{D}(\to) \to \underline{C}(\to).$$

Assume now that

$$\left[\begin{array}{l} F_1 : \underline{C} \to \underline{D} \\ \\ G_1 : \underline{D} \to \underline{C}, \end{array}\right. \qquad \left[\begin{array}{l} F_2 : \underline{C} \to \underline{D} \\ \\ G_2 : \underline{D} \to \underline{C} \end{array}\right.$$

are adjoint pairs.

———— β generates a natural transformation

$$\beta_{1,2} : F_1 \to F_2.$$

Proof: $\forall\, A \in \mathrm{Ob}\, \underline{C}$

$$\left[\begin{array}{ll} & (\mu_2)_A : A \longrightarrow G_2F_2A \\ \Rightarrow & \\ & F_1(\mu_2)_A : F_1A \longrightarrow F_1G_2F_2A \end{array}\right.$$

$$\left[\begin{array}{l} \beta_{F_2A}:G_2F_2A \longrightarrow G_1F_2A \\ \Rightarrow \\ F_1\beta_{F_2A}:F_1G_2F_2A \longrightarrow F_1G_1F_2A \end{array}\right.$$

$$\left[\begin{array}{l} (\nu_1)_X:F_1G_1X \longrightarrow X \\ \Rightarrow \\ (\nu_1)_{F_2A}:F_1G_1F_2A \longrightarrow F_2A. \end{array}\right.$$

Put

$$(\beta_{1,2})_A = (\nu_1)_{F_2A} \circ F_1\beta_{F_2A} \circ F_1(\mu_2)_A.$$

———— α generates a natural transformation

$$\alpha_{2,1}:G_2 \to G_1.$$

Proof: $\forall\ X \in \text{Ob}\ \underline{D}$

$$\left[\begin{array}{l} (\mu_1)_A:A \longrightarrow G_1F_1A \\ \Rightarrow \\ (\mu_1)_{G_2X}:G_2X \longrightarrow G_1F_1G_2X \end{array}\right.$$

$$\left[\begin{array}{l} \alpha_{G_2X}:F_1G_2X \longrightarrow F_2G_2X \\ \Rightarrow \\ G_1\alpha_{G_2X}:G_1F_1G_2X \longrightarrow G_1F_2G_2X \end{array}\right.$$

$$\left[\begin{array}{l} (\nu_2)_X : F_2 G_2 X \longrightarrow X \\ \Rightarrow \\ G_1(\nu_2)_X : G_1 F_2 G_2 X \longrightarrow G_1 X. \end{array}\right.$$

Put

$$(\alpha_{2,1})_X = G_1(\nu_2)_X \circ G_1 \alpha_{G_2 X} \circ (\mu_1)_{G_2 X}.$$

0.18.13 LEMMA Suppose that $\alpha = \beta_{1,2}$ and $\beta = \alpha_{2,1}$ -- then

$$\left[\begin{array}{l} \alpha_\bullet \\ \beta^\bullet \end{array}\right.$$

are an adjoint pair.

Accordingly, under these conditions, there is a one-to-one correspondence between the commutative squares

$$\begin{tikzcd} F_1 B \coprod_{F_1 A} F_2 A \arrow[r] \arrow[d, "\alpha_\bullet f"'] & X \arrow[d, "g"] \\ F_2 B \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & G_2 X \arrow[d, "\beta^\bullet g"] \\ B \arrow[r] & G_2 Y \times_{G_1 Y} G_1 X \end{tikzcd}$$

and their fillers.

0.19 *FUNCTORIALITY*

Let $\underline{C}$ be a category. Consider its arrow category $\underline{C}(\rightarrow)$ -- then there are functors

$$\left[\begin{array}{l} \text{dom}:\underline{C}(\rightarrow) \longrightarrow \underline{C} \\ \text{cod}:\underline{C}(\rightarrow) \longrightarrow \underline{C} \end{array}\right.$$

that project to the domain and codomain respectively and a natural transformation $\Xi:\mathrm{dom} \to \mathrm{cod}$, viz. $\Xi_f = f$.

[Note: There is also an embedding functor $E:\underline{C} \to \underline{C}(\to)$. On objects, $EX = \mathrm{id}_X$ and on morphisms,

$$E(X \xrightarrow{f} Y) = (f,f): \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \quad .]$$

0.19.1 DEFINITION A w.f.s. (L,R) on $\underline{C}$ is <u>functorial</u> if there are functors

$$\begin{cases} L:\underline{C}(\to) \longrightarrow \underline{C}(\to) \\ R:\underline{C}(\to) \longrightarrow \underline{C}(\to) \end{cases}$$

such that

$$\begin{cases} \mathrm{dom} \circ L = \mathrm{dom} \\ \mathrm{cod} \circ R = \mathrm{cod} \end{cases} \quad \& \ \mathrm{cod} \circ L = \mathrm{dom} \circ R$$

and $\forall\ f \in \mathrm{Mor}\ \underline{C}$, $f = Rf \circ Lf$ with $Lf \in L$ and $Rf \in R$.

<u>N.B.</u> Put

$$F = \mathrm{cod} \circ L = \mathrm{dom} \circ R.$$

Then there are natural transformations

$$\begin{cases} \lambda \in \mathrm{Nat}(\mathrm{dom},F) \\ \rho \in \mathrm{Nat}(F,\mathrm{cod}) \end{cases} : \quad \Xi = \rho \circ \lambda$$

and the factorization of $f \in \text{Mor}\,\underline{C}$ is given by

$$\begin{tikzcd}
\text{dom } f \arrow[r, "\Xi_f = f"] \arrow[d, "\lambda_f"'] & \text{cod } f \arrow[d, "\rho_f"] \\
Ff \arrow[r, equal] & Ff.
\end{tikzcd}$$

[Note: Let $(\phi,\psi):(X,f,Y) \to (X',f',Y')$ be a morphism in $\underline{C}(\to)$, so

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "\psi"'] & Y'
\end{tikzcd}$$

commutes -- then the diagram

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\lambda_f"'] & X' \arrow[d, "\lambda_{f'}"] \\
\bullet \arrow[r, "F(\phi,\psi)"] \arrow[d, "\rho_f"'] & \bullet \arrow[d, "\rho_{f'}"] \\
Y \arrow[r, "\psi"'] & Y'
\end{tikzcd}$$

commutes.]

0.19.2 DEFINITION The triple (F,λ,ρ) is called a <u>functorial realization</u> of the w.f.s. (L,R).

0.19.3 EXAMPLE Let $\underline{C}$ be a model category. Suppose that the w.f.s.

$$(\mathrm{cof}, \mathcal{W} \cap \mathrm{fib}) \qquad (\text{cf. } 0.18.2)$$

is functorial -- then $\forall\ X \xrightarrow{f} Y$ there is a commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[d] \\
X' \arrow[r, "F(\mathrm{id}_\emptyset, f)"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

where $\left[\begin{array}{l} X' \\ Y' \end{array}\right.$ are cofibrant and the arrows $\left[\begin{array}{l} X' \to X \\ Y' \to Y \end{array}\right.$ are acyclic fibrations. The assignment $X \to X'$ is called the <u>cofibrant replacement functor</u>, denote it by $\underline{L}$, thus by construction, there is a natural transformation $\underline{L} \xrightarrow{\Xi} \mathrm{id}_{\underline{C}}$ and $\forall\ X$, $\Xi_X : \underline{L}X \to X$ is an acyclic fibration.

- $\forall\ f \in L$, the lifting problem

$$\begin{tikzcd}
X \arrow[r, "\lambda_f"] \arrow[d, "f"'] & \bullet \arrow[d, "\rho_f"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

has a solution s, thus $\lambda_f = s \circ f$, $\rho_f \circ s = \mathrm{id}$.

• $\forall\, g \in \mathcal{R}$, the lifting problem

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "\lambda_g"'] & W \arrow[d, "g"] \\ \bullet \arrow[r, "\rho_g"'] & Z \end{tikzcd}$$

has a solution t, thus $\rho_g = g \circ t$, $t \circ \lambda_g = \mathrm{id}$.

0.19.4 NOTATION Given a functional realization (F,λ,ρ) of the w.f.s. $(\mathcal{L},\mathcal{R})$, let

$$\left[\begin{array}{l} \mathcal{L}_F = \{f : \exists\, s \text{ st } \lambda_f = s \circ f,\ \rho_f \circ s = \mathrm{id}\} \\ \\ \mathcal{R}_F = \{g : \exists\, t \text{ st } \rho_g = g \circ t,\ t \circ \lambda_g = \mathrm{id}\}. \end{array}\right.$$

If $f \in \mathcal{L}_F$, $g \in \mathcal{R}_F$, then the lifting problem

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "v"'] & \bullet \end{tikzcd}$$

can be solved by taking $w = t \circ F(u,v) \circ s$.

0.19.5 LEMMA We have

$$\left[\begin{array}{l} \mathcal{L} = \mathcal{L}_F \\ \\ \mathcal{R} = \mathcal{R}_F. \end{array}\right.$$

0.20 *COFIBRANTLY GENERATED W.F.S.*

Let $\underline{C}$ be a cocomplete category.

0.20.1 NOTATION Let $C \subset \text{Mor } \underline{C}$ be a class of morphisms -- then by $\text{cell } C$ we shall understand the smallest stable class containing C.

0.20.2 NOTATION Let $C \subset \text{Mor } \underline{C}$ be a class of morphisms -- then by $\text{cof } C$ we shall understand the smallest retract stable class containing C.

0.20.3 LEMMA We have

$$C \subset \text{cell } C \subset \text{cof } C \subset \text{LLP}(\text{RLP}(C)) \quad (\text{cf. } 0.18.10).$$

0.20.4 LEMMA Suppose that $\underline{C}$ is presentable -- then for every set $I \subset \text{Mor } \underline{C}$,

$$\text{cof } I = \text{LLP}(\text{RLP}(I)).$$

0.20.5 EXAMPLE Let $\underline{C}$ be a small category and let $M \subset \text{Mor } \hat{\underline{C}}$ be the class of monomorphisms -- then there exists a set $\text{M} \subset M$ such that $M = \text{LLP}(\text{RLP}(\text{M}))$, hence $M = \text{cof M}$ ($\hat{\underline{C}}$ being presentable).

(1) Take $\underline{C} = \underline{1}$ -- then $\hat{\underline{1}} \approx \underline{\text{SET}}$ and we can let $\text{M} = \{\emptyset \to \star\}$.

(2) Take $\underline{C} = \underline{\Delta}$ -- then $\hat{\underline{\Delta}} \approx \underline{\text{SISET}}$ and we can let $\text{M} = \{\dot{\Delta}[n] \to \Delta[n] : n \geq 0\}$.

0.20.6 NOTATION Given a class $C \subset \text{Mor } \underline{C}$, let $\underline{C}$ be the full subcategory of $\underline{C}(\to)$ having C as its objects.

0.20.7 LEMMA Suppose that $\underline{C}$ is presentable (hence that $\underline{C}(\to)$ is presentable) -- then for every set $I \subset \text{Mor } \underline{C}$, $\underline{\text{RLP}(I)}$ is an accessible subcategory of $\underline{C}(\to)$.

0.20.8 REMARK In general, $\underline{\text{cof I}} \subset \underline{C}(\to)$ is not accessible.

0.20.9 DEFINITION Let $\underline{C}$ be a cocomplete category -- then $\underline{C}$ is said to admit <u>the small object argument</u> if it has the following property: Given any set $I \subset \text{Mor } \underline{C}$, the pair

$$(\text{LLP}(\text{RLP}(I)), \text{RLP}(I))$$

is a functorial w.f.s. on $\underline{C}$.

[Note: We have

$$\text{RLP}(\text{LLP}(\text{RLP}(I))) = \text{RLP}(I).]$$

0.20.10 CRITERION Let $\underline{C}$ be a cocomplete category. Assume: $\forall\ X \in \text{Ob } \underline{C}$, there exists a regular cardinal κ_X such that X is κ_X-definite -- then $\underline{C}$ admits the small object argument.

<u>N.B.</u> In particular, every presentable category admits the small object argument.

0.20.11 REMARK <u>TOP</u> is not presentable, hence does not fall within the purview of 0.20.9. Nevertheless, <u>TOP</u> does admit the small object argument (Garner[†]).

0.20.12 REMARK If $\underline{C}$ is presentable, then in general, $\underline{C}^{\text{OP}}$ is not presentable, thus it is not automatic that $\underline{C}^{\text{OP}}$ admits the small object argument.

[Note: If $\underline{C}$ and $\underline{C}^{\text{OP}}$ are both presentable, then $\text{Mor}(X,Y)$ has at most one element for each pair $X,Y \in \text{Ob } \underline{C}$.]

0.20.13 DEFINITION Let (L,R) be a w.f.s. on a cocomplete category $\underline{C}$ -- then (L,R) is <u>cofibrantly generated</u> if there exists a set $I \subset L$ such that

$$R = \text{RLP}(I) \quad (\Rightarrow L = \text{LLP}(\text{RLP}(I))).$$

[Note: We shall refer to I as a <u>generating set</u> for (L,R).]

[†] arXiv:0712.0724

<u>N.B.</u> Accordingly, if $\underline{C}$ admits the small object argument, then a cofibrantly generated w.f.s. (L,R) on $\underline{C}$ is necessarily functorial.

0.20.14 DEFINITION Let $\underline{C}$ be a cocomplete model category -- then $\underline{C}$ is <u>cofibrantly generated</u> if the w.f.s.

$$\begin{cases} (\text{cof}, \mathcal{W} \cap \text{fib}) \\ (\mathcal{W} \cap \text{cof}, \text{fib}) \end{cases}$$

are cofibrantly generated with generating sets $\begin{cases} I \\ J \end{cases}$.

Here are a few examples.

0.20.15 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}$ (Quillen Structure) -- then $\underline{C}$ is cofibrantly generated.

[Let I be the set of inclusions $S^{n-1} \to D^n$ $(n \geq 0,\ D^0 = \{0\}$ and $S^{-1} = \emptyset)$ and let J be the set of inclusions $i_0: [0,1]^n \to [0,1]^n \times [0,1]$ $(n \geq 0)$.]

0.20.16 EXAMPLE Take $\underline{C} = \underline{\text{SISET}}$ (Kan Structure) -- then $\underline{C}$ is cofibrantly generated.

[Let I be the set of inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ and let J be the set of inclusions $\Lambda[k,n] \to \Delta[n]$ $(0 \leq k \leq n,\ n \geq 1)$.]

0.20.17 EXAMPLE Take $\underline{C} = \underline{\text{CAT}}$ (Internal Structure) -- then $\underline{C}$ is cofibrantly generated.

[In addition to the categories $\underline{0}$, $\underline{1}$, and $\underline{2}$, let $\underline{d2}$ be the discrete category with two objects, and let $\underline{p2}$ be the category with two objects and two parallel arrows -- then the canonical functors

$$\begin{cases} u:\underline{0} \longrightarrow \underline{1} \\ v:\underline{d2} \longrightarrow \underline{2} \\ w:\underline{p2} \longrightarrow \underline{2} \end{cases}$$

are cofibrations and we can take $I = \{u,v,w\}$. Turning to J, let $\underline{\underline{iso}}_2$ denote the category with objects a,b and arrows id_a, id_b, $a \xrightarrow{\alpha} b$, $b \xrightarrow{\beta} a$, where $\alpha \circ \beta = id_b$, $\beta \circ \alpha = id_a$ -- then we can take $J = \{\pi\}$, where $\pi:\underline{1} \to \underline{\underline{iso}}_2$ $(\pi(*) = a)$.]

0.20.18 EXAMPLE Take $\underline{C} = \underline{CAT}$ (External Structure) -- then $\underline{C}$ is cofibrantly generated.

[Let I be the set of arrows cat $Sd^2\dot{\Delta}[n] \to$ cat $Sd^2\Delta[n]$ $(n \geq 0)$ and let J be the set of arrows cat $Sd^2\Lambda[k,n] \to$ cat $Sd^2\Delta[n]$ $(0 \leq k \leq n,\ n \geq 1)$.]

0.20.19 EXAMPLE Take $\underline{C} = \underline{EQU}$ (Larusson Structure) -- then $\underline{C}$ is cofibrantly generated.

[One can take $I = \{f,g\}$, $J = \{h\}$, where $f:\emptyset \to \{*\}$, g is the identity map from $\{a,b\}$ (discrete partition) to $\{a,b\}$ (indiscrete partition), and $h:\{*\} \to \{a,b\}$ (indiscrete partition) sends $*$ to a.]

0.20.20 EXAMPLE Take $\underline{C} = \underline{CAT}$ and let L be the class whose elements are the full functors -- then the pair $(L, \mathrm{RLP}(L))$ is a w.f.s. which is not cofibrantly generated, thus there are model categories that are presentable but not cofibrantly generated (apply 0.18.4).

0.20.21 REMARK The Strøm structure on $\underline{\text{TOP}}$ is not cofibrantly generated (Raptis[†]).

0.20.22 LEMMA If S is a set and if

$$\mathcal{W}_s, \ \mathrm{cof}_s, \ \mathrm{fib}_s$$

is a cofibrantly generated model structure on $\underline{C}_s$ $(s \in S)$ with generating sets

sets $\begin{bmatrix} I_s \\ \\ J_s \end{bmatrix}$, then the model structure on $\underline{C} = \prod_s \underline{C}_s$ per 0.1.29 is cofibrantly

generated with generating sets

$$\begin{bmatrix} I = \bigcup\limits_{s \in S} (I_s \times \prod\limits_{t \neq s} \mathrm{id}_{\emptyset_t}) \\ \\ J = \bigcup\limits_{s \in S} (J_s \times \prod\limits_{t \neq s} \mathrm{id}_{\emptyset_t}), \end{bmatrix}$$

where $\mathrm{id}_{\emptyset_t}$ is the identity map of the initial object $\emptyset_t$ of $\underline{C}_t$.

0.21 *CISINSKI*[††] *THEORY*

Let $\underline{C}$ be a small category -- then the class $M \subset \mathrm{Mor}\ \hat{\underline{C}}$ of monomorphisms is retract stable and the pair $(M, \mathrm{RLP}(M))$ is a w.f.s. on $\hat{\underline{C}}$ (cf. 0.18.9).

[Note: For the record, recall that a morphism Ξ in $\hat{\underline{C}}$ is a monomorphism iff $\forall\, X \in \mathrm{Ob}\ \underline{C}$, Ξ_X is a monomorphism in $\underline{\text{SET}}$.]

N.B. Elements of $\mathrm{RLP}(M)$ are called trivial fibrations.

---

† *Homology, Homotopy Appl.* 12 (2010), 211-230.

†† *Astérisque* 308 (2006).

0.21.1 DEFINITION A cofibrantly generated model structure on $\hat{\underline{C}}$ is said to be a Cisinski structure if the cofibrations are the monomorphisms.

[Note: The acyclic fibrations of a Cisinski structure on $\hat{\underline{C}}$ are the trivial fibrations.]

0.21.2 EXAMPLE Take $\underline{C} = \underline{\Delta}$ -- then the Kan structure on SISET is a Cisinski structure (cf. 0.20.16).

0.21.3 LEMMA A Cisinski structure on $\hat{\underline{C}}$ is determined by its class of fibrant objects (cf. 0.1.11).

0.21.4 DEFINITION Consider a category pair $(\hat{\underline{C}}, W)$ -- then W is a $\hat{\underline{C}}$-localizer provided the following conditions are met.

(1) W satisfies the 2 out of 3 condition (cf. 2.3.13).

(2) W contains $\mathrm{RLP}(M)$.

(3) $W \cap M$ is a stable class.

N.B. If

$$W, \mathrm{cof} = M, \mathrm{fib} = \mathrm{RLP}(W \cap M)$$

is a model structure on $\hat{\underline{C}}$, then W is a $\hat{\underline{C}}$-localizer.

Let $C \subset \mathrm{Mor}\, \hat{\underline{C}}$ -- then the $\hat{\underline{C}}$-localizer generated by C, denoted $W(C)$, is the intersection of all the $\hat{\underline{C}}$-localizers containing C. The minimal $\hat{\underline{C}}$-localizer is $W(\emptyset)$ ($\emptyset$ the empty set of morphisms).

0.21.5 DEFINITION A $\hat{\underline{C}}$-localizer is admissible if it is generated by a set of morphisms of $\hat{\underline{C}}$.

0.21.6 EXAMPLE $\mathrm{Mor}\, \hat{\underline{C}}$ is an admissible $\hat{\underline{C}}$-localizer. In fact,

$$W(\{\emptyset_{\hat{\underline{C}}} \to *_{\hat{\underline{C}}}\}) = \text{Mor } \hat{\underline{C}}.$$

0.21.7 THEOREM Let $(\hat{\underline{C}},W)$ be a category pair -- then W is an admissible $\hat{\underline{C}}$-localizer iff there exists a cofibrantly generated model structure on $\hat{\underline{C}}$ whose class of weak equivalences are the elements of W and whose cofibrations are the monomorphisms.

[Note: The cofibrantly generated model structure on $\hat{\underline{C}}$ determined by W is left proper (but it need not be right proper).]

0.21.8 SCHOLIUM The map

$$W \to W,\ M,\ \text{RLP}(W \cap M)$$

induces a bijection between the class of admissible $\hat{\underline{C}}$-localizers and the class of Cisinski structures on $\hat{\underline{C}}$.

[Note: The partially ordered class of $\hat{\underline{C}}$-localizers has a maximal element and a minimal element. Furthermore, if I is a set and if W_i $(i \in I)$ is an admissible $\hat{\underline{C}}$-localizer, then the intersection $\underset{i\in I}{\cap} W_i$ is an admissible $\hat{\underline{C}}$-localizer.]

0.21.9 REMARK It follows a posteriori that the stable class $W \cap M$ is retract stable. In addition, W is necessarily saturated, i.e., $W = \bar{W}$ (cf. 2.3.20).

[Note: Every $\hat{\underline{C}}$-localizer is the filtered union over the class of the admissible $\hat{\underline{C}}$-localizers contained therein, thus, by a simple argument, is saturated.]

0.21.10 EXAMPLE Consider SISET (Joyal Structure) — then W is the class of categorical weak equivalences and is an admissible $\hat{\underline{\Delta}}$-localizer:

$$W = W(\{I[n] \to \Delta[n]: n \geq 0\}).$$

Therefore the Joyal structure is cofibrantly generated.

[Here $I[n]$ is the simplicial subset of $\Delta[n]$ generated by the edges $(k, k+1)$ $(0 \leq k \leq n-1)$ (take $I[0] = \Delta[1]$), so there is a pushout square

$$\begin{array}{ccc} \Delta[0] & \longrightarrow & \Delta[1] \\ \downarrow & & \downarrow \\ I[n] & \longrightarrow & I[n+1].] \end{array}$$

[Note: The Kan structure on $\underline{\text{SISET}}$ is cofibrantly generated and its $\hat{\underline{\Delta}}$-localizer is generated by the maps $\Delta[n] \to \Delta[0]$ $(n \geq 0)$.]

0.21.11 REMARK The HG-Structure on $\underline{\text{SISET}}$ is cofibrantly generated, thus its $\hat{\underline{\Delta}}$-localizer is admissible.

0.21.12 DEFINITION The Cisinski structure on $\hat{\underline{C}}$ corresponding to $W(\emptyset)$ is called the <u>minimal monic model structure</u> on $\hat{\underline{C}}$.

0.21.13 EXAMPLE Take $\underline{C} = \underline{1}$ -- then $\hat{\underline{1}} \approx \underline{\text{SET}}$ and $W(\emptyset)$ is the class

$$\{\emptyset \to \emptyset\} \cup \{f:X \to Y \ (X \neq \emptyset)\}.$$

0.21.14 LEMMA The minimal monic model structure on $\hat{\underline{C}}$ is proper.

0.21.15 EXAMPLE Take $\underline{C} = \underline{\Delta}$ -- then the minimal monic model structure on $\underline{\text{SISET}}$ has fewer weak equivalences than the Joyal structure (cf. 0.4.1).

0.21.16 NOTATION Given an admissible $\hat{\underline{C}}$-localizer W and a small category $\underline{I}$, denote by $W_{\underline{I}} \subset \text{Mor}[\underline{I},\hat{\underline{C}}]$ the class of morphisms $\Xi:F \to G$ such that $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is in W.

0.21.17 THEOREM The category $[\underline{I},\hat{\underline{C}}]$ carries a cofibrantly generated model structure whose weak equivalences are the elements of $W_{\underline{I}}$ and whose cofibrations are the monomorphisms.

[Identifying $[\underline{I},\hat{\underline{C}}]$ with the category of presheaves on $\underline{I}^{OP} \times \underline{C}$, observe that $W_{\underline{I}}$ is admissible and then invoke 0.21.7.]

[Note: If $\Xi:F \to G$ is a fibration in this model structure, then $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a fibration in the model structure on $\hat{\underline{C}}$ per W (but, in general, not conversely).]

0.21.18 EXAMPLE Take $\underline{C} = \underline{\Delta}$ and consider $\underline{\underline{SISET}}$ in its Kan structure (hence the admissible $\hat{\underline{\Delta}}$-localizer W is the class of simplicial weak equivalences) — then for any $\underline{I}$, the specialization of 0.21.17 to this situation gives rise to structure R on $[\underline{I},\underline{\underline{SISET}}]$ (cf. 0.16).

0.22 *MODEL FUNCTORS*

Let $\underline{C}$ and $\underline{C}'$ be model categories.

0.22.1 DEFINITION A left adjoint functor $F:\underline{C} \to \underline{C}'$ is a <u>left model functor</u> if F preserves cofibrations and acyclic cofibrations.

0.22.2 DEFINITION A right adjoint functor $F':\underline{C}' \to \underline{C}$ is a <u>right model functor</u> if F' preserves fibrations and acyclic fibrations.

0.22.3 LEMMA Suppose that

$$\left[\begin{array}{l} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{array}\right.$$

are an adjoint pair -- then F is a left model functor iff F' is a right model functor.

0.22.4 DEFINITION A <u>model pair</u> is an adjoint situation (F,F'), where F is a left model functor and F' is a right model functor.

0.22.5 EXAMPLE Consider the setup

$$\underline{\text{SISET}}\ (\text{Joyal Structure}) \underset{\text{ner}}{\overset{\text{cat}}{\rightleftarrows}} \underline{\text{CAT}}\ (\text{Internal Structure}).$$

Then (cat, ner) is a model pair.

[Note: The inclusion $\iota:\underline{\text{GRD}} \to \underline{\text{CAT}}$ admits a left adjoint $\pi_1:\underline{\text{CAT}} \to \underline{\text{GRD}}$ and a right adjoint $\text{iso}:\underline{\text{CAT}} \to \underline{\text{GRD}}$. This being so, consider the setup

$$\underline{\text{SISET}}\ (\text{Kan Structure}) \underset{\text{ner} \circ \iota \circ \text{iso}}{\overset{\iota \circ \pi_1 \circ \text{cat}}{\rightleftarrows}} \underline{\text{CAT}}\ (\text{Internal Structure}).$$

Then $(\iota \circ \pi_1 \circ \text{cat}, \text{ner} \circ \iota \circ \text{iso})$ is a model pair.]

0.22.6 EXAMPLE Consider the setup

$$\underline{\text{TOP}}(\text{Quillen Structure}) \underset{\text{id}_{\underline{\text{TOP}}}}{\overset{\text{id}_{\underline{\text{TOP}}}}{\rightleftarrows}} \underline{\text{TOP}}(\text{Strøm Structure}).$$

Then $(\text{id}_{\underline{\text{TOP}}}, \text{id}_{\underline{\text{TOP}}})$ is a model pair (take $F' = \text{id}_{\underline{\text{TOP}}}$).

0.22.7 LEMMA The adjoint situation (F,F') is a model pair iff F preserves cofibrations and F' preserves fibrations.

0.22.8 LEMMA The adjoint situation (F,F') is a model pair iff F preserves acyclic cofibrations and F' preserves acyclic fibrations.

Recall now that $\underline{C}_{cof}$ is a cofibration category and $\underline{C}'_{fib}$ is a fibration category, the setup of 2.2.6 thus becoming

$$\begin{tikzcd} \underline{C}_{cof} \arrow[r, "\iota"] & \underline{C} \arrow[r, shift left, "F"] & \underline{C}' \arrow[l, shift left, "F'"] & \underline{C}'_{fib}. \arrow[l, "\iota'"'] \end{tikzcd}$$

0.22.9 SCHOLIUM

- To ensure the existence of (LF,ν_F), it suffices to require that F send acyclic cofibrations between cofibrant objects to weak equivalences.
- To ensure the existence of $(RF',\mu_{F'})$, it suffices to require that F' send acyclic fibrations between fibrant objects to weak equivalences.

So, if the adjoint situation (F,F') is a model pair, then the functors

$$\begin{cases} LF:\underline{HC} \to \underline{HC'} \\ RF':\underline{HC'} \to \underline{HC} \end{cases}$$

exist and are an adjoint pair.

0.22.10 EXAMPLE Fix a model category $\underline{C}$, let $\underline{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet\, 2$, and equip $[\underline{I},\underline{C}]$ with its model category structure per 0.1.12. Let $\text{colim}:[\underline{I},\underline{C}] \to \underline{C}$ be the functor that on objects assigns to each 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$ it pushout P:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & P. \end{tikzcd}$$

Then colim has a right adjoint, viz. the constant diagram functor $K:\underline{C} \to [\underline{I},\underline{C}]$. But it is obvious that K preserves fibrations and acyclic fibrations. Therefore the adjoint situation (colim, K) is a model pair, thus $\left[\begin{array}{l} L\text{colim} \\ RK \end{array}\right.$ exist and $(L\text{colim}, RK)$ is an adjoint pair.

[Note: The story for pullbacks is analogous.]

Given a model category $\underline{C}$ and objects A,B in $\underline{C}$, the categories $A\backslash\underline{C}$, $\underline{C}/B$ are again model categories, a morphism in either case being declared a weak equivalence, cofibration, or fibration if it is such when viewed in $\underline{C}$ alone.

0.22.11 EXAMPLE Let $\underline{C}$ be a model category and let $X,Y \in \text{Ob } \underline{C}$ -- then each $f:X \to Y$ induces a functor

$$f_!:X\backslash\underline{C} \to Y\backslash\underline{C}$$

which sends an object $X \to Z$ of $X\backslash\underline{C}$ to its pushout along f:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d] \\ Y \arrow[r] & P. \end{tikzcd}$$

Moreover, $f_!$ is a left adjoint for the functor

$$f^*:Y\backslash\underline{C} \to X\backslash\underline{C}$$

which sends an object $Y \to W$ of $Y\backslash\underline{C}$ to its precomposition with f and it is immediate that f* preserves fibrations and acyclic fibrations:

$$f^*\left[\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "\zeta"'] & Y \arrow[d, "\zeta'"] \\ W \arrow[r, "g"'] & W' \end{tikzcd}\right] = \begin{tikzcd} X \arrow[r, equal] \arrow[d, "\zeta \circ f"'] & X \arrow[d, "\zeta' \circ f"] \\ W \arrow[r, "g"'] & W \end{tikzcd} .$$

Therefore the adjoint situation $(f_!, f^*)$ is a model pair, thus $\left[\begin{array}{l} Lf_! \\ \\ Rf^* \end{array}\right.$ exist and $(Lf_!, Rf^*)$ is an adjoint pair.

[Note: The story for $\underline{C}/X$, $\underline{C}/Y$ is analogous.]

0.22.12 EXAMPLE Define a functor $\text{Ч}:\underline{\Delta} \to \underline{\text{SISET}}$ by the rule $\text{Ч}[n] = \text{ner } \pi_1[n]$ -- then

$$\left[\begin{array}{l} \Gamma_{\text{Ч}}:\underline{\text{SISET}} \longrightarrow \underline{\text{SISET}} \\ \\ \sin_{\text{Ч}}:\underline{\text{SISET}} \longrightarrow \underline{\text{SISET}} \end{array}\right.$$

is an adjoint pair. But

$$\Gamma_{\text{Ч}}:\underline{\text{SISET}} \text{ (Kan Structure)} \longrightarrow \underline{\text{SISET}} \text{ (Joyal Structure)}$$

is a left model functor. Therefore the adjoint situation $(\Gamma_{\text{Ч}}, \sin_{\text{Ч}})$ is a model pair, thus $\left[\begin{array}{l} L\Gamma_{\text{Ч}} \\ \\ R\sin_{\text{Ч}} \end{array}\right.$ exist and $(L\Gamma_{\text{Ч}}, R\sin_{\text{Ч}})$ is an adjoint pair.

0.22.13 EXAMPLE In the notation of 0.7,

$$\left[\begin{array}{l} F_{gr}:\underline{\text{SISET}} \longrightarrow \underline{\text{SIGR}} \\ \\ U:\underline{\text{SIGR}} \longrightarrow \underline{\text{SISET}} \end{array}\right.$$

is an adjoint pair. Since F_{gr} preserves cofibrations and U preserves fibrations, it follows that $\left[\begin{array}{l} LF_{gr} \\ RU \end{array}\right.$ exist and (LF_{gr}, RU) is an adjoint pair.

A model pair (F,F') is a <u>model equivalence</u> if the adjoint pair (LF,RF') is an adjoint equivalence of homotopy categories.

0.22.14 LEMMA The adjoint pair

$$\left[\begin{array}{l} LF:\underline{HC} \to \underline{HC'} \\ \\ RF':\underline{HC'} \to \underline{HC} \end{array}\right.$$

per

$$\underline{C}_{cof} \xrightarrow{\iota} \underline{C} \underset{F'}{\overset{F}{\rightleftarrows}} \underline{C}' \xleftarrow{\iota'} \underline{C}'_{fib}$$

is an adjoint equivalence of homotopy categories if

$$\forall \left[\begin{array}{l} X \in \text{Ob } \underline{C}_{cof} \\ \\ X' \in \text{Ob } \underline{C}'_{fib}, \end{array}\right.$$

an arrow

$$\phi \in \text{Mor}(FX,X')$$

is a weak equivalence iff its adjoint

$$\psi \in \text{Mor}(X,F'X')$$

is a weak equivalence.

[This is a special case of 1.7.3.]

N.B. Since

$$\begin{cases} LF \\ RF' \end{cases}$$

are an adjoint pair, the left derived functor LF is an equivalence iff the right derived functor RF' is an equivalence.

0.22.15 EXAMPLE Take $\underline{EQU}$ as in 0.15 and equip $\underline{SET}$ with its model structure per 0.1.5, hence the weak equivalences are the bijections and

$$\begin{cases} \text{cof} = \text{Mor }\underline{SET} \\ \text{fib} = \text{Mor }\underline{SET}. \end{cases}$$

Let $Q:\underline{EQU} \to \underline{SET}$ be the functor that on objects sends $(X,\sim_X)$ to $X/\sim_X$ — then Q has a right adjoint $Q':\underline{SET} \to \underline{EQU}$ that on objects endows a set with its discrete partition. It is clear that Q preserves cofibrations and Q' preserves fibrations. Therefore the adjoint situation (Q,Q') is a model pair, thus $\begin{cases} LQ \\ RQ' \end{cases}$ exist and (LQ,RQ') is an adjoint pair. Since the arrow of adjunction

$$\mu_{(X,\sim_X)}:(X,\sim_X) \to Q'Q(X,\sim_X)$$

is the projection $X \to X/\sim_X$, an arrow

$$\phi \in \text{Mor}(Q\,(X,\sim_X),X')$$

is a bijection iff its adjoint

$$\psi \in \text{Mor}((X,\sim_X),Q'X')$$

is a bijection on quotients, so the adjoint pair (LQ,RQ') is an adjoint equivalence of homotopy categories:

$$\underline{\mathrm{HEQU}} \rightleftarrows \underline{\mathrm{HSET}},$$

where $\underline{\mathrm{HSET}}$ is isomorphic to $\underline{\mathrm{SET}}$ itself (cf. 1.1.8).

0.22.16 EXAMPLE In the theory above, take $\underline{C} = \underline{\mathrm{SISET}}$ (Kan Structure), $\underline{C}' = \underline{\mathrm{TOP}}$ (Quillen Structure) and let $F = |\ |$, $F' = \sin$ — then from the definitions, $|\ |$ preserves cofibrations and sin preserves fibrations, thus the adjoint situation $(|\ |, \sin)$ is a model pair which, in fact, is a model equivalence. Therefore the adjoint pair $(L|\ |, R\sin)$ is an adjoint equivalence of homotopy categories:

$$\underline{\mathrm{HSISET}} \rightleftarrows \underline{\mathrm{HTOP}}.$$

[We shall sketch the classical argument. Consider the bijection of adjunction

$$\Xi_{X,Y}:C(|X|,Y) \to \mathrm{Nat}(X,\sin Y),$$

so $\Xi_{X,Y}f$ is the composition $X \to \sin|X| \xrightarrow{\sin f} \sin Y$ — then the arrow $X \to \sin|X|$ is a simplicial weak equivalence. Proof: The diagram

$$\begin{tikzcd} {|X|} \arrow[r] \arrow[d, "\mathrm{id}_{|X|}"'] & {|\sin|X||} \arrow[d] \\ {|X|} \arrow[r, equal] & {|X|} \end{tikzcd}$$

commutes and the vertical arrow on the right is a weak homotopy equivalence. Consequently, $\Xi_{X,Y}f$ is a simplicial weak equivalence iff $\sin f$ is a simplicial weak equivalence. But there is a commutative diagram

$$\begin{tikzcd} {|\sin|X||} \arrow[r, "|\sin f|"] \arrow[d] & {|\sin Y|} \arrow[d] \\ {|X|} \arrow[r, "f"'] & Y \end{tikzcd} .$$

And the vertical arrows are weak homotopy equivalences, hence sin f is a simplicial weak equivalence iff f is a weak homotopy equivalence. Finally, then, $\Xi_{X,Y}f$ is a simplicial weak equivalence iff f is a weak homotopy equivalence and 0.22.14 is applicable.]

[Note: All objects in SISET are cofibrant and all objects in TOP are fibrant.]

0.22.17 REMARK Let HCW be the homotopy category of CW complexes -- then HCW is equivalent to HTOP (TOP in its Quillen structure).

[Note: There are two points to be kept in mind.

(1) If K and L are CW complexes and if $f:K \to L$ is a weak homotopy equivalence, then f is a homotopy equivalence.

(2) If X is a topological space, then there exists a CW complex K and a weak homotopy equivalence $f:K \to X$.]

0.23 *PROPRIETY*

Let $\underline{C}$ be a model category.

0.23.1 DEFINITION A weak equivalence $X \xrightarrow{f} Y$ is proper to the left if for every cofibration $X \to Z$ the arrow $Z \to Z \coprod_X Y$ is a weak equivalence.

N.B. $\underline{C}$ is left proper iff all its weak equivalences are proper to the left.

0.23.2 LEMMA A weak equivalence $X \xrightarrow{f} Y$ is proper to the left iff the model pair $(f_!,f^*)$ of 0.22.11 is a model equivalence or, equivalently, iff the functor $Rf^*:\underline{H}(Y\backslash\underline{C}) \to \underline{H}(X\backslash\underline{C})$ is an equivalence.

0.23.3 THEOREM Let $\underline{C}$ be a model category -- then $\underline{C}$ is left proper iff for every weak equivalence $X \xrightarrow{f} Y$ the functor $Rf^*:\underline{H}(Y\backslash\underline{C}) \to \underline{H}(X\backslash\underline{C})$ is an equivalence.

0.23.4 REMARK The upshot is that "left proper" can be formulated without the use of cofibrations. So if $\mathcal{W}$, cof, fib is a model structure on $\underline{C}$ which is left proper, then so is any other model structure $\mathcal{W}$, cof', fib'.

[Note: The story for "right proper" is analogous.]

0.24 *TRANSFER OF STRUCTURE*

Let $\underline{C}$ be a cofibrantly generated model category with generating sets $\begin{cases} I \\ J \end{cases}$, thus

$$\begin{cases} \mathcal{W} \cap \mathrm{fib} = \mathrm{RLP}(I) \\ \mathrm{fib} = \mathrm{RLP}(J). \end{cases}$$

Let $\underline{C}'$ be a finitely complete and finitely cocomplete category. Suppose that

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

are an adjoint pair.

- Assume:

$$(\mathrm{LLP}(\mathrm{RLP}(FI)), \mathrm{RLP}(FI))$$

is a w.f.s. on $\underline{C}'$.

- Assume:

$$(\mathrm{LLP}(\mathrm{RLP}(FJ)), \mathrm{RLP}(FJ))$$

is a w.f.s. on $\underline{C}'$.

Suppose further that

$$F'(\mathrm{LLP}(\mathrm{RLP}(FJ))) \subset \mathcal{W}.$$

Put

$$\begin{cases} \mathcal{W}' = \{f' \in \mathrm{Mor}\,\underline{C}' : F'f' \in \mathcal{W}\} \\ \mathrm{fib}' = \{f' \in \mathrm{Mor}\,\underline{C}' : F'f' \in \mathrm{fib}\} \end{cases}$$

and set

$$\mathrm{cof}' = \mathrm{LLP}(\mathcal{W}' \cap \mathrm{fib}').$$

0.24.1 THEOREM The data

$$\mathcal{W}', \mathrm{cof}', \mathrm{fib}'$$

defines a cofibrantly generated model structure on $\underline{C}'$ with generating sets $\begin{cases} FI \\ FJ \end{cases}$.

PROOF One has only to note that from the assumptions

$$\begin{cases} \mathcal{W} \cap \mathrm{fib}' = \mathrm{RLP}(FI) \\ \mathrm{fib}' = \mathrm{RLP}(FJ) \end{cases}$$

and

$$\begin{cases} \mathrm{cof}' = \mathrm{LLP}(\mathrm{RLP}(FI)) \\ \mathcal{W}' \cap \mathrm{cof}' = \mathrm{LLP}(\mathrm{RLP}(FJ)). \end{cases}$$

[Note: The detail that is not quite immediate is the relation

$$\mathcal{W}' \cap \text{cof}' = \text{LLP}(\text{RLP}(FJ)).$$

However, by hypothesis,

$$F'(\text{LLP}(\text{RLP}(FJ))) \subset \mathcal{W},$$

so

$$\text{LLP}(\text{RLP}(FJ)) \subset \mathcal{W}' \cap \text{cof}'.$$

Conversely, given $f':X' \to Y'$ in $\mathcal{W}' \cap \text{cof}'$, write $f' = \rho \circ \lambda$, where $\lambda:X' \to Z'$ is in $\text{LLP}(\text{RLP}(FJ))$ and $\rho:Z' \to Y'$ is in $\text{RLP}(FJ)$ -- then

$$f',\lambda \in \mathcal{W}' \Rightarrow \rho \in \mathcal{W}'$$

$$\Rightarrow \rho \in \mathcal{W}' \cap \text{RLP}(FJ) = \mathcal{W}' \cap \text{fib}'.$$

But since $f' \in \text{cof}'$, the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\lambda"] \arrow[d, "f'"'] & Z' \arrow[d, "\rho"] \\ Y' \arrow[r, equal] & Y' \end{tikzcd}$$

admits a filler $r:Y' \to Z'$, thus the commutative diagram

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d, "f'"'] & X' \arrow[r, equal] \arrow[d, "\lambda"] & X' \arrow[d, "f'"] \\ Y' \arrow[r, "r"'] & Z' \arrow[r, "\rho"'] & Y' \end{tikzcd}$$

exhibits f' as a retract of λ, implying thereby that $f' \in \text{LLP}(\text{RLP}(FJ))$.]

N.B. The adjoint situation (F,F') is a model pair (for by construction, F' is a right model functor), thus $\left[\begin{array}{l} LF \\ RF' \end{array}\right.$ exist and (LF,RF') is an adjoint pair.

0.24.2 EXAMPLE Take

$$\left[\begin{array}{l} \underline{C} = \underline{SISET} \\ \\ \underline{C}' = \underline{CAT} \end{array}\right. \quad \text{and} \quad \left[\begin{array}{l} F = \text{cat} \circ \text{Sd}^2 \\ \\ F' = \text{Ex}^2 \circ \text{ner}. \end{array}\right.$$

Then $\underline{C}$, $\underline{C}'$ are presentable and (F,F') is an adjoint pair. Moreover, all the assumptions of 0.24.1 are satisfied and the resulting cofibrantly generated model structure on CAT is its external structure.

- $\forall\ X \in \text{Ob}\ \underline{SISET}$, the arrow of adjunction

$$X \to \text{Ex}^2 \circ \text{ner} \circ \text{cat} \circ \text{Sd}^2 X$$

is a simplicial weak equivalence.

- $\forall\ \Phi \in \text{Mor}\ \underline{CAT}$, ner Φ is a simplicial weak equivalence iff $\text{Ex}^2 \circ \text{ner}\ \Phi$ is a simplicial weak equivalence.

Consider now the bijection of adjunction

$$\Xi_{X,\underline{C}}:\text{Mor}(\text{cat} \circ \text{Sd}^2 X,\underline{C}) \to \text{Mor}(X,\text{Ex}^2 \circ \text{ner}\ \underline{C}),$$

so $\Xi_{X,\underline{C}}\Phi$ is the composition

$$X \to \text{Ex}^2 \circ \text{ner} \circ \text{cat} \circ \text{Sd}^2 X \xrightarrow{\text{Ex}^2 \circ \text{ner}\ \Phi} \text{Ex}^2 \circ \text{ner}\ \underline{C}.$$

Then $\Xi_{X,\underline{C}}\Phi$ is a simplicial weak equivalence iff Φ is a simplicial weak equivalence. So, in view of 0.22.14, the model pair (F,F') is a model equivalence, i.e., the adjoint pair (LF,RF') is an adjoint equivalence of homotopy categories:

$$\underline{\mathrm{HSISET}} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \underline{\mathrm{HCAT}}.$$

[Note: The main reason for working with (cat $\circ$ Sd^2, Ex^2 $\circ$ ner) rather than (cat,ner) (or (cat $\circ$ Sd, Ex $\circ$ ner)) is that the arrow of adjunction $X \to \mathrm{ner}(\mathrm{cat}\, X)$ (or $X \to \mathrm{Ex} \circ \mathrm{ner} \circ \mathrm{cat} \circ \mathrm{Sd}\, X$) need not be a simplicial weak equivalence.]

0.24.3 REMARK Recall first that there are natural simplicial weak equivalences

$$\left[\begin{array}{l} \mathrm{ner}(\mathrm{gro}_{\underline{\Delta}}\, X) \to X \\ \\ \mathrm{gro}_{\underline{\Delta}}(\mathrm{ner}\, \underline{C}) \to \underline{C}. \end{array}\right.$$

- In $\underline{\mathrm{CAT}}$, let $\mathcal{W}_\infty$ denote the class of simplicial weak equivalences, i.e., the class of functors $F:\underline{C} \to \underline{D}$ such that $|\mathrm{ner}\, F|:B\underline{C} \to B\underline{D}$ is a homotopy equivalence.

N.B. $\mathcal{W}_\infty$ is the class of weak equivalences per $\underline{\mathrm{CAT}}$ (External Structure) and

$$\mathcal{W}_\infty^{-1}\underline{\mathrm{CAT}} = \underline{\mathrm{HCAT}}.$$

- In $\underline{\mathrm{SISET}}$, let W_∞ denote the class of simplicial weak equivalences, i.e., the class of simplicial maps $f:X \to Y$ such that $|f|:|X| \to |Y|$ is a homotopy equivalence.

N.B. W_∞ is the class of weak equivalences per $\underline{\mathrm{SISET}}$ (Kan Structure) and

$$W_\infty^{-1}\underline{\mathrm{SISET}} = \underline{\mathrm{HSISET}}.$$

Since $\mathrm{ner}\, \mathcal{W}_\infty \subset W_\infty$, there is a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{CAT}} \arrow[r, "\mathrm{ner}"] \arrow[d] & \underline{\mathrm{SISET}} \arrow[d] \\ \underline{\mathrm{HCAT}} \arrow[r, "\overline{\mathrm{ner}}"'] & \underline{\mathrm{HSISET}} \end{tikzcd}$$

and since $\mathrm{gro}_{\underline{\Delta}}\, W_\infty \subset \mathcal{W}_\infty$, there is a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{SISET}} \arrow[r, "\mathrm{gro}_{\underline{\Delta}}"] \arrow[d] & \underline{\mathrm{CAT}} \arrow[d] \\ \underline{\mathrm{HSISET}} \arrow[r, "\overline{\mathrm{gro}_{\underline{\Delta}}}"'] & \underline{\mathrm{HCAT}}. \end{tikzcd}$$

Taking into account the natural isomorphisms

$$\begin{cases} \overline{\mathrm{ner}} \circ \overline{\mathrm{gro}_{\underline{\Delta}}} \to \mathrm{id} \\ \overline{\mathrm{gro}_{\underline{\Delta}}} \circ \overline{\mathrm{ner}} \to \mathrm{id}, \end{cases}$$

it follows that ner induces an equivalence

$$\underline{\mathrm{HCAT}} \to \underline{\mathrm{HSISET}}$$

of homotopy categories.

N.B. Take TOP in its Quillen structure, SISET in its Kan structure, and CAT in its external structure -- then HCW is equivalent to HTOP (cf. 0.22.17), HTOP is equivalent to HSISET (cf. 0.22.16), and HSISET is equivalent to HCAT (by the above).

[Note: Let [CAT] be the category with Ob[CAT] = Ob CAT and whose morphisms are isomorphism classes of functors (i.e., in [CAT], Mor($\underline{I}$,$\underline{J}$) is the set of isomorphism classes of objects in $[\underline{I},\underline{J}]$) -- then the canonical projection

$$\underline{\mathrm{CAT}} \to [\underline{\mathrm{CAT}}]$$

is a localization of $\underline{\mathrm{CAT}}$ at the class W whose elements are the equivalences of small categories, thus when $\underline{\mathrm{CAT}}$ is equipped with its internal structure,

$$\underline{\mathrm{HCAT}} = [\underline{\mathrm{CAT}}].$$

Given a small category $\underline{I}$, write $\underline{I}_{\mathrm{dis}}$ for the discrete category underlying $\underline{I}$ -- then for any cocomplete category $\underline{C}$, the forgetful functor $U:[\underline{I},\underline{C}] \to [\underline{I}_{\mathrm{dis}},\underline{C}]$ has a left adjoint that sends X to fr X, where

$$\mathrm{fr}\, Xj = \coprod_{i \in \mathrm{Ob}\, \underline{I}} \mathrm{Mor}(i,j)\cdot Xi.$$

0.24.4 EXAMPLE Take $\underline{C} = \underline{\mathrm{SISET}}$ (Kan Structure) and consider the adjoint pair

$$\begin{cases} \mathrm{fr}:[\underline{I}_{\mathrm{dis}},\underline{\mathrm{SISET}}] \to [\underline{I},\underline{\mathrm{SISET}}] \\ \\ U:[\underline{I},\underline{\mathrm{SISET}}] \to [\underline{I}_{\mathrm{dis}},\underline{\mathrm{SISET}}]. \end{cases}$$

Then $[\underline{I}_{\mathrm{dis}},\underline{\mathrm{SISET}}]$ is a cofibrantly generated model category (cf. 0.20.22) and all the assumptions leading to 0.24.1 are satisfied ($F = \mathrm{fr}$, $F' = U$). The resulting cofibrantly generated model structure on $[\underline{I},\underline{\mathrm{SISET}}]$ is structure L (cf. 0.16).

0.24.5 LEMMA Let $\underline{G},\underline{H} \in \mathrm{Ob}\, \underline{\mathrm{GRD}}$, $f:\underline{G} \to \underline{H}$ a morphism -- then f is a simplicial weak equivalence iff f is an equivalence.

0.24.6 LEMMA Let $\underline{G},\underline{H} \in \mathrm{Ob}\, \underline{\mathrm{GRD}}$, $f:\underline{G} \to \underline{H}$ a morphism -- then $\mathrm{Ex}^2 \circ \mathrm{ner}\, f$ is a

Kan fibration iff ner f is a Kan fibration iff f has the RLP w.r.t. $\pi:\underline{1} \to \underline{\text{iso}}_2$ (cf. 0.20.16).

0.24.6 SCHOLIUM The external and internal model structures on $\underline{\text{CAT}}$ restrict to the same model structure on $\underline{\text{GRD}}$.

0.25 *COMBINATORIAL MODEL CATEGORIES*

Let $\underline{C}$ be a cofibrantly generated model category.

0.25.1 DEFINITION $\underline{C}$ is combinatorial if in addition $\underline{C}$ is presentable (hence complete and cocomplete).

Suppose that $\underline{C}$ is combinatorial -- then there exist sets

$$\left[\begin{array}{l} I \subset \text{cof} \\ J \subset \mathcal{W} \cap \text{cof} \end{array}\right.$$

such that

$$\left[\begin{array}{l} \mathcal{W} \cap \text{fib} = \text{RLP}(I) \\ \qquad \text{fib} = \text{RLP}(J). \end{array}\right.$$

0.25.2 REMARK The cofibrantly generated w.f.s.

$$\left[\begin{array}{l} (\text{cof}, \mathcal{W} \cap \text{fib}) \\ (\mathcal{W} \cap \text{cof}, \text{fib}) \end{array}\right.$$

are functorial ($\underline{C}$ being presentable) and the functors

$$\left[\begin{array}{l} L:\underline{C}(\to) \to \underline{C}(\to) \\ R:\underline{C}(\to) \to \underline{C}(\to) \end{array}\right.$$

can be taken accessible.

N.B. Recall that

$$\underline{C} \text{ presentable} \Rightarrow \underline{C}(\rightarrow) \text{ presentable.}$$

0.25.3 LEMMA Suppose that $\underline{C}$ is combinatorial -- then

$$\begin{cases} \underline{W} \cap \underline{\mathrm{fib}} \\ \underline{\mathrm{fib}} \end{cases}$$

are accessible subcategories of $\underline{C}(\rightarrow)$.

[This is an application of 0.20.7.]

0.25.4 LEMMA Suppose that $\underline{C}$ is combinatorial -- then $\underline{W}$ is an accessible subcategory of $\underline{C}(\rightarrow)$.

PROOF Work with

$$\begin{cases} L:\underline{C}(\rightarrow) \rightarrow \underline{C}(\rightarrow) \\ R:\underline{C}(\rightarrow) \rightarrow \underline{C}(\rightarrow) \end{cases}$$

per $(W \cap \mathrm{cof}, \mathrm{fib})$ and note that

$$\underline{W} = R^{-1}(\underline{W} \cap \underline{\mathrm{fib}}).$$

We turn now to the "recognition principle" for combinatorial model categories.

Thus fix a presentable category $\underline{C}$, a class $W \subset \mathrm{Mor}\ \underline{C}$, and a set $I \subset \mathrm{Mor}\ \underline{C}$. Make the following assumptions.

(1) W satisfies the 2 out of 3 condition (cf. 2.3.13).

(2) $\underline{W} \subset \underline{C}(\rightarrow)$ is an accessible subcategory of $\underline{C}(\rightarrow)$.

(3) The class RLP(I) is contained in W.

(4) The intersection $W \cap \mathrm{cof}\ I$ is a stable class.

<u>N.B.</u> The closure of $\mathcal{W}$ under the formation of retracts is automatic (cf. (2)).

0.25.5 THEOREM Under the preceding hypotheses, $\underline{C}$ is a combinatorial model category with weak equivalences $\mathcal{W}$, cofibrations cof I, fibrations RLP($\mathcal{W} \cap$ cof I).

The key is to construct a set $J \subset \mathcal{W} \cap \text{cof } I$ such that $\text{cof } J = \mathcal{W} \cap \text{cof } I$. Granting this for the moment, it is not difficult to check that $\underline{C}$ is in fact a model category, the remaining claim being that

$$\left[\begin{array}{l} \mathcal{W} \cap \text{fib} = \text{RLP}(I) \\ \quad\ \ \text{fib} = \text{RLP}(J). \end{array}\right.$$

But

$$\begin{aligned} \mathcal{W} \cap \text{fib} &= \text{RLP}(\text{cof}) \\ &= \text{RLP}(\text{LLP}(\text{RLP}(I))) \quad \text{(cf. 20.4)} \\ &= \text{RLP}(I) \end{aligned}$$

and

$$\begin{aligned} \text{fib} &= \text{RLP}(\mathcal{W} \cap \text{cof } I) \\ &= \text{RLP}(\text{cof } J) \\ &= \text{RLP}(\text{LLP}(\text{RLP}(J))) \quad \text{(cf. 20.4)} \\ &= \text{RLP}(J). \end{aligned}$$

There are two steps in the construction of J.

0.25.6 LEMMA Suppose that $J \subset \mathcal{W} \cap \text{cof } I$ is a set with the following property: Every commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & B, \end{tikzcd}$$

where

$$\begin{cases} (X \to Y) \in I \\ (A \to B) \in \mathcal{W}, \end{cases}$$

can be factored as a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & Z \arrow[r] & B, \end{tikzcd}$$

where

$$(W \to Z) \in J.$$

Then

$$\text{cof } J = \mathcal{W} \cap \text{cof } I.$$

[It suffices to show that every $f \in \mathcal{W}$ admits a factorization as $h \circ g$, where $g \in \text{cell } J$ and $h \in \text{RLP}(I)$. To this end, fix a regular cardinal κ such that the domains of the elements of I are κ-definite and proceed by transfinite induction.]

Since $\underline{\mathcal{W}}$ is an accessible subcategory of $\underline{C}(\to)$, the inclusion functor $\underline{\mathcal{W}} \to \underline{C}(\to)$ satisfies the solution set condition: Given any object $X \to Y$ in Mor $\underline{C}$, there exists a source

$$\begin{tikzcd} X \arrow[r, "u_i"] \arrow[d] & X_i \arrow[d] \\ Y \arrow[r, "v_i"'] & Y_i \end{tikzcd} \qquad ((X_i \to Y_i) \in \mathcal{W})$$

such that for every commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & B \end{tikzcd} \qquad ((A \to B) \in \mathcal{W}),$$

there is an i, an arrow

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & A \arrow[d] \\ Y_i \arrow[r] & B \end{tikzcd}$$

in $\underline{C}(\rightarrow)$, and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u_i"] \arrow[d] & X_i \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "v_i"'] & Y_i \arrow[r] & B. \end{tikzcd}$$

0.25.7 LEMMA There exists a set $J \subset \mathcal{W} \cap \text{cof } I$ which has the property set forth in 0.25.6.

PROOF Start with a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & B, \end{tikzcd}$$

where

$$\left[\begin{array}{l} (X \rightarrow Y) \in I \\ (A \rightarrow B) \in \mathcal{W}, \end{array}\right.$$

and factor it as above

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & Y_i \arrow[r] & B. \end{tikzcd}$$

So, to draw the desired conclusion, it suffices to factor the square on the left by an element of $\mathcal{W} \cap \text{cof } I$. For this purpose, form the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] \\ Y \arrow[r] & Y \sqcup_X X_i \end{tikzcd}$$

and note that the arrow $X_i \to Y \sqcup_X X_i$ is in cof I. Next, factor the arrow $Y \sqcup_X X_i \to Y_i$ as an element $Y \sqcup_X X_i \to Z_i$ of cof I followed by an element $Z_i \to Y_i$ of RLP(I) (permissible since $\underline{C}$ admits the small object argument) -- then the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r, equal] \arrow[d] & X_i \arrow[d] \\ Y \arrow[r] & Z_i \arrow[r] & Y_i \end{tikzcd}$$

factors the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] \\ Y \arrow[r] & Y_i \end{tikzcd}$$

by an arrow $X_i \to Z_i$ in $\mathcal{W} \cap \text{cof } I$.

[Note: To check the last point, introduce some labels:

$$X_i \xrightarrow{w_i} Y_i$$

and

$$X_i \xrightarrow{f_i} Y \sqcup_X X_i \xrightarrow{\phi_i} Z_i \xrightarrow{\psi_i} Y_i.$$

Then

$$w_i = \psi_i \circ \phi_i \circ f_i.$$

But

$$\psi_i \in \mathrm{RLP}(I) \subset \mathcal{W} \Rightarrow \phi_i \circ f_i \in \mathcal{W}.$$

On the other hand,

$$f_i \in \mathrm{cof}\ I, \phi_i \in \mathrm{cof}\ I \Rightarrow \phi_i \circ f_i \in \mathrm{cof}\ I.]$$

0.25.8 EXAMPLE Take $\underline{C} = \underline{\mathrm{SISET}}$, let $\mathcal{W}$ be the class of categorical weak equivalences, and let I be the set of inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ -- then this data satisfies the assumptions of 0.25.5, which thus provides a route to the construction of the Joyal structure on $\underline{\mathrm{SISET}}$.

[Note: I am unaware of a specific description of "J".]

0.25.9 REMARK Working within the framework of 0.21, let $\underline{C}$ be a small category and let $W \subset \mathrm{Mor}\ \hat{\underline{C}}$ be an admissible $\hat{\underline{C}}$-localizer -- then

$$W, \mathcal{M}, \mathrm{RLP}(W \cap \mathcal{M})$$

is a cofibrantly generated model structure on $\hat{\underline{C}}$, thus is combinatorial ($\hat{\underline{C}}$ being presentable). Therefore $\underline{W}$ is an accessible subcategory of $\hat{\underline{C}}(\to)$ (cf. 0.25.4). To reverse matters, fix a set $M \subset \mathcal{M}: \mathcal{M} = \mathrm{cof}\ M$ (cf. 0.20.5) and suppose that $\underline{W} \subset \mathrm{Mor}\ \hat{\underline{C}}$ is a class satisfying assumptions (1) through (4) above (with I replaced by M) -- then

$$\begin{aligned} \mathrm{RLP}(\mathcal{M}) &= \mathrm{RLP}(\mathrm{cof}\ M) \\ &= \mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(M))) \qquad (\text{cf. } 0.20.4) \\ &= \mathrm{RLP}(M) \subset W, \end{aligned}$$

so W is a $\hat{\underline{C}}$-localizer. But the cofibrantly generated model structure on $\hat{\underline{C}}$ produced by 0.25.5 has W for its weak equivalences and M for its cofibrations. Accordingly, on the basis of 0.21.7, W is necessarily admissible.

0.25.10 THEOREM Keep I fixed and let $\mathcal{W}_k$ $(k \in K)$ be a set of classes of morphisms of $\underline{C}$. Suppose that $\forall\ k \in K$, the pair $(\mathcal{W}_k, I)$ satisfies assumptions (1) through (4) above — then $\underline{C}$ is a combinatorial model category with weak equivalences $\bigcap_{k \in K} \mathcal{W}_k$, cofibrations cof I, fibrations RLP$(\bigcap_{k \in K} \mathcal{W}_k \cap \text{cof } I)$.

[The point here is that an intersection of a set of accessible subcategories is an accessible subcategory.]

0.26 *DIAGRAM CATEGORIES*

Fix a small category $\underline{I}$.

0.26.1 DEFINITION Let $\underline{C}$ be a model category and suppose that $\Xi \in \text{Mor}[\underline{I}, \underline{C}]$, say $\Xi : F \to G$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\ i \in \text{Ob } \underline{I}$, $\Xi_i : Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a <u>levelwise fibration</u> if $\forall\ i \in \text{Ob } \underline{I}$, $\Xi_i : Fi \to Gi$ is a fibration in $\underline{C}$.

- Ξ is a <u>projective cofibration</u> if it has the LLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise fibration.

0.26.2 DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise fibrations, and projective cofibrations is called the <u>projective structure</u> on $[\underline{I}, \underline{C}]$.

Question: Is the projective structure a model structure on $[\underline{I},\underline{C}]$?

0.26.3 EXAMPLE Let $\underline{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \underset{2}{\bullet}$ — then the model structure on $[\underline{I},\underline{C}]$ per 0.1.12 is the projective structure.

0.26.4 EXAMPLE Suppose that $(I,\leq)$ is a finite nonempty directed set of cardinality ≥ 2 — then the model structure on $[\underline{I},\underline{C}]$ per 0.17 is the projective structure.

0.26.5 THEOREM Suppose that $\underline{C}$ is a combinatorial model category — then for every $\underline{I}$, the projective structure on $[\underline{I},\underline{C}]$ is a model structure that, moreover, is combinatorial.

0.26.6 EXAMPLE Take $\underline{C}$ = <u>SISET</u> in its Kan structure -- then the projective structure on $[\underline{I},\underline{\mathrm{SISET}}]$ is a combinatorial model structure (it coincides with structure L (cf. 0.16)).

0.26.7 DEFINITION Let $\underline{C}$ be a model category and suppose that $\Xi \in \mathrm{Mor}[\underline{I},\underline{C}]$, say $\Xi:F \to G$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a <u>levelwise cofibration</u> if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a cofibration in $\underline{C}$.

- Ξ is an <u>injective fibration</u> if it has the RLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise cofibration.

0.26.8 DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise cofibrations, and injective fibrations is called the <u>injective structure</u> on $[\underline{I},\underline{C}]$.

Question: Is the injective structure a model structure on $[\underline{I},\underline{C}]$?

0.26.9 EXAMPLE Let $\underline{I}$ be the category $1 \bullet \xrightarrow{a} \underset{3}{\bullet} \xleftarrow{b} \bullet\, 2$ -- then the model structure on $[\underline{I},\underline{C}]$ per 0.1.12 is the injective structure.

0.26.10 EXAMPLE Let $\underline{C}$ be a small category -- then $\hat{\underline{C}}$ is presentable and the Cisinski structures on $\hat{\underline{C}}$ are in a one-to-one correspondence with the class of admissible $\hat{\underline{C}}$-localizers. Each Cisinski structure is cofibrantly generated and the model structure on $[\underline{I},\hat{\underline{C}}]$ per 0.21.17 is the injective structure.

[Note: Recall that here monomorphisms are levelwise.]

0.26.11 THEOREM Suppose that $\underline{C}$ is a combinatorial model category -- then for every $\underline{I}$, the injective structure on $[\underline{I},\underline{C}]$ is a model structure that, moreover, is combinatorial.

0.26.12 EXAMPLE Take $\underline{C} = \underline{\text{SISET}}$ -- then the injective structure on $[\underline{I},\underline{\text{SISET}}]$ is a combinatorial model structure (it coincides with structure R (cf. 0.16)).

0.26.13 LEMMA Take $\underline{C}$ combinatorial -- then

$$\underline{C} \text{ left proper} \Rightarrow \left[\begin{array}{l} [\underline{I},\underline{C}] \text{ (Projective Structure)} \\ \\ [\underline{I},\underline{C}] \text{ (Injective Structure)} \end{array}\right. \qquad \text{left proper}$$

and

$$\underline{C} \text{ right proper} \Rightarrow \left[\begin{array}{l} [\underline{I},\underline{C}] \text{ (Projective Structure)} \\ \\ [\underline{I},\underline{C}] \text{ (Injective Structure)} \end{array}\right. \quad \text{right proper.}$$

N.B.

- Every projective cofibration is necessarily levelwise, hence is a cofibration in the injective structure.
- Every injective fibration is necessarily levelwise, hence is a fibration in the projective structure.

0.26.14 LEMMA Take $\underline{C}$ combinatorial and consider the setup

$$[\underline{I},\underline{C}] \text{ (Projective Structure)} \underset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\overset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\rightleftarrows}} [\underline{I},\underline{C}] \text{ (Injective Structure)}.$$

Then $(\mathrm{id}_{[\underline{I},\underline{C}]}, \mathrm{id}_{[\underline{I},\underline{C}]})$ is a model equivalence.

PROOF The weak equivalences are the same and ... .

0.26.15 REMARK If $\underline{C}$ and $\underline{C}'$ are combinatorial and if

$$\underline{C} \underset{F'}{\overset{F}{\rightleftarrows}} \underline{C}'$$

is a model pair, then composition with F and F' determines a model pair

$$\underline{[I},\underline{C}] \underset{F'_{\underline{I}}}{\overset{F_{\underline{I}}}{\rightleftarrows}} [\underline{I},\underline{C}']$$

w.r.t. either the projective structure or the injective structure.

Let $\underline{I}$ and $\underline{J}$ be small categories, $K:\underline{I} \to \underline{J}$ a functor, and take $\underline{C}$ combinatorial -- then $\underline{C}$ is complete and cocomplete, so the functor $K^*:[\underline{J},\underline{C}] \to [\underline{I},\underline{C}]$ has a right adjoint

$$K_{\dagger}:[\underline{I},\underline{C}] \to [\underline{J},\underline{C}]$$

and a left adjoint

$$K_!:[\underline{I},\underline{C}] \to [\underline{J},\underline{C}].$$

0.26.16 LEMMA Consider the setup

$$[\underline{I},\underline{C}] \text{ (Projective Structure)} \underset{K^*}{\overset{K_!}{\rightleftarrows}} [\underline{J},\underline{C}] \text{ (Projective Structure)}.$$

Then $(K_!,K^*)$ is a model pair.

PROOF K^* preserves levelwise weak equivalences and levelwise fibrations.

0.26.17 LEMMA Consider the setup

$$[\underline{J},\underline{C}] \text{ (Injective Structure)} \underset{K_{\dagger}}{\overset{K^*}{\rightleftarrows}} [\underline{I},\underline{C}] \text{ (Injective Structure)}.$$

Then $(K^*,K_\dagger)$ is a model pair.

PROOF K^* preserves levelwise weak equivalences and levelwise cofibrations.

0.26.18 THEOREM The model pairs

$$\begin{cases} (K_!,K^*) \\ (K^*,K_\dagger) \end{cases}$$

are model equivalences if K is an equivalence of categories.

Since K^* preserves levelwise weak equivalences, there is a commutative diagram

$$\begin{tikzcd} {[\underline{J},\underline{C}]} \arrow[r, "K^*"] \arrow[d] & {[\underline{I},\underline{C}]} \arrow[d] \\ \underline{H}[\underline{J},\underline{C}] \arrow[r, "\overline{K^*}"'] & \underline{H}[\underline{I},\underline{C}] \end{tikzcd}$$

and adjoint pairs

$$\begin{cases} LK_! \\ \overline{K^*} \end{cases} \text{(Projective Structure)}, \quad \begin{cases} \overline{K^*} \\ RK_\dagger \end{cases} \text{(Injective Structure)}.$$

0.26.19 DEFINITION The functor

$$LK_!:\underline{H}[\underline{I},\underline{C}] \to \underline{H}[\underline{J},\underline{C}]$$

is called the <u>homotopy colimit of K</u>.

[Note: Take $\underline{J} = \underline{1}$ -- then in this case, $LK_!$ is called the <u>homotopy colimit functor</u> and is denoted by $\text{hocolim}_{\underline{I}}$.]

0.26.20 DEFINITION The functor

$$RK_{\dagger}:\underline{H}[\underline{I},\underline{C}] \to \underline{H}[\underline{J},\underline{C}]$$

is called the homotopy limit of K.

[Note: Take $\underline{J} = \underline{1}$ — then in this case, $RK_{\dagger}$ is called the homotopy limit functor and is denoted by $\mathrm{holim}_{\underline{I}}$.]

Is it true that for every small category $\underline{I}$ and model category $\underline{C}$, the functor category $[\underline{I},\underline{C}]$ admits a model structure whose weak equivalences are the levelwise weak equivalences? As far as I can tell, this is an open question. But some information is available. Thus let $\underline{C}(\mathrm{cof})$ stand for $\underline{C}$ viewed as a cofibration category and let $\underline{C}(\mathrm{fib})$ stand for $\underline{C}$ viewed as a fibration category -- then $[\underline{I},\underline{C}(\mathrm{cof})]$ in its injective structure is a homotopically cocomplete cofibration category (cf. 2.5.3) and $[\underline{I},\underline{C}(\mathrm{fib})]$ in its projective structure is a homotopically complete fibration category (cf. 2.5.6). Furthermore, since every model category is a weak model category, 2.7.5 and 2.7.6 are applicable and serve to equip $[\underline{I},\underline{C}]$ with two weak model structures.

0.27 *REEDY THEORY*

Let $\underline{I}$ be a small category.

0.27.1 DEFINITION $\underline{I}$ is said to be a direct category if there exists a function $\mathrm{deg}:\mathrm{Ob}\,\underline{I} \to \mathbf{Z}_{\geq 0}$ such that for any nonidentity morphism $i \xrightarrow{\delta} j$, we have $\mathrm{deg}(i) < \mathrm{deg}(j)$.

0.27.2 EXAMPLE The category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet\ 2$ is a direct category.

0.27.3 THEOREM Suppose that $\underline{C}$ is a cocomplete model category -- then for every direct category $\underline{I}$, the projective structure on $[\underline{I},\underline{C}]$ is a model structure.

0.27.4 DEFINITION $\underline{I}$ is said to be an <u>inverse category</u> if there exists a function $\deg:\mathrm{Ob}\ I \to Z_{\geq 0}$ such that for any nonidentity morphism $i \xrightarrow{\delta} j$, we have $\deg(i) > \deg(j)$.

0.27.5 EXAMPLE The category $1 \bullet \xrightarrow{a} \underset{3}{\bullet} \xleftarrow{b} \bullet\ 2$ is an inverse category.

0.27.6 THEOREM Suppose that $\underline{C}$ is a complete model category -- then for every inverse category $\underline{I}$, the injective structure on $[\underline{I},\underline{C}]$ is a model structure.

0.27.7 DEFINITION Let $\underline{I}$ be direct and let $i \in \mathrm{Ob}\ \underline{I}$ -- then the <u>latching</u> category $\partial(\underline{I}/i)$ is the full subcategory of $\underline{I}/i$ containing all the objects except for the identity map of i.

If $\underline{I}$ is direct, then $\partial(\underline{I}/i)$ is also direct with $\deg(i' \xrightarrow{f} i) = \deg(i')$, thus all the objects of $\partial(\underline{I}/i)$ have degree $< \deg(i)$.

0.27.8 LEMMA Suppose that $\underline{I}$ is direct -- then for any morphism $f:i' \to i$, there is a canonical isomorphism

$$\partial(\partial(\underline{I}/i)/f) \approx \partial(\underline{I}/i')$$

of categories.

0.27.9 DEFINITION Let $\underline{I}$ be inverse and let $i \in \mathrm{Ob}\ \underline{I}$ -- then the <u>matching</u> <u>category</u> $\partial(i\backslash\underline{I})$ is the full subcategory of $i\backslash\underline{I}$ containing all the objects except for the identity map of i.

If $\underline{I}$ is inverse, then $\partial(i\backslash\underline{I})$ is also inverse with $\deg(i \xrightarrow{f} i') = \deg(i')$, thus all the objects of $\partial(i\backslash\underline{I})$ have degree $< \deg(i)$.

0.27.10 LEMMA Suppose that $\underline{I}$ is inverse — then for any morphism $f:i \to i'$, there is a canonical isomorphism

$$\partial(f\backslash\partial(i\backslash\underline{I})) \approx \partial(i'\backslash\underline{I})$$

of categories.

0.27.11 DEFINITION Fix a cocomplete category $\underline{C}$, a direct category $\underline{I}$, and an $i \in \mathrm{Ob}\,\underline{I}$. Let

$$\partial U/i:\partial(\underline{I}/i) \to \underline{I}$$

be the forgetful functor — then the latching functor L_i is the composite

$$[\underline{I},\underline{C}] \xrightarrow{(\partial U/i)^*} [\partial(\underline{I}/i),\underline{C}] \xrightarrow{\mathrm{colim}} \underline{C}.$$

N.B. Given $F \in \mathrm{Ob}[\underline{I},\underline{C}]$, the latching object of F at i is L_iF and the latching morphism of F at i is the canonical arrow $L_iF \to Fi$.

0.27.12 THEOREM Suppose that $\underline{C}$ is a cocomplete model category — then for any direct category $\underline{I}$, a morphism $\Xi:F \to G$ in $[\underline{I},\underline{C}]$ is a cofibration (acyclic cofibration) in the projective structure (cf. 0.27.3) iff $\forall\ i \in \mathrm{Ob}\,\underline{I}$, the induced morphism

$$Fi \coprod_{L_iF} L_iG \to Gi$$

is a cofibration (acyclic cofibration) in $\underline{C}$.

0.27.13 DEFINITION Fix a complete category $\underline{C}$, an inverse category $\underline{I}$, and an $i \in \text{Ob } \underline{I}$. Let

$$\partial i\backslash U : \partial(i\backslash\underline{I}) \to \underline{I}$$

be the forgetful functor -- then the <u>matching functor</u> M_i is the composite

$$[\underline{I},\underline{C}] \xrightarrow{(\partial i\backslash U)^*} [\partial(i\backslash\underline{I}),\underline{C}] \xrightarrow{\lim} \underline{C}.$$

<u>N.B.</u> Given $F \in \text{Ob}[\underline{I},\underline{C}]$, the <u>matching object</u> of F at i is M_iF and the <u>matching morphism</u> of F at i is the canonical arrow $Fi \to M_iF$.

0.27.14 THEOREM Suppose that $\underline{C}$ is a complete model category -- then for any inverse category $\underline{I}$, a morphism $\Xi:F \to G$ in $[\underline{I},\underline{C}]$ is a fibration (acyclic fibration) in the injective structure (cf. 0.27.6) iff $\forall\ i \in \text{Ob } \underline{I}$, the induced morphism

$$Fi \to M_iF \times_{M_iG} Gi$$

is a fibration (acyclic fibration) in $\underline{C}$.

0.27.15 DEFINITION A small category $\underline{I}$ is said to be a <u>Reedy category</u> if the following conditions are satisfied.

- There exist subcategories $\left[\begin{array}{l} \vec{\underline{I}} \\ \\ \overleftarrow{\underline{I}} \end{array}\right.$ with $\left[\begin{array}{l} \text{Ob } \vec{\underline{I}} = \text{Ob } \underline{I} \\ \\ \text{Ob } \overleftarrow{\underline{I}} = \text{Ob } \underline{I} \end{array}\right.$ such that

every $f \in \text{Mor } \underline{I}$ admits a unique factorization $f = \vec{f} \circ \overleftarrow{f}$, where $\vec{f} \in \text{Mor } \vec{\underline{I}}$ and $\overleftarrow{f} \in \text{Mor } \overleftarrow{\underline{I}}$.

- There exists a function $\deg:\text{Ob } \underline{I} \to \mathbb{Z}_{\geq 0}$ such that

$$\left[\begin{array}{l} \forall\ i \xrightarrow{\delta} j \in \text{Mor } \vec{\underline{I}}\ (\delta \neq \text{id}),\ \deg(i) < \deg(j) \\ \\ \forall\ i \xrightarrow{\delta} j \in \text{Mor } \overleftarrow{\underline{I}}\ (\delta \neq \text{id}),\ \deg(j) < \deg(i). \end{array}\right.$$

<u>N.B.</u> Therefore $\vec{\underline{I}}$ is a direct category and $\overleftarrow{\underline{I}}$ is an inverse category.

[Note: Conversely, every direct category is a Reedy category and every inverse category is a Reedy category.]

0.27.16 REMARK The only isomorphisms in a Reedy category are the identities.

0.27.17 REMARK The notion of Reedy category is not invariant under the equivalence of categories.

0.27.18 LEMMA If $\underline{I}$ is a Reedy category, then $\underline{I}^{OP}$ is a Reedy category:

$$\begin{cases} \overrightarrow{\underline{I}^{OP}} = (\overleftarrow{\underline{I}})^{OP} \\ \overleftarrow{\underline{I}^{OP}} = (\overrightarrow{\underline{I}})^{OP}. \end{cases}$$

0.27.19 LEMMA If $\underline{I}$ and $\underline{J}$ are Reedy categories, then $\underline{I} \times \underline{J}$ is a Reedy category:

$$\begin{cases} \overrightarrow{\underline{I} \times \underline{J}} = \overrightarrow{\underline{I}} \times \overrightarrow{\underline{J}} \\ \overleftarrow{\underline{I} \times \underline{J}} = \overleftarrow{\underline{I}} \times \overleftarrow{\underline{J}}. \end{cases}$$

0.27.20 EXAMPLE $\underline{\Delta}$ is a Reedy category: $\deg([n]) = n$ with

$$\begin{cases} \overrightarrow{\underline{\Delta}} \text{ the injective maps} \\ \overleftarrow{\underline{\Delta}} \text{ the surjective maps.} \end{cases}$$

Fix a Reedy category $\underline{I}$.

0.27.21 DEFINITION Let $F \in \mathrm{Ob}[\underline{I},\underline{C}]$, where $\underline{C}$ is complete and cocomplete.

- The <u>latching object</u> of F at i is L_iF, where L_i is computed per $\partial(\vec{\underline{I}}/i)$, and the <u>latching morphism</u> of F at i is the canonical arrow $L_iF \to Fi$.

- The <u>matching object</u> of F at i is M_iF, where M_i is computed per $\partial(i\backslash\overleftarrow{\underline{I}})$, and the <u>matching morphism</u> of F at i is the canonical arrow $Fi \to M_iF$.

0.27.22 EXAMPLE Take $\underline{I} = \underline{\Delta}^{OP}$ and given a simplical object X in <u>SIC</u> $(= [\underline{\Delta}^{OP},\underline{C}])$, put

$$\begin{cases} \mathrm{sk}^{(n)}X = \mathrm{sk}^{(n)}(\mathrm{tr}^{(n)}X) \\ \mathrm{cosk}^{(n)}X = \mathrm{cosk}^{(n)}(\mathrm{tr}^{(n)}X). \end{cases}$$

Then

$$L_nX(= L_{[n]}X) = (\mathrm{sk}^{(n-1)}X)_n$$

and

$$M_nX(= M_{[n]}X) = (\mathrm{cosk}^{(n-1)}X)_n.$$

[Note: Therefore L_0X is an initial object in $\underline{C}$ and M_0X is a final object in $\underline{C}$.]

0.27.23 DEFINITION Let $\underline{C}$ be a complete and cocomplete model category and suppose that $\Xi \in \mathrm{Mor}[I,C]$, say $\Xi: F \to G$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i: Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a <u>Reedy cofibration</u> if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, the induced morphism

$$Fi \coprod_{L_iF} L_iG \to Gi$$

is a cofibration in $\underline{C}$.

- Ξ is a Reedy fibration if $\forall\ i \in \text{Ob}\ \underline{I}$, the induced morphism

$$Fi \to M_iF \times_{M_iG} Gi$$

is a fibration in $\underline{C}$.

0.27.24 LEMMA Suppose that $\Xi:F \to G$ is a Reedy cofibration -- then $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a cofibration in $\underline{C}$.

[Note: In addition, the induced morphism $L_i\Xi:L_iF \to L_iG$ of latching objects is a cofibration in $\underline{C}$ which is acyclic if Ξ is a levelwise weak equivalence.]

0.27.25 LEMMA Suppose that $\Xi:F \to G$ is a Reedy fibration -- then $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a fibration in $\underline{C}$.

[Note: In addition, the induced morphism $M_i\Xi:M_iF \to M_iG$ of matching objects is a fibration in $\underline{C}$ which is acyclic if Ξ is a levelwise weak equivalence.]

0.27.26 APPLICATION Every projective cofibration is a Reedy cofibration and every injective fibration is a Reedy fibration.

0.27.27 DEFINITION The triple consisting of the classes of levelwise weak equivalences, Reedy cofibrations, and Reedy fibrations is called the Reedy structure on $[\underline{I},\underline{C}]$.

0.27.28 THEOREM The Reedy structure on $[\underline{I},\underline{C}]$ is a model structure. And

$\underline{C}$ left proper => $[\underline{I},\underline{C}]$ (Reedy Structure) left proper

$\underline{C}$ right proper => $[\underline{I},\underline{C}]$ (Reedy Structure) right proper.

[Note: Let $\Xi \in \text{Mor}[\underline{I},\underline{C}]$, say $\Xi:F \to G$.

- Ξ is both a levelwise weak equivalence and a Reedy cofibration iff $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow

$$Fi \coprod_{L_iF} L_iG \to Gi$$

is an acyclic cofibration in $\underline{C}$.

- Ξ is both a levelwise weak equivalence and a Reedy fibration iff $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow

$$Fi \to M_iF \times_{M_iG} Gi$$

is an acyclic fibration in $\underline{C}$.]

0.27.29 REMARK It follows from 0.27.12 that if $\underline{I}$ is direct, then

$$[\underline{I},\underline{C}]\ \text{(Projective Structure)} = [\underline{I},\underline{C}]\ \text{(Reedy Structure)}$$

and it follows from 0.27.14 that if $\underline{I}$ is inverse, then

$$[\underline{I},\underline{C}]\ \text{(Injective Structure)} = [\underline{I},\underline{C}]\ \text{(Reedy Structure)}.$$

0.27.30 THEOREM Suppose that $\underline{C}$ is combinatorial — then $[\underline{I},\underline{C}]$ (Reedy Structure) is combinatorial.

0.27.31 LEMMA Take $\underline{C}$ combinatorial and consider the setup

$$[\underline{I},\underline{C}]\ \text{(Projective Structure)} \underset{\text{id}_{[\underline{I},\underline{C}]}}{\overset{\text{id}_{[\underline{I},\underline{C}]}}{\rightleftarrows}} [\underline{I},\underline{C}]\ \text{(Reedy Structure)}.$$

Then $(\mathrm{id}_{[\underline{I},\underline{C}]}, \mathrm{id}_{[\underline{I},\underline{C}]})$ is a model equivalence.

[Working from left to right, the weak equivalences are the same and every projective cofibration is a Reedy cofibration.]

0.27.32 LEMMA Take $\underline{C}$ combinatorial and consider the setup

$$[\underline{I},\underline{C}] \text{ (Reedy Structure) } \underset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\overset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\rightleftarrows}} [\underline{I},\underline{C}] \text{ (Injective Structure).}$$

Then $(\mathrm{id}_{[\underline{I},\underline{C}]}, \mathrm{id}_{[\underline{I},\underline{C}]})$ is a model equivalence.

[Working from right to left, the weak equivalences are the same and every injective fibration is a Reedy fibration.]

0.27.33 EXAMPLE Take $\underline{I} = \underline{\Delta}$, $\underline{C} = \underline{\mathrm{SISET}}$ -- then every projective cofibration is a Reedy cofibration (cf. 0.27.26) and the containment is strict since, e.g., $Y_{\underline{\Delta}}$ is a cosimplicial object in $\hat{\underline{\Delta}}$ which is cofibrant in the Reedy structure but not in the projective structure (a.k.a. structure L).

0.27.34 THEOREM If $\underline{I}$ and $\underline{J}$ are Reedy categories, then for any complete and cocomplete model category $\underline{C}$,

$$[\underline{I} \times \underline{J},\underline{C}] \text{ (Reedy Structure)}$$

is the same as

$$[\underline{I},[\underline{J},\underline{C}] \text{ (Reedy Structure)}] \text{ (Reedy Structure).}$$

Let $\underline{I}$ be a Reedy category, $\underline{C}$ a complete and cocomplete model category, and let $K:\underline{C} \to [\underline{I},\underline{C}]$ be the constant diagram functor. Equip $[\underline{I},\underline{C}]$ with the Reedy structure.

0.27.35 LEMMA The adjoint situation $(K,\lim_{\underline{I}})$ is a model pair iff $\forall\ i \in \text{Ob}\ \underline{I}$, the latching category $\partial(\vec{\underline{I}}/i)$ is either connected or empty.

0.27.36 REMARK Let $\underline{I}$ be a small category, $\underline{C}$ a combinatorial model category -- then $[\underline{I},\underline{C}]$ admits a model structure such that the adjoint situation $(K,\lim_{\underline{I}})$ is a model equivalence.

0.27.37 LEMMA The adjoint situation $(\text{colim}_{\underline{I}},K)$ is a model pair iff $\forall\ i \in \text{Ob}\ \underline{I}$, the matching category $\partial(i\backslash\overleftarrow{\underline{I}})$ is either connected or empty.

0.27.38 REMARK Let $\underline{I}$ be a small category, $\underline{C}$ a combinatorial model category -- then $[\underline{I},\underline{C}]$ admits a model structure such that the adjoint situation $(\text{colim}_{\underline{I}},K)$ is a model equivalence.

0.27.39 EXAMPLE Take $\underline{I} = \underline{\Delta}^{\text{OP}}$ to realize 0.27.35 and take $\underline{I} = \underline{\Delta}$ to realize 0.27.37.

The theory outlined above is "classical" and certain important examples do not fall within its scope, e.g. Segal's category $\underline{\Gamma}$ or Connes's category $\underline{\Lambda}$. To accommodate these (and others of significance) it is necessary to extend the notion of Reedy category so as to allow for nontrivial isomorphisms (cf. 0.27.16). For a systematic account, consult Berger-Moerdijk†.

† arXiv:0809.3341

0.28 *EXAMPLE:* $\underline{\Gamma\text{SISET}_*}$

$\underline{\Gamma}$ is the category whose objects are the finite sets $\underline{n} \equiv \{0,1,\ldots,n\}$ $(n \geq 0)$ with base point 0 and whose morphisms are the base point preserving maps.

[Note: Suppose that $\gamma:\underline{m} \to \underline{n}$ is a morphism in $\underline{\Gamma}$ -- then the partition

$$\coprod_{0 \leq j \leq n} \gamma^{-1}(j) = \underline{m}$$

of $\underline{m}$ determines a permutation $\theta:\underline{m} \to \underline{m}$ such that $\gamma \circ \theta$ is order preserving. Therefore γ has a unique factorization of the form $\alpha \circ \sigma$, where $\alpha:\underline{m} \to \underline{n}$ is order preserving and $\sigma:\underline{m} \to \underline{m}$ is a base point preserving permutation which is order preserving in the fibers of γ.]

Write $\underline{\Gamma\text{SISET}_*}$ for the full subcategory of $[\underline{\Gamma},\underline{\text{SISET}_*}]$ whose objects are the $X:\underline{\Gamma} \to \underline{\text{SISET}_*}$ such that $X_0 = *$ $(X_n = X(\underline{n}))$.

0.28.1 EXAMPLE Let G be an abelian semigroup with unit. Using additive notation, view G^n as the set of base point preserving functions $\underline{n} \to G$ -- then the rule $X_n = \text{si } G^n$ defines an object in $\underline{\Gamma\text{SISET}_*}$. Here the arrow $G^m \to G^n$ attached to $\gamma:\underline{m} \to \underline{n}$ sends $(g_1,\ldots,g_m)$ to $(\bar{g}_1,\ldots,\bar{g}_n)$, where $\bar{g}_j = \sum_{\gamma(i)=j} g_i$ if $\gamma^{-1}(j) \neq \emptyset$, $\bar{g}_j = 0$ if $\gamma^{-1}(j) = \emptyset$.

Let $S_n(\underline{\text{SISET}_*})$ be the category whose objects are the pointed simplicial left S_n-sets -- then $S_n(\underline{\text{SISET}_*})$ is a model category (cf. 0.8).

[Note: The group of base point preserving permutations $\underline{n} \to \underline{n}$ is S_n and for any X in $\underline{\Gamma\text{SISET}_*}$, X_n is a pointed simplicial left S_n-set.]

Let $\underline{\Gamma}_n$ be the full subcategory of $\underline{\Gamma}$ whose objects are the $\underline{m}$ ($m \leq n$). Assigning to the symbol $\underline{\Gamma_n\text{SISET}_*}$ the obvious interpretation, one can follow the usual procedure and introduce $\text{tr}^{(n)}:\underline{\Gamma\text{SISET}_*} \to \underline{\Gamma_n\text{SISET}_*}$ and its left (right) adjoint $\text{sk}^{(n)}$ $(\text{cosk}^{(n)})$.

0.28.2 NOTATION Given an X in $\underline{\Gamma\text{SISET}_*}$, put

$$\begin{cases} \text{sk}^{(n)}X = \text{sk}^{(n)}(\text{tr}^{(n)}X) \\ \text{cosk}^{(n)}X = \text{cosk}^{(n)}(\text{tr}^{(n)}X) \end{cases}$$

and write

$$\begin{cases} L_nX \; (= L_{\underline{n}}X) = (\text{sk}^{(n-1)}X)_n \\ M_nX \; (= M_{\underline{n}}X) = (\text{cosk}^{(n-1)}X)_n \end{cases}$$

for the

$$\begin{cases} \text{latching} \\ \text{matching} \end{cases}$$

objects of X at $\underline{n}$ (cf. 0.27.22).

0.28.3 DEFINITION Suppose that $f \in \text{Mor}\,\underline{\Gamma\text{SISET}_*}$, say $f:X \to Y$.

- f is a weak equivalence if $\forall\, n \geq 1$, $f_n:X_n \to Y_n$ is a weak equivalence in $S_n(\underline{\text{SISET}_*})$.

- f is a cofibration if $\forall\, n \geq 1$, the induced morphism $X_n \coprod_{L_nX} L_nY \to Y_n$ is a cofibration in $S_n(\underline{\text{SISET}_*})$.

- f is a fibration if $\forall\, n \geq 1$, the induced morphism $X_n \to M_n X \times_{M_n Y} Y_n$ is a fibration in $S_n(\underline{\text{SISET}}_*)$.

Call these choices the <u>Reedy structure</u> on $\underline{\Gamma\text{SISET}}_*$.

0.28.4 THEOREM $\underline{\Gamma\text{SISET}}_*$ in the Reedy structure is a proper model category.

0.29 *BISIMPLICIAL SETS*

The category $[\underline{\Delta}^{OP},\underline{\text{SISET}}]$ carries three proper combinatorial model structures:

- The projective structure (= structure L) (cf. 0.26.6)
- The Reedy structure
- The injective structure (= structure R) (cf. 0.26.12).

0.29.1 LEMMA The projective structure is not the same as the Reedy structure but the Reedy structure is the same as the injective structure (hence all objects in the Reedy structure are cofibrant).

Given a category $\underline{C}$, write <u>BISIC</u> for the functor category $[(\underline{\Delta} \times \underline{\Delta})^{OP},\underline{C}]$ — then by definition, a <u>bisimplicial object</u> in $\underline{C}$ is an object in <u>BISIC</u>.

0.29.2 EXAMPLE Suppose that $\underline{C}$ has finite products and let $\begin{cases} X \\ Y \end{cases}$ be simplicial objects in C — then the assignment $([n],[m]) \to X_n \times Y_m$ defines a bisimplicial object $X \underline{\times} Y$ in $\underline{C}$.

Specialize to $\underline{C} = \underline{\text{SET}}$ — then an object in <u>BISISET</u> is called a <u>bisimplicial set</u> and a morphism in <u>BISISET</u> is called a <u>bisimplicial map</u>. Given a bisimplicial set X, put $X_{n,m} = X([n],[m])$ — then there are horizontal operators

$$\begin{cases} d_i^h : X_{n,m} \to X_{n-1,m} \\ s_i^h : X_{n,m} \to X_{n+1,m} \end{cases} \quad (0 \le i \le n)$$

and vertical operators

$$\begin{cases} d_j^v : X_{n,m} \to X_{n,m-1} \\ s_j^v : X_{n,m} \to X_{n,m+1} \end{cases} \quad (0 \le j \le m).$$

The horizontal operators commute with the vertical operators, the simplicial identities are satisfied horizontally and vertically, and thanks to the Yoneda lemma, $\mathrm{Nat}(\Delta[n,m],X) \approx X_{n,m}$, where $\Delta[n,m] = \Delta[n] \underline{\times} \Delta[m]$.

[Note: Every simplicial set X can be regarded as a bisimplicial set by trivializing its structure in either the horizontal or vertical direction, i.e., $X_{n,m} = X_m$ or $X_{n,m} = X_n$.]

0.29.3 EXAMPLE Every functor $T:\underline{\Delta} \to \underline{\mathrm{CAT}}$ gives rise to a functor $X_T:\underline{\mathrm{CAT}} \to \underline{\mathrm{BISISET}}$ by writing

$$X_T\underline{I}([n],[m]) = \mathrm{ner}_n([T[m],\underline{I}])$$

or still,

$$\begin{aligned} &\mathrm{ner}[T[m],\underline{I}]([n]) \\ &\approx \mathrm{Nat}(\Delta[n],\mathrm{ner}[T[m],\underline{I}]) \\ &\approx \mathrm{Nat}(\mathrm{ner}[n],\mathrm{ner}[T[m],\underline{I}]) \\ &\approx \mathrm{Mor}([n],[T[m],\underline{I}]) \\ &\approx \mathrm{Mor}([n] \times T[m],\underline{I}) \end{aligned}$$

$$\approx \mathrm{Mor}(T[m] \times [n], \underline{I})$$

$$\approx \mathrm{Mor}(T[m], [[n], \underline{I}])$$

$$\approx (S_T[[n], \underline{I}])_m,$$

S_T the singular functor.

0.29.4 REMARK There are two canonical identifications

$$\underline{\mathrm{BISISET}} \approx [\underline{\Delta}^{\mathrm{OP}}, \underline{\mathrm{SISET}}]$$

that send a bisimplicial set X to the cofunctors

$$\begin{cases} [n] \to X_{n,*} \\ [m] \to X_{*,m}. \end{cases}$$

Each bisimplicial map $f:X \to Y$ induces simplicial maps

$$\begin{cases} f_{n,*}:X_{n,*} \to Y_{n,*} \\ f_{*,m}:X_{*,m} \to Y_{*,m} \end{cases}$$

and it can happen that $\forall\, n$, $f_{n,*}$ is a simplicial weak equivalence but for some m, $f_{*,m}$ is not a simplicial weak equivalence.

[Take $X_{n,*} = \Delta[1]$, $Y_{n,m} = \{*\}$ and let f be the unique bisimplicial map from X to Y — then $\forall\, n$, $f_{n,*}:X_{n,*} \to Y_{n,*}$ is the simplicial map $\Delta[1] \to \Delta[0]$, which is a simplicial weak equivalence, but $f_{*,0}:X_{*,0} \to Y_{*,0}$ is the simplicial map $\Delta[0] \coprod \Delta[0] \to \Delta[0]$, which is not a simplicial weak equivalence.]

[Note: The projective (injective) structure on $[\underline{\Delta}^{\mathrm{OP}}, \underline{\mathrm{SISET}}]$ gives rise to two model structures on $\underline{\text{BISISET}}$. In the one, a bisimplicial map $f:X \to Y$ is a weak equivalence if $\forall$ n, $f_{n,*}:X_{n,*} \to Y_{n,*}$ is a simplicial weak equivalence and in the other, a bisimplicial map $f:X \to Y$ is a weak equivalence if $\forall$ m, $f_{*,m}:X_{*,m} \to Y_{*,m}$ is a simplicial weak equivalence. The point then is that these model structures are not the same.]

0.29.5 LEMMA Let X be a bisimplicial set -- then

$$X \approx \int^{[n]}\int^{[m]} \text{Mor}(\text{—},([n],[m])) \cdot X_{n,m}$$

and

$$X \approx \int_{[n]}\int_{[m]} (X_{n,m})^{\text{Mor}(([n],[m]),\text{—})}.$$

[These formulas are instances of the integral Yoneda lemma.]

[Note: Here Mor is computed per $\underline{\Delta} \times \underline{\Delta}$ (and not $(\underline{\Delta} \times \underline{\Delta})^{\text{OP}}$).]

Using the notation of Kan extensions, take $\underline{C} = \underline{\Delta}^{\text{OP}}$, $\underline{D} = \underline{\Delta}^{\text{OP}} \times \underline{\Delta}^{\text{OP}}$ ($\approx (\underline{\Delta} \times \underline{\Delta})^{\text{OP}}$), $\underline{S} = \underline{\text{SET}}$, and let K be the diagonal $\underline{\Delta}^{\text{OP}} \to \underline{\Delta}^{\text{OP}} \times \underline{\Delta}^{\text{OP}}$ -- then the functor $K^*:\underline{\text{BISISET}} \to \underline{\text{SISET}}$ is denoted by dia, thus

$$(\text{dia } X)_n = X([n],[n]) = X_{n,n}$$

the operators being

$$\begin{cases} d_i = d_i^h d_i^v = d_i^v d_i^h \\ s_i = s_i^h s_i^v = s_i^v s_i^h. \end{cases}$$

0.29.6 EXAMPLE Let X,Y be simplicial sets -- then

$$\mathrm{dia}(X \underline{\times} Y) = X \times Y \quad (\Rightarrow \mathrm{dia}\ \Delta[n,m] = \Delta[n] \times \Delta[m]).$$

0.29.7 LEMMA Let X be a bisimplicial set -- then

$$\mathrm{dia}\ X \approx \int^{[n]}\int^{[m]} (\mathrm{Mor}(—,[n]) \times \mathrm{Mor}(—,[m])) \cdot X_{n,m}$$

$$\approx \int^{[n]} \mathrm{Mor}(—,[n]) \times X_{n,*}$$

$$\approx \int^{[m]} \mathrm{Mor}(—,[m]) \times X_{*,m}$$

and

$$\mathrm{dia}\ X \approx \int_{[n]}\int_{[m]} (X_{n,m})^{\mathrm{Mor}([n],—) \times \mathrm{Mor}([m],—)}$$

$$\approx \int_{[n]} (X_{n,*})^{\mathrm{Mor}([n],—)}$$

$$\approx \int_{[m]} (X_{*,m})^{\mathrm{Mor}([m],—)}.$$

0.29.8 DEFINITION The simplicial set

$$\int^{[n]} \mathrm{Mor}(—,[n]) \times X_{n,*}$$

$$\approx \int^{[n]} X_n \times \Delta[n] \quad (X_n \equiv X_{n,*})$$

is called the <u>realization</u> of X, written $|X|$.

[Note: Its geometric realization is the coend

$$\int^{[n]} |X_n| \times \Delta^n.]$$

0.29.9 LEMMA Let $f:X \to Y$ be a bisimplicial map. Assume: $\forall\ n,\ f_{n,*}:X_{n,*} \to Y_{n,*}$ is a simplicial weak equivalence -- then $|f|:|X| \to |Y|$ is a simplicial weak equivalence, thus $\text{dia}\ f:\text{dia}\ X \to \text{dia}\ Y$ is a simplicial weak equivalence.

0.29.10 LEMMA Let $f:X \to Y$ be a bisimplicial map. Assume: $\text{dia}\ f:\text{dia}\ X \to \text{dia}\ Y$ is a Kan fibration -- then

$$\left[\begin{array}{l} \forall\ n,\ f_{n,*}:X_{n,*} \to Y_{n,*} \\ \\ \forall\ m,\ f_{*,m}:X_{*,m} \to Y_{*,m} \end{array}\right.$$

are Kan fibrations.

[The converse is false, i.e., it can happen that

$$\left[\begin{array}{l} \forall\ n,\ f_{n,*}:X_{n,*} \to Y_{n,*} \\ \\ \forall\ m,\ f_{*,m}:X_{*,m} \to Y_{*,m} \end{array}\right.$$

are Kan fibrations but $\text{dia}\ f:\text{dia}\ X \to \text{dia}\ Y$ is not a Kan fibration. In fact, there are bisimplicial sets X such that the $X_{n,*}, X_{*,m}$ are Kan complexes but dia X is not a Kan complex.]

The functor $\text{dia}:\underline{\text{BISISET}} \to \underline{\text{SISET}}$ has a left adjoint

$$\text{dia}_!:\underline{\text{SISET}} \to \underline{\text{BISISET}}$$

and a right adjoint

$$\text{dia}_+:\underline{\text{SISET}} \to \underline{\text{BISISET}}.$$

- Let A be a simplicial set -- then

$$\begin{aligned} &(\text{dia}_!A)([n],[m]) \\ &\quad = \int^{[k]} \text{Mor}_{\underline{\Delta}^{\text{OP}} \times \underline{\Delta}^{\text{OP}}}(K[k],([n],[m])) \cdot A[k] \end{aligned}$$

$$= \int^{[k]} \mathrm{Mor}_{\underline{\Delta}^{OP} \times \underline{\Delta}^{OP}}(([k],[k]),([n],[m])) \cdot A_k$$

$$= \int^{[k]} \mathrm{Mor}_{\underline{\Delta} \times \underline{\Delta}}(([n],[m]),([k],[k])) \cdot A_k$$

$$= \int^{[k]} (\mathrm{Mor}([n],[k]) \times \mathrm{Mor}([m],[k])) \cdot A_k.$$

[Note: To run a reality check, let X be a bisimplicial set and compute:

$$\mathrm{Mor}(A,\mathrm{dia}\ X) = \mathrm{Nat}(A,\mathrm{dia}\ X)$$

$$\approx \int_{[k]} \mathrm{Mor}(A[k],\mathrm{dia}\ X([k]))$$

$$\approx \int_{[k]} \mathrm{Mor}(A_k,\int_{[n]}\int_{[m]} (X_{n,m})^{\mathrm{Mor}([n],[k]) \times \mathrm{Mor}([m],[k])})$$

$$\approx \int_{[n]}\int_{[m]}\int_{[k]} \mathrm{Mor}(A_k \times \mathrm{Mor}([n],[k]) \times \mathrm{Mor}([m],[k]),X_{n,m})$$

$$\approx \int_{[n]}\int_{[m]} \mathrm{Mor}(\int^{[k]} (\mathrm{Mor}([n],[k]) \times \mathrm{Mor}([m],[k])) \cdot A_k,X_{n,m})$$

$$\approx \mathrm{Nat}(\mathrm{dia}_! A,X) = \mathrm{Mor}(\mathrm{dia}_! A,X).]$$

0.29.11 EXAMPLE Take $A = \Delta[n]$ -- then

$$\mathrm{dia}_! \Delta[n] \approx \Delta[n,n] \quad (= \Delta[n] \underline{\times} \Delta[n]).$$

[For any bisimplicial set X, we have

$$\mathrm{Mor}(\mathrm{dia}_! \Delta[n],X) \approx \mathrm{Mor}(\Delta[n],\mathrm{dia}\ X) \approx X_{n,n}.$$

On the other hand,

$$\mathrm{Mor}(\Delta[n,n],X) \approx X_{n,n}.]$$

• Let A be a simplicial set — then

$$(\mathrm{dia}_{\dagger}A)([n],[m])$$

$$= \int_{[k]} (A[k])^{\mathrm{Mor}_{\underline{\Delta}^{\mathrm{OP}} \times \underline{\Delta}^{\mathrm{OP}}}(([n],[m]),K[k])}$$

$$= \int_{[k]} (A_k)^{\mathrm{Mor}_{\underline{\Delta}^{\mathrm{OP}} \times \underline{\Delta}^{\mathrm{OP}}}(([n],[m]),([k],[k]))}$$

$$= \int_{[k]} (A_k)^{\mathrm{Mor}_{\underline{\Delta} \times \underline{\Delta}}(([k],[k]),([n],[m]))}$$

$$= \int_{[k]} (A_k)^{\mathrm{Mor}([k],[n]) \times \mathrm{Mor}([k],[m])}$$

$$= \int_{[k]} (A_k)^{\Delta[n][k] \times \Delta[m][k]}$$

$$= \int_{[k]} \mathrm{Mor}(\Delta[n][k] \times \Delta[m][k],A_k)$$

$$\approx \mathrm{Nat}(\Delta[n] \times \Delta[m],A) = \mathrm{Mor}(\Delta[n] \times \Delta[m],A).$$

[Note: To run a reality check, let X be a bisimplicial set and compute:

$$\mathrm{Mor}(\mathrm{dia}\ X,A) = \mathrm{Nat}(\mathrm{dia}\ X,A)$$

$$\approx \int_{[k]} \mathrm{Mor}(\mathrm{dia}\ X([k]),A[k])$$

$$\approx \int_{[k]} \mathrm{Mor}(\int^{[n]}\int^{[m]} (\mathrm{Mor}([k],[n]) \times \mathrm{Mor}([k],[m])) \cdot X_{n,m},A_k)$$

$$\approx \int_{[n]}\int_{[m]}\int_{[k]} \mathrm{Mor}(X_{n,m} \times \Delta[n][k] \times \Delta[m][k],A_k)$$

$$\approx \int_{[n]}\int_{[m]}\int_{[k]} \mathrm{Mor}(X_{n,m},(A_k)^{\Delta[n][k] \times \Delta[m][k]})$$

$$\approx \int_{[n]}\int_{[m]} \mathrm{Mor}(X_{n,m},\int_{[k]}(A_k)^{\Delta[n][k] \times \Delta[m][k]})$$

$$\approx \int_{[n]}\int_{[m]} \mathrm{Mor}(X_{n,m},\mathrm{Mor}(\Delta[n] \times \Delta[m],A))$$

$$\approx \mathrm{Nat}(X,\mathrm{dia}_{\dagger}A) = \mathrm{Mor}(X,\mathrm{dia}_{\dagger}A).]$$

0.30 *THE $\bar{W}$-CONSTRUCTION*

Using the notation of Kan extensions, take $\underline{C} = \underline{\Delta}^{OP} \times \underline{\Delta}^{OP}$ $(\approx (\underline{\Delta} \times \underline{\Delta})^{OP})$, $\underline{D} = \underline{\Delta}^{OP}$, $\underline{S} = \underline{SET}$, and let K be the ordinal sum $\underline{\Delta}^{OP} \times \underline{\Delta}^{OP} \to \underline{\Delta}^{OP}$ (i.e., $([n],[m]) \to [n+m+1]$) -- then the functor $K^*:\underline{SISET} \to \underline{BISISET}$ is denoted by dec, thus

$$(\mathrm{dec}\ X)([n],[m]) = X_{n+m+1},$$

the operations being

$$\begin{bmatrix} d_i^h = d_i:X_{n+m+1} \to X_{n+m} \ (0 \le i \le n) \\ s_i^h = s_i:X_{n+m+1} \to X_{n+1+m+1} \ (0 \le i \le n) \end{bmatrix}$$

and

$$\begin{bmatrix} d_j^v = d_{n+1+j}:X_{n+m+1} \to X_{n+m} \quad (0 \le j \le m) \\ s_j^v = s_{n+1+j}:X_{n+m+1} \to X_{n+m+1+1} \quad (0 \le j \le m). \end{bmatrix}$$

0.30.1 EXAMPLE We have

$$(\mathrm{dec}\ \Delta[n])([k],[n-k]) = \Delta[n]_{n+1} \quad (0 \le k \le n).$$

Put $\bar{W} = \mathrm{dec}_{\dagger}$, hence

$$\bar{W}:\underline{BISET} \to \underline{SISET}.$$

N.B. For any bisimplicial set X,

$$(\bar{W}X)_n = \{(x_{0,n},\ldots,x_{n,0}) \in \prod_{k=0}^{n} X_{k,n-k} : d_0^v x_{k,n-k} = d_{k+1}^h x_{k+1,n-k-1} \quad (0 \leq k < n)\}.$$

And the

$$\begin{cases} d_i : (\bar{W}X)_n \to (\bar{W}X)_{n-1} \\ \\ s_i : (\bar{W}X)_n \to (\bar{W}X)_{n+1} \end{cases} \qquad (0 \leq i \leq n)$$

are the prescriptions

$$\begin{cases} d_i \underline{x} = (d_i^v x_{0,n},\ldots,d_1^v x_{i-1,n-i+1}, d_i^h x_{i+1,n-i-1},\ldots,d_i^h x_{n,0}) \\ \\ s_i \underline{x} = (s_i^v x_{0,n},\ldots,s_0^v x_{i,n-i}, s_i^h x_{i,n-i},\ldots,s_i^h x_{n,0}), \end{cases}$$

where

$$\underline{x} = (x_{0,n},\ldots,x_{n,0}).$$

[Note: To shorten matters, the elements of $(\bar{W}X)_n$ can be regarded as $(n+1)$-tuples

$$(x_0,\ldots,x_n) \in \prod_{k=0}^{n} X_{k,n-k}$$

such that

$$d_0^v x_k = d_{k+1}^h x_{k+1} \quad (0 \leq k < n).]$$

0.30.2 LEMMA The rule that assigns to each bisimplicial set X the simplicial map

$$\Xi_X : \text{dia } X \to \bar{W}X$$

given by

$$(\Xi_X)x = ((d_1^h)^n x, (d_2^h)^{n-1} d_0^v x, \ldots, (d_{i+1}^h)^{n-i} (d_0^v)^i x, \ldots, (d_0^v)^n x) \ (x \in X_{n,n})$$

defines a natural transformation

$$\Xi : \mathrm{dia} \to \bar{W}.$$

0.30.3 THEOREM For every X,

$$\Xi_X : \mathrm{dia}\ X \to \bar{W}X$$

is a simplicial weak equivalence.

0.30.4 DEFINITION A bisimplicial map $f:X \to Y$ is a <u>diagonal weak equivalence</u> if dia f is a simplicial weak equivalence.

[Note: Recalling that $\left[\begin{array}{l} |X| \\ |Y| \end{array}\right.$ are the realizations of $\left[\begin{array}{l} X \\ Y \end{array}\right.$ (cf. 0.29.8), there is a commutative diagram

$$\begin{tikzcd} ||X|| \arrow[r, "||f||"] \arrow[d, "\approx"'] & ||Y|| \arrow[d, "\approx"] \\ |\mathrm{dia}\ X| \arrow[r, "|\mathrm{dia}\ f|"'] & |\mathrm{dia}\ Y|, \end{tikzcd}$$

so f is a diagonal weak equivalence iff $|f|$ is a simplicial weak equivalence.]

0.30.5 LEMMA Let $f:X \to Y$ be a bisimplicial map -- then f is a diagonal weak equivalence iff $\bar{W}f:\bar{W}X \to \bar{W}Y$ is a simplicial weak equivalence.

PROOF Consider the commutative diagram

$$\begin{tikzcd} \mathrm{dia}\ X \arrow[r, "\Xi_X"] \arrow[d, "\mathrm{dia}\ f"'] & \bar{W}X \arrow[d, "\bar{W}f"] \\ \mathrm{dia}\ Y \arrow[r, "\Xi_Y"'] & \bar{W}Y \end{tikzcd}$$

and quote 0.30.3.

0.31 BISISET: *MOERDIJK STRUCTURE*

Given a bisimplicial map $f:X \to Y$, call f a weak equivalence if f is a diagonal weak equivalence, a fibration if dia f is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations — then with these choices, BISISET is a proper combinatorial model category.

N.B. Every cofibration in the Moerdijk structure is a monomorphism.

0.31.1 REMARK The Moerdijk structure on BISISET is not the same as the induced projective or injective structures. This is because the weak equivalences in these structures are necessarily weak equivalences in the Moerdijk structure (cf. 0.29.9) but not conversely.

0.31.2 LEMMA Consider the setup

$$\text{SISET (Kan Structure)} \underset{\text{dia}}{\overset{\text{dia}_!}{\rightleftarrows}} \text{BISISET (Moerdijk Structure)}.$$

Then $(\text{dia}_!,\text{dia})$ is a model pair.

[One has only to note that by construction, dia is a right model functor.]

Therefore $\left[\begin{array}{l}\text{Ldia}_! \\ \\ \text{Rdia}\end{array}\right.$ exist and $(\text{Ldia}_!,\text{Rdia})$ is an adjoint pair.

0.31.3 LEMMA The model pair $(\text{dia}_!,\text{dia})$ is a model equivalence.

Therefore the adjoint pair $(\mathrm{Ldia}_!, \mathrm{Rdia})$ is an adjoint equivalence of homotopy categories:

$$\underline{\mathrm{HSISET}} \rightleftarrows \underline{\mathrm{HBISISET}} .$$

There is another proper combinatorial model structure on $\underline{\mathrm{BISISET}}$ that is analogous to the Moerdijk structure, the role of "dia" being played by "$\bar{W}$". Thus the weak equivalences are again the diagonal weak equivalences but now a bisimplicial map $f:X \to Y$ is a fibration if $\bar{W}f$ is a Kan fibration and a cofibration if it has the LLP w.r.t. acyclic fibrations.

[Note: We shall refer to this model structure on $\underline{\mathrm{BISISET}}$ as the $\underline{\bar{W}\text{-structure}}$.]

<u>N.B.</u> Every cofibration in the $\bar{W}$-structure is a monomorphism.

0.32.2 LEMMA Let $f:X \to Y$ be a bisimplicial map. Assume: dia f is a Kan fibration — then $\bar{W}f$ is a Kan fibration.

Therefore

$$\mathrm{cof}(\bar{W}\text{-Structure}) \subset \mathrm{cof}(\text{Moerdijk Structure}).$$

0.32 $\underline{\mathrm{BISISET}}$: *OTHER MODEL STRUCTURES*

0.32.1 NOTATION Let

$$\mathcal{M} \subset \mathrm{Mor}\ \underline{\mathrm{BISISET}}$$

be the class of monomorphisms and let $M \subset \mathcal{M}$ be the set of inclusions

$$\dot{\Delta}[n] \underline{\times} \Delta[m] \cup \Delta[n] \underline{\times} \dot{\Delta}[n] \to \Delta[n] \underline{\times} \Delta[m].$$

0.32.2 LEMMA We have

$$\mathcal{M} = \mathrm{LLP}(\mathrm{RLP}(M)) \qquad (\text{cf. } 0.20.5).$$

0.32.3 THEOREM There is a model structure on $\underline{\text{BISISET}}$ in which the weak equivalences are the diagonal weak equivalences and the cofibrations are the monomorphisms.

[Note: This structure is proper and combinatorial.]

0.32.4 THEOREM There is a model structure on $\underline{\text{BISISET}}$ in which the weak equivalences are the bisimplicial maps $f:X \to Y$ such that $\forall$ n,

$$f_{n,*}:X_{n,*} \to Y_{n,*}$$

is a simplicial weak equivalence and the cofibrations are the monomorphisms.

[Note: This structure is proper and combinatorial.]

0.32.5 THEOREM There is a model structure on $\underline{\text{BISISET}}$ in which the weak equivalences are the bisimplicial maps $f:X \to Y$ such that $\forall$ m,

$$f_{*,m}:X_{*,m} \to Y_{*,m}$$

is a categorical weak equivalence and the cofibrations are the monomorphisms.

[Note: This structure is left proper and combinatorial.]

0.33 *MODEL LOCALIZATION*

Let $\underline{C}$ be a model category and let $\mathcal{C} \subset \text{Mor } \underline{C}$ be a class of morphisms.

0.33.1 DEFINITION A model localization of $\underline{C}$ at $\mathcal{C}$ is a pair $(\underline{L}_{\mathcal{C}}\underline{C}, L_{\mathcal{C}})$, where $\underline{L}_{\mathcal{C}}\underline{C}$ is a model category and $L_{\mathcal{C}}:\underline{C} \to \underline{L}_{\mathcal{C}}\underline{C}$ is a left model functor such that $\forall$ $f \in \mathcal{C}$, $\text{L}L_{\mathcal{C}}\text{L}_{\mathcal{W}}f$ is an isomorphism in $\underline{\text{HL}_{\mathcal{C}}\text{C}}$, $(\underline{L}_{\mathcal{C}}\underline{C}, L_{\mathcal{C}})$ being initial among all pairs having this property, i.e., for any model category $\underline{C}'$ and for any left model functor $F:\underline{C} \to \underline{C}'$ such that $\forall$ $f \in \mathcal{C}$, $\text{LFL}_{\mathcal{W}}f$ is an isomorphism in $\underline{\text{HC}}'$, there exists a unique left model functor $\bar{F}:\underline{L}_C\underline{C} \to \underline{C}'$ such that $F = \bar{F} \circ L_C$.

0.33.2 EXAMPLE Take $C = W$ and let $\underline{L}_C\underline{C} = \underline{C}$, $L_C = \text{id}_{\underline{C}}$ — then the pair $(\underline{C}, \text{id}_{\underline{C}})$ is a model localization of $\underline{C}$ at W.

Given $\underline{C}$ and C, the central question is the existence of the pair $(\underline{L}_C\underline{C}, L_C)$ (uniqueness up to isomorphism is clear) and for this it will be necessary to impose some conditions on $\underline{C}$ and C.

Assume:

- $\underline{C}$ is left proper and combinatorial.
- C is a set.

0.33.3 NOTATION Let W_C be the smallest class subject to:

(1) W_C contains W and C.

(2) W_C satisfies the 2 out of 3 condition (cf. 2.3.13).

(3) $W_C \cap \text{cof}$ is a stable class.

0.33.4 THEOREM Under the preceding hypotheses, $\underline{C}$ is a left proper combinatorial model category with weak equivalences W_C, cofibrations cof, fibrations $\text{RLP}(W_C \cap \text{cof})$.

[The proof hinges on 0.25.5, the key point being that $\underline{W_C} \subset \underline{C}(\to)$ is an accessible subcategory of $\underline{C}(\to)$.]

Write $\underline{L}_C\underline{C}$ for $\underline{C}$ equipped with the model structure per 0.33.4 and let $L_C = \text{id}_{\underline{C}}$.

0.33.5 THEOREM The pair $(\underline{L}_C\underline{C}, L_C)$ is a model localization of $\underline{C}$ at C.

[Let $F:\underline{C} \to \underline{C}'$ be a left model functor. Since $F = F \circ L_C$, it suffices to check that F is a left model functor when viewed as a functor from $\underline{L}_C\underline{C}$ to $\underline{C}'$. The fact that F preserves cofibrations is obvious, the fact the F preserves acyclic cofibrations being slightly less so.]

0.33.6 DEFINITION A <u>presentation</u> of a model category $\underline{C}$ is a small category $\underline{I}$, a set $S \subset \text{Mor}[\underline{I},\underline{\text{SISET}}]$, and a model equivalence

$$\underline{L}_S[\underline{I},\underline{\text{SISET}}] \text{ (Projective Structure)} \to \underline{C}.$$

[Note: Recall that

$$[\underline{I},\underline{\text{SISET}}] \text{ (Projective Structure)}$$

is a left proper combinatorial model category (cf. 0.26.6 and 0.26.13), so $L_S \ldots$ makes sense.]

0.33.7 THEOREM† Every combinatorial model category has a presentation.

0.33.8 NOTATION Given a small category $\underline{I}$, let $\underline{\text{PREI}} = [\underline{I}^{\text{OP}},\underline{\text{SET}}]$ $(= \hat{\underline{I}})$ and put

$$\underline{\text{SPREI}} = [\underline{I}^{\text{OP}},\underline{\text{SISET}}].$$

<u>N.B.</u> There is a canonical arrow

$$\underline{I} \xrightarrow{Y_{\underline{I}}} \underline{\text{PREI}} \xrightarrow{\text{si}_*} \underline{\text{SPREI}}$$

which will be denoted by $sY_{\underline{I}}$.

0.33.9 RAPPEL Let $\underline{C}$ be a cocomplete category — then for every $T \in \text{Ob}[\underline{I},\underline{C}]$

---

† Dugger, *Adv. Math.* <u>164</u> (2001), 177-201.

there exists $\Gamma_T \in \mathrm{Ob}[\hat{\underline{I}},\underline{C}]$ such that $T \approx \Gamma_T \circ Y_{\underline{I}}$.

0.33.10 LEMMA Suppose that $\underline{C}$ is a cocomplete model category and let $T:\underline{I} \to \underline{C}$ be a functor -- then there exists a functor $s\Gamma_T:\underline{\mathrm{SPRE I}} \to \underline{C}$ and a natural transformation

$$\Xi: s\Gamma_T \circ sY_{\underline{I}} \to T$$

such that $\forall\ i \in \mathrm{Ob}\ \underline{I}$,

$$\Xi_i:(s\Gamma_T \circ sY_{\underline{I}})_i \to T_i$$

is a weak equivalence.

0.34 *MIXING*

Let $\underline{C}$ be a finitely complete and finitely cocomplete category. Suppose that $\underline{C}$ carries two model structures

$$\left[\begin{array}{l} M_1:W_1,\mathrm{cof}_1,\mathrm{fib}_1 \\ \\ M_2:W_2,\mathrm{cof}_2,\mathrm{fib}_2. \end{array}\right.$$

0.34.1 THEOREM Assume

$$\left[\begin{array}{l} W_1 \subset W_2 \\ \\ \mathrm{fib}_1 \subset \mathrm{fib}_2. \end{array}\right.$$

Then

$$W_2,\mathrm{LLP}(W_2 \cap \mathrm{fib}_1),\mathrm{fib}_1$$

is a model structure on $\underline{C}$ which is left (right) proper if this is the case of M_2.

0.34.2 DEFINITION The model structure arising from 0.34.1 is said to be <u>mixed</u>.

0.34.3 EXAMPLE Take $\underline{C} = \underline{TOP}$ -- then $\underline{TOP}$ carries its Strøm structure and its Quillen structure. Since a homotopy equivalence is a weak homotopy equivalence and since a Hurewicz fibration is a Serre fibration, there is a mixed model structure on $\underline{TOP}$ whose weak equivalences are the weak homotopy equivalences and whose fibrations are the Hurewicz fibrations.

[Note: We shall refer to this model structure on $\underline{TOP}$ as the <u>Cole structure</u>. Consider the setup

$$\underline{TOP}\ (\text{Cole Structure}) \underset{\mathrm{id}_{\underline{TOP}}}{\overset{\mathrm{id}_{\underline{TOP}}}{\rightleftarrows}} \underline{TOP}\ (\text{Strøm Structure}).$$

Then $(\mathrm{id}_{\underline{TOP}}, \mathrm{id}_{\underline{TOP}})$ is a model pair.]

0.34.4 LEMMA X is cofibrant in the mixed model structure iff X is cofibrant in model structure M_1 and there exists an arrow $w_1: X' \to X$, where $w_1 \in \mathcal{W}_1$ and X' is cofibrant in model structure M_2.

0.34.5 EXAMPLE Consider the Cole structure on $\underline{TOP}$ -- then every cofibrant X is necessarily a CW space. In fact, for such an X, $\exists$ an arrow $w: X' \to X$, where w is a homotopy equivalence and X' is cofibrant in the Quillen structure. But X' is a CW space (cf. 0.2.1), hence the same holds for X.

0.35 *HOMOTOPY PULLBACKS*

Let $\underline{C}$ be a right proper model category -- then a commutative diagram

$$\begin{tikzcd} W \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in $\underline{C}$ is said to be a homotopy pullback if for some factorization $Y \xrightarrow{\sim} \bar{Y} \longrightarrow\!\!\!\!\to Z$ of g, the induced morphism $W \to X \times_Z \bar{Y}$ is a weak equivalence. This definition is essentially independent of the choice of the factorization of g since any two such factorizations

$$\left[\begin{array}{l} Y \xrightarrow{\sim} \bar{Y}' \longrightarrow\!\!\!\!\to Z \\ Y \xrightarrow{\sim} \bar{Y}'' \longrightarrow\!\!\!\!\to Z \end{array}\right.$$

lead to a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, equal] & X \times_Z \bar{Y}' \\ W \arrow[r] \arrow[d, equal] & \bullet \arrow[u, "\sim"'] \arrow[d, "\sim"] \\ W \arrow[r] & X \times_Z \bar{Y}'' \end{tikzcd}$$

and it does not matter whether one factors g or f.

[Note: The dual notion is homotopy pushout.]

0.35.1 LEMMA A pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback provided g is a fibration.

[Take $\bar{Y} = Y$ and factor g as $Y \xrightarrow{\mathrm{id}_Y} Y \xrightarrow{g} Z$.]

0.35.2 LEMMA A commutative diagram

$$\begin{tikzcd} W \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z, \end{tikzcd}$$

where f is a weak equivalence, is a homotopy pullback iff the arrow $W \xrightarrow{\eta} Y$ is a weak equivalence.

PROOF Factor g as $Y \xrightarrow{\bar{f}} \bar{Y} \xrightarrow{\bar{g}} Z$ and form the commutative diagram

$$\begin{tikzcd} W \arrow[rr, "\eta"] \arrow[d, equal] & & Y \arrow[d, "\bar{f}"] \\ W \arrow[r, "\rho"] \arrow[d, "\xi"'] & X \times_Z \bar{Y} \arrow[r, "\bar{\eta}"] \arrow[d, "\bar{\xi}"'] & \bar{Y} \arrow[d, "\bar{g}"] \\ X \arrow[r, equal] & X \arrow[r, "f"'] & Z, \end{tikzcd}$$

where ρ is the induced morphism and $\bar{\xi}, \bar{\eta}$ are the projections -- then the claim is that ρ is a weak equivalence iff η is a weak equivalence. Since $\underline{C}$ is right proper and $\bar{g}$ is a fibration, it follows that $\bar{\eta}$ is a weak equivalence. But $\bar{f} \circ \eta = \bar{\eta} \circ \rho$ and $\bar{f}$ is a weak equivalence. Therefore

$$\begin{bmatrix} \rho \text{ w.e.} \Rightarrow \bar{\eta} \circ \rho \text{ w.e.} \Rightarrow \bar{f} \circ \eta \text{ w.e.} \Rightarrow \eta \text{ w.e.} \\ \eta \text{ w.e.} \Rightarrow \bar{f} \circ \eta \text{ w.e.} \Rightarrow \bar{\eta} \circ \rho \text{ w.e.} \Rightarrow \rho \text{ w.e.} \end{bmatrix}$$

0.35.3 COMPOSITION LEMMA Consider the commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

in a right proper model category $\underline{C}$. Suppose that both the squares are homotopy pullbacks -- then the rectangle is a homotopy pullback. Conversely, if the rectangle and the second square are homotopy pullbacks, then the first square is a homotopy pullback.

0.35.4 LEMMA Suppose that $\underline{C}$ is a right proper model category. Let $Y \xrightarrow{g} Z$ be an arrow in $\underline{C}$ -- then the following conditions are equivalent.

(1) For every arrow $X \xrightarrow{f} Z$, the pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback.

(2) For every weak equivalence $X' \xrightarrow{u} X$ and for every arrow $X \xrightarrow{f} Z$, the arrow

$$X' \times_Z Y \xrightarrow{v} X \times_Z Y$$

in the commutative diagram

$$\begin{array}{ccccc} X' \times_Z Y & \xrightarrow{v} & X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow g \\ X' & \xrightarrow[u]{} & X & \xrightarrow[f]{} & Z \end{array}$$

is a weak equivalence.

PROOF

(1) => (2) The assumptions, in conjunction with 0.35.3, imply that the square

$$\begin{array}{ccc} X' \times_Z Y & \xrightarrow{v} & X \times_Z Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[u]{} & X \end{array}$$

is a homotopy pullback. Therefore v is a weak equivalence (cf. 0.35.2).

(2) => (1) Given an arrow $X \xrightarrow{f} Z$, factor it as $X \xrightarrow{\sim} \bar{X} \longrightarrow\!\!\!\!\to Z$ and consider the commutative diagram

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\sim} & \bar{X} \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow g \\ X & \xrightarrow{\sim} & \bar{X} & \longrightarrow\!\!\!\!\to & Z \end{array} .$$

Then the first square is a homotopy pullback (cf. 0.35.2), as is the second square (cf. 0.35.1). Therefore the rectangle is a homotopy pullback (cf. 0.35.3).

0.35.5 DEFINITION Let $\underline{C}$ be a model category -- then an arrow $Y \xrightarrow{g} Z$ in $\underline{C}$ is said to be a <u>homotopy fibration</u> if in any commutative diagram

$$\begin{array}{ccccc} X' \times_Z Y & \xrightarrow{v} & X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow g \\ X' & \xrightarrow[u]{} & X & \xrightarrow[f]{} & Z \end{array},$$

v is a weak equivalence whenever u is a weak equivalence.

N.B. If $\underline{C}$ is right proper, then every fibration is a homotopy fibration but, in general, there will be homotopy fibrations that are not fibrations.

0.35.6 EXAMPLE Take $\underline{C}$ = $\underline{\text{TOP}}$ (Strøm Structure) — then fibration = Hurewicz fibration. On the other hand, the pullback square

$$\begin{array}{ccc} P & \xrightarrow{\eta} & Y \\ \xi \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a homotopy pullback provided g is a Dold fibration.

[Note: Recall that Hurewicz => Dold but Dold ≠> Hurewicz.]

0.35.7 EXAMPLE Take $\underline{C}$ = $\underline{\text{SISET}}$ (Kan Structure) — then fibration = Kan fibration and the fibrant objects are the Kan complexes. Still, for every simplicial set Y, the arrow $Y \to *$ is a homotopy fibration.

0.35.8 LEMMA The class of homotopy fibrations is closed under composition and the formation of retracts and is pullback stable.

CHAPTER X: ANALYSIS IN CAT

A: FIBERED CATEGORIES

A: FIBERED CATEGORIES

A.1 *GROTHENDIECK FIBRATIONS*

Let $\underline{C}$ and $\underline{D}$ be categories and let $F:\underline{C} \to \underline{D}$ be a functor.

A.1.1 DEFINITION Given $Y \in \text{Ob } \underline{D}$, the <u>fiber</u> $\underline{C}_Y$ of F over Y is the subcategory of $\underline{C}$ whose objects are the $X \in \text{Ob } \underline{C}$ such that $FX = Y$ and whose morphisms are the arrows $f \in \text{Mor } \underline{C}$ such that $Ff = \text{id}_Y$.

[Note: In general, $\underline{C}_Y$ is not full and it may very well be the case that Y and Y' are isomorphic, yet $\underline{C}_Y = \underline{0}$ and $\underline{C}_{Y'} \neq \underline{0}$ (cf. A.1.20).]

<u>N.B.</u> There is a pullback square

$$\begin{tikzcd}
\underline{C}_Y \arrow[r] \arrow[d] & \underline{C} \arrow[d, "F"] \\
\underline{1} \arrow[r, "K_Y"'] & \underline{D}.
\end{tikzcd}$$

A.1.2 NOTATION Given $X,X' \in \text{Ob } \underline{C}_Y$, let $\text{Mor}_Y(X,X')$ stand for the set of morphisms $X \to X'$ in $\underline{C}_Y$.

A.1.3 DEFINITION Let $X,X' \in \text{Ob } \underline{C}$ and let $u \in \text{Mor}(X,X')$ — then u is <u>pre-horizontal</u> if $\forall$ morphism $w:X_0 \to X'$ of $\underline{C}$ such that $Fw = Fu$, there exists a unique morphism $v \in \text{Mor}_{FX}(X_0,X)$ such that $u \circ v = w$:

$$\begin{tikzcd} X \arrow[r, "u"] & X' \\ X_0 \arrow[u, "v"] \arrow[r, equal] & X_0 \arrow[u, "w"'] \end{tikzcd}.$$

[Note: Let

$$\mathrm{Mor}_u(X_0,X') = \{w \in \mathrm{Mor}(X_0,X'):Fw = Fu\}.$$

Then there is an arrow

$$\mathrm{Mor}_{FX}(X_0,X) \to \mathrm{Mor}_u(X_0,X'),$$

viz. $v \to u \circ v$ (in fact, $F(u \circ v) = Fu \circ Fv = Fu \circ \mathrm{id}_{FX} = Fu$) and the condition that u be prehorizontal is that $\forall\ X_0 \in \underline{C}_{FX}$, this arrow is bijective.]

A.1.4 DEFINITION Let $X,X' \in \mathrm{Ob}\ \underline{C}$ and let $u \in \mathrm{Mor}(X,X')$ — then u is <u>preophorizontal</u> if $\forall$ morphism $w:X \to X_0$ of $\underline{C}$ such that $Fw = Fu$, there exists a unique morphism $v \in \mathrm{Mor}_{FX'}(X',X_0)$ such that $v \circ u = w$:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "w"'] & X' \arrow[d, "v"] \\ X_0 \arrow[r, equal] & X_0 \end{tikzcd}.$$

[Note: Let

$$\mathrm{Mor}_u(X,X_0) = \{w \in \mathrm{Mor}(X,X_0):Fw = Fu\}.$$

Then there is an arrow

$$\mathrm{Mor}_{FX'}(X',X_0) \to \mathrm{Mor}_u(X,X_0),$$

viz. $v \to v \circ u$ (in fact, $F(v \circ u) = Fv \circ Fu = \mathrm{id}_{FX'} \circ Fu = Fu$) and the condition that u be preophorizontal is that $\forall\ X_0 \in \underline{C}_{FX'}$, this arrow is bijective.]

A.1.5 LEMMA The isomorphisms in $\underline{C}$ are prehorizontal (preophorizontal).

A.1.6 REMARK The composite of two prehorizontal (preophorizontal) morphisms need not be prehorizontal (preophorizontal).

A.1.7 DEFINITION The functor $F:\underline{C} \to \underline{D}$ is a <u>Grothendieck prefibration</u> if for any object $X' \in \text{Ob}\ \underline{C}$ and any morphism $g:Y \to FX'$, there exists a prehorizontal morphism $u:X \to X'$ such that $Fu = g$.

A.1.8 DEFINITION The functor $F:\underline{C} \to \underline{D}$ is a <u>Grothendieck preopfibration</u> if for any object $X \in \text{Ob}\ \underline{C}$ and any morphism $g:FX \to Y$, there exists a preophorizontal morphism $u:X \to X'$ such that $Fu = g$.

A.1.9 LEMMA The functor $F:\underline{C} \to \underline{D}$ is a Grothendieck prefibration iff $\forall\ Y \in \text{Ob}\ \underline{D}$, the canonical functor

$$\underline{C}_Y \to Y\backslash\underline{C} \quad (X \to (\text{id}_Y, X))$$

has a right adjoint.

A.1.10 LEMMA The functor $F:\underline{C} \to \underline{D}$ is a Grothendieck preopfibration iff $\forall\ Y \in \text{Ob}\ \underline{D}$, the canonical functor

$$\underline{C}_Y \to \underline{C}/Y \quad (X \to (X, \text{id}_Y))$$

has a left adjoint.

A.1.11 DEFINITION Let $X, X' \in \text{Ob}\ \underline{C}$ and let $u \in \text{Mor}(X, X')$ — then u is <u>horizontal</u>

if $\forall$ morphism $w:X_0 \to X'$ of $\underline{C}$ and $\forall$ factorization

$$Fw = Fu \circ x \quad (x \in \mathrm{Mor}(FX_0,FX)),$$

there exists a unique morphism $v:X_0 \to X$ such that $Fv = x$ and $u \circ v = w$.
Schematically:

$$\begin{tikzcd} X_0 \arrow[r, dashed, "v"'] \arrow[rr, bend left, "w"] & X \arrow[r, "u"'] & X' \end{tikzcd}, \quad \begin{tikzcd} FX_0 \arrow[r, "x"'] \arrow[rr, bend left, "Fw"] & FX \arrow[r, "Fu"'] & FX' \end{tikzcd}.$$

<u>N.B.</u> If u is horizontal, then u is prehorizontal. Proof: For $Fw = Fu \Rightarrow FX_0 = FX$, so we can take $x = \mathrm{id}_{FX}$, hence $Fv = \mathrm{id}_{FX} \Rightarrow v \in \mathrm{Mor}_{FX}(X_0,X)$.

A.1.12 DEFINITION Let $X,X' \in \mathrm{Ob}\ \underline{C}$ and let $u \in \mathrm{Mor}(X,X')$ -- then u is <u>ophorizontal</u> if $\forall$ morphism $w:X \to X_0$ of $\underline{C}$ and $\forall$ factorization

$$Fw = x \circ Fu \quad (x \in \mathrm{Mor}(FX',FX_0)),$$

there exists a unique morphism $v:X' \to X_0$ such that $Fv = x$ and $v \circ u = w$.
Schematically:

$$\begin{tikzcd} X \arrow[r, "u"'] \arrow[rr, bend left, "w"] & X' \arrow[r, dashed, "v"'] & X_0 \end{tikzcd}, \quad \begin{tikzcd} FX \arrow[r, "Fu"'] \arrow[rr, bend left, "Fw"] & FX' \arrow[r, "x"'] & FX_0 \end{tikzcd}.$$

<u>N.B.</u> If u is ophorizontal, then u is preophorizontal. Proof: For $Fw = Fu \Rightarrow FX_0 = FX'$, so we can take $x = \mathrm{id}_{FX'}$, hence $Fv = \mathrm{id}_{FX'} \Rightarrow v \in \mathrm{Mor}_{FX'}(X',X_0)$.

A.1.13 DEFINITION The functor $F:\underline{C} \to \underline{D}$ is a <u>Grothendieck fibration</u> if for any object $X' \in \mathrm{Ob}\ \underline{C}$ and any morphism $g:Y \to FX'$, there exists a horizontal morphism $u:X \to X'$ such that $Fu = g$.

N.B. If $\tilde{u}:\tilde{X} \to X'$ is another horizontal morphism such that $F\tilde{u} = g$, then $\exists$ a unique isomorphism $f \in \operatorname{Mor} \underline{C}_Y$ such that $\tilde{u} = u \circ f$.

[We have

$$\overline{\tilde{X} \overset{v}{\dashrightarrow} X \xrightarrow{u} X'}^{\tilde{u}}, \quad \overline{F\tilde{X} \xrightarrow{id_Y} FX \xrightarrow{Fu} FX'}^{F\tilde{u}}$$

$$\overline{X \overset{\tilde{v}}{\dashrightarrow} \tilde{X} \xrightarrow{\tilde{u}} X'}^{u}, \quad \overline{FX \xrightarrow{id_Y} F\tilde{X} \xrightarrow{F\tilde{u}} FX'}^{Fu}.$$

Here

$$\left[\begin{array}{l} Fv = id_Y \ \& \ u \circ v = \tilde{u} \\ \\ F\tilde{v} = id_Y \ \& \ \tilde{u} \circ \tilde{v} = u. \end{array}\right.$$

Therefore

$$\left[\begin{array}{l} \tilde{u} \circ \tilde{v} \circ v = u \circ v = \tilde{u} \\ \\ u \circ v \circ \tilde{v} = \tilde{u} \circ \tilde{v} = u, \end{array}\right.$$

so

$$\left[\begin{array}{l} \tilde{v} \circ v = id_{\tilde{X}} \\ \\ v \circ \tilde{v} = id_X.] \end{array}\right.$$

A.1.14 DEFINITION The functor $F:\underline{C} \to \underline{D}$ is a Grothendieck opfibration if for any object $X \in \operatorname{Ob} \underline{C}$ and any morphism $g:FX \to Y$, there exists an ophorizontal morphism $u:X \to X'$ such that $Fu = g$.

N.B. If $\tilde{u}:X \to \tilde{X}'$ is another ophorizontal morphism such that $F\tilde{u} = g$, then $\exists$

a unique isomorphism $f \in \text{Mor}\, \underline{C}_Y$ such that $\tilde{u} = f \circ u$ (cf. supra).

A.1.15 LEMMA The functor $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration iff the functor $F^{OP}:\underline{C}^{OP} \to \underline{D}^{OP}$ is a Grothendieck opfibration.

A.1.16 EXAMPLE The functor $p_{\underline{C}}:\underline{C} \to \underline{1}$ is a Grothendieck fibration.

[Note: The functor $\underline{0} \to \underline{C}$ is a Grothendieck fibration (all requirements are satisfied vacuously).]

A.1.17 EXAMPLE The codomain functor

$$\text{cod}:[\underline{2},\underline{C}] \ (\approx \underline{C}(\to)) \to \underline{C}$$

is a Grothendieck fibration provided $\underline{C}$ has pullbacks.

[Note: The fiber $[\underline{2},\underline{C}]_X$ of cod over $X \in \text{Ob}\, \underline{C}$ can be identified with $\underline{C}/X$.]

A.1.18 EXAMPLE Given groups $\begin{bmatrix} G \\ H \end{bmatrix}$, denote by $\begin{bmatrix} \underline{G} \\ \underline{H} \end{bmatrix}$ the groupoids having a single object $*$ with $\begin{bmatrix} \text{Mor}_{\underline{G}}(*,*) = G \\ \\ \text{Mor}_{\underline{H}}(*,*) = H \end{bmatrix}$ — then a group homomorphism $\phi:G \to H$ can be regarded as a functor $\underline{\phi}:\underline{G} \to \underline{H}$ and, as such, $\underline{\phi}$ is a Grothendieck fibration iff ϕ is surjective.

[Note: The fiber $\underline{G}_*$ of $\underline{\phi}$ over $*$ "is" Ker ϕ.]

A.1.19 EXAMPLE Let $U:\underline{TOP} \to \underline{SET}$ be the forgetful functor -- then U is a Grothendieck fibration. To see this, consider a morphism $g:Y \to UX'$, where Y is a set and X' is a topological space. Denote by X the topological space that arises by equipping Y with the initial topology per g (i.e., with the smallest topology such that g is continuous when viewed as a function from Y to X') -- then for any topological space X_0, a function $X_0 \to X$ is continuous iff the composition $X_0 \to X \to X'$ is continuous, from which it follows that the arrow $X \to X'$ is horizontal.

[Note: The fiber $\underline{\mathrm{TOP}}_Y$ of U over Y is the partially ordered set of topologies on Y thought of as a category.]

A.1.20 REMARK Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration. Let $Y,Y' \in \mathrm{Ob}\,\underline{D}$ and let $\psi:Y \to Y'$ be an isomorphism -- then $\underline{C}_{Y'} = \underline{0} \Rightarrow \underline{C}_Y = \underline{0}$.

[To get a contradiction, assume $\exists\ X \in \mathrm{Ob}\,\underline{C}:FX = Y$. Since $\psi^{-1}:Y' \to Y = FX$, $\exists$ a horizontal $u':X' \to X$ such that $Fu' = \psi^{-1}$, hence $FX' = Y'$.]

A.1.21 LEMMA The isomorphisms in $\underline{C}$ are horizontal (ophorizontal).

A.1.22 LEMMA Let $u \in \mathrm{Mor}(X,X')$, $u' \in \mathrm{Mor}(X',X'')$. Assume: u' is horizontal -- then $u' \circ u$ is horizontal iff u is horizontal.

[Note: Therefore the class of horizontal morphisms is closed under composition (cf. A.1.6).]

A.1.23 LEMMA Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration. Let $u \in \mathrm{Mor}(X,X')$ be horizontal. Assume: Fu is an isomorphism -- then u is an isomorphism.

PROOF In the definition of horizontal, take $X_0 = X'$, $w = \mathrm{id}_{X'}$, and consider the factorization

$$Fw = \mathrm{id}_{FX'} = Fu \circ (Fu)^{-1} \quad (x = (Fu)^{-1}).$$

Choose $v:X' \to X$ accordingly, thus $u \circ v = \mathrm{id}_{X'}$, so v is a right inverse for u. But thanks to A.1.21 and A.1.22, v is horizontal. Since $Fv = (Fu)^{-1}$, the argument can be repeated to get a right inverse for v. Therefore u is an isomorphism.

A.1.24 RAPPEL Consider $\underline{\mathrm{CAT}}$ (Internal Structure) (cf. 0.12) -- then by definition, a functor $F:\underline{C} \to \underline{D}$ is a fibration if $\forall$ $X \in \mathrm{Ob}\,\underline{C}$ and $\forall$ isomorphism $\psi:FX \to Y$ in $\underline{D}$, $\exists$ an isomorphism $\phi:X \to X'$ in $\underline{C}$ such that $F\phi = \psi$. Equivalently, a functor $F:\underline{C} \to \underline{D}$ is a fibration iff $\forall$ $X' \in \mathrm{Ob}\,\underline{C}$ and $\forall$ isomorphism $\psi:Y \to FX'$ in $\underline{D}$, $\exists$ an isomorphism $\phi:X \to X'$ in $\underline{C}$ such that $F\phi = \psi$.

[Note: In this connection, observe that F is a fibration iff F^{OP} is a fibration.]

A.1.25 THEOREM Let $\underline{C}$ and $\underline{D}$ be small categories -- then a Grothendieck fibration $F:\underline{C} \to \underline{D}$ is a fibration in $\underline{\mathrm{CAT}}$ (Internal Structure).

PROOF Let $\psi:Y \to FX'$ be an isomorphism in $\underline{D}$ -- then there exists a horizontal morphism $\phi:X \to X'$ such that $F\phi = \psi$. But, in view of A.1.23, ϕ is necessarily an isomorphism in $\underline{C}$.

[Note: The same conclusion obtains if instead $F:\underline{C} \to \underline{D}$ is a Grothendieck opfirbration.]

Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration.

A.1.26 LEMMA Consider any object $X' \in \mathrm{Ob}\,\underline{C}$ and any morphism $g:Y \to FX'$. Suppose that $\tilde{u}:\tilde{X} \to X'$ is prehorizontal and $F\tilde{u} = g$ -- then $\tilde{u}$ is horizontal.

PROOF Choose a horizontal $u:X \to X'$ such that $Fu = g$ -- then u is prehorizontal so $\exists$ a unique isomorphism $f \in \text{Mor}\ \underline{C}_Y$ such that $\tilde{u} = u \circ f$. Therefore $\tilde{u}$ is horizontal (cf. A.1.21 and A.1.22).

A.1.27 THEOREM Let $F:\underline{C} \to \underline{D}$ be a functor -- then F is a Grothendieck fibration iff

1. $\forall\ X' \in \text{Ob}\ \underline{C}$ and $\forall\ g \in \text{Mor}(Y,FX')$, $\exists$ a prehorizontal $\tilde{u} \in \text{Mor}(\tilde{X},X'):F\tilde{u} = g$;

2. The composition of two prehorizontal morphisms is prehorizontal.

PROOF The conditions are clearly necessary (for point 2, cf. A.1.26 and recall A.1.22). Turning to the sufficiency, one has only to prove that the $\tilde{u}$ of point 1 is actually horizontal. Consider a morphism $w:X_0 \to X'$ of $\underline{C}$ and a factorization

$$Fw = F\tilde{u} \circ x \quad (x \in \text{Mor}(FX_0,F\tilde{X})).$$

Then there is a prehorizontal $\tilde{u}_0 \in \text{Mor}(\tilde{X}_0,\tilde{X}):F\tilde{u}_0 = x$ ($\Rightarrow F\tilde{X}_0 = FX_0$). Here

$$\tilde{X}_0 \xrightarrow{\tilde{u}_0} \tilde{X} \xrightarrow{\tilde{u}} X'$$

and

$$F(\tilde{u} \circ \tilde{u}_0) = F\tilde{u} \circ F\tilde{u}_0 = F\tilde{u} \circ x = Fw.$$

But $\tilde{u} \circ \tilde{u}_0$ is prehorizontal, thus there exists a unique morphism $\tilde{v}_0 \in \text{Mor}_{F\tilde{X}_0}(X_0,\tilde{X}_0)$ such that $\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$:

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r, "\tilde{u} \circ \tilde{u}_0"] & X' \\
X_0 \arrow[u, "\tilde{v}_0"] \arrow[r, equal] & X_0. \arrow[u, "w"']
\end{tikzcd}$$

Put $v = \tilde{u}_0 \circ \tilde{v}_0$ — then $Fv = F\tilde{u}_0 \circ F\tilde{v}_0 = F\tilde{u}_0 \circ \mathrm{id}_{F\tilde{X}_0} = F\tilde{u}_0 = x$ and $\tilde{u} \circ v =$

$\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$. To establish that v is unique, let $v':X_0 \to \tilde{X}$ be another morphism with $Fv' = x$ and $\tilde{u} \circ v' = w$. Since $\tilde{u}_0$ is prehorizontal and since $Fv' = x = F\tilde{u}_0$, the diagram

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r, "\tilde{u}_0"] & \tilde{X} \\
X_0 \arrow[u, dotted, "v''"] \arrow[r, equal] & X_0 \arrow[u, "v'"']
\end{tikzcd}$$

admits a unique filler $v'' \in \mathrm{Mor}_{F\tilde{X}_0}(X_0,\tilde{X}_0):u_0 \circ v'' = v'$. Finally

$$\tilde{u} \circ \tilde{u}_0 \circ v'' = \tilde{u} \circ v' = w$$

$$\Rightarrow v'' = \tilde{v}_0 \Rightarrow v = \tilde{u}_0 \circ \tilde{v}_0 = \tilde{u}_0 \circ v'' = v'.$$

A.1.28 THEOREM Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration. Let

$$\left[\begin{array}{l} L = \text{the morphisms rendered invertible by } F \\ R = \text{the horizontal morphisms.} \end{array}\right.$$

Then the pair (L,R) is a w.f.s. on $\underline{C}$.

A.1.29 EXAMPLE Assume that $\underline{C}$ has pullbacks and work with $\mathrm{cod}:\underline{C}(\to) \to \underline{C}$ (cf. A.1.17). Consider a morphism $(\phi,\psi):(X,f,Y) \to (X',f',Y')$ in $\underline{C}(\to)$, so

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "\psi"'] & Y'
\end{tikzcd}$$

commutes -- then (ϕ,ψ) is horizontal iff this square is a pullback square. Therefore the category $\underline{C}(\rightarrow)$ admits a w.f.s. (L,R) in which R is the class of pullback squares. On the other hand, $(\phi,\psi) \in L$ iff ψ is invertible.

Fix a category $\underline{D}$ -- then by $\underline{\mathbf{FIB}}(\underline{D})$ we shall understand the metacategory whose objects are the pairs $(\underline{C},F)$, where $F:\underline{C} \rightarrow \underline{D}$ is a Grothendieck fibration, and whose morphisms $\Phi:(\underline{C},F) \rightarrow (\underline{C}',F')$ are the functors $\Phi:\underline{C} \rightarrow \underline{C}'$ that send horizontal arrows to horizontal arrows subject to $F' \circ \Phi = F$.

[Note: Such a Φ is called a <u>fibered functor</u> from $\underline{C}$ to $\underline{C}'$.]

<u>N.B.</u> $\forall\ Y \in \text{Ob}\ \underline{D}$, Φ restricts to a functor $\Phi_Y:C_Y \rightarrow C_Y'$.

A.1.30 EXAMPLE Take $\underline{D} = \underline{1}$ -- then $\underline{\mathbf{FIB}}(\underline{1})$ is $\mathbf{CAT}$.

A.1.31 DEFINITION Suppose that $F:\underline{C} \rightarrow \underline{D}$ and $F':\underline{C}' \rightarrow \underline{D}$ are Grothendieck fibrations -- then a fibered functor $\Phi:\underline{C} \rightarrow \underline{C}'$ is said to be an <u>equivalence</u> if there exists a fibered functor $\Phi':\underline{C}' \rightarrow \underline{C}$ and natural isomorphisms

$$\begin{bmatrix} \Phi' \circ \Phi \rightarrow \text{id}_{\underline{C}} \\ \\ \Phi \circ \Phi' \rightarrow \text{id}_{\underline{C}'} \end{bmatrix}.$$

A.1.32 LEMMA The fibered functor $\Phi:\underline{C} \rightarrow \underline{C}'$ is an equivalence iff $\forall\ Y \in \text{Ob}\ \underline{D}$, the functor $\Phi_Y:\underline{C}_Y \rightarrow \underline{C}_Y'$ is an equivalence of categories.

Because of A.1.15, in so far as the theory is concerned, it suffices to deal with Grothendieck fibrations. Still, Grothendieck opfibrations are pervasive (cf. B.2.6). Here is a specific instance.

A.1.33 EXAMPLE Let $\underline{C}$ be a category -- then the <u>twisted arrow category</u> $\underline{C}(\sim>)$ of $\underline{C}$ is the category whose objects are the arrows $f:X \to Y$ of $\underline{C}$ and whose morphisms $f \to f'$ are the pairs (ϕ,ψ): $\begin{cases} \phi \in \mathrm{Mor}(X',X) \\ \psi \in \mathrm{Mor}(Y,Y') \end{cases}$ for which the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\uparrow & & \downarrow\psi \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

commutes, thus

$$\mathrm{id}_f = (\mathrm{id}_X,\mathrm{id}_Y) \quad , \ (\phi',\psi') \circ (\phi,\psi) = (\phi \circ \phi',\psi' \circ \psi).$$

Denote by $\begin{cases} s_{\underline{C}} \\ t_{\underline{C}} \end{cases}$ the canonical projections

$$\begin{cases} \underline{C}(\sim>) \to \underline{C}^{\mathrm{OP}} \\ \underline{C}(\sim>) \to \underline{C}, \end{cases}$$

hence

$$\begin{cases} s_{\underline{C}}f = \mathrm{dom}\ f & s_{\underline{C}}(\phi,\psi) = \phi \\ t_{\underline{C}}f = \mathrm{cod}\ f, & t_{\underline{C}}(\phi,\psi) = \psi, \end{cases}$$

and $\begin{cases} s_{\underline{C}} \\ t_{\underline{C}} \end{cases}$ are Grothendieck opfibrations.

[Note: The functor

$$A:\underline{C}(\sim>) \rightarrow \underline{C}^{OP}(\sim>)$$

that sends f to f and (ϕ,ψ) to (ψ,ϕ) is an isomorphism of categories and

$$\begin{cases} s_{\underline{C}^{OP}} \circ A = t_{\underline{C}} \\ t_{\underline{C}^{OP}} \circ A = s_{\underline{C}}.] \end{cases}$$

<u>N.B.</u> If $F:\underline{C} \rightarrow \underline{D}$ is a functor, then the prescription

$$\begin{cases} f \rightarrow Ff \\ (\phi,\psi) \rightarrow (F\phi,F\psi) \end{cases}$$

defines a functor rendering the diagram

$$\begin{tikzcd} \underline{C}^{OP} \arrow[d, "F^{OP}"'] & \underline{C}(\sim>) \arrow[l, "s_{\underline{C}}"'] \arrow[r, "t_{\underline{C}}"] \arrow[d] & \underline{C} \arrow[d, "F"] \\ \underline{D}^{OP} & \underline{D}(\sim>) \arrow[l, "s_{\underline{D}}"] \arrow[r, "t_{\underline{D}}"'] & \underline{D} \end{tikzcd}$$

commutative.

A.1.34 REMARK To relativise the preceding setup, let $\underline{C},\underline{D}$ be categories and let $F:\underline{C} \rightarrow \underline{D}$ be a functor -- then $\underline{F}(\sim>)$ is the category whose objects are the triples (X,f,Y), where $X \in \text{Ob } \underline{C}$, $Y \in \text{Ob } \underline{D}$, $f:Y \rightarrow FX$, and whose morphisms $(X,f,Y) \rightarrow (X',f',Y')$ are the pairs (ϕ,ψ): $\begin{cases} \phi \in \text{Mor}(X,X') \\ \psi \in \text{Mor}(Y',Y) \end{cases}$ for which the square

$$\begin{array}{ccc} Y & \xrightarrow{f} & FX \\ {\scriptstyle\psi}\uparrow & & \downarrow{\scriptstyle F\phi} \\ Y' & \xrightarrow[f']{} & FX' \end{array}$$

commutes, thus

$$\mathrm{id}_{(X,f,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)\ ,\ (\phi',\psi') \circ (\phi,\psi) = (\phi' \circ \phi, \psi \circ \psi').$$

Denote by $\left[\begin{array}{l} s_F \\ t_F \end{array}\right.$ the canonical projections

$$\left[\begin{array}{l} \underline{F}(\sim>) \to \underline{D}^{OP} \\ \underline{F}(\sim>) \to \underline{C}, \end{array}\right.$$

hence

$$\left[\begin{array}{l} s_F(X,f,Y) = Y \\ t_F(X,f,Y) = X, \end{array}\right. \qquad \left[\begin{array}{l} s_F(\phi,\psi) = \psi \\ t_F(\phi,\psi) = \phi, \end{array}\right.$$

and $\left[\begin{array}{l} s_F \\ t_F \end{array}\right.$ are Grothendieck opfibrations.

[Note: Take $\underline{C} = \underline{D}$, $F = \mathrm{id}_{\underline{C}}$, and switch the labeling of the data to get $\underline{\mathrm{id}}_{\underline{C}}(\sim>) = \underline{C}(\sim>)$.]

A.2 CLOSURE PROPERTIES

A.2.1 LEMMA If $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are Grothendieck fibrations, then so is their composition $G \circ F:\underline{C} \to \underline{E}$.

A.2.2 LEMMA The projection functor

$$\underline{C} \times \underline{D} \to \underline{D}$$

is a Grothendieck fibration.

A.2.3 LEMMA If $F:\underline{C} \to \underline{D}$ and $F':\underline{C}' \to \underline{D}'$ are Grothendieck fibrations, then the product functor

$$F \times F':\underline{C} \times \underline{C}' \to \underline{D} \times \underline{D}'$$

is a Grothendieck fibration.

A.2.4 LEMMA If

$$\begin{tikzcd}
\underline{C}' \arrow[r] \arrow[d, "F'"'] & \underline{C} \arrow[d, "F"] \\
\underline{D}' \arrow[r] & \underline{D}
\end{tikzcd}$$

is a pullback square in $\mathbf{CAT}$, then

$$F \text{ a Grothendieck fibration} \Rightarrow F' \text{ a Grothendieck fibration.}$$

A.2.5 EXAMPLE Let $\underline{A}$ be a category, $\alpha:\underline{A} \to \underline{C}$ a functor -- then there is a pullback square

$$\begin{tikzcd}
|\mathrm{id}_{\underline{C}},\alpha| \arrow[r] \arrow[d, "g\ell\ \alpha"'] & \underline{C}(\to) \arrow[d, "\mathrm{cod}"] \\
\underline{A} \arrow[r, "\alpha"'] & \underline{C}
\end{tikzcd}$$

and $g\ell\ \alpha$ is a Grothendieck fibration.

A.2.6 LEMMA Let $F:\underline{C} \to \underline{D}$ be a Grothendieck fibration and let $\underline{I}$ be a small category -- then

$$F_*:[\underline{I},\underline{C}] \to [\underline{I},\underline{D}]$$

is a Grothendieck fibration.

A.2.7 EXAMPLE Define $\langle \underline{I},\underline{C} \rangle$ by the pullback square

$$\begin{tikzcd}
\langle \underline{I},\underline{C} \rangle \arrow[r] \arrow[d] & {[\underline{I},\underline{C}]} \arrow[d, "F_*"] \\
\underline{D} \arrow[r, "K"'] & {[\underline{I},\underline{D}]}.
\end{tikzcd}$$

Then the arrow $\langle \underline{I},\underline{C} \rangle \to \underline{D}$ is a Grothendieck fibration.

[Note: Let $Y \in \text{Ob } \underline{D}$ -- then the objects of the fiber $\langle \underline{I},\underline{C} \rangle_Y$ are those functors $\Delta:\underline{I} \to \underline{C}$ such that $F_*\Delta = KY$ (the constant diagram functor at Y).]

A.3 *CATEGORIES FIBERED IN GROUPOIDS*

A.3.1 DEFINITION Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration -- then $\underline{C}$ is <u>fibered in groupoids by F</u> if $\forall\, Y \in \text{Ob } \underline{D}$, $\underline{C}_Y$ is a groupoid.

A.3.2 RAPPEL Let G be a topological group, X a topological space. Suppose that X is a free right G-space: $\begin{bmatrix} X \times G \to X \\ (x,g) \to x \cdot g \end{bmatrix}$ -- then X is said to be <u>principal</u> provided that the continuous bijection $\theta:X \times G \to X \times_{X/G} X$ defined by $(x,g) \to (x,x \cdot g)$ is a homeomorphism.

Let G be a topological group -- then an X in <u>TOP</u>/B is said to be a <u>principal G-space over B</u> if X is a principal G-space, B is a trivial G-space, the projection $X \to B$ is open, surjective, and equivariant, and G operates transitively on the fibers. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d] \\ X/G \arrow[r] & B \end{tikzcd}$$

and the arrow $X/G \to B$ is a homeomorphism.

A.3.3 NOTATION Let

$$\underline{\mathrm{PRIN}}_{B,G}$$

be the category whose objects are the principal G-spaces over B and whose morphisms are the equivariant continuous functions over B, thus

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\ B \arrow[r, equal] & B, \end{tikzcd}$$

with ϕ equivariant.

A.3.4 FACT Every morphism in $\underline{\mathrm{PRIN}}_{B,G}$ is an isomorphism.

A.3.5 EXAMPLE Let G be a topological group -- then the <u>classifying stack</u> of G is the category $\underline{\mathrm{PRIN}}(G)$ whose objects are the principal G-spaces $X \to B$ and whose morphisms $(\phi,f):(X \to B) \to (X' \to B')$ are the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\ B \arrow[r, "f"'] & B', \end{tikzcd}$$

where ϕ is equivariant. Define now a functor $F:\underline{\mathrm{PRIN}}(G) \to \underline{\mathrm{TOP}}$ by $F(X \to B) = B$ and $F(\phi,f) = f$ — then F is a Grothendieck fibration. Moreover, $\underline{\mathrm{PRIN}}(G)$ is fibered in groupoids by F:

$$\underline{\mathrm{PRIN}}(G)_B = \underline{\mathrm{PRIN}}_{B,G},$$

which is a groupoid by A.3.4.

A.3.6 LEMMA If $\underline{C}$ is fibered in groupoids by F, then every morphism in $\underline{C}$ is horizontal.

PROOF Let $f \in \mathrm{Mor}(X,X')$ $(X,X' \in \mathrm{Ob}\,\underline{C})$, thus $Ff:FX \to FX'$, so one can find a horizontal $u_0:X_0 \to X'$ such that $Fu_0 = Ff$. But u_0 is necessarily prehorizontal, hence there exists a unique morphism $v \in \mathrm{Mor}_{FX_0}(X,X_0)$ such that $u \circ v = f$:

$$\begin{tikzcd} X_0 \arrow[r, "u"] & X' \\ X \arrow[u, "v"] \arrow[r, equal] & X. \arrow[u, "f"'] \end{tikzcd}$$

Since u is horizontal and v is an isomorphism, it follows that f is horizontal (cf. A.1.21 and A.1.22).

N.B. Suppose that

$\underline{C}$ is fibered in groupoids by F

$\underline{C}'$ is fibered in groupoids by F'.

Then every functor $\Phi:\underline{C} \to \underline{C}'$ such that $F' \circ \Phi = F$ is automatically a fibered functor from $\underline{C}$ to $\underline{C}'$.

A.3.7 LEMMA Let $F:\underline{C} \to \underline{D}$ be a functor. Assume: Every arrow in $\underline{C}$ is horizontal and for any morphism $g:Y \to FX'$, there exists a morphism $u:X \to X'$ such that $Fu = g$ — then F is a Grothendieck fibration and $\underline{C}$ is fibered in groupoids by F.

PROOF The conditions obviously imply that F is a Grothendieck fibration. Consider now an arrow $f:X \to X'$ of $\underline{C}_Y$ for some $Y \in \mathrm{Ob}\,\underline{D}$ — then f is horizontal, so there exists a unique morphism $v \in \mathrm{Mor}_Y(X',X)$ $(FX = Y = FX')$ such that $f \circ v = \mathrm{id}_{X'}$:

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ X' \arrow[u, "v"] \arrow[r, equal] & X' \arrow[u, "\mathrm{id}_{X'}"'] \end{tikzcd} \qquad \text{(cf. A.1.3).}$$

Therefore every arrow in $\underline{C}_Y$ has a right inverse. But this means in particular that v must have a right inverse, thus f is invertible.

Let $F:\underline{C} \to \underline{D}$ be a Grothendieck fibration. Denote by $\underline{C}_{hor}$ the subcategory of $\underline{C}$ whose objects are the objects of $\underline{C}$ and whose morphisms are the horizontal arrows of $\underline{C}$. Put

$$F_{hor} = F|\underline{C}_{hor}.$$

A.3.8 LEMMA $F_{hor}:\underline{C}_{hor} \to \underline{D}$ is a Grothendieck fibration and $\underline{C}_{hor}$ is fibered in groupoids by F_{hor}.

A.3.9 RAPPEL A category is said to be <u>discrete</u> if all its morphisms are identities.

[Note: Functors between discrete categories correspond to functions on their underlying classes.]

A.3.10 EXAMPLE Every class is a discrete category and every set is a small discrete category.

A.3.11 LEMMA A category $\underline{C}$ is equivalent to a discrete category iff $\underline{C}$ is a groupoid with the property that $\forall\ X,X' \in \text{Ob}\ \underline{C}$, there is at most one morphism from X to X'.

Every discrete category is, of course, a groupoid. So, if $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration, then the statement that $\underline{C}$ is "fibered in discrete categories by F" (or, in brief, that $\underline{C}$ is <u>discretely fibered by F</u>) is a special case of A.3.1.

A.3.12 EXAMPLE Given a category $\underline{C}$, $\forall\ X \in \text{Ob}\ \underline{C}$, the canonical functor $U_X:\underline{C}/X \to \underline{C}$ is a Grothendieck fibration. Moreover, $\underline{C}/X$ is discretely fibered by U_X ($\forall\ Y \in \text{Ob}\ \underline{C}$, the fiber $(\underline{C}/X)_Y$ is the discrete groupoid whose set of objects is $\text{Mor}(Y,X)$).

A.3.13 LEMMA Let $F:\underline{C} \to \underline{D}$ be a functor -- then $\underline{C}$ is discretely fibered by F iff for any morphism $g:Y \to FX'$, there exists a unique morphism $u:X \to X'$ such that $Fu = g$.

PROOF Assume first that $\underline{C}$ is discretely fibered by F, choose $u:X \to X'$ per g and consider a second arrow $\tilde{u}:\tilde{X} \to X'$ per g — then $F\tilde{u} = Fu$. Since u is horizontal (cf. A.3.6), thus is prehorizontal, there exists a unique morphism $v \in \text{Mor}_{FX}(\tilde{X},X)$ sucht that $u \circ v = \tilde{u}$:

$$\begin{tikzcd} X \arrow[r, "u"] & X' \\ \tilde{X} \arrow[u, "v"] \arrow[r, equal] & \tilde{X}. \arrow[u, "\tilde{u}"'] \end{tikzcd}$$

But the fiber $\underline{C}_{FX}$ is discrete, hence $X = \tilde{X}$ and v is the identity, so $\tilde{u} = u$. In the other direction, consider a setup

$$\begin{tikzcd} X_0 \arrow[r, dotted] \arrow[rr, bend left, "w"] & X \arrow[r, "u"'] & X' \end{tikzcd}, \quad \begin{tikzcd} FX_0 \arrow[r, "x"'] \arrow[rr, bend left, "Fw"] & FX \arrow[r, "Fu"'] & FX. \end{tikzcd}$$

With "x" playing the role of "g", let $v:X_0 \to X$ be the unique morphism such that $Fv = x$ -- then

$$\begin{cases} u \circ v:X_0 \to X' \Rightarrow F(u \circ v):FX_0 \to FX' \\ w:X_0 \to X' \Rightarrow F(w):FX_0 \to FX'. \end{cases}$$

Accordingly, by uniqueness, $u \circ v = w$. Therefore every arrow in $\underline{C}$ is horizontal which implies that $\underline{C}$ is fibered in groupoids by F (cf. A.3.7). That the fibers are discrete is clear.

A.4 *CLEAVAGES AND SPLITTINGS*

Let $F:\underline{C} \to \underline{D}$ be a Grothendieck fibration.

A.4.1 CONSTRUCTION Suppose that $g:Y \to Y'$ is an arrow in $\underline{D}$.

Case 1: $\underline{C}_{Y'} = \underline{0}$ -- then take $g^*:\underline{C}_{Y'} \to \underline{C}_Y$ as the canonical inclusion.

Case 2: $\underline{C}_{Y'} \neq \underline{0}$ -- then for each $X' \in \text{Ob } \underline{C}_{Y'}$, choose a horizontal $u:X \to X'$ and define $g^*:\underline{C}_{Y'} \to \underline{C}_Y$ as follows.

- On an object X', let $g^*X' = X$.
- On a morphism $\phi:X' \to \tilde{X}'$, noting that $F(\phi \circ u) = F\phi \circ Fu = id_{Y'} \circ Fu = g = F\tilde{u}$, let $g^*\phi$ be the unique filler for the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] & \tilde{X}' \\ X \arrow[u, dotted, "g^*\phi"] \arrow[r, equal] & X. \arrow[u, "\phi \circ u"'] \end{tikzcd}$$

A.4.2 LEMMA $g^*:\underline{C}_{Y'} \to \underline{C}_Y$ is a functor.

Needless to say, the construction of g^* hinges on the choice of the horizontal $u:X \to X'$.

A.4.3 DEFINITION A <u>cleavage</u> for F is a function σ which assigns to each pair (g,X'), where $g:Y \to FX'$, a horizontal morphism $u = \sigma(g,X')$ $(u:X \to X')$ such that $Fu = g$.

<u>N.B.</u> The axiom of choice for classes implies that every Grothendieck fibration has a cleavage.

A.4.4 REMARK If $\underline{C}$ is discretely fibered by F, then there is only one cleavage for F (cf. A.3.13).

Consider now a pair (F,σ), where $F:\underline{C} \to \underline{D}$ is a Grothendieck fibration and σ is a cleavage for F — then there is an association $\Sigma_{F,\sigma}$

$$Y \longrightarrow \underline{C}_Y,\ (Y \xrightarrow{g} Y') \longrightarrow (\underline{C}_{Y'} \xrightarrow{g^*} \underline{C}_Y)$$

from $\underline{D}^{OP}$ to **CAT** that, however, is not necessarily a functor for more or less obvious reasons. Still, we do have:

- $\forall\ Y$, there is an isomorphism $\varepsilon_Y:\mathrm{id}_Y^* \to \mathrm{id}_{\underline{C}_Y}$ of functors $\underline{C}_Y \to \underline{C}_Y$.

- $\forall\ Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$, there is an isomorphism $\alpha_{g,g'}:g^* \circ g'^* \to (g' \circ g)^*$ of functors $\underline{C}_{Y''} \to \underline{C}_{Y'}$.

A.4.5 DEFINITION A cleavage σ is split if the following conditions are satisfied.

1. $\sigma(\mathrm{id}_{FX'},X') = \mathrm{id}_{X'}$.

2. $\sigma(g' \circ g,X'') = \sigma(g',X'') \circ \sigma(g,g'^*X'')$.

[Note: A Grothendieck fibration is split if it has a cleavage that splits or, in brief, has a splitting.]

A.4.6 EXAMPLE In the notation of A.1.18, assume that $\phi:G \to H$ is surjective, hence that $\underline{\phi}:\underline{G} \to \underline{H}$ is a Grothendieck fibration -- then a cleavage σ for $\underline{\phi}$ is a subset K of G which maps bijectively onto H and $\underline{\phi}$ is split iff K is a subgroup of G. Therefore $\underline{\phi}$ is split iff ϕ is a retract, i.e., iff $\exists$ a homomorphism $\psi:H \to G$ such that $\phi \circ \psi = \mathrm{id}_H$.

A.4.7 LEMMA The association

$$\Sigma_{F,\sigma}:\underline{D}^{OP} \to \mathbf{CAT}$$

is a functor iff F is split.

N.B. It is a fact that every Grothendieck fibration is equivalent to a split Grothendieck fibration.

A.4.8 REMARK In the world of Grothendieck opfibrations, the term cleavage is replaced by opcleavage but there is no "op" in front of split or splittings.

B: INTEGRATION

B: INTEGRATION

B.1 *REALIZATION OF PRESHEAVES*

Given a small category $\underline{C}$, let $\gamma:\underline{C} \to \underline{CAT}$ be the functor that sends X to $\underline{C}/X$ -- then the realization functor Γ_γ assigns to each F in $\hat{\underline{C}}$ its Grothendieck construction:

$$\Gamma_\gamma F \approx \mathrm{gro}_{\underline{C}}\, F.$$

[Note: Recall that $\gamma \approx \Gamma_\gamma \circ Y_{\underline{C}}$, thus $\forall\, X \in \mathrm{Ob}\, \underline{C}$,

$$\gamma X = \underline{C}/X \approx \Gamma_\gamma h_X.]$$

B.1.1 LEMMA The projection

$$\pi_F:\mathrm{gro}_{\underline{C}}\, F \to \underline{C}$$

is a Grothendieck fibration and $\mathrm{gro}_{\underline{C}}\, F$ is discretely fibered by π_F.

In the sequel, we shall write $\underline{C}/F$ in place of $\mathrm{gro}_{\underline{C}}\, F$ and organize matters functorially.

B.1.2 NOTATION Given $F \in \mathrm{Ob}\, \hat{\underline{C}}$, let $\underline{C}/F$ be the small category whose objects are the pairs (X,s), where $X \in \mathrm{Ob}\, \underline{C}$ and $s \in \mathrm{Nat}(h_X,F) \longleftrightarrow FX$, and whose morphisms $(X,s) \to (Y,t)$ are the arrows $f:X \to Y$ such that $th_f = s$.

B.1.3 NOTATION Given $F,G \in \mathrm{Ob}\, \hat{\underline{C}}$ and $\Xi:F \to G$, let

$$\underline{C}/\Xi:\underline{C}/F \to \underline{C}/G$$

be the functor that sends (X,s) to $(X,\Xi \circ s)$.

B.1.4 NOTATION Let

$$i_{\underline{C}}:\hat{\underline{C}} \to \underline{\mathrm{CAT}}$$

be the functor defined on objects by

$$F \to \underline{C}/F$$

and on morphisms by

$$\Xi \to \underline{C}/\Xi.$$

Let $*_{\hat{\underline{C}}}$ be a final object in $\hat{\underline{C}}$ — then $i_{\underline{C}}(*_{\hat{\underline{C}}}) = \underline{C}$, so there is a factorization

$$\begin{tikzcd}
\hat{\underline{C}} \arrow[r, "i_{\underline{C}}"] \arrow[d, "j_{\underline{C}}"'] & \underline{\mathrm{CAT}} \\
\underline{\mathrm{CAT}}/\underline{C} \arrow[r, equal] & \underline{\mathrm{CAT}}/\underline{C}, \arrow[u, "U_{\underline{C}}"']
\end{tikzcd}$$

$U_{\underline{C}}$ the forgetful functor.

B.1.5 LEMMA The functor

$$j_{\underline{C}}:\hat{\underline{C}} \to \underline{\mathrm{CAT}}/\underline{C}$$

is fully faithful.

B.1.6 LEMMA The functor

$$i_{\underline{C}}:\hat{\underline{C}} \to \underline{\mathrm{CAT}}$$

is faithful.

[The forgetful functor

$$U_{\underline{C}}:\underline{\mathrm{CAT}}/\underline{C} \to \underline{\mathrm{CAT}}$$

is faithful.]

B.1.7 LEMMA The functor

$$j_{\underline{C}}:\hat{\underline{C}} \to \underline{CAT}/\underline{C}$$

preserves limits and colimits.

B.1.8 LEMMA The functor

$$i_{\underline{C}}:\hat{\underline{C}} \to \underline{CAT}$$

preserves colimits.

[The forgetful functor

$$U_{\underline{C}}:\underline{CAT}/\underline{C} \to \underline{CAT}$$

preserves colimits.]

B.1.9 LEMMA The functor

$$i_{\underline{C}}:\hat{\underline{C}} \to \underline{CAT}$$

preserves pullbacks.

[The forgetful functor

$$U_{\underline{C}}:\underline{CAT}/\underline{C} \to \underline{CAT}$$

preserves pullbacks.]

<u>N.B.</u> Therefore $i_{\underline{C}}$ preserves monomorphisms.

[Note: In any category, $A \xrightarrow{f} B$ is a monomorphism iff

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle id_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback square.]

B.1.10 LEMMA The functor

$$i^*_{\underline{C}}:\underline{CAT} \to \hat{\underline{C}}$$

that sends $\underline{I}$ to $F_{\underline{I}}$, where

$$F_{\underline{I}}(X) = \mathrm{Mor}(\underline{C}/h_X, \underline{I}) \qquad (X \in \mathrm{Ob}\ \underline{C}),$$

is a right adjoint for $i_{\underline{C}}$.

[Note: Let

$$\left[\begin{array}{l} \mu:\mathrm{id}_{\hat{\underline{C}}} \to i^*_{\underline{C}} i_{\underline{C}} \\ \\ \nu:i_{\underline{C}} i^*_{\underline{C}} \to \mathrm{id}_{\underline{CAT}} \end{array}\right.$$

be the arrows of adjunction.

- Given F,

$$\mu_F:F \to i^*_{\underline{C}} i_{\underline{C}} F,$$

i.e.,

$$\mu_F:F \to F_{\underline{C}/F}.$$

But $\mathrm{Nat}(h_X, F) \longleftrightarrow FX$ and

$$\mu_F(X):\mathrm{Nat}(h_X, F) \to \mathrm{Mor}(\underline{C}/h_X, \underline{C}/F)$$

is the map that sends s to $\underline{C}/s$.

- Given $\underline{I}$,

$$\nu_{\underline{I}}:i_{\underline{C}} i^*_{\underline{C}} \underline{I} \to \underline{I},$$

i.e.,

$$\nu_{\underline{I}}:\underline{C}/F_{\underline{I}} \to \underline{I}.$$

An object in $\underline{C}/F_{\underline{I}}$ is a pair (X,s), where $X \in \text{Ob } \underline{C}$ and $s \in \text{Nat}(h_X, F_{\underline{I}}) \longleftrightarrow F_{\underline{I}}(X) = \text{Mor}(\underline{C}/h_X, \underline{I})$. But $\underline{C}/h_X = \underline{C}/X$ and

$$\nu_{\underline{I}}(X, \underline{C}/X \xrightarrow{s} \underline{I}) = s(X, \text{id}_X).]$$

B.1.11 DEFINITION Let $\underline{C}$ be a small category -- then a <u>sieve</u> in $\underline{C}$ is a full subcategory $\underline{U}$ of $\underline{C}$ with the following property:

$$\text{cod } f \in \text{Ob } \underline{U} \Rightarrow \text{dom } f \in \text{Ob } \underline{U} \quad (f \in \text{Mor } \underline{C}).$$

B.1.12 LEMMA The functors $F:\underline{C} \rightarrow [1]$ are in a one-to-one correspondence with the sieves in $\underline{C}$ via the map $F \rightarrow F^{-1}(0)$.

B.1.13 EXAMPLE Put $L_{\underline{C}} = i_{\underline{C}}^*[1]$ -- then for any F in $\hat{\underline{C}}$, there are functorial bijections

$$\begin{aligned}\text{Mor}(F, L_{\underline{C}}) &= \text{Mor}(F, i_{\underline{C}}^*[1]) \\ &\approx \text{Mor}(i_{\underline{C}}F, [1]) \\ &\approx \text{Mor}(\underline{C}/F, [1]) \\ &\approx \{\text{sieves in } \underline{C}/F\} \approx \text{Sub}_{\hat{\underline{C}}}\, F,\end{aligned}$$

the symbol on the RHS standing for the subobjects of F. Therefore $L_{\underline{C}}$ represents $\text{Sub}_{\hat{\underline{C}}}$.

[Note: $L_{\underline{C}}$ is called the <u>object of Lawvere</u>.]

B.1.14 THEOREM For any small category $\underline{C}$, the canonical arrow

$$\widehat{\underline{C}/F} \rightarrow \hat{\underline{C}}/F$$

is an equivalence.

Specialize, taking $\underline{C} = \underline{\Delta}$ and $F = X$ (a simplicial set) — then the objects of $\underline{\Delta}/X$ are the pairs $([n],x)$ $(x \in X_n)$ and

$$\underline{\Delta}/X = \text{gro}_{\underline{\Delta}}\, X,$$

the simplex category of X.

Given a small category $\underline{I}$, consider the composite

$$\hat{\underline{I}} \xrightarrow{j_{\underline{\Delta}}} \underline{\text{CAT}}/\underline{I} \xrightarrow{\text{ner}} \underline{\text{SISET}}/\text{ner}\ \underline{I}.$$

Since ner is fully faithful, it follows from B.1.5 that ner $\circ$ $j_{\underline{\Delta}}$ is fully faithful.

B.1.15 LEMMA Let $F \in \text{Ob}\ \hat{\underline{I}}$ -- then

$$\text{ner}_n\ \underline{I}/F \approx \coprod_{i_0 \to \cdots \to i_n} Fi_n.$$

[Note: This isomorphism is natural in n.]

Let

$$N_{\underline{I}}:\hat{\underline{I}} \to \underline{\text{SISET}}/\text{ner}\ \underline{I}$$

be the functor defined by

$$N_{\underline{I}}(F)_n = (\coprod_{i_0 \to \cdots \to i_n} Fi_n \longrightarrow \coprod_{i_0 \to \cdots \to i_n} *).$$

Then

$$N_{\underline{I}} \approx \text{ner} \circ j_{\underline{\Delta}},$$

hence $N_{\underline{I}}$ is fully faithful.

B.1.16 DEFINITION The composite

$$\hat{\underline{I}} \xrightarrow{N_{\underline{I}}} \underline{\text{SISET}}/\text{ner } \underline{I} \xrightarrow{U_{\underline{I}}} \underline{\text{SISET}}$$

is called the <u>simplicial replacement functor</u>.

In B.1.14, let $\underline{C} = \underline{\Delta}$, $F = \text{ner } \underline{I}$ -- then

$$(\underline{\Delta}/\text{ner } \underline{I})^{\wedge} \to \hat{\underline{\Delta}}/\text{ner } \underline{I} = \underline{\text{SISET}}/\text{ner } \underline{I}.$$

[Note: To explicate matters, let

$$F:(\underline{\Delta}/\text{ner } \underline{I})^{\text{OP}} \to \underline{\text{SET}}$$

be a presheaf — then the object $X \xrightarrow{\pi} \text{ner } \underline{I}$ corresponding to F is given in degree n by

$$X_n = \coprod_{\Delta[n] \xrightarrow{\alpha} \text{ner } \underline{I}} F\alpha,$$

where

$$\pi_n(a) = \alpha_n(\text{id}_{[n]}) \qquad (a \in F\alpha).]$$

B.1.17 RAPPEL For any small category $\underline{I}$, there is a natural simplicial weak equivalence

$$\underline{\Delta}/\text{ner } \underline{I} \ (= \text{gro}_{\underline{\Delta}} \text{ ner } \underline{I}) \to \underline{I}.$$

<u>N.B.</u> The induced functor

$$\hat{\underline{I}} \to (\underline{\Delta}/\text{ner } \underline{I})^{\wedge} \to \underline{\text{SISET}}/\text{ner } \underline{I}$$

is $N_{\underline{I}}$.

B.2 THE FUNDAMENTAL CONSTRUCTION

Let $\underline{I}$ be a small category, $F:\underline{I} \to \underline{CAT}$ a functor.

B.2.1 DEFINITION The integral of F over $\underline{I}$, denoted $\underline{INT}_{\underline{I}}F$, is the category whose objects are the pairs (i,X), where $i \in \mathrm{Ob}\,\underline{I}$ and $X \in \mathrm{Ob}\,Fi$, and whose morphisms are the arrows $(\delta,f):(i,X) \to (j,Y)$, where $\delta \in \mathrm{Mor}(i,j)$ and $f \in \mathrm{Mor}((F\delta)X,Y)$ (composition is given by

$$(\delta',f') \circ (\delta,f) = (\delta' \circ \delta,\ f' \circ (F\delta')f)).$$

B.2.2 NOTATION Let

$$\Theta_F:\underline{INT}_{\underline{I}}F \to \underline{I}$$

be the functor that sends (i,X) to i and (δ,f) to δ.

B.2.3 LEMMA The fiber of Θ_F over i is isomorphic to the category Fi.

PROOF Define

$$\iota_i:Fi \to \underline{INT}_{\underline{I}}F$$

by

$$\begin{cases} \iota_i X = (i,X) \quad (X \in \mathrm{Ob}\,Fi) \\ \iota_i f = (\mathrm{id}_i,f) \quad (f \in \mathrm{Mor}\,Fi). \end{cases}$$

[Note: There is a natural transformation

$$\xi_\delta:\iota_i \to \iota_j \circ F\delta,$$

viz.

$$\xi_{\delta,X} = (\delta,\mathrm{id}_{(F\delta)X}):(i,X) \to (j,(F\delta)X).$$

And

$$\xi_{\delta' \circ \delta} = (\xi_{\delta'} F\delta) \circ \xi_{\delta}, \xi_{id_i} = id_{1_i}.]$$

<u>N.B.</u> There is a pullback square

$$\begin{tikzcd} Fi \arrow[r] \arrow[d] & \underline{INT}_{\underline{I}}F \arrow[d, "\Theta_F"] \\ \underline{1} \arrow[r, "K_i"'] & \underline{I} \end{tikzcd} .$$

B.2.4 LEMMA The preophorizontal morphisms are the (δ,f), where f is an isomorphism.

[Note: The composition of two preophorizontal morphisms is therefore preophorizontal.]

B.2.5 LEMMA Θ_F is a Grothendieck preopfibration.

B.2.6 THEOREM Θ_F is a Grothendieck opfibration.

PROOF In view of B.2.4 and B.2.5, one has only to cite A.1.27.

B.2.7 LEMMA Θ_F is a split Grothendieck opfibration.

PROOF Define σ_F by

$$\sigma_F(\delta,(i,X)) = (\delta, id_{F\delta X}):(i,X) \to (j,F\delta X).$$

B.2.8 EXAMPLE If $F_{\underline{J}}:\underline{I} \to \underline{CAT}$ is the constant functor with value $\underline{J}$, then $\underline{INT}_{\underline{I}}F_{\underline{J}}$ is isomorphic to $\underline{I} \times \underline{J}$.

[Note: In particular

$$\underline{\mathrm{INT}}_{\underline{I}}F_{\underline{1}} \approx \underline{I}.]$$

B.2.9 EXAMPLE Given a small category $\underline{I}$, let

$$H_{\underline{I}}:\underline{I}^{\mathrm{OP}} \times \underline{I} \rightarrow \underline{\mathrm{CAT}}$$

be the functor $(j,i) \rightarrow \mathrm{Mor}(j,i)$, where the set $\mathrm{Mor}(j,i)$ is regarded as a discrete category -- then

$$\underline{\mathrm{INT}}_{\underline{I}^{\mathrm{OP}} \times \underline{I}} H_{\underline{I}}$$

can be identified with $\underline{I}(\sim>)$ (cf. A.1.33), $\Theta_{H_{\underline{I}}}$ becoming the functor

$$(s_{\underline{I}},t_{\underline{I}}):\underline{I}(\sim>) \rightarrow \underline{I}^{\mathrm{OP}} \times \underline{I}.$$

Let $F,G:\underline{I} \rightarrow \underline{\mathrm{CAT}}$ be functors, $\Xi:F \rightarrow G$ a natural transformation.

B.2.10 DEFINITION The <u>integral of Ξ over $\underline{I}$</u>, denoted $\underline{\mathrm{INT}}_{\underline{I}}\Xi$, is the functor

$$\underline{\mathrm{INT}}_{\underline{I}}F \rightarrow \underline{\mathrm{INT}}_{\underline{I}}G$$

defined by the prescription

$$\begin{cases} (\underline{\mathrm{INT}}_{\underline{I}}\Xi)(i,X) = (i,\Xi_i X) \\ (\underline{\mathrm{INT}}_{\underline{I}}\Xi)(\delta,f) = (\delta,\Xi_j f). \end{cases}$$

[Note: Since $f:(F\delta)X \rightarrow Y \in \mathrm{Mor}\ Fj$, it follows that

$$\Xi_j f:\Xi_j(F\delta)X \rightarrow \Xi_j Y \in \mathrm{Mor}\ Gj.$$

But there is a commutative diagram

$$\begin{array}{ccc} Fi & \xrightarrow{\Xi_i} & Gi \\ {\scriptstyle F\delta}\downarrow & & \downarrow{\scriptstyle G\delta} \\ Fj & \xrightarrow[\Xi_j]{} & Gj, \end{array}$$

so

$$(\delta, \Xi_j f):(i, \Xi_i X) \to (j, \Xi_j Y)$$

is a morphism in $\underline{\mathrm{INT}}_{\underline{I}}G$.]

Obviously,

$$\Theta_G \circ \underline{\mathrm{INT}}_{\underline{I}}\Xi = \Theta_F$$

and, in fact,

$$\underline{\mathrm{INT}}_{\underline{I}}\Xi:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{\mathrm{INT}}_{\underline{I}}G$$

is an opfibered functor.

B.2.11 LEMMA The association

$$\begin{cases} F \to (\underline{\mathrm{INT}}_{\underline{I}}F, \Theta_F) \\ \Xi \to \underline{\mathrm{INT}}_{\underline{I}}\Xi \end{cases}$$

defines a functor

$$\underline{\mathrm{INT}}_{\underline{I}}:[\underline{I}, \underline{\mathrm{CAT}}] \to \underline{\mathrm{CAT}}/\underline{I}.$$

[Note: Suppose that $\underline{I}$ and $\underline{J}$ are small categories and $K:\underline{J} \to \underline{I}$ is a functor -- then there is an induced functor

$$K^*:[\underline{I},\underline{CAT}] \to [\underline{J},\underline{CAT}]$$

and $\forall\ F:\underline{I} \to \underline{CAT}$, there is a pullback square

$$\begin{tikzcd} \underline{INT}_{\underline{J}}K^*F \arrow[r] \arrow[d, "\Theta_{K^*F}"'] & \underline{INT}_{\underline{I}}F \arrow[d, "\Theta_F"] \\ \underline{J} \arrow[r, "K"'] & \underline{I} \end{tikzcd} .]$$

Let

$$\Gamma_{\underline{I}}:\underline{CAT}/\underline{I} \to [\underline{I},\underline{CAT}]$$

be the functor given on objects $(\underline{A},p)$ $(p:\underline{A} \to \underline{I})$ by

$$\Gamma_{\underline{I}}(\underline{A},p)i = \underline{A}/i.$$

[Note: There is a pullback square

$$\begin{tikzcd} \Gamma_{\underline{I}}(\underline{A},p)i \arrow[r] \arrow[d] & \underline{A} \arrow[d, "p"] \\ \underline{I}/i \arrow[r] & \underline{I} \end{tikzcd} .]$$

B.2.12 LEMMA† $\Gamma_{\underline{I}}$ is a left adjoint for $\underline{INT}_{\underline{I}}$.

PROOF It suffices to exhibit natural transformations

$$\begin{cases} \mu \in \text{Nat}(\text{id}_{\underline{CAT}/\underline{I}}, \underline{INT}_{\underline{I}} \circ \Gamma_{\underline{I}}) \\ \nu \in \text{Nat}(\Gamma_{\underline{I}} \circ \underline{INT}_{\underline{I}}, \text{id}_{[\underline{I},\underline{CAT}]}) \end{cases}$$

† Nico, *Houston J. Math.* 9 (1983), 71-99.

such that

$$\begin{cases} (\underline{\mathrm{INT}}_{\underline{I}}\nu) \circ (\mu\underline{\mathrm{INT}}_{\underline{I}}) = \mathrm{id}_{\underline{\mathrm{INT}}_{\underline{I}}} \\ (\nu\Gamma_{\underline{I}}) \circ (\Gamma_{\underline{I}}\mu) = \mathrm{id}_{\Gamma_{\underline{I}}}. \end{cases}$$

$\underline{\mu}$: Let $(\underline{A},p)$ be an object of $\underline{\mathrm{CAT}}/\underline{I}$. To define a functor

$$\mu_{(\underline{A},p)}:(\underline{A},p) \to \underline{\mathrm{INT}}_{\underline{I}}\Gamma_{\underline{I}}(\underline{A},p)$$

over $\underline{I}$, note that the objects of $\underline{\mathrm{INT}}_{\underline{I}}\Gamma_{\underline{I}}(\underline{A},p)$ are the triples $(i,a,pa \xrightarrow{\phi} i)$, where $i \in \mathrm{Ob}\,\underline{I}$, $a \in \mathrm{Ob}\,\underline{A}$, $\phi \in \mathrm{Mor}\,\underline{I}$ and the morphisms of $\underline{\mathrm{INT}}_{\underline{I}}\Gamma_{\underline{I}}(\underline{A},p)$ are the arrows

$$(\delta,f):(i,a,pa \xrightarrow{\phi} i) \to (i',a',pa' \xrightarrow{\phi'} i'),$$

where $\delta \in \mathrm{Mor}(i,i')$ and $f:a \to a'$ is a morphism of $\underline{A}$ for which the diagram

$$\begin{tikzcd} pa \arrow[r, "pf"] \arrow[d, "\phi"'] & pa' \arrow[d, "\phi'"] \\ i \arrow[r, "\delta"'] & i' \end{tikzcd}$$

commutes. This said, let

$$\begin{cases} \mu_{(\underline{A},p)}a = (pa,a,pa \xrightarrow{\mathrm{id}_{pa}} pa) \\ \mu_{(\underline{A},p)}f = (pf,f):(pa,a,\mathrm{id}_{pa}) \to (pa',a',\mathrm{id}_{pa'}). \end{cases}$$

$\underline{\nu}$: Let F be an object of $[\underline{I},\underline{\mathrm{CAT}}]$. To define a natural transformation

$$\nu_F:\Gamma_{\underline{I}}\underline{\mathrm{INT}}_{\underline{I}}F \to F$$

or still, to define a functor

$$\nu_{F,i}:\underline{\underline{\mathrm{INT}}}_{\underline{I}}F/i \to Fi$$

functorial in i, note that the objects of $\underline{\underline{\mathrm{INT}}}_{\underline{I}}F/i$ are the triples $(i',X',i' \xrightarrow{\delta'} i)$, where $i' \in \mathrm{Ob}\,\underline{I}$, $X' \in Fi'$, $\delta' \in \mathrm{Mor}\,\underline{I}$ and the morphisms of $\underline{\underline{\mathrm{INT}}}_{\underline{I}}F/i$ are the arrows

$$(\delta,f):(i',X',i' \xrightarrow{\delta'} i) \to (i'',X'',i'' \xrightarrow{\delta''} i),$$

where $\delta \in \mathrm{Mor}(i',i'')$ and $f:(F\delta)X' \to X''$ is a morphism of Fi'' for which the diagram

$$\begin{tikzcd} i' \arrow[r, "\delta"] \arrow[d, "\delta'"'] & i'' \arrow[d, "\delta''"] \\ i \arrow[r, equal] & i \end{tikzcd}$$

commutes. This said, let

$$\begin{cases} \nu_{F,i}(i',X',i' \xrightarrow{\delta'} i) = (F\delta')X' \\ \nu_{F,i}(\delta,f) = (F\delta'')f:(F\delta')X' \to (F\delta'')X''. \end{cases}$$

The verification that μ and ν have the requisite properties is straightforward.

B.2.13 REMARK Given small categories $\underline{I}$, $\underline{J}$ and a functor $K:\underline{I} \to \underline{J}$, let

$$\underline{\mathrm{CAT}}/K:\underline{\mathrm{CAT}}/\underline{I} \to \underline{\mathrm{CAT}}/\underline{J}$$

be the induced functor -- then the functor

$$\Gamma_{\underline{J}} \circ \underline{\mathrm{CAT}}/K:\underline{\mathrm{CAT}}/\underline{I} \to \underline{\mathrm{CAT}}/\underline{J} \to [\underline{J},\underline{\mathrm{CAT}}]$$

is a left adjoint for the functor

$$\underline{\mathrm{INT}}_{\underline{I}} \circ K^*:[\underline{J},\underline{\mathrm{CAT}}] \to [\underline{I},\underline{\mathrm{CAT}}] \to \underline{\mathrm{CAT}}/\underline{I},$$

the proof being an easy extension of the preceding considerations (take $\underline{I} = \underline{J}$, $K = \mathrm{id}_{\underline{I}}$ to recover B.2.12).

The category $\underline{\mathrm{INT}}_{\underline{I}}F$ has a universal mapping property.

B.2.14 THEOREM Fix a small category $\underline{C}$. Suppose given functors $\phi_i:Fi \to \underline{C}$ ($i \in \mathrm{Ob}\ \underline{I}$) and natural transformations $\Xi_\delta:\phi_i \to \phi_j \circ F\delta$ ($i \xrightarrow{\delta} j \in \mathrm{Mor}\ \underline{I}$) such that

$$\Xi_{\delta' \circ \delta} = (\Xi_{\delta'} F\delta) \circ \Xi_\delta,\ \Xi_{\mathrm{id}_i} = \mathrm{id}_{\phi_i}.$$

Then there exists a unique functor

$$\Phi:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{C}$$

such that

$$\begin{cases} \phi_i = \Phi \circ \iota_i & (\iota_i:Fi \to \underline{\mathrm{INT}}_{\underline{I}}F) \\ \Xi_\delta = \Phi\xi_\delta & (\xi_\delta:\iota_i \to \iota_j \circ F\delta) \end{cases} \qquad \text{(cf. B.2.3).}$$

PROOF Define Φ by

$$\begin{cases} \Phi(i,X) = \phi_i X & (X \in \mathrm{Ob}\ Fi) \\ \Phi(\delta,f) = \phi_j f \circ \Xi_{\delta,X}. \end{cases}$$

[Note: As regards the definition of $\Phi(\delta,f)$, observe that

$$\Xi_{\delta,X}:\phi_i X \to \phi_j F\delta X.$$

On the other hand, $f:(F\delta)X \to Y$, where $(F\delta)X, Y \in \mathrm{Ob}\, Fj$, so

$$\phi_j f:\phi_j(F\delta)X \to \phi_j Y,$$

thus

$$\Phi(\delta,f):\Phi(i,X)\,(=\phi_i X) \to \Phi(j,Y)\,(=\phi_j Y)$$

as desired.]

B.2.15 EXAMPLE Consider the natural sink $\{\ell_i:Fi \to \mathrm{colim}_{\underline{I}} F\}$, hence $\ell_i = \ell_j \circ F\delta$ -- then there exists a unique functor

$$K_F:\underline{\mathrm{INT}}_{\underline{I}} F \to \mathrm{colim}_{\underline{I}} F$$

such that

$$\begin{cases} \ell_i = K_F \circ \iota_i \\ \mathrm{id}_{\ell_i} = K_F \xi_\delta. \end{cases}$$

[Note: Spelled out,

$$\begin{cases} K_F(i,X) = \ell_i X \\ K_F(\delta,f) = \ell_j f.] \end{cases}$$

Let $\underline{C}$ be a small category, $F:\underline{I} \to \hat{\underline{C}}$ a functor -- then

$$\underline{I} \xrightarrow{F} \hat{\underline{C}} \xrightarrow{i_{\underline{C}}} \underline{\mathrm{CAT}}$$

and there is an arrow

$$K_{i_{\underline{C}}F}:\underline{\mathrm{INT}}_{\underline{I}}\,i_{\underline{C}}F \to \mathrm{colim}_{\underline{I}}\,i_{\underline{C}}F$$

$$\approx i_{\underline{C}}\mathrm{colim}_{\underline{I}}F \quad (\text{cf. B.1.8})$$

$$= \underline{C}/\mathrm{colim}_{\underline{I}}F.$$

B.2.16 LEMMA $K_{i_{\underline{C}}F}$ is a Grothendieck fibration.

Let (X,s) be an object of $\underline{C}/\mathrm{colim}_{\underline{I}}F$ (so $X \in \mathrm{Ob}\,\underline{C}$ and $s:h_X \to \mathrm{colim}_{\underline{I}}F$) -- then the fiber

$$(\underline{\mathrm{INT}}_{\underline{I}}\,i_{\underline{C}}F)_{(X,s)}$$

of $K_{i_{\underline{C}}F}$ over (X,s) admits an external description. In fact, $\forall$ i in Ob $\underline{I}$, there is an arrow $i_{\underline{C}}\ell_i:\underline{C}/Fi \to \underline{C}/\mathrm{colim}_{\underline{I}}F$ and $\forall\ \delta:i \to j$ in Mor $\underline{I}$, there is an arrow

$$(\underline{C}/Fi)_{(X,s)} \to (\underline{C}/Fj)_{(X,s)}.$$

Write

$$(i_{\underline{C}}F)_{(X,s)}:\underline{I} \to \underline{\mathrm{CAT}}$$

for the functor thus determined.

B.2.17 LEMMA We have

$$(\underline{\mathrm{INT}}_{\underline{I}}\,i_{\underline{C}}F)_{(X,s)} \approx \underline{\mathrm{INT}}_{\underline{I}}\,(i_{\underline{C}}F)_{(X,s)}.$$

[The verification is tautological.]

B.3 *THE CANONICAL EQUIVALENCE*

Fix a small category $\underline{D}$ -- then by $\underline{\mathrm{SO}}(\underline{D})$ we shall understand the category whose objects are the triples $(\underline{C},F,\sigma)$, where $\underline{C}$ is small and $F:\underline{C} \to \underline{D}$ is a split Grothendieck opfibration with splitting σ, and whose morphisms $\Phi:(\underline{C},F,\sigma) \to (\underline{C}',F',\sigma')$ are the functors $\Phi:\underline{C} \to \underline{C}'$ such that for any object $X \in \mathrm{Ob}\ \underline{C}$ and any morphism $g:FX \to Y$,

$$\Phi(\sigma(g,X)) = \sigma'(g,\Phi X)$$

subject to $F' \circ \Phi = F$.

<u>N.B.</u> $\forall\ Y \in \mathrm{Ob}\ \underline{D}$, Φ restricts to a functor $\Phi_Y:\underline{C}_Y \to \underline{C}'_Y$.

Define now the association

$$\Sigma_{F,\sigma}:\underline{D} \to \underline{\mathrm{CAT}}$$

as in A.4.7 (recast for opfibrations) — then $\Sigma_{F,\sigma}$ is a functor (σ being split).

B.3.1 NOTATION Let

$$\Sigma_{\underline{D}}:\underline{\mathrm{SO}}(\underline{D}) \to [\underline{D},\underline{\mathrm{CAT}}]$$

be the functor given on an object $(\underline{C},F,\sigma)$ by

$$\Sigma_{\underline{D}}(C,F,\sigma) = \Sigma_{F,\sigma}$$

and on a morphism

$$\Phi:(\underline{C},F,\sigma) \to (\underline{C}',F',\sigma')$$

by

$$(\Sigma_{\underline{D}}\Phi)_Y = \Phi_Y.$$

[Note: The tacit assumption is that

$$\Sigma_{\underline{D}}\Phi \in \mathrm{Nat}(\Sigma_{F,\sigma},\Sigma_{F',\sigma'}).$$

But, from the definitions,

$$\left[\begin{array}{l} \Sigma_{F,\sigma}Y = \underline{C}_Y \\ \\ \Sigma_{F',\sigma'}Y = \underline{C}'_Y \end{array}\right.$$

and for any $g:Y \to Y'$, there is a commutative diagram

$$\begin{tikzcd} \underline{C}_Y \arrow[r, "\Phi_Y"] \arrow[d, "\Sigma_{F,\sigma}g"'] & \underline{C}'_Y \arrow[d, "\Sigma_{F',\sigma'}g"] \\ \underline{C}_{Y'} \arrow[r, "\Phi_{Y'}"'] & \underline{C}'_{Y'} \end{tikzcd} \quad .]$$

Matters can be reversed. Thus let $G:\underline{D} \to \underline{CAT}$ be a functor -- then

$$\Theta_G:\underline{INT}_{\underline{D}}G \to \underline{D}$$

is a split Grothendieck opfibration with splitting σ_G (cf. B.2.7), so the triple

$$(\underline{INT}_{\underline{D}}G,\Theta_G,\sigma_G)$$

is an object in $\underline{SO}(\underline{D})$. Furthermore, if $\Omega:G \to G'$ is a natural transformation, then

$$\underline{INT}_{\underline{D}}\Omega:(\underline{INT}_{\underline{D}}G,\Theta_G,\sigma_G) \to (\underline{INT}_{\underline{D}}G',\Theta_{G'},\sigma_{G'})$$

is a morphism in $\underline{SO}(\underline{D})$.

Accordingly, these considerations lead to a functor

$$\underline{INT}_{\underline{D}}:[\underline{D},\underline{CAT}] \to \underline{SO}(\underline{D}).$$

B.3.2 THEOREM The categories $\underline{SO}(\underline{D})$, $[\underline{D},\underline{CAT}]$ are equivalent:

$$\begin{cases} \underline{SO}(\underline{D}) \xrightarrow{\Sigma_{\underline{D}}} [\underline{D},\underline{CAT}] \\ [\underline{D},\underline{CAT}] \xrightarrow[\underline{INT}_{\underline{D}}]{} \underline{SO}(\underline{D}) \end{cases}$$

with

$$\begin{cases} \Sigma_{\underline{D}} \circ \underline{INT}_{\underline{D}} \approx \mathrm{id}_{[\underline{D},\underline{CAT}]} \\ \underline{INT}_{\underline{D}} \circ \Sigma_{\underline{D}} \approx \mathrm{id}_{\underline{SO}(\underline{D})}. \end{cases}$$

B.4 *COINTEGRALS*

Let $\underline{I}$ be a small category, $F:\underline{I}^{OP} \to \underline{CAT}$ a functor.

B.4.1 DEFINITION The <u>cointegral of F over $\underline{I}$</u>, denoted $\overline{INT}_{\underline{I}}F$, is the category whose objects are the pairs (i,X), where $i \in \mathrm{Ob}\ \underline{I}$ and $X \in \mathrm{Ob}\ Fi$, and whose morphisms are the arrows $(\delta,f):(i,X) \to (j,Y)$, where $\delta \in \mathrm{Mor}(i,j)$ and $f \in \mathrm{Mor}(X,(F\delta)Y)$ (composition is given by

$$(\delta',f') \circ (\delta,f) = (\delta' \circ \delta,(F\delta)f' \circ f)).$$

B.4.2 REMARK Let $\underline{C}$ be a small category and suppose that $F \in \mathrm{Ob}\ \hat{\underline{C}}$ -- then $F:\underline{C}^{OP} \to \underline{SET}$. Thinking of $\underline{SET}$ as a subcategory of $\underline{CAT}$ (every set is a small category when viewed discretely), it follows that

$$\overline{INT}_{\underline{I}}F = \mathrm{gro}_{\underline{C}}\ F = \underline{C}/F.$$

B.4.3 NOTATION Let

$$\bar{\Theta}_F:\overline{INT}_{\underline{I}}F \to \underline{I}$$

be the functor that sends (i,X) to i and (δ,f) to δ.

B.4.4 THEOREM $\bar{\Theta}_F$ is a split Grothendieck fibration.

What has been said about integrals can be said about cointegrals, thus no additional elaboration on this score is necessary.

B.4.5 LEMMA We have

$$\overline{\mathrm{INT}}_{\underline{I}}F = (\underline{\mathrm{INT}}_{\underline{I}^{OP}}OP \circ F)^{OP}$$

and

$$\bar{\Theta}_F = (\Theta_{OP \circ F})^{OP}.$$

[Note:

$$\Theta_{OP \circ F}:\underline{\mathrm{INT}}_{\underline{I}^{OP}}OP \circ F \rightarrow \underline{I}^{OP}$$

=>

$$(\Theta_{OP \circ F})^{OP}:(\underline{\mathrm{INT}}_{\underline{I}^{OP}}OP \circ F)^{OP} \rightarrow \underline{I}.]$$

<u>N.B.</u> F^{OP} is not the same as $OP \circ F$.

B.4.6 REMARK The involution

$$OP:\underline{CAT} \rightarrow \underline{CAT}$$

induces an isomorphism

$$OP_*:[\underline{I}^{OP},\underline{CAT}] \rightarrow [\underline{I}^{OP},\underline{CAT}]$$

and there is a commutative diagram

$$\begin{tikzcd}
{[\underline{I}^{OP},\underline{CAT}]} \arrow[r, "\overline{\underline{INT}}_{\underline{I}}"] \arrow[d, "OP_*"'] & \underline{CAT}/\underline{I} \\
{[\underline{I}^{OP},\underline{CAT}]} \arrow[r, "\underline{INT}_{\underline{I}^{OP}}"'] & \underline{CAT}/\underline{I}^{OP} \arrow[u, "OP/\underline{I}"']
\end{tikzcd}$$

Let $\underline{I}$ and $\underline{J}$ be small categories, $F:\underline{I}^{OP} \times \underline{J} \to \underline{CAT}$ a functor -- then there are functors

$$\begin{cases} \underline{INT}_{\underline{J}}F:\underline{I}^{OP} \to \underline{CAT} \\ \overline{\underline{INT}}_{\underline{I}}F:\underline{J} \longrightarrow \underline{CAT} \end{cases}$$

arising from term-by-term operations and in this context

$$\begin{cases} \Theta_F:\underline{INT}_{\underline{J}}F \to \underline{J} \\ \bar{\Theta}_F:\overline{\underline{INT}}_{\underline{I}}F \to \underline{I} \end{cases}$$

are natural transformations (treat the targets as constant functors).

B.4.7 LEMMA There is a commutative diagram

$$\begin{tikzcd}
\overline{\underline{INT}}_{\underline{I}}\underline{INT}_{\underline{J}}F \arrow[r, "\approx"] \arrow[d, "\overline{\underline{INT}}_{\underline{I}}\Theta_F"'] & \underline{INT}_{\underline{J}}\overline{\underline{INT}}_{\underline{I}}F \arrow[d, "\underline{INT}_{\underline{J}}\bar{\Theta}_F"] \\
\underline{I} \times \underline{J} \arrow[r, "\approx"'] & \underline{J} \times \underline{I}
\end{tikzcd} .$$

B.4.8 NOTATION Given functors

$$\left[\begin{array}{l} F:\underline{I} \to \underline{\mathrm{CAT}} \\ \\ G:\underline{I}^{\mathrm{OP}} \to \underline{\mathrm{CAT}}, \end{array}\right.$$

define $\overline{\underline{\mathrm{INT}}}_{\underline{I}}(F,G)$ by the pullback square

$$\begin{tikzcd}
\overline{\underline{\mathrm{INT}}}_{\underline{I}}(F,G) \arrow[r, "q_G"] \arrow[d, "p_F"'] & \overline{\underline{\mathrm{INT}}}_{\underline{I}}G \arrow[d, "\bar{\Theta}_G"] \\
\underline{\mathrm{INT}}_{\underline{I}}F \arrow[r, "\Theta_F"'] & \underline{I} \ .
\end{tikzcd}$$

<u>N.B.</u> Using the notation of B.2.8,

$$\left[\begin{array}{l} \overline{\underline{\mathrm{INT}}}_{\underline{I}}(F,G_{\underline{1}}) \approx \underline{\mathrm{INT}}_{\underline{I}}F \\ \\ \overline{\underline{\mathrm{INT}}}_{\underline{I}}(F_{\underline{1}},G) \approx \overline{\underline{\mathrm{INT}}}_{\underline{I}}G. \end{array}\right.$$

B.4.9 LEMMA The functor p_F is a Grothendieck fibration and the functor q_G is a Grothendieck opfibration (cf. A.2.4).

B.5 *ISOMORPHIC REPLICAS*

Let $\underline{I}$ be a small category.

B.5.1 NOTATION Given functors

$$\left[\begin{array}{l} F:\underline{I} \to \underline{\mathrm{CAT}} \\ \\ G:\underline{I}^{\mathrm{OP}} \to \underline{\mathrm{CAT}}, \end{array}\right.$$

put

$$G \otimes_{\underline{I}} F = \int^i Gi \times Fi,$$

an object of $\underline{CAT}$.

[Note: One can realize $G \otimes_{\underline{I}} F$ as

$$\mathrm{coeq}(\coprod_{i \to j} Gj \times Fi \rightrightarrows \coprod_i Gi \times Fi).]$$

N.B. It is clear that $- \otimes_{\underline{I}} -$ is functorial in F and G and behaves in the obvious way w.r.t. a functor $\underline{I} \to \underline{J}$.

B.5.2 EXAMPLE Let G be constant with value $\underline{1}$ -- then

$$\underline{1} \otimes_{\underline{I}} F \approx \mathrm{colim}_{\underline{I}} F.$$

Specialize and take for G the functor $\underline{I}^{OP} \to \underline{CAT}$ that sends i to $i\backslash\underline{I}$ -- then the assignment $(i,j) \to i\backslash\underline{I} \times Fj$ defines a diagram $\underline{I}^{OP} \times \underline{I} \to \underline{CAT}$.

B.5.3 CONSTRUCTION $\forall\ i \in \mathrm{Ob}\ \underline{I}$, there is a canonical functor

$$f_i : i\backslash\underline{I} \times Fi \to \underline{INT}_{\underline{I}} F.$$

- Define f_i on an object $(i \xrightarrow{\delta} j, X)$ $(X \in \mathrm{Ob}\ Fi)$ by

$$f_i(i \xrightarrow{\delta} j, X) = (j, (F\delta)X).$$

[Note:

$$i \xrightarrow{\delta} j \Rightarrow Fi \xrightarrow{F\delta} Fj$$

$$\Rightarrow (F\delta)X \in \mathrm{Ob}\ Fj.]$$

- Define f_i on a morphism

$$(i \xrightarrow{\delta} j, X) \xrightarrow{(\lambda,f)} (i \xrightarrow{\delta'} j', X'),$$

where $\lambda : j \to j'$ $(\lambda \circ \delta = \delta')$ and $f : X \to X'$ $(f \in \mathrm{Mor}\, Fi)$, by

$$f_i(\lambda, f) = (\lambda, (F\delta')f) : (j, (F\delta)X) \to (j', (F\delta')X').$$

[Note:

$$\left[\begin{array}{l} F\delta : Fi \to Fj \\ \\ F\delta' : Fi \to Fj' \end{array}\right. \Rightarrow \left[\begin{array}{l} (F\delta)X \xrightarrow{(F\delta)f} (F\delta)X' \\ \\ (F\delta')X \xrightarrow[(F\delta')f]{} (F\delta')X'. \end{array}\right.$$

But

$$\lambda \circ \delta = \delta' \Rightarrow F\lambda \circ F\delta = F\delta'.$$

Therefore

$$(F\delta')f : (F\lambda)(F\delta)X \to (F\delta')X'.]$$

B.5.4 LEMMA The collection

$$\{f_i : i\backslash \underline{I} \times Fi \to \underline{\mathrm{INT}}_{\underline{I}} F\}$$

is a dinatural sink: $\forall\, i \xrightarrow{\delta} j$ in $\mathrm{Mor}\, \underline{I}$, there is a commutative diagram

$$\begin{tikzcd}
i\backslash \underline{I} \times Fi \arrow[r, "f_i"] & \underline{\mathrm{INT}}_{\underline{I}} F \\
j\backslash \underline{I} \times Fi \arrow[u] \arrow[r] & j\backslash \underline{I} \times Fj. \arrow[u, "f_j"']
\end{tikzcd}$$

B.5.5 LEMMA Suppose that $\{\gamma_i : i\backslash\underline{I} \times Fi \to \Gamma\}$ is a dinatural sink ($\Gamma \in \text{Ob } \underline{CAT}$) -- then there is a unique functor $\phi : \underline{INT}_{\underline{I}}F \to \Gamma$ such that $\gamma_i = \phi \circ f_i$ for all $i \in \text{Ob } \underline{I}$.

[The verification is elementary but fastidious.]

B.5.6 SCHOLIUM We have

$$-\backslash\underline{I} \otimes_{\underline{I}} F \approx \underline{INT}_{\underline{I}}F.$$

[Note: Let $K:\underline{I} \to \underline{J}$ be a functor -- then for all $G \in \text{Ob } [\underline{J},\underline{CAT}]$,

$$-\backslash\underline{I} \otimes_{\underline{J}} G \approx \underline{INT}_{\underline{I}}K^*G,$$

where in this context $-\backslash\underline{I}$ sends j to $j\backslash\underline{I}$.

B.5.7 REMARK If $F:\underline{I}^{OP} \to \underline{CAT}$, then

$$F \otimes_{\underline{I}} \underline{I}/- \approx \overline{INT}_{\underline{I}}F.$$

[Note: Let $K:\underline{I} \to \underline{J}$ be a functor -- then for all $G \in \text{Ob } [\underline{J}^{OP},\underline{CAT}]$,

$$G \otimes_{\underline{J}} \underline{I}/- \approx \overline{INT}_{\underline{I}}(K^{OP})^*G,$$

where in this context $\underline{I}/-$ sends j to $\underline{I}/j$.]

B.6 *HOMOTOPICAL MACHINERY*

Recall:

- In SISET, a simplicial weak equivalence is a simplicial map $f:X \to Y$ such that $|f|:|X| \to |Y|$ is a homotopy equivalence.
- In $\underline{CAT}$, a simplicial weak equivalence is a functor $F:\underline{C} \to \underline{D}$ such that $|\text{ner } F|:B\underline{C} \to B\underline{D}$ is a homotopy equivalence.

N.B. Therefore a functor $F:\underline{C} \to \underline{D}$ is a simplicial weak equivalence iff ner F:ner $\underline{C} \to$ ner $\underline{D}$ is a simplicial weak equivalence.

B.6.1 LEMMA If $F:\underline{C} \to \underline{D}$ is a functor and if ner F:ner $\underline{C} \to$ ner $\underline{D}$ is simplicially homotopic to a simplicial weak equivalence, then $F:\underline{C} \to \underline{D}$ is a simplicial weak equivalence.

B.6.2 NOTATION Let $\mathcal{W}_\infty$ denote the class of simplicial weak equivalences in $\underline{\mathrm{CAT}}$ (a.k.a. the class of weak equivalences per $\underline{\mathrm{CAT}}$ (External Structure) (cf. 0.13)).

B.6.3 EXAMPLE Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck prefibration -- then $\forall\ Y \in \mathrm{Ob}\ \underline{D}$, the canonical functor $\underline{C}_Y \to Y\backslash\underline{C}$ is a simplicial weak equivalence (cf. A.1.9).

B.6.4 EXAMPLE Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck preopfibration -- then $\forall\ Y \in \mathrm{Ob}\ \underline{D}$, the canonical functor $\underline{C}_Y \to \underline{C}/Y$ is a simplicial weak equivalence (cf. A.1.10).

B.6.5 THEOREM Fix a small category $\underline{I}$ and let

$$\left[\begin{array}{l} \underline{C} \xrightarrow{p} \underline{I} \\ \\ \underline{D} \xrightarrow{q} \underline{I} \end{array}\right.$$

be objects in $\underline{\mathrm{CAT}}/\underline{I}$. Suppose that $\Phi:(\underline{C},p) \to (\underline{D},q)$ is a morphism in $\underline{\mathrm{CAT}}/\underline{I}$ ($q \circ \Phi = p$) such that $\forall\ i \in \mathrm{Ob}\ \underline{I}$, the arrow

$$\Phi/i:\underline{C}/i \to \underline{D}/i$$

is a simplicial weak equivalence -- then Φ is a simplicial weak equivalence.

PROOF

- The elements of $\text{ner}_n\, \underline{C}/i$ are the pairs

$$((X_0 \to \cdots \to X_n), pX_n \to i),$$

where $pX_n \to i$ is a morphism in $\underline{I}$. This said, define a bisimplicial set $T_{\underline{C}}$ by

$$T_{\underline{C}}([n],[m]) = \{((X_0 \to \cdots \to X_n), pX_n \to i_0), i_0 \to \cdots \to i_m\}.$$

- The elements of $\text{ner}_n\, \underline{D}/i$ are the pairs

$$((Y_0 \to \cdots \to Y_n), qY_n \to i),$$

where $qY_n \to i$ is a morphism in $\underline{I}$. This said, define a bisimplicial set $T_{\underline{D}}$ by

$$T_{\underline{D}}([n],[m]) = \{((Y_0 \to \cdots \to Y_n), qY_n \to i_0, i_0 \to \cdots \to i_m\}.$$

Then there is a map

$$T\Phi : T_{\underline{C}} \to T_{\underline{D}}$$

of bisimplicial sets given on vertexes by

$$T\Phi((X_0 \to \cdots \to X_n), pX_n \to i_0), i_0 \to \cdots \to i_m)$$

$$= ((\Phi X_0 \to \cdots \to \Phi X_n), q\Phi X_n \to i_0, i_0 \to \cdots \to i_m).$$

Fixing the second variable leads to a commutative diagram

$$\begin{tikzcd}
T_{\underline{C}}(-,[m]) \arrow[r] \arrow[d] & T_{\underline{D}}(-,[m]) \arrow[d] \\
\coprod\limits_{i_0 \to \cdots \to i_m} \text{ner}\, \underline{C}/i_0 \arrow[r] & \coprod\limits_{i_0 \to \cdots \to i_m} \text{ner}\, \underline{D}/i_0 .
\end{tikzcd}$$

By hypothesis, the horizontal arrow on the bottom is a simplicial weak equivalence.

Since the vertical arrows are isomorphisms, it follows that the horizontal arrow on the top is a simplicial weak equivalence. Therefore

$$\text{dia } T\Phi : \text{dia } T_{\underline{C}} \to \text{dia } T_{\underline{D}}$$

is a simplicial weak equivalence. On the other hand,

$$\left[\begin{array}{l} T_{\underline{C}}([n], —) \approx \coprod\limits_{X_0 \to \cdots \to X_n} \text{ner } pX_n\backslash\underline{I} \\[2ex] T_{\underline{D}}([n], —) \approx \coprod\limits_{Y_0 \to \cdots \to Y_n} \text{ner } qY_n\backslash\underline{I} \end{array}\right.$$

and since

$$\left[\begin{array}{l} pX_n\backslash\underline{I} \\[1ex] qY_n\backslash\underline{I} \end{array}\right.$$

have initial objects, the arrows

$$\left[\begin{array}{l} \coprod\limits_{X_0 \to \cdots \to X_n} \text{ner } pX_n\backslash\underline{I} \longrightarrow \coprod\limits_{X_0 \to \cdots \to X_n} \text{ner } \underline{1} \\[2ex] \coprod\limits_{Y_0 \to \cdots \to Y_n} \text{ner } qY_n\backslash\underline{I} \longrightarrow \coprod\limits_{Y_0 \to \cdots \to Y_n} \text{ner } \underline{1} \end{array}\right.$$

are simplicial weak equivalences. Therefore

$$\left[\begin{array}{l} \text{dia } T_{\underline{C}} \to \text{ner } \underline{C} \\[1ex] \text{dia } T_{\underline{D}} \to \text{ner } \underline{D} \end{array}\right.$$

are simplicial weak equivalences. Form now the commutative diagram

$$\begin{tikzcd} \text{dia } T_{\underline{C}} \arrow[r, "\text{dia } T\Phi"] \arrow[d] & \text{dia } T_{\underline{D}} \arrow[d] \\ \text{ner } \underline{C} \arrow[r, "\text{ner } \Phi"'] & \text{ner } \underline{D} \end{tikzcd}$$

to conclude that ner Φ is a simplicial weak equivalence.

B.6.6 APPLICATION Let $\underline{C},\underline{D}$ be small categories and let $F:\underline{C} \to \underline{D}$ be a functor. Assume: $\forall\ Y \in \text{Ob } \underline{D}$, the arrow $\underline{C}/Y \to \underline{1}$ is a simplicial weak equivalence -- then F is a simplicial weak equivalence.

[In B.6.5, take $\underline{I} = \underline{D}$, $p = F$, $q = \text{id}_{\underline{D}}$:

$$\begin{tikzcd} \underline{C} \arrow[r, "F"] \arrow[d, "p = F"'] & \underline{D} \arrow[d, "q = \text{id}_{\underline{D}}"] \\ \underline{D} \arrow[r, equal] & \underline{D}. \end{tikzcd}$$

With F playing the role of Φ, consider the diagram

$$\begin{tikzcd} \underline{C}/Y \arrow[r, "F/Y"] \arrow[d] & \underline{D}/Y \arrow[d] \\ \underline{1} \arrow[r, equal] & \underline{1}. \end{tikzcd}$$

The vertical arrow on the left is a simplicial weak equivalence (by assumption), while the vertical arrow on the right is a simplicial weak equivalence ($\underline{D}/Y$ has a final object). Therefore F/Y is a simplicial weak equivalence. As this is true of all $Y \in \text{Ob } \underline{D}$, it remains only to quote B.6.5.]

B.6.7 EXAMPLE Suppose that $F:\underline{C} \to \underline{D}$ is a Grothendieck preopfibration.

Assume: $\forall\ Y \in \text{Ob}\ \underline{D}$, $\underline{C}_Y$ is contractible -- then F is a simplicial weak equivalence.

[Bearing in mind B.6.4, consider the diagram

$$\begin{tikzcd} \underline{C}_Y \arrow[r] \arrow[d] & \underline{C}/Y \arrow[d] \\ \underline{1} \arrow[r, equal] & \underline{1} \end{tikzcd} .]$$

B.6.8 LEMMA Fix a small category $\underline{I}$ and let

$$\left[\begin{array}{l} \underline{C} \xrightarrow{p} \underline{I} \\ \\ \underline{D} \xrightarrow[q]{} \underline{I} \end{array}\right.$$

be Grothendieck preopfibrations. Suppose that $\Phi:(\underline{C},p) \to (\underline{D},q)$ is a morphism in $\underline{\text{CAT}}/\underline{I}$ $(q \circ \Phi = p)$ such that $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow of restriction

$$\Phi_i:\underline{C}_i \to \underline{D}_i$$

is a simplicial weak equivalence -- then Φ is a simplicial weak equivalence.

PROOF The horizontal arrows in the commutative diagram

$$\begin{tikzcd} \underline{C}_i \arrow[r] \arrow[d, "\Phi_i"'] & \underline{C}/i \arrow[d, "\Phi/i"] \\ \underline{D}_i \arrow[r] & \underline{D}/i \end{tikzcd}$$

are simplicial weak equivalences (cf. B.6.4), thus Φ/i is a simplicial weak equivalence from which the assertion (cf. B.6.5).

B.6.9 LEMMA Let

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\eta} & \underline{Y} \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle g} \\ \underline{X} & \xrightarrow[f]{} & \underline{Z} \end{array}$$

be a pullback square in $\underline{\text{CAT}}$. Suppose that f is a Grothendieck fibration and that for all $z \in \text{Ob}\ \underline{Z}$, the category $\underline{Y}/z$ is contractible -- then for all $x \in \text{Ob}\ \underline{X}$, the category $\underline{P}/x$ is contractible, hence ξ is a simplicial weak equivalence (cf. B.6.6).

B.6.10 LEMMA Let

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\eta} & \underline{Y} \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle g} \\ \underline{X} & \xrightarrow[f]{} & \underline{Z} \end{array}$$

be a pullback square in $\underline{\text{CAT}}$. Suppose that f is a Grothendieck fibration and g is a Grothendieck opfibration with contractible fibers -- then ξ is a simplicial weak equivalence.

PROOF The assumption on g implies that the $\underline{Y}/z$ are contractible (cf. B.6.4), hence that the $\underline{P}/x$ are contractible (cf. B.6.9). But ξ is a Grothendieck opfibration (cf. A.2.4), thus its fibers are contractible (cf. B.6.4), so ξ is a simplicial weak equivalence (cf. B.6.7).

What follows next is a list of results that dualize B.6.5 - B.6.10.

B.6.11 THEOREM Fix a small category $\underline{I}$ and let

$$\begin{bmatrix} \underline{C} \xrightarrow{p} \underline{I} \\ \\ \underline{D} \xrightarrow{q} \underline{I} \end{bmatrix}$$

be objects in $\underline{\underline{CAT}}/\underline{I}$. Suppose that $\Phi:(\underline{C},p) \rightarrow (\underline{D},q)$ is a morphism in $\underline{\underline{CAT}}/\underline{I}$ $(q \circ \Phi = p)$ such that $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow

$$i\backslash\Phi:i\backslash\underline{C} \rightarrow i\backslash\underline{D}$$

is a simplicial weak equivalence -- then Φ is a simplicial weak equivalence.

B.6.12 APPLICATION Let $\underline{C},\underline{D}$ be small categories and let $F:\underline{C} \rightarrow \underline{D}$ be a functor. Assume: $\forall\ Y \in \text{Ob}\ \underline{D}$, the arrow $Y\backslash\underline{C} \rightarrow \underline{1}$ is a simplicial weak equivalence -- then F is a simplicial weak equivalence.

B.6.13 EXAMPLE Suppose that $F:\underline{C} \rightarrow \underline{D}$ is a Grothendieck prefibration. Assume: $\forall\ Y \in \text{Ob}\ \underline{D}$, $\underline{C}_Y$ is contractible -- then F is a simplicial weak equivalence.

B.6.14 LEMMA Fix a small category $\underline{I}$ and let

$$\begin{cases} \underline{C} \xrightarrow{p} \underline{I} \\ \underline{D} \xrightarrow{q} \underline{I} \end{cases}$$

be Grothendieck prefibrations. Suppose that $\Phi:(\underline{C},p) \rightarrow (\underline{D},q)$ is a morphism in $\underline{\underline{CAT}}/\underline{I}$ $(q \circ \Phi = p)$ such that $\forall\ i \in \text{Ob}\ \underline{I}$, the arrow of restriction

$$\Phi_i:\underline{C}_i \rightarrow \underline{D}_i$$

is a simplicial weak equivalence -- then Φ is a simplicial weak equivalence.

B.6.15 LEMMA Let

$$\begin{tikzcd} \underline{P} \arrow[r, "\eta"] \arrow[d, "\xi"'] & \underline{Y} \arrow[d, "g"] \\ \underline{X} \arrow[r, "f"'] & \underline{Z} \end{tikzcd}$$

be a pullback square in $\underline{\mathrm{CAT}}$. Suppose that f is a Grothendieck opfibration and that for all $z \in \mathrm{Ob}\ \underline{Z}$, the category $z\backslash\underline{Y}$ is contractible -- then for all $x \in \mathrm{Ob}\ \underline{X}$, the category $x\backslash\underline{P}$ is contractible, hence ξ is a simplicial weak equivalence (cf. B.6.12).

B.6.16 LEMMA Let

$$\begin{tikzcd} \underline{P} \arrow[r, "\eta"] \arrow[d, "\xi"'] & \underline{Y} \arrow[d, "g"] \\ \underline{X} \arrow[r, "f"'] & \underline{Z} \end{tikzcd}$$

be a pullback square in $\underline{\mathrm{CAT}}$. Suppose that f is a Grothendieck opfibration and g is a Grothendieck fibration with contractible fibers -- then ξ is a simplicial weak equivalence.

B.7 *INVARIANCE THEORY*

Let $\underline{I}$ be a small category.

B.7.1 THEOREM Suppose given functors $F,F':\underline{I} \to \underline{\mathrm{CAT}}$ and $\Xi \in \mathrm{Nat}(F,F')$. Assume: $\forall\ i \in \mathrm{Ob}\ \underline{I}$,

$$\Xi_i:Fi \to F'i$$

is a simplicial weak equivalence -- then

$$\underline{\mathrm{INT}}_{\underline{I}}\Xi:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{\mathrm{INT}}_{\underline{I}}F'$$

is a simplicial weak equivalence.

PROOF The arrows

$$\begin{cases} \Theta_F:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{I} \\ \\ \Theta_{F'}:\underline{\mathrm{INT}}_{\underline{I}}F' \to \underline{I} \end{cases}$$

are Grothendieck opfibrations (cf. B.2.6) and

$$\Theta_{F'} \circ \underline{\mathrm{INT}}_{\underline{I}}\Xi = \Theta_F.$$

Moreover, $\forall\ i \in \mathrm{Ob}\ \underline{I}$,

$$\begin{cases} (\underline{\mathrm{INT}}_{\underline{I}}F)_i \approx Fi \\ \\ (\underline{\mathrm{INT}}_{\underline{I}}F')_i \approx F'i \end{cases} \qquad \text{(cf. B.2.3)}$$

with

$$(\underline{\mathrm{INT}}_{\underline{I}}\Xi)_i \longleftrightarrow \Xi_i.$$

That $\underline{\mathrm{INT}}_{\underline{I}}\Xi$ is a simplicial weak equivalence thus follows from B.6.8.

B.7.2 REMARK Consider $\underline{\mathrm{CAT}}$ in its external structure -- then $\underline{\mathrm{CAT}}$ is combinatorial, as is $[\underline{I},\underline{\mathrm{CAT}}]$ when equipped with its projective structure (cf. 0.26.5). Since the weak equivalences per $[\underline{I},\underline{\mathrm{CAT}}]$ are levelwise, the composite

$$[\underline{I},\underline{\mathrm{CAT}}] \xrightarrow{\underline{\mathrm{INT}}_{\underline{I}}} \underline{\mathrm{CAT}}/\underline{I} \xrightarrow{U_{\underline{I}}} \underline{\mathrm{CAT}}$$

induces a functor

$$\underline{\mathrm{int}}_{\underline{I}}:\underline{H}[\underline{I},\underline{\mathrm{CAT}}] \to \underline{\mathrm{HCAT}}$$

at the level of homotopy categories (cf. B.7.1). But it is not difficult to see that $\underline{\text{int}}_{\underline{I}}$ is a left adjoint for the functor

$$\underline{\text{HCAT}} \to \underline{H}[\underline{I},\underline{\text{CAT}}]$$

associated with the arrow $p_{\underline{I}}:\underline{I} \to \underline{1}$. Therefore

$$\underline{\text{int}}_{\underline{I}} = \text{hocolim}_{\underline{I}} \quad \text{(cf. 0.26.19).}$$

B.7.3 THEOREM Suppose given functors $F,F':\underline{I} \to \underline{\text{CAT}}$ and $\Xi \in \text{Nat}(F,F')$ plus functors $G,G':\underline{I}^{\text{OP}} \to \underline{\text{CAT}}$ and $\Omega \in \text{Nat}(G,G')$. Assume: $\forall\ i \in \text{Ob}\ \underline{I}$,

$$\left[\begin{array}{l} \Xi_i:Fi \to F'i \\ \\ \Omega_i:Gi \to G'i \end{array}\right.$$

are simplicial weak equivalences -- then the induced arrow

$$\Xi|\Omega:\overline{\underline{\text{INT}}}_{\underline{I}}(F,G) \to \overline{\underline{\text{INT}}}_{\underline{I}}(F',G')$$

is a simplicial weak equivalence.

PROOF There is a commutative diagram

$$\begin{tikzcd}
\overline{\underline{\text{INT}}}_{\underline{I}}(F,G) \arrow[r, "\Xi|\text{id}"] \arrow[d, "\Xi|\Omega"'] & \overline{\underline{\text{INT}}}_{\underline{I}}(F',G) \arrow[d, "\text{id}|\Omega"] \\
\overline{\underline{\text{INT}}}_{\underline{I}}(F',G') \arrow[r, equal] & \overline{\underline{\text{INT}}}_{\underline{I}}(F',G')
\end{tikzcd}$$

from which the factorization

$$\Xi|\Omega = \text{id}|\Omega \circ \Xi|\text{id}$$

and the claim is that $\Xi|\text{id}$ and $\text{id}|\Omega$ are simplicial weak equivalences. In view of

B.4.9, the projections

$$\begin{cases} q_G:\overline{\mathrm{INT}}_{\underline{I}}(F,G) \to \overline{\mathrm{INT}}_{\underline{I}}G \\ q_G':\overline{\mathrm{INT}}_{\underline{I}}(F',G) \to \overline{\mathrm{INT}}_{\underline{I}}G \end{cases}$$

are Grothendieck opfibrations and

$$q_G' \circ \Xi|\mathrm{id} = q_G.$$

The objects of $\overline{\mathrm{INT}}_{\underline{I}}G$ are the pairs (i,y), where $i \in \mathrm{Ob}\,\underline{I}$ and $Y \in \mathrm{Ob}\,Gi$, and from the definitions,

$$\begin{cases} \overline{\mathrm{INT}}_{\underline{I}}(F,G)_{(i,Y)} \approx Fi \\ \overline{\mathrm{INT}}_{\underline{I}}(F',G)_{(i,Y)} \approx F'i \end{cases}$$

with

$$(\Xi|\mathrm{id})_{(i,Y)} \longleftrightarrow \Xi_i.$$

That $\Xi|\mathrm{id}$ is a simplicial weak equivalence thus follows from B.6.8. And analogously for $\mathrm{id}|\Omega$ (use B.6.14).

B.8 *HOMOTOPY COLIMITS*

Let $(\underline{C}_1,W_1)$, $(\underline{C}_2,W_2)$ be category pairs, where W_1,W_2 satisfy the 2 out of 3 condition. Suppose that

$$\begin{cases} F:\underline{C}_1 \to \underline{C}_2 \\ G:\underline{C}_2 \to \underline{C}_1 \end{cases}$$

are an adjoint pair with arrows of adjunction

$$\left[\begin{array}{l}\mu: \mathrm{id}_{\underline{C}_1} \to G \circ F \\ \\ \nu: F \circ G \to \mathrm{id}_{\underline{C}_2}.\end{array}\right.$$

8.8.1 LEMMA The following conditions are equivalent.

(1) $W_1 = F^{-1}(W_2)$ and $\forall\ X_2 \in \mathrm{Ob}\ \underline{C}_2$, the arrow $\nu_{X_2}: FGX_2 \to X_2$ is in W_2.

(2) $W_2 = G^{-1}(W_1)$ and $\forall\ X_1 \in \mathrm{Ob}\ \underline{C}_1$, the arrow $\mu_{X_1}: X_1 \to GFX_1$ is in W_1.

PROOF

- <u>(1) => (2)</u> Given $X_1 \in \mathrm{Ob}\ \underline{C}_1$, we have

$$\nu_{FX_1} \circ F\mu_{X_1} = \mathrm{id}_{FX_1}.$$

But $\nu_{FX_1} \in W_2$, $\mathrm{id}_{FX_1} \in W_2$, so, since W_2 satisfies the 2 out of 3 condition, $F\mu_{X_1} \in W_2$, hence $\mu_{X_1} \in W_1$. There remains the contention that $W_2 = G^{-1}(W_1)$. Given an arrow $f_2: X_2 \to Y_2$ in Mor $\underline{C}_2$, consideration of the commutative diagram

$$\begin{array}{ccc} FGX_2 & \xrightarrow{\nu_{X_2}} & X_2 \\ {\scriptstyle FGf_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ FGY_2 & \longrightarrow & Y_2 \end{array}$$

implies that $f_2 \in W_2$ iff $FGf_2 \in W_2$. However, by hypothesis, $FGf_2 \in W_2$ iff $Gf_2 \in W_1$.

- <u>(2) => (1)</u> ... .

B.8.2 LEMMA Suppose that the equivalent conditions of B.8.1 are in force -- then

$$\begin{cases} F\mathcal{W}_1 \subset \mathcal{W}_2 \\ G\mathcal{W}_2 \subset \mathcal{W}_1, \end{cases}$$

thus

$$\begin{cases} F:(\underline{C}_1,\mathcal{W}_1) \to (\underline{C}_2,\mathcal{W}_2) \\ G:(\underline{C}_2,\mathcal{W}_2) \to (\underline{C}_1,\mathcal{W}_1) \end{cases}$$

are morphisms of category pairs, so there are unique functors

$$\begin{cases} \bar{F}:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2 \\ \bar{G}:\mathcal{W}_2^{-1}\underline{C}_2 \to \mathcal{W}_1^{-1}\underline{C}_1 \end{cases}$$

for which the diagrams

$$\begin{tikzcd} \underline{C}_1 \arrow[r, "F"] \arrow[d, "L_{\mathcal{W}_1}"'] & \underline{C}_2 \arrow[d, "L_{\mathcal{W}_2}"] \\ \mathcal{W}_1^{-1}\underline{C}_1 \arrow[r, "\bar{F}"'] & \mathcal{W}_2^{-1}\underline{C}_2 \end{tikzcd} \qquad \begin{tikzcd} \underline{C}_2 \arrow[r, "G"] \arrow[d, "L_{\mathcal{W}_2}"'] & \underline{C}_1 \arrow[d, "L_{\mathcal{W}_1}"] \\ \mathcal{W}_2^{-1}\underline{C}_2 \arrow[r, "\bar{G}"'] & \mathcal{W}_1^{-1}\underline{C}_1 \end{tikzcd}$$

commute (cf. 1.4.5).

B.8.3 LEMMA Suppose that the equivalent conditions of B.8.1 are in force --

then

$$\left[\begin{array}{l} \bar{F}:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2 \\ \\ \bar{G}:\mathcal{W}_2^{-1}\underline{C}_2 \to \mathcal{W}_1^{-1}\underline{C}_1 \end{array}\right.$$

are an adjoint pair (cf. 1.7.1) and the induced arrows of adjunction

$$\left[\begin{array}{l} \bar{\mu}:\mathrm{id}_{\mathcal{W}_1^{-1}\underline{C}_1} \to \bar{G} \circ \bar{F} \\ \\ \bar{\nu}:\bar{F} \circ \bar{G} \to \mathrm{id}_{\mathcal{W}_2^{-1}\underline{C}_2} \end{array}\right.$$

are natural isomorphisms, thus the adjoint situation $(\bar{F},\bar{G},\bar{\mu},\bar{\nu})$ is an adjoint equivalence of metacategories.

[Note: Bear in mind that

$$\left[\begin{array}{l} \forall\ X_2 \in \mathrm{Ob}\ \underline{C}_2,\ L_{\mathcal{W}_2}\nu_{X_2} \text{ is an isomorphism in } \mathcal{W}_2^{-1}\underline{C}_2 \\ \\ \forall\ X_1 \in \mathrm{Ob}\ \underline{C}_1,\ L_{\mathcal{W}_1}\mu_{X_1} \text{ is an isomorphism in } \mathcal{W}_1^{-1}\underline{C}_1.] \end{array}\right.$$

Let $\underline{I}$ be a small category.

- Denote by $\mathcal{W}_{\infty,\underline{I}}$ the levelwise simplicial weak equivalences in Mor $[\underline{I},\underline{CAT}]$,

i.e., the $\Xi \in \mathrm{Nat}(F,F')$ such that $\forall\ i \in \mathrm{Ob}\ \underline{I}$,

$$\Xi_i:Fi \to F'i$$

is a simplicial weak equivalence.

- Denote by $\mathcal{W}_\infty/\underline{I}$ the local simplicial weak equivalences in Mor $\underline{CAT}/\underline{I}$, i.e., the $\Phi \in \text{Mor}((\underline{C},p),(\underline{D},q))$ such that $\forall\ i \in \text{Ob}\ \underline{I}$,

$$\Phi/i:\underline{C}/i \rightarrow \underline{D}/i$$

is a simplicial weak equivalence.

Recall now the setup of B.2.12 which produced an adjoint pair

$$\begin{cases} \Gamma_{\underline{I}}:\underline{\underline{CAT}}/\underline{I} \rightarrow [\underline{I},\underline{\underline{CAT}}] \\ \\ \underline{\underline{INT}}_{\underline{I}}:[\underline{I},\underline{\underline{CAT}}] \rightarrow \underline{\underline{CAT}}/\underline{I}. \end{cases}$$

The claim then is that the equivalent conditions figuring in B.8.1 are realized by this data.

B.8.4 LEMMA We have

$$\mathcal{W}_\infty/\underline{I} = \Gamma_{\underline{I}}^{-1}(\mathcal{W}_{\infty,\underline{I}}).$$

PROOF For $\Phi \in \Gamma_{\underline{I}}^{-1}(\mathcal{W}_{\infty,\underline{I}}) \iff \Gamma_{\underline{I}}\Phi \in \mathcal{W}_{\infty,\underline{I}}$. And $\Gamma_{\underline{I}}\Phi = \Phi/—$.

B.8.5 LEMMA Let $F \in \text{Ob}[\underline{I},\underline{\underline{CAT}}]$ — then $\forall\ i \in \text{Ob}\ \underline{I}$, the functor

$$\nu_{F,i}:\underline{\underline{INT}}_{\underline{I}}F/i \rightarrow Fi \quad (\text{cf. B.2.12})$$

is a simplicial weak equivalence.

PROOF It suffices to show that $\nu_{F,i}$ admits a right adjoint

$$\rho_{F,i}:Fi \rightarrow \underline{\underline{INT}}_{\underline{I}}F/i.$$

Definition:

$$\begin{cases} \rho_{F,i}X = (i,X,i \xrightarrow{\text{id}_i} i) \quad (X \in \text{Ob}\ Fi) \\ \\ \rho_{F,i}f = (\text{id}_i,f) \quad (f \in \text{Mor}\ Fi). \end{cases}$$

Therefore the first condition of B.8.1 is satisfied and, as a consequence, B.8.3 is applicable.

B.8.6 THEOREM The adjoint pair

$$\begin{bmatrix} \overline{\Gamma_{\underline{I}}} \\ \\ \overline{\underline{INT}_{\underline{I}}} \end{bmatrix}$$

is an adjoint equivalence of categories:

$$\begin{bmatrix} \overline{\Gamma_{\underline{I}}}:(W_\infty/\underline{I})^{-1}\underline{CAT}/\underline{I} \to W_{\infty,\underline{I}}^{-1}[\underline{I},\underline{CAT}] \\ \\ \overline{\underline{INT}_{\underline{I}}}:W_{\infty,\underline{I}}^{-1}[\underline{I},\underline{CAT}] \to (W_\infty/\underline{I})^{-1}\underline{CAT}/\underline{I}. \end{bmatrix}$$

Let $\underline{I}$ and $\underline{J}$ be small categories, $K:\underline{I} \to \underline{J}$ a functor.

B.8.7 LEMMA The functor

$$K^*:[\underline{J},\underline{CAT}] \to [\underline{I},\underline{CAT}]$$

sends $W_{\infty,\underline{J}}$ to $W_{\infty,\underline{I}}$:

$$K^*W_{\infty,\underline{J}} \subset W_{\infty,\underline{I}}.$$

PROOF If $\Omega \in W_{\infty,\underline{J}}$, then $\forall\, j \in \mathrm{Ob}\,\underline{J}$, Ω_j is a simplicial weak equivalence, so $\forall\, i \in \mathrm{Ob}\,\underline{I}$,

$$(K^*\Omega)_i = \Omega_{Ki}$$

is a simplicial weak equivalence.

Therefore

$$K^*:([\underline{J},\underline{CAT}],W_{\infty,\underline{J}}) \to ([\underline{I},\underline{CAT}],W_{\infty,\underline{I}})$$

is a morphism of category pairs, thus there is a unique functor

$$\overline{K^*}:W_{\infty,\underline{J}}^{-1}[\underline{J},\underline{CAT}] \to W_{\infty,\underline{I}}^{-1}[\underline{I},\underline{CAT}]$$

for which the diagram

$$\begin{tikzcd}
{[\underline{J},\underline{CAT}]} \arrow[r, "K^*"] \arrow[d, "L_{W_{\infty,\underline{J}}}"'] & {[\underline{I},\underline{CAT}]} \arrow[d, "L_{W_{\infty,\underline{I}}}"] \\
W_{\infty,\underline{J}}^{-1}[\underline{J},\underline{CAT}] \arrow[r, "\overline{K^*}"'] & W_{\infty,\underline{I}}^{-1}[\underline{I},\underline{CAT}]
\end{tikzcd}$$

commutes.

Now take $\underline{CAT}$ in its external structure. Since $\underline{CAT}$ is combinatorial, the functor categories

$$\left[\begin{array}{l} [\underline{I},\underline{CAT}] \\ \\ [\underline{J},\underline{CAT}] \end{array}\right.$$

in their projective structure are also combinatorial (cf. 0.26.5) and we have an instance of the setup of 0.26.16:

$$[\underline{I},\underline{CAT}] \text{ (Projective Structure) } \underset{K_*}{\overset{K_!}{\rightleftarrows}} [\underline{J},\underline{CAT}] \text{ (Projective Structure).}$$

Therefore $\overline{K^*}$ admits a left adjoint

$$LK_!:\underline{H}[\underline{I},\underline{CAT}] \to \underline{H}[\underline{J},\underline{CAT}],$$

the homotopy colimit of K (cf. 0.26.19), the explication of which will be carried out below.

B.8.8 LEMMA The functor

$$\underline{CAT}/K:\underline{CAT}/\underline{I} \to \underline{CAT}/\underline{J}$$

sends $W_\infty/\underline{I}$ to $W_\infty/\underline{J}$:

$$\underline{CAT}/KW_\infty/\underline{I} \subset W_\infty/\underline{J}.$$

PROOF Consider

$$\begin{tikzcd}
\underline{C} \arrow[r, "\Phi"] \arrow[d, "p"'] & \underline{D} \arrow[d, "q"] \\
\underline{I} \arrow[r, equal] \arrow[d, "K"'] & \underline{I} \arrow[d, "K"] \\
\underline{J} \arrow[r, equal] & \underline{J}
\end{tikzcd}$$

where $q \circ \Phi = p$ and $\forall\ i \in \mathrm{Ob}\ \underline{I}$,

$$\Phi/i:\underline{C}/i \to \underline{D}/i$$

is a simplicial weak equivalence, the claim being that $\forall\ j \in \mathrm{Ob}\ \underline{J}$,

$$\Phi/j:\underline{C}/j \to \underline{D}/j$$

is a simplicial weak equivalence. To see this, form the commutative diagram

$$\begin{tikzcd}
\underline{C}/j \arrow[r, "\Phi/j"] \arrow[d, "p/j"'] & \underline{D}/j \arrow[d, "q/j"] \\
\underline{I}/j \arrow[r, equal] & \underline{I}/j
\end{tikzcd}$$

and let (i,g) be an object of $\underline{I}/j$ $(g:Ki \rightarrow j)$ -- then

$$\begin{cases} (\underline{C}/j)/(i,g) \approx \underline{C}/i \\ (\underline{D}/j)/(i,g) \approx \underline{D}/i \end{cases}$$

and

$$(\Phi/j)/(i,g) \longleftrightarrow \Phi/i.$$

Consequently,

$$\Phi/j:\underline{C}/j \rightarrow \underline{D}/j$$

is a simplicial weak equivalence (cf. B.6.5).

Therefore

$$\underline{CAT}/K:\underline{CAT}/\underline{I} \rightarrow \underline{CAT}/\underline{J}$$

is a morphism of category pairs, thus there is a unique functor

$$\overline{\underline{CAT}/K}:(W_\infty/\underline{I})^{-1}\underline{CAT}/\underline{I} \rightarrow (W_\infty/\underline{J})^{-1}\underline{CAT}/\underline{J}$$

for which the diagram

$$\begin{tikzcd}
\underline{CAT}/\underline{I} \arrow[r, "\underline{CAT}/K"] \arrow[d, "L_{W_\infty/\underline{I}}"'] & \underline{CAT}/\underline{J} \arrow[d, "L_{W_\infty/\underline{J}}"] \\
(W_\infty/\underline{I})^{-1}\underline{CAT}/\underline{I} \arrow[r, "\overline{\underline{CAT}/K}"] & (W_\infty/\underline{J})^{-1}\underline{CAT}/\underline{J}
\end{tikzcd}$$

commutes.

B.8.9 NOTATION Write $K_!$ for the composite

$$\Gamma_{\underline{J}} \circ \underline{CAT}/K \circ \underline{INT}_{\underline{I}},$$

so

$$K(!):[\underline{I},\underline{CAT}] \to [\underline{J},\underline{CAT}].$$

[Note: $K(!)$ is not to be confused with $K_!$ (the left adjoint of K^*).]

B.8.10 NOTATION Write $LK(!)$ for the composite

$$\overline{\Gamma_{\underline{J}}} \circ \overline{\underline{CAT}/K} \circ \overline{\underline{INT}_{\underline{I}}},$$

so

$$LK(!):\underline{H}[\underline{I},\underline{CAT}] \to \underline{H}[\underline{J},\underline{CAT}].$$

B.8.11 THEOREM $LK(!)$ is a left adjoint for $\overline{K^*}$, thus $LK(!)$ "is" $LK_!$.

PROOF Start with the adjoint pair

$$\left[\begin{array}{l} \overline{\Gamma_{\underline{J}}} \circ \overline{\underline{CAT}/K} \\ \\ \overline{\underline{INT}_{\underline{I}}} \circ \overline{K^*} \end{array}\right. \qquad \text{(cf. B.2.13).}$$

Then

$$\left[\begin{array}{l} \forall\ X \in \text{Ob}\ \underline{H}[\underline{I},\underline{CAT}] \\ \\ \forall\ Y \in \text{Ob}\ \underline{H}[\underline{J},\underline{CAT}], \end{array}\right.$$

$$\begin{aligned} &\text{Mor}(LK(!)X,Y) \\ &\quad = \text{Mor}(\overline{\Gamma_{\underline{J}}} \circ \overline{\underline{CAT}/K} \circ \overline{\underline{INT}_{\underline{I}}}\ X,Y) \\ &\quad \approx \text{Mor}(\overline{\underline{INT}_{\underline{I}}}\ X,\ \overline{\underline{INT}_{\underline{I}}} \circ \overline{K^*}\ Y) \\ &\quad \approx \text{Mor}(\overline{\Gamma_{\underline{I}}} \circ \overline{\underline{INT}_{\underline{I}}}\ X,\ \overline{K^*}Y) \end{aligned}$$

(cf. B.8.6)

$$\approx \mathrm{Mor}(\mathrm{id}_{\underline{H}[\underline{I},\underline{CAT}]}X,\overline{K^*Y})$$

$$= \mathrm{Mor}(X,\overline{K^*Y}).$$

B.8.12 SCHOLIUM The composite

$$\overline{\Gamma_{\underline{J}}} \circ \overline{\underline{CAT}/K} \circ \overline{\underline{INT}_{\underline{I}}}$$

is the homotopy colimit of K.

B.8.13 EXAMPLE Take $\underline{J} = \underline{1}$ and let $K = p_{\underline{I}}$ (the canonical arrow $\underline{I} \to \underline{1}$) — then $p_{\underline{I}}^*:\underline{CAT} \to [\underline{I},\underline{CAT}]$ is the constant diagram functor and its left adjoint $p_{\underline{I}!}$ is $\mathrm{colim}_{\underline{I}}:[\underline{I},\underline{CAT}] \to \underline{CAT}$, thus

$$\mathrm{hocolim}_{\underline{I}} = L\ \mathrm{colim}_{\underline{I}},$$

and $\forall\ F \in \mathrm{Ob}[\underline{I},\underline{CAT}]$,

$$\mathrm{hocolim}_{\underline{I}}F = \underline{INT}_{\underline{I}}F \quad \text{(cf. B.7.2).}$$

E.g.: Suppose that $F = F_{\underline{J}}$ (cf. B.2.8) — then

$$\mathrm{hocolim}_{\underline{I}}F_{\underline{J}} = \underline{INT}_{\underline{I}}F_{\underline{J}} \approx \underline{I} \times \underline{J}.$$

[Note: Given $F \in \mathrm{Ob}[\underline{I},\underline{CAT}]$, put $NF = \mathrm{ner} \circ F$, so $NF:\underline{I} \to \underline{SISET}$. Denote by $\underline{\coprod}\ NF$ the bisimplicial set for which

$$(\underline{\coprod}\ NF)\ ([n],[m])$$

are the pairs of strings

$$(i_0 \xrightarrow{\delta_0} i_1 \to \cdots \to i_{n-1} \xrightarrow{\delta_{n-1}} i_n, X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_{m-1} \xrightarrow{f_{m-1}} X_m),$$

where the $X_k \in \mathrm{Ob}\ Fi_0$ and the $f_k \in \mathrm{Mor}(Fi_0,Fi_0)$ $(0 \le k \le m)$, supplied with the evident horizontal and vertical operations. Using B.2.14, one can show that for any small category $\underline{C}$,

$$\mathrm{Mor}(\mathrm{dia} \coprod \mathrm{NF}, \mathrm{ner}\,\underline{C}) \approx \mathrm{Mor}(\underline{\mathrm{INT}}_{\underline{I}}F, \underline{C})$$

from which,

$$\mathrm{Mor}(\mathrm{cat}\ \mathrm{dia} \coprod \mathrm{NF}, \underline{C}) \approx \mathrm{Mor}(\underline{\mathrm{INT}}_{\underline{I}}F, \underline{C}),$$

thus

$$\mathrm{cat}\ \mathrm{dia} \coprod \mathrm{NF} \approx \underline{\mathrm{INT}}_{\underline{I}}F.$$

On the other hand, there is an arrow of adjunction

$$\begin{aligned} \mathrm{dia} \coprod \mathrm{NF} &\longrightarrow \mathrm{ner}\ \mathrm{cat}\ \mathrm{dia} \coprod \mathrm{NF} \\ &\xrightarrow{\approx} \mathrm{ner}\ \underline{\mathrm{INT}}_{\underline{I}}F \end{aligned}$$

and Thomason[†] proved that it is a simplicial weak equivalence.]

Keeping still to the assumption that $K:\underline{I} \to \underline{J}$ is a functor, there is an arrow of adjunction

$$\mathrm{LK}(!)\overline{K^*} \to \mathrm{id}_{\underline{H}[J,\underline{CAT}]} \qquad (\text{cf. B.8.11})$$

and

$$\begin{aligned} \underline{I} \xrightarrow{K} \underline{J} \longrightarrow \underline{1} &\Rightarrow p_{\underline{I}} = p_{\underline{J}} \circ K \\ &\Rightarrow \mathrm{L}p_{\underline{I}}(!) \circ \overline{p_{\underline{I}}^*} \\ &= \mathrm{L}p_{\underline{J}}(!) \circ \mathrm{LK}(!) \circ \overline{K^*} \circ \overline{p_{\underline{J}}^*} \end{aligned}$$

[†] *Math. Proc. Cambridge Philos. Soc.* 85 (1979), 91-109.

$$\to Lp_{\underline{J}}(!) \circ \mathrm{id}_{\underline{H}[J,\underline{CAT}]} \circ \overline{p_{\underline{J}}^*}$$

$$= Lp_{\underline{J}}(!) \circ \overline{p_{\underline{J}}^*}.$$

B.8.14 LEMMA The functor $K:\underline{I} \to \underline{J}$ is a simplicial weak equivalence iff the natural transformation

$$Lp_{\underline{I}}(!) \circ \overline{p_{\underline{I}}^*} \to Lp_{\underline{J}}(!) \circ \overline{p_{\underline{J}}^*}$$

is a natural isomorphism.

PROOF Given a small category $\underline{C}$, the arrow

$$\begin{aligned} L_{\mathcal{W}_\infty}(\underline{I} \times \underline{C}) &= \underline{I} \times \underline{C} \\ &\approx (Lp_{\underline{I}}(!) \circ \overline{p_{\underline{I}}^*})(L_{\mathcal{W}_\infty}\underline{C}) \\ &\to (Lp_{\underline{J}}(!) \circ \overline{p_{\underline{J}}^*})(L_{\mathcal{W}_\infty}\underline{C}) \\ &\approx \underline{J} \times \underline{C} = L_{\mathcal{W}_\infty}(\underline{J} \times \underline{C}) \end{aligned}$$

is precisely $L_{\mathcal{W}_\infty}(K \times \mathrm{id}_{\underline{C}})$ which is an isomorphism iff $K \times \mathrm{id}_{\underline{C}}$ is a simplicial weak equivalence ($\mathcal{W}_\infty$ is saturated (cf. 2.3.20)).

[Note: The product of two simplicial weak equivalences is a simplicial weak equivalence. On the other hand, if $\forall\ \underline{C}$, $K \times \mathrm{id}_{\underline{C}}$ is a simplicial weak equivalence, then K is a simplicial weak equivalence (take $\underline{C} = \underline{1}$).]

The position of the adjoint pair

$$\left[\begin{array}{l} \Gamma_{\underline{I}} \\ \underline{\mathrm{INT}}_{\underline{I}} \end{array}\right.$$

is clarified if $\underline{\mathrm{CAT}}$ is equipped with its internal structure (cf. 0.12) (which is inherited by $\underline{\mathrm{CAT}}/\underline{I}$) and $[\underline{I},\underline{\mathrm{CAT}}]$ is given the associated projective structure (thus the weak equivalences are levelwise as are the fibrations).

B.8.15 LEMMA The adjoint situation $(\Gamma_{\underline{I}},\underline{\mathrm{INT}}_{\underline{I}})$ is a model pair.

PROOF If $F,G \in \mathrm{Ob}[\underline{I},\underline{\mathrm{CAT}}]$, if $\Xi \in \mathrm{Nat}(F,G)$, and if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is an equivalence of categories, then the opfibered functor

$$\underline{\mathrm{INT}}_{\underline{I}}\Xi:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{\mathrm{INT}}_{\underline{I}}G$$

is an equivalence (cf. A.1.32). Accordingly, we have only to show that $\underline{\mathrm{INT}}_{\underline{I}}$ preserves fibrations. So suppose that $\Xi:F \to G$ is a levelwise fibration, the claim being that

$$\underline{\mathrm{INT}}_{\underline{I}}\Xi:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{\mathrm{INT}}_{\underline{I}}G$$

is a fibration in $\underline{\mathrm{CAT}}/\underline{I}$ (Internal Structure). To establish this, let $(i,X) \in \mathrm{Ob}\ \underline{\mathrm{INT}}_{\underline{I}}F$ and let $\psi:(\underline{\mathrm{INT}}_{\underline{I}}\Xi)(i,X) \to (j,Y)$ be an isomorphism in $\underline{\mathrm{INT}}_{\underline{I}}G$ -- then

$$(\underline{\mathrm{INT}}_{\underline{I}}\Xi)(i,X) = (i,\Xi_i X)$$

and $\psi = (\delta,g)$, where $\delta:i \to j$ is an isomorphism in $\underline{I}$ and $g:(G\delta)\Xi_i X\ (= \Xi_j(F\delta)X) \to Y$ is an isomorphism in Gj. Since $\Xi_j:Fj \to Gj$ is a fibration, $\exists$ an isomorphism $\gamma:(F\delta)X \to X'$ in Fj such that $\Xi_j\gamma = g$. Now put $\phi = (\delta,\gamma)$, thus $\phi:(i,X) \to (j,X')$ and

$$(\underline{\mathrm{INT}}_{\underline{I}}\Xi)\phi = (\delta,\Xi_j\gamma) = (\delta,g) = \psi.$$

B.8.16 REMARK If $\underline{I}$ is a groupoid, then the model pair $(\Gamma_{\underline{I}},\underline{\mathrm{INT}}_{\underline{I}})$ is a model equivalence.

C: CORRESPONDENCES

C: CORRESPONDENCES

C.1 *FUNDAMENTAL LOCALIZERS*

Suppose that $(\underline{CAT}, W)$ is a category pair, where $W \subset \text{Mor } \underline{CAT}$ is weakly saturated (cf. 2.3.14).

[Note: Therefore W contains the isomorphisms of $\underline{CAT}$.]

C.1.1 DEFINITION W is a fundamental localizer provided:

(1) If $\underline{I} \in \text{Ob } \underline{CAT}$ admits a final object, then the canonical arrow $p_{\underline{I}}:\underline{I} \to \underline{1}$ is in W.

(2) If $\underline{I} \in \text{Ob } \underline{CAT}$, if

$$\begin{bmatrix} \underline{C} \xrightarrow{p} \underline{I} \\ \\ \underline{D} \xrightarrow{q} \underline{I} \end{bmatrix}$$

are objects in $\underline{CAT}/\underline{I}$, and if $\Phi:(\underline{C},p) \to (\underline{D},q)$ is a morphism in $\underline{CAT}/\underline{I}$ $(q \circ \Phi = p)$ such that $\forall\ i \in \text{Ob } \underline{I}$, the arrow

$$\Phi/i:\underline{C}/i \to \underline{D}/i$$

is in W, then Φ is in W.

C.1.2 EXAMPLE The class W_{tr} consisting of all the elements of Mor $\underline{CAT}$ is a fundamental localizer, the trivial fundamental localizer.

C.1.3 EXAMPLE The class W_{gr} consisting of $id_{\underline{0}}:\underline{0} \to \underline{0}$ and all functors $F:\underline{I} \to \underline{J}$, where $\underline{I} \neq \underline{0}$ and $\underline{J} \neq \underline{0}$, is a fundamental localizer, the coarse fundamental localizer.

N.B. If $\mathcal{W}$ is a fundamental localizer and if

$$\mathcal{W}_{gr} \subset \mathcal{W} \subset \mathcal{W}_{tr},$$

then either $\mathcal{W} = \mathcal{W}_{gr}$ or $\mathcal{W} = \mathcal{W}_{tr}$ (cf. C.5.2).

C.1.4 EXAMPLE $\mathcal{W}_\infty$ is a fundamental localizer.

[$\mathcal{W}_\infty$ is saturated (being the weak equivalences for CAT (External Structure), so 2.3.20 can be cited), hence $\mathcal{W}_\infty$ is weakly saturated (cf. 2.3.15).

Ad (1): If $\underline{I}$ has a final object, then $\underline{I}$ is contractible and the canonical arrow $p_{\underline{I}}:\underline{I} \to \underline{1}$ is a simplicial weak equivalence.

Ad (2): This is B.6.5 verbatim.]

C.1.5 RAPPEL If X and Y are simplicial sets and if $f:X \to Y$ is a simplicial map, then f is an n-equivalence $(n \geq 0)$ if $\pi_0(f):\pi_0(X) \to \pi_0(Y)$ is bijective and if $\forall\ x \in X_0$, f induces an isomorphism

$$\pi_k(X,x) \to \pi_k(Y,f(x)) \qquad (1 \leq k \leq n)$$

of homotopy groups.

C.1.6 EXAMPLE The class $\mathcal{W}_n$ $(n \geq 0)$ consisting of those functors $F:\underline{I} \to \underline{J}$ such that ner F:ner $\underline{I} \to$ ner $\underline{J}$ is an n-equivalence is a fundamental localizer.

N.B. We have

$$\mathcal{W}_\infty \subset \mathcal{W}_n \subset \mathcal{W}_m \subset \mathcal{W}_0 \subset \mathcal{W}_{gr} \subset \mathcal{W}_{tr} \qquad (m \leq n)$$

and

$$\mathcal{W}_\infty = \bigcap_{n\geq 0} \mathcal{W}_n.$$

C.1.7 EXAMPLE Given a fundamental localizer W, form the derivator $D_{(\underline{CAT},W)}$ (cf. 3.2.1) -- then

$$W_{D(\underline{CAT},W)} \quad \text{(cf. 3.5.2)}$$

coincides with W (cf. C.1.13).

[Note: Fundamental localizers are necessarily saturated (cf. C.9.3).]

C.1.8 REMARK Suppose that D is a right (left) homotopy theory -- then W_D is a fundamental localizer (cf. 3.5.17).

Let $C \subset \text{Mor } \underline{CAT}$ -- then the fundamental localizer generated by C, denoted $W(C)$, is the intersection of all the fundamental localizers containing C. The <u>minimal fundamental localizer</u> is $W(\emptyset)$ ($\emptyset$ the empty set of morphisms).

<u>N.B.</u> It turns out that $W(\emptyset) = W_\infty$ (cf. C.7.1).

C.1.9 DEFINITION A fundamental localizer is <u>admissible</u> if it is generated by a set of morphisms of $\underline{CAT}$.

C.1.10 EXAMPLE W_{tr} is an admissible fundamental localizer. In fact,

$$W(\{\underline{0} \to \underline{1}\}) = W_{tr}.$$

C.1.11 EXAMPLE W_{gr} is an admissible fundamental localizer. In fact,

$$W(\{\underline{1} \coprod \underline{1} \to 1\}) = W_{gr} \quad \text{(cf. C.5.4)}.$$

The formal aspects of "fundamental localizer theory" are spelled out in sections C.2 and C.3 below. Here I want to point out that certain important results that were stated and proved earlier for $W = W_\infty$ are true for any W. In particular: This is the case of B.7.1, B.8.6, and B.8.11.

C.1.12 EXAMPLE Take $\mathcal{W} = \mathcal{W}_0$ — then $\forall$ $\underline{I} \in \text{Ob}\ \underline{\text{CAT}}$, π_0 induces an isomorphism

$$\mathcal{W}_{0,\underline{I}}^{-1}[\underline{I},\underline{\text{CAT}}] \to [\underline{I},\underline{\text{SET}}].$$

If $K:\underline{I} \to \underline{J}$ is a functor, then

$$\overline{K^*}:\mathcal{W}_{0,\underline{J}}^{-1}[\underline{J},\underline{\text{CAT}}] \to \mathcal{W}_{0,\underline{I}}^{-1}[\underline{I},\underline{\text{CAT}}]$$

is identified with the functor

$$K^*:[\underline{J},\underline{\text{SET}}] \to [\underline{I},\underline{\text{SET}}]$$

and the functor

$$LK(!):\mathcal{W}_{0,\underline{I}}^{-1}[\underline{I},\underline{\text{CAT}}] \to \mathcal{W}_{0,\underline{J}}^{-1}[\underline{J},\underline{\text{CAT}}]$$

is identified with the functor

$$K_!:[\underline{I},\underline{\text{SET}}] \to [\underline{J},\underline{\text{SET}}].$$

C.1.13 REMARK Since $\mathcal{W}$ is saturated (cf. C.9.3), B.8.14 goes through with no change.

C.2 *SORITES*

Fix a fundamental localizer $\mathcal{W}$.

C.2.1 DEFINITION A functor $F:\underline{I} \to \underline{J}$ is <u>aspherical</u> if $\forall$ $j \in \text{Ob}\ \underline{J}$, the functor

$$F/j:\underline{I}/j \to \underline{J}/j$$

is in $\mathcal{W}$.

[Note: It then follows that F itself is in $\mathcal{W}$ (specialize condition (2) of C.1.1 in the obvious way (cf. B.6.6)).]

C.2.2 DEFINITION An object $\underline{I} \in \text{Ob}\ \underline{CAT}$ is <u>aspherical</u> if $p_{\underline{I}}:\underline{I} \to \underline{1}$ is aspherical (or, equivalently, if $p_{\underline{I}}:\underline{I} \to \underline{1}$ is in W).

[Note: Condition (1) of C.1.1 thus says that if $\underline{I}$ admits a final object, then $\underline{I}$ is aspherical.]

C.2.3 REMARK If $W \neq W_{tr}$, then

$$\underline{I} \text{ aspherical} \Rightarrow \underline{I} \neq \underline{0} \quad \text{(cf. C.5.1)}.$$

C.2.4 LEMMA The functor $F:\underline{I} \to \underline{J}$ is aspherical iff $\forall\ j \in \text{Ob}\ \underline{J}$, the category $\underline{I}/j$ is aspherical.

PROOF Since $\underline{J}/j$ has a final object, it is aspherical, thus the arrow $\underline{J}/j \to \underline{1}$ is in W. This said, consider the commutative diagram

$$\begin{tikzcd}
\underline{I}/j \arrow[r, "F/j"] \arrow[d, "p_{\underline{I}/j}"'] & \underline{J}/j \arrow[d, "p_{\underline{J}/j}"] \\
\underline{1} \arrow[r, equal] & \underline{1}
\end{tikzcd}.$$

C.2.5 LEMMA Suppose that the functor $F:\underline{I} \to \underline{J}$ admits a right adjoint $G:\underline{J} \to \underline{I}$ -- then F is aspherical.

PROOF $\forall\ i \in \text{Ob}\ \underline{I}$ and $\forall\ j \in \text{Ob}\ \underline{J}$, we have

$$\text{Mor}(Fi,j) \approx \text{Mor}(i,Gj).$$

Therefore the category $\underline{I}/j$ is isomorphic to the category $\underline{I}/Gj$. But $\underline{I}/Gj$ has a final object, thus $\underline{I}/Gj$ is aspherical, hence the same is true of $\underline{I}/j$ and one may then quote C.2.4.

C.2.6 EXAMPLE An equivalence of small categories is aspherical.

C.2.7 LEMMA If $\underline{I} \in \mathrm{Ob}\ \underline{CAT}$ admits an initial object i_0, then $\underline{I}$ is aspherical.

PROOF The functor $p_{\underline{I}}:\underline{I} \to \underline{1}$ is a right adjoint for the functor $K_{i_0}:\underline{1} \to \underline{I}$. Therefore K_{i_0} is aspherical (cf. C.2.5). But $p_{\underline{I}} \circ K_{i_0} = \mathrm{id}_{\underline{1}}$, thus $p_{\underline{I}}:\underline{I} \to \underline{1}$ is aspherical, i.e., $\underline{I}$ is aspherical.

C.2.8 LEMMA Let $\underline{C},\underline{D}$ be small categories, $F:\underline{C} \to \underline{D}$ a functor. Assume: F is a Grothendieck preopfibration -- then F is aspherical iff $\forall\ Y \in \mathrm{Ob}\ \underline{D}$, the fiber $\underline{C}_Y$ is aspherical.

PROOF The canonical functor

$$\underline{C}_Y \to \underline{C}/Y \qquad (X \to (X,\mathrm{id}_Y))$$

has a left adjoint $\underline{C}/Y \to \underline{C}_Y$ (cf. A.1.10), which is therefore aspherical (cf. C.2.5). Taking into account C.2.4, consider the commutative diagram

$$\begin{tikzcd}
\underline{C}/Y \arrow[r] \arrow[d] & \underline{C}_Y \arrow[d] \\
\underline{1} \arrow[r, equal] & \underline{1}
\end{tikzcd} .$$

C.2.9 LEMMA Let $F:\underline{I} \to \underline{J}$ be a functor -- then F is in $\mathcal{W}$ iff $F^{OP}:\underline{I}^{OP} \to \underline{J}^{OP}$ is in $\mathcal{W}$.

PROOF Consider the commutative diagram

$$\begin{tikzcd}
\underline{I}^{OP} \arrow[d, "F^{OP}"'] & \underline{I}(\sim>) \arrow[l, "s_{\underline{I}}"'] \arrow[r, "t_{\underline{I}}"] \arrow[d] & \underline{I} \arrow[d, "F"] \\
\underline{J}^{OP} & \underline{J}(\sim>) \arrow[l, "s_{\underline{J}}"] \arrow[r, "t_{\underline{J}}"'] & \underline{J}
\end{tikzcd} \qquad \text{(cf. A.1.33).}$$

Here the arrows $s_{\underline{I}}$, $t_{\underline{I}}$, $s_{\underline{J}}$, $t_{\underline{J}}$ are Grothendieck opfibrations and since their fibers admit an initial object, it follows from C.2.7 and C.2.8 that $s_{\underline{I}}$, $t_{\underline{I}}$, $s_{\underline{J}}$, $t_{\underline{J}}$ are aspherical, hence are in $\mathcal{W}$ (cf. C.2.1). Accordingly, if F is in $\mathcal{W}$, then the unlabeled vertical arrow is in $\mathcal{W}$, which implies that F^{OP} is in $\mathcal{W}$ and conversely.

C.2.10 APPLICATION Let $\underline{I} \in \text{Ob}\ \underline{CAT}$ — then $\underline{I}$ is aspherical iff $\underline{I}^{OP}$ is aspherical.

C.2.11 LEMMA Let $F:\underline{I} \to \underline{J}$ be a functor. Assume: F is a Grothendieck prefibration and $\forall\ j \in \text{Ob}\ \underline{J}$, the fiber $\underline{I}_j$ is aspherical — then F is in $\mathcal{W}$.

[The functor $F^{OP}:\underline{I}^{OP} \to \underline{J}^{OP}$ is a Grothendieck preopfibration and $\forall\ j \in \text{Ob}\ \underline{J}$, $(\underline{I}^{OP})_j = (\underline{I}_j)^{OP}$.]

C.2.12 LEMMA Suppose that $\underline{I}$ is aspherical -- then $\forall\ \underline{J}$, the projection $\underline{I} \times \underline{J} \to \underline{J}$ is in $\mathcal{W}$.

PROOF It suffices to show that $\forall\ j \in \text{Ob}\ \underline{J}$, the category $(\underline{I} \times \underline{J})/j$ is aspherical (cf. C.2.4). But

$$(\underline{I} \times \underline{J})/j \approx \underline{I} \times (\underline{J}/j)$$

and there is a commutative diagram

$$\begin{tikzcd}
\underline{I} \times (\underline{J}/j) \arrow[r] \arrow[d] & \underline{I} \arrow[d] \\
\underline{1} \arrow[r, equal] & \underline{1}
\end{tikzcd}$$

so, since $p_{\underline{I}}:\underline{I} \to \underline{1}$ is aspherical by hypothesis, one has only to prove that the arrow $\underline{I} \times (\underline{J}/j) \to \underline{I}$ is in $\mathcal{W}$. And to this end, it suffices to show that $\forall\ i \in \text{Ob}\ \underline{I}$, the category

$$(\underline{I} \times (\underline{J}/j))/i$$

is aspherical (cf. C.2.4). But

$$(\underline{I} \times (\underline{J}/j))/i \approx \underline{I}/i \times \underline{J}/j$$

and the category on the RHS admits a final object, hence is aspherical.

C.2.13 LEMMA If $\Phi:\underline{C} \to \underline{D}$ is in W, then $\forall$ $\underline{I}$, the arrow

$$\underline{C} \times \underline{I} \xrightarrow{\Phi \times \mathrm{id}_{\underline{I}}} \underline{D} \times \underline{I}$$

is in W.

[This is the relative version of C.2.12 (take $\underline{C} = \underline{I}$, $\underline{I} = \underline{J}$, $\underline{D} = \underline{1}$, $\Phi = p_{\underline{I}}$) and its proof runs along similar lines.]

C.2.14 LEMMA If $\underline{I} \in \mathrm{Ob}\ \underline{CAT}$, if

$$\begin{cases} \underline{C} \xrightarrow{p} \underline{I} \\ \underline{D} \xrightarrow{q} \underline{I} \end{cases}$$

are objects in $\underline{CAT}/\underline{I}$, and if $\Phi:(\underline{C},p) \to (\underline{D},q)$ is a morphism in $\underline{CAT}/\underline{I}$ ($q \circ \Phi = p$) which is aspherical, then $\forall$ $i \in \mathrm{Ob}\ \underline{I}$, the arrow

$$\Phi/i:\underline{C}/i \to \underline{D}/i$$

is aspherical.

C.2.15 LEMMA If $\underline{I} \in \mathrm{Ob}\ \underline{CAT}$, if

$$\begin{cases} \underline{C} \xrightarrow{p} \underline{I} \\ \underline{D} \xrightarrow{q} \underline{I} \end{cases}$$

are objects in $\underline{CAT}/\underline{I}$, and if $\Phi:(\underline{C},p) \to (\underline{D},q)$ is a morphism in $\underline{CAT}/\underline{I}$ $(q \circ \Phi = p)$ which is aspherical, then p is aspherical iff q is aspherical.

PROOF Given $i \in \mathrm{Ob}\,\underline{I}$, consider the commutative diagram

$$\begin{tikzcd} \underline{C}/i \arrow[r, "\Phi/i"] \arrow[d, "p/i"'] & \underline{D}/i \arrow[d, "q/i"] \\ \underline{I}/i \arrow[r, equal] & \underline{I}/i \end{tikzcd} .$$

Then Φ/i is aspherical (cf. C.2.14), hence is in $\mathcal{W}$. Therefore p/i is in $\mathcal{W}$ iff q/i is in $\mathcal{W}$, so p is aspherical iff q is aspherical.

C.2.16 DEFINITION Let $F:\underline{I} \to \underline{J}$ be in $\mathcal{W}$ — then F is <u>universally in $\mathcal{W}$</u> if for every pullback square

$$\begin{tikzcd} \underline{I}' \arrow[r] \arrow[d, "F'"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}' \arrow[r] & \underline{J} \end{tikzcd} ,$$

F' is in $\mathcal{W}$.

C.2.17 EXAMPLE If $p_{\underline{I}}:\underline{I} \to \underline{1}$ is in $\mathcal{W}$, then $p_{\underline{I}}$ is universally in $\mathcal{W}$ (cf. C.2.12) and conversely.

C.2.18 LEMMA If $F:\underline{I} \to \underline{J}$ is universally in $\mathcal{W}$, then F is aspherical.

PROOF $\forall\ j \in \mathrm{Ob}\,\underline{J}$, there is a pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] & \underline{J} \end{tikzcd} .$$

C.3 *STABILITY*

Fix a fundamental localizer W.

C.3.1 LEMMA If $\underline{I}_k$ $(k = 1,\ldots,n)$ are aspherical, then so is their product

$$\prod_{k=1}^{n} \underline{I}_k.$$

PROOF Take $n = 2$ — then the projection $\underline{I}_1 \times \underline{I}_2 \to \underline{I}_2$ is in W (cf. C.2.12). But $p_{\underline{I}_2}:\underline{I}_2 \to \underline{1}$ is in W, thus

$$p_{\underline{I}_1 \times \underline{I}_2}:\underline{I}_1 \times \underline{I}_2 \to \underline{1}$$

is in W.

C.3.2 LEMMA If

$$F_k:\underline{I}_k \to \underline{J}_k \quad (k = 1,\ldots,n)$$

are aspherical, then so is their product

$$\prod_{k=1}^{n} F_k: \prod_{k=1}^{n} \underline{I}_k \to \prod_{k=1}^{n} \underline{J}_k.$$

PROOF Take $n = 2$ and let $(j_1,j_2) \in \text{Ob } \underline{J}_1 \times \underline{J}_2$ -- then

$$(\underline{I}_1 \times \underline{I}_2)/(j_1,j_2) \approx \underline{I}_1/j_1 \times \underline{I}_2/j_2.$$

But the product on the RHS is aspherical (cf. C.3.1), thus $F_1 \times F_2$ is aspherical (cf. C.2.4).

C.3.3 LEMMA If

$$F_k:\underline{I}_k \to \underline{J}_k \quad (k = 1,\ldots,n)$$

are in W, then so is their product

$$\prod_{k=1}^{n} F_k: \prod_{k=1}^{n} \underline{I}_k \to \prod_{k=1}^{n} \underline{J}_k.$$

PROOF Take n = 2, decompose

$$F_1 \times F_2 : \underline{I}_1 \times \underline{I}_2 \to \underline{J}_1 \times \underline{J}_2$$

as the composition

$$\underline{I}_1 \times \underline{I}_2 \xrightarrow{F_1 \times \mathrm{id}_{\underline{I}_2}} \underline{J}_1 \times \underline{I}_2 \xrightarrow{\mathrm{id}_{\underline{J}_1} \times F_2} \underline{J}_1 \times \underline{J}_2,$$

and apply C.2.13.]

C.3.4 LEMMA If S is a set and if $\forall\ s \in S$, $F_s:\underline{I}_s \to \underline{J}_s$ is in W, then so is their coproduct

$$\coprod_s F_s: \coprod_s \underline{I}_s \to \coprod_s \underline{J}_s.$$

PROOF Let $F = \coprod F_s$ and let

$$\begin{cases} \underline{I} = \coprod_s \underline{I}_s \\ \underline{J} = \coprod_s \underline{J}. \end{cases}$$

Then there is a commutative diagram

$$\begin{array}{ccc} \underline{I} & \xrightarrow{F} & \underline{J} \\ \downarrow & & \downarrow \\ \mathrm{dis}\ S & = & \mathrm{dis}\ S \end{array}$$

and $\forall\ s \in \mathrm{Ob\ dis}\ S$, the arrow $F/s:\underline{I}/s \to \underline{J}/s$ can be identified with the arrow

$F_s:\underline{I}_s \to \underline{J}_s$. Therefore F is in $\mathcal{W}$ (recall condition (2) of C.1.1).

C.3.5 LEMMA Suppose that $\underline{I}$ is a filtered category and $F,G:\underline{I} \to \underline{CAT}$ are functors. Let $\Xi:F \to G$ be a natural transformation with the property that $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is in $\mathcal{W}$ — then

$$\text{colim}\ \Xi:\text{colim}\ F \to \text{colim}\ G$$

is in $\mathcal{W}$.

C.3.6 REMARK It follows that $\mathcal{W}$ is closed under the formation of retracts (take for $\underline{I}$ the category with one object and two morphisms $\{\text{id}_{\underline{I}},p\}$, where $p^2 = p$).

[Note: This is also a corollary to the fact that $\mathcal{W}$ is saturated (cf. C.9.3).]

C.3.7 LEMMA Suppose that $\left[\begin{array}{l}\underline{C}\\ \underline{D}\end{array}\right.$ are small categories. Let $F,G:\underline{C} \to \underline{D}$ be functors, $\Xi:F \to G$ a natural transformation -- then F is in $\mathcal{W}$ iff G is in $\mathcal{W}$.

PROOF Pass to the functor

$$\Xi_H:\underline{C} \times [1] \to \underline{D}$$

and denote by

$$\left[\begin{array}{l}e_0:[0] \to [1]\\ \\ e_1:[0] \to [1]\end{array}\right.$$

the obvious arrows -- then

$$\underline{C} \approx \underline{C} \times [0] \xrightarrow{\text{id}_{\underline{C}} \times e_0} \underline{C} \times [1] \xrightarrow{\Xi_H} \underline{D}$$

$$\underline{C} \approx \underline{C} \times [0] \xrightarrow[\text{id}_{\underline{C}} \times e_1]{} \underline{C} \times [1] \xrightarrow[\Xi_H]{} \underline{D}$$

with

$$\begin{cases} F = \Xi_H \circ (\mathrm{id}_{\underline{C}} \times e_0) \\ G = \Xi_H \circ (\mathrm{id}_{\underline{C}} \times e_1). \end{cases}$$

Since [1] has a final object, it is aspherical, thus the projection

$$\underline{C} \times [1] \xrightarrow{\mathrm{pr}} \underline{C}$$

is in $\mathcal{W}$ (cf. C.2.12). But

$$\mathrm{pr}\circ(\mathrm{id}_{\underline{C}} \times e_0) = \mathrm{id}_{\underline{C}} = \mathrm{pr}\circ(\mathrm{id}_{\underline{C}} \times e_1),$$

so

$$\begin{cases} \mathrm{id}_{\underline{C}} \times e_0 \\ \mathrm{id}_{\underline{C}} \times e_1 \end{cases}$$

are in $\mathcal{W}$. Therefore $F(G)$ is in $\mathcal{W}$ iff Ξ_H is in $\mathcal{W}$.

C.4 *SEGMENTS*

Fix a fundamental localizer $\mathcal{W}$.

C.4.1 DEFINITION A <u>segment</u> in <u>CAT</u> is a triple $(\mathcal{H},\partial_0,\partial_1)$ where $\mathcal{H} \in \mathrm{Ob}\ \underline{\mathrm{CAT}}$ is aspherical and $\partial_0,\partial_1:\underline{1} \to \mathcal{H}$ are morphisms in <u>CAT</u>.

C.4.2 EXAMPLE The triple $([1],e_0,e_1)$ figuring in C.3.7 is a segment.

Given a segment $(\mathcal{H},\partial_0,\partial_1)$ and a small category $\underline{C}$, let $\mathrm{pr}:\underline{C} \times \mathcal{H} \to \underline{C}$ be the projection -- then pr is in $\mathcal{W}$ (cf. C.2.12).

C.4.3 LEMMA $\forall\ \underline{C} \in \text{Ob}\ \underline{\text{CAT}}$, the morphisms

$$\begin{cases} \text{id}_{\underline{C}} \times \partial_0 \\ \text{id}_{\underline{C}} \times \partial_1 \end{cases}$$

are in $\mathcal{W}$.

PROOF One has only to note that

$$\text{pr} \circ (\text{id}_{\underline{C}} \times \partial_0) = \text{id}_{\underline{C}} = \text{pr} \circ (\text{id}_{\underline{C}} \times \partial_1).$$

C.4.4 DEFINITION Let $(\mathcal{H},\partial_0,\partial_1)$ be a segment in $\underline{\text{CAT}}$. Suppose that $\begin{cases} \underline{C} \\ \underline{D} \end{cases}$ are samll categories and let $F,G:\underline{C} \to \underline{D}$ be functors -- then F,G are said to be <u>$\mathcal{H}$-homotopic</u> if $\exists$ a morphism $H:\underline{C} \times \mathcal{H} \to \underline{D}$ such that

$$\begin{cases} F = H \circ (\text{id}_{\underline{C}} \times \partial_0) \\ \\ G = H \circ (\text{id}_{\underline{C}} \times \partial_1) \end{cases} \qquad (\underline{C} \approx \underline{C} \times \underline{1}).$$

C.4.5 LEMMA If $F,G:\underline{C} \to \underline{D}$ are $\mathcal{H}$-homotopic, then $L_{\mathcal{W}}F = L_{\mathcal{W}}G$.

PROOF Since $L_{\mathcal{W}}\text{pr}$ is an isomorphism in $\mathcal{W}^{-1}\underline{\text{CAT}}$,

$$L_{\mathcal{W}}\text{pr} \circ L_{\mathcal{W}}(\text{id}_{\underline{C}} \times \partial_0) = \text{id}_{L_{\mathcal{W}}\underline{C}} = L_{\mathcal{W}}\text{pr} \circ L_{\mathcal{W}}(\text{id}_{\underline{C}} \times \partial_1)$$

=>

$$L_{\mathcal{W}}(\text{id}_{\underline{C}} \times \partial_0) = L_{\mathcal{W}}(\text{id}_{\underline{C}} \times \partial_1).$$

Therefore

$$L_{\mathcal{W}}F = L_{\mathcal{W}}H \circ L_{\mathcal{W}}(\mathrm{id}_{\underline{C}} \times \partial_0) = L_{\mathcal{W}}H \circ L_{\mathcal{W}}(\mathrm{id}_{\underline{C}} \times \partial_1) = L_{\mathcal{W}}G.$$

[Note: It follows that F and G are homotopic in the sense of 1.3.1.]

C.4.6 LEMMA If $F,G:\underline{C} \to \underline{D}$ are $\mathcal{H}$-homotopic, then F is in $\mathcal{W}$ iff G is in $\mathcal{W}$.

PROOF In view of C.4.3, F(G) is in $\mathcal{W}$ iff H is in $\mathcal{W}$.

C.4.7 LEMMA Suppose that $\mathrm{id}_{\underline{C}}$ is $\mathcal{H}$-homotopic to $K_X \circ p_{\underline{C}}$ ($\exists\ X \in \mathrm{Ob}\ \underline{C}$) -- then $\underline{C}$ is aspherical.

PROOF Because $(\underline{C},\mathcal{W})$ is a category pair, $\mathrm{id}_{\underline{C}}$ is in $\mathcal{W}$, thus $K_X \circ p_{\underline{C}}$ is in $\mathcal{W}$ (cf. C.4.6). On the other hand, the composition

$$\underline{1} \xrightarrow{K_X} \underline{C} \xrightarrow{p_{\underline{C}}} \underline{1}$$

is $\mathrm{id}_{\underline{1}}$. So, since $\mathcal{W}$ is weakly saturated, $p_{\underline{C}}$ is in $\mathcal{W}$, i.e., $\underline{C}$ is aspherical.

C.4.8 THEOREM Suppose that $\Xi \in \mathrm{Nat}(\mathrm{id}_{\underline{C}},K_X \circ p_{\underline{C}})$ ($\exists\ X \in \mathrm{Ob}\ \underline{C}$) -- then $\underline{C}$ is aspherical.

PROOF In fact, $\mathrm{id}_{\underline{C}}$ is $\mathcal{H}$-homotopic to $K_X \circ p_{\underline{C}}$, where

$$(\mathcal{H},\partial_0,\partial_1) = ([1],e_0,e_1).$$

[Note: Bear in mind that [1] has a final object, hence is aspherical.]

C.4.9 EXAMPLE Consider the category $\underline{\Delta}/\underline{I}$ which is defined and discussed on pp. 28-30 of MATTERS SIMPLICIAL -- then, under the assumption that $\underline{I}$ has a final object i_0, we exhibited

$$\left[\begin{array}{l} \alpha \in \mathrm{Nat}(\mathrm{id}_{\underline{\Delta}/\underline{I}}, F) \\ \\ \beta \in \mathrm{Nat}(K_0, F). \end{array}\right.$$

Here

$$K_0 = K_{(0,K_{i_0})} \circ p_{\underline{\Delta}/\underline{I}}.$$

So, with

$$(\mathcal{H}, \partial_0, \partial_1) = ([1], e_0, e_1),$$

$\mathrm{id}_{\underline{\Delta}/\underline{I}}$ is $\mathcal{H}$-homotopic to F via α_H and K_0 is $\mathcal{H}$-homotopic to F via β_H. Therefore F is in $\mathcal{W}$, thus K_0 is in $\mathcal{W}$ (cf. C.4.6). Reasoning now as in C.4.7, the conclusion is that $p_{\underline{\Delta}/\underline{I}}$ is in $\mathcal{W}$ or still, that $\underline{\Delta}/\underline{I}$ is aspherical.

C.5 *STRUCTURE THEORY*

C.5.1 LEMMA If $\mathcal{W}$ is a fundamental localizer and if $\mathcal{W} \neq \mathcal{W}_{tr}$, then

$$\underline{I} \text{ aspherical} \Rightarrow \underline{I} \neq \underline{0}.$$

PROOF Suppose that $\underline{0}$ is aspherical. Since $\forall\ \underline{I} \in \mathrm{Ob}\ \underline{\mathrm{CAT}}$, there is a pullback square

$$\begin{array}{ccc} \underline{0} & \xrightarrow{\mathrm{id}_{\underline{0}}} & \underline{0} \\ \downarrow & & \downarrow p_{\underline{0}} \\ \underline{I} & \xrightarrow[p_{\underline{I}}]{} & \underline{1} \end{array},$$

it follows that the arrow $\underline{0} \to \underline{I}$ is in $\mathcal{W}$ (cf. C.2.17), hence $p_{\underline{I}}$ is in $\mathcal{W}$, i.e.,

$\underline{I}$ is aspherical. But this means that every morphism $F:\underline{C} \to \underline{D}$ in $\underline{CAT}$ is in W (write $p_{\underline{C}} = p_{\underline{D}} \circ F$), so $W = W_{tr}$, a contradiction.

C.5.2 APPLICATION If W is a fundamental localizer and if $W \supset W_{gr}$, then $W = W_{tr}$ or $W = W_{gr}$.

[Suppose that the containment $W \supset W_{gr}$ is proper, hence that there exists an arrow $\underline{0} \to \underline{I}$ in W ($\underline{I} \neq \underline{0}$). Consider the commutative diagram

$$\begin{tikzcd} \underline{I} \arrow[r, "p_{\underline{I}}"] & \underline{1} \\ \underline{0} \arrow[u] \arrow[r, equal] & \underline{0} \arrow[u, "p_{\underline{0}}"'] \end{tikzcd} .$$

Then $p_{\underline{I}}$ is in W_{gr}, thus is in W. Therefore $p_{\underline{0}}$ is in W or still, $\underline{0}$ is aspherical, so $W = W_{tr}$.]

C.5.3 LEMMA If W is a fundamental localizer and if $W \neq W_{tr}, W_{gr}$, then

$$\underline{I} \text{ aspherical} = \underline{I} \neq \underline{0} \ \& \ \#\pi_0(\underline{I}) = 1.$$

PROOF Owing to C.5.1, one has only to show that $\underline{I}$ is connected. Suppose false -- then there is a decomposition $\underline{I} = \underline{I}_0 \coprod \underline{I}_1$, where $\underline{I}_0, \underline{I}_1 \neq \underline{0}$. Choose $i_0 \in \text{Ob } \underline{I}_0$, $i_1 \in \text{Ob } \underline{I}_1$ and let

$$\left[\begin{array}{l} \partial_0 : \underline{1} \to \underline{I} \\ \partial_1 : \underline{1} \to \underline{I} \end{array}\right.$$

be the corresponding constant functors

$$\begin{cases} K_{i_0}:\underline{1} \to \underline{I} \\ K_{i_1}:\underline{1} \to \underline{I}. \end{cases}$$

Then $(\underline{I},\partial_0,\partial_1)$ is a segment ($\underline{I}$ being aspherical by assumption). Take now $\underline{C} \in \text{Ob } \underline{\underline{CAT}}$ ($\underline{C} \neq \underline{0}$) and fix $X \in \text{Ob } \underline{C}$. Denote by

$$\begin{cases} p_0:\underline{C} \times \underline{I}_0 \to \underline{C} \\ p_1:\underline{C} \times \underline{I}_1 \to \underline{C} \end{cases}$$

the projections and define

$$H:\underline{C} \times \underline{I} = (\underline{C} \times \underline{I}_0) \coprod (\underline{C} \times \underline{I}_1) \to \underline{C}$$

by

$$\begin{cases} H|(\underline{C} \times \underline{I}_0) = p_0 \\ H|(\underline{C} \times \underline{I}_1) = K_X \circ p_{\underline{C}} \circ p_1. \end{cases}$$

Then $\text{id}_{\underline{C}}$ is $\underline{I}$-homotopic to $K_X \circ p_{\underline{C}}$, thus $\underline{C}$ is aspherical (cf. C.4.7). Therefore every functor between nonempty categories is in W, so $W \supset W_{gr}$, a contradiction.

C.5.4 APPLICATION We have

$$W(\{\underline{1} \coprod \underline{1} \to \underline{1}\}) = W_{gr}.$$

[Per $W(\{\underline{1} \coprod \underline{1} \to \underline{1}\})$, $\underline{1} \coprod \underline{1}$ is aspherical, thus arguing as in C.5.3, one finds that every functor between nonempty categories is in $W(\{\underline{1} \coprod \underline{1} \to \underline{1}\})$, so

$$W(\{\underline{1} \coprod \underline{1} \to \underline{1}\}) \supset W_{gr}.$$

On the other hand, $\underline{1} \coprod \underline{1} \to \underline{1}$ is in W_{gr}, so

$$W_{gr} \supset W(\{\underline{1} \coprod \underline{1} \to \underline{1}\}).]$$

C.5.5 LEMMA If $\mathcal{W}$ is a fundamental localizer and if $\mathcal{W} \neq \mathcal{W}_{tr}, \mathcal{W}_{gr}$, then $\mathcal{W} \subset \mathcal{W}_0$.

[Note: Recall that $\mathcal{W}_0$ consists of those $F:\underline{I} \to \underline{J}$ such that $\pi_0(F):\pi_0(\underline{I}) \to \pi_0(\underline{J})$ is bijective.]

C.6 *PASSAGE TO PRESHEAVES*

Fix a fundamental localizer $\mathcal{W}$.

C.6.1 DEFINITION Let $\underline{C}$ be a small category. Given $F,G \in \text{Ob } \hat{\underline{C}}$ and $\Xi:F \to G$, call Ξ a <u>$\mathcal{W}$-equivalence</u> if

$$\underline{C}/\Xi:\underline{C}/F \to \underline{C}/G$$

is in $\mathcal{W}$.

C.6.2 NOTATION Write $\mathcal{W}_{\hat{\underline{C}}}$ for the class of $\mathcal{W}$-equivalences in $\text{Mor } \hat{\underline{C}}$, thus

$$\mathcal{W}_{\hat{\underline{C}}} = i_{\underline{C}}^{-1}\mathcal{W}.$$

[Note: It is clear that $(\hat{\underline{C}}, \mathcal{W}_{\hat{\underline{C}}})$ is a category pair and $\mathcal{W}_{\hat{\underline{C}}}$ satisfies the 2 out of 3 condition. Moreover,

$$i_{\underline{C}}:(\hat{\underline{C}}, \mathcal{W}_{\hat{\underline{C}}}) \to (\underline{\mathbf{CAT}}, \mathcal{W})$$

is a morphism of category pairs, thus there is a functor

$$\overline{i_{\underline{C}}}:\mathcal{W}_{\hat{\underline{C}}}^{-1}\hat{\underline{C}} \to \mathcal{W}^{-1}\underline{\mathbf{CAT}} \quad \text{(cf. 1.4.5).]}$$

C.6.3 REMARK To resolve a small matter of consistency, take $\mathcal{W} = \mathcal{W}_\infty$ and let $\underline{C} = \underline{\Delta}$ — then a simplicial map $f:X \to Y$ is a simplicial weak equivalence iff

$\mathrm{gro}_{\underline{\Delta}}\, f:\mathrm{gro}_{\underline{\Delta}}\, X \to \mathrm{gro}_{\underline{\Delta}}\, Y$ is a simplicial weak equivalence or still, in different but equivalent notation, iff $i_{\underline{\Delta}} f:\underline{\Delta}/X \to \underline{\Delta}/Y$ is a simplicial weak equivalence. Therefore

$$W_\infty = i_{\underline{\Delta}}^{-1} \mathcal{W}_\infty \quad \text{(cf. 0.24.3)}.$$

C.6.4 LEMMA $\mathcal{W}_{\hat{\underline{C}}}$ is weakly saturated.

C.6.5 LEMMA $\mathcal{W}_{\hat{\underline{C}}}$ is closed under the formation of retracts.

PROOF Suppose that Ξ is a retract of Ω, say

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d, "\Xi"'] & G \arrow[r, "\rho"] \arrow[d, "\Omega"'] & F \arrow[d, "\Xi"'] \\ F' \arrow[r, "\iota'"'] & G' \arrow[r, "\rho'"'] & F', \end{tikzcd}$$

where $\rho \circ \iota = \mathrm{id}_F$, $\rho' \circ \iota' = \mathrm{id}_{F'}$, and $\Omega \in \mathcal{W}_{\hat{\underline{C}}}$ -- then $i_{\underline{C}}\Xi$ is a retract of $i_{\underline{C}}\Omega$. But $i_{\underline{C}}\Omega \in \mathcal{W}$ and $\mathcal{W}$ is closed under the formation of retracts (cf. C.3.6), so $i_{\underline{C}}\Xi \in \mathcal{W}$ or still, $\Xi \in \mathcal{W}_{\hat{\underline{C}}}$.

C.6.6 THEOREM $\mathcal{W}_{\hat{\underline{C}}} \cap M$ is a stable class.

C.6.7 REMARK Recall the definition of $\hat{\underline{C}}$-localizer (cf. 0.21.4) -- then $\mathcal{W}_{\hat{\underline{C}}}$ satisfies conditions (1) and (3). However condition (2), which here would read "every morphism of presheaves having the RLP w.r.t. the class $M \subset \text{Mor}\ \hat{\underline{C}}$ of monomorphisms is in $W_{\hat{\underline{C}}}$", need not be true (for a characterization, cf. C.9.1).

C.6.8 LEMMA $W_{\hat{\underline{C}}} \cap M$ is a retract stable class.

[Both $W_{\hat{\underline{C}}}$ and M are stable under the formation of retracts.]

C.6.9 APPLICATION Let

$$J \subset W_{\hat{\underline{C}}} \cap M$$

be a set of morphisms — then

$$\text{cof } J = \text{LLP}(\text{RLP}(J)) \subset W_{\hat{\underline{C}}} \cap M \quad (\text{cf. } 0.20.4).$$

[Note: Bear in mind that $\hat{\underline{C}}$ is presentable.]

C.7 *MINIMALITY*

Our objective in this section is to establish the following result (conjectured by Grothendieck and proved by Cisinski[†]).

C.7.1 THEOREM If W is a fundamental localizer, then

$$W_\infty \subset W.$$

Postponing the details for now, if W is a fundamental localizer, then $\underline{\Delta}/\underline{I}$

[†] *Cahiers Topologie Géom. Différentielle* XLV-2 (2004), 109–140.

is aspherical provided $\underline{I}$ has a final object (cf. C.4.9).

N.B. From the definitions,

$$\underline{\Delta}/\underline{I} = \underline{\Delta}/\text{ner } \underline{I} = \text{gro}_{\underline{\Delta}} \text{ ner } \underline{I} = i_{\underline{\Delta}}\text{ner } \underline{I}.$$

E.g.:

$$\underline{\Delta}/[n] = i_{\underline{\Delta}}\Delta[n].$$

Write

$$\tau_{\underline{I}}:\underline{\Delta}/\underline{I} \to \underline{I}$$

for the functor that sends (m,u) to $u(m)$.

C.7.2 LEMMA A functor $F:\underline{I} \to \underline{J}$ induces a functor

$$\underline{\Delta}/F:\underline{\Delta}/\underline{I} \to \underline{\Delta}/\underline{J} \quad ((m,u) \to (m,F \circ u))$$

and the diagram

$$\begin{tikzcd} \underline{\Delta}/\underline{I} \arrow[r, "\underline{\Delta}/F"] \arrow[d, "\tau_{\underline{I}}"'] & \underline{\Delta}/\underline{J} \arrow[d, "\tau_{\underline{J}}"] \\ \underline{I} \arrow[r, "F"'] & \underline{J} \end{tikzcd}$$

commutes.

C.7.3 LEMMA The functor

$$\tau_{\underline{I}}:\underline{\Delta}/\underline{I} \to \underline{I}$$

is aspherical.

PROOF $\forall\, i \in \text{Ob } \underline{I}$,

$$(\underline{\Delta}/\underline{I})/i \approx \underline{\Delta}/(\underline{I}/i).$$

But $\underline{I}/i$ has a final object, so $\underline{\Delta}/(\underline{I}/i)$ is aspherical (cf. C.4.9), from which the assertion (cf. C.2.4).

C.7.4 LEMMA We have

$$\mathcal{W} = \text{ner}^{-1} i_{\underline{\Delta}}^{-1} \mathcal{W},$$

i.e.,

$$\mathcal{W} = \text{ner}^{-1} \mathcal{W}_{\hat{\underline{\Delta}}}.$$

PROOF Suppose that $F:\underline{I} \to \underline{J}$ is a functor -- then in the commutative diagram

$$\begin{tikzcd} \underline{\Delta}/\underline{I} \arrow[r, "\underline{\Delta}/F"] \arrow[d, "\tau_{\underline{I}}"'] & \underline{\Delta}/\underline{J} \arrow[d, "\tau_{\underline{J}}"] \\ \underline{I} \arrow[r, "F"'] & \underline{J} \end{tikzcd},$$

the vertical arrows are aspherical (cf. C.7.3), hence are in $\mathcal{W}$. Therefore F is in $\mathcal{W}$ iff $\underline{\Delta}/F$ is in $\mathcal{W}$ or still, F is in $\mathcal{W}$ iff $i_{\underline{\Delta}}\text{ner } F$ is in $\mathcal{W}$.

C.7.5 THEOREM If $\mathcal{W}$ is a fundamental localizer, then

$$W_\infty \subset \mathcal{W}_{\hat{\underline{\Delta}}} \ (= i_{\underline{\Delta}}^{-1}\mathcal{W}).$$

Admit this result momentarily -- then

$$C.7.5 \Rightarrow C.7.1.$$

Proof:

$$\mathcal{W}_\infty = \text{ner}^{-1} i_{\underline{\Delta}}^{-1} \mathcal{W}_\infty \quad \text{(cf. C.7.4)}$$

$$= \text{ner}^{-1}W_\infty \quad (\text{cf. C.6.3})$$

$$\subset \text{ner}^{-1}i_{\underline{\Delta}}^{-1}\mathcal{W} \quad (\text{cf. C.7.5})$$

$$= \mathcal{W} \quad (\text{cf. C.7.4}).$$

To deal with C.7.5, take an $f \in W_\infty$ and using the Kan structure on $\hat{\underline{\Delta}}$ (= $\underline{\text{SISET}}$), factor f as the composite of an acyclic cofibration and a Kan fibration (which is then necessarily acyclic).

C.7.6 FACT Acyclic cofibrations are in $\mathcal{W}_{\hat{\underline{\Delta}}}$.

[Let J be the set of inclusions $\Lambda[k,n] \to \Delta[n]$ $(0 \leq k \leq n, n \geq 1)$ -- then J is contained in $\mathcal{W}_{\hat{\underline{\Delta}}} \cap \mathcal{M}$ (cf. infra), hence

$$\text{cof } J = \text{LLP}(\text{RLP}(J)) \subset \mathcal{W}_{\hat{\underline{\Delta}}} \cap \mathcal{M} \quad (\text{cf. C.6.9}).$$

But cof J is precisely the class of acyclic cofibrations (cf. 0.20.15).]

[Note: The categories $i_{\underline{\Delta}}\Lambda[k,n]$, $i_{\underline{\Delta}}\Delta[n]$ are aspherical, thus the arrow

$$i_{\underline{\Delta}}\Lambda[k,n] \to i_{\underline{\Delta}}\Delta[n]$$

is in $\mathcal{W}$.]

C.7.7 LEMMA For every simplicial set X, the projection $X \times \Delta[1] \to X$ is in $\mathcal{W}_{\hat{\underline{\Delta}}}$.

PROOF It suffices to show that the functor

$$i_{\underline{\Delta}}(X \times \Delta[1]) \to i_{\underline{\Delta}}X$$

is aspherical and for this, we shall apply C.2.4. So let $([n],s)$ be an object of $i_{\underline{\Delta}}X$ -- then

$$(\underline{\Delta}/(X \times \Delta[1]))/([n],s)$$

$$\approx \underline{\Delta}/(\Delta[n] \times \Delta[1])$$

$$\approx \underline{\Delta}/(\mathrm{ner}[n] \times \mathrm{ner}[1])$$

$$\approx \underline{\Delta}/\mathrm{ner}([n] \times [1]).$$

Since the category $[n] \times [1]$ has a final object,

$$\underline{\Delta}/\mathrm{ner}([n] \times [1]) \equiv \underline{\Delta}/([n] \times [1])$$

is aspherical (cf. C.4.9).

C.7.8 FACT Acyclic Kan fibrations are in $\mathcal{W}_{\hat{\underline{\Delta}}}$.

[Let $p:X \to B$ be an acyclic Kan fibration. Because $\emptyset \to B$ is a cofibration, the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

has a filler $s:B \to X$, hence $p \circ s = \mathrm{id}_B$. We then claim that $s \circ p$ is in $\mathcal{W}_{\hat{\underline{\Delta}}}$ which, in view of C.6.4, will imply that p is in $\mathcal{W}_{\hat{\underline{\Delta}}}$. To see this, denote by

$$\phi:X \coprod X \to X$$

the arrow arising from consideration of

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_0"] \arrow[d, "\mathrm{id}_X"'] & X \coprod X & X \arrow[l, "\mathrm{in}_1"'] \arrow[d, "s \circ p"] \\ X \arrow[rr, equal] & & X \end{tikzcd}.$$

Proceed next from

$$\begin{tikzcd}
X \times \Delta[0] \approx X \arrow[r, "in_0"] \arrow[d, "id_X \times e_0"'] & X \coprod X & X \approx X \times \Delta[0] \arrow[l, "in_1"'] \arrow[d, "id_X \times e_1"] \\
X \times \Delta[0] \arrow[rr, equal] & & X \times \Delta[0]
\end{tikzcd}$$

to get a cofibration

$$X \coprod X \xrightarrow{h} X \times \Delta[1].$$

Let

$$H:X \times \Delta[1] \rightarrow X$$

be a filler for the commutative diagram

$$\begin{tikzcd}
X \coprod X \arrow[rr, "\phi"] \arrow[d, "h"'] & & X \arrow[d, "p"] \\
X \times \Delta[1] \arrow[r, "pr"'] & X \arrow[r, "p"'] & B .
\end{tikzcd}$$

Then H is a simplicial homotopy between id_X and $s \circ p$. But $pr \in \mathcal{W}_{\underline{\hat{\Delta}}}$ (cf. C.7.7). Therefore, arguing as in C.3.7,

$$id_X \in \mathcal{W}_{\underline{\hat{\Delta}}} \Rightarrow s \circ p \in \mathcal{W}_{\underline{\hat{\Delta}}}.]$$

C.8 *TEST CATEGORIES*

Fix a fundamental localizer $\mathcal{W}$.

C.8.1 EXAMPLE Take $\mathcal{W} = \mathcal{W}_{tr}$ -- then $\mathcal{W}^{-1}\underline{CAT}$ is equivalent to $\underline{1}$.

C.8.2 EXAMPLE Take $\mathcal{W} = \mathcal{W}_{gr}$ -- then $\mathcal{W}^{-1}\underline{CAT}$ is equivalent to [1].

C.8.3 EXAMPLE Take $\mathcal{W} = \mathcal{W}_0$ -- then $\mathcal{W}^{-1}\underline{CAT}$ is equivalent to $\underline{SET}$.

C.8.4 EXAMPLE Take $\mathcal{W} = \mathcal{W}_\infty$ -- then $\mathcal{W}^{-1}\underline{CAT}$ is equivalent to $\underline{HCW}$.

C.8.5 LEMMA Let $\underline{C}$ be a small category. Assume: The arrow

$$\overline{i_{\underline{C}}}:\mathcal{W}_{\hat{\underline{C}}}^{-1}\hat{\underline{C}} \to \mathcal{W}^{-1}\underline{CAT}$$

is an equivalence of metacategories — then $\underline{C}$ is aspherical.

PROOF To prove that $p_{\underline{C}}:\underline{C} \to \underline{1}$ is in $\mathcal{W}$, consider the commutative diagram

$$\begin{tikzcd}
\hat{\underline{C}} \arrow[r, "i_{\underline{C}}"] \arrow[d, "L_{\mathcal{W}_{\hat{\underline{C}}}}"'] & \underline{CAT} \arrow[d, "L_{\mathcal{W}}"] \\
\mathcal{W}_{\hat{\underline{C}}}^{-1}\hat{\underline{C}} \arrow[r, "\overline{i_{\underline{C}}}"'] & \mathcal{W}^{-1}\underline{CAT}.
\end{tikzcd}$$

Then it need only be shown that $L_{\mathcal{W}}p_{\underline{C}}$ is an isomorphism ($\mathcal{W}$ being saturated (cf. C.9.3)).

From the definitions, $i_{\underline{C}}(*_{\hat{\underline{C}}}) = \underline{C}$. And

$$\begin{aligned} L_{\mathcal{W}}(\underline{C}) &= (L_{\mathcal{W}} \circ i_{\underline{C}})(*_{\hat{\underline{C}}}) \\ &= (\overline{i_{\underline{C}}} \circ L_{\mathcal{W}_{\hat{\underline{C}}}})(*_{\hat{\underline{C}}}). \end{aligned}$$

But $L_{\mathcal{W}_{\hat{\underline{C}}}}(*_{\hat{\underline{C}}})$ is a final object in $\mathcal{W}_{\hat{\underline{C}}}^{-1}\hat{\underline{C}}$ (cf. 1.9.2) and since $\overline{i_{\underline{C}}}$ is, by hypothesis, an equivalence, hence sends final objects to final objects, it follows that $L_{W}(\underline{C})$ is a final object in $W^{-1}\underline{CAT}$. However $L_{W}(\underline{1})$ is also a final object in $W^{-1}\underline{CAT}$ (cf. 1.9.2), so

$$L_{W}p_{\underline{C}}:L_{W}(\underline{C}) \to L_{W}(\underline{1})$$

is an isomorphism.

C.8.6 DEFINITION Let $\underline{C}$ be a small category -- then $\underline{C}$ is said to satisfy <u>condition $\mathfrak{C}$</u> if $\forall\ \underline{I} \in \text{Ob}\ \underline{CAT}$, the arrow of adjunction

$$\nu_{\underline{I}}:i_{\underline{C}}i_{\underline{C}}^{*}\underline{I} \to \underline{I}$$

is in W.

C.8.7 REMARK Let

$$\begin{bmatrix} \underline{C}_1 = \hat{\underline{C}}\ , & W_1 = W_{\hat{\underline{C}}} \\ \\ \underline{C}_2 = \underline{CAT}, & W_2 = W \end{bmatrix}$$

and

$$\begin{bmatrix} F = i_{\underline{C}} \\ \\ G = i_{\underline{C}}^{*}. \end{bmatrix}$$

Then under the supposition that $\underline{C}$ satisfies condition $\mathfrak{C}$, condition (1) of B.8.1 is in force (by definition, $W_{\hat{\underline{C}}} = i_{\underline{C}}^{-1}W$). Therefore

$$W = (i_{\underline{C}}^{*})^{-1}W_{\hat{\underline{C}}}$$

and $\forall$ F $\in$ Ob $\hat{\underline{C}}$, the arrow of adjunction

$$\mu_F : F \to i^*_{\underline{C}} i_{\underline{C}} F$$

is in $W_{\hat{\underline{C}}}$. Furthermore

$$\left[\begin{array}{l} \overline{i_{\underline{C}}} : W^{-1}_{\hat{\underline{C}}}\hat{\underline{C}} \longrightarrow W^{-1}\underline{\mathrm{CAT}} \\ \\ \overline{i^*_{\underline{C}}} : W^{-1}\mathrm{CAT} \longrightarrow W^{-1}_{\hat{\underline{C}}}\hat{\underline{C}} \end{array}\right.$$

are an adjoint pair and the adjoint situation $(\overline{i_{\underline{C}}}, \overline{i^*_{\underline{C}}}, \bar{\mu}, \bar{\nu})$ is an adjoint equivalence of metacategories.

C.8.8 CRITERION Given $\underline{C}$ $\in$ Ob $\underline{\mathrm{CAT}}$, to verify condition $\mathfrak{T}$ for an arbitrary W, it suffices to verify condition $\mathfrak{T}$ for W_∞ (cf. C.7.1).

C.8.9 LEMMA If $\underline{C}$ satisfies condition $\mathfrak{T}$, then $\underline{C}$ is aspherical.

[This is implied by C.8.5, in conjunction with what was said above.]

C.8.10 DEFINITION A small category $\underline{C}$ is a <u>local test category</u> if $\forall$ X $\in$ Ob $\underline{C}$, $\underline{C}$/X satisfies condition $\mathfrak{T}$.

<u>N.B.</u> If $\underline{C}$ is a local test category, then $\forall$ X $\in$ Ob $\underline{C}$, $\underline{C}$/X is a local test category.

C.8.11 LEMMA If $\underline{C}$ is a local test category, then $\forall$ F $\in$ Ob $\hat{\underline{C}}$, $\underline{C}$/F is a local test category.

PROOF Given $(X,s) \in \mathrm{Ob}\ \underline{C}/F$, there is a canonical isomorphism

$$(\underline{C}/F)/(X,s) \approx \underline{C}/X.$$

[Note: This property is characteristic: If $\underline{C}$ is a small category such that $\forall\ F \in \mathrm{Ob}\ \hat{\underline{C}}$, $\underline{C}/F$ is a local test category, then $\underline{C}$ is a local test category.]

C.8.12 DEFINITION A small category $\underline{C}$ is a <u>test category</u> if

(1) $\underline{C}$ is a local test category

and

(2) $\underline{C}$ satisfies condition $\mathfrak{C}$.

<u>N.B.</u> If $\underline{C}$ is a test category, then the arrow

$$\overline{i_{\underline{C}}}: W_{\hat{\underline{C}}}^{-1}\hat{C} \to W^{-1}\underline{\mathrm{CAT}}$$

is an equivalence of metacategories.

C.8.13 LEMMA Suppose that $\underline{C}$ is a local test category -- then $\underline{C}$ is a test category iff $\underline{C}$ is aspherical.

C.8.14 EXAMPLE Take $W = W_{tr}$ -- then every small category is a test category.

C.8.15 EXAMPLE Take $W = W_{gr}$ -- then the test categories are the small nonempty categories.

[In view of C.5.1, a small category $\underline{C}$ is aspherical iff it is nonempty.]

C.8.16 LEMMA Suppose that $\underline{C}$ admits a final object -- then $\underline{C}$ is a local test category iff $\underline{C}$ is a test category.

C.8.17 LEMMA A small category $\underline{C}$ is a local test category iff $\forall\ X \in \text{Ob}\ \underline{C}$, the category $\underline{C}/X$ is a test category.

C.8.18 RAPPEL Given a small category $\underline{C}$, $M \subset \text{Mor}\ \hat{\underline{C}}$ is the class of monomorphisms and the elements of RLP(M) are called the trivial fibrations (cf. 0.21).

C.8.19 THEOREM Let $\underline{C}$ be a small category -- then $\underline{C}$ is a local test category iff

$$\text{RLP}(M) \subset \mathcal{W}_{\hat{\underline{C}}}.$$

C.8.20 EXAMPLE $\underline{\Delta}$ is a test category. Thus note first that $\underline{\Delta}$ has a final object (viz. [0]), hence is aspherical. So, to establish that $\underline{\Delta}$ is a local test category, it is enough to prove that $\underline{\Delta}$ is a test category per $\mathcal{W}_\infty$ (cf. C.8.8). To see this, consider $\hat{\underline{\Delta}}$ in its Kan structure -- then M is the class of cofibrations, RLP(M) is the class of acyclic Kan fibrations, and

$$(\mathcal{W}_\infty)_{\hat{\underline{\Delta}}} = i_{\underline{\Delta}}^{-1}\mathcal{W}_\infty = W_\infty \quad \text{(cf. C.6.3).}$$

Therefore

$$\text{RLP}(M) \subset (\mathcal{W}_\infty)_{\hat{\underline{\Delta}}}$$

and C.8.19 is applicable.

[Note: Here $i_{\underline{\Delta}} = \text{gro}_{\underline{\Delta}}$ and there is a commutative diagram

$$\begin{tikzcd} \underline{\text{SISET}} \arrow[r, "\text{gro}_{\underline{\Delta}}"] \arrow[d] & \underline{\text{CAT}} \arrow[d] \\ \underline{\text{HSISET}} \arrow[r, "\overline{\text{gro}}_{\underline{\Delta}}"'] & \underline{\text{HCAT}} \end{tikzcd} \quad \text{(cf. 0.24.3),}$$

where $\overline{\mathrm{gro}_{\underline{\Delta}}}$ is an equivalence of homotopy categories.]

C.8.21 REMARK $\underline{\Delta}_M$ is aspherical and satisfies condtion $\mathfrak{C}$. Still, if $W \neq W_{tr}, W_{gr}$, then $\underline{\Delta}_M$ is not a local test category.

[Suppose that $\underline{\Delta}_M$ is a local test category -- then the same is true of $\underline{\Delta}_M/[0] \approx \underline{1}$. But $\forall\ \underline{I} \in \mathrm{Ob}\ \underline{\mathrm{CAT}}$, $i_{\underline{1}} i_{\underline{1}}^* \underline{I} = \underline{I}_{dis}$ (the discrete category with objects those of $\underline{I}$). In particular: The discrete category $\{0,1\} = i_{\underline{1}} i_{\underline{1}}^* [1]$ would be aspherical ([1] is aspherical and the arrow $\{0,1\} \xrightarrow{\nu_{\underline{1}}} [1]$ is in W). This, however, is possible only if $W = W_{tr}$ or $W = W_{gr}$ (cf. C.5.3).]

C.8.22 LEMMA Suppose that $\underline{C}$ is a local test category — then for every small category $\underline{D}$, the product $\underline{C} \times \underline{D}$ is a local test category.

C.8.23 LEMMA Suppose that $\underline{C}$ is a test category — then for every small aspherical category $\underline{D}$, the product $\underline{C} \times \underline{D}$ is a test category.

[Recall that the product of two aspherical categories is aspherical (cf. C.3.1).]

C.8.24 EXAMPLE $\underline{\Delta} \times \underline{\Delta}$ is a test category.

C.9 *CISINSKI THEORY* (bis)

Fix a fundamental localizer W.

C.9.1 THEOREM Let $\underline{C}$ be a small category — then $\underline{C}$ is a local test category iff $W_{\hat{\underline{C}}}$ is a $\hat{\underline{C}}$-localizer.

PROOF Taking into account C.6.7, one has only to quote C.8.19.

C.9.2 LEMMA Let $F:\underline{I} \to \underline{J}$ be a morphism in $\underline{CAT}$ -- then F is in $\mathcal{W}$ iff $i_{\underline{\Delta}}^* F$ is in $\mathcal{W}_{\hat{\underline{\Delta}}}$.

PROOF Owing to C.8.20, $\underline{\Delta}$ is a test category, hence satisfies condition $\mathfrak{T}$ (cf. C.8.12). Therefore

$$\mathcal{W} = (i_{\underline{\Delta}}^*)^{-1}\mathcal{W}_{\hat{\underline{\Delta}}} \quad \text{(cf. C.8.7).}$$

Consequently,

$$F \in \mathcal{W} \iff F \in (i_{\underline{\Delta}}^*)^{-1}\mathcal{W}_{\hat{\underline{\Delta}}} \iff i_{\underline{\Delta}}^* F \in \mathcal{W}_{\hat{\underline{\Delta}}}.$$

C.9.3 $\mathcal{W}$ is saturated: $\mathcal{W} = \overline{\mathcal{W}}$.

PROOF Since

$$i_{\underline{\Delta}}^*:(\underline{CAT},\mathcal{W}) \to (\hat{\underline{\Delta}},\mathcal{W}_{\hat{\underline{\Delta}}})$$

is a morphism of category pairs (cf. C.9.2), there is a commutative diagram

$$\begin{tikzcd}
\underline{CAT} \arrow[r, "i_{\underline{\Delta}}^*"] \arrow[d, "L_{\mathcal{W}}"'] & \hat{\underline{\Delta}} \arrow[d, "L_{\mathcal{W}_{\hat{\underline{\Delta}}}}"] \\
\mathcal{W}^{-1}\underline{CAT} \arrow[r, "\overline{i_{\underline{\Delta}}^*}"'] & \mathcal{W}_{\hat{\underline{\Delta}}}^{-1}\hat{\underline{\Delta}}
\end{tikzcd} .$$

Suppose now that $L_{\mathcal{W}}F$ is an isomorphism in $\mathcal{W}^{-1}\underline{CAT}$ — then $\overline{i_{\underline{\Delta}}^*}L_{\mathcal{W}}F$ is an isomorphism in $W_{\hat{\underline{\Delta}}}^{-1}\hat{\underline{\Delta}}$ or still, $L_{W_{\hat{\underline{\Delta}}}} i_{\underline{\Delta}}^{*}F$ is an isomorphism in $W_{\hat{\underline{\Delta}}}^{-1}\hat{\underline{\Delta}}$. But $W_{\hat{\underline{\Delta}}}$ is a $\hat{\underline{\Delta}}$-localizer (cf. C.9.1), hence is saturated (cf. 0.21.9). Therefore $i_{\underline{\Delta}}^{*}F \in W_{\hat{\underline{\Delta}}}$ or still, $F \in W$.

C.9.4 REMARK The functor

$$\overline{i_{\underline{\Delta}}}:W_{\hat{\underline{\Delta}}}^{-1}\hat{\underline{\Delta}} \to W^{-1}\underline{\mathrm{CAT}}$$

is conservative.

C.9.5 THEOREM Suppose that W is an admissible fundamental localizer and $\underline{C}$ is a local test category — then $\hat{\underline{C}}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of $W_{\hat{\underline{C}}}$ and whose cofibrations are the monomorphisms:

$$W_{\hat{\underline{C}}},\ \mathrm{cof} = M,\ \mathrm{fib} = \mathrm{RLP}(W_{\hat{\underline{C}}} \cap M).$$

The central point is to establish that $W_{\hat{\underline{C}}}$ (which is a $\hat{\underline{C}}$-localizer (cf. C.9.1)) is necessarily admissible (for then one can cite 0.21.7). This is done in two steps.

Step 1: Prove it in the special case when $\underline{C} = \underline{\Delta}$.

[Note: If $W_{\hat{\underline{\Delta}}}$ is an accessible subcategory of $\hat{\underline{\Delta}}(\to)$, then $W_{\hat{\underline{\Delta}}}$ is necessarily admissible (cf. 0.25.9) but accessibility is not an a priori property.]

Step 2: Finesse the general case.

N.B. The composition

$$\mathrm{ner} \circ i_{\underline{C}}:\hat{\underline{C}} \to \hat{\underline{\Delta}}$$

preserves colimits and monomorphisms. In addition,

$$(\text{ner} \circ i_{\underline{C}})^{-1} \mathcal{W}_{\hat{\underline{\Delta}}} = \mathcal{W}_{\hat{\underline{C}}}.$$

C.9.6 LEMMA Let $\underline{C}_1, \underline{C}_2$ be small categories and let $F:\hat{\underline{C}}_1 \to \hat{\underline{C}}_2$ be a functor that preserves colimits and monomorphisms. Suppose that W_2 is a $\hat{\underline{C}}_2$-localizer and that $W_1 = F^{-1}W_2$ is a $\hat{\underline{C}}_1$-localizer — then

$$W_2 \text{ admissible} \Rightarrow W_1 \text{ admissible}.$$

[The argument is a lengthy workout in set-theoretic gymnastics.]

C.9.7 RAPPEL Let $\underline{C}$ be a small category — then the Cisinski structures on $\hat{\underline{C}}$ are left proper (but not necessarily right proper).

C.9.8 DEFINITION An admissible fundamental localizer $\mathcal{W}$ is <u>proper</u> if for every test category $\underline{C}$, $\mathcal{W}_{\hat{\underline{C}}}$ is proper, i.e., if the Cisinski structure on $\hat{\underline{C}}$ determined by $\mathcal{W}_{\hat{\underline{C}}}$ is proper.

C.9.9 LEMMA If $\mathcal{W}_{\hat{\underline{\Delta}}}$ is proper, then $\mathcal{W}$ is proper.

C.9.10 EXAMPLE The minimal fundamental localizer $\mathcal{W}_\infty$ is admissible (being equal to $\mathcal{W}(\emptyset)$) and proper.

[In fact,

$$(\mathcal{W}_\infty)_{\hat{\underline{\Delta}}} = i_{\underline{\Delta}}^{-1} \mathcal{W}_\infty = W_\infty$$

and the Cisinski structure on $\hat{\underline{\Delta}}$ determined by W_∞ is the Kan structure which is proper (cf. 0.3).]

C.9.11 REMARK It turns out that if W is proper, then for every local test category $\underline{C}$, $W_{\hat{\underline{C}}}$ is proper.

C.9.12 THEOREM Suppose that W is an admissible fundamental localizer. Let $\underline{C},\underline{C}'$ be local test categories and let $F:\underline{C} \to \underline{C}'$ be an aspherical functor. Equip

$$\left[\begin{array}{l} \hat{\underline{C}} \text{ with its Cisinski structure per } W_{\hat{\underline{C}}} \\ \\ \hat{\underline{C}}' \text{ with its Cisinski structure per } W_{\hat{\underline{C}}'}. \end{array}\right.$$

Then the adjoint situation

$$((F^{OP})^*, (F^{OP})_{+})$$

is a model pair that, moreover, is a model equivalence.

C.9.13 DEFINITION A <u>Thomason cofibration</u> is a cofibration in $\underline{CAT}$ (External Structure).

C.9.14 THEOREM Suppose that W is an admissible fundamental localizer — then $\underline{CAT}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of W and whose cofibrations are the Thomason cofibrations.

<u>N.B.</u> The proof is an elaboration of that used to equip $\underline{CAT}$ with its external structure (cf. 0.24.2).

C.9.15 REMARK The cofibrantly generated model structure on $\underline{CAT}$ determined by W is left proper and is right proper iff W is proper.

C.10 *CRITERIA*

Fix a fundamental localizer $\mathcal{W}$.

C.10.1 LEMMA Let $\underline{C}$ be a small category. Assume: $\forall\ \underline{I} \in \text{Ob}\ \underline{\text{CAT}}$ which admits a final object, the category

$$\underline{C}/i_{\underline{C}}^{*}\underline{I}$$

is aspherical -- then $\underline{C}$ satisfies condition $\mathfrak{C}$.

PROOF For any $\underline{I} \in \text{Ob}\ \underline{\text{CAT}}$, the arrow of adjunction

$$\nu_{\underline{I}}: i_{\underline{C}} i_{\underline{C}}^{*}\underline{I} \to \underline{I}$$

is aspherical, hence is in $\mathcal{W}$ (cf. C.2.1). In fact, $\forall\ i \in \text{Ob}\ \underline{I}$,

$$(i_{\underline{C}} i_{\underline{C}}^{*}\underline{I})/i \approx i_{\underline{C}} i_{\underline{C}}^{*}(\underline{I}/i)$$

and $\underline{I}/i$ has a final object. Now apply C.2.4.

C.10.2 DEFINITION Let $\underline{C}$ be a small category -- then a presheaf $F \in \text{Ob}\ \hat{\underline{C}}$ is said to satisfy the $\underline{\Omega\text{-condition}}$ if $\forall\ X \in \text{Ob}\ \underline{C}$, the category $\underline{C}/(h_X \times F)$ is aspherical.

[Note: If $\underline{C}$ admits a final object $*_{\underline{C}}$, then $h_{*_{\underline{C}}}$ is a final object for $\hat{\underline{C}}$, hence $\forall\ F \in \text{Ob}\ \hat{\underline{C}}$, $h_{*_{\underline{C}}} \times F \approx F$.]

<u>N.B.</u> Given an $X \in \text{Ob}\ \underline{C}$ and an $F \in \text{Ob}\ \hat{\underline{C}}$, let $F|(\underline{C}/X)$ be the presheaf induced by F on $\underline{C}/X$ — then

$$(\underline{C}/X)/(F|(\underline{C}/X)) \approx \underline{C}/h_X \times F).$$

C.10.3 LEMMA Let $\underline{C}$ be a small category. Assume: $\forall\ \underline{I} \in \text{Ob}\ \underline{CAT}$ which admits a final object, the presheaf $i_{\underline{C}}^*\underline{I}$ satisfies the Ω-condition -- then $\underline{C}$ is a local test category.

PROOF The claim is that $\forall\ X \in \text{Ob}\ \underline{C}$, $\underline{C}/X$ satisfies condition $\mathfrak{C}$ (cf. C.8.10). To establish this, it suffices to show that $\forall\ \underline{I} \in \text{Ob}\ \underline{CAT}$ which admits a final object, the category

$$(\underline{C}/X)/i_{\underline{C}/X}^*\underline{I}$$

is aspherical (cf. C.10.1). But

$$\begin{aligned}(\underline{C}/X)/i_{\underline{C}/X}^*\underline{I} \\ &\approx (\underline{C}/X)/(i_{\underline{C}}^*\underline{I}\,|\,(\underline{C}/X)) \\ &\approx \underline{C}/(h_X \times i_{\underline{C}}^*\underline{I})\end{aligned}$$

and the latter is aspherical by assumption.

C.10.4 CRITERION Let $\underline{C}$ be a small category. Assume: $i_{\underline{C}}^*[1]$ satisfies the Ω-condition -- then $\underline{C}$ is a local test category.

C.10.5 REMARK Using this criterion, Maltsiniotis† has given a direct elementary demonstration of the fact that $\underline{\Delta}$ is a local test category (cf. C.8.20).

[Note: Here $i_{\underline{\Delta}}^*[1] = \text{ner}\ [1] = \Delta[1]$, so it is a question of proving that $\underline{\Delta}/(\Delta[n] \times \Delta[1])$ is aspherical for all $n \geq 0$.]

Let $\underline{C}$ be a small category, $\iota:\underline{C} \to \underline{CAT}$ a functor -- then the nerve of ι is the

---

† *Astérisque* 301 (2005), 49-50.

functor

$$\mathrm{ner}_\iota:\underline{\mathrm{CAT}} \to \hat{\underline{C}}$$

defined by

$$\mathrm{ner}_\iota(\underline{I})(X) = \mathrm{Mor}(\iota X,\underline{I}) \quad (X \in \mathrm{Ob}\ \underline{C}).$$

<u>N.B.</u> If $\iota:\underline{C} \to \underline{\mathrm{CAT}}$ is the functor $X \to \underline{C}/X$, then $\underline{C}/X \approx \underline{C}/h_X$ and

$$\mathrm{Mor}(\iota X,\underline{I}) \approx \mathrm{Mor}(\underline{C}/h_X,\underline{I}).$$

Therefore

$$\mathrm{ner}_\iota \approx i^*_{\underline{C}} \quad \text{(cf. B.1.10).}$$

C.10.6 EXAMPLE Take $\underline{C} = \underline{\Delta}$ and let ι be the inclusion $\underline{\Delta} \to \underline{\mathrm{CAT}}$ -- then $\forall\ [n] \in \mathrm{Ob}\ \underline{\Delta}$,

$$\mathrm{ner}_\iota(\underline{I})([n]) = \mathrm{Mor}([n],\underline{I}) = \mathrm{ner}_n\underline{I}.$$

C.10.7 DEFINITION The functor $\iota:\underline{C} \to \underline{\mathrm{CAT}}$ satisfies the <u>finality hypothesis</u> if $\forall\ X \in \mathrm{Ob}\ \underline{C}$, ιX has a final object e_X.

C.10.8 EXAMPLE The inclusion $\underline{\Delta} \to \underline{\mathrm{CAT}}$ satisfies the finality hypothesis: $n \in \mathrm{Ob}\ [n]$ is a final object for $[n]$.

C.10.9 LEMMA Suppose that $\iota:\underline{C} \to \underline{\mathrm{CAT}}$ satisfies the finality hypothesis -- then there is a natural transformation

$$\Pi: i_{\underline{C}} \circ \mathrm{ner}_\iota \longrightarrow \mathrm{id}_{\underline{\mathrm{CAT}}}.$$

PROOF Let $\underline{I} \in \mathrm{Ob}\ \underline{\mathrm{CAT}}$ and recall that

$$i_{\underline{C}} \circ \mathrm{ner}_\iota\underline{I}$$

is the small category whose objects are the pairs (X,s), where $X \in \mathrm{Ob}\ \underline{C}$ and

$s:\iota X \to \underline{I}$ is a functor, and whose morphisms $(X,s) \to (Y,t)$ are the arrows $f:X \to Y$ such that $t \circ \iota(f) = s$ (cf. B.1.2). This said, define the functor

$$\Pi_{\underline{I}}:i_{\underline{C}} \circ \mathrm{ner}_{\iota}\underline{I} \to \underline{I}$$

on objects by

$$\Pi_{\underline{I}}(X,s) = s(e_X)$$

and on morphisms by

$$\Pi_{\underline{I}}(f) = s(e_X) \xrightarrow{f_{X,Y}} t(e_Y).$$

Explicated:

$$\iota(f):\iota X \to \iota Y$$

$$\Rightarrow$$

$$\iota(f)(e_X) \in \mathrm{Ob}\ \iota Y$$

$$\Rightarrow$$

$$\iota(f)(e_X) \xrightarrow{\exists !} e_Y$$

$$\Rightarrow$$

$$t(\iota(f)(e_X)) \xrightarrow{t(\exists !)} t(e_Y).$$

But

$$s(e_X) = t(\iota(f)(e_X)),$$

so

$$f_{X,Y} = t(\exists !).$$

C.10.10 EXAMPLE Take $\underline{C} = \underline{\Delta}$ and let ι be the inclusion $\underline{\Delta} \to \underline{\mathbf{CAT}}$ -- then

$\forall\ \underline{I} \in \text{Ob}\ \underline{\text{CAT}}$, $\pi_{\underline{I}}$ is the canonical arrow

$$\text{gro}_{\underline{\Delta}}(\text{ner}\ \underline{I}) \to \underline{I}.$$

C.10.11 **LEMMA** Suppose that $\iota:\underline{C} \to \underline{\text{CAT}}$ satisfies the finality hypothesis — then the following conditions are equivalent:

(1) $\forall\ \underline{I} \in \text{Ob}\ \underline{\text{CAT}}$ which admits a final object, the category

$$\underline{C}/\text{ner}_\iota \underline{I}$$

is aspherical.

(2) $\forall\ \underline{I} \in \text{Ob}\ \underline{\text{CAT}}$, the functor

$$\pi_{\underline{I}}:i_{\underline{C}} \circ \text{ner}_\iota \underline{I} \to \underline{I}$$

is in $\mathcal{W}$.

(3) $\forall\ \underline{I} \in \text{Ob}\ \underline{\text{CAT}}$, the functor

$$\pi_{\underline{I}}:i_{\underline{C}} \circ \text{ner}_\iota \underline{I} \to \underline{I}$$

is aspherical.

PROOF It is clear that (3) => (2) (cf. C.2.1). As for (2) => (1), bear in mind that

$$i_{\underline{C}} \circ \text{ner}_\iota \underline{I} = \underline{C}/\text{ner}_\iota \underline{I}$$

and consider the commutative diagram

$$\begin{tikzcd} \underline{C}/\text{ner}_\iota \underline{I} \arrow[r] \arrow[d, "\pi_{\underline{I}}"'] & \underline{1} \arrow[d, equal] \\ \underline{I} \arrow[r] & \underline{1}. \end{tikzcd}$$

Since $\underline{I}$ has a final object, the arrow $\underline{I} \to \underline{1}$ is in $\mathcal{W}$. Therefore the arrow

$$\underline{C}/\text{ner}_1\underline{I} \to \underline{1}$$

is in W, i.e.,

$$\underline{C}/\text{ner}_1\underline{I}$$

is aspherical. Finally, (1) => (3). To see this, it suffices to show that $\forall\ i \in \text{Ob}\ \underline{I}$, the category

$$(\underline{C}/\text{ner}_1\underline{I})/i$$

is aspherical (cf. C.2.4). But

$$(\underline{C}/\text{ner}_1\underline{I})/i \approx \underline{C}/\text{ner}_1(\underline{I}/i)$$

and $\underline{I}/i$ has a final object.

C.10.12 REMARK Maintain the assumptions of C.10.11 -- then

$$\text{ner}_1:(\underline{\text{CAT}},W) \to (\hat{\underline{C}},W_{\hat{\underline{C}}})$$

is a morphism of category pairs, thus there is a functor

$$\overline{\text{ner}_1}:W^{-1}\underline{\text{CAT}} \to W_{\hat{\underline{C}}}^{-1}\hat{\underline{C}} \quad \text{(cf. 1.4.5)}$$

and a natural isomorphism

$$\overline{i_{\underline{C}}} \circ \overline{\text{ner}_1} \to \text{id}_{W^{-1}\underline{\text{CAT}}}.$$

[Note: The last point requires additional argumentation and is not an a priori part of the overall picture. One is then led to ask: Is $\overline{\text{ner}_1}$ an equivalence? The answer is affirmative if $\underline{C}$ satisfies condtion $\mathfrak{C}$ (under this supposition, $\overline{i_{\underline{C}}}$ is an equivalence (cf. C.8.7).]

C.10.13 LEMMA Suppose that $\iota:\underline{C} \to \underline{CAT}$ satisfies the finality hypothesis. Assume: $\forall\ \underline{I} \in \text{Ob}\ \underline{CAT}$ which admits a final object, the presheaf $\text{ner}_\iota \underline{I}$ satisfies the Ω-condition -- then $\underline{C}$ is a local test category.

C.10.14 CRITERION Suppose that $\iota:\underline{C} \to \underline{CAT}$ satisfies the finality hypothesis. Assume: $\text{ner}_\iota [1]$ satisfies the Ω-condition -- then $\underline{C}$ is a local test category.

N.B. If $\iota:\underline{C} \to \underline{CAT}$ is the functor $X \to \underline{C}/X$, then ι satisfies the finality hypothesis. Therefore C.10.13 encompasses C.10.3 and C.10.14 encompasses C.10.4.

C.10.15 REMARK Keeping to the setup of C.10.13, assume in addition that $\underline{C}$ admits a final object -- then $\underline{C}$ is aspherical, hence is a test category (cf. C.8.13), so by definition, $\underline{C}$ satisfies condition $\mathfrak{C}$. On the other hand, $\forall\ \underline{I} \in \text{Ob}\ \underline{CAT}$,

$$h_{*_{\underline{C}}} \times \text{ner}_\iota \underline{I} \approx \text{ner}_\iota \underline{I},$$

thus

$$\underline{C}/\text{ner}_\iota \underline{I}$$

is aspherical. Therefore

$$\overline{\text{ner}_\iota}:\mathcal{W}^{-1}\underline{CAT} \to \mathcal{W}_{\hat{\underline{C}}}^{-1}\hat{\underline{C}}$$

is an equivalence of categories (cf. C.10.12).

C.10.16 EXAMPLE Take $\mathcal{W} = \mathcal{W}_\infty$, $\underline{C} = \underline{\Delta}$, $\iota:\underline{\Delta} \to \underline{CAT}$ the inclusion, $\text{ner}_\iota = \text{ner}$, and $i_{\underline{\Delta}} = \text{gro}_{\underline{\Delta}}$ -- then

$$\overline{\text{ner}}:\mathcal{W}_\infty^{-1}\underline{CAT} \to W_\infty^{-1}\hat{\underline{\Delta}}$$

is an equivalence of categories and there are natural isomorphisms

$$\left[\begin{array}{l} \overline{\mathrm{gro}_{\underline{\Delta}}} \circ \overline{\mathrm{ner}} \longrightarrow \mathrm{id}_{W_\infty^{-1}\underline{\mathrm{CAT}}} \\ \\ \overline{\mathrm{ner}} \circ \overline{\mathrm{gro}_{\underline{\Delta}}} \longrightarrow \mathrm{id}_{W_\infty^{-1}\hat{\underline{\Delta}}} \end{array}\right. \qquad \text{(cf. 0.24).}$$

D: LOCAL ISSUES

D.1 *A LOCAL CRITERION*

D.2 *FAILURE OF UBIQUITY*

D.3 *THEOREM B => THEOREM B*

D: LOCAL ISSUES

D.1 A LOCAL CRITERION

D.1.1 DEFINITION Let $\mathcal{W}$ be a fundamental localizer -- then a functor $F:\underline{I} \to \underline{J}$ is locally constant if for every morphism $j \to j'$ in $\underline{J}$, the functor

$$\underline{I}/j \to \underline{I}/j'$$

is in $\mathcal{W}$.

D.1.2 EXAMPLE If $F:\underline{I} \to \underline{J}$ is aspherical, then F is locally constant. To see this, consider the commutative diagram

$$\begin{tikzcd}
\underline{I}/j \arrow[r, "F/j"] \arrow[d] & \underline{J}/j \arrow[d] \\
\underline{I}/j' \arrow[r, "F/j'"'] & \underline{J}/j'.
\end{tikzcd}$$

Then the horizontal arrows are in $\mathcal{W}$ (F being aspherical). Furthermore, both $\left|\begin{array}{l}\underline{J}/j \\ \\ \underline{J}/j'\end{array}\right.$ have final objects, thus are aspherical. Therefore the arrow $\underline{J}/j \to \underline{J}/j'$ is in $\mathcal{W}$, hence the arrow $\underline{I}/j \to \underline{I}/j'$ is in $\mathcal{W}$.

D.1.3 EXAMPLE Let $F:\underline{I} \to \underline{\mathrm{CAT}}$ be a functor with the property that for all morphisms $i \xrightarrow{\delta} j$ in $\underline{I}$, the functor $Fi \xrightarrow{F\delta} Fj$ is in $\mathcal{W}$ -- then the Grothendieck opfibration

$$\Theta_F:\underline{\mathrm{INT}}_{\underline{I}}F \to \underline{I}$$

is locally constant.

D.1.4 THEOREM Take $\underline{\mathrm{CAT}}$ in its external structure and let $W = W_\infty$. Suppose that $F:\underline{I} \to \underline{J}$ is locally constant -- then $\forall\ j \in \mathrm{Ob}\ \underline{J}$, the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] & \underline{J} \end{tikzcd}$$

is a homotopy pullback.

[This is Cisinski's formulation of Quillen's "Theorem B" (cf. D.3.3 ff.).]

D.1.5 REMARK Within the setting of D.1.4, the converse is valid, a corollary being that the locally constant functors (per W_∞) are composition stable.

D.1.6 RAPPEL In a right proper model category $\underline{C}$, a commutative diagram

$$\begin{tikzcd} W \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd},$$

where f is a weak equivalence, is a homotopy pullback iff the arrow $W \xrightarrow{\eta} Y$ is a weak equivalence (cf. 0.35.2).

D.1.7 APPLICATION Take $\underline{\mathrm{CAT}}$ in its external structure and let $W = W_\infty$. Suppose that $F:\underline{I} \to \underline{J}$ is locally constant and a simplicial weak equivalence -- then $F:\underline{I} \to \underline{J}$ is aspherical.

[According to D.1.4, $\forall\ j \in \mathrm{Ob}\ \underline{J}$, the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] & \underline{J} \end{tikzcd}$$

is a homotopy pullback. But $\underline{CAT}$ (External Structure) is right proper, so the contention is implied by D.1.6.]

D.1.8 THEOREM Suppose that $\mathcal{W} \subset \mathcal{W}_0$ (cf. C.5.5) is a fundamental localizer. Assume: Every locally constant functor in $\mathcal{W}$ is aspherical -- then $\mathcal{W} = \mathcal{W}_\infty$.

Since $\mathcal{W}_\infty \subset \mathcal{W}$ (cf. C.7.1), it suffices to show that

$$\mathcal{W}_{\hat{\underline{\Delta}}} = i_{\underline{\Delta}}^{-1}\mathcal{W} \subset W_\infty.$$

Proof:

$$\begin{aligned} \mathcal{W} &= \text{ner}^{-1}\mathcal{W}_{\hat{\underline{\Delta}}} \quad (\text{cf. C.7.4}) \\ &\subset \text{ner}^{-1}W_\infty \\ &= \text{ner}^{-1}\ i_{\underline{\Delta}}^{-1}\mathcal{W}_\infty \quad (\text{cf. C.6.3}) \\ &= \mathcal{W}_\infty \quad (\text{cf. C.7.4}). \end{aligned}$$

D.1.9 LEMMA Let $p:X \to Y$ be a Kan fibration. Assume: $p \in \mathcal{W}_{\hat{\underline{\Delta}}}$ -- then $p \in W_\infty$.

Granted this result, it is easy to conclude matters. Thus given $f \in \mathcal{W}_{\hat{\underline{\Delta}}}$, write $f = p_f \circ i_f$, where i_f is an acyclic cofibration and p_f is a Kan fibration. So:

$$\left.\begin{array}{l} i_f \in W_\infty \subset W_{\hat{\underline{\Delta}}} \\ \\ f \in W_{\hat{\underline{\Delta}}} \end{array}\right] \Rightarrow p_f \in W_{\hat{\underline{\Delta}}} \Rightarrow p_f \in W_\infty \Rightarrow f \in W_\infty.$$

N.B. For use below, recall that

$$i_{\underline{\Delta}}:\hat{\underline{\Delta}} \to \underline{\text{CAT}}$$

preserves pullbacks (cf. B.1.9).

D.1.10 DEFINITION Let W be a $\hat{\underline{\Delta}}$-localizer — then a simplicial map $p:X \to Y$ is locally constant if given any diagram

$$\begin{tikzcd} \Delta[n] \times_Y X \arrow[r, "g"] \arrow[d] & \Delta[m] \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f"'] & \Delta[m] \arrow[r] & Y \end{tikzcd},$$

the arrow g is in W.

D.1.11 LEMMA A simplicial map $p:X \to Y$ is locally constant iff for any diagram

$$\begin{tikzcd} K \times_Y X \arrow[r, "g"] \arrow[d] & L \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ K \arrow[r, "f"'] & L \arrow[r] & Y \end{tikzcd}$$

with $f \in W_\infty$, there follows $g \in W$.

D.1.12 LEMMA Take $\hat{\underline{\Delta}}$ in its Kan structure and let $W = W_\infty$ — then $p:X \to Y$ is locally constant iff for every simplicial map $Z \to Y$, the pullback square

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow p \\ Z & \longrightarrow & Y \end{array}$$

is a homotopy pullback.

D.1.13 APPLICATION If $p:X \to Y$ is a Kan fibration, then p is locally constant (per W_∞) (cf. D.1.12). So, in the notation of D.1.11,

$$f \in W_\infty \Rightarrow g \in W_\infty \quad \text{(via propriety)}.$$

But $W_\infty \subset \mathcal{W}_{\hat{\underline{\Delta}}}$ (cf. C.7.5). Therefore p is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$).

D.1.14 LEMMA Take $W = \mathcal{W}_{\hat{\underline{\Delta}}}$ -- then a simplicial map $p:X \to Y$ is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$) iff $i_{\underline{\Delta}}p:i_{\underline{\Delta}}X \to i_{\underline{\Delta}}Y$ is locally constant (per $\mathcal{W}$).

PROOF Let $([n],s)$, $([m],t)$ be objects in $\underline{\Delta}/Y$ -- then a morphism $([n],s) \to ([m],t)$ corresponds to a diagram

$$\Delta[n] \to \Delta[m] \to Y$$

of simplicial sets and the pullback squares

$$\begin{array}{ccccc} \Delta[n] \times_Y X & \longrightarrow & \Delta[m] \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta[n] & \longrightarrow & \Delta[m] & \longrightarrow & Y \end{array}$$

in $\underline{\text{SISET}}$ induce pullback squares

$$\begin{tikzcd}
\underline{\Delta}/(\Delta[n] \times_Y X) \arrow[r] \arrow[d] & \underline{\Delta}/(\Delta[m] \times_Y X) \arrow[r] \arrow[d] & \underline{\Delta}/X \arrow[d, "\underline{\Delta}/p"] \\
\underline{\Delta}/\Delta[n] \arrow[r] & \underline{\Delta}/\Delta[m] \arrow[r] & \underline{\Delta}/Y
\end{tikzcd}$$

in $\underline{\mathrm{CAT}}$. The functor

$$(\underline{\Delta}/X)/([n],s) \longrightarrow (\underline{\Delta}/X)/([m],t)$$

is therefore isomorphic to the functor

$$\underline{\Delta}/(\Delta[n] \times_Y X) \longrightarrow \underline{\Delta}/(\Delta[m] \times_Y X).$$

In particular: If $p:X \to Y$ is a Kan fibration, then $i_{\underline{\Delta}}p:i_{\underline{\Delta}}X \to i_{\underline{\Delta}}Y$ is locally constant (per $\mathcal{W}$) (for p is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$) (cf. D.1.13)).

D.1.15 LEMMA Let $p:X \to Y$ be a simplicial map. Assume: p is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$) and in $\mathcal{W}_{\hat{\underline{\Delta}}}$ -- then for any pullback square

$$\begin{tikzcd}
X' = Y' \times_Y X \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' \arrow[r] & Y
\end{tikzcd},$$

p' is in $\mathcal{W}_{\hat{\underline{\Delta}}}$.

PROOF Pass to the pullback square

$$\begin{tikzcd}
i_{\underline{\Delta}}X' \arrow[r] \arrow[d, "i_{\underline{\Delta}}p'"'] & i_{\underline{\Delta}}X \arrow[d, "i_{\underline{\Delta}}p"] \\
i_{\underline{\Delta}}Y' \arrow[r] & i_{\underline{\Delta}}Y
\end{tikzcd}$$

in $\underline{\mathrm{CAT}}$ -- then $i_{\underline{\Delta}}p$ is locally constant (per $\mathcal{W}$ (cf. D.1.14) and in $\mathcal{W}$, thus is aspherical (by hypothesis) (cf. D.1.8). The claim is that $i_{\underline{\Delta}}p'$ is in $\mathcal{W}$ and for this, it will be enough to prove that $i_{\underline{\Delta}}p'$ is aspherical. Abusing the notation, let $y' \in \mathrm{Ob}\, i_{\underline{\Delta}}Y'$ and let $y \in \mathrm{Ob}\, i_{\underline{\Delta}}Y$ be its image. Consider the diagram

$$\begin{tikzcd}
i_{\underline{\Delta}}X'/y' \arrow[r] \arrow[d] & i_{\underline{\Delta}}X/y \arrow[r] \arrow[d] & i_{\underline{\Delta}}X \arrow[d, "i_{\underline{\Delta}}p"] \\
i_{\underline{\Delta}}Y'/y' \arrow[r] & i_{\underline{\Delta}}Y/y \arrow[r] & i_{\underline{\Delta}}Y
\end{tikzcd}$$

of pullback squares. Because $i_{\underline{\Delta}}p$ is aspherical, the arrow

$$i_{\underline{\Delta}}X/y \to i_{\underline{\Delta}}Y/y$$

is in $\mathcal{W}$. On the other hand, both $i_{\underline{\Delta}}Y'/y'$ and $i_{\underline{\Delta}}Y/y$ have final objects, hence the arrow

$$i_{\underline{\Delta}}Y/y' \to i_{\underline{\Delta}}Y/y$$

is in $\mathcal{W}_\infty \subset \mathcal{W}$. Now apply ner to get a diagram

$$\begin{tikzcd}
\mathrm{ner}\, i_{\underline{\Delta}}X'/y' \arrow[r] \arrow[d] & \mathrm{ner}\, i_{\underline{\Delta}}X/y \arrow[r] \arrow[d] & \mathrm{ner}\, i_{\underline{\Delta}}X \arrow[d, "\mathrm{ner}\, i_{\underline{\Delta}}p"] \\
\mathrm{ner}\, i_{\underline{\Delta}}Y'/y' \arrow[r] & \mathrm{ner}\, i_{\underline{\Delta}}Y/y \arrow[r] & \mathrm{ner}\, i_{\underline{\Delta}}Y
\end{tikzcd}$$

of pullback squares in $\underline{\mathrm{SISET}}$. Since $\mathrm{ner}\, i_{\underline{\Delta}}p$ is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$) and since the arrow

$$\mathrm{ner}\, i_{\underline{\Delta}}Y/y' \to \mathrm{ner}\, i_{\underline{\Delta}}Y/y$$

is in W_∞, it follows that the arrow

$$\mathrm{ner}\ i_{\underline{\Delta}}X'/y' \to \mathrm{ner}\ i_{\underline{\Delta}}X/y$$

is in $W_{\hat{\underline{\Delta}}}$ (cf. D.1.11). Therefore the arrow

$$i_{\underline{\Delta}}X'/y' \to i_{\underline{\Delta}}X/y$$

is in W (cf. C.7.4), which implies that the arrow

$$i_{\underline{\Delta}}X'/y' \to i_{\underline{\Delta}}Y'/y'$$

is in W, so $i_{\underline{\Delta}}p'$ is aspherical.

Consequently, if $p:X \to Y$ is a Kan fibration and if p is in $W_{\hat{\underline{\Delta}}}$, then for any pullback square

$$\begin{tikzcd}
X' = Y' \times_Y X \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' \arrow[r] & Y ,
\end{tikzcd}$$

p' is in $W_{\hat{\underline{\Delta}}}$.

D.1.16 EXAMPLE Let X be a Kan complex. Suppose that the arrow $X \to \Delta[0]$ is in $W_{\hat{\underline{\Delta}}}$ -- then the projections

$$\begin{cases} pr_1:X \times X \to X \\ pr_2:X \times X \to X \end{cases}$$

are in $W_{\hat{\underline{\Delta}}}$.

[Consider the pullback square

$$\begin{tikzcd} X \times X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d] \\ X \arrow[r] & \Delta[0] \end{tikzcd} \quad .]$$

D.1.17 LEMMA Suppose that $f:X \to Y$ is in $W_{\hat{\underline{\Delta}}}$ -- then $\pi_0(f):\pi_0(X) \to \pi_0(Y)$ is bijective.

PROOF Consider the commutative diagram

$$\begin{tikzcd} \text{ner } i_{\underline{\Delta}}X \arrow[r] \arrow[d, "\text{ner } i_{\underline{\Delta}}f"'] & X \arrow[d, "f"] \\ \text{ner } i_{\underline{\Delta}}Y \arrow[r] & Y \end{tikzcd} \qquad \begin{matrix} (i_{\underline{\Delta}}X = \text{gro}_{\underline{\Delta}}\, X) \\ \\ (i_{\underline{\Delta}}Y = \text{gro}_{\underline{\Delta}}\, Y). \end{matrix}$$

Since the horizontal arrows are simplicial weak equivalences, $\pi_0(f)$ is bijective iff $\pi_0(\text{ner } i_{\underline{\Delta}}f)$ is bijective. But $i_{\underline{\Delta}}f \in W$, so $\pi_0(i_{\underline{\Delta}}f)$ is bijective (recall that by hypothesis, $W \subset W_0$ (cf. D.1.8)), hence $\pi_0(\text{ner } i_{\underline{\Delta}}f)$ is bijective.

D.1.18 RAPPEL Let X be a Kan complex -- then the arrow $X \to \Delta[0]$ is a simplicial weak equivalence iff X is connected, nonempty, and $\forall\, x \in X_0$ & $\forall\, n \geq 1$, $\pi_n(X,x)$ is trivial.

D.1.19 LEMMA Let X be a Kan complex. Assume: The arrow $X \to \Delta[0]$ is in $W_{\hat{\Delta}}$ -- then the arrow $X \to \Delta[0]$ is in W_∞.

PROOF Owing to D.1.17, $\#\pi_0(X) = 1$, thus X is nonempty. This said, fix $x \in X_0$. Since X is Kan, the canonical arrow

$$\mathrm{map}(\Delta[1],X) \xrightarrow{q} \mathrm{map}(\dot{\Delta}[1],X) \approx X \times X$$

is a Kan fibration and the vertical arrows in the diagram

$$\begin{tikzcd}
\Omega(X,x) \arrow[r] \arrow[d] & \Theta(X,x) \arrow[r] \arrow[d] & \mathrm{map}(\Delta[1],X) \arrow[d, "q"] \\
\Delta[0] \arrow[r, "x"'] & X \arrow[r, "(\mathrm{id}_X, x)"'] \arrow[d] & X \times X \arrow[d, "\mathrm{pr}_2"] \\
 & \Delta[0] \arrow[r, "x"'] & X
\end{tikzcd}$$

are Kan fibrations. The composite

$$\mathrm{map}(\Delta[1],X) \to X$$

is an acyclic Kan fibration, hence is in $W_{\underline{\hat{\Delta}}}$ (cf. C.7.5). On the other hand, $\mathrm{pr}_2 : X \times X \to X$ is in $W_{\underline{\hat{\Delta}}}$ (cf. D.1.16). Therefore q is in $W_{\underline{\hat{\Delta}}}$. But q is also locally constant (per $W_{\underline{\hat{\Delta}}}$) (cf. D.1.13). Therefore the arrow $\Omega(X,x) \to \Delta[0]$ is in $W_{\underline{\hat{\Delta}}}$. Proceeding from here by iteration, one obtains a sequence $\{\Omega^n(X,x)\}$ of Kan complexes such that $\forall\, n \geq 1$, the arrow $\Omega^n(X,x) \to \Delta[0]$ is in $W_{\underline{\hat{\Delta}}}$. And $\forall\, n \geq 1$, $\#\pi_n(X,x) = 1$. That the arrow $X \to \Delta[0]$ is in W_∞ is then implied by D.1.18.

[Note: In the above, ΘX is the mapping space of (X,x) and ΩX is the loop space of (X,x):

$$\left[\begin{array}{l} \Theta X \approx \mathrm{map}_*(\Delta[1],X) \\ \Omega X \approx \mathrm{map}_*(\Delta[1]/\dot{\Delta}[1],X).] \end{array}\right.$$

D.1.20 LEMMA Let $p:X \to Y$ be a Kan fibration. Assume: $p \in \mathcal{W}_{\hat{\underline{\Delta}}}$ -- then $p \in W_\infty$ (cf. D.1.9).

PROOF First, $\pi_0(p):\pi_0(X) \to \pi_0(Y)$ is bijective (cf. D.1.17). Therefore it need only be shown that $\forall\ x \in X_0$ and $\forall\ n \geq 1$,

$$\pi_n(X,x) \approx \pi_n(Y,y) \quad (y = p(x)).$$

To this end, recall that the fiber X_y of p over y is the Kan complex defined by the pullback square

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[0] \arrow[r, "y"'] & Y \end{tikzcd}$$

Since p is locally constant (per $\mathcal{W}_{\hat{\underline{\Delta}}}$) (cf. D.1.13) and in $\mathcal{W}_{\hat{\underline{\Delta}}}$ (by hypothesis), the arrow $X_y \to \Delta[0]$ is in $\mathcal{W}_{\hat{\underline{\Delta}}}$ (cf. D.1.15), hence is in W_∞ (cf. D.1.19). So, $\forall\ n \geq 1$, $\pi_n(X_y,x)$ is trivial (cf. D.1.18). Conclude by applying the long exact sequence in homotopy.

D.2 *FAILURE OF UBIQUITY*

Fix a proper fundamental localizer $\mathcal{W} \subset \mathcal{W}_0$ (cf. C.5.5) and equip $\underline{\mathrm{CAT}}$ with the cofibrantly generated model structure determined by $\mathcal{W}$ (cf. C.9.14) (itself necessarily right proper (cf. C.9.15)).

D.2.1 THEOREM Assume: For every locally constant functor $F:\underline{I} \to \underline{J}$ and

$\forall\ j \in \mathrm{Ob}\ \underline{J}$, the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] & \underline{J} \end{tikzcd}$$

is a homotopy pullback -- then $\mathcal{W} = \mathcal{W}_\infty$.

PROOF If $F:\underline{I} \to \underline{J}$ is locally constant and in $\mathcal{W}$, then $\forall\ j \in \mathrm{Ob}\ \underline{J}$,

$$F/j:\underline{I}/j \to \underline{J}/j$$

is in $\mathcal{W}$ (cf. D.1.7). Therefore F is aspherical and one can quote D.1.8.

Moral: In the world of proper fundamental localizers $\mathcal{W} \subset \mathcal{W}_0$, $\mathcal{W}_\infty$ is characterized by the validity of "Theorem B".

D.3 *THEOREM B => THEOREM B*

Take $\underline{\mathrm{SISET}}$ in its Kan structure and $\underline{\mathrm{CAT}}$ in its external structure.

D.3.1 CRITERION A commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

of simplicial sets is a homotopy pullback (per W_∞) iff the commutative diagram

$$\begin{tikzcd} i_{\underline{\Delta}}X \arrow[r] \arrow[d] & i_{\underline{\Delta}}Y \arrow[d] \\ i_{\underline{\Delta}}X' \arrow[r] & i_{\underline{\Delta}}Y' \end{tikzcd}$$

of small categories is a homotopy pullback (per W_∞).

D.3.2 LEMMA The functor

$$\mathrm{ner}:\underline{\mathrm{CAT}} \to \underline{\mathrm{SISET}}$$

preserves homotopy pullbacks.

PROOF Suppose that

$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[d] & \underline{D} \arrow[d] \\ \underline{C}' \arrow[r] & \underline{D}' \end{tikzcd}$$

is a homotopy pullback in $\underline{\mathrm{CAT}}$ — then the claim is that

$$\begin{tikzcd} \mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & \mathrm{ner}\,\underline{D} \arrow[d] \\ \mathrm{ner}\,\underline{C}' \arrow[r] & \mathrm{ner}\,\underline{D}' \end{tikzcd}$$

is a homotopy pullback in $\underline{\mathrm{SISET}}$ and for this, it need only be shown that

$$\begin{tikzcd} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D} \arrow[d] \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \arrow[r] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D}' \end{tikzcd}$$

is a homotopy pullback in $\underline{\mathrm{CAT}}$ (cf. D.3.1). To begin with, $i_{\underline{\Delta}} = \mathrm{gro}_{\underline{\Delta}}$, thus there are simplicial weak equivalences

$$\left[\begin{array}{l} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \to \underline{C} \\ \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \to \underline{C}', \end{array}\right. \qquad \left[\begin{array}{l} i_{\underline{\Delta}}\mathrm{ner}\,\underline{D} \to \underline{D} \\ \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{D}' \to \underline{D}'. \end{array}\right.$$

• Consider the commutative diagram

$$\begin{tikzcd} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & \underline{C} \arrow[r] \arrow[d] & \underline{D} \arrow[d] \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \arrow[r] & \underline{C}' \arrow[r] & \underline{D}'. \end{tikzcd}$$

Then the first square is a homotopy pullback (cf. 0.35.2), as is the second square (by hypothesis). Therefore the rectangle

$$\begin{tikzcd} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & \underline{D} \arrow[d] \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \arrow[r] & \underline{D}' \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.3).

• Consider the commutative diagram

$$\begin{tikzcd} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D} \arrow[r] \arrow[d] & \underline{D} \arrow[d] \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \arrow[r] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D}' \arrow[r] & \underline{D}'. \end{tikzcd}$$

Then the rectangle is a homotopy pullback (by the above), as is the second square (cf. 0.35.2). Therefore the first square

$$\begin{tikzcd} i_{\underline{\Delta}}\mathrm{ner}\,\underline{C} \arrow[r] \arrow[d] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D} \arrow[d] \\ i_{\underline{\Delta}}\mathrm{ner}\,\underline{C}' \arrow[r] & i_{\underline{\Delta}}\mathrm{ner}\,\underline{D}' \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.3).

D.3.3 THEOREM B Let $\underline{I},\underline{J} \in \mathrm{Ob}\,\underline{\mathbf{CAT}}$ and let $F:\underline{I} \to \underline{J}$ be a functor. Assume: F is locally constant -- then $\forall\ j \in \mathrm{Ob}\ \underline{J}$, the pullback square

$$\begin{tikzcd} \mathrm{ner}\ \underline{I}/j \arrow[r] \arrow[d, "\mathrm{ner}\ F/j"'] & \mathrm{ner}\ \underline{I} \arrow[d, "\mathrm{ner}\ F"] \\ \mathrm{ner}\ \underline{J}/j \arrow[r] & \mathrm{ner}\ \underline{J} \end{tikzcd}$$

is a homotopy pullback.

[In view of D.3.2, this is immediate (cf. D.1.4).]

To complete the picture, we shall outline an approach to D.1.4.

D.3.4 Let $\underline{C}$ be a small category, $F:\underline{C} \to \underline{\mathrm{CAT}}$ a functor. Assume: For every arrow $f:X \to Y$ in $\underline{C}$, $Ff:FX \to FY$ is a simplicial weak equivalence -- then the Grothendieck opfibration

$$\Theta_F:\underline{\mathrm{INT}}_{\underline{C}}F \to \underline{C}$$

is a homotopy fibration (cf. 0.35.5).

D.3.5 EXAMPLE Let $\underline{J}$ be a small category. Consider the functor

$$\begin{cases} \underline{J} \to \underline{\mathrm{CAT}} \\ j \to \underline{J}/j. \end{cases}$$

Then $\underline{J}/j$ has a final object, hence is contractible. So, for every morphism $j \to j'$ in $\underline{J}$, the arrow $\underline{J}/j \to \underline{J}/j'$ is a simplicial weak equivalence. Therefore the Grothendieck opfibration

$$\Theta_{\underline{J}/—}:\underline{\mathrm{INT}}_{\underline{J}}\underline{J}/— \to \underline{J}$$

is a homotopy fibration.

D.3.6 EXAMPLE Let $\underline{I},\underline{J}$ be small categories, $F:\underline{I} \to \underline{J}$ a locally constant functor. Consider the functor

$$\begin{cases} \underline{J} \to \underline{\mathrm{CAT}} \\ j \to \underline{I}/j. \end{cases}$$

Then by definition, for every morphism $j \to j'$ in $\underline{J}$, the functor

$$\underline{I}/j \to \underline{I}/j'$$

is a simplicial weak equivalence. Therefore the Grothendieck opfibration

$$\Theta_{\underline{I}/\text{—}}:\underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \to \underline{J}$$

is a homotopy fibration.

[Note: Needless to say, D.3.5 is a special case of D.3.6 (take $\underline{I} = \underline{J}$ and $F = \mathrm{id}_{\underline{J}}$).]

D.3.7 RAPPEL Given a small category $\underline{C}$ and a functor $F:\underline{C} \to \underline{\mathrm{CAT}}$, there is a canonical arrow

$$K_F:\underline{\mathrm{INT}}_{\underline{C}}F \to \mathrm{colim}_{\underline{C}}F \quad \text{(cf. B.2.15).}$$

D.3.8 LEMMA If $\underline{I},\underline{J}$ are small categories and if $F:\underline{I} \to \underline{J}$ is a functor, then

$$K_{\underline{I}/\text{—}}:\mathrm{INT}_{\underline{J}}\underline{I}/\text{—} \to \mathrm{colim}_{\underline{J}}\underline{I}/\text{—} = \underline{I}$$

is a Grothendieck fibration with contractible fibers.

D.3.9 REMARK It follows that

$$K_{\underline{I}/\text{—}}:\underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \to \mathrm{colim}_{\underline{J}}\underline{I}/\text{—} = \underline{I}$$

is a simplicial weak equivalence (cf. B.6.13).

Here now is the data for the proof of D.1.4:

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \arrow[r, "K_{\underline{I}/\text{—}}"] \arrow[d] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] \arrow[d] & \underline{\mathrm{INT}}_{\underline{J}}\underline{J}/\text{—} \arrow[r, "K_{\underline{J}/\text{—}}"'] \arrow[d, "\Theta_{\underline{J}/\text{—}}"] & \underline{J} \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} & \end{tikzcd}$$

Each of the squares in this commutative diagram is a pullback square and the composition

$$\underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \longrightarrow \underline{\mathrm{INT}}_{\underline{J}}\underline{J}/\text{—} \xrightarrow{\Theta_{\underline{J}/\text{—}}} \underline{J}$$

is $\Theta_{\underline{I}/\text{—}}$.

- Since $\Theta_{\underline{J}/\text{—}}$ is a homotopy fibration (cf. D.3.5), the pullback square

$$\begin{tikzcd} \underline{J}/j \arrow[r] \arrow[d] & \underline{\mathrm{INT}}_{\underline{J}}\underline{J}/\text{—} \arrow[d] \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.4).

- Since $\Theta_{\underline{I}/\text{—}}$ is a homotopy fibration (cf. D.3.6), the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d] & \underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \arrow[d, "\Theta_{\underline{I}/\text{—}}"] \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.4).

Therefore the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \arrow[d] \\ \underline{J}/j \arrow[r] & \underline{\mathrm{INT}}_{\underline{J}}\underline{J}/\text{—} \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.3).

• Since $\left[\begin{array}{l} K_{\underline{I}}/\text{—} \\ \\ K_{\underline{J}}/\text{—} \end{array}\right.$ are simplicial weak equivalences (cf. D.3.9), the pullback square

$$\begin{tikzcd} \underline{\mathrm{INT}}_{\underline{J}}\underline{I}/\text{—} \arrow[r, "K_{\underline{I}}/\text{—}"] \arrow[d] & \underline{I} \arrow[d] \\ \underline{\mathrm{INT}}_{\underline{J}}\underline{J}/\text{—} \arrow[r, "K_{\underline{J}}/\text{—}"'] & \underline{J} \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.2).

Therefore the pullback square

$$\begin{tikzcd} \underline{I}/j \arrow[r] \arrow[d, "F/j"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}/j \arrow[r] & \underline{J} \end{tikzcd}$$

is a homotopy pullback (cf. 0.35.3), the contention of D.1.4.

CHAPTER 1: DERIVED FUNCTORS

CHAPTER 1: DERIVED FUNCTORS

1.1 *LOCALIZATION*

Let $\underline{C}$ be a category and let $W \subset \text{Mor } \underline{C}$ be a class of morphisms.

1.1.1 DEFINITION $(\underline{C},W)$ is a category pair if W is closed under composition and contains the identities of $\underline{C}$, the elements of W then being referred to as the weak equivalences.

E.g.: If $W_{\min}$ is the class of identities of $\underline{C}$ and if $W_{\max}$ is $\text{Mor } \underline{C}$ itself, then $(\underline{C},W_{\min})$ and $(\underline{C},W_{\max})$ are category pairs.

[Note: An intermediate possibility is to take for W the class of isomorphisms of $\underline{C}$.]

N.B. A category pair can be regarded as a subcategory of $\underline{C}$ with the same objects.

1.1.2 DEFINITION Given a category pair $(\underline{C},W)$, a localization of $\underline{C}$ at W is a pair $(W^{-1}\underline{C}, L_W)$, where $W^{-1}\underline{C}$ is a metacategory and $L_W:\underline{C} \to W^{-1}\underline{C}$ is a functor such that $\forall\, w \in W$, $L_W w$ is an isomorphism, $(W^{-1}\underline{C}, L_W)$ being initial among all pairs having this property, i.e., for any metacategory $\underline{D}$ and for any functor $F:\underline{C} \to \underline{D}$ such that $\forall\, w \in W$, Fw is an isomorphism, there exists a unique functor $\bar{F}:W^{-1}\underline{C} \to \underline{D}$ such that $F = \bar{F} \circ L_W$.

1.1.3 THEOREM Localizations of $\underline{C}$ at W exist and are unique up to isomorphism. Moreover, there is a representative $(W^{-1}\underline{C}, L_W)$ having the same objects as $\underline{C}$ and for which L_W is the identity on objects.

1.1.4 EXAMPLE Take $\underline{C} = \underline{TOP}$ and let $W \subset \mathrm{Mor}\,\underline{C}$ be the class of homotopy equivalences -- then $W^{-1}\underline{C} = \underline{HTOP}$.

1.1.5 DETAILS What follows is an outline of the proof of 1.1.3.

Step 1: Given $X,Y \in \mathrm{Ob}\,\underline{C}$, a word

$$\omega = (X,X_1,\ldots,X_{2n-1},Y)$$

connecting X to Y is a finite chain of objects and morphisms of the form

$$X \xrightarrow{f_1} X_1 \xleftarrow{w_1} X_2 \xrightarrow{f_2} \bullet \cdots \bullet \xleftarrow{w_{n-1}} X_{2n-2} \xrightarrow{f_n} X_{2n-1} \xleftarrow{w_n} Y$$

in which $\longrightarrow$ and $\longleftarrow$ alternate and the w_i are in W. Write $\Omega(X,Y)$ for the class of all words connecting X to Y.

Step 2: Two words $\omega,\omega' \in \Omega(X,Y)$ are deemed equivalent ($\omega \sim \omega'$) if there is a finite sequence

$$\omega = \omega_1,\omega_2,\ldots,\omega_n = \omega'$$

of words with the property that each ω_i is obtained from ω_{i-1} (or from ω_{i+1}) by one of the following operations.

(a) Replace

$$\bullet \xrightarrow{f} \bullet \xleftarrow{\mu} \bullet \xrightarrow{g} \bullet \xleftarrow{\nu} \bullet \qquad (\mu,\nu \in W)$$

in ω_{i-1} (or ω_{i+1}) by

$$\bullet \xrightarrow{uf} \bullet \xleftarrow{v\nu} \bullet$$

if there is a commutative diagram in $\underline{C}$

$$\begin{tikzcd} \bullet \arrow[d, "u"'] & \bullet \arrow[l, "\mu"'] \arrow[r, "g"] & \bullet \arrow[d, "v"] \\ \bullet \arrow[rr, equal] & & \bullet \end{tikzcd}$$

with $v\nu$ in W.

(b) Replace

$$\bullet \xleftarrow{\mu} \bullet \xrightarrow{f} \bullet \xleftarrow{\nu} \bullet \xrightarrow{g} \bullet \qquad (\mu,\nu \in \mathcal{W})$$

in ω_{i-1} (or ω_{i+1}) by

$$\bullet \xleftarrow{\mu u} \bullet \xrightarrow{gv} \bullet$$

if there is a commutative diagram in $\underline{C}$

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet & \bullet \arrow[l, "\nu"'] \\ \bullet \arrow[u, "u"] \arrow[rr, equal] & & \bullet \arrow[u, "v"'] \end{tikzcd}$$

with μu in $\mathcal{W}$.

(c) Replace

$$\bullet \xrightarrow{f_1} \bullet \xleftarrow{id} \bullet \xrightarrow{f_2} \bullet$$

in ω_{i-1} (or ω_{i+1}) by

$$\bullet \xrightarrow{f_2 f_1} \bullet$$

or vice-versa.

(d) Replace

$$\bullet \xleftarrow{w_1} \bullet \xrightarrow{id} \bullet \xleftarrow{w_2} \bullet$$

in ω_{i-1} (or ω_{i+1}) by

$$\bullet \xleftarrow{w_1 w_2} \bullet$$

or vice-versa.

Step 4: Given words

$$\begin{cases} \omega = (X,X_1,\ldots,X_{2n-1},Y) \\ \omega' = (Y,Y_1,\ldots,Y_{2m-1},Z), \end{cases}$$

let

$$\omega * \omega' = (X,X_1,\ldots,X_{2n-1},Y,Y_1,\ldots,Y_{2m-1},Z).$$

Then the $*$-product is associative and the equivalence class of $\omega * \omega'$ depends only on that of ω and ω'.

Step 5: Now stipulate that the metacategory $W^{-1}\underline{C}$ has for its objects those of $\underline{C}$ and for its morphisms from X to Y the elements $[\omega] \in \Omega(X,Y)/\sim$. Here composition is defined by

$$[\omega'] \circ [\omega] = [\omega * \omega']$$

and the identity in $\Omega(X,Y)/\sim$ is

$$[X \xrightarrow{\mathrm{id}_X} X \xleftarrow{\mathrm{id}_X} X].$$

As for the functor $L_W:\underline{C} \to W^{-1}\underline{C}$, on objects

$$L_W X = X$$

and on morphisms

$$L_W f = [X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y].$$

Step 6: Given a word $\omega \in \Omega(X,Y)$, suppose that its morphisms in either direction are elements of W -- then $[\omega]$ is an isomorphism in $W^{-1}\underline{C}$, its inverse being represented by ω written in reverse order. In particular: $\forall\ w \in W$, $L_W w$

is an isomorphism.

Step 7: Let $F:\underline{C} \to \underline{D}$ be a functor such that $\forall\, w \in \mathcal{W}$, Fw is an isomorphism. Define $\bar{F}:\mathcal{W}^{-1}\underline{C} \to \underline{D}$ on the $X \in \mathrm{Ob}\, \mathcal{W}^{-1}\underline{C} = \mathrm{Ob}\, \underline{C}$ by $\bar{F}X = FX$ and given a word

$$\omega = (X, X_1, \ldots, X_{2n-1}, Y),$$

put

$$\bar{F}\omega = F(w_n)^{-1} \circ Ff_n \circ \cdots \circ F(w_1)^{-1} \circ Ff_1.$$

Then

$$\omega \sim \omega' \Rightarrow \bar{F}\omega = \bar{F}\omega'.$$

Therefore the assignment

$$[\omega] \to \bar{F}\omega$$

is welldefined. And $\bar{F}:\mathcal{W}^{-1}\underline{C} \to \underline{D}$ is a functor.

Step 8: $\forall\, X \in \mathrm{Ob}\, \underline{C}$,

$$(\bar{F} \circ L_{\mathcal{W}})X = \bar{F}L_{\mathcal{W}}X = \bar{F}X = FX$$

and $\forall\, f \in \mathrm{Mor}(X,Y)$,

$$\begin{aligned}(\bar{F} \circ L_{\mathcal{W}})f &= \bar{F}L_{\mathcal{W}}f \\ &= \bar{F}[X \xrightarrow{f} Y \xleftarrow{id_Y} Y] \\ &= F(id_Y)^{-1} \circ Ff \\ &= (id_{FY})^{-1} \circ Ff = Ff.\end{aligned}$$

Modulo uniqueness (which will be left to the reader), the proof is thus complete.

1.1.6 REMARK In general, the $\Omega(X,Y)/\sim$ need not be sets and $\mathcal{W}^{-1}\underline{C}$ need not be isomorphic to a category (but it will be if $\underline{C}$ is small).

1.1.7 LEMMA Every word

$$\omega = (X,X_1,\ldots,X_{2n-1},Y)$$

is equivalent to

$$(X \xrightarrow{f_1} X_1 \xleftarrow{id_1} X_1) * (X_1 \xrightarrow{id_1} X_1 \xleftarrow{w_1} X_2) * \cdots$$

$$* (X_{2n-2} \xrightarrow{f_n} X_{2n-1} \xleftarrow{id_{2n-1}} X_{2n-1}) * (X_{2n-1} \xrightarrow{id_{2n-1}} X_{2n-1} \xleftarrow{w_n} Y).$$

Therefore

$$[\omega] = (L_{\mathcal{W}}w_n)^{-1} \circ L_{\mathcal{W}}f_n \circ \cdots \circ (L_{\mathcal{W}}w_1)^{-1} \circ L_{\mathcal{W}}f_1.$$

1.1.8 LEMMA Suppose that $(\underline{C},\mathcal{W})$ is a category pair whose weak equivalences are isomorphisms -- then $L_{\mathcal{W}}:\underline{C} \to \mathcal{W}^{-1}\underline{C}$ is an isomorphism.

PROOF $\forall\, w \in \mathcal{W}$, $id_{\underline{C}}w$ is an isomorphism, hence there is a unique functor $\Phi:\mathcal{W}^{-1}\underline{C} \to \underline{C}$ and a factorization $id_{\underline{C}} = \Phi \circ L_{\mathcal{W}}$. Meanwhile, $L_{\mathcal{W}} = L_{\mathcal{W}} \circ id_{\underline{C}} =$ $L_{\mathcal{W}} \circ (\Phi \circ L_{\mathcal{W}}) = (L_{\mathcal{W}} \circ \Phi) \circ L_{\mathcal{W}} \Rightarrow L_{\mathcal{W}} \circ \Phi = id_{\mathcal{W}^{-1}\underline{C}}$.

1.1.9 DEFINITION Let $(\underline{C},\mathcal{W})$ be a category pair — then the <u>saturation</u> $\overline{\mathcal{W}}$ of $\mathcal{W}$ is the class of morphisms of $\underline{C}$ which are sent by $L_{\mathcal{W}}$ to isomorphisms in $\mathcal{W}^{-1}\underline{C}$.

<u>N.B.</u> $(\underline{C},\overline{\mathcal{W}})$ is a category pair.

1.10 LEMMA There is a canonical isomorphism

$$\mathcal{W}^{-1}\underline{C} \to \bar{\mathcal{W}}^{-1}\underline{C}$$

of metacategories.

PROOF Since $\mathcal{W} \subset \bar{\mathcal{W}}$, there is a unique functor $\Delta:\mathcal{W}^{-1}\underline{C} \to \bar{\mathcal{W}}^{-1}\underline{C}$ such that $L_{\bar{\mathcal{W}}} = \Delta \circ L_{\mathcal{W}}$. On the other hand, $L_{\mathcal{W}}\bar{w}$ is an isomorphism for all $\bar{w} \in \bar{\mathcal{W}}$, so there is a unique functor $\bar{\Delta}:\bar{\mathcal{W}}^{-1}\underline{C} \to \mathcal{W}^{-1}\underline{C}$ such that $L_{\mathcal{W}} = \bar{\Delta} \circ L_{\bar{\mathcal{W}}}$. Therefore

$$\left[\begin{aligned} L_{\bar{\mathcal{W}}} &= \Delta \circ L_{\mathcal{W}} = \Delta \circ \bar{\Delta} \circ L_{\bar{\mathcal{W}}} \\ L_{\mathcal{W}} &= \bar{\Delta} \circ L_{\bar{\mathcal{W}}} = \bar{\Delta} \circ \Delta \circ L_{\mathcal{W}} \end{aligned}\right.$$

$$\Rightarrow$$

$$\left[\begin{aligned} \Delta \circ \bar{\Delta} &= \mathrm{id}_{\bar{\mathcal{W}}^{-1}\underline{C}} \\ \bar{\Delta} \circ \Delta &= \mathrm{id}_{\mathcal{W}^{-1}\underline{C}}. \end{aligned}\right.$$

1.11 LEMMA Let $(\underline{C},\mathcal{W})$ be a category pair -- then for every metacategory $\underline{D}$, the precomposition arrow

$$[\mathcal{W}^{-1}\underline{C},\underline{D}] \to [\underline{C},\underline{D}]$$

corresponding to $L_{\mathcal{W}}$ induces an isomorphism from $[\mathcal{W}^{-1}\underline{C},\underline{D}]$ onto the full submetacategory $[\underline{C},\underline{D}]_{\mathcal{W}}$ of $[\underline{C},\underline{D}]$ whose objects are the functors $F:\underline{C} \to \underline{D}$ such that $\forall\ w \in \mathcal{W}$, Fw is an isomorphism of $\underline{D}$.

1.2 *CALCULUS OF FRACTIONS*

Let $(\underline{C},\mathcal{W})$ be a category pair -- then under certain conditions, the description of the localization $(\mathcal{W}^{-1}\underline{C}, L_{\mathcal{W}})$ can be simplified.

1.2.1 DEFINITION $\mathcal{W}$ is said to admit a <u>calculus of left fractions</u> if

(LF_1) Given a 2-source $X' \xleftarrow{w} X \xrightarrow{f} Y$ $(w \in \mathcal{W})$, there exists a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "w"'] & Y \arrow[d, "w'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd},$$

where $w' \in \mathcal{W}$;

(LF_2) Given $f,g:X \to Y$ and $w_1:X' \to X$ $(w_1 \in \mathcal{W})$ such that $f \circ w_1 = g \circ w_1$, there exists $w_2:Y \to Y'$ $(w_2 \in \mathcal{W})$ such that $w_2 \circ f = w_2 \circ g$.

[Note: Reverse the arrows to define "calculus of right fractions".]

1.2.2 REMARK If $\mathcal{W}$ admits a calculus of left fractions, then every morphism in $\mathcal{W}^{-1}\underline{C}$ can be represented in the form $(L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f$ (cf. 1.1.7).

1.2.3 LEMMA Suppose that $\forall\ (w,w'):w' \circ w \in \mathcal{W}$ & $w \in \mathcal{W} \Rightarrow w' \in \mathcal{W}$ -- then $\mathcal{W}$ admits a calculus of left fractions if every 2-source $X' \xleftarrow{w} X \xrightarrow{f} Y$ $(w \in \mathcal{W})$ can be completed to a weak pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "w"'] & Y \arrow[d, "w'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd},$$

where $w' \in \mathcal{W}$.

1.3 *HOMOTOPY*

1.3.1 DEFINITION Let $(\underline{C},\mathcal{W})$ be a category pair — then morphisms $f,g:X \to Y$ in $\underline{C}$ are <u>homotopic</u> (written $f \simeq g$) if $L_{\mathcal{W}}f = L_{\mathcal{W}}g$.

1.3.2 REMARK If $\mathcal{W}$ admits a calculus of left fractions, then $f \simeq g \Rightarrow \exists\, w \in \mathcal{W}: w \circ f = w \circ g$.

The homotopy relation $\simeq$ is an equivalence relation on $\text{Mor}(X,Y)$ and one writes $[X,Y]$ for $\text{Mor}(X,Y)/\simeq$.

Suppose that $f \simeq g:X \to Y$ — then for $u:X' \to X$, $f \circ u \simeq g \circ u$ and for $v:Y \to Y'$, $v \circ f \simeq v \circ g$. Consequently, there is a category $\underline{HO}_{\mathcal{W}}\underline{C}$ whose objects are those of $\underline{C}$ and whose morphisms from X to Y are the quotients $\text{Mor}(X,Y)/\simeq$. Moreover, there is a functor $\underline{HO}_{\mathcal{W}}\underline{C} \to \mathcal{W}^{-1}\underline{C}$ and $L_{\mathcal{W}}$ factors as the composition $\underline{C} \to \underline{HO}_{\mathcal{W}}\underline{C} \to \mathcal{W}^{-1}\underline{C}$.

1.3.3 DEFINITION A morphism $f:X \to Y$ is a <u>homotopy equivalence</u> if there exists a morphism $g:Y \to X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$.

Write $E(\mathcal{W})$ for the class of f that are homotopy equivalences — then $E(\mathcal{W}) \subset \overline{\mathcal{W}}$ (cf. 1.1.9).

1.3.4 LEMMA $E(\mathcal{W}) = \overline{\mathcal{W}}$ iff $L_{\mathcal{W}}:\underline{C} \to \mathcal{W}^{-1}\underline{C}$ is full.

PROOF Suppose first that $L_{\mathcal{W}}$ is full, the claim then being that $\overline{\mathcal{W}} \subset E(\mathcal{W})$. But $\forall\, f \in \overline{\mathcal{W}}$, $L_{\mathcal{W}}f$ has an inverse and $(L_{\mathcal{W}}f)^{-1} = L_{\mathcal{W}}g$ for some g, thus $f \in E(\mathcal{W})$. Turning to the converse, recall that a generic morphism $[\omega]$ in $\mathcal{W}^{-1}\underline{C}$ can be factored:

$$[\omega] = (L_{\mathcal{W}}w_n)^{-1} \circ L_{\mathcal{W}}f_n \circ \cdots \circ (L_{\mathcal{W}}w_1)^{-1} \circ L_{\mathcal{W}}f_1 \quad \text{(cf. 1.1.7)}.$$

However, $\forall$ i

$$w_i \in \mathcal{W} \subset \bar{\mathcal{W}} = E(\mathcal{W}),$$

hence

$$(L_{\mathcal{W}} w_i)^{-1} = L_{\mathcal{W}} z_i$$

for some $z_i \in \mathcal{W}$. Therefore

$$[\omega] = L_{\mathcal{W}}(z_n \circ f_n \circ \cdots \circ z_1 \circ f_1),$$

so $L_{\mathcal{W}}$ is full.

1.4 *TOTALITY*

If $(\underline{C},\mathcal{W})$ is a category pair and if $F:\underline{C} \to \underline{D}$ is a functor such that $\forall$ $w \in \mathcal{W}$, Fw is an isomorphism, then there is a commutative diagram

$$\begin{tikzcd} \underline{C} \arrow[r, equal] \arrow[d, "L_{\mathcal{W}}"'] & \underline{C} \arrow[d, "F"] \\ \mathcal{W}^{-1}\underline{C} \arrow[r, "\bar{F}"'] & \underline{D} \end{tikzcd} \quad \text{(cf. 1.1.2).}$$

1.4.1 DEFINITION Let $(\underline{C},\mathcal{W})$ be a category pair but let $F:\underline{C} \to \underline{D}$ be arbitrary -- then a <u>right derived functor</u> of F is a left Kan extension of F along $L_{\mathcal{W}}$, hence is a pair $(\underline{L}_{L_{\mathcal{W}}}F,\mu_F)$, where $\underline{L}_{L_{\mathcal{W}}}F:\mathcal{W}^{-1}\underline{C} \to \underline{D}$ is a functor and $\mu_F \in \text{Nat}(F,\underline{L}_{L_{\mathcal{W}}}F \circ L_{\mathcal{W}})$, with the following property: $\forall$ $F' \in \text{Ob}[\mathcal{W}^{-1}\underline{C},\underline{D}]$ and $\forall$ $\alpha \in \text{Nat}(F,F' \circ L_{\mathcal{W}})$, there is a unique $\beta \in \text{Nat}(\underline{L}_{L_{\mathcal{W}}}F,F')$ such that $\alpha = \beta L_{\mathcal{W}} \circ \mu_F$.

1.4.2 NOTATION To simplify, let

$$\mathbf{R}F = \underline{\mathrm{L}}_{\mathrm{L}_{\mathcal{W}}} F$$

if no confusion is likely. So we have

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, "\mu_F"'] & F \arrow[d, "\alpha"] \\ \mathbf{R}F \circ \mathrm{L}_{\mathcal{W}} \arrow[r, "\beta \mathrm{L}_{\mathcal{W}}"'] & F' \circ \mathrm{L}_{\mathcal{W}}. \end{tikzcd}$$

1.4.3 DEFINITION A right derived functor $\mathbf{R}F$ of F is said to be <u>absolute</u> if for every functor $\Phi:\underline{D} \to \underline{D}'$, the pair $(\Phi \circ \mathbf{R}F, \Phi\mu_F)$ is a left Kan extension of $\Phi \circ F$ along $\mathrm{L}_{\mathcal{W}}$.

1.4.4 EXAMPLE If $F:\underline{C} \to \underline{D}$ is a functor such that $\forall\, w \in \mathcal{W}$, Fw is an isomorphism, then $(\bar{F}, \mathrm{id}_F)$ is an absolute right derived functor of F (cf. 1.11).

1.4.5 DEFINITION A morphism

$$F:(\underline{C}_1,\mathcal{W}_1) \to (\underline{C}_2,\mathcal{W}_2)$$

of category pairs is a functor $F:\underline{C}_1 \to \underline{C}_2$ such that $F\mathcal{W}_1 \subset \mathcal{W}_2$, thus there is a unique functor $\bar{F}:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2$ for which the diagram

$$\begin{tikzcd} \underline{C}_1 \arrow[r, "F"] \arrow[d, "\mathrm{L}_{\mathcal{W}_1}"'] & \underline{C}_2 \arrow[d, "\mathrm{L}_{\mathcal{W}_2}"] \\ \mathcal{W}_1^{-1}\underline{C}_1 \arrow[r, "\bar{F}"'] & \mathcal{W}_2^{-1}\underline{C}_2 \end{tikzcd}$$

commutes (cf. 1.1.2).

1.4.6 DEFINITION Let $(\underline{C}_1,\mathcal{W}_1)$, $(\underline{C}_2,\mathcal{W}_2)$ be category pairs but let $F:\underline{C}_1 \to \underline{C}_2$ be arbitrary -- then a <u>total right derived functor</u> of F is a right derived functor of $L_{\mathcal{W}_2} \circ F$, which, to minimize the notational load, will be denoted as above by (RF,μ_F) although in this context $RF:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2$ and $\mu_F \in \mathrm{Nat}(L_{\mathcal{W}_2} \circ F,\ RF \circ L_{\mathcal{W}_1})$, so $\forall\ F' \in \mathrm{Ob}\ [\mathcal{W}_1^{-1}\underline{C}_1,\ \mathcal{W}_2^{-1}\underline{C}_2]$ and $\forall\ \alpha \in \mathrm{Nat}(L_{\mathcal{W}_2} \circ F,\ F' \circ L_{\mathcal{W}_1})$, there is a unique $\beta \in \mathrm{Nat}(RF,F')$ such that $\alpha = \beta L_{\mathcal{W}_1} \circ \mu_F$.

<u>N.B.</u> The designation "absolute" total right derived functor is to be assigned the obvious interpretation.

1.4.7 EXAMPLE If

$$F:(\underline{C}_1,\mathcal{W}_1) \to (\underline{C}_2,\mathcal{W}_2)$$

is a morphism of category pairs, then $(\bar{F},\mathrm{id}_{L_{\mathcal{W}_2} \circ F})$ is an absolute total right derived functor of F.

1.4.8 REMARK The terms left derived functor, absolute left derived functor, total left derived functor, absolute total left derived functor are dual, as is the notation: (LF,ν_F).

1.5 *EXISTENCE*

Suppose that $(\underline{C}_1,\mathcal{W}_1)$, $(\underline{C}_2,\mathcal{W}_2)$ are category pairs and $F:\underline{C}_1 \to \underline{C}_2$ is a functor — then the problem is to find conditions which ensure that F possesses an absolute total right derived functor (RF,μ_F).

1.5.1 DEFINITION Let

$$K:(\underline{C}_0,\mathcal{W}_0) \to (\underline{C}_1,\mathcal{W}_1)$$

be a morphism of category pairs -- then K is <u>resolvable to the right</u> if $\forall\, X_1 \in \text{Ob}\, \underline{C}_1$, $\exists\, X_0 \in \text{Ob}\, \underline{C}_0$ and an arrow $w_1:X_1 \to KX_0$, where $w_1 \in \mathcal{W}_1$.

<u>N.B.</u> Fix $X_1 \in \text{Ob}\, \underline{C}_1$ — then the category of K-resolutions to the right of X_1 has for its objects the arrows $w_1:X_1 \to KX_0$, where $w_1 \in \mathcal{W}$, a morphism

$$(X_1 \xrightarrow{w_1} KX_0) \longrightarrow (X_1 \xrightarrow{w_1'} KX_0')$$

being an arrow $w_0:X_0 \to X_0'$, where $w_0 \in \mathcal{W}_0$, such that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, "w_1"'] & X_1 \arrow[d, "w_1'"] \\
KX_0 \arrow[r, "Kw_0"'] & KX_0'
\end{tikzcd}$$

commutes.

Let $(\underline{C}_1,\mathcal{W}_1)$ be a category pair -- then a <u>derivability structure to the right</u> on $(\underline{C}_1,\mathcal{W}_1)$ consists of a morphism

$$K:(\underline{C}_0,\mathcal{W}_0) \to (\underline{C}_1,\mathcal{W}_1)$$

of category pairs, where K is resolvable to the right, plus additional conditions on the data that serve to imply the validity of the following assertion.

1.5.2 THEOREM Fix a derivability structure to the right on $(\underline{C}_1,\mathcal{W}_1)$ -- then for any category pair $(\underline{C}_2,\mathcal{W}_2)$ and any functor $F:\underline{C}_1 \to \underline{C}_2$ such that

$$F \circ K : (\underline{C}_0, \mathcal{W}_0) \to (\underline{C}_2, \mathcal{W}_2)$$

is a morphism of category pairs, F admits an absolute total right derived functor (RF, μ_F).

1.5.3 ADDENDA $\forall\ X_1 \in \text{Ob}\ \underline{C}_1$ and $\forall\ w_1 : X_1 \to KX_0$ $(w_1 \in \mathcal{W}_1)$,

$$L_{\mathcal{W}_2}(Fw_1) : L_{\mathcal{W}_2}FX_1 \to L_{\mathcal{W}_2}FKX_0.$$

On the other hand,

$$(\mu_F)_{X_1} : L_{\mathcal{W}_2}FX_1 \to RFL_{\mathcal{W}_1}X_1.$$

This said, the existence of a derivability structure to the right on $(\underline{C}_1, \mathcal{W}_1)$ implies that there is a canonical isomorphism

$$RFL_{\mathcal{W}_1}X_1 \to L_{\mathcal{W}_2}FKX_0$$

in $\mathcal{W}_2^{-1}\underline{C}_2$ and a commutative diagram

$$\begin{tikzcd}
L_{\mathcal{W}_2}FX_1 \arrow[r, "(\mu_F)_{X_1}"] \arrow[d, equal] & RFL_{\mathcal{W}_1}X_1 \arrow[d, "\approx"] \\
L_{\mathcal{W}_2}FX_1 \arrow[r, "L_{\mathcal{W}_2}(Fw_1)"'] & L_{\mathcal{W}_2}FKX_0
\end{tikzcd},$$

where canonical refers to the category of K-resolutions to the right of X_1:

$$\begin{tikzcd} X_1 \arrow[r, equal] \arrow[d, "w_1"'] & X_1 \arrow[d, "w_1'"] \\ KX_0 \arrow[r, "Kw_0"'] & KX_0' \end{tikzcd} \longrightarrow \begin{tikzcd} RFX_1 \arrow[r, equal] \arrow[d, "\approx"'] & RFX_1 \arrow[d, "\approx"] \\ FKX_0 \arrow[r, "FKw_0"'] & FKX_0' \end{tikzcd}.$$

The specific choice of the conditions figuring in a derivability structure to the right depends on the details of the situation at hand and on ones ultimate objective. Accordingly, foregoing any pretence of a general theoretical study, we shall zero in on just one particular instance that will be of use in the sequel.

1.5.4 DEFINITION Let $(\underline{C}_1, \mathcal{W}_1)$ be a category pair — then a <u>right approximation</u> to $(\underline{C}_1, \mathcal{W}_1)$ is a morphism

$$K: (\underline{C}_0, \mathcal{W}_0) \rightarrow (\underline{C}_1, \mathcal{W}_1)$$

of category pairs, where K is resolvable to the right, such that for any 2-source $KX_0 \xleftarrow{w_1} X_1 \xrightarrow{f_1} KX_0'$ $(w_1 \in \mathcal{W}_1)$, there is a 2-source $X_0 \xleftarrow{w_0} X_0'' \xrightarrow{f_0} X_0'$ $(w_0 \in \mathcal{W}_0)$ and an arrow $w_1'': X_1 \rightarrow KX_0''$ $(w_1'' \in \mathcal{W}_1)$ leading to a commutative diagram

$$\begin{tikzcd} KX_0 \arrow[d, equal] & X_1 \arrow[l, "w_1"'] \arrow[r, "f_1"] \arrow[d, "w_1''"'] & KX_0' \arrow[d, equal] \\ KX_0 & KX_0'' \arrow[l, "Kw_0"] \arrow[r, "Kf_0"'] & KX_0' \end{tikzcd}.$$

In addition, if $(\tilde{w}_0, \tilde{f}_0, \tilde{w}'_1)$ is another choice, then

$$L_{W_0} f_0 \circ (L_{W_0} w_0)^{-1} = L_{W_0} \tilde{f}_0 \circ (L_{W_0} \tilde{w}_0)^{-1}.$$

1.5.5 THEOREM A right approximation

$$K: (\underline{C}_0, W_0) \to (\underline{C}_1, W_1)$$

to $(\underline{C}_1, W_1)$ is a derivability structure to the right on $(\underline{C}_1, W_1)$.

[For the proof, which we shall omit, consult Radulescu-Banu†.]

Therefore the existence of a right approximation to $(\underline{C}_1, W_1)$ forces 1.5.2 and 1.5.3. But here there is a bonus.

1.5.6 THEOREM The induced functor

$$\bar{K}: W_0^{-1}\underline{C}_0 \to W_1^{-1}\underline{C}_1$$

is an equivalence of metacategories.

1.5.7 REMARK The terms resolvable to the left, derivability structure to the left, left approximation are dual.

1.6 *COMPOSITION*

The result in question is this.

1.6.1 THEOREM Let $(\underline{C}_1, W_1)$, $(\underline{C}', W')$, (C_2, W_2) be category pairs. Suppose that

---

† arXiv:math/0610009

$$\left[\begin{array}{l} K:(\underline{C}_0,W_0) \to (\underline{C}_1,W_1) \\ \\ K':(\underline{C}_0',W_0') \to (\underline{C}',W') \end{array}\right.$$

are derivability structures to the right. Let $F:\underline{C}_1 \to \underline{C}'$, $F':\underline{C}' \to \underline{C}_2$, and $F_0:\underline{C}_0 \to \underline{C}_0'$ be functors. Assume:

$$\left[\begin{array}{l} K'F_0 = FK \\ \\ F_0W_0 \subset W_0' \\ \\ F'K'W_0' \subset W_2. \end{array}\right.$$

Then F, F', and $F'' = F' \circ F$ admit absolute total right derived functors (RF,μ_F), $(RF',\mu_{F'})$, and $(RF'',\mu_{F''})$. Furthermore

$$RF'' \approx RF' \circ RF.$$

PROOF First of all

$$\left[\begin{array}{l} FKW_0 = K'F_0W_0 \subset K'W_0' \subset W' \\ \\ F'K'W_0' \subset W_2 \\ \\ F''KW_0 = F'FKW_0 \subset F'K'W_0' \subset W_2. \end{array}\right.$$

So, thanks to 1.5.2, (RF,μ_F), $(RF',\mu_{F'})$, and $(RF'',\mu_{F''})$ exist. Next, by universality, $\exists$ a unique

$$\Xi \in \mathrm{Nat}(RF'',RF' \circ RF)$$

such that

$$(RF'\mu_F) \circ (\mu_{F'}F) = \Xi L_{W_1} \circ \mu_{F''},$$

and to conclude that

$$RF'' \approx RF' \circ RF,$$

it need only be shown that $\forall\ X_1 \in \mathrm{Ob}\ \underline{C}_1$,

$$\Xi_{X_1}: RF''X_1 \to RF'(RFX_1)$$

is an isomorphism. Choose $X_0 \in \mathrm{Ob}\ \underline{C}_0$ and $w_1: X_1 \to KX_0$ $(w_1 \in \mathcal{W}_1)$. Owing to 1.5.3, in $\mathcal{W}'^{-1}\underline{C}'$,

$$RFX_1 \approx FKX_0$$

and in $\mathcal{W}_2^{-1}\underline{C}_2$,

$$RF''X_1 \approx F''KX_0 = F'FKX_0.$$

But

$$FKX_0 = K'F_0X_0$$

and

$$\mathrm{id}_{K'F_0X_0}: FKX_0 \to K'F_0X_0.$$

Therefore, by 1.5.3 again, in $\mathcal{W}_2^{-1}\underline{C}_2$,

$$RF'FKX_0 \approx F'K'F_0X_0 = F'FKX_0.$$

Consequently,

$$\begin{aligned} RF''X_1 &\approx RF'FKX_0 \\ &\approx RF'(RFX_1), \end{aligned}$$

which, if unraveled, is Ξ_{X_1}.

1.7 *ADJOINTS*

Let $(\underline{C}_1, \mathcal{W}_1)$, $(\underline{C}_2, \mathcal{W}_2)$ be category pairs. Suppose that

$$\begin{cases} F:\underline{C}_1 \to \underline{C}_2 \\ G:\underline{C}_2 \to \underline{C}_1 \end{cases}$$

are an adjoint pair with arrows of adjunction

$$\begin{cases} \mu:\mathrm{id}_{\underline{C}_1} \to G \circ F \\ \nu:F \circ G \to \mathrm{id}_{\underline{C}_2}. \end{cases}$$

Assume:

$$\begin{cases} F \text{ admits an absolute total left derived functor } (LF, \nu_F) \\ G \text{ admits an absolute total right derived functor } (RG, \mu_G). \end{cases}$$

1.7.1 THEOREM The functors

$$\begin{cases} LF:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2 \\ RF:\mathcal{W}_2^{-1}\underline{C}_2 \to \mathcal{W}_1^{-1}\underline{C}_1 \end{cases}$$

are an adjoint pair and one can choose the arrows of adjunction

$$\begin{cases} \underline{\underline{\mu}}:\mathrm{id}_{\mathcal{W}_1^{-1}\underline{C}_1} \longrightarrow RG \circ LF \\ \underline{\underline{\nu}}:LF \circ RG \longrightarrow \mathrm{id}_{\mathcal{W}_2^{-1}\underline{C}_2} \end{cases}$$

so that the diagrams

$$\begin{tikzcd}
RG \circ L_{W_2} \circ F & RG \circ LF \circ L_{W_1} \arrow[l, "(RG)\nu_F"'] \\
L_{W_1} \circ G \circ F \arrow[u, "\mu_G F"] & L_{W_1} \arrow[l, "L_{W_1}\mu"] \arrow[u, "\underline{\underline{\mu}} L_{W_1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
LF \circ L_{W_1} \circ G \arrow[r, "(LF)\mu_G"] \arrow[d, "\nu_F G"'] & LF \circ RG \circ L_{W_2} \arrow[d, "\underline{\underline{\nu}} L_{W_2}"] \\
L_{W_2} \circ F \circ G \arrow[r, "L_{W_2}\nu"'] & L_{W_2}
\end{tikzcd}$$

commute.

Before establishing the existence of $\left[\begin{array}{l} \underline{\underline{\mu}} \\ \underline{\underline{\nu}} \end{array}\right.$, it will be best to review the definitions.

- (RG, μ_G) is an absolute total right derived functor of G, thus is an absolute right derived functor of $L_{W_1} \circ G$.

- (LF, ν_F) is an absolute total left derived functor of F, thus is an absolute left derived functor of $L_{W_2} \circ F$.

Therefore

- $(LF \circ RG, (LF)\mu_G)$ is a right derived functor of $LF \circ L_{W_1} \circ G$.

- $(RG \circ LF, (RG)\nu_F)$ is a left derived functor of $RG \circ L_{W_2} \circ F$.

Next, by universality,

- If $\Phi_2:\mathcal{W}_2^{-1}\underline{C}_2 \to \mathcal{W}_2^{-1}\underline{C}_2$ is a functor and if

$$\Xi_2 \in \mathrm{Nat}(LF \circ L_{\mathcal{W}_1} \circ G, \Phi_2 \circ L_{\mathcal{W}_2}),$$

then there exists a unique

$$\Xi_2' \in \mathrm{Nat}(LF \circ RG, \Phi_2)$$

such that

$$\Xi_2 = \Xi_2' L_{\mathcal{W}_2} \circ (LF)\mu_G.$$

- If $\Phi_1:\mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_1^{-1}\underline{C}_1$ is a functor and if

$$\Xi_1 \in \mathrm{Nat}(\Phi_1 \circ L_{\mathcal{W}_1}, RG \circ L_{\mathcal{W}_2} \circ F),$$

then there exists a unique

$$\Xi_1' \in \mathrm{Nat}(\Phi_1, RG \circ LF)$$

such that

$$\Xi_1 = (RG)\nu_F \circ \Xi_1' L_{\mathcal{W}_1}.$$

Now specialize and take

$$\left[\begin{array}{l} \Phi_2 = \mathrm{id}_{\mathcal{W}_2^{-1}\underline{C}_2} \\ \\ \Phi_1 = \mathrm{id}_{\mathcal{W}_1^{-1}\underline{C}_1} \end{array}\right.$$

and let

$$\left[\begin{array}{l} \Xi_2 = L_{\mathcal{W}_2}\nu \circ \nu_F G: LF \circ L_{\mathcal{W}_1} \circ G \xrightarrow{\nu_F G} L_{\mathcal{W}_2} \circ F \circ G \xrightarrow{L_{\mathcal{W}_1}\nu} L_{\mathcal{W}_2} \\ \\ \Xi_1 = \mu_G F \circ L_{\mathcal{W}_1}\mu: L_{\mathcal{W}_1} \xrightarrow{L_{\mathcal{W}_1}\mu} L_{\mathcal{W}_1} \circ G \circ F \xrightarrow{\mu_G F} RG \circ L_{\mathcal{W}_2} \circ F. \end{array}\right.$$

Then there exist unique

$$\left[\begin{array}{l} \underline{\underline{\nu}} \in \mathrm{Nat}(LF \circ RG, \mathrm{id}_{\mathcal{W}_2^{-1}\underline{C}_2}) \\ \\ \underline{\underline{\mu}} \in \mathrm{Nat}(\mathrm{id}_{\mathcal{W}_1^{-1}\underline{C}_1}, RG \circ LF) \end{array}\right.$$

such that

$$\left[\begin{array}{l} L_{\mathcal{W}_2}\nu \circ \nu_F G = \underline{\underline{\nu}} L_{\mathcal{W}_2} \circ (LF)\mu_G \\ \\ \mu_G F \circ L_{\mathcal{W}_1}\mu = (RG)\nu_F \circ \underline{\underline{\mu}} L_{\mathcal{W}_1}, \end{array}\right.$$

thus with these choices the diagrams in 1.7.1 are commutative but, of course, one still has to prove that $\left[\begin{array}{l} \underline{\underline{\mu}} \\ \underline{\underline{\nu}} \end{array}\right.$ are in fact arrows of adjunction. I.e.:

$$\left[\begin{array}{l} (RG)\underline{\underline{\nu}} \circ \underline{\underline{\mu}}(RG) = \mathrm{id}_{RG} \\ \\ \underline{\underline{\nu}}(LF) \circ (LF)\underline{\underline{\mu}} = \mathrm{id}_{LF}. \end{array}\right.$$

We shall verify the first of these relations, the argument for the second being analogous.

To begin with

$$\mathrm{id}_{RG}L_{\mathcal{W}_2} \circ \mu_G = \mu_G.$$

Proof:

$$\mu_G \in \mathrm{Nat}(L_{\mathcal{W}_1} \circ G, RG \circ L_{\mathcal{W}_2})$$

$\Rightarrow$

$$(\mu_G)_{X_2} : L_{\mathcal{W}_1}GX_2 \to RGL_{\mathcal{W}_2}X_2.$$

Meanwhile

$$(\mathrm{id}_{RG}L_{W_2} \circ \mu_G)_{X_2} = (\mathrm{id}_{RG}L_{W_2})_{X_2} \circ (\mu_G)_{X_2}$$

$$= ((L_{W_2})^* \, \mathrm{id}_{RG})_{X_2} \circ (\mu_G)_{X_2}$$

$$= (\mathrm{id}_{RG})_{L_{W_2}X_2} \circ (\mu_G)_{X_2}$$

$$= \mathrm{id}_{RGL_{W_2}X_2} \circ (\mu_G)_{X_2} = (\mu_G)_{X_2}.$$

Since id_{RG} is characterized by this property, it will be enough to show that

$$((RG)\underline{\underline{\nu}} \circ \underline{\underline{\mu}}(RG))L_{W_2} \circ \mu_G = \mu_G.$$

Starting from the LHS, write

$$((RG)\underline{\underline{\nu}} \circ \underline{\underline{\mu}}(RG))L_{W_2} \circ \mu_G$$

$$= ((RG)\underline{\underline{\nu}})L_{W_2} \circ (\underline{\underline{\mu}}(RG))L_{W_2} \circ \mu_G$$

$$= ((RG)\underline{\underline{\nu}})L_{W_2} \circ \underline{\underline{\mu}}(RG \circ L_{W_2}) \circ \mu_G$$

$$= ((RG)\underline{\underline{\nu}})L_{W_2} \circ (RG \circ LF)\mu_G \circ \underline{\underline{\mu}}(L_{W_1} \circ G)$$

$$= RG(\underline{\underline{\nu}}L_{W_2} \circ (LF)\mu_G) \circ \underline{\underline{\mu}}(L_{W_1} \circ G)$$

$$= RG(L_{W_2}\nu \circ \nu_F G) \circ \underline{\underline{\mu}}(L_{W_1} \circ G)$$

$$= (RG \circ L_{W_2})\nu \circ ((RG)\nu_F)G \circ \underline{\underline{\mu}}(L_{W_1} \circ G)$$

$$= (RG \circ L_{W_2})\nu \circ ((RG)\nu_F \circ \underline{\underline{\mu}}L_{W_1})G)$$

$$= (RG \circ L_{\mathcal{W}_2})\nu \circ (\mu_G F \circ L_{\mathcal{W}_1}\mu)G$$

$$= (RG \circ L_{\mathcal{W}_2})\nu \circ \mu_G(F \circ G) \circ (L_{\mathcal{W}_1}\mu)G$$

$$= \mu_G \circ (L_{\mathcal{W}_1} \circ G)\nu \circ (L_{\mathcal{W}_1}\mu)G$$

$$= \mu_G \circ L_{\mathcal{W}_1}((G\nu) \circ (\mu G))$$

$$= \mu_G \circ L_{\mathcal{W}_1}(\mathrm{id}_G)$$

$$= \mu_G \circ \mathrm{id}_{L_{\mathcal{W}_1}} \circ G$$

$$= \mu_G.$$

<u>N.B.</u> Hidden within the preceding chain of equalities are two commutative diagrams.

<u>**#1**</u>:

$$\begin{tikzcd}
L_{\mathcal{W}_1} \circ G \arrow[r, "\underline{\underline{\mu}}(L_{\mathcal{W}_1} \circ G)"] \arrow[d, "\mu_G"'] & RG \circ LF \circ L_{\mathcal{W}_1} \circ G \arrow[d, "(RG \circ LF)\mu_G"] \\
RG \circ L_{\mathcal{W}_2} \arrow[r, "\underline{\underline{\mu}}(RG \circ L_{\mathcal{W}_2})"'] & RG \circ LF \circ RG \circ L_{\mathcal{W}_2}.
\end{tikzcd}$$

Let

$$\left[\begin{array}{l} A = \mathrm{id}_{\mathcal{W}_1^{-1}\underline{C}_1} \\ \\ B = RG \circ LF. \end{array}\right.$$

Fix $X \in \mathrm{Ob}\, \underline{C}_2$, let

$$\left[\begin{array}{l} Y = L_{W_1} G X \\ \\ Z = R G L_{W_2} X, \end{array}\right.$$

and consider

$$\begin{tikzcd} AY \arrow[r, "\underline{\underline{\mu}}_Y"] \arrow[d, "A(\mu_G)_X"'] & BY \arrow[d, "B(\mu_G)_X"] \\ AZ \arrow[r, "\underline{\underline{\mu}}_Z"'] & BZ. \end{tikzcd}$$

Then $\underline{\underline{\mu}} \in \mathrm{Nat}(A,B)$, thus the diagram commutes.

<u>#2</u>:

$$\begin{tikzcd} L_{W_1} \circ G \arrow[r, "\mu_G"] & RG \circ L_{W_2} \\ L_{W_1} \circ G \circ F \circ G \arrow[u, "(L_{W_1} \circ G)\nu"] \arrow[r, "\mu_G(F \circ G)"'] & RG \circ L_{W_2} \circ F \circ G. \arrow[u, "(RG \circ L_{W_2})\nu"'] \end{tikzcd}$$

Let

$$\left[\begin{array}{l} A = L_{W_1} \circ G \\ \\ B = RG \circ L_{W_2}. \end{array}\right.$$

Fix $X \in \text{Ob}\ \underline{C}_2$ and consider

$$\begin{tikzcd} AX \arrow[rr, "(\mu_G)_X"] & & BX \\ AFGX \arrow[u, "A\nu_X"] \arrow[rr, "(\mu_G)_{FGX}"'] & & BFGX. \arrow[u, "B\nu_X"'] \end{tikzcd}$$

Then $\mu_G \in \text{Nat}(A,B)$, thus the diagram commutes.

1.7.2 THEOREM Let $(\underline{C}_1,\mathcal{W}_1)$, $(\underline{C}_2,\mathcal{W}_2)$ be category pairs. Suppose that

$$\begin{cases} F:\underline{C}_1 \to \underline{C}_2 \\ G:\underline{C}_2 \to \underline{C}_1 \end{cases}$$

are an adjoint pair. Assume:

$$\begin{cases} (\underline{C}_\ell,\mathcal{W}_\ell) \xrightarrow{L} (\underline{C}_1,\mathcal{W}_1) \text{ is a left approximation} \\ (\underline{C}_2,\mathcal{W}_2) \xleftarrow{K} (\underline{C}_r,\mathcal{W}_r) \text{ is a right approximation} \end{cases}$$

and

$$\begin{cases} FL\mathcal{W}_\ell \subset \mathcal{W}_2 \\ GK\mathcal{W}_r \subset \mathcal{W}_1. \end{cases}$$

Then the conclusions of 1.7.1 obtain (cf. 1.5.5).

1.7.3 LEMMA Suppose that for

$$\forall \begin{bmatrix} X_\ell \in \text{Ob } \underline{C}_\ell \\ \\ X_r \in \text{Ob } \underline{C}_r, \end{bmatrix}$$

an arrow

$$\phi \in \text{Mor}(FLX_\ell, KX_r)$$

is a weak equivalence iff its adjoint

$$\psi \in \text{Mor}(LX_\ell, GKX_r)$$

is a weak equivalence -- then the adjoint situation

$$(LF, RG, \underline{\underline{\mu}}, \underline{\underline{\nu}})$$

is an adjoint equivalence of metacategories.

1.8 *PARTIAL ADJOINTS*

Let $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ be categories (or metacategories).

1.8.1 DEFINITION Consider a diagram

$$\begin{tikzcd} \underline{A} \arrow[r, "F_1"] \arrow[d, "T_1"'] & \underline{B} \\ \underline{C} & \underline{D} \arrow[l, "F_2"] \arrow[u, "T_2"'] \end{tikzcd}$$

of functors -- then F_1, F_2 is a <u>partial adjoint</u> w.r.t. T_1, T_2 if it is possible to

to assign to each ordered pair $\begin{bmatrix} A \in \text{Ob } \underline{A} \\ D \in \text{Ob } \underline{D} \end{bmatrix}$ a bijective map

$$\Xi_{A,D}:\mathrm{Mor}(F_1A,T_2D) \to \mathrm{Mor}(T_1A,F_2D)$$

which is functorial in A and D.

N.B. Take $\underline{A} = \underline{C}$, $\underline{B} = \underline{D}$, $T_1 = \mathrm{id}_{\underline{A}}$, $T_2 = \mathrm{id}_{\underline{B}}$ to reduce to the usual scenario.

1.8.2 LEMMA If T_1 has a right adjoint S_1 and T_2 has a left adjoint S_2, then S_2F_1 is a left adjoint for S_1F_2.

PROOF In fact,

$$\begin{aligned}\mathrm{Mor}(S_2F_1A,D) &\approx \mathrm{Mor}(F_1A,T_2D)\\ &\approx \mathrm{Mor}(T_1A,F_2D)\\ &\approx \mathrm{Mor}(A,S_1F_2D).\end{aligned}$$

1.8.3 LEMMA If S_1,T_1 and S_2,T_2 are adjoint equivalences, then F_1S_1 is a left adjoint for F_2S_2.

PROOF In fact,

$$\begin{aligned}\mathrm{Mor}(F_1S_1C,B) &\approx \mathrm{Mor}(F_1S_1C,T_2S_2B)\\ &\approx \mathrm{Mor}(T_1S_1C,F_2S_2B)\\ &\approx \mathrm{Mor}(C,F_2S_2B).\end{aligned}$$

Let $(C_1,\mathcal{W}_1)$, $(C_2,\mathcal{W}_2)$ be category pairs. Assume:

$$\left[\begin{array}{l}(\underline{C}_\ell,\mathcal{W}_\ell) \xrightarrow{L} (\underline{C}_1,\mathcal{W}_1) \text{ is a left approximation}\\[1em] (\underline{C}_2,\mathcal{W}_2) \xleftarrow{K} (\underline{C}_r,\mathcal{W}_r) \text{ is a right approximation.}\end{array}\right.$$

Suppose further that

$$\begin{cases} \Phi_\ell : (\underline{C}_\ell, W_\ell) \to (\underline{C}_2, W_2) \\ \Phi_r : (\underline{C}_r, W_r) \to (\underline{C}_1, W_1) \end{cases}$$

are morphisms of category pairs. Arrange the data:

$$\begin{tikzcd} \underline{C}_\ell \arrow[r, "\Phi_\ell"] \arrow[d, "L"'] & \underline{C}_2 \\ \underline{C}_1 & \underline{C}_r . \arrow[l, "\Phi_r"] \arrow[u, "K"'] \end{tikzcd}$$

1.8.4 THEOREM If Φ_ℓ, Φ_r is a partial adjoint w.r.t. L, K, then $\bar{\Phi}_\ell$, $\bar{\Phi}_r$ is a partial adjoint w.r.t. $\bar{L}$, $\bar{K}$:

$$\begin{tikzcd} W_\ell^{-1}\underline{C}_\ell \arrow[r, "\bar{\Phi}_\ell"] \arrow[d, "\bar{L}"'] & W_2^{-1}\underline{C}_2 \\ W_1^{-1}\underline{C}_1 & W_r^{-1}\underline{C}_r , \arrow[l, "\bar{\Phi}_r"] \arrow[u, "\bar{K}"'] \end{tikzcd}$$

thus

$$\forall \begin{cases} X_\ell \in \mathrm{Ob}\, W_\ell^{-1}\underline{C}_\ell \\ X_r \in \mathrm{Ob}\, W_r^{-1}\underline{C}_r , \end{cases}$$

$$\mathrm{Mor}(\bar{\Phi}_\ell X_\ell, \bar{K}X_r) \approx \mathrm{Mor}(\bar{L}X_\ell, \bar{\Phi}_r X_r).$$

1.8.5 REMARK Recall that

$$\left[\begin{array}{l} \bar{L}: W_\ell^{-1}\underline{C}_\ell \to W_1^{-1}\underline{C}_1 \\ \\ \bar{K}: W_r^{-1}\underline{C}_r \to W_2^{-1}\underline{C}_2 \end{array}\right.$$

are equivalences of metacategories (cf. 1.5.6), thus $\bar{L}, \bar{K}$ is part of an adjoint equivalence, say

$$\left[\begin{array}{l} \bar{L}': W_1^{-1}\underline{C}_1 \to W_\ell^{-1}\underline{C}_\ell \\ \\ \bar{K}': W_2^{-1}\underline{C}_2 \to W_r^{-1}\underline{C}_r. \end{array}\right.$$

Let

$$\left[\begin{array}{l} V_1 = \bar{\Phi}_\ell \circ \bar{L}' \\ \\ V_2 = \bar{\Phi}_r \circ \bar{K}'. \end{array}\right.$$

Then

$$\left[\begin{array}{l} V_1: W_1^{-1}\underline{C}_1 \to W_2^{-1}\underline{C}_2 \\ \\ V_2: W_2^{-1}\underline{C}_2 \to W_1^{-1}\underline{C}_1 \end{array}\right.$$

are an adjoint pair (cf. 1.8.3).

1.8.6 LEMMA Suppose that

$$\forall \left[\begin{array}{l} X_\ell \in \mathrm{Ob}\,\underline{C}_\ell \\ \\ X_r \in \mathrm{Ob}\,\underline{C}_r. \end{array}\right.$$

an arrow

$$\phi \in \operatorname{Mor}(\Phi_\ell X_\ell, KX_r)$$

is a weak equivalence iff its partial adjoint

$$\psi \in \operatorname{Mor}(LX_\ell, \Phi_r X_r)$$

is a weak equivalence — then

$$\begin{cases} V_1 \circ V_2 \approx \mathrm{id}_{W_2^{-1}\underline{C}_2} \\ \\ V_2 \circ V_1 \approx \mathrm{id}_{W_1^{-1}\underline{C}_1}, \end{cases}$$

hence V_1 and V_2 are mutually inverse equivalences.

1.9 *PRODUCTS*

Let

$$(\underline{C}_i, W_i) \quad (i = 1,\ldots,n)$$

be category pairs.

1.9.1 LEMMA The canonical functor

$$\Big(\prod_{i=1}^{n} W_i\Big)^{-1} \prod_{i=1}^{n} \underline{C}_i \to \prod_{i=1}^{n} W_i^{-1}\underline{C}_i$$

is an isomorphism of metacategories.

PROOF By induction, it suffices to treat the case when $n = 2$. But bearing in mind 1.11, for every metacategory $\underline{D}$, there are functorial bijections

$$\operatorname{Mor}(W_1^{-1}\underline{C}_1 \times W_2^{-1}\underline{C}_2, \underline{D})$$

$$\approx \ \mathrm{Mor}(\mathcal{W}_1^{-1}\underline{C}_1, [\mathcal{W}_2^{-1}\underline{C}_2, \underline{D}])$$

$$\approx \ \mathrm{Mor}(\mathcal{W}_1^{-1}\underline{C}_1, [\underline{C}_2, \underline{D}]_{\mathcal{W}_2})$$

$$\approx \ \mathrm{Mor}[\underline{C}_1, [\underline{C}_2, \underline{D}]_{\mathcal{W}_2}]_{\mathcal{W}_1}$$

$$\approx \ \mathrm{Mor}[\underline{C}_1 \times \underline{C}_2, \underline{D}]_{\mathcal{W}_1 \times \mathcal{W}_2}$$

$$\approx \ \mathrm{Mor}((\mathcal{W}_1 \times \mathcal{W}_2)^{-1}(\underline{C}_1 \times \underline{C}_2), \underline{D}).$$

<u>N.B.</u> Therefore the functor

$$L_{\mathcal{W}_1} \times L_{\mathcal{W}_2} : \underline{C}_1 \times \underline{C}_2 \to \mathcal{W}_1^{-1}\underline{C}_1 \times \mathcal{W}_2^{-1}\underline{C}_2$$

is a localization of $\underline{C}_1 \times \underline{C}_2$ at $\mathcal{W}_1 \times \mathcal{W}_2$.

1.9.2 LEMMA Let $(\underline{C},\mathcal{W})$ be a category pair -- then $L_{\mathcal{W}}$ sends final objects in $\underline{C}$ to final objects in $\mathcal{W}^{-1}\underline{C}$.

1.9.3 LEMMA Let $(\underline{C},\mathcal{W})$ be a category pair. Assume: $\underline{C}$ has binary products and $\mathcal{W}$ is stable under the formation of products of pairs of arrows -- then $\mathcal{W}^{-1}\underline{C}$ has binary products.

PROOF Since $\underline{C}$ has binary products, the diagonal functor $\Delta_{\underline{C}}:\underline{C} \to \underline{C} \times \underline{C}$ has a right adjoint $\Pi_{\underline{C}}:\underline{C} \times \underline{C} \to \underline{C}$. In addition,

$$\left[\begin{array}{l} \Delta_{\underline{C}}:(\underline{C},\mathcal{W}) \to (\underline{C} \times \underline{C}, \mathcal{W} \times \mathcal{W}) \\ \\ \Pi_{\underline{C}}:(\underline{C} \times \underline{C}, \mathcal{W} \times \mathcal{W}) \to (\underline{C},\mathcal{W}) \end{array}\right.$$

are morphisms of category pairs, so

$$\begin{cases} \overline{\Delta_{\underline{C}}}: W^{-1}\underline{C} \to (W \times W)^{-1}(\underline{C} \times \underline{C}) \\ \\ \overline{\Pi_{\underline{C}}}: (W \times W)^{-1}(\underline{C} \times \underline{C}) \to W^{-1}\underline{C} \end{cases}$$

exist (cf. 1.4.5) and constitute an adjoint pair (cf. 1.7.1). But

$$(W \times W)^{-1}(\underline{C} \times \underline{C}) \approx W^{-1}\underline{C} \times W^{-1}\underline{C} \quad \text{(cf. 1.9.1)}$$

and under this isomorphism, $\overline{\Delta_{\underline{C}}}$ is identified with the diagonal functor

$$W^{-1}\underline{C} \to W^{-1}\underline{C} \times W^{-1}\underline{C},$$

which thus has a right adjoint, viz. the functor corresponding to $\overline{\Pi_{\underline{C}}}$. Therefore $W^{-1}\underline{C}$ has binary products.

[Note: $L_W: \underline{C} \to W^{-1}\underline{C}$ preserves binary products: $\forall\ X,Y \in \text{Ob}\ \underline{C}$,

$$L_W(X \times Y) \approx L_W X \times L_W Y.]$$

1.9.4 SCHOLIUM Let $(\underline{C}, W)$ be a category pair -- then $W^{-1}\underline{C}$ has finite products if $\underline{C}$ has a final object and binary products and if W is stable under the formation of products of pairs of arrows.

1.9.5 REMARK What has been said above for products admits the obvious reformulation in terms of coproducts.

CHAPTER 2: COFIBRATION CATEGORIES

CHAPTER 2: COFIBRATION CATEGORIES

2.1 *THE SETUP*

Consider a triple $(\underline{C},\mathcal{W},\text{cof})$, where $\underline{C}$ is a category with an initial object $\emptyset$ and

$$\begin{cases} \mathcal{W} \subset \operatorname{Mor} \underline{C} \\ \text{cof} \subset \operatorname{Mor} \underline{C} \end{cases}$$

are two composition closed classes of morphisms termed

$$\begin{cases} \underline{\text{weak equivalences}} \text{ (denoted } \xrightarrow{\sim}) \\ \underline{\text{cofibrations}} \text{ (denoted } \rightarrowtail). \end{cases}$$

Agreeing to call an object X <u>cofibrant</u> if the arrow $\emptyset \to X$ is a cofibration and a morphism $f:X \to Y$ an <u>acyclic cofibration</u> if it is both a weak equivalence and a cofibration, $\underline{C}$ is then said to be a <u>cofibration category</u> provided that the following axioms are satisfied.

(COF - 1) The initial object $\emptyset$ is cofibrant.

(COF - 2) All isomorphisms are weak equivalences and all isomorphisms with a cofibrant domain are cofibrations.

(COF - 3) Given composable morphisms f,g, if any two of $f,g,g \circ f$ are weak equivalences, so is the third.

(COF - 4) Every 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is a cofibration (acyclic cofibration) and Z,Y are cofibrant, admits a pushout $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$, where η is a cofibration (acyclic cofibration):

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P. \end{tikzcd}$$

(COF - 5) Every morphism with a cofibrant domain can be written as the composite of a cofibration and a weak equivalence.

N.B. $(\underline{C}, \mathcal{W})$ is a category pair.

2.1.1 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}$ -- then $\underline{\text{TOP}}$ is a cofibration category if weak equivalence = homotopy equivalence, cofibration = cofibration. All objects are cofibrant.

2.1.2 REMARK Given a cofibration category $\underline{C}$, denote by $\underline{C}_{\text{cof}}$ the full subcategory of $\underline{C}$ consisting of the cofibrant objects -- then $\underline{C}_{\text{cof}}$ is a cofibration category.

[Note: $\underline{C}_{\text{cof}}$ has finite coproducts (but this need not be true of $\underline{C}$). Proof: For cofibrant X and Y, consider the pushout square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \coprod Y, \end{tikzcd}$$

and observe that all arrows are cofibrations.]

2.1.3 DEFINITION Let $\underline{C}$ be a cofibration category -- then $\underline{C}$ is said to be homotopically cocomplete when the following conditions are met.

(H - 1) If $f_i:X_i \to Y_i$ $(i \in I)$ is a set of cofibrations with X_i cofibrant $\forall$ i, then the coproducts $\coprod_i X_i$, $\coprod_i Y_i$ exist, are cofibrant, and $\coprod_i f_i$ is a cofibration which is acyclic if this is the case of the f_i.

(H - 2) Let

$$X_0 \overset{f_0}{\rightarrowtail} X_1 \overset{f_1}{\rightarrowtail} X_2 \overset{f_2}{\rightarrowtail} \cdots$$

be a countable sequence of cofibrations (acyclic cofibrations) with X_0 cofibrant -- then $\operatorname{colim} X_n$ exists and the canonical arrow $X_0 \to \operatorname{colim} X_n$ is a cofibration (acyclic cofibration).

There is also the notion of a fibration category, the definition of which, to dispel any possible misunderstanding, will be provided in detail.

[Note: For the most part, the focus in the sequel will be on cofibration categories, the results for fibration categories being invariably dual.]

Consider a triple $(\underline{C},\mathcal{W},\text{fib})$, where $\underline{C}$ is a category with final object $*$ and

$$\begin{cases} \mathcal{W} \subset \operatorname{Mor} \underline{C} \\ \text{fib} \subset \operatorname{Mor} \underline{C} \end{cases}$$

are two composition closed classes of morphisms termed

$$\begin{cases} \underline{\text{weak equivalences}} \text{ (denoted } \xrightarrow{\sim} \text{)} \\ \underline{\text{fibrations}} \text{ (denoted } \twoheadrightarrow \text{).} \end{cases}$$

Agreeing to call an object X <u>fibrant</u> if the arrow $X \to *$ is a fibration and a morphism $f:X \to Y$ an <u>acyclic fibration</u> if it is both a weak equivalence and a fibration, $\underline{C}$ is then said to be a <u>fibration category</u> provided that the following axioms are satisfied.

(FIB – 1) The final object $*$ is fibrant.

(FIB – 2) All isomorphisms are weak equivalences and all isomorphisms with a fibrant codomain are fibrations.

(FIB – 3) Given composable morphisms f,g, if any two of $f,g,g \circ f$ are weak equivalences, so is the third.

(FIB – 4) Every 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, where g is a fibration (acyclic fibration) and X,Z fibrant, admits a pullback $X \xleftarrow{\xi} P \xrightarrow{\eta} Y$, where ξ is a fibration (acyclic fibration):

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

(FIB – 5) Every morphism with a fibrant codomain can be written as the composite of a weak equivalence and a fibration.

<u>N.B.</u> $(\underline{C}, \mathcal{W})$ is a category pair.

2.1.4 EXAMPLE Take $\underline{C} = \underline{TOP}$ –– then $\underline{TOP}$ is a fibration category if weak equivalence = homotopy equivalence, fibration = Hurewicz fibration. All objects are fibrant.

2.1.5 REMARK Given a fibration category $\underline{C}$, denote by $\underline{C}_{fib}$ the full subcategory of $\underline{C}$ consisting of the fibrant objects –– then $\underline{C}_{fib}$ is a fibration category.

[Note: $\underline{C}_{fib}$ has finite products (but this need not be true of $\underline{C}$). Proof: For fibrant X and Y, consider the pullback square

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

and observe that all arrows are fibrations.]

2.1.6 DEFINITION Let $\underline{C}$ be a fibration category -- then $\underline{C}$ is said to be <u>homotopically complete</u> when the following conditions are met.

(H - 1) If $f_i : X_i \to Y_i$ $(i \in I)$ is a set of fibrations with Y_i fibrant $\forall$ i, then the products $\prod_i X_i$, $\prod_i Y_i$ exist, are fibrant, and $\prod_i f_i$ is a fibration which is acyclic if this is the case of the f_i.

(H - 2) Let

$$\cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

be a countable sequence of fibrations (acyclic fibrations) with X_0 fibrant -- then $\lim X_n$ exists and the canonical arrow $\lim X_n \to X_0$ is a fibration (acyclic fibration).

2.1.7 REMARK In the terminology of Cisinski, a cofibration category is a category which is <u>derivable to the right</u> and a fibration category is a category which is <u>derivable to the left</u>.

There is a short list of technical facts which are formal consequences of the axioms. Since the proofs run parallel to their analogs in model category theory, they can be safely omitted.

2.1.8 LEMMA Let $\underline{C}$ be a cofibration category and let $f:X \to Y$ be a map between cofibrant objects — then f can be written as a composite $r \circ f'$, where f' is a cofibration and r is a weak equivalence which is a left inverse to an acyclic cofibration s.

2.1.9 LEMMA Let $\underline{C}$ be a cofibration category. If $f_i:X_i \to Y_i$ $(i \in I)$ is a finite set of weak equivalences (cofibrations) between cofibrant objects, then $\coprod_i f_i$ is a weak equivalence (cofibration).

2.1.10 LEMMA Let $\underline{C}$ be a cofibration category. Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P. \end{tikzcd}$$

Assume: f is a cofibration and g is a weak equivalence — then ξ is a weak equivalence provided Z,Y are cofibrant.

2.2 *APPROXIMATIONS*

Let $\underline{C}$ be a cofibration category — then a <u>cofibrant approximation</u> to $\underline{C}$ is a pair $(\underline{C}_0,\Lambda_0)$, where $\underline{C}_0$ is a cofibration category and $\Lambda_0:\underline{C}_0 \to \underline{C}$ is a functor satisfying the following conditions.

(CFA − 1) All objects of $\underline{C}_0$ are cofibrant.

(CFA − 2) Λ_0 preserves initial objects and cofibrations.

(CFA − 3) A morphism $f_0 \in \text{Mor}\,\underline{C}_0$ is a weak equivalence iff $\Lambda_0 f_0 \in \text{Mor}\,\underline{C}$ is a weak equivalence.

(CFA - 4) If $X_0 \xleftarrow{f_0} Z_0 \xrightarrow{g_0} Y_0$ is a 2-source in $\underline{C}_0$, where f_0, g_0 are cofibrations, then the induced arrow

$$\Lambda_0 X_0 \coprod_{\Lambda_0 Z_0} \Lambda_0 Y_0 \to \Lambda_0 (X_0 \coprod_{Z_0} Y_0)$$

is an isomorphism.

(CFA - 5) Every $f:\Lambda_0 X_0 \to Y$ factors as $f = r \circ \Lambda_0 f_0$, where f_0 is a cofibration in $\underline{C}_0$ and r is a weak equivalence in $\underline{C}$.

<u>N.B.</u> The definition of a fibrant approximation to a fibration category is dual.

2.2.1 EXAMPLE The inclusion $\underline{C}_{cof} \xrightarrow{\iota} \underline{C}$ is a cofibrant approximation to $\underline{C}$.

If $\Lambda_0:\underline{C}_0 \to \underline{C}$ is a cofibrant approximation to $\underline{C}$, then it is clear that

$$\Lambda_0 : (\underline{C}_0, \mathcal{W}_0) \to (\underline{C}, \mathcal{W})$$

is a morphism of category pairs and Λ_0 is resolvable to the left.

2.2.2 LEMMA A cofibrant approximation to $\underline{C}$ is a left approximation to $\underline{C}$, hence is a derivability structure to the left on $\underline{C}$ (cf. 1.5.5).

2.2.3 THEOREM If $\Lambda_0:\underline{C}_0 \to \underline{C}$ is a cofibrant approximation to $\underline{C}$, then the induced functor

$$\bar{\Lambda}_0 : \mathcal{W}_0^{-1}\underline{C}_0 \to \mathcal{W}^{-1}\underline{C}$$

is an equivalence of metacategories (cf. 1.5.6).

2.2.4 THEOREM Let $\underline{C}$ be a cofibration category and let $(\underline{C}_1, \mathcal{W}_1)$ be a category pair. Suppose that $F:\underline{C} \to \underline{C}_1$ is a functor that sends acyclic cofibrations between cofibrant objects to weak equivalences -- then F admits an absolute total left derived functor (LF, ν_F).

PROOF Consider

$$\underline{C}_{cof} \xrightarrow{\iota} \underline{C} \xrightarrow{F} \underline{C}_1.$$

To apply 1.5.2, let $f:X \to Y$ be a weak equivalence, where X and Y are cofibrant -- then the claim is that $F\iota f \equiv Ff:FX \to FY$ is a weak equivalence. To see this, use 2.1.8 and write $f = r \circ f'$. Since f and r are weak equivalences, the same holds for f'. Therefore f' is an acyclic cofibration between cofibrant objects, thus by hypothesis, Ff' is a weak equivalence. On the other hand, $r \circ s = \text{id}$ and s is an acyclic cofibration between cofibrant objects, so too Fs is a weak equivalence. But this implies that Fr is a weak equivalence, hence finally Ff is a weak equivalence.

2.2.5 THEOREM Let $\underline{C}$ be a cofibration category and let $(\underline{C}_1, \mathcal{W}_1)$ be a category pair. Let $\Lambda_0:\underline{C}_0 \to \underline{C}$ be a cofibrant approximation to $\underline{C}$ and suppose that $F:\underline{C} \to \underline{C}_1$ is a functor such that $F \circ \Lambda_0$ sends acyclic cofibrations to weak equivalences -- then F admits an absolute total left derived functor (LF, ν_F).

Let $\underline{C}$ be a cofibration category with cofibrant approximation $\Lambda_0:\underline{C}_0 \to \underline{C}$ and let $\underline{C}'$ be a fibration category with fibrant approximation $\Lambda_0':\underline{C}_0' \to \underline{C}'$. Suppose that

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

are an adjoint pair, thus schematically

$$\underline{C}_0 \xrightarrow{\Lambda_0} \underline{C} \underset{F'}{\overset{F}{\rightleftarrows}} \underline{C}' \xleftarrow{\Lambda_0'} \underline{C}_0'.$$

2.2.6 THEOREM Assume that $F \circ \Lambda_0$ sends acyclic cofibrations to weak equivalences and $F' \circ \Lambda_0'$ sends acyclic fibrations to weak equivalences -- then the functors

$$\begin{cases} LF: W^{-1}\underline{C} \to W'^{-1}\underline{C}' \\ RF': W'^{-1}\underline{C}' \to W^{-1}\underline{C} \end{cases}$$

exist and are an adjoint pair.

2.3 *SATURATION*

Let $\underline{C}$ be a cofibration category.

2.3.1 DEFINITION Suppose that $X \in \mathrm{Ob}\,\underline{C}$ is cofibrant -- then a <u>cylinder object</u> for X is an object IX in $\underline{C}$ together with a diagram $X \coprod X \overset{\iota}{\rightarrowtail} IX \xrightarrow{\sim} X$ that factors the folding map $X \coprod X \xrightarrow{\nabla} X$. Write $\begin{cases} i_0: X \to IX \\ i_1: X \to IX \end{cases}$ for the arrows $\begin{cases} \iota \circ in_0 \\ \iota \circ in_1 \end{cases}$ -- then $\begin{cases} i_0 \\ i_1 \end{cases}$ are acyclic cofibrations.

N.B. Cylinder objects exist (in general, nonfunctorially).

2.3.2 EXAMPLE For any topological space X, the inclusion

$$i_0X \cup i_1X \to X \times [0,1]$$

is a closed cofibration, thus if TOP is viewed as a model category per its Strøm structure, then a choice for IX is $X \times [0,1]$. On the other hand, the inclusion

$$i_0X \cup i_1X \to X \times [0,1]$$

need not be a cofibration in the Quillen structure but it will be if X is cofibrant (e.g., if X is a CW complex).

2.3.3 DEFINITION Morphisms $f,g:X \to Y$ between cofibrant X and Y are said to be left homotopic if $\exists$ a cylinder object IX for X, an acyclic cofibration $Y \xrightarrow{w} Y'$, and a morphism $H:IX \to Y'$ such that $H \circ i_0 = w \circ f$, $H \circ i_1 = w \circ g$. Notation: $f \underset{\ell}{\simeq} g$.

2.3.4 LEMMA Suppose that $f \underset{\ell}{\simeq} g$ — then f is a weak equivalence iff g is a weak equivalence.

PROOF Say, e.g., that f is a weak equivalence. Since $H \circ i_0 = w \circ f$ and i_0 is a weak equivalence, it follows that H is a weak equivalence. But $H \circ i_1 = w \circ g$, thus g is a weak equivalence.

2.3.5 THEOREM† If $f,g:X \to Y$ are morphisms between cofibrant X and Y, then f,g are left homotopic iff they are homotopic:

$$f \underset{\ell}{\simeq} g \iff f \simeq g.$$

---

† Brown, *Trans. Amer. Math. Soc.* 186 (1973), 419-458.

2.3.6 APPLICATION Let $\underline{C}$ be a model category. Suppose that X is cofibrant and Y is fibrant — then $f \underset{\ell}{\simeq} g$ iff $\exists$ a cylinder object IX for X and a morphism $H:IX \to Y$ such that $H \circ i_0 = f$, $H \circ i_1 = g$.

[Assume first that H exists:

$$\left[\begin{array}{l} \nabla \circ in_0 = w \circ \iota \circ in_0 = id_X \\ \\ \nabla \circ in_1 = w \circ \iota \circ in_1 = id_X \end{array}\right. \qquad (\exists\ w \in W)$$

$$\Rightarrow \quad L_W(w \circ \iota \circ in_0) = L_W(w \circ \iota \circ in_1)$$

$$\Rightarrow \quad L_W(\iota \circ in_0) = L_W(\iota \circ in_1) \Rightarrow i_0 \simeq i_1$$

$$\Rightarrow H \circ i_0 \simeq H \circ i_1 \Rightarrow f \simeq g.$$

Conversely, assume that $f \simeq g$. Choose an acyclic fibration $r:Y' \to Y$ with Y' cofibrant. Since X is cofibrant, the commutative diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y' \\ \downarrow & & \downarrow r \\ X & \xrightarrow[f]{} & Y, \end{array} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & Y' \\ \downarrow & & \downarrow r \\ X & \xrightarrow[g]{} & Y \end{array}$$

admit fillers

$$\left[\begin{array}{ll} f':X \to Y' & (r \circ f' = f) \\ \\ g':X \to Y' & (r \circ g' = g). \end{array}\right.$$

But

$$\left[\begin{array}{l} L_W(r \circ f') = L_W r \circ L_W f' = L_W f \\ \\ L_W(r \circ g') = L_W r \circ L_W g' = L_W g, \end{array}\right.$$

so

$$\begin{aligned} L_W f = L_W g &\Rightarrow L_W r \circ L_W f' = L_W r \circ L_W g' \\ &\Rightarrow L_W f' = L_W g' \\ &\Rightarrow f' \simeq g' \\ &\Rightarrow f' \underset{\ell}{\simeq} g' \quad \text{(cf. 2.3.5)}. \end{aligned}$$

Using the notation of 2.3.3, fix an acyclic cofibration $Y' \xrightarrow{w'} Y''$ and a morphism $H':IX \to Y''$ such that $H' \circ i_0 = w' \circ f'$, $H' \circ i_1 = w' \circ g'$. Let $h:Y'' \to Y$ be a filler for

$$\begin{tikzcd} Y' \arrow[r, "r"] \arrow[d, "w'"'] & Y \arrow[d] \\ Y'' \arrow[r] & * \end{tikzcd}$$

and put $H = h \circ H'$ -- then

$$\left[\begin{aligned} H \circ i_0 &= h \circ H' \circ i_0 = h \circ w' \circ f' = r \circ f' = f \\ H \circ i_1 &= h \circ H' \circ i_1 = h \circ w' \circ g' = r \circ g' = g. \end{aligned}\right]$$

2.3.7 LEMMA Suppose that X and Y are cofibrant and $w:X \to Y$ is a weak equivalence -- then any $f \in \text{Mor}(X,Y)$ which is homotopic to w is necessarily a weak equivalence.

PROOF The assumption is that $L_W w = L_W f$ or still, that $w \simeq f$. But then $w \underset{\ell}{\simeq} f$ (cf. 2.3.5), so 2.3.4 is applicable.

2.3.8 THEOREM[†] Every morphism $[\omega]$ in $\mathcal{W}^{-1}\underline{C}$ between objects X and Y which are cofibrant in $\underline{C}$ can be written as a left fraction $(L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f$, where f is a cofibration and w is an acyclic cofibration:

$$[\omega] = [X \xrightarrow{f} Y' \xleftarrow{w} Y].$$

2.3.9 LEMMA Suppose that $f:X \to Y$ is a morphism in $\underline{C}$ with X and Y cofibrant -- then $L_{\mathcal{W}}f$ has a left inverse in $\mathcal{W}^{-1}\underline{C}$ iff there is a cofibration $f':Y \to Y'$ such that $f' \circ f$ is a weak equivalence.

PROOF The implication $\Leftarrow$ is obvious. In the other direction, if $[\omega] \circ L_{\mathcal{W}}f = \mathrm{id}$, write, using 2.3.8,

$$[\omega] = (L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f',$$

hence

$$L_{\mathcal{W}}w = L_{\mathcal{W}}f' \circ L_{\mathcal{W}}f$$

or still, $w \simeq f' \circ f$. But this means that $f' \circ f$ is a weak equivalence (cf. 2.3.7).

2.3.10 LEMMA Suppose that $f:X \to Y$ is a morphism in $\underline{C}$ with X and Y cofibrant -- then $L_{\mathcal{W}}f$ is an isomorphism in $\mathcal{W}^{-1}\underline{C}$ iff there are cofibrations $f':Y \to Y'$, $f'':Y' \to Y''$ such that $f' \circ f$, $f'' \circ f'$ are weak equivalences.

PROOF First, if $f' \circ f = w$ ($w \in \mathcal{W}$), then

$$L_{\mathcal{W}}f' \circ (L_{\mathcal{W}}f \circ (L_{\mathcal{W}}w)^{-1}) = \mathrm{id},$$

so $L_{\mathcal{W}}f'$ is a retraction, and second, if $f'' \circ f' = w'$ ($w' \in \mathcal{W}$), then $L_{\mathcal{W}}f'$ is a monomorphism. Therefore $L_{\mathcal{W}}f'$ is an isomorphism, hence $L_{\mathcal{W}}f$ is an isomorphism. The

---

[†] Brown, ibid.

converse follows from a double application of 2.3.9.

2.3.11 THEOREM Let $\underline{C}$ be a cofibration category and suppose that $H - 2$ is in force -- then $\mathcal{W} = \overline{\mathcal{W}}$.

PROOF It is enough to prove that a cofibration $f:X \to Y$ in $\overline{\mathcal{W}}$ between cofibrant X and Y is in $\mathcal{W}$. Using 2.3.10, construct by induction a countable sequence of cofibrations

$$X_0 \overset{f_0}{\rightarrowtail} X_1 \overset{f_1}{\rightarrowtail} X_2 \overset{f_2}{\rightarrowtail} \cdots$$

with $X_0 = X$, $X_1 = Y$, $f_0 = f$ and such that $\forall\, n \geq 0$, the composition

$$X_n \longrightarrow X_{n+1} \longrightarrow X_{n+2}$$

is an acyclic cofibration -- then there are acyclic cofibrations

$$\begin{cases} X \to \operatorname{colim} X_{2n+1} \\ Y \to \operatorname{colim} X_{2n}, \end{cases}$$

canonical isomorphisms

$$\operatorname{colim} X_{2n+1} \approx \operatorname{colim} X_n \approx \operatorname{colim} X_{2n},$$

and a commutative diagram

$$\begin{tikzcd} \operatorname{colim} X_n \arrow[r, equal] & \operatorname{colim} X_n \\ X \arrow[u] \arrow[r, "f"'] & Y. \arrow[u] \end{tikzcd}$$

Since the vertical arrows are acyclic cofibrations, it follows that f is an acyclic cofibration.

[Note: The reduction to a cofibration $f:X \to Y$ between cofibrant X and Y runs as follows.

Step 1: Fix a cofibrant X' and a weak equivalence $X' \xrightarrow{w} X$ -- then $L_{\mathcal{W}}(f \circ w) = L_{\mathcal{W}}f \circ L_{\mathcal{W}}w$, so if $f \circ w \in \mathcal{W}$, then $f \in \mathcal{W}$. One can therefore assume that the domain of f is cofibrant.

Step 2: Write $f = r \circ f'$, where f' is a cofibration with a cofibrant domain and r is a weak equivalence -- then $L_{\mathcal{W}}f = L_{\mathcal{W}}r \circ L_{\mathcal{W}}f'$, so if $f' \in \mathcal{W}$, then $f \in \mathcal{W}$. One can therefore assume that f is a cofibration with a cofibrant domain and codomain.]

2.3.12 DEFINITION Let $(\underline{C},\mathcal{W})$ be a category pair -- then $\mathcal{W}$ satisfies the 2 out of 5 condition if whenever $f,g,h \in \text{Mor}\ \underline{C}$ have the property that $g \circ f$, $h \circ g$ exist and are in $\mathcal{W}$, then f,g,h are in $\mathcal{W}$.

2.3.13 REMARK Let $(\underline{C},\mathcal{W})$ be a category pair -- then $\mathcal{W}$ satisfies the 2 out of 3 condition if for composable $f,g \in \text{Mor}\ \underline{C}$, the assumption that two of $f,g,g \circ f$ are in $\mathcal{W}$ implies that the third is in $\mathcal{W}$. This said, it is then clear that

"2 out of 5" => "2 out of 3".

[Note: In the case of a cofibration category, the 2 out of 3 condition is assumption COF - 3.]

2.3.14 DEFINITION Let $(\underline{C},\mathcal{W})$ be a category pair -- then $\mathcal{W}$ is weakly saturated if $\mathcal{W}$ satisfies the 2 out of 3 condition and has the following property:

If $\begin{cases} i:X \to Y \\ r:Y \to X \end{cases}$, if $r \circ i = \text{id}_X$, and if $i \circ r \in \mathcal{W}$, then $i,r \in \mathcal{W}$.

2.3.15 LEMMA If $\mathcal{W}$ is saturated, then $\mathcal{W}$ is weakly saturated.

PROOF That $\mathcal{W}(= \bar{\mathcal{W}})$ satisfies the 2 out of 3 condition is obvious. Suppose now that i and r are as above and write

$$L_{\mathcal{W}}(i \circ r) = L_{\mathcal{W}}i \circ L_{\mathcal{W}}r$$

to see that $L_{\mathcal{W}}i$ is an epimorphism. But

$$L_{\mathcal{W}}r \circ L_{\mathcal{W}}i = \mathrm{id}_{L_{\mathcal{W}}X}$$

and

$$\begin{aligned}
&(L_{\mathcal{W}}i \circ L_{\mathcal{W}}r) \circ L_{\mathcal{W}}i \\
&\quad = L_{\mathcal{W}}i \circ (L_{\mathcal{W}}r \circ L_{\mathcal{W}}i) \\
&\quad = L_{\mathcal{W}}i \circ L_{\mathcal{W}}(r \circ i) \\
&\quad = L_{\mathcal{W}}i \circ \mathrm{id}_{L_{\mathcal{W}}X} \\
&\quad = L_{\mathcal{W}}i = \mathrm{id}_{L_{\mathcal{W}}Y} \circ L_{\mathcal{W}}i \\
&\Rightarrow \\
&\quad L_{\mathcal{W}}i \circ L_{\mathcal{W}}r = \mathrm{id}_{L_{\mathcal{W}}Y}.
\end{aligned}$$

Therefore $i \in \mathcal{W}$ and lastly $r \in \mathcal{W}$.

2.3.16 LEMMA If $\mathcal{W}$ satisfies the 2 out of 5 condition, then $\mathcal{W}$ is weakly saturated.

PROOF Take i and r as above and consider

$$X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{i} Y.$$

2.3.17 LEMMA If $\mathcal{W}$ satisfies the 2 out of 3 condition and is closed under the formation of retracts, then $\mathcal{W}$ is weakly saturated.

PROOF Take i and r as above and note that the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{id_Y} & Y & \xrightarrow{id_Y} & Y \\ r \downarrow & & \downarrow i \circ r & & \downarrow r \\ X & \xrightarrow[i]{} & Y & \xrightarrow[r]{} & X \end{array}$$

exhibits r as a retract of $i \circ r$.

2.3.18 THEOREM Let $\underline{C}$ be a cofibration category — then the following are equivalent.

(1) $\mathcal{W}$ is weakly saturated.

(2) $\mathcal{W}$ satisfies the 2 out of 5 condition.

(3) $\mathcal{W}$ is closed under the formation of retracts.

(4) $\mathcal{W}$ is saturated.

PROOF We have (2) => (1), (3) => (1), (4) => (1), (2), (3), so the only point at issue is (1) => (4) and for this it is enough to prove that a cofibration $f:X \to Y$ in $\overline{\mathcal{W}}$ between cofibrant X and Y is in $\mathcal{W}$. Put $X_0 = X$, $X_1 = Y$ and construct a cofibration $g:X_1 \to X_2$ and a morphism $h:X_2 \to X_1$ such that $g \circ f \in \mathcal{W}$ and $h \circ g = id_{X_1}$ (see below) -- then

$$L_{\mathcal{W}}(g \circ f) = L_{\mathcal{W}}g \circ L_{\mathcal{W}}f,$$

so $g \in \overline{\mathcal{W}}$. And

$$h \circ g = id_{X_1} \Rightarrow g \circ h \circ g = g$$

$$\Rightarrow$$

$$L_{\mathcal{W}}(g \circ h) \circ L_{\mathcal{W}}g = L_{\mathcal{W}}g$$

$$\Rightarrow \quad L_{\mathcal{W}}(g \circ h) = \mathrm{id}_{L_{\mathcal{W}}X_2} = L_{\mathcal{W}}(\mathrm{id}_{X_2})$$

$$\Rightarrow \quad g \circ h \simeq \mathrm{id}_{X_2}$$

$$\Rightarrow \quad g \circ h \in \mathcal{W} \quad (\text{cf. } 2.3.7)$$

$$\Rightarrow g \in \mathcal{W} \Rightarrow f \in \mathcal{W}.$$

2.3.19 DETAILS The category $\underline{C}/Y$ is a cofibration category (via the forgetful functor $\underline{C}/Y \to Y$). Denoting by $\mathcal{W}_Y \subset \mathrm{Mor}\ \underline{C}/Y$ its class of weak equivalences, the image of the morphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

in $\mathcal{W}_Y^{-1}\underline{C}/Y$ is an isomorphism. On the other hand, $\emptyset \to Y$ is an initial object in $\underline{C}/Y$ and there are commutative diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, equal] & Y \end{tikzcd}.$$

Since $\left[\begin{array}{l} X \\ Y \end{array}\right.$ are cofibrant, the arrows $\left[\begin{array}{l} \emptyset \to X \\ \emptyset \to Y \end{array}\right.$ are cofibrations in $\underline{C}$, thus the arrows

$$\left[\begin{array}{l} (\emptyset \longrightarrow Y) \longrightarrow (X \xrightarrow{f} Y) \\ (\emptyset \longrightarrow Y) \longrightarrow (Y \xrightarrow[\mathrm{id}_Y]{} Y) \end{array}\right.$$

are cofibrations in $\underline{C}/Y$, i.e., the objects

$$\left[\begin{array}{l} X \xrightarrow{f} Y \\ \\ Y \xrightarrow[\mathrm{id}_Y]{} Y \end{array}\right.$$

are cofibrant in $\underline{C}/Y$. One can therefore apply 2.3.10 to $\underline{C}/Y$ to get a cofibration

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, "\mathrm{id}_Y"'] & Y' \arrow[d, "g'"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \qquad (g' \circ f' = \mathrm{id}_Y)$$

in $\underline{C}/Y$ such that

$$\begin{tikzcd} X \arrow[r, "f' \circ f"] \arrow[d, "f"'] & Y' \arrow[d, "g'"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a weak equivalence in $\underline{C}/Y$. So f' is a cofibration in $\underline{C}$ and $f' \circ f \in \mathcal{W}$. Reverting back to the notation of 2.3.18, let $X_0 = X$, $X_1 = Y$, $X_2 = Y'$, $g = f'$, $h = g'$ -- then

$$g \circ f = f' \circ f \in \mathcal{W}$$

and

$$h \circ g = g' \circ f' = \mathrm{id}_Y = \mathrm{id}_{X_1}.$$

2.3.20 APPLICATION Suppose that $\underline{C}$ is a model category -- then $\mathcal{W}$ is closed under the formation of retracts, hence $\mathcal{W}$ is saturated.

[Note: For us, a model category is finitely complete and finitely cocomplete, so it would be illegal in general to quote 2.3.11.]

2.3.21 THEOREM Suppose that $(\underline{C},W,\mathrm{cof})$ is a cofibration category — then $(\underline{C},\overline{W},\mathrm{cof})$ is a cofibration category.

2.4 *FIBRANT MODELS*

Let $\underline{C}$ be a cofibration category — then an object Y in $\underline{C}$ is a <u>fibrant model</u> if for any 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where Z is cofibrant and f is an acyclic cofibration, $\exists$ $h:X \to Y$ such that $h \circ f = g$.

<u>N.B.</u> If $\underline{C}$ has a final object $*$, then Y is a fibrant model iff the arrow $Y \to *$ has the RLP w.r.t. all acyclic cofibrations that have a cofibrant domain.

E.g.: The fibrant objects of a model category are fibrant models.

2.4.1 RAPPEL The functor $\underline{HO}_W\underline{C} \to W^{-1}\underline{C}$ is faithful, so $\forall$ $X,Y \in \mathrm{Ob}\ \underline{C}$, the induced map

$$[X,Y] \to \mathrm{Mor}(X,Y)$$

is injective.

2.4.2 LEMMA If X is cofibrant and Y is a fibrant model, then the induced map

$$[X,Y] \to \mathrm{Mor}(X,Y)$$

is surjective.

PROOF Let $[\omega] \in \mathrm{Mor}(X,Y)$. Fix a cofibrant Y' and a weak equivalence $w':Y' \to Y$ — then

$$(L_W w')^{-1} \circ [\omega] \in \mathrm{Mor}(X,Y'),$$

so, using 2.3.8, we can write

$$(L_{\mathcal{W}}w')^{-1} \circ [\omega] = (L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f$$

$$= [X \xrightarrow{f} Y'' \xleftarrow{w} Y'],$$

thus

$$[\omega] = L_{\mathcal{W}}w' \circ (L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f.$$

Consider the 2-source $Y'' \xleftarrow{w} Y' \xrightarrow{w'} Y$. Since by construction w is an acyclic cofibration and since Y is a fibrant model, $\exists\ \Lambda: Y'' \to Y$ such that $\Lambda \circ w = w'$. Therefore

$$[\omega] = L_{\mathcal{W}}(\Lambda \circ w) \circ (L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f$$

$$= L_{\mathcal{W}}\Lambda \circ L_{\mathcal{W}}w \circ (L_{\mathcal{W}}w)^{-1} \circ L_{\mathcal{W}}f$$

$$= L_{\mathcal{W}}(\Lambda \circ f),$$

from which the surjectivity.

2.4.3 CRITERION Let $\underline{C}$ be a cofibration category with the following property: Given any cofibrant X, $\exists$ a fibrant model X' and a weak equivalence $X \to X'$ -- then $\mathcal{W}^{-1}\underline{C}$ is a category (and not just a metacategory).

[This is implied by 2.4.2.]

2.4.4 THEOREM Suppose that $\underline{C}$ is a model category -- then $\underline{HC}$ is a category (and not just a metacategory).

2.4.5 REMARK Let $\underline{C}$ be a category. Suppose given a composition closed class $\mathcal{W} \subset \text{Mor}\ \underline{C}$ containing the isomorphisms of $\underline{C}$ such that for composable morphisms f,g, if any two of $f, g, g \circ f$ are in $\mathcal{W}$, so is the third. Problem: Does $\mathcal{W}^{-1}\underline{C}$ exist as a category? The assumption that $\mathcal{W}$ admits a calculus of left or right fractions does not suffice to resolve the issue. However, one strategy that will work is to somehow place on $\underline{C}$ the structure of a model category in which $\mathcal{W}$ appears as the class of weak equivalences.

2.5 *PRINCIPLES OF PERMANENCE*

Fix a small category $\underline{I}$.

2.5.1 DEFINITION Let $\underline{C}$ be a cofibration category and suppose that $\Xi \in \mathrm{Mor}[\underline{I},\underline{C}]$, say $\Xi : F \to G$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\, i \in \mathrm{Ob}\,\underline{I}$, $\Xi_i : Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a <u>levelwise cofibration</u> if $\forall\, i \in \mathrm{Ob}\,\underline{I}$, $\Xi_i : Fi \to Gi$ is a cofibration in $\underline{C}$.

2.5.2 DEFINITION The <u>injective structure</u> on $[\underline{I},\underline{C}]$ is the pair consisting of the levelwise weak equivalences and the levelwise cofibrations.

2.5.3 THEOREM Suppose that $\underline{C}$ is a homotopically cocomplete cofibration category — then $[\underline{I},\underline{C}]$, equipped with its injective structure, is a homotopically cocomplete cofibration category.

2.5.4 DEFINITION Let $\underline{C}$ be a fibration category and suppose that $\Xi \in \mathrm{Mor}[\underline{I},\underline{C}]$, say $\Xi : F \to G$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\, i \in \mathrm{Ob}\,\underline{I}$, $\Xi_i : Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a levelwise fibration if $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a fibration in $\underline{C}$.

2.5.5 DEFINITION The projective structure on $[\underline{I},\underline{C}]$ is the pair consisting of the levelwise weak equivalences and the levelwise fibrations.

2.5.6 THEOREM Suppose that $\underline{C}$ is a homotopically complete fibration category -- then $[\underline{I},\underline{C}]$, equipped with its projective structure, is a homotopically complete fibration category.

Let $\underline{I}$ and $\underline{J}$ be small categories, $K:\underline{I} \to \underline{J}$ a functor. Given a category pair $(\underline{C},\mathcal{W})$, let

$$\begin{cases} \mathcal{W}_{\underline{I}} = \text{the levelwise weak equivalences in } \mathrm{Mor}[\underline{I},\underline{C}] \\ \mathcal{W}_{\underline{J}} = \text{the levelwise weak equivalences in } \mathrm{Mor}[\underline{J},\underline{C}]. \end{cases}$$

(obvious definition)

Then the functor $K^*:[\underline{J},\underline{C}] \to [\underline{I},\underline{C}]$ preserves levelwise weak equivalences, so there is a commutative diagram

$$\begin{tikzcd} {[\underline{J},\underline{C}]} \arrow[r, "K^*"] \arrow[d] & {[\underline{I},\underline{C}]} \arrow[d] \\ \mathcal{W}_{\underline{J}}^{-1}[\underline{J},\underline{C}] \arrow[r, "\overline{K^*}"'] & \mathcal{W}_{\underline{I}}^{-1}[\underline{I},\underline{C}]. \end{tikzcd}$$

- If $\underline{C}$ is a cocomplete cofibration category, then K^* has a left adjoint

$$K_!:[\underline{I},\underline{C}] \to [\underline{J},\underline{C}].$$

- If $\underline{C}$ is a complete fibration category, then K^* has a right adjoint

$$K_\dagger:[\underline{I},\underline{C}] \to [\underline{J},\underline{C}].$$

2.5.7 THEOREM Suppose that $\underline{C}$ is a cocomplete cofibration category -- then $K_!$ possesses an absolute total left derived functor $(LK_!, \nu_{K_!})$ and

$$\left[\begin{array}{l} LK_! \\ \\ \overline{K^*} \end{array}\right.$$

are an adjoint pair.

[Note: The assumption that $\underline{C}$ is cocomplete can be weakened to homotopically cocomplete. Matters then become more complicated as $K_!$ need not exist. Nevertheless, it is still the case that $\overline{K^*}$ admits a left adjoint which, in an abuse of notation, is denoted by $LK_!$ and called the <u>homotopy colimit of K</u>.]

2.5.8 THEOREM Suppose that $\underline{C}$ is a complete fibration category -- then $K_\dagger$ possesses an absolute total right derived functor $(RK_\dagger, \mu_{K_\dagger})$ and

$$\left[\begin{array}{l} \overline{K^*} \\ \\ RK_\dagger \end{array}\right.$$

are an adjoint pair.

[Note: The assumption that $\underline{C}$ is complete can be weakened to homotopically complete. Matters then become more complicated as $K_\dagger$ need not exist. Nevertheless, it is still the case that $\overline{K^*}$ admits a right adjoint which, in an abuse of notation, is denoted by $RK_\dagger$ and called the <u>homotopy limit of K</u>.]

2.6 *WEAK COLIMITS*

Let $(\underline{C}, \mathcal{W})$ be a category pair -- then for any small category $\underline{I}$, there are arrows

$$\begin{array}{ccc} [\underline{I},\underline{C}] & \xrightarrow{(L_W)_*} & [\underline{I},W^{-1}\underline{C}] \\ {\scriptstyle L_{W_{\underline{I}}}}\downarrow & & \\ W_{\underline{I}}^{-1}[\underline{I},\underline{C}], & & \end{array}$$

from which an arrow

$$\mathrm{dgm}_{\underline{I}}:W_{\underline{I}}^{-1}[\underline{I},\underline{C}] \rightarrow [\underline{I},W^{-1}\underline{C}]$$

rendering the triangle commutative:

$$\mathrm{dgm}_{\underline{I}} \circ L_{W_{\underline{I}}} = (L_W)_*.$$

[Note: Given $\Xi \in \mathrm{Mor}[\underline{I},\underline{C}]$, we have

$$((L_W)_*\Xi)_i = L_W\Xi_i \quad (i \in \mathrm{Ob}\ \underline{I}).$$

And

$$\Xi \in W_{\underline{I}} \Rightarrow \Xi_i \in W \quad (i \in \mathrm{Ob}\ \underline{I}).]$$

2.6.1 LEMMA If $\underline{C}$ is a homotopically cocomplete cofibration category, then the functor $\mathrm{dgm}_{\underline{I}}$ is conservative.

Suppose that $\underline{C}$ is a homotopically cocomplete cofibration category -- then $W^{-1}\underline{C}$ has coproducts but, in general, does not have coequalizers or pushouts, thus $W^{-1}\underline{C}$ need not be cocomplete.

2.6.2 RAPPEL Let $\underline{I}$ be a small category, $\underline{C}$ a cocomplete category -- then the constant diagram functor $K:\underline{C} \to [\underline{I},\underline{C}]$ has a left adjoint, viz. $\mathrm{colim}_{\underline{I}}:[\underline{I},\underline{C}] \to \underline{C}$. So, for any diagram $\Delta:\underline{I} \to \underline{C}$, for any $X \in \mathrm{Ob}\,\underline{C}$, and for any morphism $f:\Delta \to KX$ there exists a unique morphism $g:\mathrm{colim}_{\underline{I}}\Delta \to X$ such that $f = Kg \circ \mu_\Delta$:

$$\begin{tikzcd} \Delta \arrow[r, "\mu_\Delta"] \arrow[d, "f"'] & K\,\mathrm{colim}_{\underline{I}} \arrow[d, "Kg"] \\ KX \arrow[r, equal] & KX, \end{tikzcd}$$

where $\mu_\Delta:\Delta \to K\,\mathrm{colim}_{\underline{I}}\Delta$ is the arrow of adjunction.

2.6.3 DEFINITION Let $\underline{I}$ be a small category, $\underline{C}$ a metacategory and let $\Delta:\underline{I} \to \underline{C}$ be a diagram -- then a <u>weak colimit of Δ</u>, if it exists, is an object $\mathrm{wcolim}_{\underline{I}}\Delta \in \mathrm{Ob}\,\underline{C}$ and a morphism

$$\mu_\Delta:\Delta \to K\,\mathrm{wcolim}_{\underline{I}}\Delta$$

with the property that for any other object $X \in \mathrm{Ob}\,\underline{C}$ and morphism $f:\Delta \to KX$ there exists a (not necessarily unique) morphism $g:\mathrm{wcolim}_{\underline{I}}\Delta \to X$ such that $f = Kg \circ \mu_\Delta$:

$$\begin{tikzcd} \Delta \arrow[r, "\mu_\Delta"] \arrow[d, "f"'] & K\,\mathrm{wcolim}_{\underline{I}}\Delta \arrow[d, "Kg"] \\ KX \arrow[r, equal] & KX. \end{tikzcd}$$

2.6.4 THEOREM Suppose that $\underline{C}$ is a homotopically cocomplete cofibration category.

Assume:

$$\mathrm{dgm}_{\underline{I}}:W_{\underline{I}}^{-1}[\underline{I},\underline{C}] \rightarrow [\underline{I},W^{-1}\underline{C}]$$

is full and has a representative image -- then every diagram $\Delta:\underline{I} \rightarrow W^{-1}\underline{C}$ has a weak colimit $\mathrm{wcolim}_{\underline{I}}\Delta$ which is unique up to (noncanonical) isomorphism.

PROOF Choose $\Delta' \in \mathrm{Ob}\ W_{\underline{I}}^{-1}[\underline{I},\underline{C}]:\mathrm{dgm}_{\underline{I}}\Delta' \approx \Delta$. Taking $\underline{J} = \underline{1}$ in the theory developed in 2.5, let

$$\mu_{\Delta'}:\Delta' \rightarrow \overline{K^*}LK_!\Delta'$$

be the arrow of adjunction and put

$$\mathrm{wcolim}_{\underline{I}}\Delta = \mathrm{dgm}_{\underline{J}}LK_!\Delta',$$

which can be viewed as an element of Ob $\underline{C}$ -- then there is an arrow

$$\mu_{\Delta}:\Delta \rightarrow \mathrm{dgm}_{\underline{I}}\overline{K^*}LK_!\Delta'.$$

But the diagram

$$\begin{tikzcd} W_{\underline{J}}^{-1}[\underline{J},\underline{C}] \arrow[r, "\mathrm{dgm}_{\underline{J}}", "\approx"'] \arrow[d, "\overline{K^*}"'] & {[\underline{J},W^{-1}\underline{C}]} \arrow[d, "K^*"] \\ W_{\underline{I}}^{-1}[\underline{I},\underline{C}] \arrow[r, "\mathrm{dgm}_{\underline{I}}"'] & {[\underline{I},W^{-1}\underline{C}]} \end{tikzcd}$$

commutes, so

$$\mu_{\Delta}:\Delta \longrightarrow K^*\mathrm{dgm}_{\underline{J}}LK_!\tilde{\Delta}$$

or still,

$$\mu_{\Delta}:\Delta \longrightarrow K^*\mathrm{wcolim}_{\underline{I}}\Delta$$

or still,

$$\mu_\Delta:\Delta \longrightarrow K\ \mathrm{wcolim}_{\underline{I}}\Delta \qquad (K^* \approx K).$$

Therefore the pair

$$(\mathrm{wcolim}_{\underline{I}}\Delta, \mu_\Delta)$$

is a weak colimit of Δ. If the process is repeated with $\Delta'' \in \mathrm{Ob}\ \mathcal{W}_{\underline{I}}^{-1}[\underline{I},\underline{C}]$, thus

$$\mathrm{dgm}_{\underline{I}}\Delta'' \approx \Delta,$$

then one can find an $f \in \mathrm{Mor}(\Delta',\Delta'')$ such that $\mathrm{dgm}_{\underline{I}}f$ implements the isomorphism

$$\mathrm{dgm}_{\underline{I}}\Delta' \approx \mathrm{dgm}_{\underline{I}}\Delta''.$$

But $\mathrm{dgm}_{\underline{I}}$ is conservative (cf. 2.6.1), hence f is an isomorphism. Consequently, $\mathrm{wcolim}_{\underline{I}}\Delta$ (as constructed) is unique up to (noncanonical) isomorphism.

2.6.5 DEFINITION A small category $\underline{I}$ is <u>free</u> if it is isomorphic to a category in the image of the left adjoint to the forgetful functor $U:\underline{\mathbf{CAT}} \to \underline{\mathbf{PRECAT}}$.

[Note: A finite, free category is both direct and inverse.]

2.6.6 LEMMA If $\underline{I}$ is a small category which is free and direct, then for any homotopically cocomplete cofibration category $\underline{C}$, the functor

$$\mathrm{dgm}_{\underline{I}}:\mathcal{W}_{\underline{I}}^{-1}[\underline{I},\underline{C}] \to [\underline{I},\mathcal{W}^{-1}\underline{C}]$$

is full and has a representative image.

2.6.7 EXAMPLE The categories

$$\begin{tikzcd} 1 \bullet \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \bullet\ 2 \end{tikzcd}, \qquad \begin{tikzcd} 1 \bullet & \underset{3}{\bullet} \arrow[l, "a"'] \arrow[r, "b"] & \bullet\ 2 \end{tikzcd}$$

are free and direct.

2.6.8 APPLICATION Every homotopically cocomplete cofibration category admits weak coequalizers and weak pushouts.

[Note: The story for homotopically complete fibration categories is analogous.]

2.7 WEAK MODEL CATEGORIES

Let $\underline{C}$ be a category and let $\mathcal{W}$, cof, fib be three composition closed classes of morphisms such that

$$(\underline{C},\mathcal{W},\mathrm{cof})$$

is a homotopically cocomplete cofibration category and

$$(\underline{C},\mathcal{W},\mathrm{fib})$$

is a homotopically complete fibration category.

2.7.1 DEFINITION $\underline{C}$ is said to be a <u>weak model category</u> provided that the following axioms are satisfied.

(WMC – 1) $\mathcal{W}$ is closed under the formation of retracts.

(WMC – 2) Acyclic cofibrations with cofibrant domain have the LLP w.r.t. fibrations with fibrant codomain.

(WMC – 3) Cofibrations with cofibrant domain have the LLP w.r.t. acyclic fibrations with fibrant codomain.

2.7.2 REMARK Every complete and cocomplete model category is a weak model category (but not conversely).

2.7.3 LEMMA Suppose that $\underline{C}$ is a weak model category -- then $\mathcal{W}$ is saturated (cf. 2.3.18).

2.7.4 LEMMA Suppose that $\underline{C}$ is a weak model category -- then $W^{-1}\underline{C}$ is a category (cf. 2.4.3).

Fix a small category $\underline{I}$.

2.7.5 THEOREM[†] Let $\underline{C}$ be a weak model category — then $[\underline{I},\underline{C}]$ admits a weak model structure in which the weak equivalences are the levelwise weak equivalences and the cofibrations are the levelwise cofibrations.

[Note: The description of the fibrations is somewhat involved but they are, at least, levelwise.]

2.7.6 THEOREM[†] Let $\underline{C}$ be a weak model category -- then $[\underline{I},\underline{C}]$ admits a weak model structure in which the weak equivalences are the levelwise weak equivalences and the fibrations are the levelwise fibrations.

[Note: The description of the cofibrations is somewhat involved but they are, at least, levelwise.]

2.7.7 REMARK In either weak model structure on $[\underline{I},\underline{C}]$, $W_{\underline{I}}$ is the class of weak equivalences and $W_{\underline{I}}^{-1}[\underline{I},\underline{C}]$ is a category (cf. 2.7.4).

[†] Cisinski, *Bull. Soc. Math. France* 138 (2010), 317-393.

CHAPTER 3: HOMOTOPY THEORIES

CHAPTER 3: HOMOTOPY THEORIES

3.1 *THE STAR PRODUCT*

Let $F, F' : \underline{C} \to \underline{D}$ and $G, G' : \underline{D} \to \underline{E}$ be functors; let

$$\begin{cases} \Xi \in \mathrm{Nat}(F, F') \\ \Omega \in \mathrm{Nat}(G, G'). \end{cases}$$

Then $\forall\ X \in \mathrm{Ob}\ \underline{C}$, there is a commutative diagram

$$\begin{tikzcd}
(G \circ F)_X \arrow[r, "(G\Xi)_X"] \arrow[d, "(\Omega F)_X"'] & (G \circ F')_X \arrow[d, "(\Omega F')_X"] \\
(G' \circ F)_X \arrow[r, "(G'\Xi)_X"'] & (G' \circ F')_X.
\end{tikzcd}$$

3.1.1 DEFINITION The <u>star product</u> of Ω and Ξ is defined by

$$\Omega * \Xi = G'\Xi \circ \Omega F$$

or still,

$$\Omega * \Xi = \Omega F' \circ G\Xi.$$

[Note: The star product is associative and in suggestive notation,

$$(\Omega' \circ \Omega) * (\Xi' \circ \Xi) = (\Omega' * \Xi') \circ (\Omega * \Xi).]$$

<u>N.B.</u>

$$\Omega * \Xi \in \mathrm{Nat}(G \circ F, G' \circ F').$$

3.1.2 EXAMPLE We have

$$\begin{cases} \Omega F = \Omega * \mathrm{id}_F \\ G\Xi = \mathrm{id}_G * \Xi \end{cases} \qquad \text{and } \mathrm{id}_G * \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

3.2 *DERIVATORS*

A <u>derivator</u> D is a prescription that assigns to each small category $\underline{I}$ a metacategory $D\underline{I}$, to each functor $F:\underline{I} \to \underline{J}$ a functor

$$DF:D\underline{J} \to D\underline{I},$$

and to each natural transformation $\Xi:F \to G$ a natural transformation

$$D\Xi:DG \to DF,$$

the data being subject to the following assumptions.

- For all $\underline{I}$, $D\mathrm{id}_{\underline{I}} = \mathrm{id}_{D\underline{I}}$ and given $\underline{I} \xrightarrow{F} \underline{J} \xrightarrow{G} \underline{K}$, we have

$$D(G \circ F) = DF \circ DG.$$

- For all F, $D\mathrm{id}_F = \mathrm{id}_{DF}$ and given $F \xrightarrow{\Xi} G \xrightarrow{\Omega} H$, we have

$$D(\Omega \circ \Xi) = D\Xi \circ D\Omega.$$

- If

$$\begin{tikzcd} \underline{I} \arrow[r, shift left, "F"] \arrow[r, shift right, "F'"'] & \underline{J} \arrow[r, shift left, "G"] \arrow[r, shift right, "G'"'] & \underline{K} \end{tikzcd}$$

and if

$$\begin{cases} \Xi \in \mathrm{Nat}(F,F') \\ \Omega \in \mathrm{Nat}(G,G'), \end{cases}$$

then

$$D(\Omega * \Xi) = D\Xi * D\Omega.$$

<u>N.B.</u> If D is a derivator, then its <u>opposite</u> D^{OP} is the derivator that sends $\underline{I}$ to $(D\underline{I}^{OP})^{OP}$.

3.2.1 EXAMPLE Let $(\underline{C},W)$ be a category pair. Given $\underline{I} \in \mathrm{Ob}\ \underline{CAT}$, let $W_{\underline{I}^{OP}}$ be the levelwise weak equivalences in $\mathrm{Mor}[\underline{I}^{OP},\underline{C}]$ — then

$$([\underline{I}^{OP},\underline{C}],W_{\underline{I}^{OP}})$$

is a category pair, thus it makes sense to form the localization of $[\underline{I}^{OP},\underline{C}]$ at $W_{\underline{I}^{OP}}$:

$$W_{\underline{I}^{OP}}^{-1}[\underline{I}^{OP},\underline{C}] \qquad \text{(cf. 1.1.2).}$$

Define now a derivator $D_{(\underline{C},W)}$ by first specifying that

$$D_{(\underline{C},W)}\underline{I} = W_{\underline{I}^{OP}}^{-1}[\underline{I}^{OP},\underline{C}].$$

Next, given $F:\underline{I} \to \underline{J}$, pass to $F^{OP}:\underline{I}^{OP} \to \underline{J}^{OP}$ and note that the induced functor

$$(F^{OP})^*:[\underline{J}^{OP},\underline{C}] \to [\underline{I}^{OP},\underline{C}]$$

is a morphism of category pairs (i.e., $(F^{OP})^*W_{\underline{J}^{OP}} \subset W_{\underline{I}^{OP}}$), which leads to a functor

$$\overline{(F^{OP})^*}:W^{-1}_{\underline{J}^{OP}}[\underline{J}^{OP},\underline{C}] \to W^{-1}_{\underline{I}^{OP}}[\underline{I}^{OP},\underline{C}] \qquad \text{(cf. 1.4.5)},$$

call it $D_{(\underline{C},W)}F$. Finally, from a natural transformation $\Xi:F \to G$ there results a natural transformation

$$(\Xi^{OP})^*:(G^{OP})^* \to (F^{OP})^*$$

that gives rise in turn to a natural transformation

$$D_{(\underline{C},W)}\Xi:D_{(\underline{C},W)}G \to D_{(\underline{C},W)}F$$

characterized by the property that

$$(D_{(\underline{C},W)}\Xi)L_{W_{\underline{J}^{OP}}} = L_{W_{\underline{I}^{OP}}}(\Xi^{OP})^* \qquad \text{(cf. 1.11)}.$$

[Note: Take $\underline{I} = \underline{1}$ — then

$$D_{(\underline{C},W)}\underline{1} = W^{-1}\underline{C}.]$$

3.2.2 LEMMA Let D be a derivator. Suppose that

$$\begin{cases} F:\underline{I} \to \underline{I}' \\ F':\underline{I}' \to \underline{I} \end{cases}$$

are an adjoint pair with arrows of adjunction

$$\begin{cases} \mu:\mathrm{id}_{\underline{I}} \longrightarrow F' \circ F \\ \mu':F \circ F' \longrightarrow \mathrm{id}_{\underline{I}'}. \end{cases}$$

Then

$$\begin{cases} DF:D\underline{I}' \to D\underline{I} \\ DF':D\underline{I} \to D\underline{I}' \end{cases}$$

are an adjoint pair with arrows of adjunction

$$\begin{cases} D\mu' \in \mathrm{Nat}(\mathrm{id}_{D\underline{I}'}, DF' \circ DF) \\ D\mu \in \mathrm{Nat}(DF \circ DF', \mathrm{id}_{D\underline{I}}). \end{cases}$$

PROOF Starting from

$$\begin{cases} (F'\mu') \circ (\mu F') = \mathrm{id}_{F'} \\ (\mu' F) \circ (F\mu) = \mathrm{id}_F, \end{cases}$$

we have

$$\begin{cases} \mathrm{id}_{DF'} = D\mathrm{id}_{F'} = D(\mu F') \circ D(F'\mu') = (DF')D\mu \circ D\mu'(DF') \\ \mathrm{id}_{DF} = D\mathrm{id}_F = D(F\mu) \circ D(\mu' F) = D\mu(DF) \circ (DF)D\mu', \end{cases}$$

which leads at once to the contention.

3.2.3 LEMMA Let D be a derivator. Suppose that

$$\begin{cases} F:\underline{I} \to \underline{I}' \\ F':\underline{I}' \to \underline{I} \end{cases}$$

are an adjoint pair with arrows of adjunction

$$\begin{cases} \mu:\mathrm{id}_{\underline{I}} \to F' \circ F \\ \mu':F \circ F' \to \mathrm{id}_{\underline{I}'}. \end{cases}$$

Then

$$\begin{cases} F \text{ fully faithful} \Rightarrow DF' \text{ fully faithful} \\ F' \text{ fully faithful} \Rightarrow DF \text{ fully faithful.} \end{cases}$$

PROOF E.g.: If F is fully faithful, then μ is a natural isomorphism, thus $D\mu$ is a natural isomorphism and this, in view of 3.2.2, implies that DF' is fully faithful.

3.2.4 DEFINITION A morphism $\underline{\Phi}:D \to D'$ of derivators is a pair (Φ,ϕ), where $\forall$ $\underline{I}$,

$$\Phi_{\underline{I}}:D\underline{I} \to D'\underline{I}$$

is a functor, and $\forall$ $F:\underline{I} \to \underline{J}$,

$$\phi_F:D'F \circ \Phi_{\underline{J}} \to \Phi_{\underline{I}} \circ DF$$

is a natural isomorphism, there being two conditions on $\underline{\Phi}$.

[Note: The square per ϕ_F is

$$\begin{tikzcd} D\underline{J} \arrow[r, "\Phi_{\underline{J}}"] \arrow[d, "DF"'] & D'\underline{J} \arrow[d, "D'F"] \\ D\underline{I} \arrow[r, "\Phi_{\underline{I}}"'] & D'\underline{I} \end{tikzcd}$$

.]

- Given $\underline{I} \xrightarrow{F} \underline{J} \xrightarrow{G} \underline{K}$, we have

$$\begin{cases} \phi_F:D'F \circ \Phi_{\underline{J}} \longrightarrow \Phi_{\underline{I}} \circ DF \\ \phi_G:D'G \circ \Phi_{\underline{K}} \longrightarrow \Phi_{\underline{J}} \circ DG, \end{cases}$$

from which

$$\begin{cases} \phi_F(DG):D'F \circ \Phi_{\underline{J}} \circ DG \longrightarrow \Phi_{\underline{I}} \circ DF \circ DG \\ (D'F)\phi_G:D'F \circ D'G \circ \Phi_{\underline{K}} \longrightarrow D'F \circ \Phi_{\underline{J}} \circ DG. \end{cases}$$

On the other hand,

$$\phi_{G \circ F} = D'F \circ D'G \circ \Phi_{\underline{K}} \rightarrow \Phi_{\underline{I}} \circ DF \circ DG.$$

The assumption then is that

$$\phi_{G \circ F} = \phi_F(DG) \circ (D'F)\phi_G.$$

- Given $\Xi \in \mathrm{Nat}(F,G)$, we have

$$\begin{cases} D\Xi:DG \rightarrow DF \\ D'\Xi:D'G \rightarrow D'F, \end{cases}$$

from which the square

$$\begin{tikzcd} D'G \circ \Phi_{\underline{J}} \arrow[r, "\phi_G"] \arrow[d, "D'\Xi(\Phi_{\underline{J}})"'] & \Phi_{\underline{I}} \circ DG \arrow[d, "(\Phi_{\underline{I}})D\Xi"] \\ D'F \circ \Phi_{\underline{J}} \arrow[r, "\phi_F"'] & \Phi_{\underline{I}} \circ DF \end{tikzcd}$$

and the supposition is that it commutes.

3.2.5 EXAMPLE Let

$$F:(\underline{C}_1,W_1) \rightarrow (\underline{C}_2,W_2)$$

be a morphism of category pairs (cf. 1.4.5) — then F induces a morphism

$$D_{(\underline{C}_1,\underline{W}_1)} \to D_{(\underline{C}_2,\underline{W}_2)}$$

of derivators.

Given morphisms

$$\begin{cases} \underline{\Phi}:D \to D' \\ \underline{\Phi}':D' \to D'' \end{cases}$$

of derivators, it is clear how to define their composition

$$\underline{\Phi}' \circ \underline{\Phi}:D \to D''$$

which again is a morphism of derivators, thus there is a metacategory $\underline{\text{DER}}$ whose objects are the derivators.

If now $D,D' \in \text{Ob } \underline{\text{DER}}$ and if

$$\begin{cases} \underline{\Phi}:D \to D' \\ \underline{\Psi}:D \to D' \end{cases} \in \text{Mor}(D,D'),$$

then a <u>natural transformation</u> $\underline{\Xi}:\underline{\Phi} \to \underline{\Psi}$ is the assignment to each $\underline{I}$ of a natural transformation

$$\Xi_{\underline{I}}:\Phi_{\underline{I}} \to \Psi_{\underline{I}}$$

such that $\forall$ $F:\underline{I} \to \underline{J}$, the diagram

$$\begin{tikzcd}
D'F \circ \Phi_{\underline{J}} \arrow[r, "\phi_F"] \arrow[d, "(D'F)\Xi_{\underline{J}}"'] & \Phi_{\underline{I}} \circ DF \arrow[d, "\Xi_{\underline{I}}(DF)"] \\
D'F \circ \Psi_{\underline{J}} \arrow[r, "\psi_F"'] & \Psi_{\underline{J}} \circ DF
\end{tikzcd}$$

commutes.

3.2.6 LEMMA Let

$$\underline{\Phi},\ \underline{\Psi},\ \underline{\Theta} \in \operatorname{Mor}(D,D').$$

Suppose that

$$\begin{cases} \underline{\Xi}:\underline{\Phi} \to \underline{\Psi} \\ \underline{\Omega}:\underline{\Psi} \to \underline{\Theta} \end{cases}$$

are natural transformations. Define $\underline{\Omega} \circ \underline{\Xi}$ by

$$(\Omega \circ \Xi)_{\underline{I}} = \Omega_{\underline{I}} \circ \Xi_{\underline{I}}.$$

Then $\underline{\Omega} \circ \underline{\Xi}$ is a natural transformation from $\underline{\Phi}$ to $\underline{\Theta}$.

PROOF It is a question of showing that

$$(\Omega_{\underline{I}} \circ \Xi_{\underline{I}})(DF) \circ \phi_F = \theta_F \circ (D'F)(\Omega_{\underline{J}} \circ \Xi_{\underline{J}}).$$

But

$$\begin{aligned}(\Omega_{\underline{I}} \circ \Xi_{\underline{I}})(\ F) \circ \phi_F &= \Omega_{\underline{I}}(DF) \circ \Xi_{\underline{I}}(DF) \circ \phi_F \\ &= \Omega_{\underline{I}}(DF) \circ \psi_F \circ (D'F)\Xi_{\underline{J}} \\ &= \theta_F \circ (D'F)\Omega_{\underline{J}} \circ (D'F)\Xi_{\underline{J}} \\ &= \theta_F \circ (D'F)(\Omega_{\underline{J}} \circ \Xi_{\underline{J}}).\end{aligned}$$

3.2.7 NOTATION Given derivators D,D', let $\underline{\text{HOM}}(D,D')$ stand for the metacategory whose objects are the derivator morphisms $\underline{\Phi}:D \to D'$ and whose morphisms are the natural transformations $\text{Nat}(\underline{\Phi},\underline{\Psi})$ from $\underline{\Phi}$ to $\underline{\Psi}$.

3.2.8 EXAMPLE Let 1 be the constant derivator with value $\underline{1}$ -- then for every derivator D, $\underline{\text{HOM}}(1,D)$ is equivalent to $D\underline{1}$.

3.2.9 DEFINITION Let $\underline{\Phi} \in \text{Mor}(D,D')$ — then $\underline{\Phi}$ is an <u>equivalence</u> if $\forall$ $\underline{I}$,

$$\Phi_{\underline{I}}:D\underline{I} \to D'\underline{I}$$

is an equivalence of metacategories.

3.2.10 LEMMA A morphism $\underline{\Phi}:D \to D'$ is an equivalence iff there exists a morphism $\underline{\Phi}':D' \to D$ such that $\underline{\Phi}' \circ \underline{\Phi}$ is isomorphic to id_D and $\underline{\Phi} \circ \underline{\Phi}'$ is isomorphic to $\text{id}_{D'}$.

3.2.11 EXAMPLE Let $\underline{C}$ be a complete and cocomplete model category, W its class of weak equivalences — then there are morphisms

$$\begin{cases} (\underline{C}_{\text{cof}}, W_{\text{cof}}) \to (\underline{C}, W) \\ \\ (\underline{C}_{\text{fib}}, W_{\text{fib}}) \to (\underline{C}, W) \end{cases}$$

of category pairs, hence induced morphisms

$$\begin{cases} D_{(\underline{C}_{\text{cof}}, W_{\text{cof}})} \to D_{(\underline{C}, W)} \\ \\ D_{(\underline{C}_{\text{fib}}, W_{\text{fib}})} \to D_{(\underline{C}, W)} \end{cases}$$

of derivators that, in fact, are equivalences.

3.2.12 NOTATION In 3.2.1, take for W the identities in $\underline{C}$ and write $D_{\underline{C}}$ in place of $D_{(\underline{C},W)}$, hence $\forall$ $\underline{I} \in \text{Ob}$ <u>CAT</u>,

$$D_{\underline{C}}\underline{I} = [\underline{I}^{\text{OP}}, \underline{C}].$$

3.2.13 EXAMPLE Let $(\underline{C},W)$ be a category pair — then W contains the identities of $\underline{C}$, so there is a morphism

$$D_{\underline{C}} \to D_{(\underline{C},W)}$$

of derivators.

3.2.14 EXAMPLE If $F:\underline{C} \to \underline{C}'$ is a functor and if $\underline{I} \in \text{Ob } \underline{CAT}$, then

$$F_*:[\underline{I}^{OP},\underline{C}] \to [\underline{I}^{OP},\underline{C}']$$

and there is an induced morphism $D_{\underline{C}} \to D_{\underline{C}'}$ of derivators.

3.2.15 LEMMA Suppose that $\underline{C}$ is small -- then for every derivator D, there is a canonical equivalence

$$\underline{HOM}(D_{\underline{C}},D) \to D\underline{C}^{OP}$$

of metacategories.

[Given $\underline{\Phi}:D_{\underline{C}} \to D$, let $\underline{I} = \underline{C}^{OP}$, thus

$$\Phi_{\underline{C}^{OP}}:[\underline{C},\underline{C}] \to D\underline{C}^{OP}$$

and by definition

$$\underline{\Phi} \longrightarrow \Phi_{\underline{C}^{OP}}(\text{id}_{\underline{C}}).]$$

[Note: This is the Yoneda lemma for derivators.]

3.3 *TECHNICALITIES*

3.3.1 DEFINITION Let D be a derivator.

- A functor $K:\underline{I} \to \underline{J}$ admits a <u>right homotopy Kan extension in D</u> if the functor

$$DK:D\underline{J} \to D\underline{I}$$

has a right adjoint

$$DK_{\dagger}:D\underline{I} \to D\underline{J}.$$

• A functor $K:\underline{I} \to \underline{J}$ admits a <u>left homotopy Kan extension in D</u> if the functor

$$DK:D\underline{J} \to D\underline{I}$$

has a left adjoint

$$DK_!:D\underline{I} \to D\underline{J}.$$

3.3.2 EXAMPLE Take $D = D_{\underline{C}}$ (cf. 3.2.12).

• Assume that $\underline{C}$ is complete -- then every $K:\underline{I} \to \underline{J}$ admits a right homotopy Kan extension in $D_{\underline{C}}$.

• Assume that $\underline{C}$ is cocomplete -- then every $K:\underline{I} \to \underline{J}$ admits a left homotopy Kan extension in $D_{\underline{C}}$.

3.3.3 REMARK Let $\underline{C}$ be a model category, W its class of weak equivalences -- then in the context of the derivator $D_{(\underline{C},W)}$ (cf. 3.2.1), one uses the term <u>homotopy limit of K^{OP}</u> rather than right homotopy Kan extension of K and the term <u>homotopy colimit of K^{OP}</u> rather than the term left homotopy Kan extension of K.

[Note: The explanation for the appearance of K^{OP} is to keep matters consistent. Thus suppose that $\underline{C}$ is combinatorial -- then in the notation of 0.26.19 and 0.26.20, we introduced

$$\begin{cases} LK_! \\ RK_+ \end{cases}$$

which were called

$$\begin{cases} \text{the homotopy colimit of } K \\ \text{the homotopy limit of } K \end{cases}$$

respectively. So here

$$\left[\begin{array}{l} D_{(C,W)}K_! = LK^{OP}{}_! \\ \\ D_{(C,W)}K_\dagger = RK^{OP}{}_\dagger. \end{array}\right.$$

See also 2.5.7 and 2.5.8.]

3.3.4 NOTATION Let $\underline{I} \in \text{Ob}\ \underline{\text{CAT}}$ and let $p_{\underline{I}}:\underline{I} \to \underline{1}$ be the canonical arrow.

- Suppose that $p_{\underline{I}}$ admits a right homotopy Kan extension in D -- then

$\forall\ X \in \text{Ob}\ D\underline{I}$, we let

$$\Gamma_\dagger(\underline{I},X) = Dp_{\underline{I}\dagger}X.$$

- Suppose that $p_{\underline{I}}$ admits a left homotopy Kan extension in D -- then

$\forall\ X \in \text{Ob}\ D\underline{I}$, we let

$$\Gamma_!(\underline{I},X) = Dp_{\underline{I}!}X.$$

3.3.5 DEFINITION A <u>2-diagram</u> of categories (or metacategories) is a square

$$\begin{tikzcd} \underline{A}' \arrow[r, "u"] \arrow[d, "F'"'] & \underline{A} \arrow[d, "F"] \\ \underline{B}' \arrow[r, "v"'] & \underline{B} \end{tikzcd}$$

together with a natural transformation from $F \circ u$ to $v \circ F'$ or from $v \circ F'$ to $F \circ u$.

Let D be a derivator -- then a 2-diagram

$$\begin{tikzcd} \underline{I}' \arrow[r, "u"] \arrow[d, "F'"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}' \arrow[r, "v"'] & \underline{J} \end{tikzcd} \qquad (\Xi \in \mathrm{Nat}(F \circ u, v \circ F'))$$

of small categories induces a 2-diagram

$$\begin{tikzcd} D\underline{I}' & D\underline{I} \arrow[l, "Du"'] \\ D\underline{J}' \arrow[u, "DF'"] & D\underline{J} \arrow[l, "Dv"] \arrow[u, "DF"'] \end{tikzcd}$$

of metacategories, where

$$D\Xi : D(v \circ F') \to D(F \circ u).$$

<u>N.B.</u> We have

$$\begin{cases} D(v \circ F') = DF' \circ Dv \\ D(F \circ u) = Du \circ DF. \end{cases}$$

3.3.6 CONSTRUCTION Assume that both F and F' admit a right homotopy Kan extension in D. Starting from the arrow of adjunction $DF \circ DF_{\dagger} \to \mathrm{id}_{D\underline{I}}$, proceed to

$$Du \circ DF \circ DF_{\dagger} \to Du$$

or still, using

$$D\Xi : DF' \circ Dv \to Du \circ DF,$$

to

$$DF' \circ Dv \circ DF_{\dagger} \to Du$$

or still, by adjunction, to

$$\mathrm{III}:Dv \circ DF_{\dagger} \to DF'_{\dagger} \circ Du,$$

leading thereby to another 2-diagram

$$\begin{tikzcd} D\underline{I}' & D\underline{I} \arrow[l, "Du"'] \arrow[d, "DF_{\dagger}"] \\ D\underline{I}' \arrow[d, "DF'_{\dagger}"'] & \\ D\underline{J}' & D\underline{J} \arrow[l, "Dv"] \end{tikzcd}$$

of metacategories.

[Note: The natural transformation III is called the base change morphism induced by Ξ.]

3.3.7 EXAMPLE Let $F:\underline{I} \to \underline{J}$ be a functor. Given $j \in \mathrm{Ob}\ \underline{J}$, write $\underline{I}/j$ for the comma category $|F,K_j|$, the objects of which are the pairs (i,g), where $i \in \mathrm{Ob}\ \underline{I}$, $g \in \mathrm{Mor}\ \underline{J}$, and $g:Fi \to j$. Consider the square

$$\begin{tikzcd} \underline{I}/j \arrow[r, "pro_j"] \arrow[d, "p_{\underline{I}/j}"'] & \underline{I} \arrow[d, "F"] \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} \end{tikzcd} \qquad (pro_j(i,g) = i).$$

Then there is a natural transformation

$$\Xi:F \circ pro_j \to K_j \circ p_{\underline{I}/j},$$

viz.

$$\Xi_{(i,g)} = g.$$

Assume now that F admits a right homotopy Kan extension in D and $\forall\ j \in \text{Ob}\ \underline{J}$, $p_{\underline{I}/j}$ admits a right homotopy Kan extension in D. Accordingly, on the basis of 3.3.6, there is a natural transformation

$$\text{Ш}:\mathrm{DK}_j \circ \mathrm{DF}_{\dagger} \to \mathrm{Dp}_{\underline{I}/j\dagger} \circ \mathrm{Dpro}_j.$$

[Note: From the definitions,

$$\mathrm{Dpro}_j:\mathrm{D}\underline{I} \to \mathrm{D}\underline{I}/j,$$

so $\forall\ X \in \text{Ob}\ \mathrm{D}\underline{I}$, $\mathrm{Dpro}_j X \in \text{Ob}\ \mathrm{D}\underline{I}/j$, call it X/j — then

$$\mathrm{Dp}_{\underline{I}/j\dagger} X/j = \Gamma_{\dagger}(\underline{I}/j, X/j) \quad \text{(cf. 3.3.4.]}$$

Let D be a derivator — then a 2-diagram

$$\begin{tikzcd} \underline{I}' \arrow[r, "u"] \arrow[d, "F'"'] & \underline{I} \arrow[d, "F"] \\ \underline{J}' \arrow[r, "v"'] & \underline{J} \end{tikzcd} \qquad (\Xi \in \mathrm{Nat}(v \circ F', F \circ u))$$

of small categories induces a 2-diagram

$$\begin{tikzcd} \mathrm{D}\underline{I}' & \mathrm{D}\underline{I} \arrow[l, "\mathrm{D}u"'] \\ \mathrm{D}\underline{J}' \arrow[u, "\mathrm{DF}'"] & \mathrm{D}\underline{J} \arrow[l, "\mathrm{D}v"] \arrow[u, "\mathrm{DF}"'] \end{tikzcd}$$

of metacategories, where

$$\mathrm{D}\Xi:\mathrm{D}(F \circ u) \to \mathrm{D}(v \circ F').$$

<u>N.B.</u> We have

$$\begin{cases} D(F \circ u) = Du \circ DF \\ D(v \circ F') = DF' \circ Dv. \end{cases}$$

3.3.8 CONSTRUCTION Assume that both F and F' admit a left homotopy Kan extension in D. Starting from the arrow of adjunction $\mathrm{id}_{D\underline{I}} \to DF \circ DF_!$, proceed to

$$Du \to Du \circ DF \circ DF_!$$

or still, using

$$D\Xi : Du \circ DF \to DF' \circ Dv,$$

to

$$Du \to DF' \circ Dv \circ DF_!$$

or still, by adjunction, to

$$\text{Ш} : DF'_! \circ Du \to Dv \circ DF_!$$

leading thereby to another 2-diagram

$$\begin{tikzcd} D\underline{I}' & D\underline{I} \arrow[l, "Du"'] \arrow[d, "DF_!"] \\ D\underline{J}' \arrow[from=u, "DF'_!"'] & D\underline{J} \arrow[l, "Dv"] \end{tikzcd}$$

of metacategories.

[Note: The natural transformation Ш is called the <u>base change morphism</u> induced by Ξ.]

3.3.9 EXAMPLE Let $F:\underline{I} \to \underline{J}$ be a functor. Given $j \in \mathrm{Ob}\,\underline{J}$, write $j\backslash\underline{I}$ for the comma category $|K_j,F|$, the objects of which are the pairs (g,i), where $g \in \text{Mor } \underline{J}$, $i \in \text{Ob } \underline{I}$, and $g:j \to Fi$. Consider the square

$$\begin{tikzcd} j\backslash\underline{I} \arrow[r, "{}_j\text{pro}"] \arrow[d, "p_{j\backslash\underline{I}}"'] & \underline{I} \arrow[d, "F"] \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} \end{tikzcd} \qquad ({}_j\text{pro}(g,i) = i).$$

Then there is a natural transformation

$$K_j \circ p_{j\backslash\underline{I}} \to F \circ {}_j\text{pro},$$

viz.

$$\Xi_{(g,i)} = g.$$

Assume now that F admits a left homotopy Kan extension in D and $\forall\ j \in \text{Ob } \underline{J}$, $p_{j\backslash\underline{I}}$ admits a left homotopy Kan extension in D. Accordingly, on the basis of 3.3.8, there is a natural transformation

$$Ш:Dp_{j\backslash\underline{I}!} \circ D_j\text{pro} \to DK_j \circ DF_!.$$

[Note: From the definitions,

$$D_j\text{pro}:D\underline{I} \to Dj\backslash\underline{I},$$

so $\forall\ X \in \text{Ob } D\underline{I}$, $D_j\text{pro } X \in \text{Ob } Dj\backslash\underline{I}$, call it $j\backslash X$ — then

$$Dp_{j\backslash\underline{I}!}\ j\backslash X = \Gamma_!(j\backslash\underline{I}, j\backslash X) \qquad (\text{cf. } 3.3.4).]$$

3.3.10 NOTATION Suppose that D is a derivator — then for all $\underline{I},\underline{J} \in \text{Ob } \underline{\underline{CAT}}$, there is a canonical functor

$$d_{\underline{I},\underline{J}}:D(\underline{I} \times \underline{J}) \to [\underline{I}^{OP}, D\underline{J}].$$

In fact:

1. There is a functor

$$[\underline{J},\underline{I} \times \underline{J}]^{OP} \to [D(\underline{I} \times \underline{J}),D\underline{J}].$$

2. There is a functor

$$[\underline{J},\underline{I} \times \underline{J}]^{OP} \times D(\underline{I} \times \underline{J}) \to D\underline{J}.$$

3. There is a functor

$$D(\underline{I} \times \underline{J}) \to [[\underline{J},\underline{I} \times \underline{J}]^{OP},D\underline{J}].$$

4. There is a functor

$$\underline{I} \to [\underline{J},\underline{I} \times \underline{J}]$$

or still, a functor

$$\underline{I}^{OP} \to [\underline{J},\underline{I} \times \underline{J}]^{OP}.$$

So, in conclusion, there is a functor

$$d_{\underline{I},\underline{J}}:D(\underline{I} \times \underline{J}) \to [\underline{I}^{OP},D\underline{J}].$$

Let $d_{\underline{I}} = d_{\underline{I},\underline{1}}$, thus

$$d_{\underline{I}}:D\underline{I} \to [\underline{I}^{OP},D\underline{1}].$$

[Note: If $D = D_{(\underline{C},W)}$, where $(\underline{C},W)$ is a category pair, then $d_{\underline{I}}$ is what was labeled $dgm_{\underline{I}}$ in 2.6.]

3.3.11 LEMMA Suppose that $F:\underline{I} \to \underline{J}$ — then the diagram

$$\begin{array}{ccc} D\underline{J} & \xrightarrow{d_{\underline{J}}} & [\underline{J}^{OP}, D\underline{1}] \\ {\scriptstyle DF}\downarrow & & \downarrow{\scriptstyle (F^{OP})^*} \\ D\underline{I} & \xrightarrow[d_{\underline{I}}]{} & [\underline{I}^{OP}, D\underline{1}] \end{array}$$

commutes.

3.4 *AXIOMS*

What follows is a list of conditions that a derivator D might satisfy but which are not part of the setup per se.

(DER - 1) For any finite set $\underline{I}_1, \ldots, \underline{I}_n$ of small categories, the canonical functor

$$D\left(\coprod_{k=1}^{n} \underline{I}_k\right) \to \prod_{k=1}^{n} D(\underline{I}_k)$$

induced by the inclusions

$$\underline{I}_\ell \to \coprod_{k=1}^{n} \underline{I}_k \quad (1 \le \ell \le n)$$

is an equivalence and $D\underline{0}$ is equivalent to $\underline{1}$.

(DER - 2) For any small category $\underline{I}$, the functors

$$DK_i : D\underline{I} \to D\underline{1} \quad (i \in \text{Ob } \underline{I})$$

constitute a conservative family, i.e., if $X, Y \in \text{Ob } D\underline{I}$ and if $f: X \to Y$ is a morphism such that $\forall\ i \in \text{Ob } \underline{I}$, $DK_i f$ is an isomorphism in $D\underline{1}$, then f is an isomorphism in $D\underline{I}$.

(RDER - 3) Every $F \in \text{Mor } \underline{CAT}$ admits a right homotopy Kan extension in D.

(LDER - 3) Every $F \in \text{Mor}\ \underline{\text{CAT}}$ admits a left homotopy Kan extension in D.

(RDER - 4) For any $F:\underline{I} \to \underline{J}$ and for any $j \in \text{Ob}\ \underline{J}$,

$$\text{III}:DK_j \circ DF_+ \to Dp_{\underline{I}/j+} \circ D\text{pro}_j$$

is a natural isomorphism.

(LDER - 4) For any $F:\underline{I} \to \underline{J}$ and for any $j \in \text{Ob}\ \underline{J}$,

$$\text{III}:Dp_{j\backslash\underline{I}!} \circ D_j\text{pro} \to DK_j \circ DF_!$$

is a natural isomorphism.

(DER - 5) For any finite, free category $\underline{I}$ and for any small category $\underline{J}$, the functor

$$d_{\underline{I},\underline{J}}:D(\underline{I} \times \underline{J}) \to [\underline{I}^{OP}, D\underline{J}]$$

is full and has a representative image.

N.B. Tacitly, RDER - 4 presupposes RDER - 3 and LDER - 4 presupposes LDER - 3.

3.4.1 DEFINITION Let D be a derivator.

- D is said to be a right homotopy theory if DER - 1, DER - 2, RDER - 3, and RDER - 4 are satisfied.

- D is said to be a left homotopy theory if DER - 1, DER - 2, LDER - 3, and LDER - 4 are satisfied.

N.B. D is said to be a homotopy theory if D is both a right and left homotopy theory.

3.4.2 EXAMPLE Let $\underline{C}$ be a category and take $D = D_{\underline{C}}$ (cf. 3.2.12).

- Assume that $\underline{C}$ is complete -- then $D_{\underline{C}}$ is a right homotopy theory.

• Assume that $\underline{C}$ is cocomplete -- then $D_{\underline{C}}$ is a left homotopy theory.

3.4.3 LEMMA Suppose that DER - 1 and RDER - 3 are in force -- then $\forall$ $\underline{I}$, $D\underline{I}$ has finite products.

PROOF It suffices to prove that $\underline{C}$ has binary products and a final object.

———— Recall that $D\underline{I}$ has binary products iff the diagonal functor $\Delta_{D\underline{I}}:D\underline{I} \to D\underline{I} \times D\underline{I}$ has a right adjoint. Let $\nabla_{\underline{I}}:\underline{I} \coprod \underline{I} \to \underline{I}$ be the folding map -- then there is a commutative diagram

$$\begin{tikzcd}
D\underline{I} \arrow[r, "D\nabla_{\underline{I}}"] \arrow[d, "\Delta_{D\underline{I}}"'] & D(\underline{I} \coprod \underline{I}) \arrow[d] \\
D\underline{I} \times D\underline{I} \arrow[r, equal] & D\underline{I} \times D\underline{I}.
\end{tikzcd}$$

Since $D\nabla_{\underline{I}}$ has a right adjoint and since the vertical arrow on the right is an equivalence, it follows that $\Delta_{D\underline{I}}$ has a right adjoint.

———— Recall that $D\underline{I}$ has a final object iff the functor $p_{D\underline{I}}:D\underline{I} \to \underline{1}$ has a right adjoint. Let $i_{\underline{I}}:\underline{0} \to \underline{I}$ be the insertion -- then there is a commutative diagram

$$\begin{tikzcd}
D\underline{I} \arrow[r, "Di_{\underline{I}}"] \arrow[d, "p_{D\underline{I}}"'] & D\underline{0} \arrow[d] \\
\underline{1} \arrow[r, equal] & \underline{1}.
\end{tikzcd}$$

Since $D i_{\underline{I}}$ has a right adjoint and since the vertical arrow on the right is an equivalence, it follows that $p_{D\underline{I}}$ has a right adjoint.

3.4.4 LEMMA Suppose that DER - 1 and LDER - 3 are in force -- then $\forall$ $\underline{I}$, $D\underline{I}$ has finite coproducts.

Let D be a derivator -- then for any small category $\underline{I}$ and any $i \in \operatorname{Ob} \underline{I}$, there is a commutative diagram

$$\begin{tikzcd} D\underline{I} \arrow[r, "d_{\underline{I}}"] \arrow[d, "DK_i"'] & {[\underline{I}^{OP}, D\underline{1}]} \arrow[d, "(K_i^{OP})^*"] \\ D\underline{1} \arrow[r, equal] & D\underline{1} \end{tikzcd} \qquad \text{(cf. 3.3.11).}$$

3.4.5 LEMMA The derivator D satisfies DER - 2 iff $\forall$ $\underline{I} \in \operatorname{Ob} \underline{CAT}$, the functor $d_{\underline{I}}$ is conservative.

PROOF The $(K_i^{OP})^*$ constitute a conservative family.

[Note: It is clear that the derivator $D_{\underline{C}}$ attached to a category $\underline{C}$ satisfies DER - 2 (levelwise isomorphisms are isomorphisms).]

3.5 D-*EQUIVALENCES*

Let D be a derivator. Suppose that $\underline{I},\underline{J}$ are small categories and $F:\underline{I} \to \underline{J}$ is a functor -- then upon application of D, the commutative diagram

$$\begin{tikzcd} \underline{I} \arrow[r, "F"] \arrow[d, "p_{\underline{I}}"'] & \underline{J} \arrow[d, "p_{\underline{J}}"] \\ \underline{1} \arrow[r, equal] & \underline{1} \end{tikzcd}$$

leads to a commutative diagram

$$\begin{tikzcd} D\underline{I} & D\underline{J} \arrow[l, "DF"'] \\ D\underline{1} \arrow[u, "Dp_{\underline{I}}"] \arrow[r, equal] & D\underline{1} \arrow[u, "Dp_{\underline{J}}"'] \end{tikzcd} .$$

So, for any pair $X,Y \in \mathrm{Ob}\, D\underline{1}$, there is an arrow

$$\phi_{X,Y}:\mathrm{Mor}(Dp_{\underline{J}}X, Dp_{\underline{J}}Y) \to \mathrm{Mor}(Dp_{\underline{I}}X, Dp_{\underline{I}}Y),$$

namely

$$\phi_{X,Y}f = DFf,$$

i.e.,

$$Dp_{\underline{J}}X \xrightarrow{f} Dp_{\underline{J}}Y$$

is sent by $\phi_{X,Y}$ to

$$Dp_{\underline{I}}X = DF \circ Dp_{\underline{J}}X \xrightarrow{DFf} DF \circ Dp_{\underline{J}}Y = Dp_{\underline{I}}Y.$$

3.5.1 DEFINITION A functor $F:\underline{I} \to \underline{J}$ is a <u>D-equivalence</u> if $\forall\, X,Y \in \mathrm{Ob}\, D\underline{1}$, the arrow

$$\phi_{X,Y}:\mathrm{Mor}(Dp_{\underline{J}}X, Dp_{\underline{J}}Y) \to \mathrm{Mor}(Dp_{\underline{I}}X, Dp_{\underline{I}}Y)$$

is bijective.

3.5.2 NOTATION Write $\mathcal{W}_D$ for the class of D-equivalences in Mor $\underline{\mathrm{CAT}}$.

<u>N.B.</u> It is clear that $(\underline{\mathrm{CAT}}, \mathcal{W}_D)$ is a category pair.

3.5.3 LEMMA $\mathcal{W}_D$ is saturated (that is, $\mathcal{W}_D = \overline{\mathcal{W}_D}$ (cf. 1.1.9)).

PROOF Given $X,Y \in \text{Ob } \underline{D1}$, define a functor

$$\Phi_{X,Y}:\underline{CAT} \to \underline{SET}^{OP}$$

by the specification

$$\underline{I} \to \text{Mor}(Dp_{\underline{I}}X, Dp_{\underline{I}}Y) \text{ and } F \to \phi_{X,Y}.$$

Accordingly, from the definitions, if F is a D-equivalence, then $\Phi_{X,Y}F$ is a bijection, so there is a commutative diagram

$$\begin{tikzcd} \underline{CAT} \arrow[r, equal] \arrow[d, "L_{\mathcal{W}_D}"'] & \underline{CAT} \arrow[d, "\Phi_{X,Y}"] \\ \mathcal{W}_D^{-1}\underline{CAT} \arrow[r, "\bar{\Phi}_{X,Y}"'] & \underline{SET}^{OP}. \end{tikzcd}$$

Suppose now that $L_{\mathcal{W}_D}F_0$ is an isomorphism ($F_0:\underline{I}_0 \to \underline{J}_0$) -- then $\bar{\Phi}_{X,Y}L_{\mathcal{W}_D}F_0$ is an isomorphism or still, $\Phi_{X,Y}F_0$ is a bijection. Since this is true of all $X,Y \in \text{Ob } \underline{D1}$, it follows that F_0 is a D-equivalence: $F_0 \in \mathcal{W}_D$.

N.B. It is a corollary that $\mathcal{W}_D$ is weakly saturated (cf. 2.3.15).

3.5.4 DEFINITION An object $\underline{I} \in \text{Ob } \underline{CAT}$ is D-aspherical if $p_{\underline{I}}:\underline{I} \to \underline{1}$ is a D-equivalence.

3.5.5 LEMMA $\underline{I}$ is D-aspherical iff the functor $Dp_{\underline{I}}:D\underline{1} \to D\underline{I}$ is fully faithful.

PROOF Given $X,Y \in \text{Ob } D\underline{1}$, to say that the arrow

$$\text{Mor}(X,Y) \to \text{Mor}(Dp_{\underline{I}}X, Dp_{\underline{I}}Y)$$

is bijective amounts to saying that the functor $Dp_{\underline{I}}:D\underline{1} \to D\underline{I}$ is fully faithful.

3.5.6 LEMMA Suppose that $\underline{I}$ has a final object — then $\underline{I}$ is D-aspherical.

PROOF If $\underline{I}$ has a final object, then $p_{\underline{I}}$ has a right adjoint which is necessarily fully faithful. Therefore $Dp_{\underline{I}}$ is fully faithful (cf. 3.2.3), so 3.5.5 is applicable.

3.5.7 DEFINITION A functor $F:\underline{I} \to \underline{J}$ is <u>D-aspherical</u> if $\forall$ $j \in \text{Ob } \underline{J}$, the functor

$$F/j:\underline{I}/j \to \underline{J}/j$$

is a D-equivalence.

3.5.8 LEMMA The functor $F:\underline{I} \to \underline{J}$ is D-aspherical iff $\forall$ $j \in \text{Ob } \underline{J}$, the category $\underline{I}/j$ is D-aspherical.

PROOF Since $\underline{J}/j$ has a final object, it is D-aspherical (cf. 3.5.6), thus the arrow $\underline{J}/j \to \underline{1}$ is a D-equivalence. This said, consider the commutative diagram

$$\begin{tikzcd}
\underline{I}/j \arrow[r, "F/j"] \arrow[d, "p_{\underline{I}/j}"'] & \underline{J}/j \arrow[d, "p_{\underline{J}/j}"] \\
\underline{1} \arrow[r, equal] & \underline{1}
\end{tikzcd}.$$

3.5.9 LEMMA Suppose that the functor $F:\underline{I} \to \underline{J}$ admits a right adjoint $G:\underline{J} \to \underline{I}$ — then F is D-aspherical.

PROOF $\forall$ $i \in \text{Ob } \underline{I}$ and $\forall$ $j \in \text{Ob } \underline{J}$, we have

$$\text{Mor}(Fi,j) \approx \text{Mor}(i,Gj).$$

Therefore the category $\underline{I}/j$ is isomorphic to the category $\underline{I}/Gj$. But $\underline{I}/Gj$ has a final object, thus $\underline{I}/Gj$ is D-aspherical (cf. 3.5.6), hence the same is true of $\underline{I}/j$ and one may then quote 3.5.8.

3.5.10 EXAMPLE An equivalence of small categories is D-aspherical.

Suppose that RDER - 3 is in force. Let $F:\underline{I} \to \underline{J}$ be a functor — then the commutative diagram

$$\begin{tikzcd} \underline{I} \arrow[r, "p_{\underline{I}}"] \arrow[d, "F"'] & \underline{1} \arrow[d, equal] \\ \underline{J} \arrow[r, "p_{\underline{J}}"'] & \underline{1} \end{tikzcd}$$

generates an arrow

$$Dp_{\underline{J}} \to DF_{\dagger} \circ Dp_{\underline{I}} \quad \text{(cf. 3.3.6)}$$

or still, upon postcomposing with $Dp_{\underline{J}\dagger}$, an arrow

$$\begin{aligned} Dp_{\underline{J}\dagger} \circ Dp_{\underline{J}} \to Dp_{\underline{J}\dagger} \circ DF_{\dagger} \circ Dp_{\underline{I}} \\ = D(p_{\underline{J}} \circ F)_{\dagger} \circ Dp_{\underline{I}} \\ = Dp_{\underline{I}\dagger} \circ Dp_{\underline{I}}. \end{aligned}$$

3.5.11 LEMMA Under RDER - 3, a functor $F:\underline{I} \to \underline{J}$ is a D-equivalence iff the arrow

$$Dp_{\underline{J}\dagger} \circ Dp_{\underline{J}} \to Dp_{\underline{I}\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism (in $[D\underline{1},D\underline{1}]$).

PROOF If $F:\underline{I} \to \underline{J}$ is a D-equivalence, then $\forall$ $Y,X \in \mathrm{Ob}\, D\underline{1}$, the arrow

$$\mathrm{Mor}(Dp_{\underline{J}}Y, Dp_{\underline{J}}X) \to \mathrm{Mor}(Dp_{\underline{I}}Y, Dp_{\underline{I}}X)$$

is bijective or still, by adjunction, the arrow

$$\mathrm{Mor}(Y, Dp_{\underline{J}!} \circ Dp_{\underline{J}}X) \to \mathrm{Mor}(Y, Dp_{\underline{I}!} \circ Dp_{\underline{I}}X)$$

is bijective, which implies that the arrow

$$Dp_{\underline{J}!} \circ Dp_{\underline{J}}X \to Dp_{\underline{I}!} \circ Dp_{\underline{I}}X$$

is an isomorphism. Run the argument backwards for the converse.

Henceforth it will be assumed that D satisfies DER - 2, RDER - 3, and RDER - 4.

3.5.12 LEMMA Let $F:\underline{I} \to \underline{J}$ be a functor — then the arrow

$$Dp_{\underline{J}} \to DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism (in $[D\underline{1}, D\underline{J}]$) iff $\forall$ $j \in \mathrm{Ob}\, \underline{J}$, the arrow

$$DK_j \circ Dp_{\underline{J}} \to DK_j \circ DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism (in $[D\underline{1}, D\underline{1}]$) (cf. DER - 2).

[Note: The composition $\underline{1} \xrightarrow{K_j} \underline{J} \xrightarrow{p_{\underline{J}}} \underline{1}$ is $\mathrm{id}_{\underline{1}}$, so $D(p_{\underline{J}} \circ K_j) = DK_j \circ Dp_{\underline{J}}$ is $\mathrm{id}_{D\underline{1}}$.]

3.5.13 LEMMA Let $F:\underline{I} \to \underline{J}$ be a functor. Assume: The arrow

$$Dp_{\underline{J}} \to DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism — then F is D-aspherical.

PROOF Given $j \in \text{Ob } \underline{J}$, consider the diagram

$$\begin{tikzcd} \underline{I}/j \arrow[r, "\text{pro}_j"] \arrow[d, "p_{\underline{I}/j}"'] & \underline{I} \arrow[r, "p_{\underline{I}}"] \arrow[d, "F"] & \underline{1} \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} & \end{tikzcd} \qquad \text{(cf. 3.3.7).}$$

Then

$$p_{\underline{I}} \circ \text{pro}_j = p_{\underline{I}/j} \Rightarrow D\text{pro}_j \circ Dp_{\underline{I}} = Dp_{\underline{I}/j}.$$

And, thanks to RDER - 4, there is an isomorphism

$$DK_j \circ DF_{\dagger} \to Dp_{I/j\dagger} \circ D\text{pro}_j,$$

or still, an isomorphism

$$DK_j \circ DF_{\dagger} \circ Dp_{\underline{I}} \to Dp_{\underline{I}/j\dagger} \circ D\text{pro}_j \circ Dp_{\underline{I}}$$

$$= Dp_{\underline{I}/j\dagger} \circ Dp_{\underline{I}/j}$$

or still, an isomorphism

$$\text{id}_{D\underline{1}} \to Dp_{\underline{I}/j\dagger} \circ Dp_{\underline{I}/j}.$$

But this means that $Dp_{\underline{I}/j}$ is fully faithful (the last arrow being an arrow of adjunction), hence $\underline{I}/j$ is D-aspherical (cf. 3.5.5). Since this is the case of every $j \in \text{Ob } \underline{J}$, it follows that F is D-aspherical (cf. 3.5.8).

3.5.14 LEMMA Let $F:\underline{I} \to \underline{J}$ be a functor. Assume: F is D-aspherical — then the arrow

$$Dp_{\underline{J}} \to DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism.

PROOF Owing to 3.5.8, $\forall\ j \in \mathrm{Ob}\ \underline{J}$, $\underline{I}/j$ is D-aspherical, thus the functor $Dp_{\underline{I}/j}$ is fully faithful (cf. 3.5.5). Using the notation of 3.5.13, form the commutative diagram

$$\begin{tikzcd} \mathrm{id}_{D\underline{1}} \arrow[r] \arrow[d, "\approx"'] & DK_j \circ DF_{\dagger} \circ Dp_{\underline{I}} \arrow[d, "\approx"] \\ Dp_{\underline{I}/j\dagger} \circ Dp_{\underline{I}/j} \arrow[r, equal] & Dp_{\underline{I}/j\dagger} \circ Dp_{\underline{I}/j} \end{tikzcd}$$

to see that the arrow

$$\mathrm{id}_{D\underline{1}} \longrightarrow DK_j \circ DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism. But $j \in \mathrm{Ob}\ \underline{J}$ is arbitrary, thus the arrow

$$Dp_{\underline{J}} \longrightarrow DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism (cf. 3.5.12).

3.5.15 LEMMA If $F:\underline{I} \to \underline{J}$ is D-aspherical, then F is a D-equivalence.

PROOF The arrow

$$Dp_{\underline{J}} \longrightarrow DF_{\dagger} \circ Dp_{\underline{I}}$$

is an isomorphism (cf. 3.5.14). Therefore the arrow

$$\begin{aligned} Dp_{\underline{J}\dagger} \circ Dp_{\underline{J}} \to Dp_{\underline{J}\dagger} &\circ DF_{\dagger} \circ Dp_{\underline{I}} \\ &= D(p_{\underline{J}} \circ F)_{\dagger} \circ Dp_{\underline{I}} \\ &= Dp_{\underline{I}\dagger} \circ Dp_{\underline{I}} \end{aligned}$$

is an isomorphism, so F is a D-equivalence (cf. 3.5.11).

3.5.16 REMARK Consider a commutative diagram

$$\begin{tikzcd} \underline{I} \arrow[r, "F"] \arrow[d, "u"'] & \underline{J} \arrow[d, "v"] \\ \underline{K} \arrow[r, equal] & \underline{K} \end{tikzcd}$$

of small categories. Assume: $\forall\ k \in \text{Ob}\ \underline{K}$, the arrow $\underline{I}/k \rightarrow \underline{J}/k$ is a D-equivalence -- then F is a D-equivalence.

[This is the relative version of 3.5.15 and its proof runs along similar lines.]

N.B. The developments leading to 3.5.15 and 3.5.16 were predicated on the supposition that D satisfies DER - 2, RDER - 3, and RDER - 4. The same conclusions obtain if instead D satisfies DER - 2, LDER - 3, and LDER - 4.

3.5.17 THEOREM Suppose that D is a right (left) homotopy theory -- then $\mathcal{W}_D$ is a fundamental localizer.

PROOF One has only to cite 3.5.3, 3.5.6, and 3.5.16.

3.5.18 REMARK Consequently, if D is a right (left) homotopy theory, then $\mathcal{W}_\infty \subset \mathcal{W}_D$ (cf. C.7.1).

3.5.19 LEMMA Suppose that D is a homotopy theory. Let $F:\underline{I} \rightarrow \underline{J}$ be a functor, $F^{OP}:\underline{I}^{OP} \rightarrow \underline{J}^{OP}$ its opposite -- then F is a D-equivalence iff F^{OP} is a D-equivalence (cf. C.2.9).

3.5.20 LEMMA Suppose that D is a homotopy theory. Let $F:\underline{I} \rightarrow \underline{J}$ be a functor, $F^{OP}:\underline{I}^{OP} \rightarrow \underline{J}^{OP}$ its opposite -- then F is a D-equivalence iff F^{OP} is a D^{OP}-equivalence.

3.5.21 SCHOLIUM We have

$$W_{\mathcal{D}} = W_{\mathcal{D}^{\mathrm{OP}}}$$

if $\mathcal{D}$ is a homotopy theory.

3.6 *PRINCIPAL EXAMPLES*

Recall that if $(\underline{C},W)$ is a category pair, then $\mathcal{D}_{(\underline{C},W)}$ is the derivator that sends

$$\underline{I} \in \mathrm{Ob}\ \underline{\mathrm{CAT}} \text{ to } W_{\underline{I}^{\mathrm{OP}}}^{-1}\ [\underline{I}^{\mathrm{OP}},\underline{C}] \quad (\text{cf. } 3.2.1).$$

3.6.1 THEOREM Let $\underline{C}$ be a complete model category, W its class of weak equivalences -- then $\mathcal{D}_{(\underline{C},W)}$ is a right homotopy theory.

3.6.2 THEOREM Let $\underline{C}$ be a cocomplete model category, W its class of weak equivalences -- then $\mathcal{D}_{(\underline{C},W)}$ is a left homotopy theory.

3.6.3 THEOREM Let $\underline{C}$ be a complete and cocomplete model category, W its class of weak equivalences -- then $\mathcal{D}_{(\underline{C},W)}$ is a homotopy theory.

3.6.4 EXAMPLE Using the notation of 0.24.3, ner induces an equivalence

$$\underline{\mathrm{ner}}:\mathcal{D}_{(\underline{\mathrm{CAT}},W_\infty)} \to \mathcal{D}_{(\underline{\mathrm{SISET}},W_\infty)}$$

of homotopy theories.

[Note: It is an interesting point of detail that W_∞ coincides with the class of $\mathcal{D}_{(\underline{\mathrm{CAT}},W_\infty)}$-equivalences (cf. B.8.14).]

Let $\underline{C},\underline{C}'$ be complete and cocomplete model categories. Suppose that

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

are a model pair -- then the functors

$$\begin{cases} LF:\underline{HC} \to \underline{HC'} \\ RF':\underline{HC'} \to \underline{HC} \end{cases}$$

exist and are an adjoint pair.

In general, there are arrows

$$\begin{cases} [\underline{I}^{OP},\underline{C}] \xrightarrow{F_*} [\underline{I}^{OP},\underline{C}'] \\ [\underline{I}^{OP},\underline{C}'] \xrightarrow{F'_*} [\underline{I}^{OP},\underline{C}] \end{cases}$$

and these functor categories are complete and cocomplete but there is no claim that they are model categories with weak equivalences

$$\begin{cases} \mathcal{W}_{\underline{I}^{OP}} \\ \mathcal{W}'_{\underline{I}^{OP}} \end{cases}$$

[Note: Recall, however, that they are at least weak model categories (cf. 2.7.5 and 2.7.6).]

3.6.5 THEOREM There exist

$$\begin{cases} \underline{F} \in \mathrm{Mor}(D_{(\underline{C},\mathcal{W})}, D_{(\underline{C}',\mathcal{W}')}) \\ \underline{F}' \in \mathrm{Mor}(D_{(\underline{C}',\mathcal{W}')}, D_{(\underline{C},\mathcal{W})}) \end{cases}$$

such that $\forall\ \underline{I}$,

$$F_{\underline{I}}:D_{(\underline{C},W)}\underline{I} \rightarrow D_{(\underline{C}',W')}\underline{I}$$

is the left derived functor of F_* and

$$F'_{\underline{I}}:D_{(\underline{C}',W')}\underline{I} \rightarrow D_{(\underline{C},W)}\underline{I}$$

is the right derived functor of F'_*. Moreover, $(F_{\underline{I}},F'_{\underline{I}})$ is an adjoint pair.

<u>N.B.</u> These results are due to Cisinski[†].

The assumption that $\underline{C}$ is a model category (complete, cocomplete, or both) can be substantially weakened.

3.6.6 THEOREM Let $\underline{C}$ be a homotopically complete fibration category, W its class of weak equivalences -- then $D_{(\underline{C},W)}$ is a right homotopy theory.

3.6.7 THEOREM Let $\underline{C}$ be a homotopically cocomplete cofibration category, W its class of weak equivalences -- then $D_{(\underline{C},W)}$ is a left homotopy theory.

3.6.8 THEOREM Let $\underline{C}$ be a weak model category, W its class of weak equivalences -- then $D_{(\underline{C},W)}$ is a homotopy theory.

<u>N.B.</u> These results are due to Radulescu-Banu[††].

[†] *Ann. Math. Blaise Pascal* <u>10</u> (2003), 195-244.

[††] arXiv:math/0610009

3.6.9 REMARK All the derivators $D_{(\underline{C},W)}$ arising above also verify DER - 5.

Turning to the proofs, we obviously have

$$\left[\begin{array}{l} 3.6.6 \Rightarrow 3.6.1 \\ 3.6.7 \Rightarrow 3.6.2 \\ 3.6.8 \Rightarrow 3.6.3 \end{array}\right.$$

and, of course,

$$\left[\begin{array}{l} 3.6.1 + 3.6.2 \Rightarrow 3.6.3 \\ 3.6.6 + 3.6.7 \Rightarrow 3.6.8. \end{array}\right.$$

To illustrate the main ideas, we shall consider 3.6.1, the discussion per 3.6.6 being similar but more complicated.

3.6.10 NOTATION Given a small category $\underline{I}$, let $\underline{\Delta}_M/\underline{I}$ be the category whose objects are the pairs (m,u), where $m \geq 0$ is an integer and $u:[m] \to \underline{I}$ is a functor, a morphism $(m,u) \to (n,v)$ being a morphism $f:[m] \to [n]$ of $\underline{\Delta}_M$ such that the diagram

$$\begin{tikzcd} {[m]} \arrow[r, "f"] \arrow[d, "u"'] & {[n]} \arrow[d, "v"] \\ \underline{I} \arrow[r, equal] & \underline{I} \end{tikzcd}$$

commutes.

3.6.11 LEMMA The category $\underline{\Delta}_M/\underline{I}$ is direct.

[Define deg:Ob $\underline{\Delta}_M/\underline{I} \to \underline{Z}_{\geq 0}$ by $\deg(m,u) = m$.]

Write

$$\tau_{\underline{I}}:\underline{\Delta}_M/\underline{I} \to \underline{I}$$

for the functor that sends (m,u) to u(m).

3.6.12 LEMMA A functor $F:\underline{I} \to \underline{J}$ induces a functor

$$\underline{\Delta}_M/F:\underline{\Delta}_M/\underline{I} \to \underline{\Delta}_M/\underline{J} \quad ((m,u) \to (m,F \circ u))$$

and the diagram

$$\begin{tikzcd} \underline{\Delta}_M/\underline{I} \arrow[r, "\underline{\Delta}_M/F"] \arrow[d, "\tau_{\underline{I}}"'] & \underline{\Delta}_M/\underline{J} \arrow[d, "\tau_{\underline{J}}"] \\ \underline{I} \arrow[r, "F"'] & \underline{J} \end{tikzcd}$$

commutes.

Let $\underline{C}$ be a complete model category, W its class of weak equivalences. Put

$$D = D_{(\underline{C},W)}.$$

3.6.13 LEMMA Given a small category $\underline{I}$, the functor

$$D\tau_{\underline{I}}:D\underline{I} \to D\underline{\Delta}_M/\underline{I}$$

is fully faithful and has a right adjoint

$$D\tau_{\underline{I}\dagger}:D\underline{\Delta}_M/\underline{I} \to D\underline{I}.$$

[Note: To ground this in reality, take $\underline{I} = \underline{1}$ — then $\underline{\Delta}_M/\underline{1} \approx \underline{\Delta}_M$. But $\underline{\Delta}_M$ is D-aspherical, thus the functor

$$Dp_{\underline{\Delta}_M}:D\underline{1} \to D\underline{\Delta}_M$$

is fully faithful (cf. 3.5.5). Since both $\underline{1}$ and $\underline{\Delta}_M$ are direct, the existence of $Dp_{\underline{\Delta}_M\dagger}$ is automatic (cf. 3.6.17).]

3.6.14 RAPPEL Suppose that $\underline{C}$ is a complete model category and let $\underline{I}$ be a direct category -- then $[\underline{I}^{OP},\underline{C}]$ in its injective structure is a model category (cf. 0.27.6).

<u>Ad DER - 1:</u> The canonical functor

$$D(\coprod_{k=1}^{n} \underline{I}_k) \to \prod_{k=1}^{n} D(\underline{I}_k)$$

is bijective on objects, thus it need only be shown that it is fully faithful. To this end, form the commutative diagram

$$\begin{tikzcd}
D(\coprod\limits_{k=1}^{n} \underline{I}_k) \arrow[r] \arrow[d, "D(\coprod_k \tau_{\underline{I}_k})"'] & \prod\limits_{k=1}^{n} D(\underline{I}_k) \arrow[d, "\prod_k D\tau_{\underline{I}_k}"] \\
D(\coprod\limits_{k=1}^{n} \underline{\Delta}_M/\underline{I}_k) \arrow[r] & \prod\limits_{k=1}^{n} D(\underline{\Delta}_M/\underline{I}_k).
\end{tikzcd}$$

Then the functors

$$\left[\begin{array}{l} D(\coprod_k \tau_{\underline{I}_k}) \ (= D(\tau_{\coprod_k \underline{I}_k})) \\ \qquad\qquad \prod_k D\tau_{\underline{I}_k} \end{array}\right.$$

are fully faithful (cf. 3.6.13). On the other hand,

$$[\coprod_{k=1}^{n} (\underline{\Delta}_M/\underline{I}_k)^{OP}, \underline{C}]$$

$$= \prod_{k=1}^{n} [(\underline{\Delta}_M/\underline{I}_k)^{OP}, \underline{C}]$$

and $\forall$ k,

$$[(\underline{\Delta}_M/\underline{I}_k)^{OP}, \underline{C}]$$

is a model category (cf. 3.6.14). Therefore the arrow

$$D(\coprod_{k=1}^{n} \underline{\Delta}_M/\underline{I}_k) = \underline{H} \prod_{k=1}^{n} [(\underline{\Delta}_M/\underline{I}_k)^{OP}, \underline{C}]$$

$$\longrightarrow \prod_{k=1}^{n} D(\underline{\Delta}_M/\underline{I}_k) = \prod_{k=1}^{n} \underline{H}[(\underline{\Delta}_M/\underline{I}_k)^{OP}, \underline{C}]$$

is an equivalence of categories (cf. 0.1.29).

[Note: Here $D\underline{0} = \underline{1}$.]

3.6.15 LEMMA Let $\underline{I}$ be a small category, $\underline{C}$ a model category. Suppose that $[\underline{I},\underline{C}]$ admits a model structure in which the weak equivalences are levelwise -- then the

$$DK_i : \underline{H}[\underline{I},\underline{C}] \to \underline{HC} \quad (i \in \mathrm{Ob}\ \underline{I})$$

constitute a conservative family.

PROOF Let $f:X \to Y$ be an arrow in $\underline{H}[\underline{I},\underline{C}]$. Replacing X by a cofibrant object and Y by a fibrant object, one can assume that f is an arrow in $[\underline{I},\underline{C}]$ (cf. 2.4.2). But then the result is obvious (consider $D_{[\underline{I},\underline{C}]}$).

<u>Ad DER - 2</u>: Let $\underline{I}$ be a small category and let $f \in \text{Mor } D\underline{I}$ be a morphism such that $\forall\ i \in \text{Ob } \underline{I}$, $DK_i f$ is an isomorphism in $D\underline{1}$ -- then the claim is that f is an isomorphism in $D\underline{I}$. Given $(m,u) \in \text{Ob } \underline{\Delta}_M/\underline{I}$,

$$\tau_{\underline{I}} \circ K_{(m,u)}:\underline{1} \to \underline{I}$$

equals

$$K_{u(m)}:\underline{1} \to \underline{1}.$$

And so

$$\begin{aligned} DK_{(m,u)} D\tau_{\underline{I}} f &= D(\tau_{\underline{I}} \circ K_{(m,u)})f \\ &= DK_{u(m)} f \end{aligned}$$

is an isomorphism in $D\underline{1}$. But $[(\underline{\Delta}_M/\underline{I})^{OP},\underline{C}]$ is a model category (cf. 3.6.14), hence the

$$DK_{(m,u)}:\underline{H}[(\underline{\Delta}_M/\underline{I})^{OP},\underline{C}] \to \underline{HC}((m,u) \in \text{Ob } \underline{\Delta}_M/\underline{I})$$

constitute a conservative family (cf. 3.6.15). Therefore $D\tau_{\underline{I}} f$ is an isomorphism in $D\underline{\Delta}_M/\underline{I}$, thus f is an isomorphism in $D\underline{I}$ (cf. 3.6.13) ($D\tau_{\underline{I}}$ is fully faithful, hence reflects isomorphisms).

3.6.16 REMARK The generalization of the preceding considerations is embodied in the dual of 2.6.1 (i.e., with $\underline{C}$ a homotopically complete fibration category).

3.6.17 RAPPEL Suppose that $\underline{C}$ is a complete model category. Let $\underline{I},\underline{J}$ be direct categories and let $F:\underline{I} \to \underline{J}$ be a functor. Equip

$$\left[\begin{array}{l} [\underline{I}^{OP},\underline{C}] \\ \\ [\underline{J}^{OP},\underline{C}] \end{array}\right.$$

with their injective structures (cf. 3.6.14) -- then the arrow

$$\overline{(F^{OP})^*}:\underline{H}[\underline{J}^{OP},\underline{C}] \to \underline{H}[\underline{I}^{OP},\underline{C}]$$

has a right adjoint

$$R(F^{OP})_{\dagger}:\underline{H}[\underline{I}^{OP},\underline{C}] \to \underline{H}[\underline{J}^{OP},\underline{C}] \quad \text{(cf. 0.26.17).}$$

[Note: The supposition in this citation that $\underline{C}$ is combinatorial was made there only to ensure the existence of the injective model structure, thus is not needed here. In terms of the derivator $D_{(\underline{C},W)}$, we have

$$\left[\begin{array}{l} D_{(\underline{C},W)}F = \overline{(F^{OP})^*} \\ \\ D_{(\underline{C},W)}F_{\dagger} = R(F^{OP})_{\dagger}.] \end{array}\right.$$

<u>Ad RDER - 3</u>: The claim is that for every functor $F:\underline{I} \to \underline{J}$, the functor

$$DF:D\underline{J} \to D\underline{I}$$

has a right adjoint

$$DF_{\dagger}:D\underline{I} \to D\underline{J}.$$

To establish this, form the commutative diagram

$$\begin{array}{ccc} \underline{\Delta}_M/\underline{I} & \xrightarrow{\underline{\Delta}_M/F} & \underline{\Delta}_M/\underline{J} \\ {\scriptstyle \tau_{\underline{I}}}\downarrow & & \downarrow{\scriptstyle \tau_{\underline{J}}} \\ \underline{I} & \xrightarrow[F]{} & \underline{J} \end{array}$$

and pass to the square

$$\begin{tikzcd} D\underline{\Delta}_M/\underline{I} \arrow[r, "D\underline{\Delta}_M/F_\dagger"] & D\underline{\Delta}_M/\underline{J} \arrow[d, "D\tau_{\underline{J}\dagger}"] \\ D\underline{I} \arrow[u, "D\tau_{\underline{I}}"] \arrow[r, dotted, "DF_\dagger"'] & D\underline{J} \end{tikzcd} \quad (\text{cf. } 3.6.13),$$

$DF_\dagger$ being defined as the composition

$$D\tau_{\underline{J}\dagger} \circ D\underline{\Delta}_M/F_\dagger \circ D\tau_{\underline{I}}.$$

Bearing in mind that $D\tau_{\underline{I}}$ is fully faithful (cf. 3.6.13), $DF_\dagger$ is seen to be a right adjoint for DF.

<u>Ad RDER - 4</u>: Let $F:\underline{I} \to \underline{J}$ be a functor and fix $j \in \text{Ob } \underline{J}$ -- then the claim is that the arrow

$$DK_j \circ DF_\dagger \to Dp_{\underline{I}/j\dagger} \circ D\text{pro}_j$$

is a natural isomorphism.

<u>Step 1</u>: Check that the claim holds when $\underline{I}$ is direct.

<u>Step 2</u>: Take $\underline{I}$ arbitrary and consider the 2-diagram (cf. 3.3.7)

$$\begin{tikzcd} \underline{\Delta}_M/\underline{I}/j \arrow[r, "\text{pro}_j"] \arrow[d, "p_{\underline{\Delta}_M/\underline{I}/j}"'] & \underline{\Delta}_M/\underline{I} \arrow[d, "F \circ \tau_{\underline{I}}"] \\ \underline{1} \arrow[r, "K_j"'] & \underline{J} \end{tikzcd}.$$

Then by Step 1,

$$DK_j \circ D(F \circ \tau_{\underline{I}})_{\dagger} \approx (Dp_{\underline{\Delta}_M/\underline{I}/j})_{\dagger} \circ Dpro_j.$$

Step 3: Since the functors $D\tau_{\underline{I}}$ and $D\tau_{\underline{I}/j}$ are fully faithful (cf. 3.6.13), it follows that

$$\begin{aligned} DK_j \circ DF_{\dagger} &\approx DK_j \circ DF_{\dagger} \circ D\tau_{\underline{I}\dagger} \circ D\tau_{\underline{I}} \\ &\approx DK_j \circ D(F \circ \tau_{\underline{I}})_{\dagger} \circ D\tau_{\underline{I}} \\ &\approx (Dp_{\underline{\Delta}_M/\underline{I}/j})_{\dagger} \circ Dpro_j \circ D\tau_{\underline{I}} \\ &\approx Dp_{\underline{I}/j\dagger} \circ D\tau_{\underline{I}/j\dagger} \circ Dpro_j \circ D\tau_{\underline{I}} \\ &\approx Dp_{\underline{I}/j\dagger} \circ D\tau_{\underline{I}/j\dagger} \circ D\tau_{\underline{I}/j} \circ Dpro_j \\ &\approx Dp_{\underline{I}/j\dagger} \circ Dpro_j, \end{aligned}$$

as desired.

[Note: The canonical arrow

$$\underline{\Delta}_M/(\underline{I}/j) \to (\underline{\Delta}_M/\underline{I})/j$$

is an isomorphism and the diagram

$$\begin{tikzcd} \underline{\Delta}_M/\underline{I}/j \arrow[r, "pro_j"] \arrow[d, "\tau_{\underline{I}/j}"'] & \underline{\Delta}_M/\underline{I} \arrow[d, "\tau_{\underline{I}}"] \\ \underline{I}/j \arrow[r, "pro_j"'] & \underline{I} \end{tikzcd}$$

commutes.]

3.6.18 EXAMPLE Let $\underline{C}$ be a complete model category, W its class of weak equivalences — then $D_{(\underline{C},W)}$ is a right homotopy theory (cf. 3.6.1). Given $F:\underline{I} \to \underline{J}$, write

$$\begin{cases} \operatorname{holim}_{\underline{I}^{OP}} \text{ in place of } D_{(\underline{C},W)}p_{\underline{I}\dagger} \\ \\ \operatorname{holim}_{\underline{J}^{OP}} \text{ in place of } D_{(\underline{C},W)}p_{\underline{J}\dagger}. \end{cases}$$

Then F is a $D_{(\underline{C},W)}$-equivalence iff $\forall\ X \in \operatorname{Ob} \underline{C}$ (= Ob $\underline{HC}$), the arrow

$$\operatorname{holim}_{\underline{J}^{OP}} X \to \operatorname{holim}_{\underline{I}^{OP}} X$$

is an isomorphism, there being an abuse of notation in that

$$\begin{cases} \operatorname{holim}_{\underline{J}^{OP}} \text{ operates on } D_{(\underline{C},W)}p_{\underline{J}}X \text{ (and not on X)} \\ \\ \operatorname{holim}_{\underline{I}^{OP}} \text{ operates on } D_{(\underline{C},W)}p_{\underline{I}}X \text{ (and not on X).} \end{cases}$$

3.7 *UNIVERSAL PROPERTIES*

Given categories $\underline{C}$ and $\underline{D}$, write $[\underline{C},\underline{D}]_!$ for the full subcategory of $[\underline{C},\underline{D}]$ whose objects are the $F:\underline{C} \to \underline{D}$ that preserve colimits.

3.7.1 RAPPEL Suppose that $\underline{C}$ is small and $\underline{S}$ is cocomplete -- then precomposition with $Y_{\underline{C}}:\underline{C} \to \hat{\underline{C}}$ induces an equivalence

$$[\hat{\underline{C}},\underline{S}]_! \to [\underline{C},\underline{S}]$$

of categories.

3.7.2 EXAMPLE Take $\underline{C} = \underline{1}$ -- then $\hat{\underline{1}} \approx \underline{SET}$ and there is an equivalence

$$[\underline{SET},\underline{S}]_! \to \underline{S} \quad (F \to F\{*\}),$$

hence in particular there is an equivalence

$$[\underline{SET},\underline{SET}]_! \to \underline{SET} \quad (F \to F\{*\})$$

under which $\mathrm{id}_{\underline{SET}}$ corresponds to a final object in $\underline{SET}$.

Let D,D' be homotopy theories and let $\underline{\Phi} \in \mathrm{Mor}(D,D')$ -- then given $F:\underline{I} \to \underline{J}$, there is a square

$$\begin{tikzcd} D\underline{I} \arrow[r, "\Phi_{\underline{I}}"] \arrow[d, "DF_!"'] & D'\underline{I} \arrow[d, "D'F_!"] \\ D\underline{J} \arrow[r, "\Phi_{\underline{J}}"'] & D'\underline{J} \end{tikzcd}$$

and a canonical arrow

$$D'F_! \circ \Phi_{\underline{I}} \to \Phi_{\underline{J}} \circ DF_!.$$

3.7.3 NOTATION Write $\underline{HOM}_!(D,D')$ for the full submetacategory of $\underline{HOM}(D,D')$ whose objects are the $\underline{\Phi}$ such that the arrow

$$D'F_! \circ \Phi_{\underline{I}} \to \Phi_{\underline{J}} \circ DF_!$$

is an isomorphism $\forall\ F:\underline{I} \to \underline{J}$.

Let $\underline{I}$ be a small category -- then there is a canonical arrow

$$\underline{I} \xrightarrow{sY_{\underline{I}}} \underline{SPREI} \quad (\text{cf. } 0.33.8).$$

Here

$$\underline{\mathrm{SPREI}} = [\underline{\mathrm{I}}^{\mathrm{OP}},\underline{\mathrm{SISET}}],$$

which we shall endow with its projective structure (cf. 0.26.6). Let $\mathrm{HOT}_{\underline{\mathrm{I}}}$ be the homotopy theory arising therefrom.

3.7.4 THEOREM The functor $\mathrm{sY}_{\underline{\mathrm{I}}}$ induces a morphism

$$\mathrm{D}_{\underline{\mathrm{I}}} \to \mathrm{HOT}_{\underline{\mathrm{I}}}$$

of derivators and for every homotopy theory D, there is an equivalence

$$\underline{\mathrm{HOM}}_!(\mathrm{HOT}_{\underline{\mathrm{I}}},\mathrm{D}) \to \underline{\mathrm{HOM}}(\mathrm{D}_{\underline{\mathrm{I}}},\mathrm{D})$$

of metacategories.

3.7.5 EXAMPLE Take $\underline{\mathrm{I}} = \underline{1}$ and let $\mathrm{HOT} = \mathrm{HOT}_{\underline{1}}$, thus

$$\mathrm{HOT} = \mathrm{D}_{(\underline{\mathrm{SISET}},\mathrm{W}_\infty)}.$$

Then for every homotopy theory D, there is an equivalence

$$\underline{\mathrm{HOM}}_!(\mathrm{HOT},\mathrm{D}) \to \mathrm{D}\underline{1} \quad (\Phi \to \Phi_{\underline{1}}\Delta[0])$$

of metacategories (cf. 3.2.15). Accordingly, choosing D = HOT, it follows that up to equivalence,

$$\underline{\mathrm{HOM}}_!(\mathrm{HOT},\mathrm{HOT})$$

"is"

$$\mathrm{HOT}\ \underline{1} = \mathrm{W}_\infty^{-1}\underline{\mathrm{SISET}} = \underline{\mathrm{HSISET}}.$$

Let D be a homotopy theory and let $\mathrm{C} \subset \mathrm{Mor}\ \mathrm{D}\underline{1}$ be a class of morphisms.

3.7.6 DEFINITION A <u>homotopical localization</u> of D at C is a pair $(L_C D, \underline{L_C})$, where $L_C D$ is a homotopy theory and

$$\underline{L_C}: D \to L_C D$$

is an object in $\underline{\mathrm{HOM}}_!(D, L_C D)$ such that the functor

$$L_{C\underline{1}}: D\underline{1} \to L_C D\underline{1}$$

sends the elements of C to isomorphisms in $L_C D\underline{1}$ and is universal w.r.t. this condition: For every homotopy theory D', the arrow

$$\underline{\mathrm{HOM}}_!(L_C D, D') \to \underline{\mathrm{HOM}}_{!,C}(D, D')$$

induced by $\underline{L_C}$ is an equivalence of metacategories, the symbol on the RHS standing for the full submetacategory of $\underline{\mathrm{HOM}}_!(D, D')$ whose objects $\underline{\Phi}$ have the property that the functor

$$\Phi_{\underline{1}}: D\underline{1} \to D'\underline{1}$$

sends the elements of C to isomorphisms in $D'\underline{1}$.

3.7.7 THEOREM† Let $\underline{C}$ be a left proper combinatorial model category, $C \subset \mathrm{Mor}\,\underline{C}$ a set. Form the model localization $(\underline{L}_C\underline{C}, L_C)$ of $\underline{C}$ at C per 0.33.5 — then $L_C: \underline{C} \to \underline{L}_C\underline{C}$ induces a morphism

$$D_{(\underline{C}, W)} \to D_{(\underline{L}_C\underline{C}, W_C)}$$

of homotopy theories which is a homotopical localization of $D_{(\underline{C}, W)}$ at $L_W C$ (the image of C in $D_{(\underline{C}, W)}\underline{1} = \underline{HC}$).

---

† Tabuada, arXiv:0706.2420

[Note: Therefore

$$\mathrm{L}_{\underline{L}_W C} D_{(\underline{C},W)} = D_{(\underline{L}_C \underline{C}, W_C)}.]$$

3.7.8 REMARK The homotopy theories that are equivalent to the $D_{(\underline{C},W)}$, where $\underline{C}$ is a left proper combinatorial model category, are the homotopical localizations of the $\mathrm{HOT}_{\underline{I}}$ for some small category $\underline{I}$ (cf. 0.33.7).

CHAPTER 4: SIMPLICIAL MODEL CATEGORIES

CHAPTER 4: SIMPLICIAL MODEL CATEGORIES

4.1 SISET *ENRICHMENTS*

What follows is a review of the terminology employed in enriched category theory specialized to the case when the underlying symmetric monoidal category is SISET.

4.1.1 DEFINITION An S-category $\mathfrak{M}$ consists of a class O (the objects) and a function that assigns to each ordered pair $X,Y \in O$ a simplicial set $HOM(X,Y)$ plus simplicial maps

$$C_{X,Y,Z}:HOM(X,Y) \times HOM(Y,Z) \to HOM(X,Z)$$

and

$$I_X:\Delta[0] \to HOM(X,X)$$

satisfying the following conditions.

(S-1) The diagram

$$\begin{tikzcd}
HOM(X,Y) \times (HOM(Y,Z) \times HOM(Z,W)) \arrow[r, "id \times C"] \arrow[d, "A"'] & HOM(X,Y) \times HOM(Y,W) \arrow[dd, "C"] \\
(HOM(X,Y) \times HOM(Y,Z)) \times HOM(Z,W) \arrow[d, "C \times id"'] & \\
HOM(X,Z) \times HOM(Z,W) \arrow[r, "C"'] & HOM(X,W)
\end{tikzcd}$$

commutes.

(S-2) The diagram

$$\begin{tikzcd}
\Delta[0] \times \mathrm{HOM}(X,Y) \arrow[r, "L"] \arrow[d, "I \times \mathrm{id}"'] & \mathrm{HOM}(X,Y) \arrow[d, equal] & \mathrm{HOM}(X,Y) \times \Delta[0] \arrow[l, "R"'] \arrow[d, "\mathrm{id} \times I"] \\
\mathrm{HOM}(X,X) \times \mathrm{HOM}(X,Y) \arrow[r, "C"'] & \mathrm{HOM}(X,Y) & \mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Y) \arrow[l, "C"]
\end{tikzcd}$$

commutes.

The underlying category $U\mathfrak{M}$ of an S-category $\mathfrak{M}$ has for its class of objects the class O, Mor(X,Y) being the set Nat($\Delta[0]$,HOM(X,Y)) (= $\mathrm{HOM}(X,Y)_0$). Composition

$$\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$$

is calculated from

$$\Delta[0] \approx \Delta[0] \times \Delta[0] \xrightarrow{f \times g} \mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z),$$

while I_X serves as the identity in Mor(X,X).

4.1.2 EXAMPLE Every category $\underline{C}$ can be regarded as an S-category: Replace Mor(X,Y) by

$$\mathrm{HOM}(X,Y) \equiv \mathrm{si}\ \mathrm{Mor}(X,Y).$$

The associated underlying category is then isomorphic to $\underline{C}$. In fact,

$$\begin{aligned} &\mathrm{Nat}(\Delta[0], \mathrm{si}\ \mathrm{Mor}(X,Y)) \\ &\qquad \approx \mathrm{si}\ \mathrm{Mor}(X,Y)_0 = \mathrm{Mor}(X,Y). \end{aligned}$$

4.1.3 LEMMA Fix a class O. Consider the metacategory $\mathbf{CAT}_O$ whose objects are the categories with object class O, the morphisms being the functors which are the identity on objects -- then the S-categories with object class O can be identified with the simplicial objects in $\mathbf{CAT}_O$.

[An S-category $\mathfrak{M}$ gives rise to a simplicial object $\underline{M}:\underline{\Delta}^{OP} \to \mathbf{CAT}_O$ via $[n] \to \underline{M}_n$, where for $X,Y \in \mathrm{Ob}\,\underline{M}_n = O$, $\mathrm{Mor}_{\underline{M}_n}(X,Y) = \mathrm{HOM}(X,Y)_n$. Conversely, a simplicial object $\underline{M}:\underline{\Delta}^{OP} \to \mathbf{CAT}_O$ determines an S-category $\mathfrak{M}$ if for $X,Y \in O$,

$$\mathrm{HOM}(X,Y)_n = \{f \in \mathrm{Mor}\,\underline{M}_n : \mathrm{dom}\,f = X \ \&\ \mathrm{cod}\,f = Y\}.]$$

N.B. An object of $[\underline{\Delta}^{OP},\underline{CAT}]$ corresponds to an S-category iff its underlying simplicial set of objects is a constant simplicial set, say si O for some set O.

4.1.4 CONSTRUCTION Suppose that $\mathfrak{M}$ is an S-category with object class O -- then its opposite $\mathfrak{M}^{OP}$ is the S-category defined by

- $O^{OP} = O$;
- $\mathrm{HOM}^{OP}(X,Y) = \mathrm{HOM}(Y,X)$;
- $C^{OP}_{X,Y,Z} = C_{Z,Y,X} \circ T_{\mathrm{HOM}(Y,X),\mathrm{HOM}(Z,Y)}$;
- $I^{OP}_X = I_X$.

4.1.5 CONSTRUCTION Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are S-categories with object classes O and O' -- then their product $\mathfrak{M} \times \mathfrak{M}'$ is the S-category with object class $O \times O'$ and

$$\mathrm{HOM}((X,X'),(Y,Y')) = \mathrm{HOM}(X,Y) \times \mathrm{HOM}(X',Y').$$

[Note: The definitions of

$$C_{(X,X'),(Y,Y'),(Z,Z')} \text{ and } I_{(X,X')}$$

are "what they have to be".]

4.1.6 DEFINITION Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are S-categories with object classes O and O' -- then an <u>S-functor</u> $F:\mathfrak{M} \to \mathfrak{M}'$ is the specification of a rule that assigns to each object $X \in O$ an object $FX \in O'$ and the specification of a rule that assigns to each ordered pair $X,Y \in O$ a morphism

$$F_{X,Y}:HOM(X,Y) \to HOM(FX,FY)$$

of simplicial sets such that the diagram

$$\begin{array}{ccc} HOM(X,Y) \times HOM(Y,Z) & \xrightarrow{c} & HOM(X,Z) \\ {\scriptstyle F_{X,Y} \times F_{Y,Z}} \downarrow & & \downarrow {\scriptstyle F_{X,Z}} \\ HOM(FX,FY) \times HOM(FY,FZ) & \xrightarrow[c]{} & HOM(FX,FZ) \end{array}$$

commutes and the equality $F_{X,X} \circ I_X = I_{FX}$ obtains.

[Note: The <u>underlying functor</u> $UF:U\mathfrak{M} \to U\mathfrak{M}'$ sends X to FX and $f:\Delta[0] \to HOM(X,Y)$ to $F_{X,Y} \circ f$.]

4.1.7 EXAMPLE For any S-category $\mathfrak{M}$,

$$HOM:\mathfrak{M}^{OP} \times \mathfrak{M} \to \underline{SISET}$$

is an S-functor.

<u>N.B.</u> The opposite of an S-functor $F:\mathfrak{M} \to \mathfrak{M}'$ is an S-functor $F^{OP}:\mathfrak{M}^{OP} \to \mathfrak{M}'^{OP}$.

4.1.8 NOTATION Let S-$\mathbf{CAT}$ denote the metacategory whose objects are the S-categories and whose morphisms are the S-functors between them.

4.1.9 DEFINITION Suppose that $\mathfrak{M},\mathfrak{M}'$ are S-categories and $F,G:\mathfrak{M} \to \mathfrak{M}'$ are

S-functors — then an <u>S-natural transformation</u> Ξ from F to G is a collection of simplicial maps

$$\Xi_X:\Delta[0] \to \mathrm{HOM}(FX,GX)$$

for which the diagram

$$\begin{array}{ccc}
\Delta[0] \times \mathrm{HOM}(X,Y) & \xrightarrow{\Xi_X \times G_{X,Y}} & \mathrm{HOM}(FX,GX) \times \mathrm{HOM}(GX,GY) \\
{\scriptstyle L^{-1}}\uparrow & & \downarrow{\scriptstyle C} \\
\mathrm{HOM}(X,Y) & & \mathrm{HOM}(FX,GY) \\
{\scriptstyle R^{-1}}\downarrow & & \uparrow{\scriptstyle C} \\
\mathrm{HOM}(X,Y) \times \Delta[0] & \xrightarrow[F_{X,Y} \times \Xi_Y]{} & \mathrm{HOM}(FX,FY) \times \mathrm{HOM}(FY,GY)
\end{array}$$

commutes.

[Note: Take $\mathfrak{M}'$ = <u>SISET</u> (viewed as an S-category per 4.2.1) — then here an S-natural transformation Ξ from F to G is a collection of simplicial maps

$$\Xi_X:FX \to GX$$

rendering the diagram

$$\begin{array}{ccc}
\mathrm{HOM}(X,Y) & \xrightarrow{F_{X,Y}} & \mathrm{map}(FX,FY) \\
{\scriptstyle G_{X,Y}}\downarrow & & \downarrow{\scriptstyle \Xi_Y\circ} \\
\mathrm{map}(GX,GY) & \xrightarrow[\circ\Xi_X]{} & \mathrm{map}(FX,GY)
\end{array}$$

commutative.]

4.1.10 NOTATION Given S-categories $\mathfrak{M},\mathfrak{M}'$, let $\mathrm{Mor}_S(\mathfrak{M},\mathfrak{M}')$ stand for the

S-functors $\mathfrak{M} \to \mathfrak{M}'$ and given S-functors $F,G:\mathfrak{M} \to \mathfrak{M}'$, let $\mathrm{Nat}_S(F,G)$ stand for the S-natural transformations Ξ from F to G -- then by $[\mathfrak{M},\mathfrak{M}']_S$ we shall understand the metacategory whose objects are the elements of $\mathrm{Mor}_S(\mathfrak{M},\mathfrak{M}')$ and whose morphisms are the S-natural transformations.

4.2 *MISCELLANEOUS EXAMPLES*

One way to produce S-categories is to start with a category $\underline{C}$ and then introduce

$$\mathrm{HOM}(X,Y), C_{X,Y,Z}, \text{ and } I_X,$$

subject to S-1 and S-2. In some situations, the underlying category is isomorphic to $\underline{C}$ itself but this need not be the case in general (cf. 4.2.5 infra).

4.2.1 EXAMPLE $\underline{\mathrm{SISET}}$ is an S-category if

$$\mathrm{HOM}(X,Y = \mathrm{map}(X,Y).$$

The associated underlying category is then isomorphic to $\underline{\mathrm{SISET}}$. In fact,

$$\begin{aligned} \mathrm{Nat}(\Delta[0],\mathrm{HOM}(X,Y)) &\approx \mathrm{Nat}(\Delta[0],\mathrm{map}(X,Y)) \\ &\approx \mathrm{map}(X,Y)_0 \\ &\approx \mathrm{Nat}(X,Y). \end{aligned}$$

4.2.2 EXAMPLE $\underline{\mathrm{CAT}}$ is an S-category if

$$\mathrm{HOM}(\underline{I},\underline{J}) = \mathrm{ner}[\underline{I},\underline{J}].$$

Here $C_{\underline{I},\underline{J},\underline{K}}$ is the composition

$$\begin{aligned} &\mathrm{ner}[\underline{I},\underline{J}] \times \mathrm{ner}[\underline{J},\underline{K}] \\ &\approx \mathrm{ner}([\underline{I},\underline{J}] \times [\underline{J},\underline{K}]) \to \mathrm{ner}[\underline{I},\underline{K}] \end{aligned}$$

and

$$I_{\underline{I}}:\Delta[0] \to \text{ner}[\underline{I},\underline{I}]$$

is the result of applying ner to the canonical arrow $[0] \to [\underline{I},\underline{I}]$ $(0 \to \text{id}_{\underline{I}})$.

[Note: We have

$$\text{Nat}(\Delta[0],\text{ner}[\underline{I},\underline{J}]) \approx \text{Nat}(\text{ner}[0],\text{ner}[\underline{I},\underline{J}])$$

$$\approx \text{Mor}([0],[\underline{I},\underline{J}])$$

$$\approx \text{Ob}[\underline{I},\underline{J}] \approx \text{Mor}(\underline{I},\underline{J}).$$

Therefore the associated underlying category is isomorphic to $\underline{\text{CAT}}$.]

4.2.3 EXAMPLE $\underline{\text{CGH}}$ is an S-category if HOM(X,Y) is the simplicial set which at level n is given by

$$\text{HOM}(X,Y)_n = C(X \times_k \Delta^n,Y) \quad (n \geq 0).$$

The associated underlying category is then isomorphic to $\underline{\text{CGH}}$. In fact,

$$\text{Nat}(\Delta[0],\text{HOM}(X,Y))$$

$$\approx \text{HOM}(X,Y)_0$$

$$\approx C(X \times_k \Delta[0],Y)$$

$$\approx C(X,Y).$$

4.2.4 REMARK Let $\underline{C}$ be a category with finite products. Suppose that $\Gamma:\underline{\Delta} \to \underline{C}$ is a cosimplicial object such that $\Gamma([0])$ is a final object in $\underline{C}$ -- then the prescription

$$\text{HOM}(X,Y)_n = \text{Mor}(X \times \Gamma([n]),Y) \quad (n \geq 0)$$

equips $\underline{C}$ with the structure of an S-category whose underlying category is isomorphic to $\underline{C}$.

[Note:

- Take $\underline{C} = \underline{SISET}$ and let $\Gamma([n]) = \Delta[n]$ to recover 4.2.1.
- Take $\underline{C} = \underline{CAT}$ and let $\Gamma([n]) = [n]$ to recover 4.2.2.

[$\forall\, n \geq 0$,

$$\mathrm{Mor}(\underline{I} \times [n], \underline{J}) \approx \mathrm{Mor}([n], [\underline{I}, \underline{J}]) \approx \mathrm{ner}_n [\underline{I}, \underline{J}].]$$

- Take $\underline{C} = \underline{CGH}$ and let $\Gamma([n]) = \Delta^n$ to recover 4.2.3.]

4.2.4 EXAMPLE Define a functor $\underline{\Delta}^{OP} \to$ SISET by sending $[n]$ to $\Delta[1]^n$ and

$$\begin{cases} \delta_i \text{ to } d_i \\ \\ \sigma_i \text{ to } s_i \end{cases}, \text{ where } \begin{cases} d_i(\alpha_1, \ldots, \alpha_n) = \begin{cases} (\alpha_2, \ldots, \alpha_n) & (i = 0) \\ (\alpha_1, \ldots, \max(\alpha_{i+1}, \alpha_i), \ldots, \alpha_n) & (0 < i < n) \\ (\alpha_1, \ldots, \alpha_{n-1}) & (i = n) \end{cases} \\ \\ s_i(\alpha_1, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_i, 0, \alpha_{i+1}, \ldots, \alpha_n). \end{cases}$$

Now fix a small category $\underline{C}$. Given $X, Y \in \mathrm{Ob}\,\underline{C}$, let $C = C(X,Y)$ be the cosimplicial set specified by taking for $C(X,Y)^n$ the set of all functors $F:[n + 1] \to \underline{C}$ with $F_0 = X$, $F_{n+1} = Y$ and letting

$$\begin{cases} C\delta_i : C^n \to C^{n+1} \\ \\ C\sigma_i : C^n \to C^{n-1} \end{cases}$$

be the assignments

$$\begin{cases} (f_0, \ldots, f_n) \to (f_0, \ldots, f_{i-1}, \mathrm{id}, f_i, \ldots, f_n) \\ \\ (f_0, \ldots, f_n) \to (f_0, \ldots, f_{i+1} \circ f_i, \ldots, f_n). \end{cases}$$

Put

$$\mathrm{HOM}(X,Y) = \int^{[n]} \Delta[1]^n \times C(X,Y)^n.$$

Since

$$\mathrm{HOM}(X,Y)_m = \int^{[n]} \Delta[1]^n_m \times C(X,Y)^n,$$

one can introduce a "composition" rule and a "unit" rule satisfying the axioms. The upshot, therefore, is an S-category $\mathcal{FRC}$ with $O = \mathrm{Ob}\ \underline{C}$.

[Note: The underlying category $U\mathcal{FRC}$ is the free category on Ob $\underline{C}$ having one generator for each nonidentity morphism in $\underline{C}$.]

4.3 S-<u>CAT</u>

An S-category is <u>small</u> if its class of objects is a set.

4.3.1 NOTATION Let S-<u>CAT</u> denote the category whose objects are the small S-categories and whose morphisms are the S-functors between them.

<u>N.B.</u> Typically, elements of S-<u>CAT</u> are denoted by $\mathbf{I},\mathbf{J},\mathbf{K},\ldots$ and their object sets by $|\mathbf{I}|,|\mathbf{J}|,|\mathbf{K}|,\ldots$.

4.3.2 THEOREM[†] S-<u>CAT</u> is complete and cocomplete.

4.3.3 THEOREM[††] S-<u>CAT</u> is presentable.

4.3.4 LEMMA S-<u>CAT</u> is a symmetric monoidal category (cf. 4.1.5).

Suppose that $\mathbf{I}$ is a small S-category and $\mathfrak{M}$ is an arbitrary S-category -- then

---

[†] Wolff, *J. Pure Appl. Algebra* <u>4</u> (1974), 123-135.

[††] Kelly-Lack, *Theory Appl. Categ.* <u>8</u> (2001), 555-575.

$\mathrm{Mor}_S(\mathbf{I},\mathfrak{M})$ is the object class of an S-category

$$S[\mathbf{I},\mathfrak{M}].$$

Proof: Given S-functors $F,G:\mathbf{I} \to \mathfrak{M}$, let $\mathrm{HOM}(F,G)$ be the equalizer

$$\mathrm{HOM}(F,G) \longrightarrow \prod_{i\in|\mathbf{I}|} \mathrm{HOM}(Fi,Gi) \rightrightarrows \prod_{i,j\in|\mathbf{I}|} \mathrm{map}(\mathrm{HOM}(i,j),\mathrm{HOM}(Fi,Gj))$$

in SISET.

[Note: There is an S-functor

$$E:S[\mathbf{I},\mathfrak{M}] \times \mathbf{I} \to \mathfrak{M}$$

called evaluation.]

N.B. The underlying category

$$US[\mathbf{I},\mathfrak{M}]$$

is isomorphic to $[\mathbf{I},\mathfrak{M}]_S$.

4.3.5 LEMMA If

$$F:\mathbf{I} \to \underline{\mathrm{SISET}}$$

or if

$$F:\mathbf{I}^{OP} \to \underline{\mathrm{SISET}},$$

then in SISET,

$$\mathrm{HOM}(\mathrm{HOM}(i,-),F) \approx Fi$$

or

$$\mathrm{HOM}(\mathrm{HOM}(-,i),F) \approx Fi.$$

[This is the "enriched" Yoneda lemma.]

4.3.6 LEMMA Let $\mathbf{I},\mathbf{J},\mathbf{K}$ be small S-categories -- then

$$\mathrm{Mor}_S(\mathbf{I} \times \mathbf{J},\mathbf{K}) \approx \mathrm{Mor}_S(\mathbf{I},S[\mathbf{J},\mathbf{K}]).$$

4.3.7 SCHOLIUM S-CAT is cartesian closed.

It is also true that S-CAT is an S-category.

4.3.8 CONSTRUCTION Let $\mathbf{I}$ be a small S-category. Given $n \geq 0$, define a small S-category $\mathbf{I}^{(n)}$ by stipulating that $|\mathbf{I}^{(n)}| = |\mathbf{I}|$ and

$$\mathrm{HOM}^{(n)}(i,j) = \mathrm{map}(\Delta[n],\mathrm{HOM}(i,j)).$$

Then

$$\begin{aligned}
&\mathrm{map}(\Delta[0],\mathrm{HOM}(i,j))([n]) \\
&\quad \approx \mathrm{Nat}(\Delta[0] \times \Delta[n],\mathrm{HOM}(i,j)) \\
&\quad \approx \mathrm{Nat}(\Delta[n],\mathrm{HOM}(i,j)) \\
&\quad \approx \mathrm{HOM}(i,j)_n
\end{aligned}$$

$$\Rightarrow$$

$$\mathbf{I}^{(0)} \approx \mathbf{I}.$$

And there are canonical arrows

$$\begin{cases}
\mathbf{I} \longrightarrow \mathbf{I}^{(n)} & (\Delta[n] \longrightarrow \Delta[0]) \\
\mathbf{I}^{(n)(n)} \longrightarrow \mathbf{I}^{(n)} & (\Delta[n] \xrightarrow{\mathrm{dia}} \Delta[n] \times \Delta[n]).
\end{cases}$$

Suppose now that $\mathbf{I}$ and $\mathbf{J}$ are small S-categories -- then the prescription

$$\mathrm{HOM}(\mathbf{I},\mathbf{J})_n = \mathrm{Mor}_S(\mathbf{I},\mathbf{J}^{(n)}) \qquad (n \geq 0)$$

defines a simplicial set $\mathrm{HOM}(\mathbf{I},\mathbf{J})$.

4.3.9 LEMMA Under the preceding operations, S-CAT is an S-category.

[To define

$$C_{I,J,K}:\mathrm{HOM}(I,J) \times \mathrm{HOM}(J,K) \to \mathrm{HOM}(I,K),$$

consider

$$\mathrm{Mor}_S(I,J^{(n)}) \times \mathrm{Mor}_S(J,K^{(n)}).$$

Then one arrives at

$$\mathrm{Mor}_S(I,K^{(n)})$$

via the diagram

$$\begin{tikzcd} & J \arrow[r] \arrow[d] & K^{(n)} \arrow[d] & \\ I \arrow[r] & J^{(n)} \arrow[r] & K^{(n)(n)} \arrow[r] & K^{(n)}.] \end{tikzcd}$$

Every small category $\underline{C}$ can be regarded as a small S-category (cf. 4.1.2) and this association defines a functor

$$\iota_S:\underline{\mathrm{CAT}} \to \mathrm{S}\text{-}\underline{\mathrm{CAT}}.$$

4.3.10 LEMMA The functor ι_S has a right adjoint S-$\underline{\mathrm{CAT}} \to \underline{\mathrm{CAT}}$, viz. the rule that sends a given $I \in$ Ob S-$\underline{\mathrm{CAT}}$ to its underlying category UI.

4.3.11 REMARK Given a small category $\underline{C}$ and an S-category $\mathbf{M}$, there is an isomorphism

$$[\underline{C},U\mathbf{M}] \longleftrightarrow [\iota_S\underline{C},\mathbf{M}]_S$$

of categories.

4.3.12 LEMMA The functor ι_S has a left adjoint, viz. the rule that sends a given $I \in$ Ob S-CAT to the category $\pi_0 I$ whose objects are those of I with

$$\mathrm{Mor}(i,j) = \pi_0(\mathrm{HOM}(i,j)) \quad (i,j \in |I|).$$

4.3.13 DEFINITION Let $\underline{I},\underline{J}$ be small S-categories, $F:\underline{I} \rightarrow \underline{J}$ an S-functor -- then F is a DK-equivalence if $\forall$ i,j $\in |\underline{I}|$, the simplicial map

$$F_{i,j}:\text{HOM}(i,j) \rightarrow \text{HOM}(Fi,Fj)$$

is a simplicial weak equivalence and

$$\pi_0 F:\pi_0\underline{I} \rightarrow \pi_0\underline{J}$$

is surjective on isomorphism classes.

4.3.14 EXAMPLE Let $\underline{C},\underline{D}$ be small categories -- then the DK-equivalences $\iota_S\underline{C} \rightarrow \iota_S\underline{D}$ are in a one-to-one correspondence with the equivalences $\underline{C} \rightarrow \underline{D}$.

[If X is a set, then the geometric realization of si X is X equipped with the discrete topology. And if A,B are topological spaces, each with the discrete topology, and if $\phi:A \rightarrow B$ is a homotopy equivalence, then ϕ is bijective.]

4.3.15 DEFINITION Let $\underline{I},\underline{J}$ be small S-categories, $F:\underline{I} \rightarrow \underline{J}$ an S-functor -- then F is a DK-fibration if $\forall$ i,j $\in |\underline{I}|$, the simplicial map

$$F_{i,j}:\text{HOM}(i,j) \rightarrow \text{HOM}(Fi,Fj)$$

is a fibration in SISET (Kan Structure) and

$$\pi_0 F:\pi_0\underline{I} \rightarrow \pi_0\underline{J}$$

is a fibration in CAT (Internal Structure).

4.3.16 THEOREM[†] S-CAT admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations.

[†] Bergner, *Trans. Amer. Math. Soc.* 359 (2007), 2043-2058; see also Lurie, *Annals of Math. Studies* 170 (2009), 852-863.

[Note: We shall refer to this model structure as the Bergner structure (which is therefore combinatorial (cf. 4.3.3)).]

Here are some additional facts.

- If $F: \mathbf{I} \to \mathbf{J}$ is a cofibration in the Bergner structure, then $\forall\ i,j \in |\mathbf{I}|$,

$$F_{i,j}: \mathrm{HOM}(i,j) \to \mathrm{HOM}(Fi,Fj)$$

is an injective simplicial map, thus is a cofibration in SISET (Kan Structure).

- The Bergner structure is proper (Bergner proved right proper and Lurie proved left proper).

- A small S-category $\mathbf{I}$ is fibrant in the Bergner structure iff $\forall\ i,j \in |\mathbf{I}|$, $\mathrm{HOM}(i,j)$ is a Kan complex, thus is fibrant in SISET (Kan Structure).

It is also possible to explicate the generating sets $\left[\begin{array}{l} I \\ J \end{array}\right.$, matters being simplest for I.

4.3.17 NOTATION Given a simplicial set X, let Σ_X be the small S-category with two objects a,b and

$$\left[\begin{array}{ll} \mathrm{HOM}(a,a) = \Delta[0] & \mathrm{HOM}(a,b) = X \\ \mathrm{HOM}(b,b) = \Delta[0], & \mathrm{HOM}(b,a) = \dot{\Delta}[0]. \end{array}\right.$$

4.3.18 NOTATION Let $[0]_S$ be the small S-category with one object x and $\mathrm{HOM}(x,x) = \Delta[0]$.

One can then take for I the arrows $\Sigma_{\dot{\Delta}[n]} \to \Sigma_{\Delta[n]}$ $(n \geq 0)$ plus the arrow $\emptyset \to [0]_S$ ($\emptyset$ the small S-category with no objects).

[Note: The arrows $\Sigma_{\Lambda[k,n]} \to \Sigma_{\Delta[n]}$ $(0 \leq k \leq n,\ n \geq 1)$ are part of J but the full description requires more input.]

4.3.19 DEFINITION Let

$$\mathbf{C}:\underline{\Delta} \to \text{S-}\underline{\text{CAT}}$$

be the functor that sends [n] to the small S-category whose objects are those of [n] and with

$$\text{HOM}(i,j) = \begin{cases} \Delta[1]^{j-i-1} & (j > i) \\ \Delta[0] & (j = i) \\ \dot{\Delta}[0] & (j < i). \end{cases}$$

[Note: Let $P_{i,j}$ be the poset of all subsets of $\{i,i+1,\ldots,j\}$ containing i and j (ordered by inclusion) — then the nerve of $P_{i,j}$ is isomorphic to $(\Delta[1])^{j-i-1}$ if $j > i$, $\Delta[0]$ if $j = i$, and $\dot{\Delta}[0]$ if $j < i$. Composition is defined using the pairings

$$P_{i,j} \times P_{j,k} \to P_{i,k}$$

given by taking unions.]

Bearing in mind that S-$\underline{\text{CAT}}$ is, in particular, cocomplete (cf. 4.3.2), pass from

$$\mathbf{C} \in \text{Ob}[\underline{\Delta},\text{S-}\underline{\text{CAT}}]$$

to the realization functor

$$\Gamma_{\mathbf{C}} \in \text{Ob}[\hat{\underline{\Delta}},\text{S-}\underline{\text{CAT}}],$$

thus

$$\Gamma_{\mathbf{C}}X = \int^{[n]} X_n \cdot \mathbf{C}[n]$$

and

$$|\Gamma_{\mathbf{C}}X| = X_0.$$

4.3.20 LEMMA Let $f:X \to Y$ be a simplicial map -- then f is a categorical weak equivalence iff $\Gamma_{\mathbf{C}}f:\Gamma_{\mathbf{C}}X \to \Gamma_{\mathbf{C}}Y$ is a DK-equivalence.

Denote the singular functor $\sin_{\mathbf{C}}$ by ner_S, so

$$\mathrm{ner}_S: S\text{-}\underline{\mathrm{CAT}} \to \underline{\mathrm{SISET}}$$

and

$$\mathrm{ner}_S I([n]) = \mathrm{Mor}_S(\mathbf{C}[n], I).$$

4.3.21 REMARK There is no a priori connection between $\mathrm{ner}_S I$ and $\mathrm{ner}\ UI$. On the other hand, for any small category $\underline{C}$,

$$\mathrm{ner}\ \underline{C} \approx \mathrm{ner}_S \iota_S \underline{C}.$$

4.3.22 THEOREM Consider the setup

$$\underline{\mathrm{SISET}}\ (\text{Joyal Structure}) \underset{\mathrm{ner}_S}{\overset{\Gamma_{\mathbf{C}}}{\rightleftarrows}} S\text{-}\underline{\mathrm{CAT}}\ (\text{Bergner Structure}).$$

Then $(\Gamma_{\mathbf{C}}, \mathrm{ner}_S)$ is a model equivalence, thus the adjoint pair $(L\Gamma_{\mathbf{C}}, R\mathrm{ner}_S)$ is an adjoint equivalence of homotopy categories:

$$\underline{\text{HSISET}} \begin{array}{c}\longrightarrow\\ \longleftarrow\end{array} \underline{\text{HS-CAT}}.$$

[Note: Compare this assertion with that of 0.22.5.]

4.3.23 REMARK It is not difficult to see that $\Gamma_{\mathbf{C}}$ preserves cofibrations. Accordingly, in view of 4.3.20, $(\Gamma_{\mathbf{C}},\text{ner}_S)$ is at least a model pair. However, the verification that $(\Gamma_{\mathbf{C}},\text{ner}_S)$ is actually a model equivalence lies deeper (complete details can be found in Dugger-Spivak[†]).

4.4 *SIMPLICIAL ACTIONS*

4.4.1 RAPPEL Given a category $\underline{C}$, $\underline{\text{SIC}}$ is the functor category $[\underline{\Delta}^{OP},\underline{C}]$ and a <u>simplicial object</u> in $\underline{C}$ is an object in $\underline{\text{SIC}}$.

4.4.2 DEFINITION Let $\underline{C}$ be a category. Suppose that X,Y are simplicial objects in $\underline{C}$ and let K be a simplicial set — then a <u>formality</u> $f:X|\overline{\underline{\ }}|K \to Y$ is a collection of morphisms $f_n(k):X_n \to Y_n$ in $\underline{C}$, one for each $n \geq 0$ and $k \in K_n$, such that

$$Y\alpha \circ f_n(k) = f_m((K\alpha)k) \circ X\alpha,$$

where $\alpha:[m] \to [n]$.

4.4.3 NOTATION Let

$$\text{For}(X|\overline{\underline{\ }}|K,Y)$$

be the set of formalities $f:X|\overline{\underline{\ }}|K \to Y$.

[†] arXiv:0911.0469

[Note: As it stands, $X|\square|K$ is just a symbol, not an object in $\underline{\mathrm{SIC}}$ (but see below).]

4.4.4 EXAMPLE $\mathrm{For}(X|\square|\Delta[0],Y)$ can be identified with $\mathrm{Nat}(X,Y)$.

4.4.5 LEMMA Let $\underline{C}$ be a category -- then the class of simplicial objects in $\underline{C}$ is the object class of an S-category $\underline{\mathrm{SIMC}}$.

PROOF Define $\mathrm{HOM}(X,Y)$ by the prescription

$$\mathrm{HOM}(X,Y)_n = \mathrm{For}(X|\square|\Delta[n],Y) \quad (n \geq 0).$$

[Note:

$$\begin{aligned}\mathrm{Nat}(\Delta[0],\mathrm{HOM}(X,Y)) &\approx \mathrm{HOM}(X,Y)_0 \\ &\approx \mathrm{For}(X|\square|\Delta[0],Y) \\ &\approx \mathrm{Nat}(X,Y) \quad (\text{cf. } 4.4.4).\end{aligned}$$

Therefore the underlying category $\underline{\mathrm{USIMC}}$ is isomorphic to $\underline{\mathrm{SIC}}$.]

4.4.6 DEFINITION Given a category $\underline{C}$, a <u>simplicial action</u> on $\underline{C}$ is a functor

$$|\square|:\underline{C} \times \underline{\mathrm{SISET}} \to \underline{C}$$

together with natural isomorphisms A and R, where

$$A_{X,K,L}:X|\square|(K \times L) \to (X|\square|K)|\square|L$$

and

$$R_X:X|\square|\Delta[0] \to X,$$

subject to the following assumptions.

(SA_1) The diagram

$$\begin{tikzcd}
X|\square|(K \times (L \times M)) \arrow[r, "A"] \arrow[d, "\mathrm{id}|\square|A"'] & (X|\square|K)|\square|(L \times M) \arrow[r, "A"] & ((X|\square|K)|\square|L)|\square|M \\
X|\square|((K \times L) \times M) \arrow[rr, "A"'] & & (X|\square|(K \times L))|\square|M \arrow[u, "A|\square|\mathrm{id}"']
\end{tikzcd}$$

commutes.

(SA_2) The diagram

$$\begin{tikzcd}
X|\square|(\Delta[0] \times K) \arrow[r, "A"] \arrow[d, "\mathrm{id}|\square|L"'] & (X|\square|\Delta[0])|\square|K \arrow[d, "R|\square|\mathrm{id}"] \\
X|\square|K \arrow[r, equal] & X|\square|K
\end{tikzcd}$$

commutes.

[Note: Every category admits a simplicial action, viz. the trivial simplicial action.]

<u>N.B.</u> It is automatic that the diagram

$$\begin{tikzcd}
X|\square|(K \times \Delta[0]) \arrow[r, "A"] \arrow[d, "\mathrm{id}|\square|R"'] & (X|\square|K)|\square|\Delta[0] \arrow[d, "R"] \\
X|\square|K \arrow[r, equal] & X|\square|K
\end{tikzcd}$$

commutes.

4.4.7 EXAMPLE If $|\square|$ is a simplicial action on $\underline{C}$, then for every small category $\underline{I}$, the composition

$$[\underline{I},\underline{C}] \times \underline{\mathrm{SISET}} \rightarrow [\underline{I},\underline{C}] \times [\underline{I},\underline{\mathrm{SISET}}]$$

$$\approx [\underline{I},\underline{C} \times \underline{\mathrm{SISET}}] \xrightarrow{[\underline{I},|\square|]} [\underline{I},\underline{C}]$$

is a simplicial action on $[\underline{I},\underline{C}]$.

4.4.8 THEOREM Let $\underline{C}$ be a category. Assume: $\underline{C}$ admits a simplicial action $|\overline{\underline{\ \ }}|$ -- then there is an S-category $|\overline{\underline{\ \ }}|\underline{C}$ such that $\underline{C}$ is isomorphic to the underlying category $U|\overline{\underline{\ \ }}|\underline{C}$.

PROOF Put $O = \mathrm{Ob}\,\underline{C}$ and assign to each ordered pair $X,Y \in O$ the simplicial set $\mathrm{HOM}(X,Y)$ defined by

$$\mathrm{HOM}(X,Y)_n = \mathrm{Mor}(X|\overline{\underline{\ \ }}|\Delta[n],Y) \quad (n \geq 0).$$

- Given X,Y,Z, let

$$c_{X,Y,Z}:\mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$$

be the simplicial map that sends

$$\left[\begin{array}{l} f:X|\overline{\underline{\ \ }}|\Delta[n] \to Y \\ \\ g:Y|\overline{\underline{\ \ }}|\Delta[n] \to Z \end{array}\right.$$

to the composite

$$X|\overline{\underline{\ \ }}|\Delta[n] \xrightarrow{\mathrm{id}|\overline{\underline{\ \ }}|\mathrm{dia}} X|\overline{\underline{\ \ }}|(\Delta[n] \times \Delta[n])$$

$$\xrightarrow{A} (X|\overline{\underline{\ \ }}|\Delta[n])|\overline{\underline{\ \ }}|\Delta[n] \xrightarrow{f|\overline{\underline{\ \ }}|\mathrm{id}} Y|\overline{\underline{\ \ }}|\Delta[n] \xrightarrow{g} Z.$$

- Given X, let

$$I_X:\Delta[0] \to \mathrm{HOM}(X,X)$$

be the simplicial map that sends $[n] \to [0]$ to

$$X|\overline{\underline{\ \ }}|\Delta[n] \to X|\overline{\underline{\ \ }}|\Delta[0] \xrightarrow{R} X.$$

Call $|\overline{\underline{\ \ }}|\underline{C}$ the S-category arising from this data. That $\underline{C}$ is isomorphic to the underlying category $U|\overline{\underline{\ \ }}|\underline{C}$ can be seen by considering the functor which is the identity on objects and sends a morphism $f:X \to Y$ in $\underline{C}$ to

$$X|\overline{\underline{\ }}|\Delta[0] \xrightarrow{R} X \xrightarrow{f} Y,$$

an element of

$$\mathrm{Mor}(X|\overline{\underline{\ }}|\Delta[0],Y) = \mathrm{HOM}(X,Y)_0 \approx \mathrm{Nat}(\Delta[0],\mathrm{HOM}(X,Y)).$$

<u>N.B.</u> If $|\overline{\underline{\ }}|$ is the trivial simplicial action, then

$$\mathrm{HOM}(X,Y) = \mathrm{si}\ \mathrm{Mor}(X,Y).$$

4.4.9 EXAMPLE <u>SISET</u> admits a simplicial action:

$$K|\overline{\underline{\ }}|L = K \times L.$$

Therefore

$$\mathrm{HOM}(K,L) = \mathrm{map}(K,L) \quad (\text{cf. } 4.2.1).$$

[Note: Let $\underline{I}$ be a small category -- then there is an induced simplicial action on [$\underline{I}$,<u>SISET</u>], viz.

$$(F|\overline{\underline{\ }}|K)i = Fi \times K \quad (\text{cf. } 4.4.7).$$

And

$$\mathrm{HOM}(F,G) \approx \int_i \mathrm{map}(Fi,Gi).$$

In fact,

$$\begin{aligned}\mathrm{HOM}(F,G)_n &\approx \mathrm{Nat}(F|\overline{\underline{\ }}|\Delta[n],G)\\ &\approx \int_i \mathrm{Nat}(Fi \times \Delta[n],Gi)\\ &\approx \int_i \mathrm{Nat}(\Delta[n],\mathrm{map}(Fi,Gi))\\ &\approx \mathrm{Nat}(\Delta[n],\int_i \mathrm{map}(Fi,Gi))\\ &\approx (\int_i \mathrm{map}(Fi,Gi))_n.]\end{aligned}$$

4.4.10 EXAMPLE $\underline{CGH}$ admits a simplicial action:

$$X|\overline{}|K = X \times_k |K|.$$

Therefore

$$HOM(X,Y)_n = C(X \times_k \Delta^n, Y) \quad (n \geq 0) \quad (\text{cf. } 4.2.3).$$

[Note: $\underline{CGH}$ is cartesian closed, the exponential object being $Y^X = kC(X,Y)$, where $C(X,Y)$ carries the compact open topology. Accordingly,

$$\begin{aligned} C(X \times_k \Delta^n, Y) &\approx C(\Delta^n \times_k X, Y) \\ &\approx C(\Delta^n, Y^X) \\ &\approx \sin Y^X([n]), \end{aligned}$$

so

$$HOM(X,Y) \approx \sin Y^X.]$$

4.4.11 THEOREM Let $\underline{C}$ be a category. Assume: $\underline{C}$ has coproducts -- then $\underline{SIC}$ admits a simplicial action $|\overline{}|$ such that $|\overline{}|\text{SIC}$ is isomorphic to $\underline{SIMC}$ (cf. 4.4.5).

PROOF Define $X|\overline{}|K$ by $(X|\overline{}|K)_n = K_n \cdot X_n$, thus for $\alpha:[m] \to [n]$,

$$K_n \cdot X_n \xrightarrow{X\alpha} K_n \cdot X_m \xrightarrow{K\alpha} K_m \cdot X_m.$$

The symbol $X|\overline{}|K$ also has another connotation (cf. 4.4.3). To resolve the ambiguity, note that there is a formality $\text{in}:X|\overline{}|K \to X|\overline{}|K$, where

$$\text{in}_n(k):X_n \to (X|\overline{}|K)_n$$

is the injection from X_n to $K_n \cdot X_n$ corresponding to $k \in K_n$. Moreover,

$$\text{in}^*:\text{Nat}(X|\overline{}|K,Y) \to \text{For}(X|\overline{}|K,Y)$$

is bijective and functorial. Therefore $|\overline{}|$<u>SIC</u> and <u>SIMC</u> are isomorphic.

[Note: $|\overline{}|$ is the <u>canonical</u> simplicial action on <u>SIC</u>.]

<u>N.B.</u> Take $\underline{C} = \underline{SET}$ -- then the canonical simplicial action on <u>SISET</u> is the simplicial action of 4.4.9. In fact,

$$X|\overline{}|K = X \times K$$

and

$$(X \times K)_n = X_n \times K_n \approx K_n \times X_n = K_n \cdot X_n.$$

4.4.12 DEFINITION A simplicial action $|\overline{}|$ on a category $\underline{C}$ is said to be <u>cartesian</u> if $\forall\ X \in \text{Ob}\ \underline{C}$, the functor

$$X|\overline{}|—:\underline{SISET} \to \underline{C}$$

has a right adjoint.

4.4.13 LEMMA Let $\underline{C}$ be a category. Assume: $\underline{C}$ has coproducts -- then the canonical simplicial action $|\overline{}|$ on <u>SIC</u> is cartesian.

PROOF Let HOM(X,Y) be the simplicial set figuring in the definition of <u>SIMC</u>, so

$$\text{HOM}(X,Y)_n = \text{For}(X|\overline{}|\Delta[n],Y) \quad (\text{cf. } 4.4.5).$$

Define

$$\text{ev} \in \text{For}(X|\overline{}|\text{HOM}(X,Y),Y)$$

by

$$\text{ev}_n(f) = f_n(\text{id}_{[n]}):X_n \to Y_n \quad (n \geq 0).$$

Viewing ev as "evaluation", there is an induced functorial bijection

$$\text{Nat}(K,\text{HOM}(X,Y)) \to \text{For}(X|\overline{}|K,Y).$$

But

$$\text{For}(X|\overline{}|K,Y) \approx \text{Nat}(X|\overline{}|K,Y) \quad (\text{cf. } 4.4.11).$$

Therefore $|\overline{\underline{\ }}|$ is cartesian.

4.4.14 LEMMA Suppose that the simplicial action $|\overline{\underline{\ }}|$ on $\underline{C}$ is cartesian — then $\forall\ X \in \text{Ob}\ \underline{C}$,

$$\text{HOM}(X,\text{—}):\underline{C} \to \underline{\text{SISET}}$$

is a right adjoint for

$$X|\overline{\underline{\ }}|\text{—}:\underline{\text{SISET}} \to \underline{C}.$$

PROOF The functor $X|\overline{\underline{\ }}|\text{—}$ is a left adjoint, hence preserves colimits. This said, given a simplicial set K, write

$$K \approx \text{colim}_i\ \Delta[n_i].$$

Then

$$\begin{aligned}
\text{Mor}(X|\overline{\underline{\ }}|K,Y) &\approx \text{Mor}(X|\overline{\underline{\ }}|\ \text{colim}_i\ \Delta[n_i],Y) \\
&\approx \text{Mor}(\text{colim}_i\ X|\overline{\underline{\ }}|\Delta[n_i],Y) \\
&\approx \lim_i\ \text{Mor}(X|\overline{\underline{\ }}|\Delta[n_i],Y) \\
&\approx \lim_i\ \text{HOM}(X,Y)_{n_i} \\
&\approx \lim_i\ \text{Nat}(\Delta[n_i],\text{HOM}(X,Y)) \\
&\approx \text{Nat}(\text{colim}_i\ \Delta[n_i],\text{HOM}(X,Y)) \\
&\approx \text{Nat}(K,\text{HOM}(X,Y)).
\end{aligned}$$

[Note: Here, of course, we are viewing $\underline{C}$ as an S-category per 4.4.8.]

4.4.15 DEFINITION A simplicial action $|\overline{\underline{\ }}|$ on a category $\underline{C}$ is said to be <u>closed</u> provided that it is cartesian and each of the functors $\text{—}|\overline{\underline{\ }}|K:\underline{C} \to \underline{C}$ has a right adjoint $X \to \text{hom}(K,X)$, so

$$\text{Mor}(X|\overline{\underline{\ }}|K,Y) \approx \text{Mor}(X,\text{hom}(K,Y)).$$

4.4.16 EXAMPLE The simplicial action on $\underline{\mathrm{SISET}}$ is closed (cf. 4.4.9), as is the simplicial action on $\underline{\mathrm{CGH}}$ (cf. 4.4.10).

4.4.17 EXAMPLE Take $\underline{C} = \underline{\mathrm{CAT}}$. Bearing in mind that

$$\mathrm{cat}:\underline{\mathrm{SISET}} \to \underline{\mathrm{CAT}}$$

preserves finite products, define a simplicial action

$$|\overline{}|:\underline{\mathrm{CAT}} \times \underline{\mathrm{SISET}} \to \underline{\mathrm{CAT}}$$

by the prescription

$$\underline{I}|\overline{}|K = \underline{I} \times \mathrm{cat}\ K.$$

Then

$$\begin{aligned} \mathrm{Mor}(\underline{I}|\overline{}|K,\underline{J}) &= \mathrm{Mor}(\underline{I} \times \mathrm{cat}\ K,\underline{J}) \\ &\approx \mathrm{Mor}(\mathrm{cat}\ K,[\underline{I},\underline{J}]) \\ &\approx \mathrm{Nat}(K,\mathrm{ner}[\underline{I},\underline{J}]). \end{aligned}$$

Therefore $|\overline{}|$ is cartesian and

$$\mathrm{HOM}(\underline{I},\underline{J}) = \mathrm{ner}[\underline{I},\underline{J}] \quad (\text{cf. } 4.2.2).$$

In addition, $|\overline{}|$ is closed with

$$\mathrm{hom}(K,X) = [\mathrm{cat}\ K,X].$$

4.4.18 EXAMPLE Take $\underline{C} = \underline{\mathrm{CAT}}$. Since $\pi_1 \circ \mathrm{cat}$ preserves finite products and $\iota:\underline{\mathrm{GRD}} \to \underline{\mathrm{CAT}}$ is a right adjoint, the prescription

$$\underline{I}|\overline{}|K = X \times \iota \circ \pi_1 \circ \mathrm{cat}\ K$$

defines a simplicial action

$$|\overline{}|:\underline{\mathrm{CAT}} \times \underline{\mathrm{SISET}} \to \underline{\mathrm{CAT}}.$$

Here

$$\begin{aligned}
\mathrm{Mor}(\underline{I}|\overline{\underline{\ \ }}|K,\underline{J}) &= \mathrm{Mor}(\underline{I} \times \iota \circ \pi_1 \circ \mathrm{cat}\ K,\underline{J}) \\
&\approx \mathrm{Mor}(\iota \circ \pi_1 \circ \mathrm{cat}\ K,[\underline{I},\underline{J}]) \\
&\approx \mathrm{Mor}(\pi_1 \circ \mathrm{cat}\ K,\mathrm{iso}[\underline{I},\underline{J}]) \\
&\approx \mathrm{Mor}(\mathrm{cat}\ K,\iota \circ \mathrm{iso}[\underline{I},\underline{J}]) \\
&\approx \mathrm{Nat}(K,\mathrm{ner} \circ \iota \circ \mathrm{iso}[\underline{I},\underline{J}])
\end{aligned}$$

from which it follows that $|\overline{\underline{\ \ }}|$ is cartesian and

$$\mathrm{HOM}(\underline{I},\underline{J}) = \mathrm{ner} \circ \iota \circ \mathrm{iso}[\underline{I},\underline{J}].$$

Furthermore, $|\overline{\underline{\ \ }}|$ is closed:

$$\mathrm{hom}(K,X) = [\iota \circ \pi_1 \circ \mathrm{cat}\ K,X].$$

4.4.19 LEMMA Suppose that the simplicial action $|\overline{\underline{\ \ }}|$ on $\underline{C}$ is closed — then

$$\mathrm{HOM}(X|\overline{\underline{\ \ }}|K,Y) \approx \mathrm{map}(K,\mathrm{HOM}(X,Y)) \approx \mathrm{HOM}(X,\mathrm{hom}(K,Y)).$$

4.4.20 REMARK From the perspective of enriched category theory, this just means that the S-category $|\overline{\underline{\ \ }}|\underline{C}$ is "tensored" and "cotensored" (cf. 4.7.14).

4.4.21 LEMMA Suppose that $|\overline{\underline{\ \ }}|$ is a closed simplicial action on $\underline{C}$. Assume: $K = \mathrm{colim}_i\ K_i$ — then $\forall\ X,Y \in \mathrm{Ob}\ \underline{C}$,

$$\mathrm{Mor}(X,\mathrm{hom}(\mathrm{colim}_i\ K_i,Y)) \approx \lim_i\ \mathrm{Mor}(X,\mathrm{hom}(K_i,Y)).$$

PROOF In fact,

$$\begin{aligned}
\mathrm{LHS} &\approx \mathrm{Mor}(X|\overline{\underline{\ \ }}|\mathrm{colim}_i\ K_i,Y) \\
&\approx \mathrm{Mor}(\mathrm{colim}_i\ X|\overline{\underline{\ \ }}|K_i,Y)
\end{aligned}$$

$$\approx \lim_i \operatorname{Mor}(X|\overline{}|K_i, Y) \approx \text{RHS}.$$

4.4.22 NOTATION Let $\underline{C}$ be a complete category. Given a simplicial object X in $\underline{C}$ and a simplicial set K, put

$$X \pitchfork K = \int_{[n]} (X_n)^{K_n},$$

an object in $\underline{C}$.

4.4.23 EXAMPLE In view of the integral Yoneda lemma,

$$X \approx \int_{[k]} (X_k)^{\operatorname{Mor}([k],—)}.$$

Therefore

$$\begin{aligned} X_n &\approx \int_{[k]} (X_k)^{\operatorname{Mor}([k],[n])} \\ &\approx \int_{[k]} (X_k)^{\Delta[n]([k])} \\ &\approx \int_{[k]} (X_k)^{\Delta[n]_k} \\ &\approx X \pitchfork \Delta[n]. \end{aligned}$$

[Note: We have

$$M_n X \approx X \pitchfork \dot{\Delta}[n] \quad \text{(cf. 0.27.22).}$$

And the inclusion $\dot{\Delta}[n] \to \Delta[n]$ induces the canonical arrow $X_n \to M_n X$.]

4.4.24 EXAMPLE $\forall\ X \in \operatorname{Ob} \underline{C}\ \&\ \forall\ Y \in \operatorname{Ob} \underline{\mathrm{SIC}}$,

$$\operatorname{Mor}(X, Y \pitchfork K) \approx \operatorname{Mor}(X, \int_{[n]} (Y_n)^{K_n})$$

$$\approx \int_{[n]} \mathrm{Mor}(X, (Y_n)^{K_n})$$

$$\approx \int_{[n]} \mathrm{Mor}(X, Y_n)^{K_n}$$

$$\approx \int_{[n]} \mathrm{Mor}(K_n, \mathrm{Mor}(X, Y_n)).$$

Suppose that $|\overline{}|$ is a closed simplicial action on $\underline{C}$ — then there is a functor $\underline{C} \to \underline{SIC}$ that sends an object X in $\underline{C}$ to $X^{\Delta[\ ]}$, where

$$X^{\Delta[\ ]}([n]) = \hom(\Delta[n], X).$$

4.4.25 THEOREM Suppose that $|\overline{}|$ is a closed simplicial action on $\underline{C}$. Assume: $\underline{C}$ is complete — then

$$\hom(K, X) \approx X^{\Delta[\ ]} \pitchfork K.$$

PROOF $\forall$ X,Y $\in$ Ob $\underline{C}$,

$$\mathrm{Mor}(X, Y^{\Delta[\ ]} \pitchfork K) \approx \mathrm{Mor}(X, \int_{[n]} (Y^{\Delta[\ ]})_n{}^{K_n})$$

$$\approx \mathrm{Mor}(X, \int_{[n]} \hom(\Delta[n], Y)^{K_n})$$

$$\approx \int_{[n]} \mathrm{Mor}(X, \hom(\Delta[n], Y)^{K_n})$$

$$\approx \int_{[n]} \mathrm{Mor}(X, \hom(\Delta[n], Y))^{K_n}$$

$$\approx \int_{[n]} \mathrm{Mor}(X|\overline{}|\Delta[n], Y)^{K_n}$$

$$\approx \int_{[n]} \mathrm{Mor}(K_n, \mathrm{Mor}(X|\overline{}|\Delta[n], Y))$$

$$\approx \int_{[n]} \mathrm{Mor}(K_n, \mathrm{HOM}(X, Y)_n) \quad (\text{cf. } 4.4.8)$$

$$\approx \mathrm{Nat}(K,\mathrm{HOM}(X,Y))$$

$$\approx \mathrm{map}(K,\mathrm{HOM}(X,Y))_0$$

$$\approx \mathrm{HOM}(X|\overline{\underline{\ }}|K,Y)_0 \quad (\text{cf. } 4.4.19)$$

$$\approx \mathrm{Mor}((X|\overline{\underline{\ }}|K)|\overline{\underline{\ }}|\Delta[0],Y)$$

$$\approx \mathrm{Mor}(X|\overline{\underline{\ }}|(K \times \Delta[0]),Y)$$

$$\approx \mathrm{Mor}(X|\overline{\underline{\ }}|K,Y)$$

$$\approx \mathrm{Mor}(X,\mathrm{hom}(K,Y)).$$

4.4.26 NOTATION Given a category $\underline{C}$ and a simplicial object X in $\underline{C}$, write h_X for the functor $\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}$ defined by $(h_X A)_n = \mathrm{Mor}(A,X_n)$.

[Note: For all $X,Y \in \mathrm{Ob}\ \underline{\mathrm{SIC}}$,

$$\mathrm{Nat}(X,Y) \approx \mathrm{Nat}(h_X,h_Y) \text{ (simplicial Yoneda).]}$$

4.4.27 THEOREM Let $\underline{C}$ be a category. Assume: $\underline{C}$ has coproducts and is complete -- then the canonical simplicial action $|\overline{\underline{\ }}|$ on $\underline{\mathrm{SIC}}$ is closed ($|\overline{\underline{\ }}|$ is necessarily cartesian (cf. 4.4.13)).

PROOF Given a simplicial set K, write

$$K \times \Delta[n] \approx \mathrm{colim}_i\ \Delta[n_i].$$

Then $\forall\ A \in \mathrm{Ob}\ \underline{C}$,

$$\begin{aligned}\mathrm{Nat}(K \times \Delta[n],h_X A) &\approx \lim_i \mathrm{Nat}(\Delta[n_i],h_X A)\\ &\approx \lim_i \mathrm{Mor}(A,X_{n_i})\\ &\approx \mathrm{Mor}(A,\lim_i X_{n_i})\\ &\approx \mathrm{Mor}(A,\mathrm{hom}(K,X)_n),\end{aligned}$$

where by definition,

$$\mathrm{hom}(K,X)_n = \lim_i X_{n_i}.$$

In other words, $\mathrm{hom}(K,X)_n$ represents

$$A \to \mathrm{Nat}(K \times \Delta[n], h_X A).$$

Varying n yields a simplicial object $\mathrm{hom}(K,X)$ in $\underline{C}$ with

$$h_{\mathrm{hom}(K,X)} \approx \mathrm{map}(K,h_X).$$

Agreeing to let $h_X|\overline{\underline{\ \ }}|K$ be the cofunctor $\underline{C} \to \underline{\mathrm{SISET}}$ that sends A to $h_X A \times K$, we have

$$\begin{aligned}
\mathrm{Nat}(X|\overline{\underline{\ \ }}|K,Y) &\approx \mathrm{Nat}(h_{X|\overline{\underline{\ \ }}|K}, h_Y) \\
&\approx \mathrm{Nat}(h_X|\overline{\underline{\ \ }}|K, h_Y) \\
&\approx \mathrm{Nat}(h_X, \mathrm{map}(K,h_Y)) \\
&\approx \mathrm{Nat}(h_X, h_{\mathrm{hom}(K,Y)}) \\
&\approx \mathrm{Nat}(X, \mathrm{hom}(K,Y)),
\end{aligned}$$

which proves that $|\overline{\underline{\ \ }}|$ is closed.

4.4.28 EXAMPLE The canonical simplicial action $|\overline{\underline{\ \ }}|$ on $\underline{\mathrm{SIGR}}$ or $\underline{\mathrm{SIAB}}$ is closed.

4.4.29 REMARK If $|\overline{\underline{\ \ }}|$ is a closed simplicial action on $\underline{C}$, then the composition

$$[\underline{\Delta}^{\mathrm{OP}},\underline{C}] \times \underline{\mathrm{SISET}} \to [\underline{\Delta}^{\mathrm{OP}},\underline{C}] \times [\underline{\Delta}^{\mathrm{OP}},\underline{\mathrm{SISET}}]$$

$$\approx [\underline{\Delta}^{\mathrm{OP}},\underline{C} \times \underline{\mathrm{SISET}}] \xrightarrow{[\underline{\Delta}^{\mathrm{OP}},|\overline{\underline{\ \ }}|]} [\underline{\Delta}^{\mathrm{OP}},\underline{C}]$$

is a closed simplicial action on $[\underline{\Delta}^{\mathrm{OP}},\underline{C}] \cong \underline{\mathrm{SIC}}$. When $\underline{C}$ has coproducts and is complete, the canonical simplicial action on $\underline{\text{SIC}}$ is also closed. However, in general, these two actions are not the same.

Let K be a simplicial set. Assume: $\underline{C}$ has coproducts — then K determines a functor

$$K\cdot\text{---}:\underline{C} \to \underline{\text{SIC}}$$

by writing

$$(K\cdot X)([n]) = K_n\cdot X.$$

4.4.30 LEMMA Assume: $\underline{C}$ has coproducts and is complete — then $K\cdot\text{---}$ is a left adjoint for

$$\text{---} \pitchfork K:\underline{\text{SIC}} \to \underline{C}.$$

PROOF $\forall\ X \in \text{Ob}\ \underline{C}\ \&\ \forall\ Y \in \text{Ob}\ \underline{\text{SIC}}$,

$$\begin{aligned}
\text{Nat}(K\cdot X,Y) &\approx \int_{[n]} \text{Mor}(K_n\cdot X,Y_n)\\
&\approx \int_{[n]} \text{Mor}(X,Y_n)^{K_n}\\
&\approx \int_{[n]} \text{Mor}(X,(Y_n)^{K_n})\\
&\approx \text{Mor}(X,\int_{[n]} (Y_n)^{K_n})\\
&\approx \text{Mor}(X,Y \pitchfork K).
\end{aligned}$$

4.4.31 LEMMA Assume: $\underline{C}$ has coproducts and is complete. Suppose that $K = \text{colim}_i\ K_i$ — then for every simplicial object X in $\underline{C}$,

$$X \pitchfork K \approx \lim_i\ X \pitchfork K_i.$$

PROOF Given $A \in \text{Ob } \underline{C}$, let $\underline{A} \in \text{Ob } \underline{\text{SIC}}$ be the constant simplicial object determined by A, thus

$$\begin{aligned}
\text{Mor}(A, X \pitchfork K) &\approx \text{Mor}(K \cdot A, X) \\
&\approx \text{Mor}(\underline{A}\,|\overline{\underline{\ \ }}|\,K, X) \\
&\approx \text{Mor}(\text{colim}_i\ \underline{A}\,|\overline{\underline{\ \ }}|\,K_i, X) \\
&\approx \lim_i\ \text{Mor}(\underline{A}\,|\overline{\underline{\ \ }}|\,K_i, X) \\
&\approx \lim_i\ \text{Mor}(K_i \cdot A, X) \\
&\approx \lim_i\ \text{Mor}(A, X \pitchfork K_i) \\
&\approx \text{Mor}(A, \lim_i\ X \pitchfork K_i).
\end{aligned}$$

4.4.32 LEMMA Assume: $\underline{C}$ has coproducts and is complete — then

$$\text{hom}(K,X)_n \approx X \pitchfork (K \times \Delta[n]).$$

PROOF Write

$$K \times \Delta[n] = \text{colim}_i\ \Delta[n_i].$$

Then

$$\begin{aligned}
X \pitchfork (K \times \Delta[n]) &\approx \lim_i\ X \pitchfork \Delta[n_i] \quad (\text{cf. } 4.4.31) \\
&\approx \lim_i\ X_{n_i} \quad (\text{cf. } 4.4.23) \\
&\approx \text{hom}(K,X)_n.
\end{aligned}$$

4.4.33 EXAMPLE Under the preceding assumptions on $\underline{C}$, for all simplicial sets K and L,

$$\text{hom}(K,X) \pitchfork L \approx X \pitchfork (K \times L).$$

4.5 *SMC*

4.5.1 DEFINITION A simplicial model category is a model category $\underline{C}$ equipped with a closed simplicial action $|\overline{\underline{\ \ }}|$ satisfying

(SMC) Suppose that $A \to Y$ is a cofibration and $X \to B$ is a fibration — then the arrow

$$\mathrm{HOM}(Y,X) \to \mathrm{HOM}(A,X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y,B)$$

is a Kan fibration which is a simplicial weak equivalence if $A \to Y$ or $X \to B$ is acyclic.

[Note: Associated with $|\overline{\underline{\ \ }}|$ is an S-category $|\overline{\underline{\ \ }}|\underline{C}$ such that $U|\overline{\underline{\ \ }}|\underline{C}$ is isomorphic to $\underline{C}$ (cf. 4.4.8).]

N.B.

- If A is cofibrant, then the arrow

$$\mathrm{HOM}(A,X) \to \mathrm{HOM}(A,B)$$

is a Kan fibration. Therefore the pullback square

$$\begin{tikzcd}
\mathrm{HOM}(A,X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y,B) \arrow[r] \arrow[d] & \mathrm{HOM}(Y,B) \arrow[d] \\
\mathrm{HOM}(A,X) \arrow[r] & \mathrm{HOM}(A,B)
\end{tikzcd}$$

is a homotopy pullback (cf. 0.35.1).

- If B is fibrant, then the arrow

$$\mathrm{HOM}(Y,B) \to \mathrm{HOM}(A,B)$$

is a Kan fibration. Therefore the pullback square

$$\begin{tikzcd}
\mathrm{HOM}(A,X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y,B) \arrow[r] \arrow[d] & \mathrm{HOM}(Y,B) \arrow[d] \\
\mathrm{HOM}(A,X) \arrow[r] & \mathrm{HOM}(A,B)
\end{tikzcd}$$

is a homotopy pullback (cf. 0.35.1).

4.5.2 EXAMPLE Take $\underline{C}$ = SISET (Kan Structure) and take $|\overline{\underline{\ \ }}|$ per 4.4.9 -- then $|\overline{\underline{\ \ }}|$ is closed and SISET is a simplicial model category.

[Note: SISET is also a simplicial model category if the Kan structure is replaced by the HG-structure but it is not a simplicial model category if the Kan structure is replaced by the Joyal structure.]

4.5.3 EXAMPLE Take $\underline{C}$ = CGH (Quillen Structure) and take $|\overline{\underline{\ \ }}|$ per 4.4.10 -- then $|\overline{\underline{\ \ }}|$ is closed and CGH is a simplicial model category.

4.5.4 EXAMPLE Take $\underline{C}$ = CAT (External Structure) and take $|\overline{\underline{\ \ }}|$ per 4.4.17 -- then $|\overline{\underline{\ \ }}|$ is closed and CAT is a simplicial model category.

4.5.5 EXAMPLE Take $\underline{C}$ = CAT (Internal Structure) and take $|\overline{\underline{\ \ }}|$ per 4.4.18 -- then $|\overline{\underline{\ \ }}|$ is closed and CAT is a simplicial model category.

4.5.6 REMARK It is not clear whether S-CAT (Bergner Structure) admits a closed simplicial action making it a simplicial model category.

4.5.7 EXAMPLE Take $\underline{C}$ = [$\underline{I}$,SISET] (Structure L) and take $|\overline{\underline{\ \ }}|$ per 4.4.7 -- then $|\overline{\underline{\ \ }}|$ is closed and [$\underline{I}$,SISET] is a simplicial model category.

4.5.8 LEMMA In a simplicial model category $\underline{C}$: (1) $X|\overline{\underline{\ \ }}|\Delta[0] \approx X$; (2) $\hom(\Delta[0],X) \approx X$; (3) $\emptyset|\overline{\underline{\ \ }}|K \approx \emptyset$; (4) $\hom(K,*) \approx *$; (5) $\mathrm{HOM}(\emptyset,X) \approx \Delta[0]$; (6) $\mathrm{HOM}(X,*) \approx \Delta[0]$; (7) $X|\overline{\underline{\ \ }}|\emptyset \approx \emptyset$; (8) $\hom(\emptyset,X) \approx *$.

What follows is strictly sorital... .

4.5.9 LEMMA Suppose that $|\overline{}|$ is a closed simplicial action on a model category $\underline{C}$ -- then $\underline{C}$ is a simplicial model category iff whenever $A \to Y$ is a cofibration in $\underline{C}$ and $L \to K$ is an inclusion of simplicial sets, the arrow

$$A|\overline{}|K \underset{A|\overline{}|L}{\sqcup} Y|\overline{}|L \to Y|\overline{}|K$$

is a cofibration which is acyclic if $A \to Y$ or $L \to K$ is acyclic.

4.5.10 APPLICATION Let $\underline{C}$ be a simplicial model category.

(i) Suppose that $A \to Y$ is a cofibration in $\underline{C}$ -- then for every simplicial set K, the arrow $A|\overline{}|K \to Y|\overline{}|K$ is a cofibration which is acyclic if $A \to Y$ is acyclic.

(ii) Suppose that Y is cofibrant and $L \to K$ is an inclusion of simplicial sets -- then the arrow $Y|\overline{}|L \to Y|\overline{}|K$ is a cofibration which is acyclic if $L \to K$ is acyclic.

[Note: In particular, Y cofibrant => $Y|\overline{}|K$ cofibrant.]

4.5.11 CRITERION Suppose that $|\overline{}|$ is a closed simplicial action on a model category $\underline{C}$ -- then $\underline{C}$ is a simplicial model category iff whenever $A \to Y$ is a cofibration in $\underline{C}$, the arrows

$$A|\overline{}|\Delta[n] \underset{A|\overline{}|\dot{\Delta}[n]}{\sqcup} Y|\overline{}|\dot{\Delta}[n] \to Y|\overline{}|\Delta[n] \qquad (n \geq 0)$$

are cofibrations which are acyclic if $A \to Y$ is acyclic and the arrows

$$A|\overline{}|\Delta[1] \underset{A|\overline{}|\Lambda[i,1]}{\sqcup} Y|\overline{}|\Lambda[i,1] \to Y|\overline{}|\Delta[1] \qquad (i = 0,1)$$

are acyclic cofibrations.

4.5.12 LEMMA Suppose that $|\overline{}|$ is a closed simplicial action on a model category $\underline{C}$ -- then $\underline{C}$ is a simplicial model category iff whenever $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a fibration in $\underline{C}$, the arrow

$$\text{hom}(K,X) \to \text{hom}(L,X) \times_{\text{hom}(L,B)} \text{hom}(K,B)$$

is a fibration which is acyclic if $L \to K$ or $X \to B$ is acyclic.

4.5.13 APPLICATION Let $\underline{C}$ be a simplicial model category.

(i) Suppose that $L \to K$ is an inclusion of simplicial sets and X is fibrant -- then the arrow $\text{hom}(K,X) \to \text{hom}(L,X)$ is a fibration which is acyclic if $L \to K$ is acyclic.

(ii) Suppose that $X \to B$ is a fibration in $\underline{C}$ -- then for every simplicial set K, the arrow $\text{hom}(K,X) \to \text{hom}(K,B)$ is a fibration which is acyclic if $X \to B$ is acyclic.

[Note: In particular, X fibrant => $\text{hom}(K,X)$ fibrant.]

4.5.14 CRITERION Suppose that $|\overline{}|$ is a closed simplicial action on a model category $\underline{C}$ -- then $\underline{C}$ is a simplicial model category iff whenever $X \to B$ is a fibration in $\underline{C}$, the arrows

$$\text{hom}(\Delta[n],X) \to \text{hom}(\dot{\Delta}[n],X) \times_{\text{hom}(\dot{\Delta}[n],B)} \text{hom}(\Delta[n],B) \quad (n \geq 0)$$

are fibrations which are acyclic if $X \to B$ is acyclic and the arrows

$$\text{hom}(\Delta[1],X) \to \text{hom}(\Lambda[i,1],X) \times_{\text{hom}(\Lambda[i,1],B)} \text{hom}(\Delta[1],B) \quad (i = 0,1)$$

are acyclic fibrations.

Apart from these structural formalities, there are a few things to be said about the weak equivalences.

4.5.15 LEMMA Let X,Y, and Z be objects in a simplicial model category $\underline{C}$.

(i) If $f:X \to Y$ is an acyclic cofibration and Z is fibrant, then $f^*:\mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence.

(ii) If $g:Y \to Z$ is an acyclic fibration and X is cofibrant, then $g_*:\mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence.

4.5.16 LEMMA Let X,Y, and Z be objects in a simplicial model category $\underline{C}$.

(i) If $f:X \to Y$ is a weak equivalence between cofibrant objects and Z is fibrant, then $f^*:\mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence.

(ii) If $g:Y \to Z$ is a weak equivalence between fibrant objects and X is cofibrant, then $g_*:\mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence.

4.5.17 EXAMPLE Take $\underline{C} = \underline{\mathrm{CGH}}$ (Quillen Structure) — then all objects are fibrant, so if $g:Y \to Z$ is a weak homotopy equivalence and X is cofibrant, then $g_*:\mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence. But

$$\left[\begin{array}{l} \mathrm{HOM}(X,Y) \approx \sin(Y^X) \\ \\ \mathrm{HOM}(X,Z) \approx \sin(Z^X) \end{array}\right. \qquad \text{(cf. 4.4.10)},$$

thus $g_*:Y^X \to Z^X$ is a weak homotopy equivalence.

[Note: There is a commutative diagram

$$\begin{array}{ccc} |\sin(Y^X)| & \longrightarrow & |\sin(Z^X)| \\ \downarrow & & \downarrow \\ Y^X & \longrightarrow & Z^X \end{array}$$

and the vertical arrows are weak homotopy equivalences.]

4.5.18 THEOREM Let $\underline{C}$ be a simplicial model category — then a morphism $f:X \to Y$ is a weak equivalence if for every fibrant Z, $f^*:\mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$ is a simplicial weak equivalence.

[Note: The result can also be formulated in terms of the arrows $g_*:\mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Z)$ (X cofibrant).]

4.5.19 APPLICATION Let $\underline{C}$ be a simplicial model category. Suppose that $f:X \to Y$ is a weak equivalence between cofibrant objects — then $\forall$ K,

$$f \square \mathrm{id}_K : X \square K \to Y \square K$$

is a weak equivalence between cofibrant objects (cf. 4.5.10).

[Take any fibrant Z and consider the arrow

$$\mathrm{HOM}(Y \square K, Z) \to \mathrm{HOM}(X \square K, Z)$$

or still, the arrow

$$\mathrm{HOM}(Y, \mathrm{hom}(K,Z)) \to \mathrm{HOM}(X, \mathrm{hom}(K,Z)).$$

Because $\mathrm{hom}(K,Z)$ is fibrant (cf. 4.5.13), the latter is a simplicial weak equivalence (cf. 4.5.16), hence the same is true of the former. Therefore $f \square \mathrm{id}_K$ is a weak equivalence (cf. 4.5.18).]

4.5.20 EXAMPLE Fix a small category $\underline{I}$ and view the functor category $[\underline{I}^{OP}, \underline{SISET}]$ as a simplicial model category (cf. 4.5.7). Suppose that $L \to K$ is a weak equivalence, where $L,K:\underline{I}^{OP} \to \underline{SISET}$ are cofibrant — then $\forall$ $f:\underline{I} \to \underline{SISET}$, the induced map

$$\int^i Li \times Fi \to \int^i Ki \times Fi$$

of simplicial sets is a simplicial weak equivalence.

[To see this, use 4.5.18. Thus take any fibrant Z and consider the arrow

$$\mathrm{map}(\textstyle\int^i Ki \times Fi, Z) \to \mathrm{map}(\textstyle\int^i Li \times Fi, Z),$$

i.e., the arrow

$$\textstyle\int_i \mathrm{map}(Ki \times Fi, Z) \to \textstyle\int_i \mathrm{map}(Li \times Fi, Z),$$

i.e., the arrow

$$\textstyle\int_i \mathrm{map}(Ki, \mathrm{map}(Fi, Z)) \to \textstyle\int_i \mathrm{map}(Li, \mathrm{map}(Fi, Z)),$$

i.e., the arrow

$$\mathrm{HOM}(K, \mathrm{map}(F,Z)) \to \mathrm{HOM}(L, \mathrm{map}(F,Z)) \quad (\text{cf. } 4.4.9),$$

which is a simplicial weak equivalence (cf. 4.5.16).]

[Note: Here map(F,Z) is the functor $\underline{I}^{OP} \to \underline{SISET}$ defined by $i \to \mathrm{map}(Fi,Z)$, thus map(F,Z) is a fibrant object in $[\underline{I}^{OP}, \underline{SISET}]$.]

4.6 SIC

Let $\underline{C}$ be a category. Assume: $\underline{C}$ is complete and cocomplete and there is an adjoint pair (F,G), where

$$\left[\begin{array}{l} F:\underline{SISET} \to \underline{SIC} \\ G:\underline{SIC} \to \underline{SISET}, \end{array}\right.$$

subject to the requirement that G preserves filtered colimits.

4.6.1 THEOREM Call a morphism $f:X \to Y$ a weak equivalence if Gf is a simplicial weak equivalence, a fibration if Gf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations -- then with these choices, $\underline{SIC}$ is a model category provided that every cofibration with the LLP w.r.t. fibrations is a weak equivalence (cf. infra).

<u>N.B.</u> This result is an instance of the overall theme of "transfer of structure". Thus one works with the $F\dot{\Delta}[n] \to F\Delta[n]$ $(n \geq 0)$ to show that every f can be written as the composite of a cofibration and an acyclic fibration and one works with the $F\Lambda[k,n] \to F\Delta[n]$ $(0 \leq k \leq n, n \geq 1)$ to show that every f can be written as the composite of a cofibration that has the LLP w.r.t. fibrations and a fibration. This leads to MC-5 under the assumption that every cofibration with the LLP w.r.t. fibrations is a weak equivalence, which is also needed to establish the nontrivial half of MC-4. In practice, this condition can be forced.

4.6.2 SUBLEMMA Let $\left[\begin{array}{l} X \\ Y \end{array}\right.$ be topological spaces, $f:X \to Y$ a continuous function; let $\phi:X' \to X$, $\psi:Y \to Y'$ be continuous functions. Assume: $f \circ \phi, \psi \circ f$ are weak homotopy equivalences -- then f is a weak homotopy equivalence.

4.6.3 LEMMA Suppose that there is a functor $T:\underline{\text{SIC}} \to \underline{\text{SIC}}$ and a natural transformation $\varepsilon:\text{id}_{\underline{\text{SIC}}} \to T$ such that $\forall$ X, $\varepsilon_X:X \to TX$ is a weak equivalence and $TX \to *$ is a fibration -- then every cofibration with the LLP w.r.t. fibrations is a weak equivalence.

PROOF Let $i:A \to Y$ be a cofibration with the stated properties. Fix a filler $w:Y \to TA$ for

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "i"'] & TA \arrow[d] \\ Y \arrow[r] & * . \end{tikzcd}$$

Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & \hom(\Delta[1],TY) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Pi} \\ Y & \xrightarrow[g]{} & \hom(\dot{\Delta}[1],TY), \end{array}$$

where f is the arrow

$$A \xrightarrow{i} Y \xrightarrow{\varepsilon_Y} TY \approx \hom(\Delta[0],TY) \longrightarrow \hom(\Delta[1],TY)$$

and g is the arrow

$$\begin{cases} Y \xrightarrow{\varepsilon_Y} TY \\ Y \xrightarrow{w} TA \xrightarrow{Ti} TY \end{cases} \qquad (\hom(\dot{\Delta}[1],TY) \approx TY \times TY).$$

Since GTY is fibrant and

$$\begin{cases} G\hom(\Delta[1],Y) \approx \operatorname{map}(\Delta[1],GTY) \\ G\hom(\dot{\Delta}[1],Y) \approx \operatorname{map}(\dot{\Delta}[1],GTY), \end{cases}$$

it follows that Π is a fibration, thus our diagram admits a filler

$$H:Y \to \hom(\Delta[1],TY).$$

But ε_Y is a weak equivalence, hence $Ti \circ w$ is a weak equivalence, i.e., $|GTi| \circ |Gw|$ is a weak homotopy equivalence. Assemble the data:

$$|GA| \xrightarrow{|Gi|} |GY| \xrightarrow{|Gw|} |GTA| \xrightarrow{|Gti|} |GTY|.$$

Because $|Gw| \circ |Gi| = |G\varepsilon_A|$ is a weak homotopy equivalence, one can apply the sublemma and conclude that $|Gw|$ is a weak homotopy equivalence. Therefore $|Gi|$

is a weak homotopy equivalence which means by definition that i is a weak equivalence.

4.6.4 RAPPEL Suppose that $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a Kan fibration -- then the arrow

$$\text{map}(K,X) \to \text{map}(L,X) \times_{\text{map}(L,B)} \text{map}(K,B)$$

is a Kan fibration which is a simplicial weak equivalence if this is the case of $L \to K$ or $X \to B$.

4.6.5 THEOREM Equip $\underline{\text{SIC}}$ with its model structure per 4.6.1 and let $\square$ = canonical simplicial action (cf. 4.4.11) — then $\underline{\text{SIC}}$ is a simplicial model category.

PROOF Thanks to 4.4.27, $\square$ is closed. This said, we have

$$\text{Ghom}(K,Y) \approx \text{map}(K,GY).$$

Proof:

- $$\begin{aligned}\text{Nat}(F(X \times K),Y) &\approx \text{Nat}(X \times K,GY)\\ &\approx \text{Nat}(X,\text{map}(K,GY)).\end{aligned}$$
- $$\begin{aligned}\text{Nat}(FX \square K,Y) &\approx \text{Nat}(FX,\text{hom}(K,Y))\\ &\approx \text{Nat}(X,\text{Ghom}(K,Y)).\end{aligned}$$

Let now $L \to K$ be an inclusion of simplicial sets and $X \to B$ a fibration in $\underline{\text{SIC}}$. Apply G to the arrow

$$\text{hom}(K,X) \to \text{hom}(L,X) \times_{\text{hom}(L,B)} \text{hom}(K,B)$$

to get

$$\text{Ghom}(K,X) \to \text{Ghom}(L,X) \times_{\text{Ghom}(L,B)} \text{Ghom}(K,B)$$

or still,

$$\text{map}(K,GX) \to \text{map}(L,GX) \times_{\text{map}(L,GB)} \text{map}(K,GB).$$

Taking into account 4.6.4 and the definitions, it remains only to quote 4.5.12.

4.6.6 EXAMPLE The hypotheses of 4.6.3 are trivially met if $\forall$ X, $X \to *$ is a fibration. So, for instance, $\underline{\text{SIC}}$ is a simplicial model category if $\underline{C} = \underline{\text{GR}}$ or $\underline{\text{AB}}$ (cf. 4.4.28).

4.6.7 CONSTRUCTION Retaining the supposition that $\underline{C}$ is complete and cocomplete, let us assume in addition that $\underline{C}$ has a set of separators and is cowellpowered. Given a simplicial object X in $\underline{C}$, the functor $\underline{C}^{OP} \to \underline{\text{SET}}$ defined by $A \to (\text{ExHOM}(A,X))_n$ $(n \geq 0)$ is representable (view A as a constant simplicial object). Indeed, HOM(—,X) converts colimits into limits and Ex preserves limits. The assertion is then a consequence of the special adjoint functor theorem. Accordingly, $\exists$ an object $(\text{Ex } X)_n$ in $\underline{C}$ and a natural isomorphism $\text{Mor}(A,(\text{Ex } X)_n) \approx (\text{ExHOM}(A,X))_n$. Thus there is a functor $\text{Ex}:\underline{\text{SIC}} \to \underline{\text{SIC}}$, where $\forall$ X, $\text{Ex } X([n]) = (\text{Ex } X)_n$ $(n \geq 0)$, with $\text{HOM}(A,\text{Ex } X) \approx \text{ExHOM}(A,X)$ (since $\text{HOM}(A,\text{Ex } X)_n \approx \text{Nat}(A|\overline{\underline{\ }}|\Delta[n],\text{Ex } X) \approx \text{Mor}(A,(\text{Ex } X)_n) \approx (\text{ExHOM}(A,X))_n$). Iterate to arrive at $\text{Ex}^\infty:\underline{\text{SIC}} \to \underline{\text{SIC}}$ and $\varepsilon^\infty:\text{id}_{\underline{\text{SIC}}} \to \text{Ex}^\infty$. Now fix a $P \in \text{Ob } \underline{C}$ such that $\text{Mor}(P,—):\underline{C} \to \underline{\text{SET}}$ preserves filtered colimits. Viewing P as a constant simplicial object, define $G:\underline{\text{SIC}} \to \underline{\text{SISET}}$ by $GX = \text{HOM}(P,X)$ -- then G has a left adjoint F, viz. $FK = P|\overline{\underline{\ }}|K$, and G preserves filtered colimits:

$$\begin{aligned}
(G \text{ colim } X_i)_n &\approx \text{HOM}(P,\text{colim } X_i)_n \\
&\approx \text{Nat}(P|\overline{\underline{\ }}|\Delta[n],\text{colim } X_i) \\
&\approx \text{Mor}(P,(\text{colim } X_i)_n) \\
&\approx \text{Mor}(P,\text{colim}(X_i)_n)
\end{aligned}$$

$$\begin{aligned}
&\approx \operatorname{colim} \operatorname{Mor}(P,(X_i)_n)\\
&\approx \operatorname{colim} \operatorname{Nat}(P|\underline{\ \ }|\Delta[n],X_i)\\
&\approx \operatorname{colim} \operatorname{HOM}(P,X_i)_n\\
&\approx (\operatorname{colim} GX_i)_n.
\end{aligned}$$

In 4.6.3, take $T = Ex^\infty$, $\varepsilon = \varepsilon^\infty$. Since

$$\begin{aligned}
\operatorname{HOM}(P,Ex^\infty X) &\approx \operatorname{HOM}(P,\operatorname{colim} Ex^n X)\\
&\approx \operatorname{colim} \operatorname{HOM}(P,Ex^n X)\\
&\approx Ex^\infty \operatorname{HOM}(P,X),
\end{aligned}$$

it follows that $\forall\ X$, $\varepsilon_X^\infty:X \to Ex^\infty X$ is a weak equivalence and $Ex^\infty X \to *$ is a fibration. Therefore $\underline{SIC}$ admits the structure of a simplicial model category in which a morphism $f:X \to Y$ is a weak equivalence or a fibration if this is the case of the simplicial map $f_*:\operatorname{HOM}(P,X) \to \operatorname{HOM}(P,Y)$.

4.6.7 EXAMPLE In the small object construction, take $\underline{C} = \underline{SISET}$ — then every finite simplicial set P determines a simplicial model category structure on $[\underline{\Delta}^{OP},\underline{SISET}]$.

4.6.8 RAPPEL Let $\underline{C}$ be a complete and cocomplete model category — then $\underline{SIC}$ in the Reedy structure is a model category (cf. 0.27.28).

[Note: For the record, if $f:X \to Y$ is a morphism in $\underline{SIC}$, then f is a weak equivalence if $\forall\ n$, $f_n:X_n \to Y_n$ is a weak equivalence in $\underline{C}$, a cofibration if $\forall\ n$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \to Y_n$ is a cofibration in $\underline{C}$, a fibration if $\forall$ n, the arrow $X_n \to M_nX \times_{M_nY} Y_n$ is a fibration in $\underline{C}$.]

4.6.9 LEMMA Suppose further that $\underline{C}$ is a simplicial model category. Equip $\underline{SIC}$ with the closed simplicial action derived from that on $\underline{C}$ (cf. 4.4.29) -- then $\underline{SIC}$ (Reedy Structure) is a simplicial model category.

PROOF It will be convenient to employ 4.5.9. So let $A \to Y$ be a cofibration in $\underline{SIC}$ and let $L \to K$ be an inclusion of simplicial sets -- then the claim is that the arrow

$$A|\boxed{}|K \underset{A|\boxed{}|L}{\sqcup} Y|\boxed{}|L \to Y|\boxed{}|K$$

is a cofibration which is acyclic if $A \to Y$ or $L \to K$ is acyclic. Thus fix n and consider the arrow

$$(A|\boxed{}|K \underset{A|\boxed{}|L}{\sqcup} Y|\boxed{}|L)_n \underset{L_n(A|\boxed{}|K \underset{A|\boxed{}|L}{\sqcup} Y|\boxed{}|L)}{\sqcup} L_n(Y|\boxed{}|K) \to (Y|\boxed{}|K)_n$$

or, equivalently, the arrow

$$(A_n \underset{L_nA}{\sqcup} L_nY)|\boxed{}|K \underset{(A_n \underset{L_nA}{\sqcup} L_nY)|\boxed{}|L}{\sqcup} Y_n|\boxed{}|L \to Y_n|\boxed{}|K,$$

from which one can read off the assertion.

4.6.10 REMARK Let $|\boxed{}|$ be the canonical simplicial action on $\underline{SIC}$ -- then $|\boxed{}|$ is closed (cf. 4.4.27) but it is not compatible with the Reedy Structure on $\underline{SIC}$. Specifically: If $A \to Y$ is a cofibration in $\underline{SIC}$ and $L \to K$ is an inclusion of simplicial sets, then the arrow

$$A\square K \coprod_{A\square L} Y\square L \to Y\square K$$

is a cofibration which is acyclic if $A \to Y$ is acyclic but it need not be acyclic if $L \to K$ is acyclic (take a Reedy cofibrant A and look at the arrow $A\square\Delta[0] \to A\square\Delta[1]$ (in degree 0, this is the map $A_0 \to A_0 \coprod A_0$)).

4.7 *SIMPLICIAL DIAGRAM CATEGORIES*

Let $\mathcal{I}$ be a small S-category, $\underline{C}$ a simplicial model category -- then $\underline{C}$ can be regarded as an S-category $\mathbf{C}$ ($= |\square|\underline{C}$) (cf. 4.4.8).

4.7.1 RAPPEL $[\mathcal{I},\mathbf{C}]_S$ is the category whose objects are the elements of $\mathrm{Mor}_S(\mathcal{I},\mathbf{C})$ and whose morphisms are the S-natural transformations (cf. 4.1.10).

<u>N.B.</u> Given an S-functor $F:\mathcal{I} \to \mathbf{C}$, we have

$$\mathrm{Nat}(\mathrm{HOM}(i,j),\mathrm{HOM}(Fi,Fj)) \approx \mathrm{Mor}(Fi\square\mathrm{HOM}(i,j),Fj),$$

thus the

$$F_{i,j}:\mathrm{HOM}(i,j) \to \mathrm{HOM}(Fi,Fj)$$

can equivalently be construed as morphisms

$$F_{i,j}:Fi\square\mathrm{HOM}(i,j) \to Fj$$

in $\underline{C}$. An S-natural transformation $\Xi:F \to G$ is then a collection of morphisms $\Xi_i:Fi \to Gi$ in $\underline{C}$ such that the diagram

$$
\begin{tikzcd}
Fi\,|\square|\,\mathrm{HOM}(i,j) \arrow[r, "F_{i,j}"] \arrow[d, "\Xi_i\,|\square|\,\mathrm{id}"'] & Fj \arrow[d, "\Xi_j"] \\
Gi\,|\square|\,\mathrm{HOM}(i,j) \arrow[r, "G_{i,j}"'] & Gj
\end{tikzcd}
$$

commutes.

4.7.2 DEFINITION Let $\Xi \in \mathrm{Nat}_S(F,G)$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\, i \in |\mathbf{I}|$, $\Xi_i:Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a <u>levelwise fibration</u> if $\forall\, i \in |\mathbf{I}|$, $\Xi_i:Fi \to Gi$ is a fibration in $\underline{C}$.

- Ξ is a <u>projective cofibration</u> if it has the LLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise fibration.

4.7.3 DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise fibrations, and projective cofibrations is called the <u>projective structure</u> on $[\mathbf{I},\mathbf{C}]_S$.

4.7.4 THEOREM Suppose that $\underline{C}$ is a combinatorial simplicial model category -- then for every $\mathbf{I}$, the projective structure on $[\mathbf{I},\mathbf{C}]_S$ is a model structure that, moreover, is combinatorial.

4.7.5 DEFINITION Let $\Xi \in \mathrm{Nat}_S(F,G)$.

- Ξ is a <u>levelwise weak equivalence</u> if $\forall\, i \in |\mathbf{I}|$, $\Xi_i:Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a levelwise cofibration if $\forall\ i \in |\underline{I}|$, Ξ_i:Fi $\rightarrow$ Gi is a cofibration in $\underline{C}$.

- Ξ is an injective fibration if it has the RLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise cofibration.

4.7.6 DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise cofibrations, and injective fibrations is called the injective structure on $[\underline{I},\underline{C}]_S$.

4.7.7 THEOREM Suppose that $\underline{C}$ is a combinatorial simplicial model category -- then for every $\underline{I}$, the injective structure on $[\underline{I},\underline{C}]_S$ is a model structure that, moreover, is combinatorial.

N.B.

- Every projective cofibration is necessarily levelwise, hence is a cofibration in the injective structure.

- Every injective fibration is necessarily levelwise, hence is a fibration in the projective structure.

4.7.8 REMARK The category $[\underline{I},\underline{C}]_S$ inherits a closed simplicial action from that on $\underline{C}$ and is a simplicial model category in either the projective structure or the injective structure.

[To deal with the projective structure, use 4.5.12, the claim being that $\forall\ i \in |\underline{I}|$, the arrow

$$\text{hom}(K,Xi) \rightarrow \text{hom}(L,Xi) \times_{\text{hom}(L,Bi)} \text{hom}(K,Bi)$$

is a fibration in $\underline{C}$ which is acyclic if L $\rightarrow$ K or X $\rightarrow$ B is acyclic. But this is obvious (matters are levelwise). As for the injective structure, apply 4.5.9.]

[Note: Spelled out, given $F \in \mathrm{Mor}_S(\mathbf{I},\mathbf{C})$,

$$(F|\square|K)i = Fi|\square|K$$

and

$$\begin{aligned}(F|\square|K)_{i,j}: (F|\square|K)i|\square|\mathrm{HOM}(i,j) &\approx (Fi|\square|K)|\square|\mathrm{HOM}(i,j)\\ &\approx Fi|\square|(K \times \mathrm{HOM}(i,j))\\ &\approx Fi|\square|(\mathrm{HOM}(i,j) \times K)\\ &\approx (Fi|\square|\mathrm{HOM}(i,j))|\square|K\\ &\xrightarrow{F_{i,j}|\square|\mathrm{id}} Fj|\square|K \approx (F|\square|K)j.]\end{aligned}$$

To proceed further, it will be necessary to cite some facts from enriched category theory sticking as always to the case when the underlying symmetric monoidal category is <u>SISET</u>.

The following terms will be admitted without explanation:

S-complete
S-cocomplete.

E.g.: <u>SISET</u> is S-complete and S-cocomplete.

4.7.9 RAPPEL If $\underline{I}$ is a small category, then $[\underline{I},\underline{SET}]$ is complete and cocomplete.

4.7.10 EXAMPLE If $\mathbf{I}$ is a small S-category, then $S[\mathbf{I},\underline{SISET}]$ is S-complete and S-cocomplete.

4.7.11 THEOREM Let $\underline{I}$ be a small S-category.

- If $\mathfrak{M}$ is S-complete, then $S[\underline{I},\mathfrak{M}]$ is S-complete.
- If $\mathfrak{M}$ is S-cocomplete, then $S[\underline{I},\mathfrak{M}]$ is S-cocomplete.

4.7.12 DEFINITION Let $\mathfrak{M},\mathfrak{M}'$ be S-categories and let

$$\begin{cases} F:\mathfrak{M} \to \mathfrak{M}' \\ F':\mathfrak{M}' \to \mathfrak{M} \end{cases}$$

be S-functors — then F is a <u>left S-adjoint</u> for F' and F' is a <u>right S-adjoint</u> for F if there exist isomorphisms

$$\mathrm{HOM}(FX,X') \approx \mathrm{HOM}(X,F'X')$$

natural in $X \in O$, $X' \in O'$.

[Note: Therefore $\begin{cases} UF:U\mathfrak{M} \to U\mathfrak{M}' \\ UF':U\mathfrak{M}' \to U\mathfrak{M} \end{cases}$ is an adjoint pair.]

4.7.13 EXAMPLE Let $\underline{C}$ be a simplicial model category — then the S-functor

$$X|\overline{\underline{\ }}|\text{—}:\underline{\mathrm{SISET}} \to \mathbf{C}$$

is a left S-adjoint for

$$\mathrm{HOM}(X,\text{—}):\mathbf{C} \to \underline{\mathrm{SISET}}$$

and the S-functor

$$\text{—}|\overline{\underline{\ }}|K:\mathbf{C} \to \mathbf{C}$$

is a left S-adjoint for

$$\hom(K,\text{—}):\mathbf{C} \to \mathbf{C}.$$

[The simplicial action $|\overline{\underline{\ }}|$ on $\underline{C}$ is closed, so one can quote 4.4.19.]

4.7.14 DEFINITION Let $\mathfrak{M}$ be an S-category.

- $\mathfrak{M}$ is tensored if every S-functor

$$\mathrm{HOM}(X,\text{—}):\mathfrak{M} \to \underline{\mathrm{SISET}}$$

has a left S-adjoint.

[Note: If $\mathfrak{M}$ is tensored, then $\forall$ X & $\forall$ K, there is an object $X \otimes K \in O$ and isomorphisms

$$\mathrm{HOM}(X \otimes K,Y) \approx \mathrm{map}(K,\mathrm{HOM}(X,Y)).]$$

- $\mathfrak{M}$ is cotensored if every S-functor

$$\mathrm{HOM}(\text{—},X):\mathfrak{M}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}$$

has a left S-adjoint.

[Note: If $\mathfrak{M}$ is cotensored, then $\forall$ X & $\forall$ K, there is an object $X^K \in O$ and isomorphisms

$$\mathrm{HOM}^{\mathrm{OP}}(X^K,Y) \approx \mathrm{map}(K,\mathrm{HOM}(Y,X)).]$$

4.7.15 LEMMA Let $\mathfrak{M}$ be an S-category.

- Suppose that $\mathfrak{M}$ is tensored -- then $\forall$ K, the correspondence

$$X \to X \otimes K$$

induces an S-functor $\mathfrak{M} \to \mathfrak{M}$.

- Suppose that $\mathfrak{M}$ is cotensored -- then $\forall$ K, the correspondence

$$X \to X^K$$

induces an S-functor $\mathfrak{M} \to \mathfrak{M}$.

E.g.: $\underline{\mathrm{SISET}}$ is tensored and cotensored:

$$\begin{bmatrix} X \otimes K = X \times K \\ X^K = \mathrm{map}(K,X). \end{bmatrix}$$

4.7.16 EXAMPLE Let $\mathbf{I}$ be a small S-category -- then $S[\mathbf{I},\underline{\text{SISET}}]$ is tensored and cotensored.

[Let $F:\mathbf{I} \to \underline{\text{SISET}}$ be an S-functor.

- Given K, put

$$(F \otimes K)i = Fi \times K$$

and define

$$(F \otimes K)_{i,j}:\text{HOM}(i,j) \to \text{map}((F \otimes K)i,(F \otimes K)j)$$

by

$$\text{HOM}(i,j) \xrightarrow{F_{i,j}} \text{map}(Fi,Fj)$$

$$\xrightarrow{(\text{—} \otimes K)_{Fi,Fj}} \text{map}(Fi \times K,Fj \times K).$$

- Given K, put

$$(F^K)i = \text{map}(K,Fi)$$

and define

$$(F^K)_{i,j}:\text{HOM}(i,j) \to \text{map}((F^K)i,(F^K)j)$$

by

$$\text{HOM}(i,j) \xrightarrow{F_{i,j}} \text{map}(Fi,Fj)$$

$$\xrightarrow{((\text{—})^K)_{Fi,Fj}} \text{map}(\text{map}(K,Fi),\text{map}(K,Fj)).]$$

4.7.17 EXAMPLE S-$\underline{\text{CAT}}$ is an S-category (cf. 4.3.9). As such, it is tensored and cotensored.

[The cotensored situation is this. If K is connected, then $|\mathbf{I}^K| = |\mathbf{I}|$ and

$$\text{HOM}^{(K)}(i,j) = \text{map}(K,\text{HOM}(i,j)).$$

In general,

$$I^K = \prod_{k \in \pi_0(K)} I^{K_k},$$

where K_k is a component of K, thus

$$|I^K| = |I|^{\pi_0(K)}.]$$

[Note: Take $K = \Delta[n]$ — then

$$\mathrm{HOM}^{(\Delta[n])}(i,j) = \mathrm{map}(\Delta[n], \mathrm{HOM}(i,j))$$

$$\Rightarrow$$

$$I^{\Delta[n]} = I^{(n)}.]$$

N.B. We have

$$|I \otimes K| = |I| \times \pi_0(K) = \pi_0(K) \cdot |I|.$$

4.7.18 THEOREM Let $\mathfrak{M}$ be an S-category. Assume: $\mathfrak{M}$ is tensored and cotensored.

- $\mathfrak{M}$ is S-complete iff $U\mathfrak{M}$ is complete.
- $\mathfrak{M}$ is S-cocomplete iff $U\mathfrak{M}$ is cocomplete.

4.7.19 REMARK Let $\underline{C}$ be a category. Assume: $\underline{C}$ admits a closed simplicial action $|\overline{\underline{\ \ }}|$ — then the S-category $|\overline{\underline{\ \ }}|\underline{C}$ is tensored and cotensored (cf. 4.4.20). Recalling that $U|\overline{\underline{\ \ }}|\underline{C}$ is isomorphic to $\underline{C}$, it follows that

$$\left[\begin{array}{l} |\overline{\underline{\ \ }}|\underline{C} \text{ is S-complete iff } \underline{C} \text{ is complete} \\ \\ |\overline{\underline{\ \ }}|\underline{C} \text{ is S-cocomplete iff } \underline{C} \text{ is cocomplete.} \end{array}\right.$$

[Note: This applies in particular if $\underline{C}$ is presentable.]

4.7.20 THEOREM Let $\left[\begin{array}{l} \mathbf{I} \\ \mathbf{J} \end{array}\right.$ be small S-categories and let $\mathfrak{M}$ be a tensored and cotensored S-category. Suppose that $K:\mathbf{I} \to \mathbf{J}$ is an S-functor and

$$K^*:S[\mathbf{J},\mathfrak{M}] \to S[\mathbf{I},\mathfrak{M}]$$

is the induced S-functor.

- If $\mathfrak{M}$ is S-complete, then K^* has a right adjoint

$$K_{\dagger}:S[\mathbf{I},\mathfrak{M}] \to S[\mathbf{J},\mathfrak{M}].$$

- If $\mathfrak{M}$ is S-cocomplete, then K^* has a left adjoint

$$K_{!}:S[\mathbf{I},\mathfrak{M}] \to S[\mathbf{J},\mathfrak{M}].$$

So, if $\mathfrak{M}$ is S-complete and S-cocomplete (as well as tensored and cotensored), then

$$K^* \equiv UK^*:US[\mathbf{J},\mathfrak{M}] \to US[\mathbf{I},\mathfrak{M}]$$

has a right adjoint

$$K_{\dagger} \equiv UK_{\dagger}:US[\mathbf{I},\mathfrak{M}] \to US[\mathbf{J},\mathfrak{M}]$$

and a left adjoint

$$K_{!} \equiv UK_{!}:US[\mathbf{I},\mathfrak{M}] \to US[\mathbf{J},\mathfrak{M}].$$

But

$$\left[\begin{array}{l} US[\mathbf{I},\mathfrak{M}] \approx [\mathbf{I},\mathfrak{M}]_S \\ \\ US[\mathbf{J},\mathfrak{M}] \approx [\mathbf{J},\mathfrak{M}]_S. \end{array}\right.$$

Therefore the constituents of the setup become

$$K^*:[\mathbf{J},\mathfrak{M}]_S \to [\mathbf{I},\mathfrak{M}]_S$$

and

$$\begin{cases} K_{+}:[\mathbf{I},\mathfrak{M}]_S \to [\mathbf{J},\mathfrak{M}]_S \\ K_{!}:[\mathbf{I},\mathfrak{M}]_S \to [\mathbf{J},\mathfrak{M}]_S. \end{cases}$$

Assume now that $\underline{C}$ is a combinatorial simplicial model category -- then the S-category $\mathfrak{C}$ (= $|\overline{\underline{\ }}|\underline{C}$) is tensored and cotensored, S-complete and S-cocomplete (cf. 4.7.19). The preceding machinery is thus applicable (replace $\mathfrak{M}$ by $\mathfrak{C}$). Accordingly, bearing in mind 4.7.4 and 4.7.7, we see that 0.26.16 and 0.26.17 go through with no change, i.e.,

$$\begin{cases} (K_{!},K^*) \text{ is a model pair (Projective Structure)} \\ (K^*,K_{+}) \text{ is a model pair (Injective Structure).} \end{cases}$$

4.7.21 THEOREM[†] If $K:\mathbf{I} \to \mathbf{J}$ is a DK-equivalence, then the model pairs

$$\begin{cases} (K_{!},K^*) \\ (K^*,K_{+}) \end{cases}$$

are model equivalences (cf. 0.26.18).

4.8 *REALIZATION AND TOTALIZATION*

Let $\underline{C}$ be a simplicial model category. Assume: $\underline{C}$ is complete and cocomplete.

4.8.1 DEFINITION Given an X in $\underline{SIC}$, put

$$|X| = \int^{[n]} X_n |\overline{\underline{\ }}|\Delta[n].$$

[†] Dwyer-Kan, *Annals of Math. Studies* $\underline{113}$ (1987), 180-205.

Then $|X|$ is called the <u>realization</u> of X.

<u>N.B.</u> The assignment $X \to |X|$ is a functor $\underline{\text{SIC}} \to \underline{C}$.

4.8.2 LEMMA $|\ |$ admits a right adjoint $\sin:\underline{C} \to \underline{\text{SIC}}$, where

$$\sin_n Y = \hom(\Delta[n],Y).$$

PROOF In fact,

$$\begin{aligned}\text{Mor}(|X|,Y) &\approx \text{Mor}(\textstyle\int^{[n]} X_n \,|\underline{\square}|\, \Delta[n],Y)\\ &\approx \textstyle\int_{[n]} \text{Mor}(X_n \,|\underline{\square}|\, \Delta[n],Y)\\ &\approx \textstyle\int_{[n]} \text{Mor}(X_n,\hom(\Delta[n],Y))\\ &\approx \textstyle\int_{[n]} \text{Mor}(X_n,\sin_n Y)\\ &\approx \text{Nat}(X,\sin Y).\end{aligned}$$

4.8.3 EXAMPLE Take $\underline{C} = \underline{\text{CGH}}$, thus

$$|\ |:\underline{\text{SICGH}} \to \underline{\text{CGH}}.$$

Now let X be a simplicial set thought of as a discrete simplicial space, i.e., as an object dis X of $\underline{\text{SICGH}}$ -- then

$$|\text{dis } X| \approx |X|,$$

the entity on the RHS being the geometric realization of X.

4.8.4 EXAMPLE Take $\underline{C} = \underline{\text{SISET}}$ and let X be a simplicial object in $\underline{C}$. One can fix [m] and form $|X_m^h|$, the geometric realization of $[n] \to X([n],[m])$, and one can fix [n] and form $|X_n^v|$, the geometric realization of $[m] \to X([n],[m])$. The assignments $\left[\begin{array}{l}[m] \to |X^h_m| \\ [n] \to |X^v_n|\end{array}\right.$ define simplicial objects $\left[\begin{array}{l}X^h \\ X^v\end{array}\right.$ in $\underline{\mathrm{CGH}}$ and their

realizations $\left[\begin{array}{l}|X^h| \\ |X^v|\end{array}\right.$ are homeomorphic to the geometric realization of $|X|$.

4.8.5 REMARK In 4.4, sin Y was denoted by the symbol $Y^{\Delta[\ ]}$ and there it was shown that

$$\mathrm{hom}(K,Y) \approx Y^{\Delta[\ ]} \pitchfork K \quad \text{(cf. 4.4.25).}$$

Therefore

$$M_n \sin Y = M_n Y^{\Delta[\ ]} \approx \mathrm{hom}(\dot{\Delta}[n],Y) \quad \text{(cf. 4.4.23).}$$

4.8.6 THEOREM Equip $\underline{\mathrm{SIC}}$ with its Reedy structure — then the adjoint situation $(|\ |, \sin)$ is a model pair.

PROOF It suffices to show that sin preserves fibrations and acyclic fibrations. So let $Y \to Y'$ be a fibration in $\underline{C}$ and consider the arrow

$$\sin_n Y \to M_n \sin Y \times_{M_n \sin Y'} \sin_n Y'$$

or still, the arrow

$$\mathrm{hom}(\Delta[n],Y) \to \mathrm{hom}(\dot{\Delta}[n],Y) \times_{\mathrm{hom}(\dot{\Delta}[n],Y')} \mathrm{hom}(\Delta[n],Y').$$

Then this arrow is a fibration in $\underline{C}$ that, moreover, is acyclic if $Y \to Y'$ is acyclic (cf. 4.5.12).

4.8.7 COROLLARY The realization functor

$$|\ |:\underline{\mathrm{SIC}} \text{ (Reedy Structure)} \to \underline{C}$$

preserves cofibrations and acyclic cofibrations.

4.8.8 LEMMA Let X be a simplicial object in $\underline{C}$ -- then

$$|X| \approx \operatorname{colim}_n |X|_n,$$

where

$$|X|_n = \int^{[k]} X_k |\underline{\square}| \Delta[k]^{(n)}.$$

PROOF The functors $X_n |\underline{\square}|$ — are left adjoints, hence preserve colimits, so

$$\begin{aligned}
|X| &= \int^{[n]} X_n |\underline{\square}| \Delta[n] \\
&\approx \int^{[n]} X_n |\underline{\square}| \operatorname{colim}_k \Delta[n]^{(k)} \\
&\approx \int^{[n]} \operatorname{colim}_k X_n |\underline{\square}| \Delta[n]^{(k)} \\
&\approx \operatorname{colim}_n \int^{[k]} X_k |\underline{\square}| \Delta[k]^{(n)} \\
&\approx \operatorname{colim}_n |X|_n.
\end{aligned}$$

4.8.9 LEMMA $\forall$ n > 0, there is a pushout square

$$\begin{array}{ccc}
L_n X |\underline{\square}| \Delta[n] \underset{L_n X |\underline{\square}| \dot{\Delta}[n]}{\sqcup} X_n |\underline{\square}| \dot{\Delta}[n] & \longrightarrow & |X|_{n-1} \\
\downarrow & & \downarrow \\
X_n |\underline{\square}| \Delta[n] & \longrightarrow & |X|_n
\end{array}.$$

4.8.10 LEMMA If X is a cofibrant object in $\underline{SIC}$ (Reedy Structure), then $\forall$ n > 0, the arrow $|X|_{n-1} \to |X|_n$ is a cofibration in $\underline{C}$.

PROOF The latching morphism $L_nX \rightarrow X_n$ is a cofibration in $\underline{C}$. Therefore the arrow

$$L_nX|\square|\Delta[n] \underset{L_nX|\square|\dot{\Delta}[n]}{\sqcup} X_n|\square|\dot{\Delta}[n] \longrightarrow X_n|\square|\Delta[n]$$

is a cofibration in $\underline{C}$ (cf. 4.5.9), from which the assertion.

<u>N.B.</u> If X is a cofibrant object in $\underline{SIC}$ (Reedy Structure), then both L_nX and X_n are cofibrant objects in $\underline{C}$, thus $L_nX|\square|\dot{\Delta}[n]$, $L_nX|\square|\Delta[n]$, and $X_n|\square|\dot{\Delta}[n]$ are cofibrant objects in $\underline{C}$, so

$$L_nX|\square|\Delta[n] \underset{L_nX|\square|\dot{\Delta}[n]}{\sqcup} X_n|\square|\dot{\Delta}[n]$$

is a cofibrant object in $\underline{C}$ (cf. 4.5.10).

4.8.11 LEMMA Suppose that $\left[\begin{array}{l} X \\ Y \end{array}\right.$ are cofibrant objects in $\underline{SIC}$ (Reedy Structure) and $f:X \rightarrow Y$ is a weak equivalence -- then the arrow

$$L_nX|\square|\Delta[n] \underset{L_nX|\square|\dot{\Delta}[n]}{\sqcup} X_n|\square|\dot{\Delta}[n]$$

$$\longrightarrow L_nY|\square|\Delta[n] \underset{L_nY|\square|\dot{\Delta}[n]}{\sqcup} Y_n|\square|\dot{\Delta}[n]$$

is a weak equivalence in $\underline{C}$.

PROOF The functor $L_n:\underline{SIC} \rightarrow \underline{C}$ sends acyclic cofibrations between cofibrant objects to weak equivalences, hence preserves weak equivalences between cofibrant objects (cf. 2.2.4). This said, consider the commutative diagram

$$
\begin{tikzcd}
L_nX\,\square\,\Delta[n] \arrow[d] & L_nX\,\square\,\dot{\Delta}[n] \arrow[l] \arrow[r] \arrow[d] & X_n\,\square\,\dot{\Delta}[n] \arrow[d] \\
L_nY\,\square\,\Delta[n] & L_nY\,\square\,\dot{\Delta}[n] \arrow[l] \arrow[r] & Y_n\,\square\,\dot{\Delta}[n].
\end{tikzcd}
$$

Then the horizontal arrows are cofibrations (cf. 4.5.10) and the vertical arrows are weak equivalences (cf. 4.5.19). Now apply 0.1.20.

4.8.12 THEOREM Suppose that $\left[\begin{array}{l} X \\ Y \end{array}\right.$ are cofibrant objects in $\underline{\mathrm{SIC}}$ (Reedy Structure) and $f:X \to Y$ is a weak equivalence — then $|f|:|X| \to |Y|$ is a weak equivalence.

PROOF Since $\left[\begin{array}{l} |X|_0 = X_0 \\ |Y|_0 = Y_0 \end{array}\right.$ and since $\forall\, n$, $\left[\begin{array}{l} |X|_n \longrightarrow |X|_{n+1} \\ |Y|_n \longrightarrow |Y|_{n+1} \end{array}\right.$ is a cofibration in $\underline{C}$ (cf. 4.8.10), one may view $\left[\begin{array}{l} \{|X|_n : n \geq 0\} \\ \{|Y|_n : n \geq 0\} \end{array}\right.$ as cofibrant objects in $\underline{\mathrm{FIL}}(\underline{C})$ (cf. 0.1.13). So, to prove that $|f|:|X| \to |Y|$ is a weak equivalence, it need only be shown that $\forall\, n$, $|f|_n:|X|_n \to |Y|_n$ is a weak equivalence. To this end, work with

$$
\begin{tikzcd}
X_n\,\square\,\Delta[n] \arrow[d] & L_nX\,\square\,\Delta[n] \underset{L_nX\,\square\,\dot{\Delta}[n]}{\sqcup} X_n\,\square\,\dot{\Delta}[n] \arrow[l] \arrow[r] \arrow[d] & |X|_{n-1} \arrow[d] \\
Y_n\,\square\,\Delta[n] & L_nY\,\square\,\Delta[n] \underset{L_nY\,\square\,\dot{\Delta}[n]}{\sqcup} Y_n\,\square\,\dot{\Delta}[n] \arrow[l] \arrow[r] & |Y|_{n-1}
\end{tikzcd}
$$

and use induction.

4.8.13 EXAMPLE Take $\underline{C}$ = SISET (Kan Structure) and suppose that $f:X \to Y$ is a weak equivalence, i.e., $\forall$ n, $f_n:X_n \to Y_n$ is a simplicial weak equivalence -- then $|f|:|X| \to |Y|$ is a simplicial weak equivalence.

[All simplicial objects in $\hat{\underline{\Delta}}$ are cofibrant in the Reedy structure (a.k.a. structure R).]

Let $\underline{C}$ be a simplicial model category. Assume: $\underline{C}$ is complete and cocomplete.

4.8.14 DEFINITION Given an X in COSIC, put

$$\text{tot } X = \int_{[n]} \text{hom}(\Delta[n],X_n).$$

Then tot X is called the totalization of X.

N.B. The assignment $X \to \text{tot } X$ is a functor COSIC $\to \underline{C}$.

4.8.15 LEMMA tot admits a left adjoint cosin:$\underline{C} \to$ COSIC, where

$$\text{cosin}_n Y = Y_n|\overline{\underline{\ \ }}|\Delta[n].$$

PROOF In fact,

$$\begin{aligned}\text{Mor}(Y,\text{tot } X) &\approx \text{Mor}(Y,\int_{[n]} \text{hom}(\Delta[n],X_n)) \\ &\approx \int_{[n]} \text{Mor}(Y,\text{hom}(\Delta[n],X_n)) \\ &\approx \int_{[n]} \text{Mor}(Y|\overline{\underline{\ \ }}|\Delta[n],X_n) \\ &\approx \int_{[n]} \text{Mor}(\text{cosin}_n Y,X_n) \\ &\approx \text{Nat}(\text{cosin } Y,X).\end{aligned}$$

4.8.16 EXAMPLE Take $\underline{C}$ = SISET and in 4.4.9, let $\underline{I} = \underline{\Delta}$ -- then

$$\text{HOM}(F,G) \approx \int_{[n]} \text{map}(F[n],G[n]).$$

Specialize to $\left[\begin{array}{l} F = Y_{\underline{\Delta}} \\ \\ G = X \end{array}\right.$, thus

$$\begin{aligned} \mathrm{HOM}(Y_{\underline{\Delta}},X) &\approx \textstyle\int_{[n]} \mathrm{map}(Y_{\underline{\Delta}}[n],X[n]) \\ &\approx \textstyle\int_{[n]} \mathrm{map}(\Delta[n],X_n) \\ &\approx \textstyle\int_{[n]} \mathrm{hom}(\Delta[n],X_n) \\ &\approx \mathrm{tot}\, X. \end{aligned}$$

4.8.17 EXAMPLE Given a simplicial set K and a compactly generated Hausdorff space X, let X^K be the cosimplicial object in <u>CGH</u> with $(X^K)_n = X^{K_n}$ — then $X^{|K|} \approx \mathrm{tot}\, X^K$.

4.8.18 REMARK There are obvious analogs for tot of 4.8.6 and 4.8.12: Take <u>COSIC</u> in its Reedy structure — then the adjoint situation (cosin, tot) is a model pair and if $f:X \to Y$ is a weak equivalence, where X,Y are fibrant, then $\mathrm{tot}\, f:\mathrm{tot}\, X \to \mathrm{tot}\, Y$ is a weak equivalence.

4.8.19 NOTATION Given a simplicial set K, put

$$\underline{\Delta}K = \mathrm{gro}_{\underline{\Delta}}\, K \text{ (a.k.a. } i_{\underline{\Delta}}K\ (\equiv \underline{\Delta}/K))$$

and let $\underline{\Delta}^{OP}K$ be its opposite — then there are functors

$$\Delta K:\underline{\Delta}K \to \underline{\mathrm{SISET}}$$

and

$$\Delta^{OP}K:\underline{\Delta}^{OP}K \to \underline{\mathrm{SISET}}^{OP}.$$

4.8.20 NOTATION Given a category $\underline{C}$, write K-$\underline{SIC}$ for the functor category $[\underline{\Delta}^{OP}K,\underline{C}]$ and K-$\underline{COSIC}$ for the functor category $[\underline{\Delta}K,\underline{C}]$.

4.8.21 DEFINITION A K-simplicial object in $\underline{C}$ is an object in K-$\underline{SIC}$ and a K-cosimplicial object in $\underline{C}$ is an object in K-$\underline{COSIC}$.

[Note: Take $K = \Delta[0]$ to recover $\underline{SIC}$ and $\underline{COSIC}$.]

4.8.22 LEMMA $\underline{\Delta}K$ and $\underline{\Delta}^{OP}K$ are Reedy categories.

[Note: Generalizing 0.27.39, take $\underline{I} = \underline{\Delta}^{OP}K$ to realize 0.27.35 and take $\underline{I} = \underline{\Delta}K$ to realize 0.27.37.]

Consequently, if $\underline{C}$ is a complete and cocomplete model category, then

$$\text{K-}\underline{SIC} \text{ and K-}\underline{COSIC}$$

are model categories (Reedy Structure).

Assume now that $\underline{C}$ is, in addition, a simplicial model category.

- There is a realization functor

$$|\ |_K:\text{K-}\underline{SIC} \to \underline{C}$$

that sends X to

$$|X|_K = \int^{\Delta K} X|\overline{\underline{\ \ }}|\Delta K,$$

where

$$X|\overline{\underline{\ \ }}|\Delta K:\underline{\Delta}^{OP}K \times \underline{\Delta}K \to \underline{C}$$

is the composite

$$\underline{\Delta}^{OP}K \times \underline{\Delta}K \xrightarrow{X \times \Delta K} \underline{C} \times \underline{SISET} \xrightarrow{|\overline{\underline{\ \ }}|} \underline{C}.$$

• There is a totalization functor

$$\mathrm{tot}_K : K\text{-}\underline{\mathrm{COSIC}} \to \underline{C}$$

that sends X to

$$\mathrm{tot}_K X = \int_{\underline{\Delta}K} \mathrm{hom}(\Delta K, X),$$

where

$$\mathrm{hom}(\Delta K, X) : \underline{\Delta}^{\mathrm{OP}}K \times \underline{\Delta}K \to \underline{C}$$

is the composite

$$\underline{\Delta}^{\mathrm{OP}}K \times \underline{\Delta}K \xrightarrow{\Delta^{\mathrm{OP}}K \times X} \underline{\mathrm{SISET}}^{\mathrm{OP}} \times \underline{C} \xrightarrow{\mathrm{hom}} \underline{C}.$$

Let $p_K : K \to \Delta[0]$ be the canonical arrow — then

$$\underline{\Delta}K \to \underline{\Delta}\Delta[0] = \underline{\Delta}$$

and

$$\underline{\Delta}^{\mathrm{OP}}K \to \underline{\Delta}^{\mathrm{OP}}\Delta[0] = \underline{\Delta}^{\mathrm{OP}}.$$

• The induced map

$$\underline{\mathrm{SIC}} \to K\text{-}\underline{\mathrm{SIC}}$$

has a left adjoint

$$\mathrm{lan}_K : K\text{-}\underline{\mathrm{SIC}} \to \underline{\mathrm{SIC}}$$

and there is a commutative diagram

$$\begin{tikzcd} K\text{-}\underline{\mathrm{SIC}} \arrow[r, "\mathrm{lan}_K"] \arrow[d, "|\ |_K"'] & \underline{\mathrm{SIC}} \arrow[d, "|\ |"] \\ \underline{C} \arrow[r, equal] & \underline{C} \end{tikzcd} .$$

<u>N.B.</u> $| \ |_K$ admits a right adjoint

$$\sin_K : \underline{C} \to K\text{-}\underline{SIC}$$

and the adjoint situation $(| \ |_K, \sin_K)$ is a model pair.

- The induced map

$$\underline{COSIC} \to K\text{-}\underline{COSIC}$$

has a right adjoint

$$\operatorname{ran}_K : K\text{-}\underline{COSIC} \to \underline{COSIC}$$

and there is a commutative diagram

$$\begin{tikzcd}
K\text{-}\underline{COSIC} \arrow[r, "\operatorname{ran}_K"] \arrow[d, "\operatorname{tot}_K"'] & \underline{COSIC} \arrow[d, "\operatorname{tot}"] \\
\underline{C} \arrow[r, equal] & \underline{C}.
\end{tikzcd}$$

<u>N.B.</u> tot_K admits a left adjoint

$$\operatorname{cosin}_K : \underline{C} \to K\text{-}\underline{COSIC}$$

and the adjoint situation $(\operatorname{cosin}_K, \operatorname{tot}_K)$ is a model pair.

4.8.23 THEOREM Suppose that $\begin{bmatrix} X \\ Y \end{bmatrix}$ are cofibrant objects in K-<u>SIC</u> (Reedy Structure) and $f: X \to Y$ is a weak equivalence — then $|f|_K : |X|_K \to |Y|_K$ is a weak equivalence.

4.8.24 THEOREM Suppose that $\begin{bmatrix} X \\ Y \end{bmatrix}$ are fibrant objects in K-<u>COSIC</u> (Reedy Structure) and $f: X \to Y$ is a weak equivalence — then $\operatorname{tot}_K f : \operatorname{tot}_K X \to \operatorname{tot}_K Y$ is a weak equivalence.

4.9 *HOMOTOPICAL ALGEBRA*

4.9.1 NOTATION Let $\underline{I}$ be a small category -- then

$$\underline{\Delta}/I = \underline{\Delta}/\text{ner } \underline{I} = \text{gro}_{\underline{\Delta}} \text{ ner } \underline{I} = i_{\underline{\Delta}} \text{ ner } \underline{I} = \underline{\Delta} \text{ ner } \underline{I}.$$

Abbreviate and call any of these renditions $\underline{\Delta I}$, thus $\underline{\Delta I}$ is isomorphic to the comma category

$$|\iota, K_{\underline{I}}|: \quad \begin{array}{ccc} [m] & \xrightarrow{f} & [n] \\ u \downarrow & & \downarrow v \\ \underline{I} & = & \underline{I} \end{array} \quad (\iota:\underline{\Delta} \to \underline{\text{CAT}})$$

and

$$\underline{\Delta}^{\text{OP}}\underline{I} \equiv (\underline{\Delta I})^{\text{OP}}.$$

- Define $\tau_{\underline{I}}:\underline{\Delta I} \to \underline{I}$ by

$$\tau_{\underline{I}}([m] \xrightarrow{u} \underline{I}) = u(m).$$

- Define $\sigma_{\underline{I}}:\underline{\Delta}^{\text{OP}}\underline{I} \to \underline{I}$ by

$$\sigma_{\underline{I}}([m] \xrightarrow{u} \underline{I}) = u(0).$$

4.9.2 EXAMPLE We have

$$\underline{\Delta 1} = \underline{\Delta} \text{ and } \underline{\Delta}^{\text{OP}}\underline{1} = \underline{\Delta}^{\text{OP}}.$$

4.9.3 LEMMA Let $\underline{C}$ be a complete and cocomplete model category. Suppose that $F:\underline{I} \to \underline{C}$ is a functor such that $\forall\ i \in \mathrm{Ob}\ \underline{I}$, Fi is cofibrant (fibrant) -- then $F \circ \sigma_{\underline{I}}$ $(F \circ \tau_{\underline{I}})$ is a cofibrant (fibrant) object in $[\underline{\Delta}^{OP}\underline{I},\underline{C}]$ $([\underline{\Delta}\underline{I},\underline{C}])$ (Reedy Structure) (cf. 4.8.22)).

Let $\underline{C}$ be a simplicial model category. Assume: $\underline{C}$ is complete and cocomplete. Fix a small category $\underline{I}$.

- The uncorrected homotopy colimit of a functor $F:\underline{I} \to \underline{C}$ is the coend

$$\int^{\underline{I}^{OP}} F|\overline{\underline{\ }}|\mathrm{ner}(—\backslash\underline{I}),$$

denoted

$$\mathrm{hocolim}_{\underline{I}}F.$$

- The uncorrected homotopy limit of a functor $F:\underline{I} \to \underline{C}$ is the end

$$\int_{\underline{I}} \hom(\mathrm{ner}(\underline{I}/—),F),$$

denoted

$$\mathrm{holim}_{\underline{I}}F.$$

4.9.4 EXAMPLE Take $\underline{C} = \underline{\mathrm{SISET}}$ (Kan Structure) -- then (cf. 4.5.2)

$$Fi|\overline{\underline{\ }}|\mathrm{ner}(i\backslash\underline{I}) = Fi \times \mathrm{ner}(i\backslash\underline{I})$$

and

$$\hom(\mathrm{ner}(\underline{I}/i),Fi) = \mathrm{map}(\mathrm{ner}(\underline{I}/i),Fi).$$

4.9.5 EXAMPLE Take $\underline{C} = \underline{\mathrm{CGH}}$ (Quillen Structure) -- then (cf. 4.5.3)

$$Fi|\overline{\underline{\ }}|\mathrm{ner}(i\backslash\underline{I}) = Fi \times_k B(i\backslash\underline{I})$$

and

$$\text{hom}(\text{ner}(\underline{I}/i),Fi) = Fi^{B(\underline{I}/i)} .$$

4.9.6 APPLICATION

- Let $F:\underline{I} \to \underline{\text{SISET}}$ be a functor -- then

$$\begin{aligned}|\text{hocolim}_{\underline{I}} F| &= |\textstyle\int^i Fi \times \text{ner}(i\backslash\underline{I})| \\ &\approx \textstyle\int^i |Fi \times \text{ner}(i\backslash\underline{I})| \\ &\approx \textstyle\int^i |Fi| \times_k B(i\backslash\underline{I}) \\ &\approx \text{hocom}_{\underline{I}} |F|,\end{aligned}$$

a natural homeomorphism of compactly generated Hausdorff spaces.

- Let $F:\underline{I} \to \underline{\text{CGH}}$ be a functor — then

$$\begin{aligned}\sin \text{holim}_{\underline{I}} F &= \sin \textstyle\int_i Fi^{B(\underline{I}/i)} \\ &\approx \textstyle\int_i \sin Fi^{B(\underline{I}/i)} \\ &\approx \textstyle\int_i \text{map}(\text{ner}(\underline{I}/i),\sin Fi) \\ &= \text{holim}_{\underline{I}} \sin F,\end{aligned}$$

a natural isomorphism of simplicial sets.

[Note: If K is a simplicial set and if X is a compactly generated Hausdorff space, then

$$\sin X^{|K|} \approx \text{map}(K,\sin X).$$

Proof:

$$\begin{aligned}
\sin X^{|K|}([n]) &\approx C(\Delta^n, X^{|K|}) \\
&\approx C(\Delta^n \times_k |K|, X) \\
&\approx C(|\Delta[n] \times K|, X) \\
&\approx \mathrm{Nat}(K \times \Delta[n], \sin X) \\
&\approx \mathrm{map}_n(K, \sin X).]
\end{aligned}$$

4.9.7 EXAMPLE Take $\underline{C} = \underline{CAT}$ (External Structure) — then (cf. 4.5.4)

$$\begin{aligned}
Fi|\overline{\underline{\ }}|\mathrm{ner}(i\backslash\underline{I}) &= Fi \times \mathrm{cat} \circ \mathrm{ner}(i\backslash\underline{I}) \\
&\approx Fi \times i\backslash\underline{I}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{hom}(\mathrm{ner}(\underline{I}/i), Fi) &= [\mathrm{cat} \circ \mathrm{ner}(\underline{I}/i), Fi] \\
&\approx [\underline{I}/i, Fi].
\end{aligned}$$

[Note: Therefore

$$\mathrm{hocolim}_{\underline{I}} F \approx \underline{\mathrm{INT}}_{\underline{I}} F \quad \text{(cf. B.5)},$$

a conclusion that is in agreement with B.8.13. Here is another point:

$$\begin{aligned}
\mathrm{holim}_{\underline{I}}\ \mathrm{ner} \circ F &= \textstyle\int_i \mathrm{map}(\mathrm{ner}(\underline{I}/i), \mathrm{ner}\ Fi) \\
&\approx \textstyle\int_i \mathrm{ner}[\underline{I}/i, Fi] \\
&\approx \mathrm{ner}(\textstyle\int_i [\underline{I}/i, Fi]).]
\end{aligned}$$

N.B. One can also explicate matters for $\underline{\mathrm{CAT}}$ (Internal Structure) (cf. 4.5.5).

4.9.8 REMARK The functor

$$\mathrm{hocolim}_{\underline{I}}:[\underline{I},\underline{C}] \to \underline{C}$$

has a right adjoint, viz.

$$\mathrm{hom}(\mathrm{ner}(—\backslash\underline{I}),—)$$

and the functor

$$\mathrm{holim}_{\underline{I}}:[\underline{I},\underline{C}] \to \underline{C}$$

has a left adjoint, viz.

$$—|\overline{\underline{\ }}|\mathrm{ner}(\underline{I}/—).$$

4.9.9 LEMMA Fix $F \in \mathrm{Ob}[\underline{I},\underline{C}]$ -- then

$$\mathrm{hocolim}_{\underline{I}}F \approx \int^{\underline{\Delta I}} F \circ \sigma_{\underline{I}}|\overline{\underline{\ }}|\Delta\mathrm{ner}\ \underline{I}\ \ (= |F \circ \sigma_{\underline{I}}|_{\mathrm{ner}\ \underline{I}})$$

and

$$\mathrm{holim}_{\underline{I}}F \approx \int_{\underline{\Delta I}} \mathrm{hom}(\Delta\mathrm{ner}\ \underline{I},F \circ \tau_{\underline{I}})\ \ (= \mathrm{tot}_{\mathrm{ner}\ \underline{I}}\ F \circ \tau_{\underline{I}}).$$

4.9.10 THEOREM Let $F,G:\underline{I} \to \underline{C}$ be functors and let $\Xi:F \to G$ be a natural transformation. Assume: $\forall\ i$, $\Xi_i:Fi \to Gi$ is a weak equivalence — then

$$\mathrm{hocolim}_{\underline{I}}\Xi:\mathrm{hocolim}_{\underline{I}}F \to \mathrm{hocolim}_{\underline{I}}G$$

is a weak equivalence if $\forall\ i$, $\left[\begin{array}{l} Fi \\ Gi \end{array}\right.$ is cofibrant and

$$\mathrm{holim}_{\underline{I}}\Xi:\mathrm{holim}_{\underline{I}}F \to \mathrm{holim}_{\underline{I}}G$$

is a weak equivalence if $\forall$ i, $\begin{bmatrix} Fi \\ \\ Gi \end{bmatrix}$ is fibrant.

PROOF Apply 4.8.23 and 4.8.24 (4.9.3 and 4.9.9 set the stage).

[Note: Take $\underline{C} = \underline{CAT}$ (External Structure) (cf. 4.9.7) — then 4.9.10 does not specialize to B.7.1 (the latter makes no cofibrancy assumptions).]

4.9.11 EXAMPLE Let $F:\underline{I} \to \underline{CGH}$ be a functor such that $\forall$ i, Fi is cofibrant — then there is a natural simplicial weak equivalence

$$\text{hocolim}_{\underline{I}} \sin F \to \sin \text{hocolim}_{\underline{I}} F.$$

[Consider the natural transformation $|\sin F| \to F$: $\forall$ i, $|\sin Fi|$ is cofibrant and the arrow $|\sin Fi| \to Fi$ is a weak homotopy equivalence, thus the arrow

$$\text{hocolim}_{\underline{I}} |\sin F| \to \text{hocolim}_{\underline{I}} F$$

is a weak homotopy equivalence (cf. 4.9.10). But

$$\text{hocolim}_{\underline{I}} \sin F| \approx \text{hocolim}_{\underline{I}} |\sin F| \quad \text{(cf. 4.9.6)},$$

so taking adjoints leads to the conclusion.]

[Note: In the same vein, if $F:\underline{I} \to \underline{SISET}$ is a functor such that $\forall$ i, Fi is fibrant, then there is a natural weak homotopy equivalence

$$|\text{holim}_{\underline{I}} F| \to \text{holim}_{\underline{I}} |F|.]$$

4.9.12 REMARK A corollary to 4.9.10 is the fact that

$$\text{hocolim}_{\underline{I}} F \approx |\text{lan}_{\text{ner }\underline{I}} (F \circ \sigma_{\underline{I}})|$$

and

$$\text{holim}_{\underline{I}} F \approx \text{tot ran}_{\text{ner }\underline{I}} (F \circ \tau_{\underline{I}}).$$

4.9.13 LEMMA (SIMPLICIAL REPLACEMENT) Fix $F \in \mathrm{Ob}\,[\underline{I},\underline{C}]$. Define $\coprod F$ in $\underline{\mathrm{SIC}}$ by

$$(\coprod F)_n = \coprod_{[n] \xrightarrow{f} \underline{I}} Ff0.$$

Then

$$\coprod F \approx \mathrm{lan}_{\mathrm{ner}\,\underline{I}}\,(F \circ \sigma_{\underline{I}}).$$

[Note: Therefore

$$\mathrm{hocolim}_{\underline{I}} F \approx |\coprod F|.]$$

4.9.14 LEMMA (COSIMPLICIAL REPLACEMENT) Fix $F \in \mathrm{Ob}\,[\underline{I},\underline{C}]$. Define $\prod F$ in $\underline{\mathrm{COSIC}}$ by

$$(\prod F)_n = \prod_{[n] \xrightarrow{f} \underline{I}} Ffn.$$

Then

$$\prod F \approx \mathrm{ran}_{\mathrm{ner}\,\underline{I}}\,(F \circ \tau_{\underline{I}}).$$

[Note: Therefore

$$\mathrm{holim}_{\underline{I}} F \approx \mathrm{tot} \prod F.]$$

4.9.15 EXAMPLE Given $X:\underline{\Delta}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}$, define $\mathrm{dia}\,X:\underline{\Delta}^{\mathrm{OP}} \to \underline{\mathrm{SET}}$ by

$$\mathrm{dia}\,X([n]) = X([n])([n]).$$

But also, by definition, $|X|:\underline{\Delta}^{\mathrm{OP}} \to \underline{\mathrm{SET}}$ and, up to natural isomorphism, dia and $|\ |$ are the same (both are left adjoints for sin). Now form $\coprod X$ per 4.9.13, thus

$$\coprod X:\underline{\Delta}^{OP} \to \underline{SISET}.$$

And then

$$\text{hocolim}_{\underline{\Delta}^{OP}} X \approx |\coprod X| \approx \text{dia} \coprod X.$$

APPENDIX

Recall that $\underline{I}$ is a small category and $\underline{C}$ is a simplicial model category which is both complete and cocomplete.

If $F:\underline{I} \to \underline{C}$ is a functor, then

$$\text{hocolim}_{\underline{I}} F = \int^{\underline{I}^{OP}} F|\underline{\ }|\text{ner}(-\backslash \underline{I})$$

is its uncorrected homotopy colimit and

$$\text{holim}_{\underline{I}} F = \int_{\underline{I}} \text{hom}(\text{ner}(\underline{I}/-),F)$$

is its uncorrected homotopy limit. Here we shall explain the origin of this terminology and for that it will be enough to consider $\text{hocolim}_{\underline{I}}$.

RAPPEL View $\underline{C}$ as a cofibration category and place on $[\underline{I},\underline{C}]$ its injective structure, so $[\underline{I},\underline{C}]$ is a cocomplete cofibration category (cf. 2.5.3).

Let $p_{\underline{I}}:\underline{I} \to \underline{1}$ be the canonical arrow — then $p_{\underline{I}}^*$ has a left adjoint $p_{\underline{I}!}$, viz.

$$\text{colim}_{\underline{I}}:[\underline{I},\underline{C}] \to \underline{C},$$

that in turn admits an absolute total left derived functor

$$\mathrm{Lcolim}_{\underline{I}}: \mathcal{W}_{\underline{I}}^{-1}[\underline{I},\underline{C}] \to \mathcal{W}^{-1}\underline{C} \quad (\text{cf. } 2.5.7),$$

the "true" homotopy colimit.

Now refer back to 4.9.10. Since the weak equivalences in $[\underline{I},\underline{C}]$ are levelwise and since the cofibrant objects in $[\underline{I},\underline{C}]$ are levelwise, it follows that

$$\mathrm{hocolim}_{\underline{I}}: [\underline{I},\underline{C}] \to \underline{C}$$

also admits an absolute total left derived functor

$$\mathrm{Lhocolim}_{\underline{I}}: \mathcal{W}_{\underline{I}}^{-1}[\underline{I},\underline{C}] \to \mathcal{W}^{-1}\underline{C} \quad (\text{cf. } 2.2.4).$$

And, on general grounds, if $F \in \mathrm{Ob}[\underline{I},\underline{C}]$ is cofibrant, then the natural map

$$\mathrm{Lhocolim}_{\underline{I}} F \to \mathrm{hocolim}_{\underline{I}} F$$

is an isomorphism in $\mathcal{W}^{-1}\underline{C}$.

ASSUMPTION The w.f.s.

$$(\mathrm{cof}, \mathcal{W} \cap \mathrm{fib})$$

is functorial (cf. 0.19.3).

NOTATION Given $F \in \mathrm{Ob}[\underline{I},\underline{C}]$, define $\underline{L}F$ levelwise:

$$(\underline{L}F)(i) = \underline{L}(Fi).$$

N.B. The functor

$$F \to \mathrm{hocolim}_{\underline{I}} \underline{L}F$$

is a morphism

$$([\underline{I},\underline{C}], \mathcal{W}_{\underline{I}}) \to (\underline{C}, \mathcal{W})$$

of category pairs (cf. 4.9.10), thus there is a unique functor

$$\overline{\mathrm{hocolim}_{\underline{I}} \circ \underline{L}}: W_{\underline{I}}^{-1}[\underline{I},\underline{C}] \to W^{-1}\underline{C}$$

for which the diagram

$$\begin{tikzcd}
{[\underline{I},\underline{C}]} \arrow[rr, "\mathrm{hocolim}_{\underline{I}} \circ \underline{L}"] \arrow[d, "L_{W_{\underline{I}}^{-1}}"'] & & \underline{C} \arrow[d, "L_W"] \\
W_{\underline{I}}^{-1}[\underline{I},\underline{C}] \arrow[rr, "\overline{\mathrm{hocolim}_{\underline{I}} \circ \underline{L}}"'] & & W^{-1}\underline{C}
\end{tikzcd}$$

commutes (cf. 1.4.5).

THEOREM[†] The functor

$$\overline{\mathrm{hocolim}_{\underline{I}} \circ \underline{L}}$$

"is"

$$L\mathrm{colim}_{\underline{I}}.$$

REMARK Changing the cofibrant replacement functor from $\underline{L}$ to $\underline{L}'$ leads to another model for $L\mathrm{colim}_{\underline{I}}$.

[†] Shulman, arXiv:math/0610194; see also González, arXiv:1104.0646

CHAPTER 5: CUBICAL THEORY

5.1 $\underline{|\square|}$: *DEFINITION AND PROPERTIES*

5.2 *CUBICAL SETS*

CHAPTER 5: CUBICAL THEORY

5.1 $\underline{|\square|}$: *DEFINITION AND PROPERTIES*

Given an integer $n \geq 0$, let I^n be the set-theoretic product $\{0,1\}^n$.

- For $n \geq 1$, $1 \leq i \leq n$, $\varepsilon = 0,1$, define

$$\delta_{i,\varepsilon}^n : I^{n-1} \to I^n$$

by

$$\delta_{i,\varepsilon}^n(x_1,\ldots,x_{n-1}) = (x_1,\ldots x_{i-1},\varepsilon,x_i,\ldots,x_{n-1}).$$

- For $n \geq 0$, $1 \leq i \leq n+1$, define

$$\sigma_i^n : I^{n+1} \to I^n$$

by

$$\sigma_i^n(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}).$$

5.1.1 DEFINITION $\underline{|\square|}$ is the category whose objects are the I^n and whose morphisms are generated by the $\delta_{i,\varepsilon}^n$ and the σ_i^n.

[Note: $\underline{|\square|}$ has a final object, viz. I^0.]

5.1.2 LEMMA We have

$$\left[\begin{array}{ll} \delta_{j,\eta}^n \circ \delta_{i,\varepsilon}^{n-1} = \delta_{i,\varepsilon}^n \circ \delta_{j-1,\eta}^{n-1} & (i < j) \\ \\ \sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} & (i \leq j) \end{array}\right.$$

and

$$\sigma_j^n \circ \delta_{i,\varepsilon}^{n+1} = \begin{cases} \delta_{i,\varepsilon}^n \circ \sigma_{j-1}^{n-1} & (i < j) \\ \mathrm{id}_{I^n} & (i = j) \\ \delta_{i-1,\varepsilon}^n \circ \sigma_j^{n-1} & (i > j). \end{cases}$$

N.B. In particular

$$\begin{cases} \sigma_1^0 \circ \delta_{1,0}^1 = \mathrm{id}_{I^0} \\ \sigma_1^0 \circ \delta_{1,1}^1 = \mathrm{id}_{I^0}\ . \end{cases}$$

5.1.3 LEMMA $\underline{|\square|}$ is a strict monoidal category.

[Define

$$\otimes: \underline{|\square|} \times \underline{|\square|} \to \underline{|\square|}$$

by

$$(I^m, I^n) \to I^m \otimes I^n = I^{m+n}$$

and let $e = I^0$.]

5.1.4 DEFINITION Let $(\underline{V}, \otimes, e)$ be a strict monoidal category -- then a cylinder in $\underline{V}$ is a 4-tuple (I, d_0, d_1, p), where $I \in \mathrm{Ob}\,\underline{V}$ and $d_0, d_1: e \to I$, $p: I \to e$ are morphisms of $\underline{V}$ such that

$$pd_0 = \mathrm{id}_e = pd_1.$$

5.1.5 EXAMPLE Take $\underline{V} = \underline{|\square|}$ (cf. 5.1.3) -- then $(I^1,\delta^1_{1,0},\delta^1_{1,1},\sigma^0_1)$ is a cylinder in $\underline{|\square|}$.

5.1.6 LEMMA Let $(\underline{V},\otimes,e)$ be a strict monoidal category -- then the association that sends a functor $F:\underline{|\square|} \to \underline{V}$ to the 4-tuple

$$(F(I^1),F(\delta^1_{1,0}),F(\delta^1_{1,1}),F(\sigma^0_1))$$

is a bijection between the set of strict monoidal functors from $\underline{|\square|}$ to $\underline{V}$ and the cylinders in $\underline{V}$.

5.1.7 SCHOLIUM There is a strict monoidal functor $c:\underline{|\square|} \to \underline{CAT}$ with $I^n \to [1]^n$.

[Send I^1 to $[1]$, $\delta^1_{1,0}$ to δ^1_1, $\delta^1_{1,1}$ to δ^1_0, and σ^0_1 to σ^0_0.]

5.1.8 LEMMA $\underline{|\square|}$ is a Reedy category.

[Put

$$\deg(I^n) = n$$

and let

$$\begin{cases} \overrightarrow{\underline{|\square|}} = \text{subcategory of } \underline{|\square|} \text{ generated by the } \delta^n_{i,\varepsilon} \\ \overleftarrow{\underline{|\square|}} = \text{subcategory of } \underline{|\square|} \text{ generated by the } \sigma^n_i.] \end{cases}$$

5.1.9 LEMMA $\underline{|\square|}$ is a local test category per $\mathcal{W}_\infty$.

[The functor $c:\underline{|\square|} \to \underline{CAT}$ satisfies the finality hypothesis, thus it is enough to prove that $\mathrm{ner}_c[1]$ satisfies the Ω-condition (cf. C.10.14), i.e., that the categories

$$i_{\underline{|\square|}}(|\square|(n) \times \mathrm{ner}_c[1]) = \underline{|\square|}/(|\square|(n) \times \mathrm{ner}_c[1]) \quad (n \geq 0)$$

are aspherical. But it is possible to proceed homotopically and construct an equivalence between

$$\underline{|\square|}/(|\square|(n) \times \mathrm{ner}_c[1]) \text{ and } \underline{|\square|}/|\square|(n),$$

which suffices (since $\underline{|\square|}/|\square|(n)$ has a final object, hence is aspherical).]

5.1.10 REMARK Consequently, $(W_\infty)_{\underline{\widehat{|\square|}}}$ is a $\underline{\widehat{|\square|}}$-localizer (cf. C.9.1) and C.9.5 is applicable: $\underline{\widehat{|\square|}}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of $(W_\infty)_{\underline{\widehat{|\square|}}}$ and whose cofibrations are the monomorphisms.

[Note: The $\underline{\widehat{|\square|}}$-localizer generated by the arrows $|\square|(n) \to |\square|(0)$ $(n \geq 0)$ is $(W_\infty)_{\underline{\widehat{|\square|}}}$.]

N.B. This model structure on $\underline{\widehat{|\square|}}$ is proper (cf. C.9.10).

5.2 *CUBICAL SETS*

5.2.1 DEFINITION A cubical set is a functor $X: \underline{|\square|}^{\mathrm{OP}} \to \underline{\mathrm{SET}}$.

5.2.2 NOTATION <u>CUSET</u> is the category whose objects are the cubical sets and whose morphisms are the natural transformations between them.

[Note: A morphism in <u>CUSET</u> is called a cubical map.]

The <u>cubical standard n-cube</u> is the cubical set $|\square|(n) = \mathrm{Mor}(\text{—}, I^n)$. If X is a cubical set and if $X_n = X(I^n)$, then

$$\mathrm{Mor}(|\square|(n), X) \approx X_n.$$

<u>N.B.</u> If $\alpha: I^m \to I^n$, then

$$|\square|(\alpha): |\square|(m) \to |\square|(n).$$

A <u>cubical subset</u> of a cubical set X is a cubical set Y such that Y is a subfunctor of X, i.e., $Y_n \subset X_n$ for all n and the inclusion $Y \to X$ is a cubical map.

5.2.3 DEFINITION The <u>frontier</u> of $|\square|(n)$ is the cubical subset $\partial|\square|(n)$ $(n \geq 0)$ of $|\square|(n)$ given by

$$\partial|\square|(n)(I^m) = \{f: I^m \to I^n : \exists \text{ a factorization } f: I^m \to I^k \to I^n \ (k < n)\}.$$

5.2.4 RAPPEL Suppose that $\underline{C}$ is a small category — then $\mathcal{M} \subset \mathrm{Mor}\,\hat{\underline{C}}$ is the class of monomorphisms.

5.2.5 EXAMPLE Let $\underline{C} = \underline{\Delta}$ and let

$$M = \{\dot{\Delta}[n] \to \Delta[n] : n \geq 0\}.$$

Then

$$\mathcal{M} = \mathrm{LLP}(\mathrm{RLP}(M)) = \mathrm{cof}\ M \quad (\text{cf. } 0.20.5).$$

5.2.6 LEMMA Let $\underline{C} = \underline{|\square|}$ and let

$$M = \{\partial|\square|(n) \to |\square|(n) : n \geq 0\}.$$

Then

$$M = \mathrm{LLP}(\mathrm{RLP}(M)) = \mathrm{cof}\ M.$$

N.B. Expanding on 5.1.10, one can take for "I" the set

$$\{\partial|\underline{\square}|(n) \to |\underline{\square}|(n) : n \geq 0\}.$$

5.2.7 REMARK Let $\sqcap_{i,\varepsilon}^{n}$ $(n \geq 1,\ 1 \leq i \leq n,\ \varepsilon = 0,1)$ be the cubical subset of $|\underline{\square}|(n)$ given by

$$\sqcap_{i,\varepsilon}^{n}(I^m) = \{f:I^m \to I^n : \exists \text{ a factorization } f:I^m \to I^{n-1} \xrightarrow{\alpha} I^n\ (\alpha \neq \delta_{i,\varepsilon}^{n})\}.$$

Then one can take for "J" the set

$$\{\sqcap_{i,\varepsilon}^{n} \to |\underline{\square}|(n)\}.$$

In the current setting, the machinery of Kan extensions assigns to each $T \in \mathrm{Ob}[\underline{|\underline{\square}|},\hat{\underline{\Delta}}]$ its realization functor $\Gamma_T \in \mathrm{Ob}[\underline{\widehat{|\underline{\square}|}},\hat{\underline{\Delta}}]$, itself a left adjoint for the singular functor $\sin_T:\hat{\underline{\Delta}} \to \underline{\widehat{|\underline{\square}|}}$.

Specialize and let T be the composite

$$\underline{|\underline{\square}|} \xrightarrow{c} \underline{\mathrm{CAT}} \xrightarrow{\mathrm{ner}} \hat{\underline{\Delta}}.$$

Put

$$\begin{cases} c_! = \Gamma_{\mathrm{ner}} \circ c \\ c^* = \sin_{\mathrm{ner}} \circ c^{\boldsymbol\cdot} \end{cases}$$

Then

$$\begin{cases} c_!:\underline{\widehat{|\underline{\square}|}} \to \hat{\underline{\Delta}} \\ c^*:\hat{\underline{\Delta}} \to \underline{\widehat{|\underline{\square}|}}. \end{cases}$$

So $\forall$ n,

$$c_! |\underline{\square}|(n) = \Delta[1]^n$$

and $\forall$ X $\in$ Ob $\hat{\underline{\Delta}}$,

$$(c^*X)_n = \text{Mor}(\Delta[1]^n, X).$$

5.2.8 REMARK If $\underline{C}$ is a small category, then

$$\text{ner}_c \underline{C} \approx c^*\text{ner }\underline{C}.$$

In fact,

$$\begin{aligned}(c^*\text{ner }\underline{C})_n &= \text{Mor}(\Delta[1]^n, \text{ner }\underline{C}) \\ &\approx \text{Mor}(\text{cat }\Delta[1]^n, \underline{C}) \\ &\approx \text{Mor}((\text{cat }\Delta[1])^n, \underline{C}) \\ &\approx \text{Mor}([1]^n, \underline{C}) \\ &= \text{ner}_c(\underline{C})(I^n).\end{aligned}$$

Equip $\underline{\widehat{|\underline{\square}|}}$ with its Cisinski structure and $\hat{\underline{\Delta}}$ with its Kan structure.

5.2.9 LEMMA The adjoint situation $(c_!, c^*)$ is a model pair.

More is true: The model pair $(c_!, c^*)$ is a model equivalence. Therefore the categories

$$\left[\begin{array}{l}\underline{H\widehat{|\underline{\square}|}} \\ \\ \underline{H\hat{\Delta}}\end{array}\right.$$

are canonically equivalent.

APPENDIX

CATEGORICAL BACKGROUND

TOPICS

CATEGORICAL BACKGROUND

DEFINITIONS AND NOTATION

Given a category $\underline{C}$, denote by Ob $\underline{C}$ its class of objects and by Mor $\underline{C}$ its class of morphisms. If $X,Y \in$ Ob $\underline{C}$ is an ordered pair of objects, then $\text{Mor}(X,Y)$ is the set of morphisms (or arrows) from X to Y. An element $f \in \text{Mor}(X,Y)$ is said to have <u>domain</u> X and <u>codomain</u> Y. One writes $f:X \to Y$ or $X \xrightarrow{f} Y$. Composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is denoted by $g \circ f$.

A morphism $f:X \to Y$ in a category $\underline{C}$ is said to be an <u>isomorphism</u> if there exists a morphism $g:Y \to X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. If g exists, then g is unique. It is called the <u>inverse</u> of f and is denoted by f^{-1}. Objects $X,Y \in$ Ob $\underline{C}$ are said to be <u>isomorphic</u>, written $X \approx Y$, provided there is an isomorphism $f:X \to Y$. The relation "isomorphic to" is an equivalence relation on Ob $\underline{C}$.

A functor $F:\underline{C} \to \underline{D}$ is said to be <u>faithful</u> (<u>full</u>) if for any ordered pair $X,Y \in$ Ob $\underline{C}$, the map $\text{Mor}(X,Y) \to \text{Mor}(FX,FY)$ is injective (surjective). If F is full and faithful, then F <u>reflects isomorphisms</u> or still, is <u>conservative</u>, i.e., f is an isomorphism iff Ff is an isomorphism.

A functor $F:\underline{C} \to \underline{D}$ is said to be an <u>isomorphism</u> if there exists a functor $G:\underline{D} \to \underline{C}$ such that $G \circ F = \text{id}_{\underline{C}}$ and $F \circ G = \text{id}_{\underline{D}}$. A functor is an isomorphism iff it is full, faithful, and bijective on objects. Categories $\underline{C}$ and $\underline{D}$ are said to be <u>isomorphic</u> provided there is an isomorphism $F:\underline{C} \to \underline{D}$.

[Note: An isomorphism between categories is the same as an isomorphism in the "category of categories".]

A functor $F:\underline{C} \rightarrow \underline{D}$ is said to be an <u>equivalence</u> if there exists a functor $G:\underline{D} \rightarrow \underline{C}$ such that $G \circ F \approx \mathrm{id}_{\underline{C}}$ and $F \circ G \approx \mathrm{id}_{\underline{D}}$, the symbol $\approx$ standing for natural isomorphism. A functor is an equivalence iff it is full, faithful, and has a <u>representative image</u>, i.e., for any $Y \in \mathrm{Ob}\ \underline{D}$ there exists an $X \in \mathrm{Ob}\ \underline{C}$ such that FX is isomorphic to Y. Categories $\underline{C}$ and $\underline{D}$ are said to be <u>equivalent</u> provided that there is an equivalence $F:\underline{C} \rightarrow \underline{D}$. The object isomorphism types of equivalent categories are in a one-to-one correspondence.

[Note: If F and G are injective on objects, then $\underline{C}$ and $\underline{D}$ are isomorphic (categorical "Schroeder-Bernstein").]

<u>N.B.</u> If $\underline{C},\underline{D}$ are equivalent and $\underline{D},\underline{E}$ are equivalent, then $\underline{C},\underline{E}$ are equivalent.

A category is <u>skeletal</u> if isomorphic objects are equal. Given a category $\underline{C}$, a <u>skeleton</u> of $\underline{C}$ is a full, skeletal subcategory $\bar{\underline{C}}$ for which the inclusion $\bar{\underline{C}} \rightarrow \underline{C}$ has a representative image (hence is an equivalence). Every category has a skeleton and any two skeletons of a category are isomorphic.

A category is said to be <u>discrete</u> if all its morphisms are identities. Every class is the class of objects of a discrete category.

[Note: A category is <u>small</u> if its class of objects is a set; otherwise it is <u>large</u>. A category is <u>finite</u> (<u>countable</u>) if its class of morphisms is a finite (countable) set.]

EXAMPLES

Here is a list of commonly occurring categories.

(1) <u>SET</u>, the category of sets, and $\underline{\mathrm{SET}_*}$, the category of pointed sets. If $X,Y \in \mathrm{Ob}\ \underline{\mathrm{SET}}$, then $\mathrm{Mor}(X,Y) = F(X,Y)$, the functions from X to Y, and if (X,x_0), $(Y,y_0) \in \mathrm{Ob}\ \underline{\mathrm{SET}_*}$, then $\mathrm{Mor}((X,x_0),(Y,y_0)) = F(X,x_0;Y,y_0)$, the base point preserving

functions from X to Y.

(2) TOP, the category of topological spaces, and TOP_*, the category of pointed topological spaces. If $X,Y \in \mathrm{Ob}\ \mathrm{TOP}$, then $\mathrm{Mor}(X,Y) = C(X,Y)$, the continuous functions from X to Y, and if $(X,x_0),(Y,y_0) \in \mathrm{Ob}\ \mathrm{TOP}_*$, then $\mathrm{Mor}((X,x_0),(Y,y_0)) = C(X,x_0;Y,y_0)$, the base point preserving continuous functions from X to Y.

(3) HTOP, the homotopy category of topological spaces, and HTOP_*, the homotopy category of pointed topological spaces. If $X,Y \in \mathrm{Ob}\ \mathrm{HTOP}$, then $\mathrm{Mor}(X,Y) = [X,Y]$, the homotopy classes in $C(X,Y)$, and if $(X,x_0),(Y,y_0) \in \mathrm{Ob}\ \mathrm{HTOP}_*$, then $\mathrm{Mor}((X,x_0),(Y,y_0)) = [X,x_0;Y,y_0]$, the homotopy classes in $C(X,x_0;Y,y_0)$.

(4) HAUS, the full subcategory of TOP whose objects are the Hausdorff spaces and CPTHAUS, the full subcategory of HAUS whose objects are the compact spaces.

(5) ΠX, the fundamental groupoid of a topological space X.

(6) GR, AB, RG (A-MOD or MOD-A), the category of groups, abelian groups, rings with unit (left or right A-modules, $A \in \mathrm{Ob}\ \mathrm{RG}$).

(7) 0, the category with no objects and no arrows. 1, the category with one object and one arrow. 2, the category with two objects and one arrow not the identity.

(8) CAT, the category whose objects are the small categories and whose morphisms are the functors between them.

(9) GRD, the full subcategory of CAT whose objects are the groupoids, i.e., the small categories in which every morphism is invertible.

(10) PRECAT, the category whose objects are the small precategories (a.k.a. graphs) and whose morphisms are the prefunctors between them.

EXAMPLE Every arrow $f:X \to Y$ of $\underline{C}$ appears as an arrow $f^{OP}:Y \to X$ of $\underline{C}^{OP}$. This said, define a functor $OP:\underline{CAT} \to \underline{CAT}$ on objects by

$$OP(\underline{C}) = \underline{C}^{OP}$$

and on morphisms $F:\underline{C} \to \underline{D}$ by

$$F^{OP}(Y \xrightarrow{f^{OP}} X) = (Ff)^{OP}.$$

Then

$$OP \circ OP = id_{\underline{CAT}}.$$

EXAMPLE The assignment

$$\begin{cases} \underline{TOP} \to \underline{GRD} \\ X \to \Pi X \end{cases}$$

is a functor.

[Note: A continuous function $f:X \to Y$ induces a functor $F_f:\Pi X \to \Pi Y$, viz. $F_f x = f(x)$, $F_f[\gamma] = [f \circ \gamma]$ $(\gamma \in C([0,1],X))$.]

In this book, the foundation for category theory is the "one universe" approach taken by Herrlich-Strecker[†]. The key words are "set", "class", and "conglomerate". Thus the issue is not only one of size but also of membership (every set is a class and every class is a conglomerate). Example: $\{\text{Ob } \underline{SET}\}$ is a conglomerate, not a class (the members of a class are sets).

A <u>metacategory</u> is defined in the same way as a category except that the objects and the morphisms are allowed to be conglomerates and the requirement that the conglomerate of morphisms between two objects be a set is dropped.

[†] *Category Theory*, Heldermann Verlag, 1979.

While there are exceptions, most categorical concepts have metacategorical analogs or interpretations.

[Note: Every category is a metacategory. On the other hand, it can happen that a metacategory is isomorphic to a category but is not itself a category. Still, the convention is to overlook this technical nicety and treat such a metacategory as a category.]

N.B. Additional discussion and information can be found in Shulman[†].

NOTATION $\mathbf{CAT}$, the metacategory whose objects are the categories and whose morphisms are the functors between them.

COMMA CATEGORIES

Given categories $\underline{A},\underline{B},\underline{C}$ and functors $\begin{cases} T:\underline{A} \to \underline{C} \\ S:\underline{B} \to \underline{C} \end{cases}$, the comma category $|T,S|$ is the category whose objects are the triples (X,f,Y): $\begin{cases} X \in \text{Ob } \underline{A} \\ Y \in \text{Ob } \underline{B} \end{cases}$ & $f \in \text{Mor}(TX,SY)$ and whose morphisms $(X,f,Y) \to (X',f',Y')$ are the pairs

(ϕ,ψ): $\begin{cases} \phi \in \text{Mor}(X,X') \\ \psi \in \text{Mor}(Y,Y') \end{cases}$ for which the square

$$\begin{array}{ccc} TX & \xrightarrow{f} & SY \\ {\scriptstyle T\phi}\downarrow & & \downarrow{\scriptstyle S\psi} \\ TX' & \xrightarrow[f']{} & SY' \end{array}$$

† arXiv:0810.1279

commutes. Composition is defined componentwise and the identity attached to (X,f,Y) is $(\mathrm{id}_X,\mathrm{id}_Y)$.

LEMMA There are functors

$$\begin{cases} P:|T,S| \to \underline{A} \\ Q:|T,S| \to \underline{B} \end{cases}$$

and a canonical natural transformation

$$T \circ P \to S \circ Q.$$

PROOF Let

$$\begin{cases} P(X,f,Y) = X \\ P(\phi,\psi) = \phi \end{cases}, \quad \begin{cases} Q(X,f,Y) = Y \\ Q(\phi,\psi) = \psi \end{cases}$$

and define

$$\Xi \in \mathrm{Nat}(T \circ P, S \circ Q)$$

by

$$\Xi_{(X,f,Y)} = f.$$

[Note: In general, the diagram

$$\begin{tikzcd} {|T,S|} \arrow[r, "Q"] \arrow[d, "P"'] & \underline{B} \arrow[d, "S"] \\ \underline{A} \arrow[r, "T"'] & \underline{C} \end{tikzcd}$$

does not commute.]

$(A\backslash\underline{C})$ Let $A \in \mathrm{Ob}\,\underline{C}$ and write K_A for the constant functor $\underline{1} \to \underline{C}$ with value A -- then

$$A\backslash\underline{C} \equiv |K_A,\mathrm{id}_{\underline{C}}|$$

is the category of objects under A.

($\underline{C}$/B) Let $B \in \text{Ob } \underline{C}$ and write K_B for the constant functor $\underline{1} \rightarrow \underline{C}$ with value B -- then

$$\underline{C}/B \equiv |\text{id}_{\underline{C}}, K_B|$$

is the category of objects over B.

N.B. The comma category $|K_A, K_B|$ is $\text{Mor}(A,B)$ viewed as a discrete category.

The arrow category $\underline{C}(\rightarrow)$ of $\underline{C}$ is the comma category $|\text{id}_{\underline{C}}, \text{id}_{\underline{C}}|$.

FUNCTOR CATEGORIES

Let $\begin{cases} F:\underline{C} \rightarrow \underline{D} \\ G:\underline{C} \rightarrow \underline{D} \end{cases}$ be functors -- then a natural transformation Ξ from F to G is a function that assigns to each $X \in \text{Ob } \underline{C}$ an element $\Xi_X \in \text{Mor}(FX,GX)$ such that for every $f \in \text{Mor}(X,Y)$ the square

$$\begin{tikzcd} FX \arrow[r, "\Xi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\Xi_Y"'] & GY \end{tikzcd}$$

commutes, Ξ being termed a natural isomorphism if all the Ξ_X are isomorphisms, in which case F and G are said to be naturally isomorphic, written $F \approx G$.

Given categories $\begin{cases} \underline{C} \\ \underline{D} \end{cases}$, the functor category $[\underline{C},\underline{D}]$ is the metacategory whose objects are the functors $F:\underline{C} \rightarrow \underline{D}$ and whose morphisms are the natural transformations Nat(F,G) from F to G. In general, $[\underline{C},\underline{D}]$ need not be isomorphic to a category, although this will be true if $\underline{C}$ is small.

[Note: The isomorphisms in $[\underline{C},\underline{D}]$ are the natural isomorphisms.]

<u>N.B.</u> The identity $id_F \in Nat(F,F)$ is defined by $(id_F)_X = id_{FX}$ and if $F \xrightarrow{\Xi} G$, $G \xrightarrow{\Omega} H$ are natural transformations, then $\Omega \circ \Xi : F \to H$ is the natural transformation that assigns to each X the composition $\Omega_X \circ \Xi_X : FX \to HX$.

(K^*) Let $K:\underline{A} \to \underline{C}$ be a functor -- then there is an induced functor

$$K^*:[\underline{C},\underline{D}] \to [\underline{A},\underline{D}]$$

given on objects by

$$K^*F = F \circ K$$

and on morphisms by

$$(K^*\Xi)_A = \Xi_{KA}.$$

(L_*) Let $L:\underline{D} \to \underline{B}$ be a functor -- then there is an induced functor

$$L_*:[\underline{C},\underline{D}] \to [\underline{C},\underline{B}]$$

given on objects by

$$L_*F = L \circ F$$

and on morphisms by

$$(L_*\Xi)_X = L\Xi_X.$$

Write $\begin{cases} \Xi K \\ L\Xi \end{cases}$ in place of $\begin{cases} K^*\Xi \\ L_*\Xi \end{cases}$, so $L(\Xi K) = (L\Xi)K$ -- then

$$\begin{cases} \Xi(K \circ K') = (\Xi K)K' \\ (\Xi' \circ \Xi)K = (\Xi' K) \circ (\Xi K) \end{cases} \quad \text{and} \quad \begin{cases} (L' \circ L)\Xi = L'(L\Xi) \\ L(\Xi' \circ \Xi) = (L\Xi') \circ (L\Xi), \end{cases}$$

YONEDA THEORY

Associated with any object X in a category $\underline{C}$ is the functor $\mathrm{Mor}(X,—) \in \mathrm{Ob}[\underline{C},\underline{SET}]$ and the functor $\mathrm{Mor}(—,X) \in \mathrm{Ob}[\underline{C}^{OP},\underline{SET}]$. If $F \in \mathrm{Ob}[\underline{C},\underline{SET}]$ is a functor or if $F \in \mathrm{Ob}[\underline{C}^{OP},\underline{SET}]$ is a functor, then the Yoneda lemma establishes a bijection ι_X between $\mathrm{Nat}(\mathrm{Mor}(X,—),F)$ or $\mathrm{Nat}(\mathrm{Mor}(—,X),F)$ and FX, viz. $\iota_X(\Xi) = \Xi_X(\mathrm{id}_X)$. Therefore the assignments $\begin{cases} X \to \mathrm{Mor}(X,—) \\ X \to \mathrm{Mor}(—,X) \end{cases}$ lead to functors $\begin{cases} \underline{C}^{OP} \to [\underline{C},\underline{SET}] \\ \underline{C} \to [\underline{C}^{OP},\underline{SET}] \end{cases}$ that are full, faithful, and injective on objects, the <u>Yoneda embeddings</u>. One says that F is <u>representable</u> (by X) if F is naturally isomorphic to $\mathrm{Mor}(X,—)$ or $\mathrm{Mor}(—,X)$. Representing objects are isomorphic.

EXAMPLE The forgetful functor $U:\underline{TOP} \to \underline{SET}$ is representable:

$$\forall\ X,\ \mathrm{Mor}(\{*\},X) \approx UX.$$

The forgetful functor $U:\underline{GR} \to \underline{SET}$ is representable:

$$\forall\ X,\ \mathrm{Mor}(Z,X) \approx UX.$$

The forgetful functor $U:\underline{RG} \to \underline{SET}$ is representable:

$$\forall\ X,\ \mathrm{Mor}(Z[t],X) \approx UX.$$

It is traditional to write

$$\hat{\underline{C}} = [\underline{C}^{OP},\underline{SET}]$$

and call an object of $\hat{\underline{C}}$ a <u>presheaf</u> (of sets) on $\underline{C}$.

EXAMPLE We have

$$\begin{cases} \hat{\underline{0}} = \underline{1} \\ \hat{\underline{1}} \approx \underline{SET}. \end{cases}$$

Given $X \in \text{Ob } \underline{C}$, put

$$h_X = \text{Mor}(\text{---},X).$$

Then

$$\text{Mor}(X,Y) \approx \text{Nat}(h_X,h_Y)$$

and in this notation the Yoneda embedding

$$Y_{\underline{C}}:\underline{C} \to \hat{\underline{C}}$$

sends X to h_X.

EXAMPLE Let $F:\underline{SET}^{OP} \to \underline{SET}$ be the functor that sends X to 2^X (the set of all subsets of X) and sends $f:X \to Y$ to $f^{-1}:2^Y \to 2^X$ -- then F is representable:

$$F \approx h_{\{0,1\}}.$$

EXAMPLE Let $F:\underline{TOP}^{OP} \to \underline{SET}$ be the functor that sends X to τ_X (the set of open subsets of X) and sends $f:X \to Y$ to $f^{-1}:\tau_Y \to \tau_X$ -- then F is representable:

$$F \approx h_{\{0,1\}},$$

$\{0,1\}$ being Sierpinski space.

[Note: This fails if $\underline{TOP}$ is replaced by $\underline{HAUS}$.]

MORPHISMS

A morphism $f:X \to Y$ in a category $\underline{C}$ is said to be a monomorphism if it is left cancellable with respect to composition, i.e., for any pair of morphisms $u,v:Z \to X$ such that $f \circ u = f \circ v$, there follows $u = v$.

A morphism $f:X \to Y$ in a category $\underline{C}$ is said to be an epimorphism if it is right cancellable with respect to composition, i.e., for any pair of morphisms

$u,v:Y \to Z$ such that $u \circ f = v \circ f$, there follows $u = v$.

A morphism is said to be a <u>bimorphism</u> if it is both a monomorphism and an epimorphism. Every isomorphism is a bimorphism. A category is said to be <u>balanced</u> if every bimorphism is an isomorphism. The categories <u>SET</u>, <u>GR</u>, and <u>AB</u> are balanced but the category <u>TOP</u> is not.

EXAMPLE In <u>SET</u>, <u>GR</u>, and <u>AB</u>, a morphism is a monomorphism (epimorphism) iff it is injective (surjective). In any full subcategory of <u>TOP</u>, a morphism is a monomorphism iff it is injective. In the full subcategory of $\underline{\text{TOP}}_*$ whose objects are the connected spaces, there are monomorphisms that are not injective on the underlying sets (covering projections in this category are monomorphisms). In <u>TOP</u>, a morphism is an epimorphism iff it is surjective but in <u>HAUS</u>, a morphism is an epimorphism iff it has a dense range. The homotopy class of a monomorphism (epimorphism) in <u>TOP</u> need not be a monomorphism (epimorphism) in <u>HTOP</u>. In <u>CAT</u>, a morphism is a monomorphism iff it is injective on objects and fully faithful. On the other hand, in <u>CAT</u> there are epimorphisms which are surjective on objects but which are not surjective on morphism sets.

LEMMA Let $\underline{C}$ be a small category -- then a morphism Ξ in $[\underline{C},\underline{\text{SET}}]$ is a monomorphism iff $\forall\ X \in \text{Ob}\ \underline{C}$, Ξ_X is a monomorphism in <u>SET</u>.

[Note: This can fail if <u>SET</u> is replaced by an arbitrary category $\underline{D}$.]

Given a category $\underline{C}$ and an object X in $\underline{C}$, let $M(X)$ be the class of all pairs (Y,f), where $f:Y \to X$ is a monomorphism. Two elements (Y,f) and (Z,g) of $M(X)$ are deemed equivalent if there exists an isomorphism $\phi:Y \to Z$ such that $f = g \circ \phi$. A <u>representative class of monomorphisms</u> in $M(X)$ is a subclass of $M(X)$ that is a system of representatives for this equivalence relation. $\underset{\sim}{C}$ is said to be <u>well-powered</u> provided that each of its objects has a representative class of monomorphisms which is a set.

Given a category $\underset{\sim}{C}$ and an object X in $\underset{\sim}{C}$, let $E(X)$ be the class of all pairs (Y,f), where $f:X \to Y$ is an epimorphism. Two elements (Y,f) and (Z,g) of $E(X)$ are deemed equivalent if there exists an isomorphism $\phi:Y \to Z$ such that $g = \phi \circ f$. A <u>representative class of epimorphisms</u> in $E(X)$ is a subclass of $E(X)$ that is a system of representatives for this equivalence relation. $\underset{\sim}{C}$ is said to be <u>cowell-powered</u> provided that each of its objects has a representative class of epimorphisms which is a set.

EXAMPLE <u>SET</u>, <u>GR</u>, <u>AB</u>, <u>TOP</u> (or <u>HAUS</u>) are wellpowered and cowellpowered.

THEOREM <u>CAT</u> is wellpowered and cowellpowered.

A monomorphism $f:X \to Y$ in a category $\underset{\sim}{C}$ is said to be <u>extremal</u> provided that in any factorization $f = h \circ g$, if g is an epimorphism, then g is an isomorphism.

An epimorphism $f:X \to Y$ in a category $\underset{\sim}{C}$ is said to be <u>extremal</u> provided that in any factorization $f = h \circ g$, if h is a monomorphism, then h is an isomorphism.

In a balanced category, every monomorphism (epimorphism) is extremal. In any category, a morphism is an isomorphism iff it is both a monomorphism and an extremal epimorphism iff it is both an extremal monomorphism and an epimorphism.

EXAMPLE In <u>TOP</u>, a monomorphism is extremal iff it is an embedding but in <u>HAUS</u>, a monomorphism is extremal iff it is a closed embedding. In <u>TOP</u> or <u>HAUS</u>, an epimorphism is extremal iff it is a quotient map.

A morphism $r:Y \to X$ in a category $\underset{\sim}{C}$ is called a <u>retraction</u> if there exists a morphism $i:X \to Y$ such that $r \circ i = id_X$, in which case X is said to be a retract of Y.

EXAMPLE Consider the arrow category $\underline{C}(\to)$ and suppose that $\begin{cases} f \in Mor(X,X') \\ g \in Mor(Y,Y') \end{cases}$ --

then to say that f is a retract of g means that there exists a pair

$$(i,i'): \begin{cases} i \in Mor(X,Y) \\ i' \in Mor(X',Y') \end{cases}$$

and a pair

$$(r,r'): \begin{cases} r \in Mor(Y,X) \\ r' \in Mor(Y',X') \end{cases}$$

such that

$$(r,r') \circ (i,i') = id_f$$

or still,

$$(r \circ i, r' \circ i') = (id_X, id_{X'}).$$

In other words, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "r"] \arrow[d, "g"'] & X \arrow[d, "f"'] \\ X' \arrow[r, "i'"'] & Y' \arrow[r, "r'"'] & X', \end{tikzcd}$$

where $r \circ i = id_X$, $r' \circ i' = id_{X'}$.

[Note: If g is an isomorphism and if f is a retract of g, then f is an isomorphism.]

IDEMPOTENTS

A morphism $e:X \to X$ in a category $\underline{C}$ is <u>idempotent</u> if $e \circ e = e$. An idempotent $e:X \to X$ is <u>split</u> if $\exists\ Y \in \mathrm{Ob}\ \underline{C}$ and morphisms $\phi:X \to Y$, $\psi:Y \to X$ such that $e = \psi \circ \phi$ and $\phi \circ \psi = \mathrm{id}_Y$.

EXAMPLE Every idempotent in <u>SET</u> is split.

Given a category $\underline{C}$, there is a category $\tilde{\underline{C}}$ in which idempotents split and a functor $E:C \to \tilde{C}$ that is full, faithful, and injective on objects with the following property: Every functor from $\underline{C}$ to a category in which idempotents split has an extension to $\tilde{C}$, unique up to natural isomorphism.

SEPARATION AND COSEPARATION

Given a category $\underline{C}$, a set $\mathcal{U}$ of objects in $\underline{C}$ is said to be a <u>separating set</u> if for every pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma:U \to X$ such that $f \circ \sigma \neq g \circ \sigma$. An object U in $\underline{C}$ is said to be a <u>separator</u> if $\{U\}$ is a separating set, i.e., if the functor $\mathrm{Mor}(U,—):\underline{C} \to \underline{\mathrm{SET}}$ is faithful. If $\underline{C}$ is balanced, finitely complete, and has a separating set, then $\underline{C}$ is wellpowered. Every cocomplete cowellpowered category with a separator is wellpowered and complete. If $\underline{C}$ has coproducts, then a $U \in \mathrm{Ob}\ \underline{C}$ is a separator iff each $X \in \mathrm{Ob}\ \underline{C}$ admits an epimorphism $\coprod U \to X$.

[Note: Suppose that $\underline{C}$ is small -- then the representable functors are a separating set for $[\underline{C},\underline{\mathrm{SET}}]$.]

EXAMPLE Every nonempty set is a separator for SET. SET × SET has no separators but the set $\{(\emptyset,\{0\}),\ (\{0\},\emptyset)\}$ is a separating set. Every nonempty discrete topological space is a separator for TOP (or HAUS). $\mathbb{Z}$ is a separator for GR and AB, while $\mathbb{Z}[t]$ is a separator for RG. In A-MOD, A (as a left A-module) is a separator and in MOD-A, A (as a right A-module) is a separator.

Given a category $\underline{C}$, a set $\mathcal{U}$ of objects in $\underline{C}$ is said to be a coseparating set if for every pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma:Y \to U$ such that $\sigma \circ f \neq \sigma \circ g$. An object U in $\underline{C}$ is said to be a coseparator if {U} is a coseparating set, i.e., if the cofunctor $\mathrm{Mor}(—,U):\underline{C} \to \underline{SET}$ is faithful. If $\underline{C}$ is balanced, finitely cocomplete, and has a coseparating set, then $\underline{C}$ is cowellpowered. Every complete wellpowered category with a coseparator is cowellpowered and cocomplete. If $\underline{C}$ has products, then a $U \in \mathrm{Ob}\ \underline{C}$ is a coseparator iff each $X \in \mathrm{Ob}\ \underline{C}$ admits a monomorphism $X \to \prod U$.

EXAMPLE Every set with at least two elements is a coseparator for SET. Every indiscrete topological space with at least two elements is a coseparator for TOP. $\mathbb{Q}/\mathbb{Z}$ is a coseparator for AB. None of the categories GR, RG, HAUS has a coseparating set.

INJECTIVES

Given a category $\underline{C}$, an object Q in $\underline{C}$ is said to be injective if the cofunctor $\mathrm{Mor}(—,Q):\underline{C} \to \underline{SET}$ converts monomorphisms into epimorphisms. In other words: Q is injective iff for each monomorphism $f:X \to Y$ and each morphism $\phi:X \to Q$, there exists a morphism $g:Y \to Q$ such that $g \circ f = \phi$. A product of injective objects is injective.

A category $\underline{C}$ is said to have <u>enough injectives</u> provided that for any $X \in \text{Ob } \underline{C}$, there is a monomorphism $X \to Q$, with Q injective. If a category has products and an injective coseparator, then it has enough injectives.

EXAMPLE The injective objects in the category of Banach spaces and linear contractions are, up to isomorphism, the $C(X)$, where X is an extremally disconnected compact Hausdorff space. In $\underline{AB}$, the injective objects are the divisible abelian groups (and Q/Z is an injective coseparator) but the only injective objects in $\underline{GR}$ or $\underline{RG}$ are the final objects.

SOURCES AND SINKS

A <u>source</u> in a category $\underline{C}$ is a collection of morphisms $f_i:X \to X_i$ indexed by a set I and having a common domain. An <u>n-source</u> is a source for which $\#I = n$.

A <u>sink</u> in a category $\underline{C}$ is a collection of morphisms $f_i:X_i \to X$ indexed by a set I and having a common codomain. An <u>n-sink</u> is a sink for which $\#I = n$.

LIMITS AND COLIMITS

A <u>diagram</u> in a category $\underline{C}$ is a functor $\Delta:\underline{I} \to \underline{C}$, where $\underline{I}$ is a small category, the <u>indexing category</u>. To facilitate the introduction of sources and sinks associated with Δ, we shall write Δ_i for the image in $\text{Ob } \underline{C}$ of $i \in \text{Ob } \underline{I}$.

(lim) Let $\Delta:\underline{I} \to \underline{C}$ be a diagram — then a source $\{f_i:X \to \Delta_i\}$ is said to be <u>natural</u> if for each $\delta \in \text{Mor } \underline{I}$, say $i \xrightarrow{\delta} j$, $\Delta\delta \circ f_i = f_j$. A <u>limit</u> of Δ is a natural source $\{\ell_i:L \to \Delta_i\}$ with the property that if $\{f_i:X \to \Delta_i\}$ is a natural source, then there exists a unique morphism $\phi:X \to L$ such that $f_i = \ell_i \circ \phi$ for all

$i \in \mathrm{Ob}\, \underline{I}$. Limits are essentially unique. Notation: $L = \lim_{\underline{I}} \Delta$ (or $\lim \Delta$).

(colim) Let $\Delta: \underline{I} \to \underline{C}$ be a diagram — then a sink $\{f_i: \Delta_i \to X\}$ is said to be natural if for each $\delta \in \mathrm{Mor}\, \underline{I}$, say $i \xrightarrow{\delta} j$, $f_i = f_j \circ \Delta\delta$. A colimit of Δ is a natural sink $\{\ell_i: \Delta_i \to L\}$ with the property that if $\{f_i: \Delta_i \to X\}$ is a natural sink, then there exists a unique morphism $\phi: L \to X$ such that $f_i = \phi \circ \ell_i$ for all $i \in \mathrm{Ob}\, \underline{I}$. Colimits are essentially unique. Notation: $L = \mathrm{colim}_{\underline{I}} \Delta$ (or $\mathrm{colim}\, \Delta$).

There are a number of basic constructions that can be viewed as a limit or colimit of a suitable diagram.

PRODUCTS AND COPRODUCTS

Let I be a set; let $\underline{I}$ be the discrete category with $\mathrm{Ob}\, \underline{I} = I$. Given a collection $\{X_i : i \in I\}$ of objects in $\underline{C}$, define a diagram $\Delta: \underline{I} \to \underline{C}$ by $\Delta_i = X_i$ $(i \in I)$.

(Products) A limit $\{\ell_i: L \to \Delta_i\}$ of Δ is said to be a product of the X_i. Notation: $L = \prod_i X_i$ (or X^I if $X_i = X$ for all i), $\ell_i = \mathrm{pr}_i$, the projection from $\prod_i X_i$ to X_i. Briefly put: Products are limits of diagrams with discrete indexing categories. In particular, the limit of a diagram having $\underline{0}$ for its indexing category is a final object in $\underline{C}$.

[Note: An object X in a category $\underline{C}$ is said to be final if for each object Y there is exactly one morphism from Y to X.]

(Coproducts) A colimit $\{\ell_i: \Delta_i \to L\}$ of Δ is said to be a coproduct of the X_i. Notation: $L = \coprod_i X_i$ (or $I \cdot X$ if $X_i = X$ for all i), $\ell_i = \mathrm{in}_i$, the injection from X_i to $\coprod_i X_i$. Briefly put: Coproducts are colimits of diagrams with discrete indexing categories. In particular, the colimit of a diagram having $\underline{0}$ for its indexing category is an initial object in $\underline{C}$.

[Note: An object X in a category $\underline{C}$ is said to be initial if for each object Y there is exactly one morphism from X to Y.]

EXAMPLE In the full subcategory of $\underline{TOP}$ whose objects are the locally connected spaces, the product is the product in $\underline{SET}$ equipped with the coarsest locally connected topology that is finer than the product topology. In the full subcategory of $\underline{TOP}$ whose objects are the compact Hausdorff spaces, the coproduct is the Stone-Čech compactification of the coproduct in $\underline{TOP}$.

EQUALIZERS AND COEQUALIZERS

Let $\underline{I}$ be the category $1 \bullet \underset{b}{\overset{a}{\rightrightarrows}} \bullet\ 2$. Given a pair of morphisms $u,v:X \to Y$ in $\underline{C}$, define a diagram $\Delta:\underline{I} \to \underline{C}$ by $\begin{bmatrix} \Delta_1 = X \\ \\ \Delta_2 = Y \end{bmatrix}$ & $\begin{bmatrix} \Delta a = u \\ \\ \Delta b = v \end{bmatrix}$.

(Equalizers) An equalizer in a category $\underline{C}$ of a pair of morphisms $u,v:X \to Y$ is a morphism $f:Z \to X$ with $u \circ f = v \circ f$ such that for any morphism $f':Z' \to X$ with $u \circ f' = v \circ f'$ there exists a unique morphism $\phi:Z' \to Z$ such that $f' = f \circ \phi$. The 2-source $X \xleftarrow{f} Z \xrightarrow{u \circ f} Y$ is a limit of Δ iff $Z \xrightarrow{f} X$ is an equalizer of $u,v:X \to Y$. Notation: $Z = eq(u,v)$.

[Note: Every equalizer is a monomorphism. A monomorphism is regular if it is an equalizer. A regular monomorphism is extremal.]

(Coequalizers) A <u>coequalizer</u> in a category $\underline{C}$ of a pair of morphisms $u,v:X \to Y$ is a morphism $f:Y \to Z$ with $f \circ u = f \circ v$ such that for any morphism $f':Y \to Z'$ with $f' \circ u = f' \circ v$ there exists a unique morphism $\phi:Z \to Z'$ such that $f' = \phi \circ f$. The 2-sink $Y \xrightarrow{f} Z \xleftarrow{f \circ u} X$ is a colimit of Δ iff $Y \xrightarrow{f} Z$ is a coequalizer of $u,v:X \to Y$. Notation: $Z = \mathrm{coeq}(u,v)$.

[Note: Every coequalizer is an epimorphism. An epimorphism is <u>regular</u> if it is a coequalizer. A regular epimorphism is extremal.]

REMARK There are two aspects to the notion of equalizer or coequalizer, namely: (1) Existence of f and (2) Uniqueness of ϕ. Given (1), (2) is equivalent to requiring that f be a monomorphism or an epimorphism. If (1) is retained and (2) is abandoned, then the terminology is <u>weak equalizer</u> or <u>weak coequalizer</u>. For example, $\underline{\mathrm{HTOP}}_*$ has neither equalizers nor coequalizers but does have weak equalizers and weak coequalizers.

EXAMPLE Given objects $\underline{C}$, $\underline{D}$ in $\underline{\mathrm{CAT}}$ and morphisms $F,G:\underline{C} \to \underline{D}$ in $\underline{\mathrm{CAT}}$, their equalizer $\mathrm{eq}(F,G)$ is the inclusion inc of the subcategory of $\underline{C}$ on which F,G coincide:

$$\mathrm{eq}(F,G) \xrightarrow{\mathrm{inc}} \underline{C} \underset{G}{\overset{F}{\rightrightarrows}} \underline{D},$$

where

$$\begin{cases} \mathrm{Ob}\ \mathrm{eq}(F,G) = \{X \in \mathrm{Ob}\ \underline{C} : FX = GX\} \\ \mathrm{Mor}\ \mathrm{eq}(F,G) = \{f \in \mathrm{Mor}\ \underline{C} : Ff = Gf\}. \end{cases}$$

EXAMPLE Take $\underline{C} = \underline{\mathrm{SET}}$ and consider a pair of morphisms $u,v:X \to Y$. Let $\sim$ be the equivalence relation generated by $\{(u(x),v(x)):x \in X\}$ -- then the canonical map $Y \to Y/\sim$ which assigns to each $y \in Y$ its equivalence class $[y]$ is a coequalizer of u,v.

PULLBACKS AND PUSHOUTS

Let $\underline{I}$ be the category $1 \bullet \xrightarrow{a} \underset{3}{\bullet} \xleftarrow{b} \bullet\, 2$. Given morphisms $\begin{cases} f:X \to Z \\ g:Y \to Z \end{cases}$ in $\underline{C}$, define a diagram $\Delta:\underline{I} \to \underline{C}$ by $\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \\ \Delta_3 = Z \end{cases}$ & $\begin{cases} \Delta a = f \\ \Delta b = g \end{cases}$.

(Pullbacks) Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is said to be a <u>pullback square</u> if for any 2-source $X \xleftarrow{\xi'} P' \xrightarrow{\eta'} Y$ with $f \circ \xi' = g \circ \eta'$ there exists a unique morphism $\phi:P' \to P$ such that $\xi' = \xi \circ \phi$ and $\eta' = \eta \circ \phi$. The 2-source $X \xleftarrow{\xi} P \xrightarrow{\eta} Y$ is called a <u>pullback</u> of the 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$. Notation: $P = X \times_Z Y$. Limits of Δ are pullback squares and conversely.

Let $\underline{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet\, 2$. Given morphisms $\begin{cases} f:Z \to X \\ g:Z \to Y \end{cases}$ in $\underline{C}$, define a diagram $\Delta:\underline{I} \to \underline{C}$ by $\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \\ \Delta_3 = Z \end{cases}$ & $\begin{cases} \Delta a = f \\ \Delta b = g \end{cases}$.

(Pushouts) Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

is said to be a <u>pushout square</u> if for any 2-sink $X \xrightarrow{\xi'} P' \xleftarrow{\eta'} Y$ with $\xi' \circ f = \eta' \circ g$ there exists a unique morphism $\phi: P \to P'$ such that $\xi' = \phi \circ \xi$ and $\eta' = \phi \circ \eta$. The 2-sink $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$ is called a <u>pushout</u> of the 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$. Notation: $P = X \sqcup_Z Y$. Colimits of Δ are pushout squares and conversely.

REMARK The result of dropping uniqueness in ϕ is <u>weak pullback</u> or <u>weak pushout</u>. Examples are the commutative squares that define fibration and cofibration in <u>TOP</u>.

EXAMPLE Let X and Y be topological spaces. Let $A \to X$ be a closed embedding and let $f: A \to Y$ be a continuous function — then the <u>adjunction space</u> $X \sqcup_f Y$ corresponding to the 2-source $X \leftarrow A \xrightarrow{f} Y$ is defined by the pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \sqcup_f Y, \end{tikzcd}$$

f being the <u>attaching map</u>. Agreeing to identify A with its image in X, the restriction of the projection $p: X \coprod Y \to X \sqcup_f Y$ to $\left[\begin{array}{l} X - A \\ Y \end{array}\right.$ is a homeomorphism of $\left[\begin{array}{l} X - A \\ Y \end{array}\right.$ onto an $\left[\begin{array}{l} \text{open} \\ \text{closed} \end{array}\right.$ subset of $X \sqcup_f Y$ and the images $\left[\begin{array}{l} p(X-A) \\ p(Y) \end{array}\right.$ partition $X \sqcup_f Y$.

FILTERED CATEGORIES AND FINAL FUNCTORS

Let $\underline{I} \neq \underline{0}$ be a small category -- then $\underline{I}$ is said to be filtered if

(F_1) Given any pair of objects i,j in $\underline{I}$, there exists an object k and morphisms $\begin{cases} i \to k \\ j \to k \end{cases}$;

(F_2) Given any pair of morphisms $a,b: i \to j$ in $\underline{I}$, there exists an object k and a morphism $c: j \to k$ such that $c \circ a = c \circ b$.

Every nonempty directed set $(I, \leq)$ can be viewed as a filtered category $\underline{I}$, where $\text{Ob}\ \underline{I} = I$ and $\text{Mor}(i,j)$ is a one element set when $i \leq j$ but is empty otherwise.

EXAMPLE Let $[\mathbb{N}]$ be the filtered category associated with the directed set of non-negative integers. Given a category $\underline{C}$, denote by $\underline{\text{FIL}}(\underline{C})$ the functor category $[[\mathbb{N}],\underline{C}]$ — then an object $(\underline{X},\underline{f})$ in $\underline{\text{FIL}}(\underline{C})$ is a sequence $\{X_n,f_n\}$, where $X_n \in \text{Ob}\ \underline{C}$ & $f_n \in \text{Mor}(X_n,X_{n+1})$, and a morphism $\phi: (\underline{X},\underline{f}) \to (\underline{Y},\underline{g})$ in $\underline{\text{FIL}}(\underline{C})$ is a sequence $\{\phi_n\}$, where $\phi_n \in \text{Mor}(X_n,Y_n)$ & $g_n \circ \phi_n = \phi_{n+1} \circ f_n$.

(Filtered Colimits) A filtered colimit in $\underline{C}$ is the colimit of a diagram $\Delta: \underline{I} \to \underline{C}$, where $\underline{I}$ is filtered.

(Cofiltered Limits) A cofiltered limit in $\underline{C}$ is the limit of a diagram $\Delta: \underline{I} \to \underline{C}$, where $\underline{I}$ is cofiltered.

[Note: A small category $\underline{I} \neq \underline{0}$ is said to be cofiltered provided that $\underline{I}^{\text{OP}}$ is filtered.]

EXAMPLE A Hausdorff space is compactly generated iff it is the filtered colimit in $\underline{\text{TOP}}$ of its compact subspaces. Every compact Hausdorff space is the cofiltered limit in $\underline{\text{TOP}}$ of compact metrizable spaces.

Given a small category $\underline{C}$, a <u>path</u> in $\underline{C}$ is a diagram σ of the form $X_0 \rightarrow X_1 \leftarrow \cdots \rightarrow X_{2n-1} \leftarrow X_{2n}$ $(n \geq 0)$. One says that σ <u>begins</u> at X_0 and <u>ends</u> at X_{2n}. The quotient of Ob $\underline{C}$ with respect to the equivalence relation obtained by declaring that $X' \sim X''$ iff there exists a path in $\underline{C}$ which begins at X' and ends at X'' is the set $\pi_0(\underline{C})$ of <u>components</u> of $\underline{C}$, $\underline{C}$ being called <u>connected</u> when the cardinality of $\pi_0(\underline{C})$ is one. The full subcategory of $\underline{C}$ determined by a component is connected and is maximal with respect to this property. If $\underline{C}$ has an initial object or a final object, then $\underline{C}$ is connected.

[Note: The concept of "path" makes sense in any category.]

EXAMPLE The assignment

$$\begin{cases} \underline{TOP} \rightarrow \underline{SET} \\ X \rightarrow \pi_0(\Pi X) \end{cases}$$

is a functor.

[Note: The elements of $\pi_0(\Pi X)$ are the path components of X.]

Let $\underline{I} \neq \underline{0}$ be a small category -- then $\underline{I}$ is said to be <u>pseudofiltered</u> if

(PF_1) Given any pair of morphisms $\begin{cases} a:i \rightarrow j \\ b:i \rightarrow k \end{cases}$ in $\underline{I}$, there exists an object ℓ and morphisms $\begin{cases} c:j \rightarrow \ell \\ d:k \rightarrow \ell \end{cases}$ such that $c \circ a = d \circ b$;

(PF_2) Given any pair of morphisms $a,b:i \rightarrow j$ in $\underline{I}$, there exists a morphism $c:j \rightarrow k$ such that $c \circ a = c \circ b$.

$\underline{I}$ is filtered iff $\underline{I}$ is connected and pseudofiltered. $\underline{I}$ is pseudofiltered iff its components are filtered.

Given small categories $\begin{bmatrix} \underline{I} \\ \underline{J} \end{bmatrix}$, a functor $\nabla:\underline{J} \to \underline{I}$ is said to be final provided that for every $i \in \mathrm{Ob}\,\underline{I}$, the comma category $|K_i,\nabla|$ is nonempty and connected. If $\underline{J}$ is filtered and $\nabla:\underline{J} \to \underline{I}$ is final, then $\underline{I}$ is filtered.

[Note: A subcategory of a small category is final if the inclusion is a final functor.]

Let $\nabla:\underline{J} \to \underline{I}$ be final. Suppose that $\Delta:\underline{I} \to \underline{C}$ is a diagram for which colim $\Delta \circ \nabla$ exists -- then colim Δ exists and the arrow colim $\Delta \circ \nabla \to$ colim Δ is an isomorphism. Corollary: If i is a final object in $\underline{I}$, then colim $\Delta \approx \Delta_i$.

[Note: Analogous considerations apply to limits so long as "final" is replaced throughout by "initial".]

REMARK Let $\underline{I}$ be a filtered category -- then there exists a directed set $(J,\le)$ and a final functor $\nabla:\underline{J} \to \underline{I}$.

Limits commute with limits. In other words, if $\Delta:\underline{I} \times \underline{J} \to \underline{C}$ is a diagram, then under the obvious assumptions

$$\lim_{\underline{I}} \lim_{\underline{J}} \Delta \approx \lim_{\underline{I}\times\underline{J}} \Delta \approx \lim_{\underline{J}\times\underline{I}} \Delta \approx \lim_{\underline{J}} \lim_{\underline{I}} \Delta.$$

Likewise, colimits commute with colimits. In general, limits do not commute with colimits. However, if $\Delta:\underline{I} \times \underline{J} \to \underline{SET}$ and if $\underline{I}$ is finite and $\underline{J}$ is filtered, then the arrow $\mathrm{colim}_{\underline{J}} \lim_{\underline{I}} \Delta \to \lim_{\underline{I}} \mathrm{colim}_{\underline{J}} \Delta$ is a bijection, so that in $\underline{SET}$ filtered colimits commute with finite limits.

[Note: It is also true that in $\underline{GR}$ or $\underline{AB}$, filtered colimits commute with finite limits. But, e.g., filtered colimits do not commute with finite limits in $\underline{SET}^{OP}$.]

COMPLETENESS AND COCOMPLETENESS

A category $\underline{C}$ is said to be complete (cocomplete) if for each small category $\underline{I}$, every $\Delta \in \mathrm{Ob}\,[\underline{I},\underline{C}]$ has a limit (colimit). The following are equivalent.

(1) $\underline{C}$ is complete (cocomplete).

(2) $\underline{C}$ has products and equalizers (coproducts and coequalizers).

(3) $\underline{C}$ has products and pullbacks (coproducts and pushouts).

EXAMPLE The categories SET, GR, and AB are complete and cocomplete. The same holds for TOP and TOP_* but not for HTOP and HTOP_*.

[Note: HAUS is complete; it is also cocomplete, being epireflective in TOP.]

THEOREM CAT is complete and cocomplete.

[Note: $\underline{0}$ is an initial object in CAT and $\underline{1}$ is a final object in CAT.]

A category $\underline{C}$ is said to be finitely complete (finitely cocomplete) if for each finite category $\underline{I}$, every $\Delta \in \mathrm{Ob}\,[\underline{I},\underline{C}]$ has a limit (colimit). The following are equivalent.

(1) $\underline{C}$ is finitely complete (finitely cocomplete).

(2) $\underline{C}$ has finite products and equalizers (finite coproducts and coequalizers).

(3) $\underline{C}$ has finite products and pullbacks (finite coproducts and pushouts).

EXAMPLE The full subcategory of TOP whose objects are the finite topological spaces is finitely complete and finitely cocomplete but neither complete nor cocomplete. A nontrivial group, considered as a category, is neither finitely complete nor finitely cocomplete.

If $\underline{C}$ is small and $\underline{D}$ is finitely complete and wellpowered (finitely cocomplete and cowellpowered), then $[\underline{C},\underline{D}]$ is wellpowered (cowellpowered).

EXAMPLE $\underline{\text{SET}}(\to)$, $\underline{\text{GR}}(\to)$, $\underline{\text{AB}}(\to)$, $\underline{\text{TOP}}(\to)$ (or $\underline{\text{HAUS}}(\to)$), $\underline{\text{CAT}}(\to)$ are wellpowered and cowellpowered.

[Note: The arrow category $\underline{C}(\to)$ of any category $\underline{C}$ is isomorphic to $[\underline{2},\underline{C}]$.]

PRESERVATION

Let $F:\underline{C} \to \underline{D}$ be a functor.

(a) F is said to preserve a limit $\{\ell_i:L \to \Delta_i\}$ (colimit $\{\ell_i:\Delta_i \to L\}$) of a diagram $\Delta:\underline{I} \to \underline{C}$ if $\{F\ell_i:FL \to F\Delta_i\}$ ($\{F\ell_i:F\Delta_i \to FL\}$) is a limit (colimit) of the diagram $F \circ \Delta:\underline{I} \to \underline{D}$.

(b) F is said to preserve limits (colimits) over an indexing category $\underline{I}$ if F preserves all limits (colimits) of diagrams $\Delta:\underline{I} \to \underline{C}$.

(c) F is said to preserve limits (colimits) if F preserves limits (colimits) over all indexing categories $\underline{I}$.

EXAMPLE The forgetful functor $\underline{\text{TOP}} \to \underline{\text{SET}}$ preserves limits and colimits. The forgetful functor $\underline{\text{GR}} \to \underline{\text{SET}}$ preserves limits and filtered colimits but not coproducts. The inclusion $\underline{\text{HAUS}} \to \underline{\text{TOP}}$ preserves limits and coproducts but not coequalizers. The inclusion $\underline{\text{AB}} \to \underline{\text{GR}}$ preserves limits but not colimits.

There are two rules that determine the behavior of $\left[\begin{array}{l}\text{Mor}(X,—)\\ \text{Mor}(—,X)\end{array}\right.$ with respect to limits and colimits.

(1) The functor $\text{Mor}(X,—):\underline{C} \to \underline{\text{SET}}$ preserves limits. Symbolically, therefore, $\text{Mor}(X,\lim \Delta) \approx \lim(\text{Mor}(X,—) \circ \Delta)$.

(2) The functor $\text{Mor}(—,X):\underline{C}^{\text{OP}} \to \underline{\text{SET}}$ converts colimits into limits. Symbolically, therefore, $\text{Mor}(\text{colim } \Delta,X) \approx \lim(\text{Mor}(—,X) \circ \Delta)$.

Limits and colimits in functor categories are computed "object by object". So, if $\underline{C}$ is a small category, then $\underline{D}$ (finitely) complete => $[\underline{C},\underline{D}]$ (finitely) complete and $\underline{D}$ (finitely) cocomplete => $[\underline{C},\underline{D}]$ (finitely) cocomplete.

In particular: $\hat{\underline{C}} = [\underline{C}^{OP},\underline{SET}]$ is complete and cocomplete.

[Note: An initial object $\emptyset_{\hat{\underline{C}}}$ in $\hat{\underline{C}}$ is the constant presheaf with value $\emptyset$. A final object $*_{\hat{\underline{C}}}$ in $\hat{\underline{C}}$ is the constant presheaf with value $\{*\}$.]

N.B. The Yoneda embedding $Y_{\underline{C}}:\underline{C} \rightarrow \hat{\underline{C}}$ preserves limits; it need not, however, preserve finite colimits. E.g.: Suppose that $\underline{C}$ has an initial object $\emptyset_{\underline{C}}$ -- then $h_{\emptyset_{\underline{C}}}$ and $\emptyset_{\hat{\underline{C}}}$ are not isomorphic.

EXAMPLE Let G be a nontrivial group, considered as a category $\underline{G}$ -- then the category of right G-sets is the category $[\underline{G}^{OP},\underline{SET}]$, thus is complete and cocomplete.

THEOREM Let $\underline{C}$ be a small category -- then every presheaf F is a colimit of representable presheaves: There exists a small category $\underline{I}_F$ and a functor $\Delta_F:\underline{I}_F \rightarrow \underline{C}$ such that

$$\operatorname{colim} Y_{\underline{C}} \circ \Delta_F \approx F.$$

[Let $\underline{I}_F$ be the category whose objects are the pairs (X,x), where $X \in \operatorname{Ob} \underline{C}$

and $x \in FX$, and whose morphisms $(X,x) \to (X',x')$ are the $f \in \mathrm{Mor}(X,X')$ such that $(Ff)x' = x$ -- then $\underline{I}_F$ is a small category and the assignment

$$\begin{cases} (X,x) \longrightarrow X \\ ((X,x) \xrightarrow{f} (X',x')) \to f \end{cases}$$

defines a functor $\Delta_F:\underline{I}_F \to \underline{C}$ with the stated properties. In this connection, bear in mind that

$$\mathrm{Nat}(h_X,F) \longleftrightarrow FX,$$

so each $(X,x) \in \mathrm{Ob}\ \underline{I}_F$ determines a natural transformation $\Xi_{(X,x)}:h_X \to F$ and $\forall\ f:(X,x) \to (X',x')$, we have

$$\Xi_{(X,x)} = \Xi_{(X',x')} \circ Y_{\underline{C}}(f).$$

[Note: Take $F = h_X$ -- then $\underline{I}_{h_X}$ has a final object, namely the pair (X,id_X).]

REMARK Let $\underline{C}/F = \underline{I}_F$ -- then the canonical arrow

$$\widehat{\underline{C}/F} \to \hat{\underline{C}}/F$$

is an equivalence.

[Note: Some authorities write $\mathrm{gro}_{\underline{C}}\ F$ for $\underline{I}_F$ and call it the Grothendieck construction on F.]

PRESENTABILITY

Fix a regular cardinal κ and let $\underline{I} \neq \underline{0}$ be a small category — then $\underline{I}$ is said to be <u>κ-filtered</u> if

($F_1 - \kappa$) Given any set $\{i_\alpha : \alpha \in A\}$ of objects in $\underline{I}$ with $\#A < \kappa$, there exists an object k and morphisms $i_\alpha \rightarrow k$;

($F_2 - \kappa$) Given any set $\{i \xrightarrow{f_\alpha} j : \alpha \in A\}$ of morphisms in $\underline{I}$ with $\#A < \kappa$, there exists an object k and a morphism $f: j \rightarrow k$ such that $f \circ f_\alpha$ is independent of α.

<u>N.B.</u> Take $\kappa = \aleph_0$ — then $\aleph_0$-filtered = filtered and κ-filtered => filtered.

Let $\underline{C}$ be a cocomplete category — then an object $X \in \operatorname{Ob} \underline{C}$ is <u>κ-definite</u> if $\operatorname{Mor}(X,—)$ preserves κ-filtered colimits, i.e., if for every κ-filtered category $\underline{I}$ and for every diagram $\Delta: \underline{I} \rightarrow \underline{C}$, the canonical arrow

$$\operatorname{colim}_{\underline{I}} \operatorname{Mor}(X, \Delta_i) \rightarrow \operatorname{Mor}(X, \operatorname{colim}_{\underline{I}} \Delta_i)$$

is bijective.

[Note: Obviously, if $\kappa' > \kappa$ (κ' regular), then

$$X \ \kappa\text{-definite} \Rightarrow X \ \kappa'\text{-definite.}]$$

EXAMPLE Take $\underline{C}$ = <u>SET</u> — then X is κ-definite iff $\#X < \kappa$. On the other hand, in $\underline{C}$ = <u>TOP</u>, no nondiscrete X is κ-definite.

Let $\underline{C}$ be a cocomplete category — then $\underline{C}$ is said to be <u>κ-presentable</u> if up to isomorphism, there exists a set of κ-definite objects and every object in $\underline{C}$ is a κ-filtered colimit of κ-definite objects.

<u>N.B.</u> If $\underline{C}$ is κ-presentable and if $\kappa' > \kappa$ (κ' regular), then $\underline{C}$ is κ'-presentable.

[Note: This becomes clear in view of the following characterization: A cocomplete category $\underline{C}$ is κ-presentable iff it admits a set $\{G_i\}$ of strong separators, where each G_i is κ-definite.]

EXAMPLE $\underline{\text{SET}}$ and $\underline{\text{CAT}}$ are $\aleph_0$-presentable but $\underline{\text{TOP}}$ is not κ-presentable for any κ.

In $\underline{\text{SET}}$, κ-filtered colimits commute with κ-limits.

[Note: In this context, "κ-limit" means the limit of a functor $F:\underline{C} \to \underline{\text{SET}}$, where $\underline{C}$ is a small category with $\#\text{Mor}\ \underline{C} < \kappa$.]

LEMMA Suppose that $\underline{C}$ is κ-presentable -- then $\forall\ X \in \text{Ob}\ \underline{C}$, there exists a regular cardinal κ_X such that X is κ_X-definite.

PROOF Fix a κ-filtered category $\underline{I}$ and a diagram $\Delta:\underline{I} \to \underline{C}$ such that $X = \text{colim}_{\underline{I}}\ \Delta_i$, where $\forall\ i$, Δ_i is κ-definite. Choose a regular cardinal $\kappa_X \equiv \kappa' > \kappa$ such that $\#\text{Mor}\ \underline{I} < \kappa'$ — then $\forall\ i$, Δ_i is κ'-definite and for any κ'-filtered category $\underline{I}'$ and any diagram $\Delta':\underline{I}' \to \underline{C}$, we have

$$\begin{aligned}
&\text{colim}_{\underline{I}'}\ \text{Mor}(X,\Delta'_{i'}) \\
&\quad\approx \text{colim}_{\underline{I}'}\ \text{Mor}(\text{colim}_{\underline{I}}\ \Delta_i,\Delta'_{i'}) \\
&\quad\approx \text{colim}_{\underline{I}'}\ \lim_{\underline{I}}\ \text{Mor}(\Delta_i,\Delta'_{i'}) \\
&\quad\approx \lim_{\underline{I}}\ \text{colim}_{\underline{I}'}\ \text{Mor}(\Delta_i,\Delta'_{i'}) \\
&\quad\approx \lim_{\underline{I}}\ \text{Mor}(\Delta_i,\text{colim}_{\underline{I}'}\ \Delta'_{i'})
\end{aligned}$$

$$\approx \mathrm{Mor}(\mathrm{colim}_{\underline{I}}\, \Delta_i, \mathrm{colim}_{\underline{I}'}\, \Delta'_{i'})$$

$$\approx \mathrm{Mor}(X, \mathrm{colim}_{\underline{I}'}\, \Delta'_{i'}).$$

If $\underline{C}$ is κ-presentable, then for all $A,B \in \mathrm{Ob}\,\underline{C}$, the categories $A\backslash\underline{C}$, $\underline{C}/B$ are κ-presentable.

If $\underline{C}$ is κ-presentable and if $\underline{I}$ is a small category, then $[\underline{I},\underline{C}]$ is κ-presentable and the κ-definite objects in $[\underline{I},\underline{C}]$ are the functors $\Delta:\underline{I} \to \underline{C}$ such that $\forall\ i \in \mathrm{Ob}\,\underline{I}$, Δ_i is κ-definite. So, e.g.,

$$\underline{C}\ \kappa\text{-presentable} \Rightarrow \underline{C}(\to)\ \kappa\text{-presentable}.$$

EXAMPLE If $\underline{C}$ is a small category, then

$$\hat{\underline{C}} = [\underline{C}^{\mathrm{OP}}, \underline{\mathrm{SET}}]$$

is $\aleph_0$-presentable.

[Note: Every full, reflective subcategory of $\hat{\underline{C}}$ which is closed under the formation of κ-filtered colimits is κ-presentable.]

A category $\underline{C}$ is <u>presentable</u> if it is κ-presentable for some κ. Every presentable category is complete and cocomplete, wellpowered and cowellpowered.

EXAMPLE Suppose that $\underline{C}$ is a Grothendieck category with a separator -- then $\underline{C}$ is presentable.

ACCESSIBILITY

Let κ be a regular cardinal. Suppose that $\underline{C}$ is a category which has κ-filtered colimits -- then $\underline{C}$ is said to be κ-accessible if up to isomorphism, there exists a set of κ-definite objects and every object in $\underline{C}$ is a κ-filtered colimit of κ-definite objects.

[Note: Obviously,

$$\underline{C}\ \kappa\text{-presentable} \Rightarrow \underline{C}\ \kappa\text{-accessible.}]$$

EXAMPLE The category $\underline{C}$ whose objects are the sets and whose morphisms are the injections is $\aleph_0$-accessible but not presentable.

REMARK If $\kappa' > \kappa$ (κ' regular), then it need not be true that

$$\underline{C}\ \kappa\text{-accessible} \Rightarrow \underline{C}\ \kappa'\text{-accessible.}$$

Still, there is a transitive relation $>>$ on the regular cardinals such that

$$\kappa' >> \kappa \Rightarrow \kappa' > \kappa$$

and if $\kappa' >> \kappa$, then

$$\underline{C}\ \kappa\text{-accessible} \Rightarrow \underline{C}\ \kappa'\text{-accessible.}$$

In addition, for any set K of regular cardinals, one can find a regular cardinal κ' such that $\kappa' >> \kappa$ for all $\kappa \in K$.

A category $\underline{C}$ is accessible if it is κ-accessible for some κ.

[Note: On the basis of the foregoing, there exist arbitrarily large regular cardinals κ such that $\underline{C}$ is κ-accessible.]

REMARK In an accessible category, idempotents split. On the other hand, every small category in which idempotents split is accessible.

N.B. Suppose that $\underline{C}$ is accessible -- then $\forall\ X \in \text{Ob}\ \underline{C}$, there exists a regular cardinal κ_X such that X is κ_X-definite.

LEMMA The following conditions on an accessible category $\underline{C}$ are equivalent.

(a) $\underline{C}$ is presentable.

(b) $\underline{C}$ is cocomplete.

(c) $\underline{C}$ is complete.

If $\underline{C}$ is accessible, then for all $A,B \in \text{Ob } \underline{C}$, the categories $A\backslash\underline{C}$, $\underline{C}/B$ are accessible.

If $\underline{C}$ is accessible and if $\underline{I}$ is a small category, then $[\underline{I},\underline{C}]$ is accessible.

[Note: In contrast to what happens in the presentable situation, the degree of accessibility of $[\underline{I},\underline{C}]$ may be strictly larger than that of $\underline{C}$. However, in the special case when $\underline{C} = \underline{2}$, we have

$$\underline{C}\ \kappa\text{-accessible} \Rightarrow \underline{C}(\to)\ \kappa\text{-accessible.}]$$

Suppose that $\underline{C}$ and $\underline{D}$ are κ-accessible -- then a functor $F:\underline{C} \to \underline{D}$ is <u>κ-accessible</u> if F preserves κ-filtered colimits.

[Note: F is <u>accessible</u> if it is κ-accessible for some κ.]

E.g.: If $\underline{C}$ is accessible, then the $\text{Mor}(X,—)$ $(X \in \text{Ob } \underline{C})$ are accessible.

LEMMA A functor $F:\underline{C} \to \underline{\text{SET}}$ is accessible iff F is a colimit of representable functors:

$$F = \text{colim}_{\underline{I}} \text{Mor}(X_i,—).$$

EXAMPLE Take $\underline{C} = \underline{\text{SET}}$, $\underline{D} = \underline{\text{SET}}$ and let $F:\underline{\text{SET}} \to \underline{\text{SET}}$ be the functor that sends X to 2^X (the set of all subsets of X) and sends $f:X \to Y$ to the arrow

$$\left[\begin{array}{l} 2^X \to 2^Y \\ A \to f(A) \end{array}\right. \quad \text{-- then } F \text{ is not accessible.}$$

LEMMA Let $\underline{C}$ and $\underline{D}$ be accessible categories -- then a functor $F:\underline{C} \to \underline{D}$ is accessible iff $\forall\ Y \in \text{Ob}\ \underline{D}$, the composition $\text{Mor}(Y,\text{---}) \circ F:\underline{C} \to \underline{\text{SET}}$ is accessible.

If $\{F_i : i \in I\}$ is a set of accessible functors, then there exist arbitrarily large regular cardinals κ such that each F_i is κ-accessible and preserves κ-definite objects (i.e., X κ-definite $\Rightarrow$ F_iX κ-definite).

ADJOINTS

Given categories $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$, functors $\left[\begin{array}{l} F:\underline{C} \to \underline{D} \\ G:\underline{D} \to \underline{C} \end{array}\right.$ are said to be an <u>adjoint pair</u> if the functors $\left[\begin{array}{l} \text{Mor} \circ (F^{\text{OP}} \times \text{id}_{\underline{D}}) \\ \text{Mor} \circ (\text{id}_{\underline{C}^{\text{OP}}} \times G) \end{array}\right.$ from $\underline{C}^{\text{OP}} \times \underline{D}$ to $\underline{\text{SET}}$ are naturally isomorphic, i.e., if it is possible to assign to each ordered pair $\left[\begin{array}{l} X \in \text{Ob}\ \underline{C} \\ Y \in \text{Ob}\ \underline{D} \end{array}\right.$ a bijective map $\Xi_{X,Y}:\text{Mor}(FX,Y) \to \text{Mor}(X,GY)$ which is functorial in X and Y. When this is so, F is a <u>left adjoint</u> for G and G is a <u>right adjoint</u> for F. Any two left (right) adjoints for G (F) are naturally isomorphic. Left adjoints preserve colimits; right adjoints preserve limits. In order that (F,G) be an adjoint pair, it is necessary and and sufficient that there exist natural transformations $\left[\begin{array}{l} \mu \in \text{Nat}(\text{id}_{\underline{C}}, G \circ F) \\ \nu \in \text{Nat}(F \circ G, \text{id}_{\underline{D}}) \end{array}\right.$

subject to $\begin{cases} (G\nu) \circ (\mu G) = \mathrm{id}_G \\ (\nu F) \circ (F\mu) = \mathrm{id}_F \end{cases}$. The data (F,G,μ,ν) is referred to as an adjoint situation, the natural transformations $\begin{cases} \mu:\mathrm{id}_{\underline{C}} \to G \circ F \\ \nu:F \circ G \to \mathrm{id}_{\underline{D}} \end{cases}$ being the <u>arrows of adjunction</u>.

<u>N.B.</u> $\begin{cases} \forall\ X \in \mathrm{Ob}\ \underline{C} \\ \forall\ Y \in \mathrm{Ob}\ \underline{D} \end{cases}$, we have

$$\begin{cases} X \xrightarrow{\mu_X} GFX \\ FGY \xrightarrow{\nu_Y} Y. \end{cases}$$

Therefore, when explicated, the relations $\begin{cases} (G\nu) \circ (\mu G) = \mathrm{id}_G \\ (\nu F) \circ (F\mu) = \mathrm{id}_F \end{cases}$ become

$$\begin{cases} GY \xrightarrow{\mu_{GY}} GFGY \xrightarrow{G\nu_Y} GY \\ FX \xrightarrow{F\mu_X} FGFX \xrightarrow{\nu_{FX}} FX \end{cases}$$

with

$$\begin{cases} G\nu_Y \circ \mu_{GY} = \mathrm{id}_{GY} \\ \nu_{FX} \circ F\mu_X = \mathrm{id}_{FX}. \end{cases}$$

REMARK Given an adjoint situation (F,G,μ,ν), $\forall\ X \in \mathrm{Ob}\ \underline{C}$ & $\forall\ Y \in \mathrm{Ob}\ \underline{D}$,

$$\Xi_{X,Y}:\mathrm{Mor}(FX,Y) \to \mathrm{Mor}(X,GY)$$

sends $g \in \mathrm{Mor}(FX,Y)$ to $Gg \circ \mu_X \in \mathrm{Mor}(X,GY)$, so $\forall\ f \in \mathrm{Mor}(X,GY)$ there exists a unique $g \in \mathrm{Mor}(FX,Y)$ such that $f = Gg \circ \mu_X$. Conversely, starting from

$$\Xi_{X,Y}:\mathrm{Mor}(FX,Y) \rightarrow \mathrm{Mor}(X,GY),$$

specialize and take $Y = FX$ -- then the

$$\mu_X = \Xi_{X,X}(\mathrm{id}_{FX}) \in \mathrm{Mor}(X,GFX)$$

are the components of a $\mu \in \mathrm{Nat}(\mathrm{id}_{\underline{C}},G \circ F)$.

[Note: The story for Ξ^{-1} and ν is analogous.]

LEMMA Let $\underline{I}$ be a small category, $\underline{C}$ a complete and cocomplete category -- then the constant diagram functor $K:\underline{C} \rightarrow [\underline{I},\underline{C}]$ has a left adjoint, viz. $\mathrm{colim}_{\underline{I}}:[\underline{I},\underline{C}] \rightarrow \underline{C}$, and a right adjoint, viz. $\lim_{\underline{I}}:[\underline{I},\underline{C}] \rightarrow \underline{C}$.

EXAMPLE The forgetful functor $U:\underline{GR} \rightarrow \underline{SET}$ has a left adjoint that sends a set X to the free group on X.

EXAMPLE The forgetful functor $U:\underline{TOP} \rightarrow \underline{SET}$ has a left adjoint that sends a set X to the pair (X,τ), where τ is the discrete topology, and a right adjoint that sends a set X to the pair (X,τ), where τ is the indiscrete topology.

EXAMPLE The forgetful functor $U:\underline{CAT} \rightarrow \underline{PRECAT}$ has a left adjoint that sends a precategory $\underline{G}$ to the free category generated by $\underline{G}$.

EXAMPLE Let $\pi_0:\underline{CAT} \rightarrow \underline{SET}$ be the functor that sends $\underline{C}$ to $\pi_0(\underline{C})$, the set of components of $\underline{C}$; let $\mathrm{dis}:\underline{SET} \rightarrow \underline{CAT}$ be the functor that sends X to $\mathrm{dis}\,X$, the discrete category on X; let $\mathrm{ob}:\underline{CAT} \rightarrow \underline{SET}$ be the functor that sends $\underline{C}$ to $\mathrm{Ob}\,\underline{C}$, the set of objects in $\underline{C}$; let $\mathrm{grd}:\underline{SET} \rightarrow \underline{CAT}$ be the functor that sends X to $\mathrm{grd}\,X$, the category whose objects are the elements of X and whose morphisms are the elements of $X \times X$ -- then π_0 is a left adjoint for dis, dis is a left adjoint for ob, and ob is a left adjoint for grd.

[Note: π_0 preserves finite products; it need not preserve arbitrary products.]

EXAMPLE Let $\text{iso}:\underline{\text{CAT}} \to \underline{\text{GRD}}$ be the functor that sends $\underline{C}$ to iso $\underline{C}$, the groupoid whose objects are those of $\underline{C}$ and whose morphisms are the invertible morphisms in $\underline{C}$ -- then iso is a right adjoint for the inclusion $\underline{\text{GRD}} \to \underline{\text{CAT}}$. Let $\pi_1:\underline{\text{CAT}} \to \underline{\text{GRD}}$ be the functor that sends $\underline{C}$ to $\pi_1(\underline{C})$, the <u>fundamental groupoid</u> of $\underline{C}$, i.e., the localization of $\underline{C}$ at Mor $\underline{C}$ -- then π_1 is a left adjoint for the inclusion $\underline{\text{GRD}} \to \underline{\text{CAT}}$.

EXAMPLE Suppose that $\underline{C}$ has finite products and finite coproducts -- then the diagonal functor $\Delta:\underline{C} \to \underline{C} \times \underline{C}$ has the coproduct $\coprod:\underline{C} \times \underline{C} \to \underline{C}$ as a left adjoint and the product $\times:\underline{C} \times \underline{C} \to \underline{C}$ as a right adjoint.

EXAMPLE Let $\Sigma:\underline{\text{TOP}}_* \to \underline{\text{TOP}}_*$ be the suspension functor and let $\Omega:\underline{\text{TOP}}_* \to \underline{\text{TOP}}_*$ be the loop space functor -- then (Σ,Ω) is an adjoint pair and drops to $\underline{\text{HTOP}}_*:[\Sigma X,Y] \approx [X,\Omega Y]$.

An <u>adjoint equivalence</u> of categories is an adjoint situation (F,G,μ,ν) in which both μ and ν are natural isomorphisms.

LEMMA A functor $F:\underline{C} \to \underline{D}$ is an equivalence iff F is part of an adjoint equivalence.

REMARK Replacing categories by equivalent categories need not lead to equivalent results.

COMPOSITION LAW Let

$$(F_1,G_1,\mu_1,\nu_1)\ \left|\begin{array}{l} F_1:\underline{C} \to \underline{D} \\ \\ G_1:\underline{D} \to \underline{C} \end{array}\right.$$

and

$$(F_2,G_2,\mu_2,\nu_2)\ \left|\begin{array}{l} F_2:\underline{D} \to \underline{E} \\ \\ G_2:\underline{E} \to \underline{D} \end{array}\right.$$

be adjoint situations -- then their composition is the adjoint situation

$$(F_2 \circ F_1,\ G_1 \circ G_2,\ \mu_{21},\ \nu_{12}),$$

where μ_{21} is computed as

$$\mathrm{id}_{\underline{C}} \xrightarrow{\mu_1} G_1 \circ F_1 = G_1 \circ \mathrm{id}_{\underline{D}} \circ F_1 \xrightarrow{G_1\mu_2F_1} G_1 \circ G_2 \circ F_2 \circ F_1$$

and ν_{21} is computed as

$$F_2 \circ F_1 \circ G_1 \circ G_2 \xrightarrow{F_2\nu_1G_2} F_2 \circ \mathrm{id}_{\underline{D}} \circ G_2 = F_2 \circ G_2 \xrightarrow{\nu_2} \mathrm{id}_{\underline{E}}.$$

SPECIAL ADJOINT FUNCTOR THEOREM Given a complete wellpowered category $\underline{D}$ which has a coseparating set, a functor $G:\underline{D} \to \underline{C}$ has a left adjoint iff G preserves limits.

EXAMPLE A functor from $\underline{\mathrm{SET}}$, $\underline{\mathrm{AB}}$ or $\underline{\mathrm{TOP}}$ to a category $\underline{C}$ has a left adjoint iff it preserves limits.

LEMMA Every left or right adjoint functor between accessible categories is accessible.

THE SOLUTION SET CONDITION

Let $\underline{C}$ and $\underline{D}$ be categories and let $F:\underline{C} \to \underline{D}$ be a functor — then F satisfies the solution set condition if for each $Y \in \text{Ob } \underline{D}$, there exists a source $\{g_i:Y \to FX_i\}$ such that for every $g:Y \to FX$, there is an i and an $f:X_i \to X$ such that $g = Ff \circ g_i$:

$$\begin{tikzcd} Y \arrow[r, "g_i"] \arrow[d, "g"'] & FX_i \arrow[d, "Ff"] \\ FX \arrow[r, equal] & FX. \end{tikzcd}$$

E.g.: Every accessible functor satisfies the solution set condition.

GENERAL ADJOINT FUNCTOR THEOREM Given a complete category $\underline{D}$, a functor $G:\underline{D} \to \underline{C}$ has a left adjoint iff G preserves limits and satisfies the solution set condition.

ADJOINT FUNCTOR THEOREM Given presentable categories $\underline{C}$ and $\underline{D}$, a functor $G:\underline{D} \to \underline{C}$ has a left adjoint iff G preserves limits and κ-filtered colimits for some regular cardinal κ.

A full, isomorphism closed subcategory $\underline{C}'$ of an accessible category $\underline{C}$ is accessibly embedded if there is a regular cardinal κ such that $\underline{C}'$ is closed under κ-filtered colimits.

THEOREM Let $\underline{C}$ be an accessible category and let $\underline{C}'$ be an accessibly embedded subcategory — then $\underline{C}'$ is accessible iff the inclusion functor $\underline{C}' \to \underline{C}$ satisfies the solution set condition.

A full, isomorphism closed subcategory $\underline{C}'$ of an accessible category $\underline{C}$ is said to be an accessible subcategory if $\underline{C}'$ is accessible and the inclusion functor $\iota':\underline{C}' \to \underline{C}$ is an accessible functor.

REMARK If $\underline{C}'$ is an accessible subcategory of $\underline{C}$, then $\underline{C}'$ is accessibly embedded in $\underline{C}$ and ι' satisfies the solution set condition.

If $\underline{C}$ is an accessible category and if $\{C_i : i \in I\}$ is a set of accessible subcategories, then $\bigcap_{i \in I} C_i$ is an accessible subcategory of $\underline{C}$.

If $F:\underline{C} \to \underline{D}$ is an accessible functor and if $\underline{D}'$ is an accessible subcategory of $\underline{D}$, then the inverse image $F^{-1}(\underline{D}')$ is an accessible subcategory of $\underline{C}$.

[Note: Define $F^{-1}(\underline{D}')$ by the pullback square

$$\begin{tikzcd} F^{-1}(\underline{D}') \arrow[r] \arrow[d] & \underline{D}' \arrow[d] \\ \underline{C} \arrow[r] & \underline{D}.] \end{tikzcd}$$

REFLECTORS AND COREFLECTORS

A full, isomorphism closed subcategory $\underline{D}$ of a category $\underline{C}$ is said to be a reflective (coreflective) subcategory of $\underline{C}$ if the inclusion $\underline{D} \to \underline{C}$ has a left (right) adjoint R, a reflector (coreflector) for $\underline{D}$.

[Note: A full subcategory $\underline{D}$ of a category $\underline{C}$ is isomorphism closed provided that every object in $\underline{C}$ which is isomorphic to an object in $\underline{D}$ is itself an object in $\underline{D}$.]

EXAMPLE Fix a topological space X — then the category of sheaves of sets on

X is a reflective subcategory of the category of presheaves of sets on X.

EXAMPLE The category $\underline{CG}$ of compactly generated topological spaces is a coreflective subcategory of $\underline{TOP}$, the coreflector $k:\underline{TOP} \to \underline{CG}$ sending X to kX, its compactly generated modification.

Let $\underline{D}$ be a reflective subcategory of $\underline{C}$, R a reflector for $\underline{D}$ -- then one may attach to each $X \in \text{Ob } \underline{C}$ a morphism $r_X:X \to RX$ in $\underline{C}$ with the following property: Given any $Y \in \text{Ob } \underline{D}$ and any morphism $f:X \to Y$ in $\underline{C}$, there exists a unique morphism $g:RX \to Y$ in $\underline{D}$ such that $f = g \circ r_X$. If the r_X are epimorphisms, then $\underline{D}$ is said to be an epireflective subcategory of $\underline{C}$.

EXAMPLE $\underline{AB}$ is an epireflective subcategory of $\underline{GR}$, the reflector sending X to its abelianization $X/[X,X]$.

A reflective subcategory $\underline{D}$ of a complete (cocomplete) category $\underline{C}$ is complete (cocomplete).

[Note: Let $\Delta:\underline{I} \to \underline{D}$ be a diagram in $\underline{D}$.

(1) To calculate a limit of Δ, postcompose Δ with the inclusion $\underline{D} \to \underline{C}$ and let $\{\ell_i:L \to \Delta_i\}$ be its limit in $\underline{C}$ -- then $L \in \text{Ob } \underline{D}$ and $\{\ell_i:L \to \Delta_i\}$ is a limit of Δ.

(2) To calculate a colimit of Δ, postcompose Δ with the inclusion $\underline{D} \to \underline{C}$ and let $\{\ell_i:\Delta_i \to L\}$ be its colimit in $\underline{C}$ -- then $\{r_L \circ \ell_i:\Delta_i \to RL\}$ is a colimit of Δ.]

EPIREFLECTIVE CHARACTERIZATION THEOREM If a category $\underline{C}$ is complete, wellpowered, and cowellpowered, then a full, isomorphism closed subcategory $\underline{D}$ of $\underline{C}$ is an epireflective subcategory of $\underline{C}$ iff $\underline{D}$ is closed under the formation in $\underline{C}$ of products and extremal monomorphisms.

ENDS AND COENDS

Let $\underline{I}$ be a small category, $\Delta:\underline{I}^{OP} \times \underline{I} \to \underline{C}$ a diagram.

(Ends) A source $\{f_i:X \to \Delta_{i,i}\}$ is said to be dinatural if for each $\delta \in \text{Mor } \underline{I}$, say $i \xrightarrow{\delta} j$,

$$\Delta(\text{id},\delta) \circ f_i = \Delta(\delta,\text{id}) \circ f_j.$$

An end of Δ is a dinatural source $\{e_i:E \to \Delta_{i,i}\}$ with the property that if $\{f_i:X \to \Delta_{i,i}\}$ is a dinatural source, then there exists a unique morphism $\phi:X \to E$ such that $f_i = e_i \circ \phi$ for all $i \in \text{Ob } \underline{I}$. Every end is a limit (and every limit is an end). Notation: $E = \int_i \Delta_{i,i}$ (or $\int_{\underline{I}} \Delta$).

(Coends) A sink $\{f_i:\Delta_{i,i} \to X\}$ is said to be dinatural if for each $\delta \in \text{Mor } \underline{I}$, say $i \xrightarrow{\delta} j$,

$$f_i \circ \Delta(\delta,\text{id}) = f_j \circ \Delta(\text{id},\delta).$$

A coend of Δ is a dinatural sink $\{e_i:\Delta_{i,i} \to E\}$ with the property that if $\{f_i:\Delta_{i,i} \to X\}$ is a dinatural sink, then there exists a unique morphism $\phi:E \to X$ such that $f_i = \phi \circ e_i$ for all $i \in \text{Ob } \underline{I}$. Every coend is a colimit (and every colimit is a coend). Notation: $E = \int^i \Delta_{i,i}$ (or $\int^{\underline{I}} \Delta$).

There are a number of basic constructions that can be viewed as an end or coend of a suitable diagram.

EXAMPLE Let $\underline{I}$ be a small category and let $\begin{cases} F:\underline{I} \to \underline{C} \\ G:\underline{I} \to \underline{C} \end{cases}$ be functors -- then the assignment $(i,j) \to \mathrm{Mor}(Fi,Gj)$ defines a diagram $\underline{I}^{OP} \times \underline{I} \to \underline{SET}$ and $\mathrm{Nat}(F,G)$ is the end $\int_i \mathrm{Mor}(Fi,Gi)$.

EXAMPLE Suppose that A is a ring with unit -- then a right A-module X and a left A-module Y define a diagram $A^{OP} \times A \to \underline{AB}$ (tensor product over Z) and the coend $\int^A X \otimes Y$ is $X \otimes_A Y$, the tensor product over A.

[Note: In context, view A as a category with one object.]

LEMMA Let $\underline{I}$ be a small category, $\underline{C}$ a complete and cocomplete category.

(L) Let

$$L:\underline{C} \to [\underline{I}^{OP} \times \underline{I}, \underline{C}]$$

be the functor given on objects by

$$LX(i,j) = \mathrm{Mor}(i,j) \cdot X.$$

Then L is a left adjoint for

$$\mathrm{end}:[\underline{I}^{OP} \times \underline{I},\underline{C}] \to \underline{C}.$$

(R) Let

$$R:\underline{C} \to [\underline{I}^{OP} \times \underline{I},\underline{C}]$$

be the functor given on objects by

$$RX(i,j) = X^{\mathrm{Mor}(j,i)}.$$

Then R is a right adjoint for

$$\mathrm{coend}:[\underline{I}^{OP} \times \underline{I},\underline{C}] \to \underline{C}.$$

INTEGRAL YONEDA LEMMA Let $\underline{I}$ be a small category, $\underline{C}$ a complete and cocomplete category -- then for every $F \in \text{Ob}[\underline{I}^{\text{OP}},\underline{C}]$,

$$\int^i \text{Mor}(—,i) \cdot F_i \approx F \approx \int_i Fi^{\text{Mor}(i,—)}.$$

[We shall verify the first of these relations. So take $G \in \text{Ob}[\underline{I}^{\text{OP}},\underline{C}]$ and compute:

$$\text{Nat}(\int^i \text{Mor}(—,i) \cdot Fi,G)$$

$$\approx \int_j \text{Mor}(\int^i \text{Mor}(j,i) \cdot Fi,Gj)$$

$$\approx \int_j \int_i \text{Mor}(\text{Mor}(j,i) \cdot Fi,Gj)$$

$$\approx \int_i \int_j \text{Mor}(\text{Mor}(j,i) \cdot Fi,Gj)$$

$$\approx \int_i \int_j \text{Mor}(Fi,Gj)^{\text{Mor}(j,i)}$$

$$\approx \int_i \int_j \text{Mor}(\text{Mor}(j,i),\text{Mor}(Fi,Gj))$$

$$\approx \int_i \text{Nat}(h_i,\text{Mor}(Fi,G—))$$

$$\approx \int_i \text{Mor}(Fi,Gi) \quad \text{(Yoneda lemma)}$$

$$\approx \text{Nat}(F,G).$$

Since G is arbitrary, it follows that

$$\int^i \text{Mor}(—,i) \cdot Fi \approx F.]$$

EXAMPLE If X is a simplicial set, then

$$\int^{[n]} \mathrm{Mor}(\text{—},[n]) \cdot X_n \approx X \approx \int_{[n]} (X_n)^{\mathrm{Mor}([n],\text{—})}.$$

KAN EXTENSIONS

THEOREM Given small categories $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$, a complete category $\underline{S}$, and a functor $K:\underline{C} \to \underline{D}$, the functor $K^*:[\underline{D},\underline{S}] \to [\underline{C},\underline{S}]$ has a right adjoint $K_\dagger:[\underline{C},\underline{S}] \to [\underline{D},\underline{S}]$.

Let $T \in \mathrm{Ob}[\underline{C},\underline{S}]$ -- then $K_\dagger T$ is called the <u>right Kan extension</u> of T along K. In terms of ends,

$$(K_\dagger T)Y = \int_X TX^{\mathrm{Mor}(Y,KX)}.$$

There is a canonical natural transformation $K_\dagger T \circ K \xrightarrow{\nu_T} T$. It is a natural isomorphism if K is full and faithful.

[Note: In general, the diagram

$$\begin{tikzcd} \underline{C} \arrow[r, "K"] \arrow[d, "T"'] & \underline{D} \arrow[d, "K_\dagger T"] \\ \underline{S} \arrow[r, equal] & \underline{S} \end{tikzcd}$$

does not commute.]

THEOREM Given small categories $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$, a cocomplete category $\underline{S}$, and a functor $K:\underline{C} \to \underline{D}$, the functor $K^*:[\underline{D},\underline{S}] \to [\underline{C},\underline{S}]$ has a left adjoint $K_!:[\underline{C},\underline{S}] \to [\underline{D},\underline{S}]$.

Let $T \in \mathrm{Ob}[\underline{C},\underline{S}]$ -- then $K_! T$ is called the <u>left Kan extension</u> of T along K.

In terms of coends,

$$(K_!T)Y = \int^X \mathrm{Mor}(KX,Y) \cdot TX.$$

There is a canonical natural transformation $T \xrightarrow{\mu_T} (K_!T) \circ K$. It is a natural isomorphism if K is full and faithful.

[Note: In general, the diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{K} & \underline{D} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle K_!T} \\ \underline{S} & = & \underline{S} \end{array}$$

does not commute.]

EXAMPLE Suppose that $\underline{C}$ and $\underline{D}$ are small categories and let $K:\underline{C} \to \underline{D}$ be a functor -- then $K^{OP}:\underline{C}^{OP} \to \underline{D}^{OP}$ and the precomposition functor $\hat{\underline{D}} \to \hat{\underline{C}}$ has a left adjoint $\hat{\underline{C}} \to \hat{\underline{D}}$, call if $\hat{K}$ (technically, $\hat{K} = (K^{OP})_!$). Given $X \in \mathrm{Ob}\ \underline{C}$ and $G \in \mathrm{Ob}\ \hat{\underline{D}}$, we have

$$\begin{aligned} &\mathrm{Nat}((\hat{K} \circ Y_{\underline{C}})(X),G) \\ &\approx \mathrm{Nat}(\hat{K}(h_X),G) \\ &\approx \mathrm{Nat}(h_X,G \circ K^{OP}) \\ &\approx G(KX). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathrm{Nat}((Y_{\underline{D}} \circ K)(X),G) \\ &\approx \mathrm{Nat}(h_{KX},G) \\ &\approx G(KX). \end{aligned}$$

Therefore

$$\hat{K} \circ Y_{\underline{C}} \approx Y_{\underline{D}} \circ K.$$

[Note: One can arrange matters so that

$$\hat{K} \circ Y_{\underline{C}} = Y_{\underline{D}} \circ K.]$$

REMARK The functor $K_!:[\underline{C},\underline{S}] \to [\underline{D},\underline{S}]$ preserves colimits but it need not preserve finite limits. E.g.: Take $\underline{C} = \underline{d2}$ (the discrete category with two objects), $\underline{D} = \underline{1}$, $\underline{S} = \underline{SET}$ -- then $K_!$ is the arrow

$$\underline{SET} \times \underline{SET} \to \underline{SET}$$

that sends (X,Y) to $X \coprod Y$ and coproducts do not commute with products in $\underline{SET}$.

The construction of the right (left) adjoint of K^* does not use the assumption that $\underline{D}$ is small, its role being to ensure that $[\underline{D},\underline{S}]$ is a category. For example, if $\underline{C}$ is small and $\underline{S}$ is cocomplete, then taking $K = Y_{\underline{C}}$, the functor $Y_{\underline{C}}^*:[\hat{\underline{C}},\underline{S}] \to [\underline{C},\underline{S}]$ has a left adjoint that sends $T \in \text{Ob}[\underline{C},\underline{S}]$ to $\Gamma_T \in \text{Ob}[\hat{\underline{C}},\underline{S}]$, where $T \approx \Gamma_T \circ Y_{\underline{C}}$. On an object F of $\hat{\underline{C}}$,

$$\begin{aligned}\Gamma_T F &= \int^X \text{Nat}(Y_{\underline{C}}X,F) \cdot TX \\ &\approx \int^X \text{Nat}(h_X,F) \cdot TX \\ &\approx \int^X FX \cdot TX.\end{aligned}$$

<u>N.B.</u> Γ_T is the <u>realization functor</u>; it is a left adjoint for the <u>singular functor</u> $\sin_T:\underline{S} \to \hat{\underline{C}}$ which is defined by the prescription

$$(\sin_T Y)X = \mathrm{Mor}(TX,Y).$$

[Note: The arrow of adjunction $\Gamma_T \circ S_T \to \mathrm{id}_{\underline{S}}$ is a natural isomorphism iff S_T is full and faithful.]

EXAMPLE While not reflected in the notation, the pair (Γ_T,S_T) depends, of course, on the choice of $\underline{S}$. E.g.: Take $\underline{S} = \hat{\underline{C}}$ -- then $\forall\ T \in \mathrm{Ob}[\underline{C},\hat{\underline{C}}]$,

$$\Gamma_T F \approx \mathrm{colim}(\mathrm{gro}_{\underline{C}}\, F \xrightarrow{\pi_F} \underline{C} \xrightarrow{T} \hat{\underline{C}}),$$

$\pi_F:\mathrm{gro}_{\underline{C}}\, F \to \underline{C}$ the projection. Specialize further and take $T = Y_{\underline{C}}$:

$$\Gamma_{Y_{\underline{C}}} F \in \mathrm{Ob}\ \hat{\underline{C}}$$

and $\forall\ Y \in \mathrm{Ob}\ \underline{C}$,

$$\begin{aligned}
(\Gamma_{Y_{\underline{C}}} F)Y &= \int^X FX \cdot Y_{\underline{C}}(X) \\
&\approx \int^X FX \cdot \mathrm{Mor}(Y,X) \\
&\approx \int^X FX \times \mathrm{Mor}(Y,X) \\
&\approx \int^X \mathrm{Mor}(Y,X) \times FX \\
&\approx \int^X \mathrm{Mor}(Y,X) \cdot FX \\
&\approx FY \ \text{(integral Yoneda lemma)}.
\end{aligned}$$

I.e.:

$$\Gamma_{Y_{\underline{C}}} F \approx F \approx \mathrm{colim}(\mathrm{gro}_{\underline{C}}\, F \xrightarrow{\pi_F} C \xrightarrow{Y_{\underline{C}}} \hat{C}).$$

REMARK Take $\underline{S} = \underline{CAT}$ and let $\gamma \in \mathrm{Ob}[\underline{C},\underline{CAT}]$ be the functor that sends X to $\underline{C}/X$ — then the realization functor Γ_γ assigns to each F in $\hat{\underline{C}}$ its Grothendieck construction:

$$\Gamma_\gamma F \approx \mathrm{gro}_{\underline{C}}\, F.$$

From the definitions,

$$\mathrm{Nat}(K_!T,T') \approx \mathrm{Nat}(T,K^*T') = \mathrm{Nat}(T,T' \circ K),$$

where

$$\begin{cases} T \in \mathrm{Ob}[\underline{C},\underline{S}] \\ T' \in \mathrm{Ob}[\underline{D},\underline{S}]. \end{cases}$$

So, $\forall\ \alpha \in \mathrm{Nat}(T,T' \circ K)$, there is a unique $\beta \in \mathrm{Nat}(K_!T,T')$ such that

$$\alpha = K^*\beta \circ \mu_T = \beta K \circ \mu_T.$$

Now drop the assumptions on $\begin{cases} \underline{C} \\ \underline{D} \end{cases}$ and $\underline{S}$ and suppose that they are arbitrary. Let $K:\underline{C} \to \underline{D}$ be a functor and let $T:\underline{C} \to \underline{S}$ be a functor — then a <u>left Kan extension</u> of T along K is a pair $(\underline{L}_K T,\mu_T)$, where $\underline{L}_K T:\underline{D} \to \underline{S}$ is a functor and $\mu_T \in \mathrm{Nat}(T,\underline{L}_K T \circ K)$, with the following property: $\forall\ T' \in \mathrm{Ob}[\underline{D},\underline{S}]$ and $\forall\ \alpha \in \mathrm{Nat}(T,T' \circ K)$, there is a unique $\beta \in \mathrm{Nat}(\underline{L}_K T,T')$ such that $\alpha = \beta K \circ \mu_T$. Schematically:

$$\begin{tikzcd} T \arrow[r, equal] \arrow[d, "\mu_T"'] & T \arrow[d, "\alpha"] \\ \underline{L}_K T \circ K \arrow[r, "\beta K"'] & T' \circ K. \end{tikzcd}$$

N.B. If $(\underline{L}_K'T,\mu_T')$, $(\underline{L}_K''T,\mu_T'')$ are left Kan extensions of T along K, then $\exists$ a unique natural isomorphism $\Xi:\underline{L}_K'T \rightarrow \underline{L}_K''T$ such that $\mu_T'' = \Xi K \circ \mu_T'$.

[Note: Conversely, given a left Kan extension $(\underline{L}_K'T,\mu_T')$ of T along K, a functor $\underline{L}_K''T \in \text{Ob}[\underline{D},\underline{S}]$ and a natural isomorphism $\Xi:\underline{L}_K'T \rightarrow \underline{L}_K''T$, put $\mu_T'' = \Xi K \circ \mu_T'$ -- then $(\underline{L}_K''T,\mu_T'')$ is a left Kan extension of T along K. Proof: Determine $\beta \in \text{Nat}(\underline{L}_K'T,T')$ uniquely per $\alpha \in \text{Nat}(T,T' \circ K)$ and write

$$(\beta \circ \Xi^{-1})K \circ \mu_T'' = (\beta \circ \Xi^{-1})K \circ \Xi K \circ \mu_T'$$

$$= \beta K \circ \Xi^{-1}K \circ \Xi K \circ \mu_T' = \beta K \circ (\Xi^{-1} \circ \Xi)K \circ \mu_T'$$

$$= (\beta \circ \Xi^{-1} \circ \Xi)K \circ \mu_T' = \beta K \circ \mu_T' = \alpha,$$

which settles existence. Uniqueness is clear.]

LEMMA Suppose that $K:\underline{C} \rightarrow \underline{D}$ has a right adjoint L and let

$$\begin{bmatrix} \phi:\text{id}_{\underline{C}} \rightarrow L \circ K \\ \psi:K \circ L \rightarrow \text{id}_{\underline{D}} \end{bmatrix}$$

be the arrows of adjunction -- then the pair $(T \circ L, T\phi)$ is a left Kan extension of T along K.

REMARK The notion of a right Kan extension $(\underline{R}_K T,\nu_T)$ is dual.