Stacks and non-abelian cohomoloy

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The purpose of this first lecture is to present a general formalism of higher stacks based on the theory of simplicial presheaves introduced by A. Joyal and developped by many authors after him. My main purpose will be to explain throught examples why the homotopy theory of simplicial presheaf is actually a very good model for a theory of higher stacks. For this, I will present the homotopy category of stacks and investigate its relations with the usual theory of (1)-stacks, sheaf cohomology and non-abelian cohomology.

Some references on the subjects are [Ja1], [S2], [H-S], [Ja2], [Hol], [Du].

Terminology remarks:

- The *stacks* of this talk will probably have a different flavor than usual. Indeed, instead of considering stacks from a geometrical point of view (e.g. algebraic stacks) they will be considered as *coefficients* for cohomology and will not be endowed with geometrical structure (algebraic, topological ...). In the future they will serve to study other geometrical objects (varieties, spaces, possibly other stacks ...) exactly as sheaves are used to study schemes. The theory of higher stacks should actually be understood as part of *higher topos theory*.
- The expression *non-abelian cohomology* can be quite ambiguous. One could try to make it clearer by the following observation. Abelian cohomology is certainely dual to (abelian) homology. On the other side, homology is nothing else than the abelianization of homotopy, or *abelian homotopy*. Therefore, non-abelian cohomology should really be understood as dual to homotopy. This is why homotopy theory will play an imporant role in this lecture.

In the following I will be using the homotopy theory of simplicial sets. If one is not familliar with it, one can simply replace *simplicial sets* by *topological spaces*, and *equivalences of simplicial sets* by *weak equivalences of topological spaces*. The category of sets will be denoted by Set, and the category of simplicial sets by $SSet$. As any set can be considered as a discrete simplicial set one can considered Set as embedded in $SSet$.

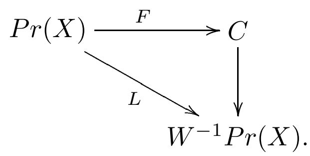
To fix the ideas we will work over the big site $T := (Top)$ of all topological spaces. However, everything will be valid over any Grothendieck site.

1 Sheaves

Let us start by a very formal construction of the category of sheaves on T. We consider first $Pr(X)$, the category of presheaves of sets on X. Inside this category one consider a certain set of morphisms W defined by the following. A morphism $f : A \longrightarrow B$ in $Pr(X)$ belongs to W if for any $X \in T$ and any point $x \in X$ the map induced on the fibers $f_x : A_x \longrightarrow B_x$ is bijective. In other words, W consists of local isomorphisms. To such a categorical data $(Pr(X), W)$, of a category and a subset of morphisms, one can construct the *localized category* $W^{-1}Pr(X)$. By definition, it is a category together with a functor

$$L : Pr(X) \longrightarrow W^{-1}Pr(X),$$

which sends elements of W into isomorphisms, and which is universal for such property. The universal property [1] means that any functor $F : Pr(X) \longrightarrow C$ sending W into isomorphisms factors in a unique (up to unique isomorphism) way throught L



It is a general fact that such a category $W^{-1}Pr(X)$ always exists, but might be quite difficult to describe (see [Ho, Def. 1.2.1]). Roughly speaking, $W^{-1}Pr(X)$ has the same objects as $Pr(X)$, and morphisms between A and B are certain equivalence classes of strings of morphisms in $Pr(X)$, $A \cdots \to A_n \leftarrow B_n \to A_{n+1} \leftarrow \dots B$, where each arrow from the right to the left belongs to W.

Let us consider now the category $Sh(X)$ of sheaves of sets on X. There exists an *associated sheaf functor* $a : Pr(X) \longrightarrow Sh(X)$ that sends precisely W to isomorphisms. This functor factors throught $W^{-1}Pr(X)$, giving rise to a functor $W^{-1}Pr(X) \longrightarrow Sh(X)$.

Proposition 1.0.1 *The above functor is an equivalence of categories.*

Another way to state this proposition is to say that the localization functor $L : Pr(X) \longrightarrow W^{-1}Pr(X)$ has a right adjoint $j : W^{-1}Pr(X) \longrightarrow Pr(X)$ which is fully faithful and whose image is the sub-category of sheaves. One can actually recognize the image of j as the category of *W-local objects* in $Pr(X)$. More precisely one can check that

$$Sh(T) = \{F \in Pr(X) / \forall (f : E \to E') \in W, \quad f^* : Hom(E', F) \to Hom(E, F) \quad \text{is bijective}\}.$$

The striking conclusion of this proposition is that one can construct the category of sheaves, up to an equivalence, without even mentioning the sheaf condition (the construction only depends on $Pr(X)$ and the set of morphisms W). In other words, *it is not necessary to know what a sheaf is in order to talk about the category of sheaves.* We will follow the same point of view in order to work with higher stacks, and therefore will define the category of stacks as a certain localized category.

2 The homotopy category of stacks

Let $SPr(T)$ be the category of presheaves of simplicial sets on T, or equivalentely of simplicial presheaves on T. In other words, an object F in $SPr(T)$ is the data of a simplicial set $F(X)$ for any topological space X, together with transitions morphisms $f^* : F(X) \longrightarrow F(Y)$ for any continuous map $f : Y \longrightarrow X$, such that $(f \circ g)^* = g^* \circ f^*$. Inside the category $SPr(T)$, one defines a set of morphism W in the following way. A morphism $f : F \longrightarrow F'$ belongs to W if for any space $X \in T$ and any point $x \in X$ the induced morphism on the fiber $f_x : F_x \longrightarrow F'_x$ is an equivalence[2]. The morphisms in W might reasonably be called *local equivalences*.

In the same way, one defines the set W^{pr} of *global equivalences*, to be the set of all morphisms $f : F \longrightarrow F'$ such that for any $X \in T$ the induced morphism $f_X : F(X) \longrightarrow F'(X)$ is an equivalence. One has $W^{pr} \subset W$.

[1] The correct way to state this universal property is to claim that for any category C, the induced functor on the categories of functors

$$L^* : \underline{Hom}(W^{-1}Pr(X), C) \longrightarrow \underline{Hom}(Pr(X), C)$$

is fully faithful, and its image consists of functors sending elements of W to isomorphisms in C.

[2] Recall that an equivalence of simplicial sets is a morphism inducing an isomorphism on all homotopy groups for all base points.

Definition 2.0.2 *1. The homotopy category of stacks (on T) is the localized category*

$$Ho(T) := W^{-1}SPr(T).$$

Objects of $Ho(T)$ will simply be called stacks, and morphisms in $Ho(T)$ will be denoted by $[-,-]$, or by $[-,-]_{Ho(T)}$.

2. The homotopy category of pre-stacks (on T) is the localized category

$$Ho^{pr}(T) := (W^{pr})^{-1}SPr(T).$$

Objects of $Ho^{pr}(T)$ will simply be called pre-stacks.

Remarks:

- One should not confuse $Ho(T)$ defined above and the homotopy category of spaces $Ho(Top)$ used in homotopy theory and obtained by inverting weak equivalences of topological spaces.

- In order to avoid confusion, we will use the expression 1-*stack* to refer to the usual notion of stacks in groupoids defined in the previous lectures (see also [L-M]). If necessary the same terminology applies to distinguish between pre-stacks and 1-pre-stacks.

As an inductive colimit of equivalences is an equivalence, one has $W^{pr} \subset W$, and therefore a natural functor

$$a : Ho^{pr}(T)(W^{pr})^{-1}SPr(T) \longrightarrow W^{-1}SPr(T) = Ho(T).$$

One can prove that a possesses a right adjoint j which is fully faithful. The image of j consists precisely of simplicial presheaves which are W-local in $Ho^{pr}(T)$ (the definition is the same as in the previous paragraph). In particular, a local equivalence between those is always a global equivalence (this is the local-to-global principle for simplicial presheaves). In other words, there is a full sub-category $SPr(T)^{desc}$ of $SPr(T)$, consisting of W-local simplicial presheaves such that $Ho(T)$ is equivalent to $(W^{pr})^{-1}SPr(T)^{desc}$ [3]. One should note that the adjunction morphism $F \longrightarrow ja(F)$ in $Ho^{pr}(T)$ has to be understood as an *associated stack construction*.

3 Sheaves vs stacks

Recall that a sheaf F on the site T is nothing else but a presheaf such that each restriction of F on a space $X \in T$ is a sheaf in the usual sense. The category of sheaves on T will be denoted by $Sh(T)$.

To a stack $F \in Ho(T)$ one associates its sheaf of connected components $\pi_0(F) \in Sh(T)$. This defines a functor $\pi_0 : Ho(T) \longrightarrow Sh(T)$, which possesses a right adjoint $j_0 : Sh(T) \longrightarrow Ho(T)$. The functor j_0 simply sends a sheaf of sets to the corresponding presheaf of discrete simplicial sets.

For the purpose of the following proposition, we recall that a stack $F \in Ho(T)$ is 0-truncated if for any space $X \in T$ and any point $x \in X$ the fiber F_x is acyclic (i.e. for any $y \in F_x$ one has $\pi_n(F_x, y) = 0$ for all $n > 0$).

Proposition 3.0.3 *The functor j_0 is fully faithful. Its image consists of* 0*-truncated stacks.*

One has the Yoneda embedding $h : T \longrightarrow Sh(T)$, sending a space X to the sheaf $Y \mapsto h_X(Y) := Hom(Y, X)$. When compose with the functor j_0 one obtains a full embedding $h : T \longrightarrow Ho(T)$. This shows that $Ho(T)$ contains the category of spaces as a full sub-category, which by definition consists of the *representable stacks*.

[3] One can actually characterize these local simplicial presheaves as objects with possess a certain descent property for hyper-coverings (this result is very unfortunately still un-published: [Du] or [Hi]). This descent property is nothing else but a homotopy analog of the sheaf condition.

4 1-Stacks vs stacks

Recall that a 1-stack of groupoids on T is by definition a fibered category in groupoids $\mathcal{X} \longrightarrow T$ which satisfies certain decsent conditions. These form a category, where morphisms are just (strictly) commutative diagrams of functors

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & \searrow \qquad \swarrow & \\ & T. & \end{array}$$

Two such morphisms f and g are homotopic if there exist a natural isomorphism from f to g, compatible with the two projections to T. Being homotopic is an equivalence relation on the set of morphisms, compatible with composition. We will denote by $Ho(1-St(T))$ the category whose objects are 1-stacks (in groupoids) over T and morphisms are homotopy classes of morphisms. We are going to embedd $Ho(1-St(T))$ into $Ho(T)$.

Even if a fibered category in groupoids $\mathcal{X}$ is not strictly speaking a presheaf of groupoids, one can always replace it, in a functorial way, by an actual presheaf of groupoids. This process is the *strictification of fibered categories* which possesses the following easy and explicit description (which is nothing else than the so called 2-Yoneda lemma). For a 1-stack $\mathcal{X}$ one defines a presheaf of groupoids $F_{\mathcal{X}}$ on T by the formula

$$F_{\mathcal{X}}(X) := \underline{Hom}_T(\underline{X}, \mathcal{X}) \; for \; any X \in T,$$

where $\underline{Hom}_T(\underline{X}, \mathcal{X})$ is the category of functors from the fibered category $\underline{X}$ represented by X to $\mathcal{X}$. It is an exercice to check this defines a (2-)functor from $1-St(T)$ to $Gpd(T)$, the category of genuine presheaves of groupoids on T.

Any groupoid G possesses a classifying simplicial set (or topological space) BG, defined in a very analog way than the usual classfying space of a group (not to be confused with the classifying stack !). By definition BG is a simplicial set whose fundamental groupoid is naturally equivalent to G and which does not have any non-trivial higher homotopy groups. This construction $G \mapsto BG$ is a functor from groupoids to simplicial sets, and therefore can be applied to the presheaf $F_{\mathcal{X}}$. Puting these constructions all together one gets a functor

$$\begin{array}{ccc} 1-St(T) & \longrightarrow & SPr(T) \longrightarrow Ho(T) \\ \mathcal{X} & \mapsto & BF_{\mathcal{X}}, \end{array}$$

which is easily checked to induce a functor on the level of homotopy categories

$$Ho(1-St(T)) \longrightarrow Ho(T).$$

Theoreme 4.0.4 *The above functor is fully faithful. Its essential image consists exactely of n-truncated stacks (i.e. stacks F such that for each $X \in T$ and each $x \in X$ the fiber F_x is 1-truncated* [4]*).*

There are some generalization of the previous theorem relating n-stacks in groupoids and n-truncated stacks (2-stacks in groupoids where introduced in [Br], and n-stacks in groupoids might be defined using any good definition of n-groupoids, see e.g. [Le]). For $n = \infty$ I do not know any definition of ∞-stacks in groupoids, but there is no doubt that an equivalence between the homotopy category of such and $Ho(T)$ exists. This last statement strongly suggests that our notion of stacks is actually a reasonnable notion of ∞-stacks in groupoids.

[4] A simplicial set or a space S is 1-truncated if it does not have non-trivial homotopy groups other than π_0 and π_1.

5 Enriched structures and non-abelian cohomology

Though the category $Ho(T)$ contains the homotopy category of 1-stacks $Ho(1-St(T))$, it is too coarse to reconstruct the 2-category $1-St(T)$. This means that $Ho(T)$ does not allow one to consider 2-morphisms. These 2-morphisms are actually a very important structure of stack theory, and neglecting them is not a good idea. Another related problem is that the category $Ho(T)$ does not have fibered products, whereas we have seen that fibered product of 1-stacks does exists in a certain way. There exists several ways to solve this problem. One is to work with the simplicial localization of Dwyer and Kan (see [D-K], and also [S1]) rather than the usual localization of categories. From this point of view, the *true homotopy category of stacks on T* is not the category $Ho(T)$ but the simplicial enriched category $LSPr(T)$[5], obtained from $SPr(T)$ by inverting the equivalences using the simplicial localization of categories. The relation between these two localization process is that $Ho(T)$ is the category of connected components of $LSPr(T)$ (i.e. they have the same object and the morphisms in $Ho(T)$ are the set of connected components of the simplicial sets of morphisms in $LSPr(T)$). One important properties of $LSPr(T)$ is that it induces a sort of *non-abelian triangulated structure* on $Ho(T)$. In particular it allows one to talk about *homotopy fiber products*, and more generally about *homotopy limits and colimits* in a very intrisic way (see [H-S, §8,14]). It also allows to reconstruct the whole 2-category $1-St(T)$ (up to an equivalence), as well as the $(n+1)$-category $n-St(T)$, but we will not go further in that direction. Though the Dwyer and Kan localization technique is a very elegant and powerful way of taking into account higher categorical structures on $Ho(T)$ we will use a more down to earth approach that will be enough for our purposes. Another solution would be to use a reasonnable *model category structure* on the category $SPr(T)$, which is a less intrinsic structure than $LSPr(T)$, but more workable in practice (working with model categories instead of simplicially enriched categories is very similar to working with sites rather than topoi).

Simplicially enriched structures and internal Hom's: For any simplicial presheaf $F \in SPr(T)$ and simplicial set $A \in SSet$, one can form a simplicial presheaf $A \times F$, defined by the formula $(A \times F)(X) := A \times F(X)$, for $X \in T$. This induces an action of the homotopy category of simplicial sets $Ho(SSet)$ on $Ho(T)$. Using this additional structure one can define for any $F, F' \in Ho(T)$ the following functor

$$\begin{array}{cccc} \mathbb{R}\underline{Hom}(F,F') : & Ho(SSet)^{op} & \longrightarrow & Set \\ & A & \mapsto & [A \times F, F']. \end{array}$$

One can prove that this functor is representable by an object $\mathbb{R}\underline{Hom}(F,F') \in Ho(SSet)$. These derived simplicial Hom's naturally define a structure of a $Ho(SSet)$-enriched category on $Ho(T)$ that will be of fundamental importance for us (in particular for the purpose of non-abelian cohomology). Note that one has

$$[F, F'] \simeq \pi_0 \mathbb{R}\underline{Hom}(F,F'),$$

for any $F, F' \in Ho(T)$.

One can also prove that the category $Ho(T)$ is cartesian closed. This means that for any pair of objects $F, F' \in Ho(T)$, the functor

$$\begin{array}{cccc} \mathbb{R}\mathcal{HOM}(F,F') : & Ho(T)^{op} & \longrightarrow & Set \\ & G & \mapsto & [G \times F, F'] \end{array}$$

is representable by an object $\mathbb{R}\mathcal{HOM}(F,F') \in Ho(T)$, called the *stack of morphisms between F and F'*.

[5] A simplicially enriched category is a category where the morphisms sets have a structure of simplicial sets compatible with the compositions maps.

Homotopy limits and colimits: Let I be a category, and consider $Ho(T \times I^{op})$, where the topology on $T \times I^{op}$ is the product of the topology of T and the trivial topology on I (i.e. a presheaf $F : T^{op} \times I \longrightarrow Set$ is a sheaf if and only if for any $i \in I$, the presheaf $F(-, i)$ on T is a sheaf). There is a natural functor $c : SPr(T) \longrightarrow SPr(T \times I^{op}) \simeq SPr(T)^I$, which sends a simplicial presheaf F to the I-diagram with values constant equal to F. In formula, for any $F \in SPr(T)$ and $(X, i) \in T \times I^{op}$, one has $c(F)(X, i) := F(X)$. The functor c sends local equivalences to local equivalences and therefore induces a functor on the homotopy categories

$$c : Ho(T) \longrightarrow Ho(T \times I^{op}).$$

One can prove that this functor possesses adjoints on the left and on the right, respectively denoted by

$$Hocolim_I : Ho(T \times I^{op}) \longrightarrow Ho(T) \qquad Holim_I : Ho(T \times I^{op}) \longrightarrow Ho(T),$$

and called *homotopy colimits* and *homotopy limits* along I. The existence of these two functors is a kind of *homotopy cocompletness and completness* property of the homotopy category of stacks. To make this assertion really meaningful one has to come back to the simplicially enriched category $LSPr(T)$, which is actually cocomplete and complete in a very reasonnable sense (see [S1, p. 29]).

An important special case is when I is the category $1 \to 0 \leftarrow 2$. Then, the homotopy limit of a diagram $F_1 \to F_0 \leftarrow F_2$ is called the *homotopy fibered product of* F_1 *and* F_2 *over* F_0. It is denoted by $F_1 \times^h_{F_0} F_2$, and is such that for any global section s, any space $X \in T$ and any $x \in X$ there exists a natural long exact sequence

$$\cdots \longrightarrow \pi_n((F_0)_x, s) \longrightarrow \pi_n((F_1)_x, s) \times \pi_n((F_2)_x, s) \longrightarrow \pi_n((F_1 \times^h_{F_0} F_2)_x, s) \longrightarrow \pi_{n-1}((F_0)_x, s) \longrightarrow \cdots.$$

In other words, the fibers of $F_1 \times^h_{F_0} F_2$ at $x \in X$ is equivalent to the homotopy fibered product of the fibers $(F_1)_x \times^h_{(F_0)_x} (F_2)_x$.

The homotopy fibered product $F_1 \times^h_{F_0} *$ is called the *homotopy fiber* of the morphism $F_1 \longrightarrow F_0$ at the global section $* \longrightarrow F_0$.

Definition 5.0.5 *A fibration sequence of stacks is a commutative diagram in $SPr(T)$ (not in $Ho(SPr(T))$)*

$$\begin{array}{ccccc} F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 \\ & \searrow & & & \uparrow \\ & & & & \bullet, \end{array}$$

such that the induced natural morphism in $Ho(T)$

$$F_0 \longrightarrow F_1 \times^h_{F_2} *$$

is an isomorphism. We will simply refer to such as a fibration sequence

$$F_0 \longrightarrow F_1 \longrightarrow F_2$$

.

As the datas of triangles in the derived categories determines a triangulated structure, one should think of the data of the fibration sequences as a kind of *non-abelian triangulated structure on* $Ho(T)$.

6 Non-abelian cohomology

We will end this talk by the definition of non-abelian cohomolgy. We also give some examples.

Definition 6.0.6 *Let F and F' be two stacks in $Ho(T)$. The simplicial set of cohomology of F with values in F' is defined to be $\mathbb{R}\underline{Hom}(F, F') \in Ho(SSet)$.*

Examples:

1. Let F be the sheaf represented by a space $X \in T$, and let us consider the constant simplicial presheaf of value $K(\mathbb{Z}, n)$[6]. Then one has

$$\pi_i \mathbb{R}\underline{Hom}(F, K(\mathbb{Z}, n)) \simeq H^{n-i}(X, \mathbb{Z}) \text{ (sheaf cohomology).}$$

In particular

$$[F, K(\mathbb{Z}, n)] \simeq H^n(X, \mathbb{Z}).$$

In case F is now a 1-stack $\mathcal{X}$ then

$$\pi_i \mathbb{R}\underline{Hom}(F, K(\mathbb{Z}, n)) \simeq H^{n-i}(\mathcal{X}, \mathbb{Z}),$$

where the right hand side is the sheaf cohomology defined by Kai Behrend in his first lecture.

These two isomorphisms generalizes to the case of sheaf cohomology with coefficients any sheaf of abelian groups on T. For any sheaf of abelian groups A on T, one conctruct the stack $K(A, n)$ that sends $Y \in T$ to $K(A(Y), n)$. Then one has

$$\pi_i \mathbb{R}\underline{Hom}(F, K(\mathbb{Z}, n)) \simeq H^{n-i}(\mathcal{X}, \mathbb{Z}).$$

2. Let F be represented by a space $X \in T$ and G any sheaf of groups on T (maybe non-abelian). Let $K(G, 1)$ be the simplicial presheaf of classfying simplicial sets associated to it. In formula one has $K(G, 1)(Y) := K(G(Y), 1)$ for any $Y \in T$. Then one has

$$[F, K(G, 1)] := \pi_0 \mathbb{R}\underline{Hom}(F, K(G, 1)) \simeq H^1(X, G),$$

where $H^1(X, G)$ denotes the set of isomorphisms classes of G-torsors over X. One can also prove that the fundamental groupoid of $\mathbb{R}\underline{Hom}(F, K(G, 1))$ is equivalent to the groupoid of G-torsors on X. Also $\pi_i \mathbb{R}\underline{Hom}(F, K(G, 1)) = 0$ for any $i > 1$.

3. Let still G be any group, $X \in T$, and let us consider the simplicial set $K(Z(G), 2)$, where $Z(G)$ is the center of G. The group of outer automorphisms $Out(G)$ acts on $Z(G)$ and therefore on the simplicial set $K(Z(G), 2)$. This action corresponds to a fibration sequence

$$K(Z(G), 2) \longrightarrow \mathcal{G}_G \longrightarrow K(Out(G), 1),$$

that defines the simplicial set $\mathcal{G}_G$ (one can check that $\mathcal{G}_G$ is equivalent to the classifying space of the group of self-equivalences of $K(G, 1)$). One proves that $[X, \mathcal{G}_G]$ is naturally in bijection with the set of equivalences classes of gerbes of group G on the space X (i.e. of 1-stacks $\mathcal{X} \longrightarrow X$ that are locally equivalent to the classifying stack BG). A more general result is true: the 2-fundamental groupoid of $\mathbb{R}\underline{Hom}(X, \mathcal{G}_G)$ is equivalent (as a 2-groupoid) to the 2-groupoid of gerbes of group G over X. This example of course generalizes to the case where G is a sheaf of groups.

From this last equivalence one can prove also reinterpret the non-abelian cohomology group $H^2(X, G)$ defined by J. Giraud in our context. More precisely, if $b \in H^1(X, Out(G))$ is a bound, then $H^2(X, G)$ is in bijection with the set $\pi_0(F)$, where F is the homotopy fiber of the projection

$$\mathbb{R}\underline{Hom}(X, \mathcal{G}_G) \longrightarrow \mathbb{R}\underline{Hom}(X, K(OUt(G), 1))$$

at the point corresponding to b.

[6]Recall that $K(\mathbb{Z}, n)$ denotes a connected simplicial set whith $\pi_i(K(\mathbb{Z}, n)) = 0$ if $i \neq n$ and $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$.

4. Let us go back to the previous example. There is a fibration sequence

$$K(Z(G),2) \longrightarrow \mathcal{G}_G \longrightarrow K(Out(G),1),$$

that induces a fibration sequence on cohomology spaces

$$\mathbb{R}\underline{Hom}(X,K(Z(G),2)) \longrightarrow \mathbb{R}\underline{Hom}(X,\mathcal{G}_G) \longrightarrow \mathbb{R}\underline{Hom}(X,K(Out(G),1)).$$

This sequence shows that $\mathbb{R}\underline{Hom}(X,\mathcal{G}_G)$ is an extension of $\mathbb{R}\underline{Hom}(X,\mathcal{G}_G)$ (which computes non-abelian cohomology in degree 1) by $\mathbb{R}\underline{Hom}(X,K(Z(G),2))$ (which computes abelian cohomology). Therefore, the previous example is simply a mixture of the first and second examples.

5. The previous decomposition of $\mathcal{G}_G$ into abelian cohomology and non-abelian cohomology in degree 1 is a general fact. Let $F \in Ho(T)$ be a pointed and connected stack (i.e. there is a morphism $* \to F$ which induces isomorphisms $* \simeq \pi_0(F_x)$ for any $X \in T$ and $x \in X$). Let us suppose that F is n-truncated for some n (i.e. $\pi_i(F_x) = 0$ for all $i > n$ and all $X \in T$, $x \in X$). Then, there exists a Postnikov tower

$$F = F_n \longrightarrow \dots F_i \longrightarrow F_{i-1} \longrightarrow \dots F_1 \longrightarrow F_0 = *,$$

such that for each i there exists a fibration sequence

$$K(\pi_i,i) \longrightarrow F_i \longrightarrow F_{i-1},$$

where π_i are certain sheaves of groups on T (in particular $F_1 \simeq K(\pi_1,1)$). These diagrams induce fibration sequences

$$\mathbb{R}\underline{Hom}(X,K(A_i,i)) \longrightarrow \mathbb{R}\underline{Hom}(X,F_i) \longrightarrow \mathbb{R}\underline{Hom}(X,F_{i-1}),$$

showing that in general the cohomology space $\mathbb{R}\underline{Hom}(X,F)$ is a successive extension of spaces of the form $\mathbb{R}\underline{Hom}(X,K(\pi_i,i))$. This is of course still valid if X is replaced by a topological 1-stack $\mathcal{X}$, or even by any other stack $F' \in Ho(T)$.

From this we deduce the very important

Whitehead principle: *Non-abelian cohomology is controlled by non-abelian cohomology in degree one (i.e. torsors theory) and usual abelian cohomology.*

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