Autonomous categories in which $A \cong A^*$ (extended abstract)

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Abstract

Recently, there has been some interest in autonomous categories (such as compact closed categories) in which the objects are self-dual, in the sense that $A \cong A^*$, or even $A = A^*$, for all objects A. In this talk, we investigate which coherence conditions should be required of such a category. We also investigate what graphical language could be used to reason about such a category.

1 Introduction

1.1 Self-duality

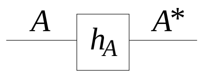
It is well-known that each finite dimensional vector space A is isomorphic, but not naturally isomorphic, to its dual space A^*. Moreover, if A is an inner product space, then there exists a natural bijection between A and A^*, but (in the case of complex inner product spaces) it is skew linear instead of linear, and therefore again A and A^* are not naturally isomorphic. On the other hand, there are autonomous categories (such as the category of finite dimensional real inner product spaces, or the category of finite sets and relations) that are equipped with a naturally arising family of isomorphisms $A \cong A^*$.

It therefore makes sense to study autonomous categories that are equipped with a family of isomorphisms $h_A : A \to A^*$. Such a family may either arise naturally from the category itself, or it might be imposed by arbitrary choice as an additional structure. In any case, it makes sense to ask which coherence conditions, if any, the isomorphisms h_A should satisfy. It also makes sense to consider a *strict* version in which $A = A^*$, and to ask whether any coherence conditions should be imposed. Finally, in light of the fact that there are sound and complete graphical languages for various notions of autonomous categories [6], it makes sense to ask whether there is a sound and complete graphical language for autonomous categories with the additional structure of $A = A^*$.

A similar structure of self-duality was recently described, in the more general context of involutive monoidal categories, by Egger [1].

1.2 Self-duality without coherence

There are two possible approaches to coherence. The first is to start with an autonomous (for example, compact closed) category, and to equip each object A with a chosen isomorphism $h_A : A \to A^*$, without requiring any naturality or coherence conditions at all. In this approach, each object is equipped with a chosen "self-duality structure", which is independent of any other object. It is akin, for example, to equipping each finite dimensional vector space with an arbitrarily and independently chosen basis. In this case, the structure chosen on, say, $A \otimes B$ or A^* does not need to bear any relationship to the structure chosen on A or B. There is an obvious sound and complete graphical language, namely the usual language of autonomous categories, extended with basic boxes

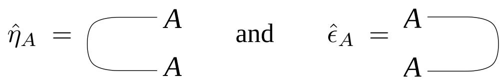


and their inverses, satisfying no special laws.

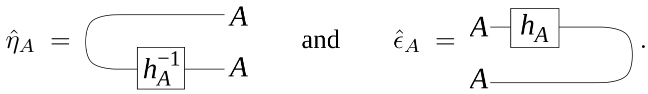
The strict version of this notion is to require $A = A^*$ for all objects without any coherence. Equivalently, for each object A, one requires a unit $\hat{\eta}_A : I \to A \otimes A$ and a counit $\hat{\epsilon}_A : A \otimes A \to I$, satisfying the usual two laws for an exact pairing, i.e.,

(1.1)

with no additional conditions imposed. Note that, in the absence of coherence conditions, it is not possible to write



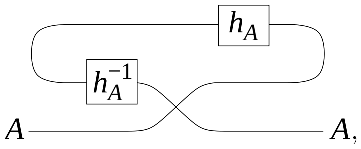
in the graphical language, because the graphical language would then validate additional laws (think of the graphical representation of $\hat{\eta}_{A\otimes B}$, for example). So even in the strict case, the only available graphical language (which is then sound and complete) is the same as in the non-strict case, i.e.,



(1.2)

In particular, even in the strict formulation there is no reason to expect any particular equations to hold between morphisms. For example, in the presence of a symmetric monoidal structure (or more generally, a pivotal structure), there is a canonical isomorphism $i_A : A \to A^{**}$, usually depicted as the identity in the graphical language. In the setting of strict self-duality without coherence, we of course have $A = A^{**}$; however, there is no reason to expect the canonical map $i_A : A \to A^{**}$ to be equal to the identity of A.

Indeed, in the graphical language, the map i_A will then have to be depicted as



which is not a diagram for the identity morphism.

1.3 Self-duality with coherence

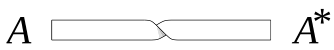
In light of the above, it seems reasonable to require the self-duality isomorphisms $h_A : A \to A^*$ to satisfy some coherence axioms. Preferably the coherence axioms should be chosen in such a way that (a) they are valid in relevant examples, and (b) there is a good graphical language.

Note that the assumption $A \cong A^*$ is not as benign as it might first appear. For example, because an isomorphism $(A \otimes B)^* \cong B^* \otimes A^*$ is present in any autonomous category, we obtain a derived isomorphism

$$A \otimes B \xrightarrow{h_{A\otimes B}} (A \otimes B)^* \xrightarrow{\cong} B^* \otimes A^* \xrightarrow{h_B^{-1} \otimes h_A^{-1}} B \otimes A. \tag{1.3}$$

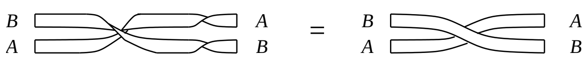
If the original category was assumed to be symmetric monoidal (or more generally, braided monoidal), one would certainly want to require (1.3) to be equal to the symmetry (or braiding). And even if the underlying category was not assumed to possess a braiding (or symmetry), the self-duality assumption forces a braiding upon us via (1.3).

The morphism (1.3) also has implications for the graphical language. The isomorphism $(A \otimes B)^* \cong B^* \otimes A^*$ is usually represented in the graphical language like an identity (and in planar autonomous categories this is the only possible choice). Therefore, because of (1.3), it is not possible to represent $h_X : X \to X^*$ as an identity in the graphical language for an arbitrary object term X. As the case $X = A \otimes B$ shows, the map h_X should instead be represented as a half-twist:



(1.4)

The morphism (1.3) then becomes:



Also note that, if $h_A : A \to A^*$ is represented in the graphical language as a half-twist, then the morphism

$$\theta_{A^*} := A^* \xrightarrow{h_{A^*}} A^{**} \xrightarrow{h_A^*} A^* \tag{1.5}$$

is represented as a full twist on A^*:

This indicates that an autonomous category with self-duality should at minimum be tortile (and possibly compact closed if the braiding is a symmetry). Or in other words, the minimal autonomous structure on which it makes sense to require a self-duality is that of a tortile category.

2 Background

We recall some well-known definitions from the theory of monoidal categories [4].

Definition (Braided monoidal category). A *braiding* on a monoidal category is a natural family of isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$, satisfying the two "hexagon axioms":

$$(\mathrm{id}_B \otimes c_{A,C}) \circ \alpha_{B,A,C} \circ (c_{A,B} \otimes \mathrm{id}_C) = \alpha_{B,C,A} \circ (c_{A,B\otimes C}) \circ \alpha_{A,B,C},$$
$$(\mathrm{id}_B \otimes c^{-1}_{C,A}) \circ \alpha_{B,A,C} \circ (c^{-1}_{B,A} \otimes \mathrm{id}_C) = \alpha_{B,C,A} \circ (c^{-1}_{B\otimes C,A}) \circ \alpha_{A,B,C}.$$

A monoidal category that is equipped with a braiding is called a *braided monoidal category*.

Definition (Balanced monoidal category). Recall that a *twist* on a braided monoidal category is a natural family of isomorphisms $\theta_A : A \to A$, satisfying $\theta_I = \mathrm{id}_I$ and such the following commutes for all A, B:

$$\begin{array}{ccc} A\otimes B & \xrightarrow{c_{A,B}} & B\otimes A \\ \Big\downarrow{\scriptstyle \theta_{A\otimes B}} & & \Big\downarrow{\scriptstyle \theta_B\otimes\theta_A} \\ A\otimes B & \xleftarrow[c_{B,A}]{} & B\otimes A. \end{array} \qquad (2.1)$$

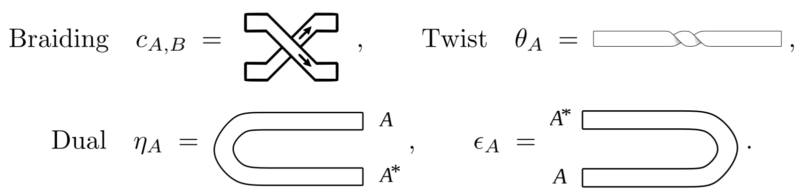
A *balanced monoidal category* is a braided monoidal category with twist.

Definition (Right autonomous category). Recall that a *right dual* for an object A in a monoidal category is given by (B, η, ϵ), where B is an object, and $\eta : I \to B \otimes A$ and $\epsilon : A \otimes B \to I$ are morphisms, such that the following two adjunction triangles commute:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A\otimes\eta} & A\otimes B\otimes A \\ & {\scriptstyle \mathrm{id}_A}\searrow & \Big\downarrow{\scriptstyle \epsilon\otimes\mathrm{id}_A} \\ & & A, \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\eta\otimes\mathrm{id}_B} & B\otimes A\otimes B \\ & {\scriptstyle \mathrm{id}_B}\searrow & \Big\downarrow{\scriptstyle \mathrm{id}_B\otimes\epsilon} \\ & & B. \end{array} \qquad (2.2)$$

The maps η and ϵ determine each other uniquely, and moreover, the triple (B, η, ϵ), if it exists, is uniquely determined by A up to isomorphism. A monoidal category is *right autonomous* if every object A has a right dual, which we then denote $(A^*, \eta_A, \epsilon_A)$.

In the graphical language [6, 3, 2], these structures are represented as follows:



In each case (braided, balanced, or right autonomous) there is a coherence theorem stating that an equation follows from the respective axioms if and only if it holds in the graphical language.

When combining a balanced structure with an autonomous structure, an additional axiom is required that relates them. In the presence of this axiom, coherence holds for the combined graphical language.

Definition (Tortile category). A *tortile category* is a balanced monoidal category which is also right autonomous and satisfies

$$\theta_{A^*} = (\theta_A)^*.$$

A *compact closed category* [5] is a special case of a tortile category satisfying $\theta_A = \mathrm{id}_A$ for all A (and therefore $c_{A,B} = c^{-1}_{B,A}$). Equivalently, a *compact closed category* [5] is a right autonomous symmetric monoidal category.

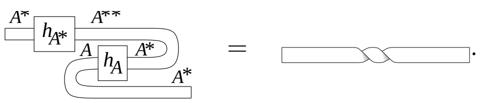
3 Self-duality structure on tortile categories

We will state the coherence axioms for the self-duality isomorphisms $h_A : A \to A^*$ in two different ways. The first formulation, given in this section, assumes that the underlying category is tortile (or compact closed, which is a special case). The coherence axioms then relate the self-duality structure to the tortile structure, along the lines of (1.3) and (1.5) in the introduction. In the next section, we will give an equivalent set of axioms assuming only right autonomous structure.

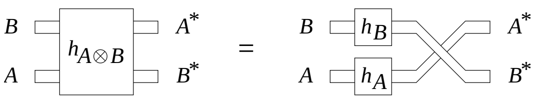
Definition. Let $\mathbf{C}$ be a tortile category. A *self-duality structure* on $\mathbf{C}$ is given by a family of morphisms $h_A : A \to A^*$, one for each object A, satisfying the following five axioms (for all A and B):

(T1) h_A is an isomorphism.

(T2) $h^*_A \circ h_{A^*} = \theta_{A^*} : A^* \to A^*$. Equivalently: $h_{A^*} \circ h^*_A = \theta_{A^{**}} : A^{**} \to A^{**}$.



(T3) $h_{A\otimes B} = A \otimes B \xrightarrow{h_A \otimes h_B} A^* \otimes B^* \xrightarrow{c_{A^*,B^*}} B^* \otimes A^* \xrightarrow{\cong} (A \otimes B)^*$, where the last isomorphism is the canonical one from the autonomous structure.

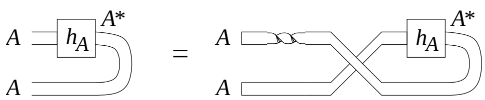


(T4) $h_I : I \xrightarrow{\cong} I^*$ is the canonical isomorphism from the autonomous structure.

(T5) $h_A = A \xrightarrow{i_A} A^{**} \xrightarrow{h^*_A} A^*$, where $i_A : A \xrightarrow{\cong} A^{**}$ is the canonical isomorphism arising from the pivotal structure, i.e.,

$$i_A = A \xrightarrow{\theta_A} A \xrightarrow{\mathrm{id}\otimes\eta_{A^*}} A \otimes A^{**} \otimes A^* \xrightarrow{c^{-1}_{A^{**},A}\otimes\mathrm{id}} A^{**} \otimes A \otimes A^* \xrightarrow{\mathrm{id}\otimes\epsilon_A} A^{**}.$$

In the graphical language of braided autonomous categories, this axiom is equivalent to:



Remark 3.1. The above axioms are sound for the graphical language, where h is represented by a half-twist as in (1.4).

Theorem 3.2. *Let $\phi : A \to B$ be any of the canonical isomorphism arising from the tortile structure. By this, we mean isomorphisms that are represented by identities in the usual graphical language, i.e., α, λ, ρ, $(A \otimes B)^* \cong B^* \otimes A^*$, $I \cong I^*$, or $i_A : A \to A^{**}$, but not for example θ or c. Then the following commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_B} \\ A^* & \xleftarrow{\phi^*} & B^*. \end{array}$$

Conjecture 3.3 (Coherence). *An equation follows from the axioms of tortile categories with self-duality structure if and only if it holds in the graphical language, where h_A is represented as a half-twist as in (1.4).*

The axioms have some interesting structure, captured in the following discussion.

Theorem 3.4. *Assuming axioms (T1)–(T4). If axiom (T5) holds for some objects A and B, then axiom (T5) also holds for A^*, $A \otimes B$, and I. Therefore, it suffices to check (T5) for a set of generators of the objects of* **C**.

Remark 3.5. In case the objects of **C** are generated freely, axioms (T2)–(T4) can be used as definitions of h_{A^*}, $h_{A\otimes B}$, and h_I, respectively. It therefore suffices, in this case, to define the isomorphisms $h_A : A \to A^*$ for object generators, in some arbitrary way subject only to axiom (T5). The definition then extends to all objects via (T2)–(T4), and gives a self-duality structure on **C** by Theorem 3.4.

Theorem 3.6. *Axiom (T1) is a consequence of (T2) and (T5). The remaining axioms are independent.*

Proof. Consider some non-trivial tortile category where the objects are (without loss of generality) generated freely from a set of generators and the operations A^*, $A \otimes B$, and I. In light of Remark 3.5, it is clearly possible to define h_A on the generators in some arbitrary way so that axiom (T5) is violated, then extend it to all objects using axioms (T2)–(T4). In this case, axioms (T1)–(T4) are valid and (T5) is not. On the other hand, define h_A on generators to satisfy (T5), and then extend the definition inductively to all objects, but modifying the definition of exactly one of h_{A^*}, $h_{A\otimes B}$, or h_I by multiplying by an additional scalar (for example, using $h_{A^*} = \phi \cdot (h_A^*)^{-1} \circ \theta_{A^*}$, where ϕ is a non-trivial scalar). It is easy to see that the extra scalar does not invalidate (T5). So in this case, exactly one of the axioms (T2)–(T4) fails, while the remaining axioms are valid. Finally, to see that (T2) and (T5) imply (T1), note that h_A^* is a split epi by (T2), hence h_A is a split mono. Since i_A is an isomorphism, it follows from (T5) that h_A^* is mono, hence iso, hence h_A is iso as well. □

Remark 3.7. Notice that axiom (T5) in particular implies that

$$A \xrightarrow{h_A} A^* \xrightarrow{(h^*)^{-1}} A^{**} \tag{3.1}$$

is a monoidal natural transformation (because it is equal to i_A, which is a monoidal natural transformation in any tortile category). As a matter of fact, if one assumes that (3.1) is a natural transformation, then the fact that it is monoidal already follows from axioms (T1)–(T4). However, this is not quite strong enough to imply that (3.1) is equal to i_A, i.e., axiom (T5).

Example 3.8 (Natural examples). The compact closed category of finite dimensional real inner product spaces possesses a self-duality, where $h_A : A \to A^*$ is given as the adjoint of the inner product $A \otimes A \to I$. The compact closed category of finite sets and relations possesses a self-duality, where $A = A^*$ and $h_A : A \to A^*$ is given by the identity relation.

Example 3.9 (Unnatural example). The compact closed category of finite dimensional complex inner product spaces (i.e., finite dimensional Hilbert spaces) can be equipped with a self-duality, but not in a canonical way. First, rename the objects (up to equivalence of categories) so that they are freely generated. Then choose the structure according to Remark 3.5.

Example 3.10. From any tortile category $\mathbf{C}$, we can construct another category $\mathbf{D}$ with self-duality. Let the objects of $\mathbf{D}$ be pairs (A, h), where A is an object of $\mathbf{C}$ and $h : A \to A^*$ is an isomorphism of $\mathbf{C}$ satisfying axiom (T5). A morphism from (A, h) to (B, h') is just a morphism from A to B in $\mathbf{C}$. Define $(A, h) \otimes (B, h') = (A \otimes B, h'')$, $(A, h)^* = (A^*, h''')$, and (I, h'''') in the unique way so that the axioms are satisfied, with the tortile structure inherited from $\mathbf{C}$. Note that the construction is non-canonical, in the sense that each object A of $\mathbf{C}$ generates potentially many non-isomorphic objects (A, h_1), $(A, h_2), \ldots$, of $\mathbf{D}$.

4 Self-duality structure on right autonomous categories

We remarked in the introduction that a self-duality structure on a right autonomous category already yields isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$ and $\theta_{A^*} : A^* \to A^*$ as in (1.3) and (1.5), which can be used as the basis for a tortile structure. Of course, one still has to require special axioms to ensure that the resulting structure is indeed tortile, and that the self-duality and tortile structures are compatible in the sense of Section 3. The result is an alternate axiomatization of self-duality structure, using only the language of right autonomous categories.

Definition. Let $\mathbf{C}$ be a right autonomous category. A *self-duality structure* on $\mathbf{C}$ is given by a family of morphisms $h_A : A \to A^*$, one for each object A, satisfying the following eight axioms (for all A, B, C, f, and g). For convenience, we express some axioms in the graphical language of right autonomous categories (which is legitimate due to the coherence theorem for right autonomous categories). We also write

$$f_\sharp = A^* \xrightarrow{h_A^{-1}} A \xrightarrow{f} B \xrightarrow{h_B} B^*.$$

(A1) h_A is an isomorphism.

(A2) $h_I : I \xrightarrow{\cong} I^*$ is the canonical isomorphism from the autonomous structure.

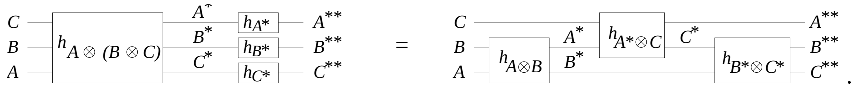
(A3) $(f^*)_\sharp = (f_\sharp)^*$.

(A4) The following commutes, where the vertical isomorphisms are the canonical ones from the autonomous structure:

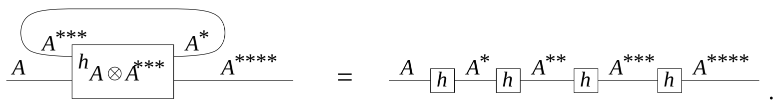
$$\begin{array}{ccc} (A\otimes B)^* & \xrightarrow{(f\otimes g)_\sharp} & (A'\otimes B')^* \\ \cong\downarrow & & \downarrow\cong \\ B^*\otimes A^* & \xrightarrow{g_\sharp\otimes f_\sharp} & B'^*\otimes A'^*. \end{array}$$

(A5) $\alpha^* = (\alpha_\sharp)^{-1}$, where $\alpha : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$ is the associativity isomorphism from the monoidal structure. Equivalently, $h_{A\otimes(B\otimes C)}$ is equal to $h_{(A\otimes B)\otimes C}$ modulo associativity.

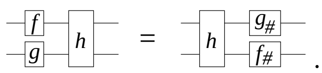
(A6)



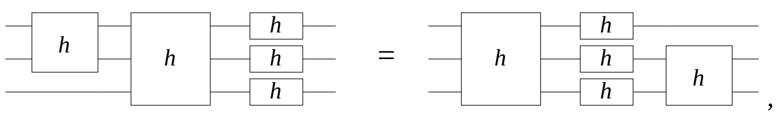
(A7)



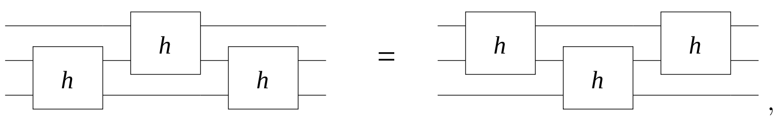
We remark the following. First, $(-)_\sharp$ is a covariant functor whose object part is $(-)^*$. Second, $h_A : A \to A^*$ is a natural transformation with respect to $\sharp$, by definition: $h_B \circ f = f_\sharp \circ h_A$. We also have $h_{A^*} = (h_A)_\sharp$. Axiom (A4) is equivalent to the following, which is a componentwise naturality for $h_{A\otimes B}$:

 (4.1)

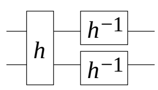
Note that this naturality implies



and therefore with (A6),

 (4.2)

We define

$c_{A,B} =$ 

$$\theta_A = A \xrightarrow{h_A} A^* \xrightarrow{h_{A^*}} A^{**} \xrightarrow{h_A^*} A^* \xrightarrow{h_A^{-1}} A.$$

Then $c_{A,B}$ satisfies the two hexagon axioms by (A6), (A5), and (4.2), giving a braided structure. Similarly, one must verify that θ_A defines a balanced structure and that the remaining axioms are satisfied.

Theorem 4.1. *A self-duality structure on a right autonomous category yields a tortile structure satisfying the axioms of Section 3. Conversely, any self-duality structure on a tortile category satisfies the axioms of Section 4. The two constructions are mutually inverse, establishing a one-to-one correspondence between tortile categories with a self-duality structure (in the sense of Section 3) and autonomous categories with a self-duality structure (in the sense of Section 4).*

Remark 4.2. The property (4.2) is a version of the Yang-Baxter equation for braids. The axiom (A7) is a version of yanking.

Remark 4.3. Axiom (A5) states that the associativity map satisfies the condition of Theorem 3.2. The corresponding properties for the other maps of Theorem 3.2 follows from the remaining axioms. Most of them follow from axiom (A7) and coherence for braided autonomous categories.

Remark 4.4. A useful consequence of (A7) and (A4) is $f^{**} = f_{\sharp\sharp}$. Equivalently, $A \xrightarrow{h_A} A^* \xrightarrow{h_{A^*}} A^{**}$ is a natural transformation.

Remark 4.5 (Induced dagger structure). As the presence of the functor $(-)_\sharp$ suggests, a self-duality structure on a right autonomous (or tortile, or compact closed) category induces a dagger structure: namely, for $f : A \to B$, define $f^\sharp : B \to A$ to be the unique morphism for $(f^\sharp)_\sharp = f^*$. Then the properties of Theorem 3.2 mean precisely that the canonical isomorphisms mentioned there are *unitary* with respect to the induced dagger structure.

However, the induced dagger structure is not usually the one that is useful in the example categories. In particular, for $f : I \to I$, we always have $f^\sharp = f$, whereas for the "natural" dagger (say arising from linear adjoints in finite dimensional Hilbert spaces) we have that $f^\dagger$ is the complex conjugate of f. Therefore, in finite dimensional Hilbert spaces, equipped by brute force with a self-duality structure as in Examples 3.9 or 3.10, the induced dagger structure is never the canonical one.

Moreover, if $\mathbf{C}$ is tortile with self-duality, then the induced dagger structure $(-)^\sharp$ is not a dagger tortile structure. Namely, we have $(c_{A,B})^\sharp = c_{B,A}$ and $(\theta_A)^\sharp = \theta_A$, whereas a dagger tortile category should satisfy $(c_{A,B})^\dagger = c^{-1}_{A,B}$ and $(\theta_A)^\dagger = \theta^{-1}_A$. In other words, $c_{A,B}$ and θ_A fail to be unitary with respect to the induced dagger structure. For these reasons, we refrain from using the usual notation $(-)^\dagger$ for these unnatural dagger structures.

On the other hand, in "natural" examples of self-duality, such as *real* inner product spaces, the induced dagger does coincide with the usual dagger.

5 Strict self-duality

Recall that every self-duality structure $h_A : A \to A^*$ on a right autonomous category induces a strict autonomous structure

$$\hat{\eta}_A : I \to A \otimes A, \quad \hat{\epsilon}_A : A \otimes A \to I,$$

namely via $\hat{\eta}_A = (h_A^{-1} \otimes \mathrm{id}_A) \circ \eta_A$ and $\hat{\epsilon}_A = \epsilon_A \circ (\mathrm{id}_A \otimes h_A)$ (as displayed graphically in (1.2). It is therefore an obvious idea to axiomatize the self-duality directly in terms of such $\hat{\eta}_A$ and $\hat{\epsilon}_A$, rather than passing via a pre-existing autonomous structure and isomorphisms h_A. This question will be addressed in the full version of this article.

6 Conclusions

We have proposed a set of coherence conditions for autonomous categories in which $A \cong A^*$. We have given two equivalent formulations of the coherence conditions. All the listed conditions are sound for an obvious graphical language, and are satisfied in relevant examples. We conjecture that the conditions are also complete. One area where self-duality arises is in the study of categories with chosen Frobenius algebra structures on each object. It remains to be seen how the coherence conditions discussed here can be generalized to Frobenius algebras.

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