ALGEBRAIC K-THEORY OF FINITELY PRESENTED RING SPECTRA

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1. Galois extensions

Let S be the sphere spectrum. An S-algebra A is a monoid $(A, \mu\colon A \wedge A \to A, \eta\colon S \to A)$ in a good symmetric monoidal category of spectra, such as the S-modules of Elmendorf, Kriz, Mandell and May [EKMM], the symmetric spectra of Jeff Smith [HSS], or the simplicial functors of Manos Lydakis [Ly]. When A is commutative there is also the notion of an A-algebra $(B, \mu\colon B \wedge_A B \to B, \eta\colon A \to B)$.

La $A \to B$ be a map of commutative S-algebras. (Make the necessary cofibrancy and fibrancy assumptions.) Let G be a grouplike topological monoid acting on B through A-algebra maps.

Definition. $A \to B$ is a G-Galois extension if

(1) $G \simeq \pi_0(G)$ is finite,
(2) $A \simeq B^{hG} = F(EG_+, B)^G$, and
(3) $B \wedge_A B \simeq F(G_+, B)$.

$A \to B$ is a G-pro-Galois extension if G is a filtered limit $G = \lim_\alpha G_\alpha$, B is a filtered colimit $B = \operatorname{colim}_\alpha B_\alpha$ and $A \to B_\alpha$ is a G_α-Galois extension for each α. Then $A \simeq B^{hG}$ and $B \wedge_A B \simeq F(G_+, B)$ where the homotopy fixed points and function spectra are formed in a continuous sense.

Examples.

(1) The trivial G-Galois extension $A \to B = F(G_+, A)$ takes A to constant maps from G.
(2) When $R \to T$ is a G-Galois extension of commutative rings, the map of Eilenberg–Mac Lane ring spectra $HR \to HT$ is a G-Galois extension (of commutative S-algebras).
(3) Complexification $KO \to KU$ is a C_2-Galois extension, and inclusion of the p-local Adams summand $L \to KU_{(p)}$ is a $(\mathbb{Z}/p)^*$-Galois extension.
(4) More generally $EO_n \to E_n$ is a G-Galois extension when $EO_n = E_n^{hG}$ for G a maximal finite subgroup of $G_n = S_n \rtimes C_n$. Here $C_n = \operatorname{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic of order n and S_n is the nth Morava stabilizer group of automorphisms of a height n formal group law defined over $\mathbb{F}_{p^n}$. The Lubin–Tate spectrum E_n has homotopy $E_{n*} = \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$, where each u_i has degree 0 and u has degree 2, and G_n acts on E_n through S-algebra maps,

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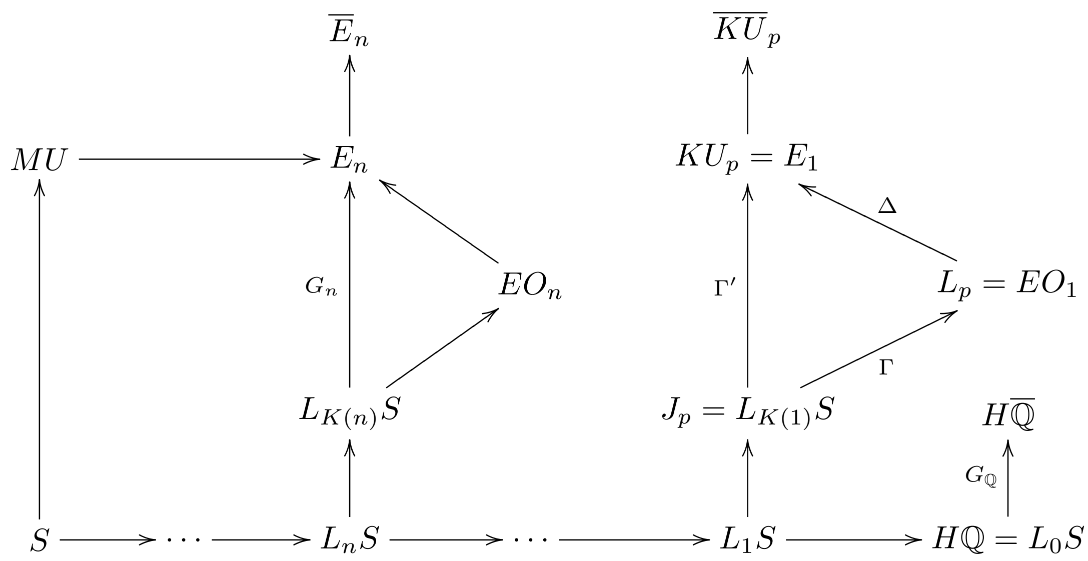
cf. the works of Morava, Hopkins–Miller and Goerss–Hopkins [Mo], [Re], [GH].

(5) The inclusion $J_p \to KU_p$ is a $\mathbb{Z}_p^*$-pro-Galois extension, with $k \in \mathbb{Z}_p^*$ acting as the Adams operation ψ^k.

(6) More generally $L_{K(n)}S \to E_n$ is (most likely) a G_n-pro-Galois extension. The assertion $L_{K(n)}S \simeq E_n^{hG_n}$ is a version of the Morava change of rings theorem, and the equivalence $E_n \wedge_{L_{K(n)}S} E_n \simeq F(G_{n+}, E_n)$ is a variation on a result of Devinatz–Hopkins. (Needs some further checking.)



(When the maximal finite subgroup of G_n used to form EO_n is not normal, EO_n will not be Galois over $L_{K(n)}S$.)

Proposition. *Let $A \to B$ be G-Galois, M an A-module, N a B-module.*

(1) *B is strongly dualizable as an A-module. So $F_A(B, A) \wedge_A M \simeq F_A(B, M)$. We write $D_A(B) = F_A(B, A)$ for the A-dual of B.*

(2) *B is self-dual as an A-module. So $B \simeq D_A(B)$.*

(3) *A is B-complete, i.e., the map $A \to C(A \to B)$ to the totalization of the cosimplicial spectrum $[q] \mapsto B \wedge_A \cdots \wedge_A B$ with $(q+1)$ copies of B is a homotopy equivalence.*

(4) *$B \wedge_A N \simeq F(G_+, N)$.*

(5) *$N \wedge G_+ \simeq F_A(B, N)$. In particular $B \wedge G_+ \simeq F_A(B, B)$.*

Question. What can be said when $B = A \wedge X$ for a spectrum X ? Is X suitably self-dual ?

Question. Is B faithfully flat as an A-module ? That is, does $B \wedge_A M \simeq *$ imply that $M \simeq *$?

This holds in many cases, including the trivial Galois extension, Galois extensions of commutative rings, $KO \to KU$, $L \to KU_{(p)}$, $EO_2 \to E_2$ and $J_p \to KU_p$.

For example, if M is a KO-module with $KU \wedge_{KO} M \simeq *$ then from the cofiber sequence $\Sigma KO \to KO \to KU$ we get that $\eta \colon \Sigma M \to M$ is a homotopy equivalence. But η is nilpotent, so $M \simeq *$. A similar argument works for $EO_2 \to E_2 \simeq EO_2 \wedge DA(1)$.

Definition. A commutative S-algebra A is connected if we can only factor A as $A \simeq A' \times A''$ as commutative S-algebras when A' or A'' is contractible.

Definition. A connected commutative S-algebra A is separably closed if it admits no connected G-Galois extension $A \to B$ with $\pi_0(G)$ nontrivial. We write $\overline{A}$ for a separable closure of A.

2. Étale maps

Example. Let $F \to E$ be a G-Galois extension of number fields. Then the map of number rings $\mathcal{O}_F \to \mathcal{O}_E$ is G-Galois if and only if $F \to E$ is unramified, i.e., if and only if $\mathcal{O}_F \to \mathcal{O}_E$ is an étale map.

Definition. A map $A \to B$ of S-algebras is formally étale if the topological André–Quillen homology $TAQ(B/A) \simeq *$ is contractible.

One definition of $TAQ(B/A)$ is as the B-module spectrum with nth space $S^n \otimes B$ with the tensor product formed in the category of commutative A-algebras.

The lifts in the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \vee M \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

where M is a C-module and $M \to C \vee M \to C$ a square-zero extension, are the A-linear derivations $\mathrm{Der}_A(B, M)$ of B with values in M, and $\mathrm{Der}_A(B, M) \simeq F_B(TAQ(B/A), M)$. Dually to the unique lifting property of covering spaces this space is always contractible precisely when $A \to B$ is formally étale.

The following criterion is useful.

Proposition. *$TAQ(B/A) \simeq *$ if and only if $B \simeq HH^A(B)$.*

Here $HH^A(B)$ is the realization of the simplicial spectrum $[q] \mapsto B \wedge_A \cdots \wedge_A B$ with $(q+1)$ copies of B and Hochschild-type face maps. The special case $THH(B) = HH^S(B)$ is the topological Hochschild homology of B.

Proof. There is a spectral sequence from the symmetric B-algebra of $TAQ(B/A)$ to $HH^A(B)$, which when $TAQ(B/A) \simeq *$ collapses to $B \simeq HH^A(B)$.

Conversely the identity $TAQ(HH^A(B)/B) \simeq \Sigma TAQ(B/A)$ shows that $B \simeq HH^A(B)$ implies $TAQ(B/A) \simeq *$. □

Proposition. *A G-Galois extension $A \to B$ is formally étale.*

Proof. $B \wedge_A B \simeq F(G_+, B)$ is a product of copies of B, so contains B as a retract as a $B \wedge_A B$-module. The composite $B \to HH^A(B) \simeq \mathrm{Tor}^{B\wedge_A B}(B, B) \to \mathrm{Tor}^{B\wedge_A B}(B \wedge_A B, B) \simeq B$ is an equivalence, and the right hand map is a split injection. Hence all the maps are homotopy equivalences. □

The transitivity sequence for TAQ can be applied to show that $A \to B$ is formally étale (if and?) only if $B \wedge_A THH(A) \simeq THH(B)$. Compare Geller and Weibel [GW].

This much indicates that we have the beginnings of a good theory.

3. Galois descent in algebraic K-theory

Let E be an S-algebra. Two important invariants of the category of E-module spectra is the algebraic K-theory $K(E)$ and the topological Hochschild homology $THH(E)$. When E is commutative, these are also commutative S-algebras.

What are the global structural properties of these invariants ?

Galois descent problem. La $A \to B$ be a G-Galois extension of commutative S-algebras. Does $K(A) \to K(B)^{hG}$ induce an equivalence (with suitable coefficients, in sufficiently high degrees) ?

This is known to hold for $A \to B$ a Galois extension of finite fields by Quillen [Q1], for p-complete algebraic K-theory of p-local number fields (p odd) by Hesselholt and Madsen [HM2], and for 2-local algebraic K-theory of number fields or 2-local number fields by Voevodsky [V] and Rognes–Weibel [RW].

The separably closed case. Is $K(\overline{A})$ simple to describe when $\overline{A}$ is separably closed ?

Theorem (Quillen, Suslin).

(1) $K(\overline{\mathbb{F}}_p)_p \simeq H\mathbb{Z}_p$.
(2) $K(\overline{\mathbb{Q}})_p \simeq ku_p$.

Note that $p^{-1}H\mathbb{Z}_p$ may deserve the name E_0, and $v_1^{-1}ku_p = KU_p = E_1$.

Questions. What is a separable closure $\overline{E}_n$ of E_n, or equivalently of $L_{K(n)}S$? What does the "fundamental theorem of algebra" say in such an S-algebra ?

What is $\overline{S}$? If $\overline{S} = S$ this is the S-algebra version of Minkowski's theorem $\overline{\mathbb{Z}} = \mathbb{Z}$, saying that every number ring other than $\mathbb{Z}$ is ramified somewhere.

I stated something like the following conjecture at Schloß Ringberg in January 1999.

Optimistic Conjecture. *The k-connected covers of $K(\overline{E}_n)_p$ and E_{n+1} are homotopy equivalent for k sufficiently large.*

This would allow the recursive definition $E_{n+1} = L_{K(n+1)}K(\overline{E}_n)$, in the category of commutative S-algebras.

When Galois descent holds, we get a spectral sequence

$$E^2_{st} = H^{-s}(G; K_t(B)) \Longrightarrow K_{s+t}(A)$$

converging with suitable coefficients and in sufficiently high degrees. Then the complexity of $K(A)$ gets split between the group cohomology of G and the algebraic K-theory of B. When $B = \overline{A}$ is separably closed, and if $K(\overline{A})$ has a simple form, then the complexity is all in the cohomology of the absolute Galois group $G_A = \mathrm{Gal}(\overline{A}/A)$.

Conversely, if we can somehow compute $K(A)$ we may estimate $H^*(G_A; -)$ and $K(\overline{A})$. (Differentials in the descent spectral sequence tend to make this harder.) We shall elaborate on this in two examples later.

In the Hopkins–Miller example we are looking at spectral sequences

$$E^2_{st} = H^{-s}(G_n; K_t(E_n)) \Longrightarrow K_{s+t}(L_{K(n)}S)\,.$$

The relation between $L_n S$ and $L_{K(n)} S$ is illuminated by Hopkins' chromatic splitting conjecture. Letting n grow, we can hope to compare $K(S) = A(*)$ with $\lim_n K(L_n S)$, but it is not clear how algebraic K-theory interacts with the limit in the chromatic tower.

4. Localization sequences

Here is one strategy for how to compute algebraic K-theory. The maps $\mathbb{F}_p = \mathbb{Z}_p/p \leftarrow \mathbb{Z}_p \to p^{-1}\mathbb{Z}_p = \mathbb{Q}_p$ induce a cofiber sequence of spectra $K(\mathbb{F}_p) \to K(\mathbb{Z}_p) \to K(\mathbb{Q}_p)$ due to Quillen [Q2].

Let ku_p be the connective p-complete topological K-theory spectrum, and ℓ_p its Adams summand. So $ku_{p*} = \mathbb{Z}_p[u]$ with $|u| = 2$, and $\ell_p = \mathbb{Z}_p[v_1]$ with $|v_1| = 2p-2$.

There are analogous maps $H\mathbb{Z}_p = ku_p/u \leftarrow ku_p \to u^{-1}ku_p = KU_p$ and $H\mathbb{Z}_p = \ell_p/v_1 \leftarrow \ell_p \to v_1^{-1}\ell_p = L_p$ inducing diagrams $K(\mathbb{Z}_p) \to K(ku_p) \to K(KU_p)$ and $K(\mathbb{Z}_p) \to K(\ell_p) \to K(L_p)$.

Question. Are these cofiber sequences of spectra ? This requires identifying the algebraic K-theory of u-torsion ku_p-modules or v_1-torsion ℓ_p-modules with $K(\mathbb{Z}_p)$.

Note that $K(\mathbb{Z}_p)$ is known, by calculations of Bökstedt and Madsen for p odd [BM] and by Rognes for $p = 2$ [R]. Thus if these diagrams are cofiber sequences then it suffices to compute $K(ku_p)$ or $K(\ell_p)$, and the transfer map from $K(\mathbb{Z}_p)$, in order to compute $K(KU_p)$ or $K(L_p)$. This would in turn give an estimate on G_{KU_p} or G_{L_p}, and thus a hint about the structure of $\overline{KU}_p$.

The spectra ku_p and ℓ_p are connective. This makes it significantly easier to compute their algebraic K-theory, due to the possibility of comparing with topological cyclic homology.

5. Topological cyclic homology

We briefly recall the topological cyclic homology of an S-algebra E, first constructed in [BHM] by Bökstedt, Hsiang and Madsen.

$THH(E) = HH^S(E)$ is the geometric realization of the simplicial spectrum $[q] \mapsto E \wedge E \wedge \cdots \wedge E$ with $(q+1)$ copies of E and Hochschild-type face maps. This is a cyclic object in the sense of Connes, and $THH(E)$ admits an S^1-action. Let $C_{p^n} \subset S^1$ be the cyclic group of order p^n. Then $TC(E;p)$ is formed as a homotopy limit:

$$TC(E;p) =$$

$$\operatorname{holim}\left(\cdots \overset{R}{\underset{F}{\rightrightarrows}} THH(E)^{C_{p^n}} \overset{R}{\underset{F}{\rightrightarrows}} THH(E)^{C_{p^{n-1}}} \overset{R}{\underset{F}{\rightrightarrows}} \cdots \overset{R}{\underset{F}{\rightrightarrows}} THH(E) \right)$$

The maps R and F are called restriction and Frobenius maps, respectively, by analogy with similar maps among Witt rings of finite length.

The cyclotomic trace map is a natural transformation $trc\colon K(E) \to TC(E;p)$, and the composite with the canonical map $\beta\colon TC(E;p) \to THH(E)$ is the Dennis–Bökstedt trace map $tr = \beta \circ trc\colon K(E) \to THH(E)$.

Theorem (Hesselholt–Madsen, Dundas, McCarthy). *Let E be a connective S-algebra with $\pi_0(E)$ a finite module over the Witt vectors of a perfect field of* *characteristic p, e.g. a finite $\mathbb{Z}_p$-module, then trc: $K(E) \to TC(E;p)$ identifies $K(E)_p$ with the connective cover of $TC(E;p)_p$.*

In general $TC(E;p)_p$ is (-2)-connected, so the homotopy cofiber of trc has the form $\Sigma^{-1}HA$ for a known group A.

$H_*(THH(E);\mathbb{F}_p)$ is generally quite accessible through the Bökstedt spectral sequence

$$E^2_{s*} = HH_s^{\mathbb{F}_p}(H_*(E;\mathbb{F}_p)) \Longrightarrow H_*(THH(E);\mathbb{F}_p)\,.$$

Supposing E is commutative, this is a spectral sequence of $H_*(E;\mathbb{F}_p)$-algebras and A_*-comodules, where A_* is the dual Steenrod algebra.

We will eventually want to pass over the (inverse) limit defining $TC(E;p)$. One cannot expect to do this in homology, since the correspondence

$$H_*(TC(E;p);\mathbb{F}_p) \to \operatorname*{Rlim}_{n,R,F} \big(H_*(THH(E)^{C_{p^n}};\mathbb{F}_p)\big)$$

rarely is an equivalence.

But limits interact well with homotopy, even with finite coefficients, i.e., with coefficients in a finite CW-spectrum V. Let $V_*(X) = \pi_*(V \wedge X)$ be the V-homotopy of X.

Examples.

(1) $V = S = V(-1)$ gives ordinary homotopy.
(2) $V = S/p = V(0)$ (the mod p Moore spectrum) gives mod p homotopy.
(3) For p odd the Smith–Toda complex $V(1)$ is the homotopy cofiber of the Adams map $v_1\colon \Sigma^{2p-2}V(0) \to V(0)$ inducing multiplication by v_1 in BP-homology and an isomorphism in topological K-theory. Then $V(1)$-homotopy may be thought of as mod p and v_1 homotopy.

So we should choose V to match E so as to make $V_*(THH(E))$ computable from $H_*(THH(E);\mathbb{F}_p)$. Presumably we can then also determine $V_*(THH(E)^{C_{p^n}})$ for all $n \geq 1$, and by forming the algebraic limit we obtain $V_*(TC(E;p))$. This is essentially $V_*(K(E)_p)$ by the cited theorem.

In turn, knowing the V-homotopy of $TC(E;p)$ suffices to detect, if not to construct, a completed version of $TC(E;p)$. If $X \to Y$ induces $V_*(X) \cong V_*(Y)$ then $X \simeq Y$ if $H_*(V)$ is infinite, and $X_p \simeq Y_p$ if $H_*(V)$ contains nontrivial p-torsion.

Example. Bökstedt and Madsen considered the case $E = H\mathbb{Z}_p$, p odd, using $V = S/p = V(0)$. Using the mod p homotopy of $THH(\mathbb{Z}_p)$ they computed the mod p homotopy of $TC(\mathbb{Z}_p;p)$, and thus of $K(\mathbb{Z}_p)$ and $K(\mathbb{Q}_p)$. Then they (essentially) produced a map

$$j_p \vee \Sigma j_p \vee \Sigma ku_p \to K(\mathbb{Q}_p)_p$$

inducing an isomorphism between the computed mod p homotopy groups, and could conclude that the map is a homotopy equivalence.

Variants of this argument go through for $p = 2$, cf. [R].

6. Finitely presented spectra

The extraction of V-homotopy $V_*(THH(E))$ from homology $H_*(THH(E);\mathbb{F}_p)$ is most plausible when $H_*(V \wedge THH(E);\mathbb{F}_p)$ has tiny projective dimension as an A_*-comodule, e.g. when it is free, i.e., when $V \wedge THH(E)$ is a wedge of suspensions of $H\mathbb{F}_p$. For E commutative and V a ring spectrum, $V \wedge THH(E)$ is a module spectrum over $V \wedge E$, so this happens when $V \wedge E$ is a wedge of suspensions of $H\mathbb{F}_p$.

A related notion was considered by Mahowald and Rezk [MR]:

Definition. A bounded below, p-complete spectrum E is finitely presented (an fp-spectrum) if $H^*(E;\mathbb{F}_p)$ is finitely presented as an A-module. Equivalently there is a nontrivial finite CW spectrum F such that $\pi_*(F \wedge E) = F_*(E)$ is finite. Then there is a unique integer n, called the fp-type of E, such that $F_*(E)$ is infinite if F has chromatic type $\leq n$ ($K(n)_*(F) \neq 0$), and $F_*(E)$ is finite if F has chromatic type $> n$ ($K(n)_*(F) = 0$).

We may also define a more refined notion:

Definition. E has pure fp-type n if furthermore $F_*(E)$ is a free finitely generated $P(v)$-module for some finite CW spectrum F of chromatic type n, with v_n-map $v\colon \Sigma^d F \to F$. (Then the mapping cone $V = C_v$ has chromatic type $(n+1)$ and $V_*(E)$ is finite.)

These definitions are well behaved by thick subcategory considerations.

When E is a finitely presented ring spectrum of fp-type n we choose a finite CW ring spectrum V (of chromatic type $n+1$) making $V_*(E)$ as simple as possible. Then $V_*(THH(E))$ can be (relatively) easily read off from $H_*(V \wedge THH(E);\mathbb{F}_p) \cong H_*(V;\mathbb{F}_p) \otimes H_*(THH(E);\mathbb{F}_p)$, which is now a $H_*(V \wedge E;\mathbb{F}_p)$-module. Then proceed as before to determine $V_*(THH(E)^{C_{p^n}})$ and pass to the limit to obtain $V_*(TC(E;p))$.

Examples.

(1) For $E = H\mathbb{F}_p$ of fp-type -1 use $V = S$. Hesselholt and Madsen [HM1] computed $TC(\mathbb{F}_p;p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1} H\mathbb{Z}_p$ recovering Quillen's result $K(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$. The answer has pure fp-type 0, i.e., has no p-torsion.

(2) For $E = H\mathbb{Z}_p$ of fp-type 0 use $V = S/p = V(0)$, at least for p odd. Bökstedt and Madsen [BM1], [BM2] computed the mod p homotopy of $TC(\mathbb{Z}_p;p)$ and deduce $K(\mathbb{Z}_p)_p \simeq j_p \vee \Sigma j_p \vee \Sigma^3 ku_p$. Then answer has pure fp-type 1, i.e., its mod p homotopy has no v_1-torsion. Similar results hold for $p = 2$ by [R].

(3) For $E = \ell_p = BP\langle 1\rangle_p$ of fp-type 1 use $V = V(1)$ for $p \geq 5$. Ausoni and Rognes [AR] computed the mod p and v_1 homotopy of $TC(\ell_p;p)$, and similarly for $K(\ell_p)_p$. The result has pure fp-type 2, i.e., its $V(1)$-homotopy is a free finitely generated $P(v_2)$-module on $4p+4$ generators.

(4) Other fp-spectra of fp-type 1 include ku_p, ko_p and j_p.

(5) The connective topological modular forms spectrum eo_2 with $H^*(eo_2;\mathbb{F}_2) = A//A(2)$ has fp-type 2.

(6) The spectrum $E = BP\langle n\rangle_p$ has fp-type n, but is not known to be a commutative S-algebra for $n \geq 2$. The nth Smith–Toda complex $V(n)$ with $BP_*(V(n)) = BP_*/(p,\dots,v_{n-1})$ makes $V(n) \wedge BP\langle n\rangle_p \simeq H\mathbb{F}_p$, but is not known to exist for $n \geq 4$. (But other chromatic type $(n+1)$ ring spectra certainly exist.)

7. Algebraic K-theory of topological K-theory

Theorem (Ausoni–Rognes). *For $p \geq 5$ let $\ell_p = BP\langle 1\rangle_p$ be the Adams summand* *of connective p-complete topological K-theory and let $V(1)$ be the Smith–Toda complex. Let $v_2 = [\tau_2] \in \pi_{2p^2-2}V(1)$. Then*

$$\begin{aligned} V(1)_*(TC(\ell_p;p)) \cong\ & E(\lambda_1,\lambda_2,\partial)\otimes P(v_2) \\ & \oplus E(\lambda_2)\{t^e\lambda_1 \mid 0<e<p\}\otimes P(v_2) \\ & \oplus E(\lambda_1)\{t^{ep}\lambda_2 \mid 0<e<p\}\otimes P(v_2) \end{aligned}$$

is a free $P(v_2)$-module on $4p+4$ generators. Here $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ and $|t| = -2$.

There is an exact sequence

$$0 \to \Sigma^{2p-3}\mathbb{F}_p \xrightarrow{a} V(1)_*K(\ell_p) \xrightarrow{trc} V(1)_*TC(\ell_p;p) \xrightarrow{\partial} \Sigma^{-1}\mathbb{F}_p \to 0$$

determining the $V(1)$-homotopy of $K(\ell_p)$.

Corollary. *$TC(\ell_p;p)_p$ is a finitely presented spectrum of pure fp-type 2.*

In this sense $TC(\ell_p;p)$ is like eo_2, or $BP\langle 2\rangle_p$ if the latter exists.

Recall that $K(\mathbb{Q}_p)_p$ has mod p homotopy a free $P(v_1)$-module on $p+3$ generators, where

$$p+3 = \sum_{i=1}^{p-1}\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p} H^n(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p);\mathbb{F}_p(i))\,,$$

and $K(\mathbb{Q}_p)_p$ is constructed from $p+3$ copies of $BP\langle 1\rangle_p = \ell_p$ up to extensions involving Adams operations. A more precise statement can be obtained by taking the degrees of the $P(v_1)$-module generators into account.

Likewise we get that the cofiber of the transfer map $K(\mathbb{Z}_p) \to K(\ell_p)$, which most likely is $K(L_p)$, has $V(1)$-homotopy a free $P(v_2)$-module on $4p+4$ generators, where we estimate

$$4p+4 = \sum_{i=1}^{p^2-1}\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p} H^n(\mathrm{Gal}(\overline{L}_p/L_p;\mathbb{F}_{p^2}(i))\,,$$

and $K(L_p)$ is constructed from $4p+4$ copies of $BP\langle 2\rangle_p$, up to extensions involving $BP\langle 2\rangle_p$-operations. Again a more precise statement can be obtained by taking the degrees of the $P(v_2)$-module generators into account.

Moral. Algebraic K-theory of topological K-theory is a form of elliptic cohomology.

These calculations generalize to determine $V(n)_*K(BP\langle n\rangle_p)$ if $BP\langle n\rangle_p$ exists as a commutative S-algebra and $V(n)$ exists as a ring spectrum, in which case the result is of pure fp-type $n+1$. Hence we are led to the following:

Chromatic red-shift problem. *Let E be an S-algebra of pure fp-type n. Does $TC(E;p)$ have pure fp-type $n+1$?*

So far this is known to be correct for $E = Hk$ with k a finite extension of $\mathbb{F}_p$, for $E = HA$ with A the valuation ring of a finite extension of $\mathbb{Q}_p$, and for $E = \ell_p$. One might also consider $E = S$ as a limiting case, of infinite fp-type. Then $TC(S;p)$ contains $S \simeq THH(S)$ as a retract, so in this case the fp-type of the result is $\infty + 1 = \infty$.

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