LECTURES ON HIGHER TOPOS THEORY (LEEDS, JUNE 2019)

CHARLES REZK

These are notes for my lectures in a workshop on "Higher topos theory and univalent foundations", Leeds, June 2019. Higher topoi are relevant to homotopy type theory: it is believed (proved?) that all ∞-topoi serve as models for univalent type theories. In other words, homotopy type theory should function as an "internal language" for ∞-topoi.

I describe ∞-topoi mainly from the point of view of geometry and homotopy theory, leaving for others to make direct connections to type theory.

Note: not everything in these notes made it into the actual lectures. Which was probably for the best. There is also an appendix which includes material that I decided wasn't appropriate for the talks, but might be of interest.

I had fun giving these talks! I want to thank the organizers (Nicola Gambino, Nima Rasekh, Karol Szumiło) for giving me the opportunity.

Lecture 1: What is it?

An approximate answer: **∞-topos** is the higher-categorical generalization of the notion of a topological space

Topological spaces.

Topological space: (X, Open_X) consisting of a set X and a collection $\mathrm{Open}_X \subseteq \mathcal{P}X$ of "open subsets" of X, where Open_X is required to be closed under arbitrary unions and finite intersections.

In particular, Open_X is closed under unions and intersections of empty collections, so $\varnothing, X \in \mathrm{Open}_X$.

Continuous map $f\colon X \to Y$: a function f on point-sets such that inverse image preserves open sets: $f^{-1}\colon \mathrm{Open}_Y \to \mathrm{Open}_X$.

Locales. To effect a higher categorical generalization, we first try to reformulate the above definition entirely in terms of categories: we want to *identify* a topological space X with its category (poset) Open_X of open sets. This works pretty well, but not perfectly.

Locale $\mathcal{O}$: poset which

- has arbitrary coproducts and finite products, and is such that
- products distribute over arbitrary coproducts: $U \times (\coprod_i V_i) = \coprod_i (U \times V_i)$.

In a poset, coproducts are called *joins*: $\coprod = \bigvee$ while products are called *meets*: $\times = \wedge$. The distributivity condition is then written $U \wedge (\bigvee_i V_i) = \bigvee_i (U \wedge V_i)$.

Example: the poset Open_X is an example of a locale. Coproduct/join = union, finite product/finite meet = finite intersection. Distributivity is immediate from properties of sets.

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Note: locales actually have arbitrary products/meets: $\bigwedge_i U_i = \bigvee_{V \leq U_i, \forall i} V$. In Open_X, $\bigwedge U_i = \mathrm{int} \bigcap_i U_i$.

Map $f \colon \mathcal{O} \to \mathcal{O}'$ of locales: functor $f^* \colon \mathcal{O}' \to \mathcal{O}$ which preserves coproducts/joins and finite products/finite meets.

Exercise ("Points" of a locale). Let $\{y_0\} = 1$ point space, and $\mathcal{O}$ an arbitrary locale. Then there is a bijective correspondence

$$\mathrm{Map}_{\mathrm{Loc}}(\mathrm{Open}_{\{y_0\}}, \mathcal{O}) \longleftrightarrow \{\, U \in \mathcal{O} \mid U \neq \top,\ U \geq V_1 \wedge V_2 \implies U \geq V_1 \text{ or } U \geq V_2 \,\},$$

which sends $f \colon \mathrm{Open}_{\{y_0\}} \to \mathcal{O}$ to $U_f := \bigvee_{f^*U = \varnothing} U$, the largest element of $\mathcal{O}$ which pulls back to $\varnothing \subset \{y_0\}$.

If $\mathcal{O} = \mathrm{Open}_X$, then the complements $C_f := X \smallsetminus U_f$ are precisely the *closed irreducible* subsets of X (i.e., closed sets which are not finite unions of proper closed subsets). The $* \xrightarrow{x} X$ which come from continuous maps correspond to $C_x = \overline{\{x\}}$, the closures of one-point subsets.

Sober topological space X: The operation $x \mapsto \overline{\{x\}}$ defines a bijection between the points of X and the closed irreducible subsets of X.

Sober spaces include: all Hausdorff spaces, all Zariski spectra of commutative rings.

Proposition. *Let X be a topological space Then* $\mathrm{Hom}_{\mathrm{Top}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Loc}}(\mathrm{Open}_Y, \mathrm{Open}_X)$ *for every topological space Y, if and only if X is sober. (See* [MLM94, IX.3]*.)*

In fact: there is an adjoint pair

$$X \mapsto \mathrm{Open}_X \colon \mathrm{Top} \rightleftarrows \mathrm{Loc} : \mathrm{Map}_{\mathrm{Loc}}(\mathrm{Open}_{\{y_0\}}, \mathcal{O}) \leftarrow\!\shortmid \mathcal{O}$$

which induces an equivalence between (i) the full subcategory of sober spaces and (ii) the full subcategory of "spacelike locales".

Since all "nice" spaces are sober, we might take this to mean that topological spaces are "basically the same thing" as locales. In these lectures I am going to treat spaces and locales as basically the same idea. However there are major caveats:

- There are locales which do not come from spaces. In fact there are non-trivial locales which have "no points". (Example: any complete boolean algebra $\mathcal{B}$ is a locale, whose points are in bijective correspondence with atoms (=minimal non-zero elements of $\mathcal{B}$), so take any atomless complete boolean algebra, e.g., Lesbegue measurable sets on a finite interval modulo measure 0 sets.)
- Both Top and Loc have arbitrary limits and colimits, but these are not necessarily preserved by the functors, even for "nice" objects. For instance, the product space $\mathbb{Q} \times \mathbb{Q}$ is not a product locale (though $\mathbb{R} \times \mathbb{R}$ is).

Sheaves of ∞-groupoids on a space/locale. We now enlarge the poset Open_X of open sets on a space X to an "envelope" of *sheaves* on X. (Everything here can be done just as easily for a locale; I will stick to spaces for familiarity of language.)

Presheaf on X with values in $\mathcal{C}$: a functor $F \colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{C}$.

Sheaf on X with values in $\mathcal{C}$: a presheaf $F\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{C}$ such that, for every open cover $\{U_i\}_{i\in I}$ of $U = \bigcup_i U_i$, the evident map

$$F(U) \to \lim_{J\in\mathcal{P}_{\mathrm{fne}}(I)} F(U_J), \qquad U_J := \bigcap_{j\in J} U_j$$

is an equivalence, where $\mathcal{P}_{\mathrm{fne}}(I)$ is the poset of finite non-empty subsets of I.

(See appendix for a comparison of this definition to the one more commonly given.)

Example (Two-fold covers). For $U = U_1 \cup U_2$ with $I = \{1,2\}$, this says

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & F(U_2) \arrow[d] \\ F(U_1) \arrow[r] & F(U_1 \cap U_2) \end{tikzcd}$$

is a pullback whenever F is a sheaf.

Example (Empty covers). The set $U = \varnothing$ has a cover with no elements: $I = \varnothing$. The sheaf condition on F implies $F(\varnothing)$ is the terminal object of $\mathcal{C}$.

Example (Sheaf of sets). If $\mathcal{C} = \mathrm{Set}$, then "$F\colon \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf" means that for any cover $U = \bigcap_{i\in I} U_i$, elements $s \in F(U)$ correspond bijectively to tuples $(s_i \in F(U_i))_{i\in I}$ such that s_i and s_j have the same image under the maps $F(U_i) \to F(U_i \cap U_j) \leftarrow F(U_j)$. Thus, you don't have to worry about n-fold intersections of U_is for $n > 2$.

Example (Representable sheaves). For each open set U, the functor $\rho_U\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Set}$ defined by $\rho_U = \mathrm{Hom}_{\mathrm{Open}_X}(-, U)$ is a sheaf of sets.

Morphism of sheaves: a natural transformation $\alpha\colon F \to F'$ of functors $F, F'\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{C}$, where F and F' are sheaves.

So what should $\mathcal{C}$ be? For our ∞-categorical generalization of topological spaces we take it to be $\mathcal{S}$, the *∞-category* of *∞-groupoids*. We thus obtain an ∞-category of such sheaves.

∞-category of sheaves of ∞-groupoids on a topological space X:

$$\mathrm{Shv}(X) \subseteq \mathrm{PSh}(\mathrm{Open}_X) = \mathrm{Fun}(\mathrm{Open}_X^{\mathrm{op}}, \mathcal{S}),$$

the full ∞-subcategory of sheaves of ∞-groupoids.

In what follows, a "sheaf" always means a sheaf of ∞-groupoids, unless otherwise specified.

At this point it is necessary to address some questions: *What is an ∞-groupoid? What is the ∞-category of ∞-groupoids? What is a functor between ∞-categories? What is the ∞-category of functors between ∞-categories.*

I will not try to answer these explicitly (I will say more in lecture 2). For now, I will rely on your intuition about homotopy type theory and "classical" category theory:

- ∞-groupoids are "homotopy types of spaces"; they behave like types in homotopy type theory.
- ∞-groupoids "generalize sets". In particular, any set can be regarded as a special type of ∞-groupoid. I write $\mathrm{Set} \subset \mathcal{S}$ for the full subcategory of sets.
- ∞-categories are like categories, except that $\mathrm{Hom}(X, Y)$ is not a set but an ∞-groupoid. A category is thus a special kind of ∞-category. Note: I'll usually write $\mathrm{Map}(X, Y)$ instead of $\mathrm{Hom}(X, Y)$ for the ∞-groupoid of morphisms from X to Y, to emphasize this.

- ∞-categories admit definitions and constructions analogous to those for categories; e.g., limits and colimits, functors and natural transformations, adjoint functors, etc.

If we only want a 1-categorical generalization of topological space, we take the ordinary category of sheaves of sets:

$$\mathrm{Shv}_{\mathrm{Set}}(X) \subseteq \mathrm{Fun}(\mathrm{Open}_X^{\mathrm{op}}, \mathrm{Set}).$$

We have a full embedding

$$\mathrm{Shv}_{\mathrm{Set}}(X) \subseteq \mathrm{Shv}(X),$$

since sets are just a kind of ∞-groupoid, and a limit of a functor to Set is the same as its limit as a functor to $\mathcal{S}$.

Exercise. Let Prop $\subset$ Set be the full subcategory spanned by the sets $\varnothing$ and $*$. Then sheaves $F\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Prop} \subset \mathrm{Set}$ with values in Prop correspond exactly to open sets in X. That is, "open sets are the same thing as sheaves of propositions". (Hint: given an open set $U \in \mathrm{Open}_X$, define a functor $\rho_U\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Prop}$ by $\rho_U(V) :=$ "$* \iff V \subseteq U$".)

Sheafification. It is easy to give examples of a presheaf which is not a sheaf on X.

Example (Constant presheaf). Fix an ∞-groupoid (e.g., a set) S. Then $F(U) := S$ defines the *constant presheaf* on X. This is not typically a sheaf, since $F(\varnothing) \neq *$ (unless $S = *$ or $X = \varnothing$).

If we replace F with F' by forcing $F'(\varnothing) = *$, then we get another problem: if $U \cap V = \varnothing$, then the sheaf condition requires $F'(U \cup V) \to F'(U) \times F'(V)$ to be an isomorphism, but $S \to S \times S$ isn't unless $S = *$ or $\varnothing$.

Sheafification of a presheaf F on a space X: a map of presheaves $\eta\colon F \to aF$ such that

(1) aF is a sheaf, and
(2) for every *sheaf* F' the map

$$\mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(aF, F') \xrightarrow{\sim} \mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(F, F')$$

defined by restriction along η is an equivalence of ∞-groupoids.

This is an example of a universal property: so any two sheafifications of F are equivalent.

Every presheaf can be "sheafified".

Proposition. *Every presheaf $F \in \mathrm{PSh}(\mathrm{Open}_X)$ admits a sheafification $F \to aF$. These fit together to give a pair of adjoint functors $a \vdash i$:*

$$i\colon \mathrm{Shv}(X) \underset{}{\overset{\twoheadleftarrow}{\rightarrowtail}} \mathrm{PSh}(\mathrm{Open}_X) : a\ ,$$

where i is fully faithful and a is a localization.

Example (Constant sheaf of sets). For an ∞-groupoid S, let $C_S := aF$ where $F(U) := S$ is the constant presheaf. Then C_S is called the **constant sheaf** with values in S.

When S is a *set*, you can show

$$C_S(U) := \{\text{continuous maps } U \to S^{\mathrm{disc}}\} = \{\text{locally constant functions } U \to S\},$$

where S^{disc} is given the discrete topology. It is clear that this defines a sheaf (because continuous functions glue along open sets).

Remark. What should the constant sheaf be if S is a general ∞-groupoid? Note that C_S "knows" about the connectivity of U when S is a set: for instance,

$$C_{\{0,1\}}(U) \approx \{\text{locally constant } U \to \{0,1\}\} \approx \{\text{open-and-closed subsets of } U\}.$$

In general, C_S "knows" something about the homotopy type of the topological space U. For instance, if the subset $U \subseteq X$ is nice ("paracompact"), then there is a homotopy equivalence

$C_S(U) \approx \mathrm{Map}_{\mathcal{S}}(hU, S)$, where hU here stands for "the homotopy type of the topological space U". (See appendix.)

The existence of sheafification is "formal": because the subcategory $\mathrm{Shv}(X) \subseteq \mathrm{PSh}(\mathrm{Open}_X)$ is closed under limits, a left adjoint to the inclusion can be constructed using an "adjoint functor theorem". (More about this later.) However, we also have the following, which is not at all formal.

Proposition. *The sheafification functor $a\colon \mathrm{PSh}(\mathrm{Open}_X) \to \mathrm{Shv}(X)$ is* left exact*, i.e., it preserves all finite limits; equivalently, it preserves (i) the terminal object, and (ii) pullbacks.*

As we will see, this is an extremely important fact. It is proved by writing down an "explicit" construction of sheafification. However, this is by no means easy to do. (See the appendix for more detail.)

Example (Sheaves on a 4-point space). Consider a space with point set $X = \{1, 2, a, b\}$ and with topology generated by open subsets $V_1 = \{1\}$, $V_2 = \{2\}$, $U_a = \{1, a, 2\}$, $U_b = \{1, b, 2\}$. The complete open set lattice has the form

$$\begin{tikzcd}
 & U_a \arrow[dl] & & V_1 \arrow[dl] & \\
X & & V_{12} \arrow[ul] \arrow[dl] & & \varnothing \arrow[ul] \arrow[dl] \\
 & U_b \arrow[ul] & & V_2 \arrow[ul] &
\end{tikzcd}$$

A presheaf $F\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{S}$ can be represented as a diagram

$$\begin{tikzcd}
 & F(U_a) \arrow[dr] & & F(V_1) \arrow[dr] & \\
F(X) \arrow[ur] \arrow[dr] & & F(V_{12}) \arrow[ur] \arrow[dr] & & F(\varnothing) \\
 & F(U_b) \arrow[ur] & & F(V_2) \arrow[ur] &
\end{tikzcd}$$

It is a sheaf exactly if $F(\varnothing) \approx *$ and the two squares are pullbacks.

Here is (approximately!) the recipe for sheafification of a presheaf of ∞-groupoids. For any F define a map $F \to aF$ so that:

- $F(S) \to aF(S)$ is iso when $S = U_a, U_b, V_1, V_2$.
- $aF(\varnothing) \approx *$.
- $aF(V_{12})$ is isomorphic to the product $F(V_1) \times F(V_2)$, with $F(V_{12}) \to aF(V_{12})$ the tautological map.
- $aF(X)$ is isomorphic to the pullback $F(U_a) \times_{aF(V_{12})} F(U_b)$, with $F(X) \to aF(X)$ the tautological map.

Note: if F is a presheaf of *sets*, then you know how to make sense of this construction, and you can show that it is correct.

Exercise. Let C_S be the constant sheaf on the 4-point space X with value an ∞-groupoid S. Show that $C_S(X) \approx \mathrm{Map}(S^1, S)$, the space of maps of a circle into S.

Definition of ∞-topos. Given a topological space X, we have described

(1) a small category Open_X of open sets,
(2) a *set* of "sheaf conditions" for presheaves on Open_X, which determine
(3) a full subcategory $\mathrm{Shv}(X) \subseteq \mathrm{PSh}(\mathrm{Open}_X) = \mathrm{Fun}(\mathrm{Open}_X^{\mathrm{op}}, \mathcal{S})$, whose objects satisfy the sheaf conditions, together with
(4) a left adjoint $a\colon \mathrm{Fun}(\mathrm{Open}_X^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Shv}(X)$ to the inclusion which is *left exact*.

∞-topos: an ∞-category $\mathcal{X}$ which is equivalent to some ∞-category $\mathrm{Shv}(\mathcal{C},\mathcal{T})$ of the following form. There is

(1) a small ∞-category $\mathcal{C}$, and
(2) a set $\mathcal{T} = \{t_i \colon T_i \to T_i'\}$ of morphisms in $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, so that
(3) $\mathrm{Shv}(\mathcal{C},\mathcal{T}) \subseteq \mathrm{PSh}(\mathcal{C})$ is the full subcategory spanned by objects F such that $\mathrm{Map}(T_i', F) \to \mathrm{Map}(T_i, F)$ is an equivalence for all $t_i \in \mathcal{T}$, and such that
(4) the inclusion functor $\mathrm{Shv}(\mathcal{C},\mathcal{T}) \to \mathrm{PSh}(\mathcal{C})$ admits a left adjoint $a\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C},\mathcal{T})$ which is left exact.

This generalizes the classical notion of topos=1-topos.

1-topos: a 1-category $\mathcal{X}$ defined exactly as an ∞-topos, except with $\mathcal{S}$ replaced by Set.

Example (Sheaves of ∞-groupoids on a space form an ∞-topos). Set $\mathcal{C} = \mathrm{Open}_X$. Given $U \in \mathrm{Open}_X$ we have a representable presheaf

$$\rho_U \colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{S}, \qquad \rho_U(V) = \mathrm{Map}_{\mathrm{Open}_X}(V, U),$$

which by the "Yoneda lemma" satisfies $\mathrm{Map}_{\mathrm{PSh}(\mathcal{C})}(\rho_U, F) \approx F(U)$. Given an open cover $\{U_i\}$ of U, we get

$$T_{\{U_i\}} := \mathrm{colim}_{J \in \mathcal{P}_{\mathrm{fne}}(I)^{\mathrm{op}}}\, \rho_{U_J} \to T'_{\{U_i\}} := \rho_U.$$

Set $\mathcal{T} = \{T_{\{U_i\}} \to T'_{\{U_i\}}\}$.

Note that $(\mathcal{C},\mathcal{T})$ is not part of the the data of an ∞-topos.

Example. Any presheaf ∞-category $\mathrm{PSh}(\mathcal{C})$ is an ∞-topos: take $\mathcal{T} = \varnothing$.

Example. $\mathcal{S} = \mathrm{PSh}(1)$ is thus an ∞-topos. This is also equivalent to $\mathrm{Shv}(\{x\})$ sheaves on the one point space, which is defined as a full subcategory of $\mathrm{PSh}(\mathcal{P}(\{x\}))$ presheaves on the power set of $\{x\}$. (The sheaf condition just says $F(\varnothing) \approx *$.)

The last example illustrates the phenomenon that an ∞-topos can have multiple presentations. Another example is sheaves on the 4-point space, which also turns out to be equivalent to $\mathrm{PSh}(\mathcal{C})$ for some $\mathcal{C}$. (Exercise: which $\mathcal{C}$?)

Geometric morphism. The notion of map between ∞-topoi that generalizes "continuous map of spaces" is

Geometric morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of ∞-topoi: a functor

$$f^*\colon \mathcal{X} \to \mathcal{Y}$$

which preserves all colimits and all finite limits.

It turns out that for locales (and hence for sober spaces), geometric morphisms of their ∞-topoi correspond exactly to maps of locales (and hence to continuous maps).

Example (Equivalence of ∞-topoi). Any equivalence of categories $f^*\colon \mathcal{X} \to \mathcal{Y}$ automatically provides a geometric morphism $f\colon \mathcal{Y} \to \mathcal{X}$, and its adjoint/inverse $g^*\colon \mathcal{Y} \to \mathcal{X}$ provides an inverse geometric morphism $g\colon \mathcal{X} \to \mathcal{Y}$. In particular, two ∞-topoi are equivalent as ∞-topoi iff they are equivalent as ∞-categories.

More on geometric morphisms in Lecture 5.

Lecture 2: Homotopy theory

Here are a few brief remarks on ∞-categories. (Some of this material was covered in a tutorial given by Karol Szumiło, so I did not discuss it in my lectures.)

Higher categories. What is an ∞-categeory (=$(\infty,1)$-category)? Like an ordinary category (=1-category), an ∞-category has a collection of objects, of morphisms, and then some additional stuff. The additional stuff of an ordinary category is not too complicated: a rule for composition of functions, identity maps for each objects, and some relations these satisfy. An ∞-category comes with an *infinite list* of higher structure.

Models for higher categories: quasicategories. There are a number of different models for ∞-categories. The main one in use is that of *quasicategories*, so you can assume I am using it, so that "∞-category" will be synonymous with "quasicategory". Few particular details about this or any other model will play a role in this talk, but a brief summary may be helpful to get a feeling for things.

A quasicategory is a particular kind of simplicial set, i.e., a functor

$$\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$

from the *simplicial indexing category* Δ, whose objects are the non-empty finite ordered sets $[n] := \{0 < 1 < \cdots < n\}$ and whose morphisms are weakly order preserving maps. Quasicategories are characterized by an "extension condition" which I won't write down. A quasicategory has:

- 0-dimensional elements $X \in \mathcal{C}_0$, which are *objects*,
- 1-dimensional elements $f \in \mathcal{C}_1$, which are *morphisms*,
- 2-dimensional elements $s \in \mathcal{C}_2$, which are "witnesses" for function compositions of the form $h = g \circ f$: each $s \in \mathcal{C}_2$ is an assertion that a triangle

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

  commutes. Unlike for an ordinary category, composition is not a function of suitable pairs of arrows: there can be more than one composite h of g with f, and there can be more than one witness that $h = g \circ f$.
- n-dimensional elements $t \in \mathcal{C}_n$, which are "witnesses" that certain n-dimensional tetrahedral diagrams "commute".

Ordinary 1-categories are in particular quasicategories, via the *nerve* construction: for an ordinary 1-category $\mathcal{C}$, the set $\mathcal{C}_n$ is exactly the set of all n-dimensional commuting tetrahedra in the category.

In an ∞-category, the 2-dimensional elements define a *composition relation* on morphisms (but not in general a composition function). Using this, you can define when a morphism is an *isomorphism* (also called *equivalence*).

Quasigroupoid: a quasicategory where all morphisms are isomorphisms, i.e., an ∞-*groupoid.* It is a theorem (Joyal), that quasigroupoids are exactly the *Kan complexes*, which are also used to model homotopy types of spaces. So

$$(\infty\text{-groupoids}) \quad \Longleftrightarrow \quad (\text{homotopy types}).$$

Here are a few basic ∞-categorical constructions, which will appear often.

- For any pair of objects $X, Y \in \mathcal{C}_0$, there is a quasigroupoid

$$\mathrm{Map}_{\mathcal{C}}(X, Y)$$

  whose objects are exactly the morphisms $f\colon X \to Y$. This is called the *mapping space.*

- A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a map of simplicial sets. This means that to specify a functor, you must not only specify what F does to objects and morphisms, but also where it sends higher dimensional elements.
- For any pair $\mathcal{C}, \mathcal{D}$ of quasicategories there is a *functor quasicategory*
$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$
whose objects are exactly the functors, and whose morphisms are *natural transformations*, and whose isomorphisms are *natural isomorphisms*.
- An *equivalence* of quasicategories is a functor which admits an inverse "up to natural isomorphism".
- A *terminal object* $T \in \mathcal{C}_0$ is one such that $\mathrm{Map}(C, T)$ is equivalent to the terminal (contractible) quasicategory for all $C \in \mathcal{C}_0$. *Initial objects* are defined similarly.
- Using this, we can define *limits* and *colimits* of functors $F\colon \mathcal{C} \to \mathcal{D}$.
  For instance, a *right cone* on F is a functor
$$F'\colon \mathcal{C}^{\triangleright} \to \mathcal{D},$$
where $\mathcal{C}^{\triangleright} \supset \mathcal{C}$ is a quasicategory obtained by "freely adjoining a terminal object" v to $\mathcal{C}$, and where $F'|\mathcal{C} = F$. There is a quasicategory $\mathcal{C}_{F/}$ whose objects are right cones on F. A *colimit* of F is an initial object of this quasicategory.
  Example: $\Lambda^2_0 \subset (\Lambda^2_0)^{\triangleright}$, which is the inclusion of the "walking cospan" inside the "walking commutative square". Colimits in this case are called *pushouts*, and the corresponding colimit cones are *pushout squares*.
  This definition of limit and colimit strictly generalizes the 1-categorical definition.
- There is a quasicategory Cat_∞, whose objects are quasicategories and whose morphisms are functors. This includes a full subcategory of quasigroupoids, usually denoted $\mathcal{S} \subset \mathrm{Cat}_\infty$.

Example. Consider the 4-point space X from the previous lecture. Recall that a presheaf $F\colon \mathrm{Open}_X \to \mathcal{C}$ can be represented as a diagram:

$$\begin{tikzcd}
 & F(U_a) \arrow[dr] & & F(V_1) \arrow[dr] & \\
F(X) \arrow[ur] \arrow[dr] & & F(V_{12}) \arrow[ur] \arrow[dr] & & F(\varnothing) \\
 & F(U_b) \arrow[ur] & & F(V_2) \arrow[ur] &
\end{tikzcd}$$

If $\mathcal{C}$ is an ordinary category, then the squares are simply commutative squares. But if $\mathcal{C}$ is an ∞-category, then there is an additional datum of commutativity for each square.

Even if you stick to quasicategories, there are a couple of other models for ∞-categories that one sometimes needs to deal with: *simplicially enriched categories*, and *relative categories* (the latter typically in the special case of a *model category*). (See the appendix for more.)

New ∞-topoi from old: slices. Given an ∞-category $\mathcal{C}$ and object X, the *slice* is an ∞-category $\mathcal{C}_{/X}$ whose:

- objects are morphisms $C \to X$ in $\mathcal{C}$ with target X, and
- morphisms are commutative triangles
$$\begin{tikzcd}
C \arrow[rr] \arrow[dr] & & C' \arrow[dl] \\
 & X &
\end{tikzcd}$$
- etc. (More precisely, in the quasicategory model: the n-dimensional elements of $\mathcal{C}_{/X}$ are those $(n+1)$-dimensional elements of $\mathcal{C}$ whose "final object" is equal to X.)

Remark: You can also model the slice as the fiber of the restriction functor $\mathrm{Fun}([1],\mathcal{C}) \xrightarrow{\langle 1\rangle^*} \mathrm{Fun}([0],\mathcal{C}) = \mathcal{C}$ over X. As a simplicial set this is not isomorphic to $C_{/X}$, but both are equivalent as ∞-categories.

Theorem (Slices of ∞-topoi)**.** *All slices of an ∞-topos are an ∞-topos.*

Example. Every slice of a presheaf category is equivalent to a presheaf category:

$$\mathrm{PSh}(\mathcal{C})_{/F} \approx \mathrm{PSh}(\mathcal{C}/F), \qquad (\mathcal{C}/F) := \mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/F}.$$

(This is straightforward to verify for 1-categories.)

Example. Let X be a topological space, and F a sheaf of *sets* on X. There is a classical notion of the *espace étalé* of F, which is a space X_F such that

$$\mathrm{Shv}(X_F) \approx \mathrm{Shv}(X)_{/F}.$$

It is easiest to describe its open sets:

$$\mathrm{Open}_{X_F} := \coprod_{U\in \mathrm{Open}_X} F(U), \qquad (U,s) \leq (U',s') \iff U \leq U',\ s'|U = s.$$

Or more abstractly,

$$\mathrm{Open}_{X_F} := (\mathrm{Open}_X/F) = \mathrm{Open}_X \times_{\mathrm{Shv}(X)} \mathrm{Shv}(X)_{/F}.$$

Here is an explicit recipe for exhibiting the slice of an ∞-topos as an ∞-topos.

$$\mathcal{X} = \mathrm{Shv}(\mathcal{C},\mathcal{T}), \qquad \mathcal{T} \subseteq \mathrm{Fun}([1],\mathrm{PSh}(\mathcal{C}))$$

and let $F \in \mathrm{Shv}(\mathcal{C},\mathcal{T}) \rightarrowtail \mathrm{PSh}(\mathcal{C})$. Let $\mathcal{T}/F \subseteq \mathrm{PSh}(\mathcal{C})_{/F}$ be the collection of all morphisms which correspond to commutative triangles

$$\begin{tikzcd} T \arrow[rr, "t"] \arrow[dr] & & T' \arrow[dl] \\ & F & \end{tikzcd}$$

with $t \in \mathcal{T}$. Then

$$\mathrm{Shv}(\mathcal{C})_{/F} \approx \mathrm{Shv}(\mathcal{C}/F, \mathcal{T}/F) \subseteq \mathrm{PSh}(\mathcal{C}/F)$$

and the left adjoint to the inclusion is left exact.

Truncation. We define an important invariant of a morphism in an ∞-category, its *truncation* level. I give a definition which makes sense in any ∞-category with finite limits.

Given a map $f\colon X \to Y$, write

$$\Delta_f := (\mathrm{id},\mathrm{id})\colon X \to X \times_Y X$$

for the associated diagonal map. Recusively define:

$$\Delta_f^n := \Delta_{\Delta_f^{n-1}}\colon X \to L^n(f),$$

so $\Delta_f^0 = f$, $\Delta_f^1 = \Delta_f$.

n**-truncated morphism** $f\colon X \to Y$**:** morphism such that $\Delta_f^{n+2}\colon X \to L^{n+2}(f)$ is iso.

n**-truncated object** X**:** object such that $X \to *$ is an n-truncated morphism.

Note: a map $f\colon X \to Y$ is n-truncated iff f is an n-truncated object in the slice $\mathcal{C}_{/Y}$. This is because pullbacks in $\mathcal{C}_{/Y}$ coincide with pullbacks in the underlying category $\mathcal{C}$.

We have the following basic cases.

- -2-truncated maps are exactly the isos.

- -1-truncated maps are called *monomorphisms.* They are f such that

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

  is a pullback square.

  In a 1-category, this is exactly the usual notion of monomorphism.
- 0-truncated objects are called *discrete.*

Example. In a 1-category, Δ_f is always a monomorphism. Therefore Δ_f^2 is always an isomorphism. So every morphism and thus every object is 0-truncated, i.e., discrete. Therefore they are also n-truncated for all $n \geq 0$.

The converse holds: if every object is 0-truncated, then the ∞-category is equivalent to a 1-category, by the following proposition.

Proposition. *An object $X \in \mathcal{C}$ is n-truncated iff* $\mathrm{Map}(C, X)$ *is an n-truncated ∞-groupoid for all C in $\mathcal{C}$.*

Using this proposition, we can generalize the definition of n-truncated object (and thus morphism) to $\mathcal{C}$ which don't have all finite limits.

Example (Truncated objects in ∞-groupoids)*.* When $f\colon X \to *$, write $L^n(X) := L^n(f)$. Thus $L^0(X) = X$, while

$$L^1(X) \approx X \times X, \qquad L^2(X) \approx X \times_{X\times X} X, \qquad L^3(X) \approx X \times_{X \times_{X\times X} X} X, \qquad \dots$$

In ∞-groupoids, this becomes:

$$L^{n+2}(X) \approx \mathrm{Map}_{\mathcal{S}}(S^{n+1}, X),$$

with Δ_f^n corresponding to $X = \mathrm{Map}(*, X) \to \mathrm{Map}(S^{n+1}, X)$ induced by restriction along $S^{n+1} \to *$, where S^{n+1} is the (homotopy type of) the $(n+1)$-dimensional sphere. This amounts to the observation that $\mathrm{Map}(S^{n+1}, X)$ is equivalent to the (homotopy) pullback of $X \to \mathrm{Map}(S^n, X) \leftarrow X$, which is a consequence of the fact that S^{n+1} the (homotopy) pushout of

$$* \leftarrow S^n \to *,$$

since $S^{n+1} \approx \mathrm{colim}[D^{n+1} \leftarrow S^n \to D^{n+1}]$ is the *suspension* of S^n .

When $n + 2 \geq 1$ we obtain a factorization

$$X \xrightarrow{\Delta_f^{n+2}} \mathrm{Map}(S^{n+1}, X) \xrightarrow{\epsilon_s} X$$

of the identity map, where ϵ_s is evaluation at some point $s \in S^{n+1}$. Thus the fibers of ϵ_s over $x \in X$ are $(n+1)$-fold based loop spaces $\Omega^{n+1}(X, x)$. Since ϵ_s is a weak equivalence iff all the homotopy fibers are contractible, we find for $n \geq -1$:

$$X \text{ is } n\text{-truncated if and only if all } \Omega^{n+1}(X, x) \text{ are contractible.}$$

That is:

$$X \text{ is } n\text{-truncated (for } n \geq -1\text{) iff all } \pi_k(X, x) \approx * \text{ for all } x \in X \text{ and } k > n.$$

Remark. Truncation can behave very differently in other ∞-categories. For instance, in spectra, for all n the only n-truncated object is the terminal object.

(Proof: use that $0 = 1$ and $\Sigma^n\Omega^n = \mathrm{Id}$, and note that if X is n-truncated then ΩX is $(n-1)$-truncated.)

Some general properties of n-truncation in ∞-categories:

- n-truncated maps are also $n+1$-truncated.

- (Slice invariance.) Given $f\colon X \to Y$ in $\mathcal{C}$, we can regard f a morphism of either $\mathcal{C}$ or of $\mathcal{C}_{/Y}$. It is n-truncated in $\mathcal{C}$ iff it is n-truncated in $\mathcal{C}_{/Y}$.
- (Base change.) The pullback of an n-truncated map along an arbitrary map is n-truncated.
- (Composition.) The composite of two n-truncated maps is n-truncated.
- (Left cancellation.) If g and gf are n-truncated, then f is n-truncated.

To prove the last two: reduce to ∞-groupoids, and by thinking about homotopy fibers reduce to $X \xrightarrow{f} Y \xrightarrow{g} Z = *$. Finally, if F_y is a homotopy fiber of f over $y \in Y$, note that the long exact sequence of homotopy groups lets you show that if Y is n-truncated, then X is n-truncated iff all F_y are.

Cotruncation and epimorphisms. There is a dual notion of *n-cotruncated map* (=n-truncated in the opposite category). That is, given $f\colon A \to B$, we can form the fold map $\nabla_f\colon B \amalg_A B \to B$ and its iterations $\nabla_f^n\colon S^n(f) \to B$, so that f is n-contruncated iff ∇_f^{n+2} is iso.

A (-1)-cotruncated map $f\colon A \to B$ is one such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

is a pushout. Such a map is an *epimorphism.*

Cotruncation in familiar ∞-categories is weird, even in ∞-groupoids.

Example (Epimorphisms of ∞-groupoids). In $\mathcal{S}$, every epimorphism $f\colon X \to Y$ induces a *bijection* on path components. (Proof: Mayer-Vietoris in homology shows $H_0(f)$ is iso, and $H_0(X) = \mathbb{Z}\{\pi_0 X\}$.)

Classification of epimorphisms in $\mathcal{S}$: for X path connected, the epimorphisms $X \to Y$ correspond to the *perfect normal subgroups* of $\pi_1(X)$, and are produced by the "Quillen plus construction". General epimorphisms are coproducts of these. (See [Rap17] for a proof.)

Furthermore, in $\mathcal{S}$ you get nothing new for $n > -1$: all n-contruncated maps are already epis.

Example (Poincaré). Let $G \leq SO(3)$ be the symmetry group of the icosahedron, and let $X = (SO(3)/G) \smallsetminus \{IG\}$, the coset space with one point removed. Then $X \to *$ is an epimorphism, but X is not contractible as it has a non-trivial fundamental group.

You can also construct this X up to homotopy as an explicit cell complex: see [Hat02, 2.38]. *Question:* Hatcher's cell complex can presumably be built in a univalent type theory. Can you give a purely type theoretic proof that it gives an epimorphism?

Orthogonality. This is a relation $f \perp g$ between pairs of maps in an ∞-category.

Orthogonal maps $f\colon A \to B$ and $g\colon X \to Y$: the square

$$\begin{tikzcd} \mathrm{Map}(B,X) \arrow[r] \arrow[d] & \mathrm{Map}(B,Y) \arrow[d] \\ \mathrm{Map}(A,X) \arrow[r] & \mathrm{Map}(B,Y) \end{tikzcd}$$

is a pullback.

It is the same as saying: in any commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists an "essentially unique" lift (i.e., the space of such is contractible).

Exercise. In Set, we have $p \perp i$ whenever p is surjective and i injective.

Exercise. We have $f \perp f$ iff f is iso.

Connectivity.

$(n+1)$**-connective map** f**:** when $f \perp g$ for all n-truncated maps g.

$(n+1)$**-connective object** X**:** when $X \to *$ is $(n+1)$-connective.

Note: type theorists usually say "'n-connected" for "$(n+1)$-connective". The "connective" terminology is used by Lurie, and I'm going to use it here.

In algebraic topology: $(n+1)$-connective maps are called "$n+1$-connected maps", while $(n+1)$-connective spaces are called "n-connected spaces"[1].

- All maps are (-1)-connective.
- A map $f\colon$ is 0-connective if and only if unique lifts exist in every diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, tail, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

  with g a monomorphism. In 1-categories, such f are called *strong epimorphisms.*

  For instance, any map f which is the coequalizer of a pair of arrows is 0-connective. In 1-categories, such maps are called *regular epimorphisms.*

  Warning: in ∞-categories, 0-connective maps are *not* generally *epimorphisms* (and in practice rarely are). Thus the 1-categorical terminology is misleading.
- In a 1-category, the only 1-connective maps are isos.

Some properties of connectivity, which in most cases are formal consequences of the analogous results for truncation.

- $(n+1)$-connective maps are also n-connective.
- (Slice invariance.) $f\colon A \to B$ is $(n+1)$-connective in $\mathcal{C}$ iff it is $(n+1)$-connective in $\mathcal{C}_{/B}$. (See appendix for proof.)
- (Cobase change.) The pushout of an $(n+1)$-connective map along an arbitrary map is $(n+1)$-connective.
- (Composition.) The composite of two $(n+1)$-connective maps is $(n+1)$-connective.
- (Right cancellation). If f and gf are $(n+1)$-connective, then g is $(n+1)$-connective.

Truncation factorization.

n**-truncation factorization of a map** $f\colon X \to Y$**:** a factorization $$X \xrightarrow{p} \tau_{\leq n}(f) \xrightarrow{i} Y$$ of $f = ip$ such that i is n-truncated and p is $(n+1)$-connective.

n**-truncation of an object** X**:** an n-truncation factorization of $X \to *$. That is, a map $$X \xrightarrow{p} \tau_{\leq n}X$$ such that $\tau_{\leq n}X$ is an n-truncated object and p is an $(n+1)$-connective map.

[1] Don't blame me it was like that when I got here.

Such factorizations, if they exist, are unique up to contractible choice, by a formal application of the definition of orthogonality. When truncation factorizations exist, they are functorial in a suitable sense.

Example (Truncation in ∞-groupoids). In algebraic topology, n-truncation can be constructed by a process of "killing homotopy groups". E.g., given X and $f\colon S^k \to X$ representing an element of $\pi_k(X,x)$, we can form $X' := X \cup_{f,S^k} D^{k+1}$. Then it turns out that $\pi_n(X,x) \to \pi_n(X',x)$ is an *isomorphism* for $n<k$ and a *surjection* for $n=k$. The proof of this uses the "cellular approximation theorem" (which is also how you can prove that $\pi_n S^k = 0$ if $k<n$).

This shows something stronger than the formal construction of truncation. Namely, in ∞-groupoids the n-truncation map induces *isomorphisms*

$$\pi_k(X,x) \to \pi_k(\tau_{\leq n}X,x) \qquad \text{for } k \leq n.$$

Thus, an ∞-groupoid X is $(n+1)$-connective (for $n+1 \geq 0$) if it is non-empty, and $\pi_k(X,x) \approx *$ for all $x \in X$ and $k < n+1$. (Classically, these are called "n-connected spaces.)

A similar argument applied to truncation of maps shows that $f\colon X \to Y$ is $(n+1)$-connective (for $n+1 \geq 0$) if the induced maps $\pi_k(X,x) \to \pi_k(Y,f(x))$ are isos for $k<n+1$ and surjections for $k=n+1$, or equivalently if the homotopy fibers of f are $(n+1)$-connective.

Example (Truncation in presheaves). In $\mathrm{PSh}(\mathcal{C})$, limits are computed "objectwise". So a presheaf F is n-truncated if and only if each of its values $F(C)$ are n-truncated ∞-groupoids. Since truncation is functorial, we can deduce that $(\tau_{\leq n}^{\mathrm{PSh}(\mathcal{C})}F)(C) \approx \tau_{\leq n}^{\mathcal{S}}(F(C))$. Similar remarks apply to truncation of maps between presheaves.

Example. Let $\mathcal{X} = \mathrm{Shv}(X)$. Since $\mathrm{Shv}(X) \subseteq \mathrm{PSh}(\mathrm{Open}_X) = \mathrm{Fun}(\mathrm{Open}_X^{\mathrm{op}}, \mathcal{S})$ is fully faithful, a sheaf F is n-truncated if and only if all its values $F(U)$ are n-truncated ∞-groupoids. Similarly for n-truncated maps of sheaves.

In particular, you find that

$$\mathrm{Shv}(X)_{\leq -1} \approx \mathrm{Open}_X, \qquad \mathrm{Shv}_{\leq 0} \approx \mathrm{Shv}(X,\mathrm{Set}),$$

where I write $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ for the full subcategory of n-truncated objects in $\mathcal{C}$.

In fact, all classical 1-topoi arise as $\mathcal{X}_{\leq 0}$ for some ∞-topos $\mathcal{X}$.

Example (Truncation in sheaves). Consider an ∞-topos of the form $\mathcal{X} = \mathrm{Shv}(\mathcal{C},\mathcal{T}) \subseteq \mathrm{PSh}(\mathcal{C})$. Given a sheaf F, consider

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \tau_{\leq n}^{\mathrm{PSh}}F \arrow[r] \arrow[dl, dashed] & a(\tau_{\leq n}^{\mathrm{PSh}}F) \arrow[dll, dashed] \\ G & & \end{tikzcd}$$

If G is an n-truncated sheaf, it is also an n-truncated presheaf, so the first extension exists up to contractible choice. But since G is a sheaf, the second extension also exists up to contractible choice. We have produced a *formula* for truncation in an ∞-topos with a given presentation: truncate as presheaves, then sheafify.

Since slices of ∞-topoi are also ∞-topoi, this immediately gives a truncation construction for morphisms.

For an ∞-category $\mathcal{C}$, let $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ be the full subcategory of n-truncated objects, so that we have a filtration

$$\{*\} \approx \mathcal{C}_{\leq -2} \subseteq \mathcal{C}_{\leq -1} \subseteq \mathcal{C}_{\leq 0} \subseteq \cdots \subseteq \mathcal{C}.$$

If truncation factorizations exist, then for each n we have an adjoint pair

$$\tau_{\leq n}\colon \mathcal{C} \rightleftarrows \mathcal{C}_{\leq n}.$$

Remark. If truncation factorizations exist in ∞-category, then the pair of classes of maps $(\{(n+1)\text{-connected}\}, \{n\text{-truncated}\})$ is an example of a *factorization system* (see [ABFJ17]).

Truncation and connectivity in ∞-topoi. Here are some properties of truncation and connectivity that hold in ∞-topoi, but not generally in an ∞-category.

Proposition. *In ∞-topoi, truncation factorizations are preserved under pullback. That is, if the large rectangle in*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tau_{\leq n}(f) \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & \tau_{\leq n}(f') \arrow[r] & Y' \end{tikzcd}$$

is a pullback , then both squares are pullbacks.

In particular, the classes of n-truncated and n-connective maps are preserved by pullback.

Proof sketch. Verify this is true in ∞-groupoids, using homotopy groups. Therefore it holds in presheaf ∞-categories $\mathrm{PSh}(\mathcal{C})$, since all needed constructions (pullback and truncation factorization) are "computed objectwise". Finally, show it holds in a left-exact localization $\mathrm{Shv}(\mathcal{C},\mathcal{T})$, using the fact that sheafification a is left-exact. □

This can be formalized by saying that the factorization system is a *modality*, after [ABFJ17].

Corollary. *In an ∞-topos, truncation preserves finite products.*

Proof. Compatibility with pullbacks implies that $X\times Y \to X\times\tau_{\leq n}Y$ is $(n+1)$-connective. Applying this on the other side and composing gives an $(n+1)$-connective map $X\times Y\to \tau_{\leq n}X\times\tau_{\leq nY}$ to an n-truncated object, whence this is the truncation factorization of $X\times Y\to *$. □

n-topoi. For any $-1\leq n+1<\infty$ we have a notion of an $(n+1)$-topos [Lur09, 6.4].

$(n+1)$**-topos:** an ∞-category $\mathcal{X}$ equivalent to some $\mathrm{Shv}^{\leq n}(\mathcal{C},\mathcal{T})$ with: (1) a small ∞-category $\mathcal{C}$, and (2) a set $\mathcal{T}=\{t_i\}$ of morphisms in $\mathrm{PSh}_{\leq n}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}_{\leq n}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})_{\leq n}$, so that (3) $\mathrm{Shv}_{\leq n}(\mathcal{C},\mathcal{T})\subseteq \mathrm{PSh}_{\leq n}(\mathcal{C})$ is the full subcategory spanned by objects F such that $\mathrm{Map}(t_i,F)$ is an equivalence for all $t_i\in\mathcal{T}$, and such that (4) the inclusion $\mathrm{Shv}_{\leq n}(\mathcal{C},\mathcal{T})\to\mathrm{PSh}_{\leq n}(\mathcal{C})$ admits a left adjoint a which is left exact.

That is: replace functors to ∞-groupoids with functors to n-groupoids (=n-truncated ∞-groupoids).

Proposition. *If $\mathcal{X}$ is an ∞-topos then the full subcategory $\mathcal{X}_{\leq n}$ of n-truncated objects is an n-topos.*

Proof. Assume $\mathcal{X} = \mathrm{Shv}(\mathcal{C},\mathcal{T})$, with adjunction $a\colon \mathrm{Shv}(\mathcal{C},\mathcal{T}) \rightleftarrows \mathrm{PSh}(\mathcal{C}) : i$. Since both i and a preserve finite limits, they preserve truncated objects and so restrict to adjoint functors $a\colon \mathrm{Shv}_{\leq n}(\mathcal{C},\mathcal{T}) \rightleftarrows \mathrm{PSh}_{\leq n}(\mathcal{C}) : i$. It is straightforward to verify that $\mathcal{X}_{\leq n} = \mathrm{Shv}_{\leq n}(\mathcal{C},\mathcal{T})$ is an n-topos. □

Examples:

- A 0-topos is exactly the same thing as a locale. This can be proved directly. Later I'll show that the subcategory $\mathcal{X}_{\leq -1}$ of any ∞-topos is a locale.
- A 1-topos is exactly the "classical" notion of a Grothendieck topos.

Homotopy groups. We work in an ∞-topos $\mathcal{X}$.

Write X^{S^n} for the target of $\Delta_f^{n+1}\colon X \to X^{S^n}$ (which I previously denoted $L^{n+1}(X)$). As noted above, in ∞-groupoids this really is $\mathrm{Map}(S^n, X)$.

For a general diagonal of $f\colon X \to Y$, there is a projection map π: projection to the first factor in

$$X \xrightarrow{\Delta_f} X \times_Y X \xrightarrow{\pi} X.$$

Applying this iteratively, we obtain a map $\pi\colon X^{S^n} \to X$.

Given a map $x\colon 1 \to X$ from the terminal object, define

$$\Omega^n(X,x) := \text{pullback of } \pi\colon X^{S^n} \to X \text{ along } x\colon 1 \to X$$

and then

$$\pi_n(X,x) := \tau_{\leq 0}\Omega^n(X,x) \in \mathcal{X}_{\leq 0}.$$

This is a based object of $\mathcal{X}$ (i.e., has a canonical map $1 \to \tau_{\leq 0}\Omega^n(X,x)$) and is a group if $n \geq 1$, abelian if $n \geq 2$. These are the "homotopy groups" of X at x, which are in fact discrete group objects of $\mathcal{X}$.

The above notion is not general enough, because it requires a choice of basepoint: an object X can easily fail to have any "points", i.e., map $1 \to X$. To repair this, for $X \in \mathcal{X}$ we "give it a tautological point" by considering

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\pi_1} X \qquad \text{in } (\mathcal{X}_{/X})_*.$$

Define the n**th homotopy sheaf** by

$$\pi_n X := \pi_n(X \times X \xrightarrow{\pi_1} X,\ \Delta) \in (\mathcal{X}_{/X})_{\leq 0}.$$

Proposition. *An object $X \in \mathcal{X}$ is n-connective if and only it is (-1)-connective and $\pi_k X \approx 1$ for all $k < n$.*

I won't prove this; see [Lur09, 6.5.1].

A map is ∞-*connected* if it is n-connected for all n, or equivalently if the relative homotopy groups of the map vanish. It turns out that an ∞-topos can contain maps which are ∞-connected but not iso. More on this later.

Lecture 3: Descent

Recall that an ∞-topos is defined to be an ∞-topos which is equivalent to some prescribed left-exact localization of a presheaf ∞-category. We would prefer to have a more intrinsic characterization of ∞-topoi. This can be realized through the concept of descent.

I will describe several forms of descent, all of which hold in an ∞-topos. There are actually two distinct properties here, which are typically called *universality* and *descent*. However I sometimes informally group them together under the umbrella of "descent" (since you rarely see descent without universality).

Coproducts are universal. In an ∞-topos, coproducts are universal.

Coproducts are universal: Given

- a collection $\{X_i\}_{i\in I}$ of objects, with
- a coproduct $X = \coprod_i X_i$, and
- a map $f\colon Y \to X$, we can form
- pullbacks

$$\begin{array}{ccc} Y_i & \xrightarrow{\alpha_i} & Y \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X \end{array}$$

- which assemble to give a map $\alpha = (\alpha_i)\colon \coprod_i Y_i \to Y$.

Then the map α is an equivalence.

Example (Empty coproduct)*.* If $I = \varnothing$, this says that for $X = 0$ and a map $Y \to 0$, we must have that $Y = 0$. That is, only an initial object can map to an initial object.

Example (Pairwise coproduct)*.* We have

$$\begin{array}{ccccc} Y_1 & \longrightarrow & Y & \longleftarrow & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 \amalg X_2 & \longleftarrow & X_2 \end{array} \qquad \Longrightarrow \qquad Y_1 \amalg Y_2 \xrightarrow{\sim} Y$$

if the squares are pullbacks. That is, an object over $X_1 \coprod X_2$ can be "reassembled" from the two "pieces" obtained by pulling back over X_1 or X_2.

You can phrase this a little more generally: given

- $\{X_i\}_{i\in I}$ and $X = \coprod_i X_i$, and
- maps $X \to S \leftarrow T$, and
- pullbacks

$$\begin{array}{ccccc} Y_i & \xrightarrow{\alpha_i} & Y & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ X_i & \longrightarrow & X & \longrightarrow & S \end{array}$$

then $\coprod_i Y_i \xrightarrow{\sim} Y$. The prior statement is the special case ($X = S$, $T = Y$), and is implied by it (by "patching of pullbacks").

This gives a convenient reformulation, in terms of the *pullback functor*: given $f\colon S \to T$ in $\mathcal{X}$, the pullback functor is defined so that

$$f^*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}, \qquad f^*(X \to S) := (X \times_S T \to T).$$

Coproducts are universal, reformulation: for every map $f\colon T \to S$ in $\mathcal{X}$, the pullback functor $f^*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ preserves coproducts.

Remark. Pulling back along the projection $Z \times X \to X$ gives

$$\coprod (Z \times X_i) \xrightarrow{\sim} Z \times \coprod X_i,$$

i.e., if coproducts are universal then finite product distributes over infinite coproduct.

Corollary. *For any ∞-topos $\mathcal{X}$, the subcategory $\mathcal{X}_{\leq -1}$ of (-1)-truncated objects is equivalent to a locale.*

Proof. It is clear that $\mathcal{O} = \mathcal{X}_{\leq -1}$ is equivalent to a poset. It has finite limits. It also has colimits, computed by taking colimits in $\mathcal{X}$ and then truncating. For instance, for $U_i \in \mathcal{O}$,

$$\coprod^{\mathcal{O}} U_i \approx \tau_{\leq -1}\Big(\coprod^{\mathcal{X}} U_i\Big).$$

It remains to prove the infinite distributive law $V \times (\coprod U_i) \approx \coprod (V \times U_i)$ in $\mathcal{O}$, but this is immediate from the the distributive law in $\mathcal{X}$ proved above, and the fact that truncation $\tau_{\leq -1} \colon \mathcal{X} \to \mathcal{O} \subseteq \mathcal{X}$ preserves colimits and finite products. □

The locale $\mathcal{O}$ is called the *underlying locale of* $\mathcal{X}$.

Example. As we have noted earlier, the underlying locale of $\mathrm{Shv}(X)$ is the open set lattice Open_X.

Let $X \in \mathcal{S}$. Then the underlying locale of $\mathcal{S}_{/X}$ is $\mathcal{P}(\pi_0 X)$, the power set of the set of path components.

Proof that coproducts are universal in an ∞*-topos.* In the 1-category of topological spaces, coproducts are universal. This gives the corresponding result for the ∞-category of ∞-groupoids, since "point-set" coproducts of spaces model ∞-categorical coproducts, as do "point-set" pullbacks along fibrations.

The result then passes to presheaves (because colimits and limits are computed objectwise), and thus to left exact localizations of presheaves (because a preserves colimits and finite limits).

Coproducts are disjoint. In an ∞-topos, coproducts are disjoint.

Coproducts are disjoint: For any pair X_1, X_2 of objects with coproduct $X = X_1 \amalg X_2$, the commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd}$$

is a pullback.

Again, this is true in any ∞-topos, by "ratcheting up" from spaces, where it is true on the point-set level.

Combined with universality of coproducts, this has pleasant consequences.

Proposition. *The pullback of* $X_k \to X_1 \amalg X_2 \leftarrow X_k$ *is* X_k*.*

Proof. I'll do the case $k = 1$. Form pullbacks

$$\begin{tikzcd} (X_1)_1 \arrow[r] \arrow[d] & X_1 \arrow[d] & (X_1)_2 \arrow[l] \arrow[d] \\ X_1 \arrow[r] & X_1 \amalg X_2 & X_2 \arrow[l] \end{tikzcd}$$

"Coproducts are universal" implies $(X_1)_1 \amalg (X_1)_2 \xrightarrow{\sim} X_1$, while "coproducts are disjoint" says $(X_1)_2 \approx 0$. Use that $Y \amalg 0 \xrightarrow{\sim} Y$ is universally true. □

You can extend this to pullbacks of summands of infinite coproducts: the pullback of

$$X_i \to \coprod_k X_k \leftarrow X_j$$

is either 0 (if $i \neq j$), or X_i (if $i = j$). (This is a consequence of the case already proved.)

Descent over coproducts.

Descent over coproducts: Given
- a collection $\{f_i\colon Y_i \to X_i\}$ of maps,
- coproducts $Y = \coprod Y_i$ and $X = \coprod X_i$, with
- induced map $f = \coprod f_i\colon Y \to X$,

the squares

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d, "f_i"'] & Y \arrow[d, "f"] \\ X_i \arrow[r] & X \end{tikzcd}$$

are pullbacks.

Proposition. *Universality and disjointness of coproducts implies descent over coproducts.*

Proof. For convenience of notation I'll just prove the case $I = \{1, 2\}$. Given $Y_i \to X_i$ for $i = 1, 2$, show that

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_1 \amalg Y_2 \arrow[d] & Y_2 \arrow[l] \arrow[d] \\ X_1 \arrow[r] & X_1 \amalg X_2 & X_2 \arrow[l] \end{tikzcd}$$

are pullbacks. To do this, let P_{ij} denote the pullback of the diagram $X_i \to X_1 \amalg X_2 \leftarrow Y_1 \amalg Y_2 \leftarrow Y_j$. Using universality of coproducts, $P_{i1} \amalg P_{i2}$ is seen to be equivalent to the pullback of $X_i \to X_1 \amalg X_2 \leftarrow Y_1 \amalg Y_2$.

Thus, it suffices to show that either $P_{ij} \to Y_j$ is an equivalence or $P_{ij} \approx 0$, depending on whether $i = j$ or $i \neq j$. To see this, note that the composite of $Y_j \to Y_1 \amalg Y_2 \to X_1 \amalg X_2$ can also be factored as $Y_j \to X_j \to X_1 \amalg X_2$, so we can reduce to the case when $Y_j \to X_j$ are identity maps, in which case we have already proved it. □

You can also show that "universality of coproducts + descent over coproducts" implies "disjointness of coproducts".

Slices over coproducts. We can reformulate "universality + descent" for coproducts in the following way. Given a coproduct $X = \coprod X_i$ of a family $\{X_i\}$, we have functors

$$(\alpha_i^*)\colon \mathcal{X}_{/X} \to \prod \mathcal{X}_{/X_i}, \qquad \amalg_I\colon \prod \mathcal{X}_{/X_i} \to \mathcal{X}_{/X}$$

defined respectively by pullback and coproduct.

Proposition. *The functors*

$$(\alpha_i^*)\colon \mathcal{X}_{/X} \leftrightarrows \prod \mathcal{X}_{/X_i} : \amalg_I$$

are inverse equivalences.

Proof. That $\coprod_I \circ (\alpha_i^*) \approx \mathrm{Id}$ is universality of coproducts; that $(\alpha_i^*) \circ \coprod_I \approx \mathrm{Id}$ is descent over coproducts. □

Universality and descent for pushouts. The following refer to an arbitrary commutative cube in $\mathcal{X}$ of the form

$$\begin{tikzcd}
& Y_0 \arrow[rr] \arrow[dl] \arrow[dd] & & Y_1 \arrow[dl] \arrow[dd] \\
Y_2 \arrow[rr, crossing over] \arrow[dd] & & Y & \\
& X_0 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\
X_2 \arrow[rr] & & X \arrow[from=uu, crossing over] &
\end{tikzcd}$$

In an ∞-topos, pushouts are universal.

Pushouts are universal: Given a commutative cube as above such that the bottom is a pushout and the sides are pullbacks, the top is a pushout. (That is, $X_1 \amalg_{X_0} X_2 \xrightarrow{\sim} X$ and $Y_i \xrightarrow{\sim} X_i \times_X Y$ for $i = 0, 1, 2$ imply $Y_1 \amalg_{Y_0} Y_2 \xrightarrow{\sim} Y$.)

This can be reformulated as: for all $f\colon T \to S$ in $\mathcal{X}$, the pullback functor $f^*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ preserves pushouts.

An ∞-topos has descent over pushouts.

Descent over pushouts: Given a commutative cube as above such that the bottom and top are pushouts and the back sides are pullbacks, the front sides are pullbacks. (That is, $X_1 \amalg_{X_0} X_2 \xrightarrow{\sim} X$, $Y_1 \amalg_{Y_0} Y_2 \xrightarrow{\sim} Y$, and $Y_0 \xrightarrow{\sim} X_0 \times_{X_i} Y_i$ for $i = 1, 2$ imply $Y_i \xrightarrow{\sim} X_i \times_X Y$ for $i = 1, 2$.)

Remark (In ∞-groupoids). That pushouts are universal is "classical". The idea is that whenever $X = U \cup V$ presents a topological space as a union of two open subsets, the corresponding commutative square

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r] & X
\end{tikzcd}$$

is also an ∞-categorical pushout of ∞-groupoids; furthermore, any ∞-categorical pushout is equivalent to one modelled as a union of open sets as above. Y.

Thus given any map $f\colon Y \to X$ whose target is such a union of open sets, we get a corresponding decomposition of Y as an ∞-categorical pushout of preimages: $Y \approx \operatorname{colim}(f^{-1}U \leftarrow f^{-1}(U \cap V) \to f^{-1}V)$. If f is a fibration, then the sides of the square are ∞-categorical pullbacks, and we get the result.

Descent for pushouts is more subtle. The earliest published reference I'm aware of is by Puppe [Pup74]. Another proof was given around the same time by May [May90], but not published until later.

Given universality and descent for pushouts in ∞-groupoids, the proof of these for a general ∞-topos is straightforward, as in the case for coproducts.

The properties "universality of pushouts + descent over pushouts", taken together are equivalent to the following.

Proposition. *Given a pushout square*

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & X_1 \arrow[d, "\beta_1"] \\
X_2 \arrow[r, "\beta_2"'] & X
\end{tikzcd}$$

in an ∞-topos $\mathcal{X}$, the induced diagram

$$\begin{tikzcd}
\mathcal{X}_{/X} \arrow[r, "\beta_1^*"] \arrow[d, "\beta_2^*"'] & \mathcal{X}_{/X_1} \arrow[d, "\alpha_1^*"] \\
\mathcal{X}_{/X_2} \arrow[r, "\alpha_2^*"'] & X_{/X_0}
\end{tikzcd}$$

is a pullback of ∞-categories.

General formulation of descent. In an ∞-topos, colimits are universal.

Colimits are universal: for all maps $f\colon T \to S$ in $\mathcal{X}$, the pullback functor $f^*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ preserves all small colimits.

This implies universality for coproducts and pushouts.

Let

$$\mathrm{Cart}(\mathcal{X}) \subseteq \mathrm{Fun}([1], \mathcal{X})$$

be the subcategory (not full, but containing all objects) whose morphisms $f' \to f$ are pullback squares

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

An ∞-topos admits descent for arbitrary diagrams.

Descent for arbitrary colimits: $\mathrm{Cart}(\mathcal{X})$ has all small colimits and the inclusion $\mathrm{Cart}(\mathcal{X}) \to \mathrm{Fun}([1], \mathcal{X})$ preserves small colimits.

Specializing to coproducts or pushouts gives the forms of descent described above.

Proof of general universality and descent in an ∞-topos. Straightforward from the coproduct and pushout cases, using that all colimits can be built from these. $\square$

The general forms of "universality" and "descent" taken together, imply the following, where "CAT_∞" means an ∞-category of large ∞-categories.

Proposition. *The functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ which on objects sends $U \mapsto \mathcal{X}_{/U}$ and on morphisms sends $(f\colon U \to V) \mapsto (f^*\colon \mathcal{X}_{/V} \to \mathcal{X}_{/U})$ is limit preserving.*

Remark. The descent property distinguishes ∞-topoi from n-topoi.

Locales (=0-topoi) have universal colimits, but descent only for the initial object in general (you can have $U_i \leq V_i$ for all i but $U_j \neq (\bigvee U_i) \cap V_j$).

1-topoi have universal colimits, but descent only for coproducts in general. A counterexample to descent for pushouts in Set:

$$\begin{tikzcd}
& S \times \{1,2\} \arrow[rr, "f"] \arrow[dl, "\pi"'] \arrow[dd] & & S \arrow[dl] \arrow[dd] \\
S \arrow[rr] \arrow[dd] & & T \arrow[dd] & \\
& \{1,2\} \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
* \arrow[rr] & & * &
\end{tikzcd}$$

where π is projection, $f|S \times \{1\} = \mathrm{id}$ but $f|S \times \{2\} = g \neq \mathrm{id}$, with pushout $T = S/(s \sim g(s))$. If you form the pushouts of the top and bottom squares of this diagrams of sets in ∞-groupoids instead of in sets, rather that $T \to *$ you get a *non-trivial* covering map $E \to S^1$ over the circle whose fiber is S.

Lecture 4: Applications of descent

Here is an immediate application of descent for pushouts.

Proposition. *In an ∞-topos, a cobase change of a monomorphism is a monomorphism.*

Proof. Descent for pushouts applied to

$$\begin{tikzcd}
& Y \arrow[rr] \arrow[dl, equal] \arrow[dd, equal] & & Y' \arrow[dl, equal] \arrow[dd, equal] \\
Y \arrow[rr] \arrow[dd, "f"'] & & Y' \arrow[dd, "f'"] & \\
& Y \arrow[rr] \arrow[dl, "f"'] & & Y' \arrow[dl, "f'"] \\
X \arrow[rr] & & X' &
\end{tikzcd}$$

where top and bottom sides are pushouts and back and left sides are pullbacks (so that f' is a monomorphism whenever f is). $\square$

Epi/mono factorization via the Čech nerve. Let Δ_+ be the *augmented simplicial indexing category*: objects= standard finite totally ordered sets $[n] = \{\, x \in \mathbb{Z} \mid 0 \le x \le n \,\}$ with $n \ge -1$. Removing the $[-1]$ = empty set gives $\Delta \subseteq \Delta_+$, the *simplicial indexing category.*

Čech nerve of map $f\colon A \to B$: a functor $C(f)\colon \Delta_+^{\mathrm{op}} \to \mathcal{X}$ with the form

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A \times_B A \times_B A \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A \times_B A \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A \xrightarrow{f} B$$

More precisely, it is the *right Kan-extension* of $\mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ along $\mathcal{C}^{\mathrm{op}} \to \Delta_+^{\mathrm{op}}$, where $\mathcal{C} \subset \Delta_+$ is the full subcategory spanned by $[-1]$ and $[0]$. Observe that there is a map of the constant functor on A into $C(f)$.

Proposition. *For $f\colon A \to B$ in an ∞-topos $\mathcal{X}$, the maps*

$$A = \operatorname{colim}_{\Delta^{\mathrm{op}}} A \to E = \operatorname{colim}_{\Delta^{\mathrm{op}}} C(f) \to B$$

are a 0-connective/(-1)-truncated factorization of f.

Proof. Without loss of generality assume $B = *$, by replacing $\mathcal{X}$ with the slice $\mathcal{X}_{/B}$.

To show $E \to *$ is mono, i.e., that $E \to E \times E$ is an equivalence, it suffices to show either projection $E \times E \to E$ is an equivalence. By universality of colimits $E \times E \approx \operatorname{colim}_{[n] \in \Delta^{\mathrm{op}}}(A^{n+1} \times E)$. It thus suffices to show the projection $A^{n+1} \times E \xrightarrow{\sim} A^{n+1}$ is an equivalence when $n \ge 0$, since then $\operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}}(A^{n+1} \times E) \xrightarrow{\sim} \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} A^{n+1} \approx E$. By an obvious induction on n we just need to show $A \times E \xrightarrow{\sim} A$.

A functor $F\colon \Delta_+^{\mathrm{op}} \to \mathcal{X}$ admits a *contracting homotopy* (or *splitting*) if it factors as a composite $\Delta_+^{\mathrm{op}} \xrightarrow{s} \Delta_0^{\mathrm{op}} \to \mathcal{X}$, where $\Delta_0 \subset \Delta$ is the subcategory with all objects, but only maps $f\colon [m] \to [n]$ such that $f(0) = 0$, while s sends $f\colon [m] \to [n]$ to $s(f) = f'\colon [m+1] \to [n+1]$ defined by $f'(0) = 0$, $f'(x+1) = f(x) + 1$. A standard fact is that if F admits a contracting homotopy, then $\operatorname{colim}_{\Delta^{\mathrm{op}}} F \xrightarrow{\sim} F([-1])$.

In particular, the functor $A \times C(f)\colon \Delta_+^{\mathrm{op}} \to \mathcal{X}$ sending $[n] \mapsto A \times C(f)([n]) = A \times A^{n+1}$ admits a contracting homotopy, whence $A \times E \approx \operatorname{colim}_{\Delta^{\mathrm{op}}}(A \times C(f)) \xrightarrow{\sim} A$.

To show $A \to E$ is 0-connective, we consider $T \in \mathcal{X}_{\leq -1}$ and show $\operatorname{Map}(E, T) \to \operatorname{Map}(A, T)$ is an equivalence. Either $\operatorname{Map}(A, T) \approx \varnothing$, in which case the conclusion is immediate since only the initial object maps to initial object in an ∞-topos, or $\operatorname{Map}(A, T) \approx *$. In this case:

$$\operatorname{Map}(E, T) \approx \operatorname{Map}(\operatorname{colim}_{\Delta^{\mathrm{op}}} C(f), T) \approx \lim_{[n]\in\Delta} \operatorname{Map}(A^{n+1}, T) \approx \lim_{\Delta} * = *,$$

as each $\operatorname{Map}(A^{n+1}, T)$ is non-empty and thus contractible, as witnessed by $A^{n+1} \to A \to T$. $\square$

Remark. 0-connected morphism in an ∞-topos are also called *effective epimorphisms*. This is a generalization of the 1-categorical notion, where $f\colon A \to B$ is effective epi if $A \times_B A \rightrightarrows A \to B$ is a coequalizer.

Warning: as noted earlier, effective epimorphisms in an ∞-topos are rarely epimorphisms.

Remark. Factorization $f = pi$ into a 0-connective p and a (-1)-truncated i is sometimes called *epi/mono-factorization.*

Corollary. *If $f\colon A \to B$ is an effective epi in an ∞-topos $\mathcal{X}$, then the pullback functor $f^*\colon \mathcal{X}_{/B} \to \mathcal{X}_{/A}$ is conservative.*

Proof. Consider a pullback square

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "g'"'] & B' \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where g' is iso. The hypothesis implies the evident map $C(f') \to C(f)$ restricts to an equivalence between functors $\Delta^{\mathrm{op}} \to \mathcal{X}$, and g is the map induced between colimits. $\square$

Presentable ∞-categories. We are going to characterize ∞-topoi: they are precisely the *presentable* ∞-categories with universal colimits and descent for all colimits.

Presentable ∞-category: an ∞-category equivalent to one of the form $\operatorname{Shv}(\mathcal{C}, \mathcal{T})$ where

(1) $\mathcal{C}$ is a small category,
(2) $\mathcal{T} = \{t_i\colon T_i \to T_i'\}$ is a set of morphisms in $\operatorname{PSh}(\mathcal{C}) = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, so that
(3) $\operatorname{PSh}(\mathcal{C})_{\mathcal{T}} \subseteq \operatorname{PSh}(\mathcal{C})$ is the full subcategory spanned by objects F such that $\operatorname{Map}(T_i', F) \to \operatorname{Map}(T_i, F)$ is an equivalence for all $t_i \in \mathcal{T}$.

These imply that the inclusion $\operatorname{Shv}(\mathcal{C}, \mathcal{T}) \rightarrowtail \operatorname{PSh}(\mathcal{C})$ admits a left adjoint $\ell\colon \operatorname{PSh}(\mathcal{C}) \twoheadrightarrow \operatorname{Shv}(\mathcal{C}, \mathcal{T})$. However, note that ℓ need not be left exact.

Remark. The 1-categorical analogue of this, with ∞-categories replaced by 1-categories and ∞-groupoids replaced by sets, is a *locally presentable category.*

Presentable ∞-categories have excellent properties:

- They have all small limits and colimits.
- They have the ideal left-adjoint-functor theorem: if $\mathcal{A}$ is presentable, any limit preserving functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{S}$ is representable by an object of $\mathcal{A}$. As a consequence, any colimit preserving functor $\mathcal{A} \to \mathcal{B}$ to any $\mathcal{B}$ admits a right adjoint.
- They also have a right-adjoint-functor theorem: a functor $\mathcal{A} \to \mathcal{B}$ between presentable ∞-categories admits a left adjoint if it is limit preserving and *accessible* (i.e., preserves λ-filtered colimits for some regular cardinal λ.).

Functors from presentable ∞-categories. A presentable ∞-category $\mathrm{Shv}(\mathcal{C},\mathcal{T})$ is, in some sense, "freely generated under colimits by $\mathcal{C}$, subject to relations from $\mathcal{T}$".

Presheaf ∞-categories are *free colimit completions*. If $\mathcal{A}$ is cocomplete, we have an equivalence

$$\begin{tikzcd}
\mathrm{Fun}^{\text{colim pres}}(\mathrm{PSh}(\mathcal{C}),\mathcal{A}) \arrow[r, tail] \arrow[dr, "\sim"'] & \mathrm{Fun}(\mathrm{PSh}(\mathcal{C}),\mathcal{A}) \arrow[d, "-\circ\rho"] \\
& \mathrm{Fun}(\mathcal{C},\mathcal{A})
\end{tikzcd}$$

in particular, functors $\mathcal{C}\to\mathcal{A}$ extend essentially uniquely to colimit preserving functors $\mathrm{PSh}(\mathcal{C})\to\mathcal{A}$. The vertical functor is restriction along the Yoneda embedding $\rho\colon\mathcal{C}\to\mathrm{PSh}(\mathcal{C})$. The inverse of the equivalence sends $F\colon\mathcal{C}\to\mathcal{A}$ to its left-Kan-extension $\widehat{F}:=\mathrm{LKan}_\rho F$ along Yoneda.

For a presentable ∞-category $\mathrm{Shv}(\mathcal{C},\mathcal{T})$ this extends to

$$\begin{tikzcd}
\mathrm{Fun}^{\text{colim pres}}(\mathrm{Shv}(\mathcal{C},\mathcal{T}),\mathcal{A}) \arrow[r, tail, "\ell^*"] & \mathrm{Fun}^{\text{colim pres}}(\mathrm{PSh}(\mathcal{C}),\mathcal{A}) \arrow[r, tail] \arrow[dr, "\sim"'] & \mathrm{Fun}(\mathrm{PSh}(\mathcal{C}),\mathcal{A}) \arrow[d, "-\circ\rho"] \\
& & \mathrm{Fun}(\mathcal{C},\mathcal{A})
\end{tikzcd}$$

where $\ell\colon\ \mathrm{PSh}(\mathcal{C})\twoheadrightarrow\colon\ \mathrm{Shv}(\mathcal{C},\mathcal{T})$ is left-adjoint to inclusion, and is the universal functor which is colimit preserving and which inverts elements of $\mathcal{T}$. Thus

$$\{\text{colimit preserving } \mathrm{Shv}(\mathcal{C},\mathcal{T})\to\mathcal{A}\}\iff\{F\colon\mathcal{C}\to\mathcal{A}\text{ st. }\widehat{F}(t)\text{ is iso for }t\in\mathcal{T}\}.$$

Example. Taking opposites and setting $\mathcal{A}=\mathcal{S}$ gives

$$\begin{tikzcd}
\mathrm{Fun}^{\text{lim pres}}(\mathrm{Shv}(\mathcal{C},\mathcal{T})^{\mathrm{op}},\mathcal{S}) \arrow[r, tail, "\ell^*"] \arrow[dr, "\sim"'] & \mathrm{Fun}^{\text{lim pres}}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}},\mathcal{S}) \arrow[r, tail] \arrow[drr, "\sim"'] & \mathrm{Fun}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}},\mathcal{S}) \arrow[r, equal] & \mathrm{Fun}(\mathrm{PSh}(\mathcal{C}^{\mathrm{op}}),\mathcal{S}) \arrow[d, "-\circ\rho"] \\
& \mathrm{Shv}(\mathcal{C},\mathcal{T}) \arrow[rr, tail] & & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})
\end{tikzcd}$$

which exhibits the observation that limit preserving $F\colon\ \mathrm{Shv}(\mathcal{C},\mathcal{T})^{\mathrm{op}}\to\mathcal{S}$ are precisely the representable functors on $\mathrm{Shv}(\mathcal{C},\mathcal{T})$. In fact, such an F is represented by $F\circ\rho\in\mathrm{Shv}(\mathcal{C},\mathcal{T})$.

Example (Internal function objects in an ∞-topos). Let $A,B\in\mathcal{X}$. The functor $F\colon\mathcal{X}^{\mathrm{op}}\to\mathcal{S}$ by

$$F(T):=\mathrm{Map}_{\mathcal{X}}(T\times A,\ B)$$

is limit preserving, since colimits are universal in an ∞-topos so $T\times(-)$ is colimit preserving. Therefore F is representable by some $\underline{\mathrm{Map}}(A,B)\in\mathcal{X}$:

$$F(T)=\mathrm{Map}_{\mathcal{X}}(T\times A,\ B)\approx\mathrm{Map}_{\mathcal{X}}(T,\ \underline{\mathrm{Map}}(A,B)).$$

Example (Equivalence objects in an ∞-topos). Note that

$$F(T)=\mathrm{Map}_{\mathcal{X}}(A,\ B)\approx\mathrm{Map}_{\mathcal{X}_{/T}}(T\times A\to T,\ T\times B\to T).$$

Let

$$F'(T):=\mathrm{Eq}_{\mathcal{X}_{/T}}(T\times A\to T,\ T\times B\to T)\rightarrowtail\mathrm{Map}_{\mathcal{X}_{/T}}(T\times A\to T,\ T\times B\to T).$$

the full subgroupoid of the mapping space spanned by equivalences. The subfunctor $F'\colon\mathcal{X}^{\mathrm{op}}\to\mathcal{S}$ of F is also limit preserving, and so is representable by an object $\underline{\mathrm{Eq}}(A,B)\in\mathcal{X}$.

Characterization of ∞-topoi.

Theorem. *$\mathcal{X}$ is an ∞-topos if and only if it is presentable, and has universal colimits and descent for colimits.*

Sketch proof.

(i) Any presentable $\mathcal{A}$ admits a presentation $\mathrm{Shv}(\mathcal{C},\mathcal{T})$ where

(a) the Yoneda functor $\rho\colon\mathcal{C}\to\mathrm{PSh}(\mathcal{C})$ factors through $\mathrm{Shv}(\mathcal{C},\mathcal{T})\subseteq\mathrm{PSh}(\mathcal{C})$ and

(b) $\mathcal{C}$ has finite limits.

(The recipe is to find a large regular cardinal κ so that $\mathcal{C}$ can be taken to be a skeleton of $\mathcal{A}^{\kappa\text{-cpt}} \subseteq \mathcal{A}$, the full subcategory of objects X such that $\mathrm{Map}_{\mathcal{A}}(X, -) \colon \mathcal{A} \to \mathcal{S}$ preserves κ-filtered colimits. While doing so we can also ensure that $\mathcal{C}$ is closed under finite limits.)

(ii) For an ∞-topos, any presentation as in (i) is such that the left adjoint $\ell \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{X}$ to inclusion is left exact. To see this, note that we have a commutative diagram (up to equivalence) of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\widehat{\rho}"'] \arrow[rr, bend left, "\rho"] & \mathrm{Shv}(\mathcal{C}, \mathcal{T}) \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \mathrm{PSh}(\mathcal{C}) \arrow[r, two heads, "\ell"] & \mathcal{X} \end{tikzcd}$$

Since $\mathcal{C}$ has finite limits, the Yoneda embedding ρ preserves these finite limits and thus so does $\widehat{\rho}$, which by the diagram is equivalent to $\ell\rho$. That ℓ is left exact follows from the following proposition applied to $F = \ell$, using that $\mathcal{C}$ has finite limits (which are necessarily preserved by ρ), and that $\mathrm{colim}_{\mathcal{C}}\, \rho$ is the terminal object of $\mathrm{PSh}(\mathcal{C})$. □

Proposition. *Let $\mathcal{X}$ be an ∞-category which has small colimits and finite limits, and which has universal colimits and descent for colimits. Then a colimit preserving functor $F \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{X}$ is left exact if and only if*

(1) *$F(*) \approx *$, and*

(2) *F preserves all pullback squares of the form*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \rho_{C_2} \arrow[d] \\ \rho_{C_1} \arrow[r] & \rho_{C_0} \end{tikzcd}$$

in $\mathrm{PSh}(\mathcal{C})$, where $\rho_C = \mathrm{Map}_{\mathcal{C}}(-, C)$.

Proof. (This is basically [Lur09, 6.1.5.2].) It suffices to show that (2) alone implies that F preserves all pullbacks.

First I'll prove an easier variant: F preserves *pairwise products* whenever $F(\rho_C \times \rho_{C'}) \xrightarrow{\sim} F(\rho_C) \times F(\rho_{C'})$.

For any $A, A' \in \mathrm{PSh}(\mathcal{C})$ we can write $A = \mathrm{colim}_{i \in I}\, \rho_{C_i}$ and $A' = \mathrm{colim}_{j \in J}\, \rho_{C'_j}$, for some functors $C \colon I \to \mathcal{C}$ and $C' \colon J \colon \mathcal{C}$ from small categories I, J. Universality of colimits in $\mathrm{PSh}(\mathcal{C})$ implies

$$A \times A' \approx (\mathrm{colim}_{i\in I}\, \rho_{C_i}) \times (\mathrm{colim}_{j \in J}\, \rho_{C'_j}) \approx \mathrm{colim}_{(i,j) \in I \times J}\, \rho_{C_i} \times \rho_{C'_j},$$

so the hypotheses on F together with universality of colimits in $\mathcal{X}$ implies

$$\mathrm{colim}_{(i,j)\in I \times J} F(\rho_{C_i} \times \rho_{C'_j}) \xrightarrow{\sim} \mathrm{colim}_{(i,j) \in I \times J} F(\rho_{C_i}) \times F(C'_j) \approx (\mathrm{colim}_{i \in I} F(\rho_{C_i})) \times (\mathrm{colim}_{j \in J} F(\rho_{C'_j})),$$

so $F(A \times A') \xrightarrow{\sim} F(A) \times F(A')$, as desired.

Now consider the evident functor

$$F_C \colon \mathrm{PSh}(\mathcal{C}_{/C}) = \mathrm{PSh}(\mathcal{C})_{/\rho_C} \to \mathcal{X}_{F(\rho_C)}$$

defined by "restricting" F. Hypothesis (2) gives exactly what we need to apply the "pairwise product" variant that we just proved to F_C, and thus we obtain a *special case*: F preserves all pullback squares in $\mathcal{X}$ of the form

$$\begin{tikzcd} P_{00} \arrow[r] \arrow[d] & P_{01} \arrow[d] \\ P_{10} \arrow[r] & P_{11} \end{tikzcd}$$

where $P_{11} \approx \rho_C$ for some object C of $\mathcal{C}$.

For a general pullback square $[1] \times [1] \to \mathcal{X} \colon k\ell \mapsto P_{k\ell}$ write $P_{11} \approx \operatorname{colim}_{\alpha \in \mathcal{A}} \rho_{C_\alpha}$, a colimit of representables over some small ∞-category $\mathcal{A}$. Write $P^\alpha_{k\ell} := P_{k\ell} \times_{P_0} \rho_{C_\alpha}$, so that $P^\alpha_{11} = \rho_{C_\alpha}$. By the special case already shown, F takes each of the pullback squares

$$\begin{tikzcd} P^\alpha_{00} \arrow[r] \arrow[d] & P^\alpha_{01} \arrow[d] \\ P^\alpha_{10} \arrow[r] & P^\alpha_{11} \end{tikzcd} \qquad \begin{tikzcd} P^\alpha_{10} \arrow[r] \arrow[d] & P^\alpha_{11} \arrow[d] & P^\alpha_{01} \arrow[l] \arrow[d] \\ P^{\alpha'}_{10} \arrow[r] & P^{\alpha'}_{11} & P^{\alpha'}_{01} \arrow[l] \end{tikzcd}$$

in $\mathrm{PSh}(\mathcal{C})$ to pullback squares in $\mathcal{X}$, where the right two squares are induced by a map $\alpha \to \alpha'$ in $\mathcal{A}$.

Universality of colimits in $\mathrm{PSh}(\mathcal{C})$ implies $\operatorname{colim}_\alpha P^\alpha_{k\ell} \xrightarrow{\sim} P_{k\ell}$, so $\operatorname{colim}_\alpha F(P^\alpha_{k\ell}) \xrightarrow{\sim} F(P_{k\ell})$, since F is colimit preserving. Now the following lemma applied to $G \colon \alpha \mapsto ((k,\ell) \mapsto F(P^\alpha_{k\ell}))$ implies that F preserves the general pullback square. $\square$

Lemma. *Suppose $\mathcal{X}$ has small colimits, finite limits, universal colimits, and descent for colimits. Let*

$$G \colon \mathcal{A} \to \operatorname{Fun}([1] \times [1], \mathcal{X})$$

be a functor from a small ∞-category such that (i) $G(\alpha)$ is a pullback square for each object α of $\mathcal{A}$, and (ii) for each morphism $\alpha \to \alpha'$ of $\mathcal{A}$, the squares

$$\begin{tikzcd} G(\alpha)(1,0) \arrow[r] \arrow[d] & G(\alpha)(1,1) \arrow[d] & G(\alpha)(0,1) \arrow[l] \arrow[d] \\ G(\alpha')(1,0) \arrow[r] & G(\alpha')(1,1) & G(\alpha')(0,1) \arrow[l] \end{tikzcd}$$

are pullback squares in $\mathcal{X}$. Then $\operatorname{colim}_{\mathcal{A}} G$ is a pullback square in $\operatorname{Fun}([1] \times [1], \mathcal{X})$.

Variant characterization: the Giraud theorem. I use notation $\langle k_0, \ldots, k_m \rangle \colon [m] \to [n]$ for the morphism in the simplicial indexing category Δ sending $j \mapsto k_j$.

Groupoid object in $\mathcal{A}$: a functor $G \colon \Delta^{\mathrm{op}} \to \mathcal{A}$ such that

(1) for all $n \geq 1$,
$$(\langle n-1, n \rangle, \ldots, \langle 0, 1 \rangle) \colon G[n] \xrightarrow{\sim} G[1] \times_{G[0]} \cdots \times_{G[0]} G[1],$$

(2)
$$(\langle 01 \rangle, \langle 02 \rangle) \colon G[2] \xrightarrow{\sim} G(1) \times_{G[0]} G[1].$$

This is also called a **Segal groupoid** in $\mathcal{A}$.

Effective groupoid object in $\mathcal{A}$: a groupoid object $G \colon \Delta^{\mathrm{op}} \to \mathcal{A}$ such that the colimit cone

$$\cdots \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} G[2] \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} G[1] \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} G[0] \xrightarrow{\pi} E$$

of G is equivalent to the Čech nerve of π. Equivalently, E is the colimit of $G|\Delta^{\mathrm{op}}$ and

$$\begin{tikzcd} G[1] \arrow[r, "\langle 1 \rangle"] \arrow[d, "\langle 0 \rangle"'] & G[0] \arrow[d, "\pi"] \\ G[0] \arrow[r, "\pi"'] & E \end{tikzcd}$$

is a pullback.

Example. A groupoid object G such that $(\langle 0 \rangle, \langle 1 \rangle) \colon G[1] \to G[0] \times G[0]$ is a monomorphism is an *equivalence relation* on $G[0]$.

Theorem (Töen-Vezzosi, Lurie). *$\mathcal{X}$ is an ∞-topos if and only if*

(1) *it is presentable,*
(2) *colimits are universal,*
(3) *coproducts are disjoint, and*
(4) *all groupoid objects in $\mathcal{X}$ are effective.*

I won't prove this. See [Lur09, 6.1.5].

Remark (Giraud theorem). This is an ∞-categorical analogue of the theorem of Giraud: a 1-category $\mathcal{X}$ is a 1-topos iff it is locally presentable, colimits are universal, coproducts are disjoint, and all *equivalence relations* are effective.

Universal families and local classes. Recall $\mathrm{Cart}(\mathcal{X}) \subseteq \mathrm{Fun}([1], \mathcal{X})$, the subcategory of the arrow category of $\mathcal{X}$ whose objects f are morphisms in $\mathcal{X}$, and whose morphisms $f \Rightarrow g$ are pullback squares in $\mathcal{X}$, with composition given by patching of pullback squares. When $\mathcal{X}$ is an ∞-topos, $\mathrm{Cart}(\mathcal{X})$ has small colimits and pullbacks, but not in general a terminal object. It turns out however that $\mathrm{Cart}(\mathcal{X})$ has "arbitrarily close approximations" to a terminal object, which are called "universal families".

Universal family in $\mathcal{X}$: map $p\colon E \to U$ which represents a (-1)-truncated object of $\mathrm{Cart}(\mathcal{X})$. Equivalently, $\mathrm{Map}_{\mathrm{Cart}(\mathcal{X})}(f, p)$ is a (-1)-truncated ∞-groupoid (=proposition) for all morphisms f in $\mathcal{X}$.

A commutative square in $\mathcal{X}$ whose bottom side is the identity map of some object B is basically the same thing as a morphism in the slice category $\mathcal{X}_{/B}$, and the square is a pullback iff the morphism is an isomorphism. Given such an object B and a choice of map $p\colon E \to U$, we obtain pullback squares of ∞-categories:

$$\begin{tikzcd}
\mathrm{Map}(B,U) \arrow[r, "\kappa_B"] \arrow[d] & (\mathcal{X}_{/B})^{\mathrm{core}} \arrow[r] \arrow[d] & \mathcal{X}_{/B} \arrow[r] \arrow[d] & \{B\} \arrow[d] \\
\mathrm{Cart}(\mathcal{X})_{/p} \arrow[r, "\text{forget}"'] & \mathrm{Cart}(\mathcal{X}) \arrow[r, hook] & \mathrm{Fun}([1],\mathcal{X}) \arrow[r, "\text{target}"'] & \mathcal{X}
\end{tikzcd}$$

where "$\mathcal{C}^{\mathrm{core}}$" denotes the maximal subgroupoid of $\mathcal{C}$.

It's not hard to check that p is a universal family iff $\mathrm{Cart}(\mathcal{X})_{/p} \to \mathrm{Cart}(\mathcal{X})$ is fully faithful, and that this is the case iff κ_B is fully faithful for every object $B \in \mathcal{X}$.

For a universal family p let $\mathcal{L}_p$ be the essential image of $\mathrm{Cart}(\mathcal{X})_{/p} \rightarrowtail \mathrm{Cart}(\mathcal{X})$. Then $\mathcal{L}_p$ is an example of a *local class*:

Local class in $\mathcal{X}$: a full subcategory $\mathcal{L} \subseteq \mathrm{Cart}(\mathcal{X})$ such that

(1) $f \Rightarrow f'$, $f' \in \mathcal{L}$, implies $f \in \mathcal{L}$, and
(2) $\mathcal{L}$ is closed under colimits in $\mathrm{Cart}(\mathcal{X})$.

In fact, for a universal family p the class $\mathcal{L}_p$ is a *bounded local class.*

Bounded local class in $\mathcal{X}$: a local class $\mathcal{L}$ such that for every $B \in \mathcal{X}$ the ∞-groupoid $(\mathcal{X}_{/B})^{\mathrm{core}}_{\mathcal{L}}$, defined as a pullback in

$$\begin{tikzcd}
(\mathcal{X}_{/B})^{\mathrm{core}}_{\mathcal{L}} \arrow[r, tail] \arrow[d] & (\mathcal{X}_{/B})^{\mathrm{core}} \arrow[r] \arrow[d] & \{B\} \arrow[d] \\
\mathcal{L} \arrow[r, tail] & \mathrm{Cart}(\mathcal{X}) \arrow[r, "\text{target}"'] & \mathcal{X}
\end{tikzcd}$$

(i.e., the full subgroupoid of $(\mathcal{X}_{/B})^{\mathrm{core}}$ spanned by elements of $\mathcal{L}$) is *essentially small*, i.e., equivalent to a small ∞-category.

This is because if $\mathcal{L} = \mathrm{Cart}(\mathcal{X})_{/p}$ then $(\mathcal{X}_{/B})^{\mathrm{core}}_{\mathcal{L}} \approx \mathrm{Map}(B, U)$, which is certainly small.

Proposition (Gepner-Kock)**.** *In an ∞-topos, every bounded local class is that of a universal family.*

Proof. Given a bounded local class $\mathcal{L} \subseteq \mathrm{Cart}(\mathcal{X})$, let $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ be the functor "defined by"

$$F(B) := (\mathcal{X}_{/B})^{\mathrm{core}}_{\mathcal{L}} = (\mathcal{X}_{/B})^{\mathrm{core}} \times_{\mathrm{Cart}(\mathcal{X})} \mathcal{L}.$$

This takes values in small ∞-groupoids exactly because $\mathcal{L}$ is bounded. Descent says that $\mathcal{X}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ defined by $B \mapsto \mathcal{X}_{/B}$ is limit preserving, from which we can derive that F is limit preserving, and therefore is representable by some object $U \in \mathcal{X}$, i.e., $F \approx \mathrm{Map}(-, U)$. The object of $(\mathcal{X}_{/U})^{\mathrm{core}}_{\mathcal{L}}$ corresponding to the identity map of U is the desired universal family $p \colon E \to U$. $\square$

Proposition. *In an ∞-topos there are "enough" universal families. That is, for every map $f \colon A \to B$ in $\mathcal{X}$, there exists a universal family p and a map $f \Rightarrow p$ in* $\mathrm{Cart}(\mathcal{X})$*.*

Proof sketch. A map $f \colon A \to B$ is *relatively κ-compact* for some regular cardinal κ if for every base-change $f' \colon A' \to B'$ of f over map $B' \to B$, if B' is κ-compact then A' is κ-compact. One can show that the class $\mathcal{L}^\kappa \subseteq \mathrm{Cart}(\mathcal{X})$ spanned by κ-compact maps is a bounded local class, and that $\bigcup \mathcal{L}^\kappa = \mathrm{Cart}(\mathcal{X})$. $\square$

Given any set $\{f_i\}$ of maps, apply this to $f = \coprod f_i$ to get a universal family p such that there exist $f_i \Rightarrow p$ in $\mathrm{Cart}(\mathcal{X})$.

Object classifiers. If we are willing to put aside issues of size (e.g., by passing to a larger universe), then there is a (large) *object classifier*. That is, the functor $B \mapsto (\mathcal{X}_{/B})^{\mathrm{core}}$ from $\mathcal{X}^{\mathrm{op}}$ to large ∞-groupoids is can be thought of as represented by a "large" object $E^{\mathrm{univ}} \to U^{\mathrm{univ}}$ of "Cart(large $\mathcal{X}$).

Univalence in ∞-topoi. Recall that $p \colon E \to U$ is a universal family iff $\mathrm{Cart}(\mathcal{X})_{/p} \to \mathrm{Cart}(\mathcal{X})$ is fully faithful, iff the base-change

$$\kappa_B \colon \mathrm{Map}(B, U) \to (\mathcal{X}_{/B})^{\mathrm{core}}$$

is fully faithful for every object $B \in \mathcal{X}$.

The map κ_B is a functor of ∞-groupoids, in which case "fully faithful" is the same as "monomorphism". To investigate whether κ_B is a monomorphism we can consider

$$\begin{tikzcd}
\mathrm{Map}(B,U) \arrow[r, "\kappa_B"] \arrow[d, "\Delta"'] & (\mathcal{X}_{/B})^{\mathrm{core}} \arrow[d, "\Delta'"] \\
\mathrm{Map}(B,U) \times \mathrm{Map}(B,U) \arrow[r, "\kappa_B \times \kappa_B"'] & (\mathcal{X}_{/B})^{\mathrm{core}} \times (\mathcal{X}_{/B})^{\mathrm{core}}
\end{tikzcd}$$

So κ_B is a monomorphism, iff for every pair $f, g \in \mathrm{Map}(B, U)$, the induced map on fibers of Δ and Δ' is an equivalence.

Given $a \colon A \to B$, $a' \colon A' \to B$, the fiber of $\mathcal{X}_{/B} \to \mathcal{X}_{/B} \times \mathcal{X}_{/B}$ over (a, a') is exactly $\mathrm{Map}_{\mathcal{X}_{/B}}(a, a')$, so the fiber of Δ' over this point is $\mathrm{Eq}_{\mathcal{X}_{/B}}(a, a')$.

So: $p \colon E \to U$ is a universal family in $\mathcal{X}$ iff for every $B \in \mathcal{X}$ and every $f, g \colon B \to U$, the induced map

$$\mathrm{Path}(\mathrm{Map}(B, U),\ f,\ g) \to \mathrm{Eq}_{\mathcal{X}_{/B}}(f^*(p),\ g^*(p))$$

is an equivalence. Both source and target are natural in the data $(B,\ f, g \colon B \to U)$, i.e., they define functors

$$(\mathcal{X}_{/U \times U})^{\mathrm{op}} \to \mathcal{S}.$$

These functors are representable by objects of $\mathcal{X}_{/U \times U}$. Unwinding this gives the following.

Proposition (Gepner-Kock)**.** *$p\colon E \to U$ is a universal family if and only if it is* univalent*, i.e., if and only if the induced horizontal map*

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "\Delta"'] & & \underline{\mathrm{Eq}}(\pi_1^*(p),\, \pi_2^*(p)) \arrow[dl] \\ & U \times U & \end{tikzcd}$$

is an equivalence, where $(\underline{\mathrm{Eq}}(a, a') \to U \times U) \in \mathcal{X}_{/U\times U}$ is the internal object representing equivalences of objects $a, b \in \mathcal{X}_{/U\times U}$, and $\pi_i\colon U \times U \to U$ are projections.

Lecture 5: Hypercompletion, Grothendieck topologies, and geometric morphisms

∞-connectivity. This is the limiting case of n-connectivity.

∞-connected map f: if f is n-connective for all n.
∞-connected object X: if X is n-connective for all n.

There is a possibly unexpected phenomenon here: an ∞-connected map is not necessarily an equivalence, even in an ∞-topos. ("The Whitehead theorem can fail in an ∞-topos".)

Example (A non-trivial ∞-connected object [DHI04])*.* Consider a locale with the following lattice of opens:

$$\begin{tikzcd}[column sep=small] & U_0^+ \arrow[dl] & & U_1^+ \arrow[dl] & & U_2^+ \arrow[dl] & & \cdots \arrow[dl] \\ V_0 & & V_1 \arrow[ul] \arrow[dl] & & V_2 \arrow[ul] \arrow[dl] & & V_3 \arrow[ul] \arrow[dl] & \\ & U_0^- \arrow[ul] & & U_1^- \arrow[ul] & & U_2^- \arrow[ul] & & \cdots \arrow[ul] \end{tikzcd}$$

So we have opensV_n, U_n^+, U_n^- for all $n \in \mathbb{Z}_{\geq 0}$. This is actually the open set lattice of a space, so I'll call it the space X. (I won't tell you what the points are because I don't need them.)

There is an "easy" recipe for sheafification: define F^+ so that $F^+(V_n) \approx F(U_n^+) \times_{F(V_{n+1})} F(U_n^-)$, and $F^+(U_n^\pm) \approx F(U_n^\pm)$. Then $aF = \operatorname{colim}(F \to F^+ \to F^{+2} \to \cdots)$, a colimit over a countable sequence, where $F^{+(n+1)} := (F^{+n})^+$.

Define $F\colon \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{S}$ by:

$$\begin{tikzcd}[column sep=small] & D^1_+ \arrow[dr] & & D^2_+ \arrow[dr] & & D^3_+ \arrow[dr] & & \cdots \\ S^0 \arrow[ur] \arrow[dr] & & S^1 \arrow[ur] \arrow[dr] & & S^2 \arrow[ur] \arrow[dr] & & S^3 \arrow[ur] \arrow[dr] & \\ & D^1_- \arrow[ur] & & D^2_- \arrow[ur] & & D^3_- \arrow[ur] & & \cdots \end{tikzcd}$$

where $S^n = D^n_+ \cup D^n_-$ is the decomposition of a unit sphere into closed hemispheres, which are homeomorphic to unit disks.

Claim. aF is ∞-connected but $aF \not\approx *$.

Proof. To show aF is ∞-connected we show $\tau^{\mathrm{Shv}}_{\leq n}(aF) \approx *$ for all n. We know truncation commutes with sheafification: $\tau^{\mathrm{Shv}}_{\leq n}(aF) \approx a(\tau^{\mathrm{PSh}}_{\leq n} F)$. We see directly that that $\tau_{\leq n}(F(V_k)) \approx *$ if $k > n$, and therefore $(\tau^{\mathrm{PSh}}_{\leq n} F)^{+(n+1)} \approx *$ for all n, which gives the desired result.

On the other hand, $\pi_0 aF(X) = \pi_0 \operatorname{colim}_{k\to\infty} \Omega^k S^k \approx \mathbb{Z}$, so it is non-trivial.

Hypercompletion.

Hypercompete object $F \in \mathcal{X}$**:** object such that

$$\mathrm{Map}(f, F) \text{ is iso for all } \infty\text{-connected maps } f \text{ in } \mathcal{X}.$$

Any n-truncated object is hypercomplete. Any non-contractible ∞-connected object F is not hypercomplete (take $f\colon F \to *$).

Write $\mathcal{X}^{\mathrm{hyp}} \subseteq \mathcal{X}$ for the full subcategory of hypercomplete objects.

Hypercomplete ∞-topos: an ∞-topos such that every object is hypercomplete, i.e., if $\mathcal{X} = \mathcal{X}^{\mathrm{hyp}}$.

Some ∞-topoi are hypercomplete, including $\mathcal{S}$ and $\mathrm{PSh}(\mathcal{C})$. But as we have seen, non-hypercomplete ∞-topoi exist.

Proposition. *There is an adjoint pair*

$$\iota\colon \mathcal{X}^{\mathrm{hyp}} \rightleftarrows \mathcal{X} : \ell$$

such that ℓ is left exact, and $\mathcal{X}^{\mathrm{hyp}}$ is an ∞-topos. Furthermore, ℓ inverts precisely the ∞-connected maps (i.e., $\ell(f)$ is an iso iff f is ∞-connected).

Example (Excisive functors). A functor is *excisive* (or *1-excisive*) if it takes pushout squares to pullback squares. Let $\mathcal{S}_*^{\mathrm{fin}} \subset \mathcal{S}_*$ be the full subcategory of based ∞-groupoids spanned by finite CW complexes. The category $\mathcal{X} = \mathrm{Fun}^{\mathrm{exc}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is an ∞-topos, with the property that $\mathcal{X}^{\mathrm{hyp}} \equiv \mathcal{S}$ but $\mathcal{X} \not\equiv \mathcal{S}$.

In fact, the full subcategory of ∞-connected objects in $\mathcal{X}$ is equivalent to the ∞-category of spectra.

Topological localizations. Consider a left exact localization

$$a\colon \mathrm{PSh}(\mathcal{C}) \rightleftarrows \mathrm{Shv}(\mathcal{C}, \mathcal{T}) : i\ .$$

Let $\overline{\mathcal{T}} = \{\, f \in \mathrm{PSh}(\mathcal{C}) \mid a(f) \text{ is iso} \,\}$, the class of maps inverted by $a\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{T})$. By construction $\mathcal{T} \subseteq \overline{\mathcal{T}}$. We call $\overline{\mathcal{T}}$ the *saturation* of $\mathcal{T}$. It is clear that such a left-exact localization determines and is determined by $\overline{\mathcal{T}}$.

I am going to show that this localization is *almost* determined by the *monomorphisms* in the saturation $\overline{\mathcal{T}}$. More precisely, I claim that the full subcategories $\mathrm{Shv}(\mathcal{C}, \mathcal{T})_{\leq n} \subseteq \mathrm{PSh}(\mathcal{C})$ of *n-truncated sheaves* only depend on the monomorphisms in $\overline{\mathcal{T}}$.

Observe that $\overline{\mathcal{T}}$

- has the 2-out-of-3 property,
- is closed under colimits in $\mathrm{Fun}([1], \mathrm{PSh}(\mathcal{C}))$,
- closed under cobase-change, and
- is closed under base-change.

Note also that $(X \to *) \in \overline{\mathcal{T}}$ iff $X \in \mathrm{Shv}(\mathcal{C}, \mathcal{T})$.

Here are some facts about $\overline{\mathcal{T}}$:

(1) For a pullback square

$$\begin{tikzcd} A' \arrow[r, "p'"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B' \arrow[r, "p"'] & B \end{tikzcd}$$

with p an effective epi (=0-connective), $g \in \overline{\mathcal{T}}$ imples $f \in \overline{\mathcal{T}}$.

Proof: The base-change p' of p is also an effective epi. The properties of $\overline{\mathcal{T}}$ imply that the induced map $C(p') \to C(p)$ on Čech nerves, restricted to any object of Δ, is in $\overline{\mathcal{T}}$. Taking colimits of this induced map gives f, which is therefore also in $\overline{\mathcal{T}}$.

(2) If $f = pi$ is an epi/mono-factorization (i.e., p is 0-connective, i is (-1)-truncated), then $f \in \overline{\mathcal{T}}$ iff $p, i \in \overline{\mathcal{T}}$.
Proof: $\Longleftarrow$ is clear. For $\Longrightarrow$: if $a(f)$ is iso then $a(p)a(f)^{-1}$ is a section of the monomorphism $a(i)$, so $a(i)$ is also iso.

(3) If p is effective epi, then $p \in \overline{\mathcal{T}}$ iff $\Delta_p \in \overline{\mathcal{T}}$.
Proof: Consider

$$\begin{tikzcd} X \arrow[r, "\Delta_p"] \arrow[dr, "\mathrm{id}"'] & X \times_Y X \arrow[r, "q_2"] \arrow[d, "q_1"] & X \arrow[d, "p"] \\ & X \arrow[r, "p"'] & Y \end{tikzcd}$$

$\Delta_p \in \overline{\mathcal{T}} \iff q_1 \in \overline{\mathcal{T}}$ since $\overline{\mathcal{T}}$ has 2-out-of-3. Then $p \in \overline{\mathcal{T}} \iff q_1 \in \overline{\mathcal{T}}$ since $\overline{\mathcal{T}}$ is closed under base-change and using (1) above

Proposition. *Let $\mathcal{T}, \mathcal{T}$ be two sets of maps which generate left exact localizations of* $\mathrm{PSh}(\mathcal{C})$. *If*

$$\overline{\mathcal{T}} \cap (\textit{monomorphisms in } \mathrm{PSh}(\mathcal{C})) = \overline{\mathcal{T}}' \cap (\textit{monomorphisms in } \mathrm{PSh}(\mathcal{C})),$$

then for any n we have

$$\overline{\mathcal{T}} \cap (n\textit{-truncated maps in } \mathrm{PSh}(\mathcal{C})) = \overline{\mathcal{T}}' \cap (n\textit{-truncated maps in } \mathrm{PSh}(\mathcal{C}))$$

In particular, the hypothesis implies

$$\mathrm{Shv}(\mathcal{C}, \mathcal{T})_{\leq n} = \mathrm{Shv}(\mathcal{C}, \mathcal{T}')_{\leq n}$$

as full subcategories of $\mathrm{PSh}(\mathcal{C})_{\leq n} \subseteq \mathrm{PSh}(\mathcal{C})$.

Proof. If $f\colon X \to Y$ is a map in $\mathrm{PSh}(\mathcal{C})$, inductively define epi/mono-factorizations of the form

$$f = f_0 = p_0 i_0, \qquad f_1 := \Delta_{p_0} = p_1 i_1, \qquad f_2 := \Delta_{p_1} = p_2 i_2, \qquad \dots, \qquad f_k := \Delta_{p_{k-1}} = p_k i_k, \qquad \dots.$$

From our previous observations, f_k n-truncated $\Longrightarrow$ p_k n-truncated $\Longrightarrow$ Δ_{p_k} $(n-1)$-truncated (using the left cancellation property of n-truncated maps and the inductive definition via diagonals). So by induction, f n-truncated $\Longrightarrow$ f_{n+2}, p_{n+2} iso.

Thus $f_k \in \overline{\mathcal{T}}$ iff $p_k, i_k \in \overline{\mathcal{T}}$ by (2), iff $f_{k+1} = \Delta_{p_k}, i_k \in \overline{\mathcal{T}}$ by (3). By induction, we have that $f \in \overline{\mathcal{T}}$ iff $f_{n+2}, i_0, \dots, i_{n+1} \in \overline{\mathcal{T}}$, and this is true iff $i_0, \dots, i_{n+1} \in \overline{\mathcal{T}}$ since f_{n+2} is iso. Thus the n-truncated maps in $\overline{\mathcal{T}}$ are determined by the monomorphisms.

For the final statement, it suffices to note that an n-truncated X in $\mathrm{PSh}(\mathcal{C})$ is an object of $\mathrm{Shv}(\mathcal{C}, \mathcal{T})$ iff $\mathrm{Map}_{\mathrm{PSh}(\mathcal{C})}(f, X)$ is iso for every *n-truncated* f in $\overline{\mathcal{T}}$. To see this, recall that $a\tau_{\leq n}^{\mathrm{PSh}(\mathcal{C})} \approx \tau_{\leq n}^{\mathrm{Shv}(\mathcal{C},\mathcal{T})} a$, where $a\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}, \mathcal{T})$ is sheafification, and the other functors are n-truncation endofunctors on $\mathrm{PSh}(\mathcal{C})$ and $\mathrm{Shv}(\mathcal{C}, \mathcal{T})$. Thus $(f\colon U \to V) \in \overline{\mathcal{T}}$ implies $\overline{f} := \tau_{\leq n}^{\mathrm{PSh}(\mathcal{C})} f \in \overline{\mathcal{T}}$. Since $\overline{f}$ is a map between n-truncated objects it is also an n-truncated map (the "left cancellation" property of n-truncated maps), and if X is an n-truncated presheaf then restriction $f^*\colon \mathrm{Map}(V, X) \to \mathrm{Map}(U, X)$ along f is the same map as restriction $\overline{f}^*\colon \mathrm{Map}(\tau_{\leq n}^{\mathrm{PSh}(\mathcal{C})} V, X) \to \mathrm{Map}(\tau_{\leq n}^{\mathrm{PSh}(\mathcal{C})} U, X)$ along $\overline{f}$. □

Topological localization: a left-exact localization which is generated by its collection of monomorphisms.

By the above remarks, a topological localization of $\mathrm{PSh}(\mathcal{C})$ is one which is generated by its collection of n-truncated morphisms, where n can be any finite value.

Since saturations are closed under colimits and base-change, since every object of $\mathrm{PSh}(\mathcal{C})$ is a colimit of representables, and since colimits are universal in $\mathrm{PSh}(\mathcal{C})$ and satisfy descent, it's not hard to show that topological localizations of $\mathrm{PSh}(\mathcal{C})$ can be generated by collections of "sieves", where a **sieve** on $\mathcal{C}$ is a monomorphism of presheaves whose target is a representable functor:

$$S \rightarrowtail \rho_C.$$

Pursuing this further, it turns out that topological localizations of $\mathrm{PSh}(\mathcal{C})$ are in bijective correspondence with *Grothendieck topologies* on the ∞-category $\mathcal{C}$. When this is a 1-category we recover the classical notion of Grothendieck topology.

Example. Presheaf categories $\mathrm{PSh}(\mathcal{C})$ are (trivially) obtained by a topological localization of a presheaf category.

Categories $\mathrm{Shv}(X)$ of ∞-groupoids on a space (or local) are topological localizations of presheaves on Open_X.

It appears to be an open question as to whether every ∞-category can be presented as a topological localization of a presheaf category. There are certainly examples (such as the "excisive functor" example $\mathrm{Fun}^{\mathrm{exc}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ above) for which no such topological presentation is known.

However, we do have the following [Lur09, 6.5.2.20].

Proposition. *Every ∞-topos $\mathcal{X}$ admits a presentation obtained as a composite of two left-exact localizations*

$$\mathcal{X} \underset{\rightarrowtail}{\twoheadleftarrow} \mathcal{X}' \;=\; \mathrm{Shv}(\mathcal{C}, \mathcal{T}) \underset{\rightarrowtail}{\twoheadleftarrow} \mathrm{PSh}(\mathcal{C})$$

where the right-hand localization is topological (and so arises from a Grothendieck topology on $\mathcal{C}$), while the left-hand localizaton is obtained by inverting some set $\mathcal{S}$ of ∞-connected maps in $\mathcal{X}'$ (this is called a cotopological localization*).*

An example of a cotopological localization is hypercompletion $\mathcal{X}^{\mathrm{hyp}} \underset{\rightarrowtail}{\twoheadleftarrow} \mathcal{X}$, which is the "maximal" cotopological localization of $\mathcal{X}$. In particular, every *hypercomplete* ∞-topos is equivalent to one of the form $\mathrm{Shv}(\mathcal{C}, \mathcal{T})^{\mathrm{hyp}}$, where $\mathcal{T}$ is a Grothendieck topology on a small ∞-category $\mathcal{C}$.

Morphisms of ∞-topoi. Recall that a geometric morphism (or just "morphism") $f\colon \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is a functor $f^*\colon \mathcal{Y} \to \mathcal{X}$ which is colimit preserving and left-exact. This functor is always the left adjoint of a pair

$$f_*\colon \mathcal{X} \leftrightarrows \mathcal{Y} : f^*.$$

We have seen several examples of such, including geometric morphisms such as $\mathrm{Shv}(\mathcal{C}, \mathcal{T}) \to \mathrm{PSh}(\mathcal{C})$ arising from left-exact localizations.

I write

$$[\mathcal{X}, \mathcal{Y}] := \mathrm{Fun}^{\text{colim pres/lex}}(\mathcal{Y}, \mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{Y}, \mathcal{X}).$$

for the ∞-category of geometric morphisms.

Étale morphism: a geometric morphism equivalent to one of the form

$$\mathcal{X}_{/A} \to \mathcal{X}_{/B}$$

associated to a morphism $f\colon A \to B$ in $\mathcal{X}$, whose left-adjoint functor $f^*\colon \mathcal{X}_{/B} \to \mathcal{X}_{/A}$ is the "pullback along f" functor: $f^*(E \to B) = (E \times_B A \to A)$.

The left-adjoint functor of an Étale morphism is also a right adjoint, so that we have functors

$$\begin{tikzcd} \mathcal{X}_{/A} \arrow[r, shift left=2, "f_\#"] \arrow[r, shift right=2, "f_*"'] & \mathcal{X}_{/B}, \arrow[l, "f^*" description] \end{tikzcd}$$

where $f_\#(D \to A) = (D \to A \xrightarrow{f} B)$. The functor f_* is such that $f_*(D \to A)$ represents a functor $(\mathcal{X}_{/B})^{\mathrm{op}} \to \mathcal{S}$ which sends $(E \to B) \mapsto \mathrm{Map}_{\mathcal{X}_{/A}}(E \times_B A \to A,\ D \to A)$. Type theorists might find the notation $\Sigma_f := f_\#$ and $\Pi_f := f_*$ suggestive.

Remark. There is an "intrinsic" characterization of étale morphisms: a geometric morphism $f\colon \mathcal{Y} \to \mathcal{X}$ is equivalent to an étale morphism iff

(1) f^* admits a left adjoint $f_\#\colon \mathcal{Y} \to \mathcal{X}$,
(2) $f_\#$ is conservative, and
(3) $f_\#$ has a "push-pull formula": $f_\#(f^*A \times_{f^*B} C) \xrightarrow{\sim} A \times_B f_\# C$.

See [Lur09, 6.3.5.11].

Constructing morphisms of ∞-topoi. Let's try to "compute" $[\mathcal{Y}, \mathcal{X}]$ when $\mathcal{X} = \mathrm{Shv}(\mathcal{C}, \mathcal{T})$. Consider a functor $f^*\colon \mathcal{X} \to \mathcal{Y}$, and the resulting diagram

$$\begin{tikzcd}
 & \mathcal{C} \arrow[d, tail, "\rho"] & \\
\mathcal{X} \arrow[r, tail, "i"] \arrow[rrd, bend right, "f^*"'] & \mathrm{PSh}(\mathcal{C}) \arrow[r, two heads, "a"] \arrow[rd, "\widehat{\phi}"] & \mathcal{X} \arrow[d, "f^*"] \\
 & & \mathcal{Y}
\end{tikzcd}$$

Because a is a colimit preserving localization, f^* is colimit preserving iff $\widehat{\phi} := f^* \circ a$ is colimit preserving, and that in such a case $\widehat{\phi}$ is the unique colimit preserving functor extending $\phi := \widehat{\phi} \circ \rho\colon \mathcal{C} \to \mathcal{Y}$. Furthermore, f^* is left-exact iff $\widehat{\phi}$ is left-exact, since both i and a preserve finite limits.

We get a diagram of fully-faithful functors:

$$\begin{tikzcd}
\mathrm{Fun}^{\text{colim pres/lex}}(\mathcal{X}, \mathcal{Y}) \arrow[r, tail] \arrow[d, "\approx"] & \mathrm{Fun}^{\text{colim pres/lex}}(\mathrm{PSh}(\mathcal{C}), \mathcal{Y}) \arrow[r, tail] \arrow[d, "\approx"] & \mathrm{Fun}^{\text{colim pres}}(\mathrm{PSh}(\mathcal{C}), \mathcal{Y}) \arrow[d, "\approx", "\circ\rho"'] \\
\left\{ \begin{array}{c} \phi \text{ such that} \\ \text{in addtion} \\ \widehat{\phi}(\mathcal{T}) \text{ are isos} \end{array} \right\} \arrow[r, tail] & \left\{ \begin{array}{c} \phi\colon \mathcal{C} \to \mathcal{Y} \text{ such that} \\ \widehat{\phi}(*) = * \text{ and } \widehat{\phi} \text{ preserves} \\ \text{pullbacks of form } \rho_{C_1} \times_{\rho_{C_0}} \rho_{C_2} \end{array} \right\} \arrow[r, tail] & \mathrm{Fun}(\mathcal{C}, \mathcal{Y})
\end{tikzcd}$$

This uses the characterization of colimit preserving and left exact $\mathrm{PSh}(\mathcal{C}) \to \mathcal{Y}$ from Lecture 4.

Proposition. *$[\mathcal{Y}, \mathrm{Shv}(\mathcal{C}, \mathcal{T})]$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{Y})$ spanned by ϕ whose colimit preserving extension $\widehat{\phi}\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{Y}$*

(1) *preserves the terminal object,*
(2) *preserves pullbacks of cospans of representables, and*
(3) *takes morphisms in $\mathcal{T}$ to isomorphisms.*

Example (Morphisms to a locale)*.* Let X be a space (or locale). Consider

$$\begin{tikzcd}
\mathrm{Open}_X \arrow[r, tail] \arrow[rrd, bend right, "\phi"'] & \mathrm{PSh}(\mathrm{Open}_X) \arrow[r, two heads] \arrow[rd, dotted, "\widehat{\phi}"] & \mathrm{Shv}(X) \arrow[d, dotted, "f^*"] \\
 & & \mathcal{Y}
\end{tikzcd}$$

A functor ϕ extends uniquely to a colimit preserving $\widehat{\phi}$.

Since Open_X already has finite limits, necessarily preserved by ρ, we have that $\widehat{\phi}$ is left-exact iff ϕ is, i.e., if

$$\phi(X) \approx *, \qquad \phi(U \cap V) \approx \phi(U) \times \phi(V).$$

(All morphisms in Open_X are mono, so all pullbacks are products.)

This implies that all $\phi(U)$ are (-1)-truncated objects ("propositions") in $\mathcal{Y}$, since $U \cap U = U$, i.e., ϕ factors $\mathrm{Open}_X \to \mathcal{Y}_{\leq -1} \subseteq \mathcal{Y}$ through the "underlying locale" of $\mathcal{Y}$.

Finally, $\widehat{\phi}$ factors through a (unique) colimit preserving f^* if for every open cover $U = \bigcup U_j$ it takes $\coprod \rho_{U_j} \to \rho_U$ to an effective epi in $\mathcal{Y}$, i.e., if $\bigvee \phi(U_j) = \phi(U)$ in $\mathcal{Y}_{\leq -1}$.

Thus

$$[\mathcal{Y}, \mathrm{Shv}(X)] \xrightarrow{\sim} \{\text{ maps } \mathcal{Y}_{\leq -1} \to \mathrm{Open}_X \text{ of locales}\}$$

When $Y = \mathrm{Shv}(Y)$ and if X is a sober space, we get that geometric morphisms $\mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ correspond exactly continuous maps $Y \to X$.

Example (Morphisms to ∞-groupoids). Since $\mathcal{S} = \mathrm{PSh}(1)$ is presheaves on the terminal category 1, to give a geometric morphism $f\colon \mathcal{X} \to \mathcal{S}$ it suffices to produce $\phi\colon 1 \to \mathcal{X}$ such that $\widehat{\phi}\colon \mathrm{PSh}(1) \to \mathcal{X}$ preserves the terminal object and preserves pullbacks of representables. Since the only representable is also the terminal object, we deduce that there is a *unique* geometric morphism $\pi\colon \mathcal{X} \to \mathcal{S}$. Thus $\mathcal{S}$ is the *terminal ∞-topos.*

Example (Morphisms to presheaves on an ∞-groupoid). Let $\mathcal{G}$ be a small ∞-groupoid, and consider geometric morphisms $\mathcal{X} \to \mathrm{PSh}(\mathcal{G})$. Since all morphisms in $\mathcal{G}$ are isomorphisms, it has all pullbacks, and every commutative square in $\mathcal{G}$ is a pullback square. As a consequnce, every functor $\mathcal{G} \to \mathcal{X}$ preserves pullbacks.

We thus learn that $[\mathcal{X}, \mathrm{PSh}(\mathcal{G})]$ is equivalent to the ∞-category of *principal $\mathcal{G}$-bundles* on $\mathcal{X}$, defined to be the full subcategory of $\mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by

$$P\colon \mathcal{G} \to \mathcal{X} \text{ such that } \operatorname{colim}_{\mathcal{G}} P \approx *.$$

When $\mathcal{G}$ has only one object and $\mathcal{X} = \mathcal{S}_{/B}$ this recovers the notion from algebraic topology of principal $\mathcal{G}$-bundle over B. See [NSS15].

Appendix: additional remarks

Lecture 1: Definition of sheaves on a space. I gave a definition of "sheaf on a space X with values in an ∞-category" which is not usually found in the literature.

However, it is equivalent. Here is a sketch of a proof.

Given an open cover $\{U_i\}_{i\in I}$ of U, define a functor

$$\mathcal{P}_{\mathrm{f}}(I)^{\mathrm{op}} \to \mathrm{PSh}(\mathrm{Open}_X)$$

by

$$J \mapsto \rho_{U_J}, \qquad U_J := \bigcap_{j\in J} U_j, \qquad \rho_{U_J}(V) = \mathrm{Hom}_{\mathrm{Open}_X}(V, U_J),$$

where $\mathcal{P}_{\mathrm{f}}(I)$ is the poset of finite subsetets of I. Restricting to the non-empty finite subsets $\mathcal{P}_{\mathrm{fne}}(I) \subseteq \mathcal{P}^{\mathrm{fin}}(I)$ and taking a colimit, we get a map

$$\rho_U \xleftarrow{\eta} S := \operatorname{colim}_{J\in\mathcal{P}_{\mathrm{fne}}(I)^{\mathrm{op}}} \rho_{U_j}$$

of presheaves. Then for any presheaf F, we have a map

$$\mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(\rho_U, F) \xrightarrow{\eta^*} \mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(S, F) \qquad \text{isomorphic to} \qquad F(U) \to \lim_{J\in\mathcal{P}_{\mathrm{fne}}(I)} F(U_J).$$

Claim. The presheaf S has values

$$S(V) \approx \begin{cases} * & \text{if } \exists\, i \in I \text{ such that } V \subseteq U_i, \\ \varnothing & \text{otherwise.} \end{cases}$$

That is, $S \to \rho_U$ is the *sieve* generated by the set of maps $\{U_i \to U\}$.

Proof of claim. We can compute the values of S "pointwise":

$$S(V) \approx \operatorname{colim}_{J\in\mathcal{P}_{\mathrm{fne}}(I)^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Open}_X}(V, U_J),$$

a colimit of a functor to ∞-groupoids. Since $\mathrm{Hom}_{\mathrm{Open}_X}(V, U_J)$ is either contractible or empty depending on whether $V \subseteq U_J$, we see that the colimit (in $\mathcal{S}$) is isomorphic to

$$\mathrm{colim}_{J \in \mathcal{P}_{\mathrm{fne}}(I_V)^{\mathrm{op}}} * \approx B(\mathcal{P}_{\mathrm{fne}}(I_V)),$$

the classifying space of the poset $\mathcal{P}_{\mathrm{fne}}(I_V)$ where $I_V = \{\, i \in I \mid V \subseteq U_i \,\}$. This classifying space is either empty or contractible depending on whether I_V is so: given $i_0 \in I_V$, a contracting homotopy is defined by a sequence of two natural transformations connecting the identity functor with a constant functor, namely $J \leq J \cup \{i_0\} \geq \{i_0\}$.

It is standard that F is a sheaf if and only if the η^* are isomorphisms for all "covering sieves", e.g., [MLM94, II.2] for set-valued sheaves, [Lur09, 6.2.2] for ∞-groupoid-valued sheaves.

Another standard statement of the sheaf condition is that the evident map

$$F(U) \to \lim_{\Delta}\left[[n] \mapsto \prod_{i_0,\dots,i_n} F(U_{i_0} \cap \cdots \cap U_{i_n})\right]$$

be an equivalence for every open cover $\{U_i\}$ of U. The target of this map is isomorphic to $\mathrm{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(S', F)$, where

$$S' := \mathrm{colim}_{\Delta^{\mathrm{op}}}\left[[n] \mapsto \coprod_{i_0,\dots,i_n} \rho_{U_{i_0}} \times_{\rho_U} \cdots \times_{\rho_U} \rho_{U_{i_n}}\right].$$

This is actually the colimit of the Čech nerve of the map $\coprod_i \rho_{U_i} \to \rho_U$, and as described in Lecture 4 this means that $\coprod_i \rho_{U_i} \to S' \to \rho_U$ is an epi/mono factorization in $\mathrm{PSh}(\mathrm{Open}_X)$, so $S' \to \rho_U$ is easily seen to be equivalent to the sieve $S \to \rho_U$ above.

Lecture 1: Constant sheaves of ∞-groupoids. Let X be a topological space, and let $\pi\colon \mathrm{Shv}(X) \to \mathrm{Shv}(*) = \mathcal{S}$ be the unique geometric morphism to the terminal ∞-topos, as described in Lecture 5. Then $\pi^*\colon \mathcal{S} \to \mathrm{Shv}(X)$ is another name for the functor which associates to an ∞-groupoid S the constant sheaf $C_S = \pi^* S$ on X with "value" S. Lurie shows [Lur09, §7.1] that for $U \in \mathrm{Open}_X$ which is *paracompact*, we have an equivalence

$$C_S(U) \approx \mathrm{Map}_{\mathcal{S}}(hU, S),$$

where the ∞-groupoid hU represents the "usual" homotopy type of the topological space U.

Why is paracompactness needed? Given an ∞-groupoid, model it by a topological space S. We get a presheaf

$$U \mapsto F(U) := \mathrm{Map}_{\mathrm{Top}}(U, S)$$

in the 1-category of topological spaces, where the right-hand side is the set of continuous maps equipped with a suitable topology, e.g., the compact-open topology. This functor thus gives a presheaf $U \mapsto hF(U)$ of ∞-groupoids; we would like to say that it is actually a *sheaf* of ∞-groupoids, representing the constant sheaf with value S.

For instance, if $U = U_0 \cup U_1$, we would like to show that $F(U)$ is equivalent to the homotopy pullback of $F(U_0) \to F(U_0 \cap U_1) \leftarrow F(U_1)$. To do this, we need a construction which takes a point in the homotopy pullback:

$$\big(f_0\colon U_0 \to S,\ f_1\colon U_1 \to S,\ H\colon (U_0 \cap U_1) \times [0,1] \to S\big), \qquad H_0 = f_0,\ H_1 = f_1,$$

to a point $f \in F(X) = \mathrm{Map}(X, S)$. The recipe is to "interpolate" between f_0 and f_1 using H, e.g., to choose a continuous $\chi\colon X \to [0,1]$ such that $\mathrm{int}(\chi^{-1}(0)) \supseteq (X \smallsetminus U_1)$, and $\mathrm{int}(\chi^{-1}(1)) \supseteq (X \smallsetminus U_0)$, which imply $\chi(x) = 0$ for $x \in X \smallsetminus U_1$ and $\chi(x) = 0$ for $x \in X \smallsetminus U_0$. Then

$$f(x) = \begin{cases} f_0(x) & \text{if } \chi(x) = 0, \\ H(x, \chi(x)) & \text{if } \chi(x) \in (0,1) \\ f_1(x) & \text{if } \chi(x) = 1, \end{cases}$$

defines $f \in F(X)$.

Existence of such a χ follows from the "paracompactness" condition, which ensures the existence of a "partition of unity" dominated by the open cover $\{U_0, U_1\}$.

Lecture 1: Sheafification. Here is a recipe for the construction of sheafification of presheaves on a topological space.

Given an open cover $\{U_i\}$ of U, define

$$F(\{U_i\}_{i\in I}) := \lim_{J\in\mathcal{P}_{\text{fne}}(I)} F(U_J).$$

A **covering sieve** for U is an open cover $\{U_i\}$ that is closed under downward inclusion: $V \subseteq U_i$ implies $V \in \{U_i\}$. (Any open cover *generates* a covering sieve.) Set

$$F^+(U) := \operatorname{colim}_{\left\{\substack{\text{covering sieves} \\ \{U_i\} \text{ of } U}\right\}} F(\{U_i\}),$$

using that open covers are ordered by refinement. There is a map $\zeta\colon F \to F^+$, corresponding to the "trivial covering sieve" generated by U.

The map $\zeta\colon F \to F^+$ is "trying to be a sheafification", but F^+ can fail to be a sheaf, so we have to iterate the construction. For any ordinal λ define $F^{+\lambda}$ and a map $F \to F^{+\lambda}$ by: $F^{+0} := F$, $F^{+\lambda+1} := (F^{+\lambda})^+$, and $F^{+\lambda} := \operatorname{colim}_{\mu<\lambda} F^{+\mu}$ if λ is a limit ordinal. Then you can show that there exists a κ (depending on the topology on X) such that $F \to F^{+\kappa}$ is a sheafification.

A key point is that the functor $F \mapsto F^+$ is itself left exact, since it is built from a limit and from a *directed colimit*, and directed colimits in $\mathcal{S}$ commute with finite limits. Likewise, the construction $F \mapsto F^{+\kappa}$ also only involves a directed colimit. Another necessary ingredient is to show that $\operatorname{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(F^+, G) \xrightarrow{\sim} \operatorname{Map}_{\mathrm{PSh}(\mathrm{Open}_X)}(F, G)$ whenever G is a sheaf.

The above recipe can be easily made precise when F is a presheaf of sets. The same idea works for presheaves of ∞-groupoids, though a precise formulation is more delicate. See [Lur09, 6.2.2], which in fact handles the case of an arbitrary Grothendieck topology.

Amusing fact: if F is a presheaf of *sets*, then $F \to F^{+2}$ is already a sheafification.

Exercise. If F is a presheaf of propositions, then $F^{+1} = F^+$ is a sheaf of propositions, and is the sheafification of F.

Lecture 2: Other models for ∞-categories. In addition to *quasicategories*, there are a number of other models for ∞-categories.

- **Simplicially enriched categories.**

  For any pair of objects X, Y in an ∞-category, we can consider the collection $\operatorname{Map}(X, Y)$ of all maps $X \to Y$. This collection should have the structure of an ∞-groupoid, and is sometimes called a "mapping space".

  This leads to a different model of ∞-categories, as *categories enriched over Kan complexes* (or a little more generally, *categories enriched over simplicial sets*). These consist of a set of objects; for each pair of objects a Kan complex (or simplicial set) $\operatorname{Map}(X, Y)$; and a composition operation $\operatorname{Map}(Y, Z) \times \operatorname{Map}(X, Y) \to \operatorname{Map}(X, Z)$ which is *strictly* associative and unital.

  Note that although a quasicategory $\mathcal{C}$ comes with associated mapping spaces $\operatorname{Map}_{\mathcal{C}}(X, Y)$ for every pair of objects, these are not the function objects of a simplicially enriched category. Instead there is a construction $C \mapsto \mathfrak{C}(C)$ which produces a Kan-enriched category from a quasicategory [Lur09, 1.1, 1.2].
- **Relative categories.** A relative category is $(W \subseteq C)$, consisting of an ordinary category C and a wide subcategory W of "weak equivalences". Although C is an ordinary category, the ∞-category this data models has as its morphisms all "zig-zags":

$$X \xleftarrow{W} U_0 \to U_1 \xleftarrow{W} U_2 \to U_3 \xleftarrow{W} \cdots \to U_k \xleftarrow{W} Y,$$

in which the left-to-right arrows must be in W. You can also express the higher structure of the ∞-category in terms of "hammocks". (See [DHKS04] for more on hammocks, and [BK12] for relative categories as a model for ∞-categories.)

The bare structure of a relative category is difficult to deal with in practice. A **model category** is a relative category equipped with additional structure: classes of maps called *fibrations* and *cofibrations*, which satisfy some axioms.

In practice, the familiar examples from algebraic topology and homological algebra arise from model categories.

- **Segal categories**, **complete Segal spaces**, **topologically enriched categories**, **A_∞-categories**, etc. See [Ber10] for more on many of these.

Lecture 2: ∞-categorical language. When you hear people talk about ∞-categories, they use the same words that you use for talking about 1-categories. Often they will even drop the "∞-" prefix when speaking about these notions. So it can be hard for the casually observer to spot what the differences are. What is often going is is that there is an ediface of higher structure which is *carefully not mentioned* by the speaker, and thus may bypass the listener entirely.

For example: A functor $F\colon C \to D$ of categories is a rule which assigns objects to objects, and morphisms to morphisms, and which is compatible with the remaining structure (i.e., preserves composition and identity maps).

A functor of ∞-categories is exactly the same thing, but now there is a lot more remaining structure to keep track of. *Note:* a model for a theory of ∞-categories gives you an explicit description of this data: e.g., a functor of quasicategories is just a map of the simplicial sets.

When speaking of functors of categories informally: we typically just describe the rule on objects and morphisms, and leave the verification of the remaining properties to the reader. Sometimes we don't even bother to describe the rule on morphisms, and leave this to be intuited.

Something very similar happens in the the ∞-categorical setting. For instance, to describe a functor between ∞-categories one might: say what it does on objects and morphisms, and leave the rest to the imagination. (In fact, the same is done when speaking of functors of 1-categories: sometimes we just give the rule on objects, and let the rule on morphisms be inferred.)

What has actually happened here is that the speaker has gestured at an idea of the definition of the desired functor, but has not actually constructed it. One may suppose that there is some hard work going on behind the scenes to actually describe the functor properly. In practice, there is "standard machinery" which can be used to produce functors of the sort that you are used to having. For instance, the construction of *limit functors* and *colimit functors*.

Limits and colimits. Recall that a limit (or colimit) of a functor is defined to be terminal (or initial) object of a suitable "slice" ∞-category. Given $\mathcal{C}$ we have the *right cone* $\mathcal{C}^{\triangleright}$ obtained by adjoining one new object v to $\mathcal{C}$, so that $\mathrm{Map}(C, v) = *$. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, we can form an ∞-category via the pullback

$$\begin{tikzcd} \mathcal{D}^{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) \arrow[d] \\ \{F\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{tikzcd}$$

Objects of $\mathcal{D}^{F/}$ are functors $\widehat{F}\colon \mathcal{C}^{\triangleright} \to \mathcal{D}$ which restrict to F: $\widehat{F}|\mathcal{C} = F$. These are called *right-cones* on F. Informally, $\widehat{F}$ consists of F together with a map $F(C) \to \widehat{F}(v)$ for each object C (and also higher commutativity). A *colimit* of F is an initial object of the slice $\mathcal{D}^{F/}$.

Being a colimit is a *property* of a right-cone, and admitting the existence of a colimit is a *property* of a functor. So we get maps

$$\mathrm{Fun}^{\exists\,\mathrm{colim}}(\mathcal{C}, \mathcal{D}) \xleftarrow{\text{restrict}} \mathrm{Fun}^{\mathrm{colimit}}(\mathcal{C}^{\triangleright}, \mathcal{D}) \xrightarrow{\text{eval at } v} \mathrm{Fun}(\{v\}, \mathcal{D}) = \mathcal{D},$$

where $\mathrm{Fun}^{\mathrm{colimit}}(\mathcal{C}^{\triangleright}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D})$ is the full subcategory of colimit-cones, and $\mathrm{Fun}^{\exists\,\mathrm{colim}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the full subcategory of functors for which a colimit exists.

It is a theorem that "restrict" is an equivalence of ∞-categories, so admits an inverse up to natural isomorphism. The composite of such an inverse with evaluation at v is a "colimit functor" $\mathrm{Fun}^{\exists\,\mathrm{colim}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$. If $\mathcal{D}$ "has all colimits of shape $\mathcal{C}$", then this is a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$.

Thus, we have obtained a colimit functor. But it is not uniquely defined as it requires a choice of inverse (though it is unique up to "contractible choice"). A more "natural" description of the colimit functor is the "zig-zag" presented above.

Lecture 2: The ∞-category of ∞-categories. The objects are ∞-categories, and the morphisms correspond to functors. The full structure is subtle to get at. Here are some options.

- The relative category ($\{\text{equivalences}\} \subseteq \mathrm{qCat}$), where qCat is the ordinary category of quasicategories.
- An enlarged version: the relative category ($\{\text{categorical equivalences}\} \subseteq \mathrm{sSet}$), where sSet is the ordinary category of simplicial sets. This relative category has an associated model structure ("Joyal model structure").
- A Kan enriched category, whose objects are quasicategories and whose function spaces are
$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{core}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$
where "core" denotes the *maximal ∞-subgroupoid*.

There are machines which turn these back into a quasicategory. The one from the third example is called Cat_∞. There is a full subcategory of ∞-groupoids, often called $\mathcal{S}$.

This is complicated, and a bit unsatisfactory. Many of the difficult technical parts of the theory revolve around maneuvers which go between different models like this. We need it, because I need to talk about presheaves of ∞-groupoids.

Lecture 2: Slice invariance of connectivity. This is the claim that $f\colon A \to B$ is $(n+1)$-connective as a morphism in $\mathcal{C}$ iff it is $(n+1)$-connective as an object of $\mathcal{C}_{/B}$. It is an immediate consequence of the following.

Proposition (Slice criterion for connectivity)**.** *A map $f\colon A \to B$ in $\mathcal{C}$ is $(n+1)$-connective if and only if $\mathrm{Map}_{\mathcal{C}_{/B}}(\mathrm{id}_B, g) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}_{/B}}(f, p)$ for all n-truncated objects $p\colon P \to B$ in $\mathcal{C}_{/B}$.*

Proof. The only if direction is a consequence of $f \perp p$, as the fiber of the map between mapping spaces over $u \in \mathrm{Map}_{\mathcal{C}_{/B}}(f, p)$ is the space of lifts in the diagram:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Conversely, suppose every such diagram as above has a unique lift. Given an n-truncated map $g\colon U \to V$ in $\mathcal{C}$, and a lifting problem of f against h, consider

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & P \arrow[r] \arrow[d, "p"] & U \arrow[d, "g"] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] \arrow[urr, dashed] & B \arrow[r] & V \end{tikzcd}$$

where the right-hand square is a pullback. A lift in the big rectangle amounts to the same thing as a lift in the left-hand square. $\square$

The above argument is a special case of a general observation about the orthogonality relation. Let $\mathcal{R}$ be a class of maps in an ∞-category $\mathcal{C}$ (with finite limits) which is closed under base change. Write $\mathcal{R}_B$ for the class of *objects* in the slice $\mathcal{C}_{/B}$ whose "underlying map" is in $\mathcal{R}$.

Proposition (Slice criterion for orthogonality)**.** *Consider a map $f\colon A \to B$ in $\mathcal{C}$, which can also be regarded as an object of $\mathcal{C}_{/B}$. The following are equivalent.*

(1) $f \perp \mathcal{R}$ *(in $\mathcal{C}$).*
(2) $\operatorname{Map}_{\mathcal{C}_{/B}}(\mathrm{id}_B, p) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}_{/B}}(f, p) \approx *$ *for all $p \in \mathcal{R}_B$.*

Lecture 5: Construction of hypercompletion. A class $\mathcal{S}$ of maps in an ∞-category $\mathcal{A}$ is a *strongly saturated class* if it

(1) has the 2-out-of-3 property.
(2) is closed under colimits in $\operatorname{Fun}([1], \mathcal{A})$, and
(3) is closed under cobase-change.

Furthermore, say that $\mathcal{S}$ is *stable under pullback* if

(4) is closed under base-change.

For instance, for any presentable left exact localization of the form $\operatorname{Shv}(\mathcal{C}, \mathcal{T}) \rightleftarrows \operatorname{PSh}(\mathcal{C})$, the saturation $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $\operatorname{PSh}(\mathcal{C})$ is a strongly saturated class which is stable under pullback.

A strongly saturated class is of *small generation* if it is the smallest saturated class containing some set $\mathcal{S}_0$.

Fact. If $\mathcal{A}$ is presentable, and $\mathcal{S}$ a strongly saturated class of small generation, you can form an adjoint pair

$$\ell\colon \mathcal{A} \rightleftarrows \mathcal{A}_{\mathcal{S}} : \iota$$

where $\mathcal{A}_{\mathcal{S}}$ is the full subcategory of $F \in \mathcal{A}$ such that $\operatorname{Map}(s, F)$ is iso for all $s \in \mathcal{S}$. Furthermore, $\mathcal{A}_{\mathcal{S}}$ is presentable, and $\mathcal{S} = \{\, f \in \operatorname{mor}\mathcal{A} \mid \ell(f) \text{ is iso} \,\}$. (See [Lur09, 5.5.4].) Furthermore, we have the following.

Proposition. *Let $\mathcal{S}$ be a strongly satruated class of strong generation. If it is also stable under pullback, then ℓ is left exact.*

Proof. Let $f\colon P \to Q$ be a map of pullback squares $P, Q\colon [1] \times [1] \to \mathcal{A}$. It suffices to show that if $f_{k\ell}\colon P_{k\ell} \to Q_{k\ell}$ is in $\mathcal{S}$ for $k\ell = 10, 01, 11$, then $f_{00} \in \mathcal{S}$. For then we can apply to $Q_{k\ell} = \ell P_{k\ell}$ for $k\ell = 10, 01, 11$, and $Q_{00} = Q_{10} \times_{Q_{11}} Q_{01}$.

To prove this, use the following diagram in which every square is a pullback, and the indicated maps are in $\mathcal{S}$ by hypothesis, together with properties (1) and (4), to show that g and then f_{00} are in $\mathcal{S}$.

$$\begin{tikzcd}
P_{10} \times_{Q_{11}} P_{01} \arrow[rr] \arrow[d] \arrow[dr, "g"] & & P_{10} \arrow[d, "\mathcal{S}"] \\
Q_{10} \times_{Q_{11}} P_{01} \arrow[r] \arrow[d] & Q_{00} \arrow[r] \arrow[d] & Q_{10} \arrow[d] \\
P_{01} \arrow[r, "\mathcal{S}"'] & Q_{01} \arrow[r] & Q_{11}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P_{00} \arrow[r] \arrow[d] \arrow[dl, bend right, "f_{00}"'] & P_{10} \arrow[rr] \arrow[d] & & P_{11} \arrow[d] \arrow[dr, equal, bend left] & \\
Q_{00} & P_{10} \times_{Q_{11}} P_{01} \arrow[l, "g"] \arrow[r] \arrow[d] & P_{11} \times_{Q_{11}} P_{10} \arrow[rr] \arrow[d] \arrow[dr] & & P_{11} \times_{Q_{11}} P_{11} \arrow[r] \arrow[d] & P_{11} \arrow[d, "\mathcal{S}"] \\
 & P_{01} \arrow[r] & P_{11} & P_{10} \arrow[r] & P_{11} \arrow[r, "\mathcal{S}"'] & Q_{11}
\end{tikzcd}$$

□

The class of n-connective maps in an ∞-topos has nearly all of these proprties: The one exception is that it is missing one part of the 2-out-of-3 property: if g and gf are n-connective (with $n \geq 1$), we can only be sure that f is $(n-1)$-connective. However, this is enough to show that the class of ∞-connected maps is a strongly saturated.

One can be even more explicit: hypercompletion is generated by a collection of "explicitly" defined maps of presheaves called *hypercovers*. This was described in [DHI04]. See also [Lur09, 6.5.3].

Lecture 5: Truncation towers and homotopy dimension. An object X in an ∞-topos $\mathcal{X}$ has a natural *truncation tower* (or *Postnikov tower*):

$$X \to \cdots \to \tau_{\leq n} X \to \tau_{\leq n-1} X \to \cdots \to \tau_{\leq -1} X \to *.$$

Say that *truncation towers are convergent* in $\mathcal{X}$ if:

(A) for every truncation tower as above, $X \to \lim_n \tau_{\leq n} X$ is an equivalence, and
(B) if $X \to \cdots \to X_n \to X_{n-1} \to \dots$ is a tower such that (i) $X \xrightarrow{\sim} \lim_n X_n$ and (ii) $X_n \to X_{n-1}$ is the universal $(n-1)$-truncation of X_n, then it is a truncation tower.

Example. In $\mathcal{S}$, and more generally in $\mathrm{PSh}(\mathcal{C})$, truncation towers are convergent.

Proposition. *If truncation towers are convergent in $\mathcal{X}$, then $\mathcal{X}$ is hypercomplete.*

Proof. The class $\mathcal{X}^{\mathrm{hyp}}$ of hypercomplete objects (which includes all n-truncated objects) is closed under limits in $\mathcal{X}$. (In fact, this uses only property (A) above.) □

Example. Property (A) fails in a general ∞-topos, e.g., in any non-hypercomplete one. Apparently property (B) also fails in general, though I don't have a convenient example to cite here.

An ∞-topos $\mathcal{X}$ has *homotopy dimension* $\leq n$ if every n-connective object F in $\mathcal{X}$ admits a section, i.e., there exists a morphism $1 \to F$ in $\mathcal{X}$.

Example. $\mathcal{S}$ has homotopy dimension $= 0$.
$\mathcal{S}_{/B}$ has homotopy dimension $\leq n$ whenever B is a CW complex of dimension n.

An ∞-topos $\mathcal{X}$ is *locally of homotopy dimension* $\leq n$ if there exists an effective epi

$$\coprod F_i \to 1$$

in $\mathcal{X}$ such that each slice $\mathcal{X}_{/F_i}$ has homotopy dimension $\leq n$.

Example. For any ∞-groupoid B, the slice $\mathcal{S}_{/B}$ is locally of homotopy dimension ≤ 0. This is because there always exists an effective epi of the form $\coprod_I 1 \to B$.

Theorem. *If $\mathcal{X}$ is locally of homotopy dimension $\leq n$, then $\mathcal{X}$ has convergent truncation towers, and thus is hypercomplete.*

See [Lur09, 7.2.1].

Lecture 5: Classifying ∞-topos for objects. In general, computing $[\mathcal{Y}, \mathrm{PSh}(\mathcal{C})]$ can be difficult. However, there's a special case with a clean answer.

Given a small ∞-category $\mathcal{C}$, we can form its *free finite limit completion* $i \colon \mathcal{C} \to \widehat{\mathcal{C}}^{\mathrm{f.lim}}$, so that for any $\mathcal{A}$ with finite limits, the evident restriction functor i^* in

$$\mathrm{Fun}(\widehat{\mathcal{C}}^{\mathrm{f.lim}}, \mathcal{A}) \supseteq \mathrm{Fun}^{\mathrm{lex}}(\widehat{\mathcal{C}}^{\mathrm{f.lim}}, \mathcal{A}) \xrightarrow{i^*} \mathrm{Fun}(\mathcal{C}, \mathcal{A})$$

gives an equivalence from the full subcategory of left-exact functors.

Combined with our general method for describing geometric morphisms, we find that

$$[\mathcal{Y}, \mathrm{PSh}(\widehat{\mathcal{C}}^{\mathrm{f.lim}})] \approx \mathrm{Fun}^{\mathrm{colim\ pres/lex}}(\mathrm{PSh}(\widehat{\mathcal{C}}^{\mathrm{f.lim}}), \mathcal{Y}) \approx \mathrm{Fun}^{\mathrm{lex}}(\widehat{\mathcal{C}}^{\mathrm{f.lim}}, \mathcal{Y}) \approx \mathrm{Fun}(\mathcal{C}, \mathcal{Y}).$$

Note that this demonstrates that an ∞-category of geometric morphisms need not be small.

Here is a "formula" for the free finite limit completion. It is actually more direct to describe the free finite *colimit* completion, since $(\widehat{\mathcal{C}}^{\mathrm{f.lim}})^{\mathrm{op}} = \widehat{\mathcal{C}^{\mathrm{op}}}^{\mathrm{f.colim}}$. In fact, $\widehat{\mathcal{D}}^{\mathrm{f.colim}}$ is equivalent to the smallest full subcategory of $\mathrm{PSh}(\mathcal{D})$ which is closed under finite colimits and contains the image of the Yoneda functor $\rho \colon \mathcal{D} \to \mathrm{PSh}(\mathcal{D})$.

Example. If $\mathcal{D} = 1$ is the terminal ∞-category, then $\widehat{\mathcal{D}}^{\text{f.colim}} \subseteq \mathrm{PSh}(1) = \mathcal{S}$ is identical with $\mathcal{S}^{\text{fin}} \subseteq \mathcal{S}$, the full subcategory spanned by finite CW-complexes. We deduce that

$$[\mathcal{Y}, \mathrm{Fun}(\mathcal{S}^{\text{fin}}, \mathcal{S})] \approx \mathcal{Y}.$$

Thus $\mathrm{Fun}(\mathcal{S}^{\text{fin}}, \mathcal{S})$ may be called the *classifying ∞-topos for objects.* (This is not the same as the "object classifier" of Lecture 4.)

Lecture 5: Geometric morphisms to a slice. Fix an ∞-topos $\mathcal{X}$ and an object U in $\mathcal{X}$, and let $\pi_U \colon \mathcal{X}_{/U} \to \mathcal{X}$ be the étale morphism associated to $U \to 1$. It is defined by $\pi_U^* X := (X \times U \xrightarrow{\text{proj}} U)$. We will describe objects of $[\mathcal{Y}, \mathcal{X}_{/U}]$.

First, suppose given a geometric morphism $g \colon \mathcal{Y} \to \mathcal{X}_{/U}$. From this we can obtain:

(1) a geometric morphism $f \colon \mathcal{Y} \to \mathcal{X}$, and
(2) a map $s \colon 1_{\mathcal{Y}} \to f^*U$ in $\mathcal{Y}$, where $1_{\mathcal{Y}}$ is the terminal object of $\mathcal{Y}$.

For (1) just take $f = \pi_U g$. For (2), consider the object

$$U_* := (U \times U \xrightarrow{\text{proj}_2} U) \approx \pi_U^* U \quad \text{in } \mathcal{X}_{/U}.$$

This is equipped with a tautological section $s_U \colon 1_{\mathcal{X}_{/U}} \to U_*$, namely the diagonal map $\Delta \colon U \to U \times U$. Let $s := g^*(s_U)$, and note that $g^*(1_{\mathcal{X}_{/U}}) \approx 1_{\mathcal{Y}}$ and $g^* U_* = g^* \pi_U^* U = f^* U$.

Conversely, given $f \colon \mathcal{Y} \to \mathcal{X}$ and $s \colon 1_{\mathcal{Y}} \to f^*U$, there is a functor $g^* \colon \mathcal{Y} \to \mathcal{X}_{/X}$, with the property that $g^*(X \to U)$ fits in a pullback square

$$\begin{tikzcd} g^*\left(\begin{smallmatrix} X \\ \downarrow \\ U \end{smallmatrix}\right) \arrow[r] \arrow[d] & f^*X \arrow[d] \\ 1_{\mathcal{Y}} \arrow[r, "s"'] & f^*U \end{tikzcd}$$

in $\mathcal{Y}$.

The above operations $g \mapsto (f, s)$ and $(f, s) \mapsto g$ are inverse to each other (up to equivalence), so the above discussion effectively gives a "calculation" of $[\mathcal{Y}, \mathcal{X}_{/U}]$, part of which says that for any object $f \colon \mathcal{Y} \to \mathcal{X}$ of $[\mathcal{Y}, \mathcal{X}]$ there is a pullback of ∞-categories:

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{Y}}(1_{\mathcal{Y}}, f^*U) \arrow[r] \arrow[d] & {[\mathcal{Y}, \mathcal{X}_{/U}]} \arrow[d, "{[-,\pi_U]}"] \\ \{f\} \arrow[r] & {[\mathcal{Y}, \mathcal{X}]} \end{tikzcd}$$

See [Lur09, 6.3.5.5-6] for details.

Taking $\mathcal{Y} = \mathcal{X}_{/V}$ and $f = \pi_V \colon \mathcal{X}_{/V} \to \mathcal{X}$, we learn that there is a correspondence

$$\left\{ \begin{tikzcd} \mathcal{X}_{/V} \arrow[rr, dotted, "f"] \arrow[dr, "\pi_V"'] & & \mathcal{X}_{/U} \arrow[dl, "\pi_U"] \\ & \mathcal{X} & \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} V \arrow[r, dotted, "g"] & U \end{tikzcd} \right\}$$

between geometric morphisms f compatible with étale morphisms π_V and π_U, and maps g in $\mathcal{X}$. In other words, the "slice of ∞-topoi over $\mathcal{X}$" contains $\mathcal{X}$ as a full subcategory.

References

[ABFJ17] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal, *A generalized Blakers-Massey Theorem* (2017), available at `arXiv:1703.09050`.

[BK12] C. Barwick and D. M. Kan, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68.

[Ber10] Julia E. Bergner, *A survey of* $(\infty, 1)$*-categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83.

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9–51.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[MLM94] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. A first introduction to topos theory; Corrected reprint of the 1992 edition.

[May90] J. P. May, *Weak equivalences and quasifibrations*, Groups of self-equivalences and related topics (Montreal, PQ, 1988), Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 91–101.

[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson, *Principal* ∞*-bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), no. 4, 749–801.

[Pup74] Volker Puppe, *A remark on "homotopy fibrations"*, Manuscripta Math. **12** (1974), 113–120.

[Rap17] George Raptis, *Some characterizations of acyclic maps* (2017), available at `arXiv:1711.08898`.

Department of Mathematics, University of Illinois, Urbana, IL

Email address: `rezk@illinois.edu`