Categorical Interpolation: Descent and the Beck-Chevalley Condition without Direct Images

Duško Pavlović

Zevenwouden 223, 3524 CR Utrecht, The Netherlands

Fibred categories have been introduced by Grothendieck (1959, 1971), as the setting for his theory of descent. The present paper contains (in section 4) a characterisation of the effective descent morphisms under an arbitrary fibred category. This essentially geometric result complements a logical analysis of the Beck-Chevalley property (section 1) – which was crucial in the well-known theorem on sufficient conditions for the descent under bifibrations, due to Bénabou-Roubaud (1970) and Beck (unpublished). We describe the notion of *interpolants* (sections 2 and 3) as the common denominator of the concepts of descent and the Beck-Chevalley property.

(For the basic notions and facts about fibred categories, the reader can consult Gray 1966, or Bénabou 1985. A survey can also be found in Pavlović 1990.)

1. The Beck-Chevalley condition/property

11. Proposition. Let $E: \mathcal{E} \longrightarrow \mathcal{B}$ be a bifibration, $Q = (f,g,s,t)$ a square in $\mathcal{B}$ such that $f \circ g = s \circ t$, and $\Theta = (\varphi,\gamma,\sigma,\vartheta)$ a square in $\mathcal{E}$ such that $\varphi \circ \gamma = \sigma \circ \vartheta$, with $E\varphi=f$, $E\gamma=g$, $E\sigma=s$ and $E\vartheta=t$. The following conditions are equivalent:

a) if ϑ and φ are cartesian <u>and</u> if σ is cocartesian <u>then</u> γ must be cocartesian;

b) if σ and γ are cocartesian <u>and</u> if ϑ is cartesian <u>then</u> φ must be cartesian;

c) if ϑ is cartesian <u>and</u> if σ is cocartesian <u>then</u> φ is cartesian <u>iff</u> γ is cocartesian.

If some inverse image functors f^* and t^* and some direct image functors $g_!$ and $s_!$ are chosen, then every square Θ over Q satisfies conditions (a-c) iff there is a canonical natural isomorphism

d) $f^* s_! \simeq g_! t^*$.

12. Definition. A square Q in the base of a bifibration $E: \mathcal{E} \to \mathcal{B}$ satisfies the *Beck-Chevalley condition* if every square Θ over Q satisfies conditions (a-c). E is said to have the *Beck-Chevalley property* if all the pullback squares in $\mathcal{B}$ satisfy the Beck-Chevalley condition. – "Beck-Chevalley" will be abbreviated to "BC".

13. Proposition. A bifibration E has the BC-property iff the cocartesian arrows are stable under those pullbacks along cartesian arrows which E preserves.

14. Sources. The Beck-Chevalley condition has arisen in the theory of descent – as developed from Grothendieck 1959. Jon Beck and Claude Chevalley studied it independently from each another. The former expressed it in the form 11(d), the latter as in 11(a). It is conspicuous that neither of them ever published anything on it. Early references are: Bénabou-Roubaud 1970, Lawvere 1970.

The proofs of propositions 11 and 13 are elementary. They can be found in my thesis (1990).

15. Logical meaning of the BC-property. Consider a fibration $E: \mathcal{E} \to \mathcal{B}$ as a "category of predicates": the base category $\mathcal{B}$ is to be thought of as a category of "sets" and "functions", while the objects and arrows of a fibre $\mathcal{E}_I$ represent "predicates" $\alpha(x^I)$ over the "set" I, and "proofs" between them. In this setting, the logical operation of substitution is interpreted by the inverse images. An inverse image functor over a "function" $t: I \to J$ in $\mathcal{B}$ can be understood as mapping

$$t^*: \mathcal{E}_J \to \mathcal{E}_I : \beta(y^J) \mapsto \beta(t(x^I)).$$

Lawvere (1969) noticed that *the quantifiers are adjoint to the substitution*:

$$\alpha(x^I) \vdash \beta(t(x^I)) \quad \Leftrightarrow \quad \exists x^I\big(t(x^I)=y^J \wedge \alpha(x^I)\big) \vdash \beta(y^J),$$

$$\beta(t(x^I)) \vdash \alpha(x^I) \quad \Leftrightarrow \quad \beta(y^J) \vdash \forall x^I\big(t(x^I)=y^J \to \alpha(x^I)\big),$$

so that the logical picture of the direct image functors ($t_! \dashv t^* \dashv t_*$) becomes

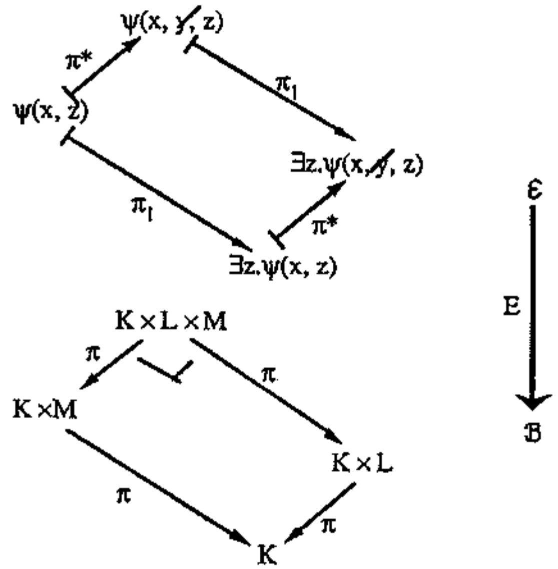
$$t_! : \mathcal{E}_I \to \mathcal{E}_J : \alpha(x^I) \mapsto \exists x^I\big(t(x^I)=y^J \wedge \alpha(x^I)\big), \text{ and}$$

$$t_* : \mathcal{E}_I \to \mathcal{E}_J : \alpha(x^I) \mapsto \forall x^I\big(t(x^I)=y^J \to \alpha(x^I)\big).$$

What does the Beck-Chevalley property mean in this context? The simplest case is when the commutative square Q consists of projection arrows. A direct image functor along a projection $\pi: K \times M \to K$ just quantifies a variable, while an inverse image functor adds a dummy:

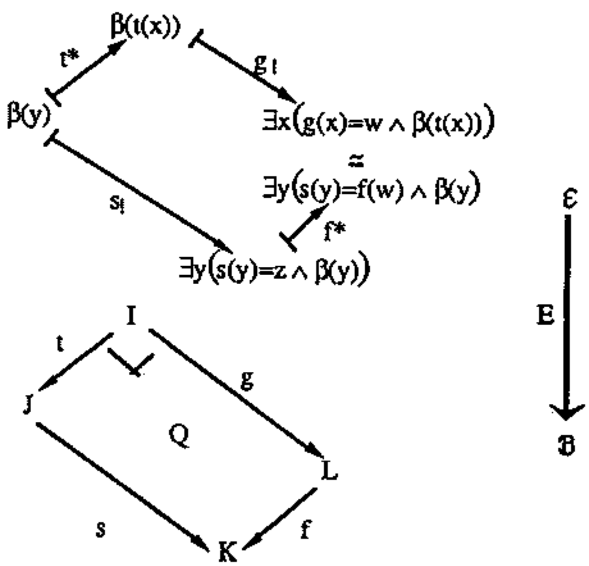
$$\pi^*: \mathcal{E}_K \longrightarrow \mathcal{E}_{K\times M}: \quad \chi(x^K) \;\longmapsto \chi(x^K, \not{z}^M)$$
$$\pi_! : \mathcal{E}_{K\times M} \longrightarrow \mathcal{E}_K : \psi(x^K, z^M) \longmapsto \exists z^M.\psi(x^K, z^M),$$
$$\pi_*: \mathcal{E}_{K\times M} \longrightarrow \mathcal{E}_K : \psi(x^K, z^M) \longmapsto \forall z^M.\psi(x^K, z^M).$$

The picture of the BC-condition is:



"The quantifier $\exists z$ and the variable y do not interfere" – says the BC-condition here. If we apply $\exists z$ on $\psi(x, \not{y}, z)$, we get the same result as when when we apply it on $\psi(x,z)$ and then add y.

Over a general pullback square Q, this picture becomes



A proof $\exists x\big(g(x)=w \wedge \beta(t(x))\big) \vdash \exists y\big(s(y)=f(w) \wedge \beta(y)\big)$ can be derived from a proof that Q is commutative:

$$\vdash f(g(x))=s(t(x)).$$

The converse proof $\exists y\big(s(y)=f(w) \wedge \beta(y)\big) \vdash \exists x\big(g(x)=w \wedge \beta(t(x))\big)$ – follows from

$$s(y)=f(w) \vdash \exists x\big(t(x)=y \wedge g(x)=w\big),$$

which tells that Q is a (weak) pullback. In this way, logic suggests the demand for an isomorphism $f^*s_!(\beta) \simeq g_!t^*(\beta)$ when Q is a pullback.

2. Interpolation condition

21. Motivation. The fact that variables do not interfere with each other can be expressed in a different way, without quantifiers:

$\alpha(x,y) \vdash \gamma(y,z) \Leftrightarrow$

there is an *interpolant* $\beta(y)$, such that $\alpha(x,y) \vdash \beta(y) \vdash \gamma(y,z)$.

At the first sight, this seems to be a *different idea* of the independence of variables. Surprisingly, it is not. We show in the sequel that the BC-property – whenever it can be expressed – is equivalent with the existence of a certain kind of interpolants.

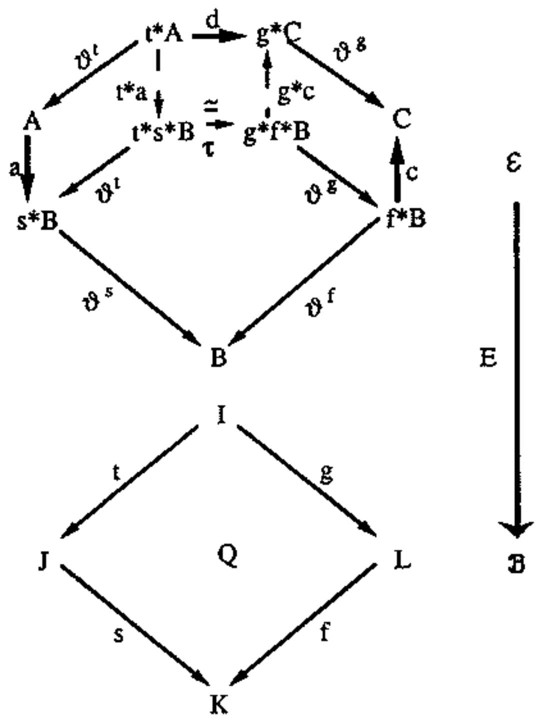
22. Notation. For a given fibration $E:\mathcal{E} \to \mathcal{B}$, an arrow $t \in \mathcal{B}(I,J)$ and an object $Y \in |\mathcal{E}_J|$ (i.e. $EY=J$), $\vartheta^{t}_{Y}: t^*Y \to Y$ denotes an *arbitrary* cartesian lifting of t at Y; if E is a cofibration (i.e. if its dual $E^o:\mathcal{E}^o \to \mathcal{B}^o$ is a fibration), then $\sigma^{t}_{X}: X \to t_!X$ will be an *arbitrary* cocartesian lifting of t at an object X over I (i.e. a cartesian lifting with respect to E^o). In general, we do not choose the whole (co)cleavages, but we do use these generic symbols for an arbitrarily chosen cartesian or cocartesian arrow. Moreover, the unique vertical arrow by which σ^{t}_{X} factorizes through $\vartheta^{t}_{t_!X}$ will be $\eta: X \to t^*t_!X$ – for the obvious reason that this arrow would be a component of the unit of the adjointness $t_! \dashv t^*$ *if* functors $t_!$ and t^* were chosen. Similarly, given a vertical arrow $q: t_!X \to Y$, we denote by $q^{\prime}: X \to t^*Y$ the unique vertical arrow such that $\vartheta^{t}_{Y} \circ q^{\prime} = q \circ \sigma^{t}_{X}$: this is the "right transpose" of q by $t_! \dashv t^*$. Given vertical $p: X \to t^*Y$, its "left transpose" ${}^{\prime}p: t_!X \to Y$ is the unique vertical arrow such that $\vartheta^{t}_{Y} \circ p = {}^{\prime}p \circ \sigma^{t}_{X}$.

The unique vertical isomorphisms between various inverse images of an object along an arrow will all be denoted by τ. E.g., if $f \circ g = s \circ t$, g^*f^*B and t^*s^*B are inverse images of B along the same arrow, and there is a unique vertical iso $\tau: t^*s^*B \to f^*g^*B$.

Note, finally, that the thick points •...• enclose the (sketches of) proofs.

23. Definition. Let $E\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration, and $Q = (f,g,s,t)$ a commutative square in $\mathcal{B}$. An (Q-)*interpolant* of an arrow $d \in \mathcal{E}_I(t^*A, g^*C)$ is a triple $\langle a,B,c \rangle$, where $B \in |\mathcal{E}_K|$, $a \in \mathcal{E}_J(A, s^*B)$, $c \in \mathcal{E}_L(f^*B, C)$, such that

$$d = g^*(c) \circ \tau \circ t^*(a).$$



A square Q in the base of a fibration E satisfies the *interpolation condition* if there is a Q-interpolant for every arrow $d \in \mathcal{E}_I(t^*A, g^*C)$.

24. Proposition. Let $E\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a bifibration. A commutative square $Q = (f,g,s,t)$ in $\mathcal{B}$ satisfies the interpolation condition <u>iff</u> the vertical arrow $\rho = \rho_A : g_!t^*A \longrightarrow f^*s_!A$ is a split mono for every $A \in |\mathcal{E}_J|$. (The arrow ρ is defined by the equation: $\vartheta^f_{s_!A} \circ \rho \circ \sigma^g_{t^*A} = \sigma^s_A \circ \vartheta^t_A$.)

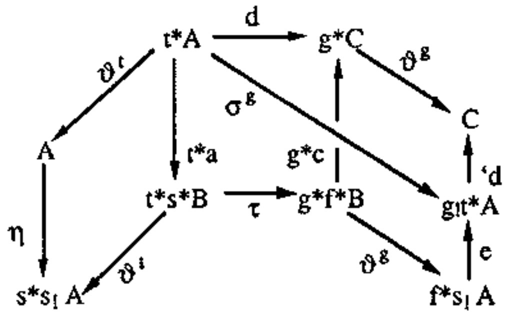
• <u>If</u>: Suppose $e \circ \rho = id$. We claim that an interpolant $\langle a,B,c \rangle$ is given by:

$B := s_!A$,

$a := \eta : A \longrightarrow s^*s_!A$,

$c := {}^{\backprime}d \circ e : f^*s_!A \longrightarrow g_!t^*A \longrightarrow C$.

On the diagram



we see that

$$\sigma^g_{t^*A} = e\circ\vartheta^g_{f^*B}\circ\tau\circ t^*(a) \Rightarrow d = g^*(c)\circ\tau\circ t^*(a).$$

But the antecedens of this implication is a consequence of the fact that $e\circ\rho = \mathrm{id}$ and of lemma 25 below.

<u>Then</u>: Let $\langle a_\eta, B_\eta, c_\eta\rangle$ be an interpolant of the "unit"

$$\eta : t^*A \longrightarrow g^*g_!t^*A,$$

(defined as in 22) and let $`a_\eta : s_!A \longrightarrow B$ be the "left transpose" of a_η. By lemma 26, the arrow

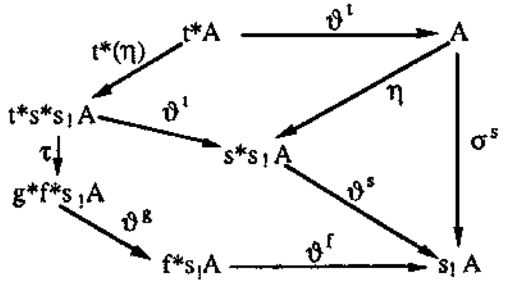
$$e := c_\eta\circ f^*(`a_\eta) : f^*s_!A \longrightarrow f^*B \longrightarrow g_!t^*A$$

is then a left inverse of ρ.•

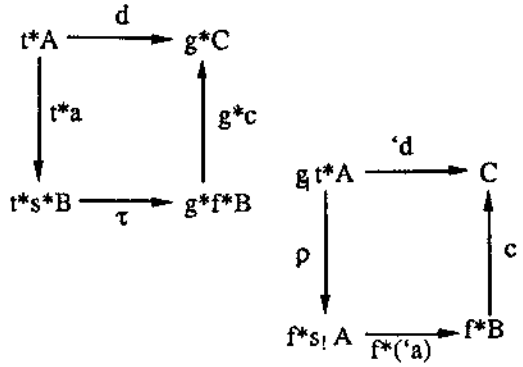
Lemmas. The following statements are true for any bifibration E.

25. $\rho\circ\sigma^g = \vartheta^g\circ\tau\circ t^*(\eta)$.

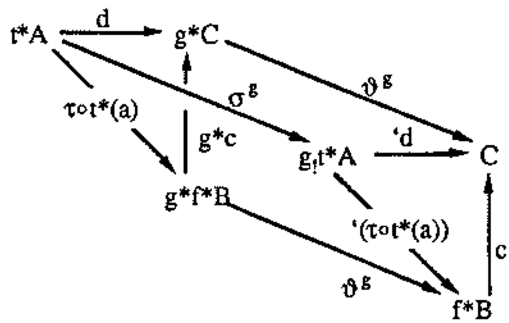
• By the definition of ρ, the left side is the unique factorisation over g of $\sigma^s_A\circ\vartheta^t_A$ through $\vartheta^f_{s_!A}$. But the right side is such a factorisation too, as the next diagram shows.



26. Each of two squares below commutes <u>iff</u> the other one does.

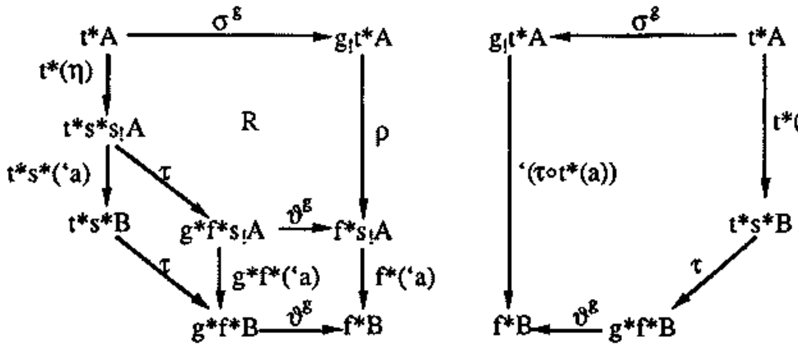


• In the following diagram, each of the triangles clearly commutes <u>iff</u> the other one does.



Thus we are done if we prove $f^*(`a)\circ\rho = `(\tau\circ t^*(a))$.

As for this equality, compare the following two diagrams:



The pentangle R commutes by lemma 25, the rest by definitions. It is easy to see that $a = s^*(`a)\circ\eta$; hence $t^*(a) = t^*s^*(`a)\circ t^*(\eta)$. The arrows $`(\tau\circ t^*(a))$ and $f^*(`a)\circ\rho$ are thus the vertical factorizations of the same arrow $\vartheta^g \circ \tau \circ \iota^*(a) = \vartheta^g \circ \tau \circ \iota^* s^*(`a) \circ \iota^*(\eta)$ through σ^g. By the uniqueness, they must be equal.•

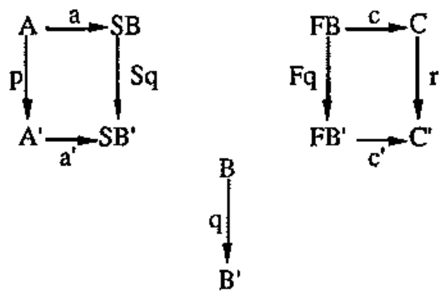
27. Corollary. For fibred preorders, the BC-condition is equivalent with the interpolation condition.

28. Remark. The connection of interpolation and the Beck-Chevalley condition in the category of Heyting algebras has been noticed by A.M. Pitts (1983a). He also studied the interpolation condition for a special sort of fibred Heyting algebras (1983b), showing how Craig's Interpolation Theorem and Beth's Definability Theorem can be presented in this setting.

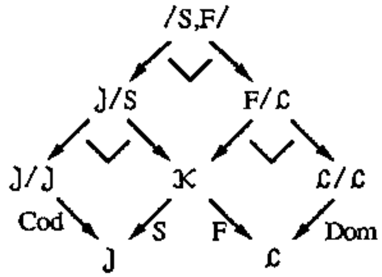
3. Uniform interpolation

31. Definition. For functors $S:\mathcal{K} \to \mathcal{J}$ and $F:\mathcal{K} \to \mathcal{C}$, the *category of interpolants* $/S,F/$ consists of:

- the triples $\langle a,B,c \rangle$, where $B \in |\mathcal{K}|$, $a \in \mathcal{J}(A, SB)$, $c \in \mathcal{C}(FB,C)$;
- a morphism $\langle a,B,c \rangle \to \langle a',B',c' \rangle$ is a triple $\langle p,q,r \rangle$, such that the squares on the following diagram commute.



32. Comments. $/S,F/$ (with obvious projections) is a certain type of lax limit of the diagram $\mathcal{J} \xleftarrow{S} \mathcal{K} \xrightarrow{F} \mathcal{C}$ in the category of categories. It can also be obtained by strict pullbacks, as the next picture shows.



Given functors $T:\mathcal{J} \longrightarrow \mathcal{I}$ and $G:\mathcal{L} \longrightarrow \mathcal{I}$, every natural transformation $\varphi:TS \longrightarrow GF$ induces a functor from the category of interpolants of S and F to the comma category of T and G:

$$R: /S,F/ \longrightarrow T/G: \langle a, B, c\rangle \longmapsto \langle A, Gc\circ\varphi_B\circ Ta, C\rangle$$
$$\langle p, q, r\rangle \longmapsto \langle p, r\rangle.$$

In the obvious sense, $\langle a,B,c\rangle$ is an interpolant of $R\langle a,B,c\rangle$, relative to φ.

33. Definition. Given a square of functors $Q=(F,G,S,T)$ as above, with a natural transformation $\varphi:TS \longrightarrow GF$, we define *initial interpolants* to be those objects of $/S,F/$ which are initial among the interpolants of the same arrow. In other words, $\langle a, B, c\rangle$ is an initial interpolant <u>if</u> for any other interpolant $\langle a',B',c'\rangle$, such that $R\langle a',B',c'\rangle = R\langle a,B,c\rangle$, there is unique $q\in \mathcal{K}(B, B')$, with

$$a' = Sq\circ a \ \underline{\text{and}}\ c'\circ Fq = c.$$

We consider, thus, the initiality in R-fibres.

We say that the *interpolation in* Q *is uniform* if an initial interpolant can be recognized by the first component. In other words, with uniform interpolation, an interpolant $\langle a, B, c\rangle$ must be initial <u>whenever</u> for every $\langle a',B',c'\rangle$, such that $R\langle a',B',c'\rangle = R\langle a,B,c\rangle$, there is unique $q\in \mathcal{K}(B, B')$, with

$$a' = Sq\circ a.$$

34. Back to fibrations. For a fibration $E: \mathcal{E} \longrightarrow \mathcal{B}$ and a commutative square $Q=(f,g,s,t)$ in $\mathcal{B}$, the categories $/s^*,f^*/$ obtained for various choices of the inverse image functors s^* and f^* are all isomorphic. Moreover, these categories are equivalent with the category of *all* the interpolants for all the possible inverse images along s and f. Similarly, the comma category t^*/g^* for some chosen t^* and g^* is equivalent with the category of triples $\langle A, d, C\rangle$, where $d: t^*A \longrightarrow g^*C$ is a vertical arrow from an *arbitrary* inverse image of A along t to an *arbitrary* inverse image of C along g.

Note that Q satisfies the interpolation condition <u>if</u> the functor $R:/s^*,f^*/ \longrightarrow t^*/g^*$ induced by the canonical isomorphism $\tau: t^*s^* \longrightarrow g^*f^*$ is a retraction, i.e. if there is a functor

$M: t^*/g^* \longrightarrow /s^*,f^*/$

such that RM=id. This functor M gives a choice of initial interpolants if it is left adjoint to R. On the other hand, the interpolation condition does not mention the arrows of t^*/g^*, so that it does not seem to imply the existence of M.

35. Terminology. We shall say that a commutative square Q in the base of a fibration satisfies the *uniform interpolation condition* if it satisfies the interpolation condition, and if the interpolation over it is uniform.

For the next proposition – which explains what is uniform about the uniform interpolation – we need the notion of *trifibration.* The notion of *bifibration* is standard: A functor $E:\mathcal{E} \longrightarrow \mathcal{B}$ is a bifibration if both E and its dual $E^o:\mathcal{E}^o \longrightarrow \mathcal{B}^o$ are fibrations. (The dual categories and functors are, of course, obtained by formally changing the directions of all morphisms.) We say that E is a *trifibration* if both $E:\mathcal{E} \longrightarrow \mathcal{B}$ and $E^{op}: \mathcal{E}^{op} \longrightarrow \mathcal{B}$ are bifibrations. The category $\mathcal{E}^{op}$, fibred over $\mathcal{B}$, is obtained by changing the direction of all the *vertical* arrows in $\mathcal{E}$. (The arrows of $\mathcal{E}^{op}$ are the equivalence classes of spans with a vertical arrow pointing at the source, and a cartesian arrow pointing at the target.)

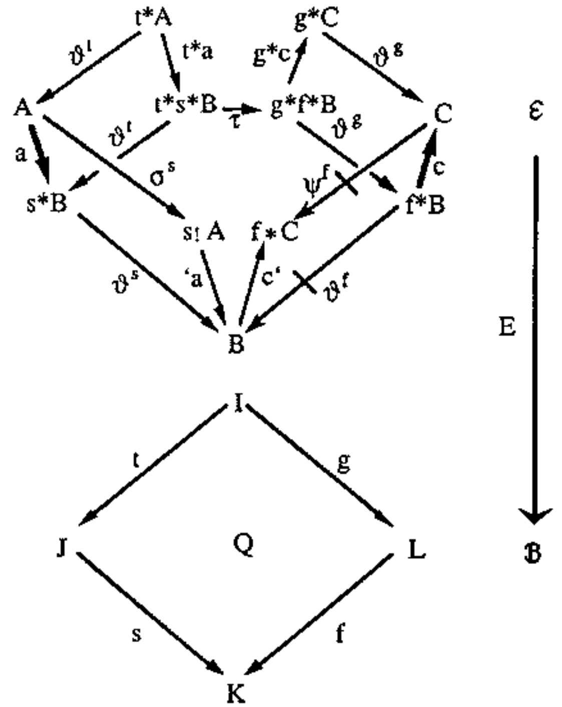
A fibration $E:\mathcal{E} \longrightarrow \mathcal{B}$ is a bifibration <u>iff</u> every inverse image functor $t^*: \mathcal{E}_J \longrightarrow \mathcal{E}_I$ has a left adjoint $t_!: \mathcal{E}_I \longrightarrow \mathcal{E}_J$. It is a trifibration <u>iff</u> there is also a right adjoint $t_*: \mathcal{E}_I \longrightarrow \mathcal{E}_J$ of t^*.

A cocartesian lifting by E^{op} of t at X is generically denoted $\psi^t: X\backslash o(\longrightarrow, \mid\ )t_*X$. (The barred arrows $\not\rightarrow$ always belong to $\mathcal{E}^{op}$.)

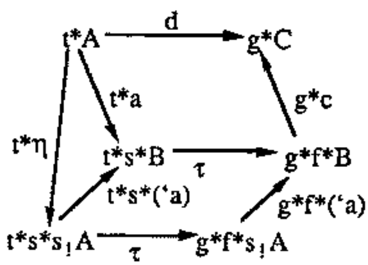
36. Proposition. Let $E: \mathcal{E} \longrightarrow \mathcal{B}$ be a trifibration, and Q = (f,g,s,t) a commutative square in $\mathcal{B}$, satisfying the interpolation condition. The interpolation is uniform <u>iff</u>

$c`\circ`a = \widetilde{c}`\circ`\widetilde{a}$

is true for any two interpolants $\langle a, B, c\rangle$ and $\langle \widetilde{a}, \widetilde{B}, \widetilde{c}\rangle$ of the same arrow.



• <u>Then</u>: If $\langle a,B,c\rangle$ is an interpolant of d, the triple $\langle \eta, s_!A, c\circ f^*(`a)\rangle$ is another interpolant of d – since $a = s^*(`a)\circ\eta$, so that the next diagram commutes.

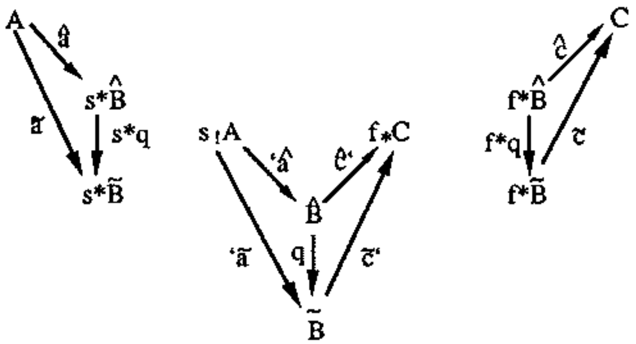


It follows from the uniformity that $\langle \eta, s_!A, c\circ f^*(`a)\rangle$ is an initial interpolant. Hence an arrow

$$\langle id_A, `\tilde{a}, id_C\rangle : \langle \eta, s_!A, c\circ f^*(`a)\rangle \longrightarrow \langle \tilde{a}, \tilde{B}, \tilde{c}\rangle$$

for each interpolant $\langle \tilde{a}, \tilde{B}, \tilde{c}\rangle$ of d.

On the other hand, for any two interpolants $\langle \hat{a}, \hat{B}, \hat{c}\rangle$, $\langle \tilde{a}, \tilde{B}, \tilde{c}\rangle$ of d, the existence of an arrow $\langle id_A, q, id_C\rangle : \langle \hat{a}, \hat{B}, \hat{c}\rangle \longrightarrow \langle \tilde{a}, \tilde{B}, \tilde{c}\rangle$ (in $/s^*,f^*/$), <u>implies</u> $\tilde{c}`\circ`\tilde{a} = \hat{c}`\circ`\hat{a}$.



Putting $\hat{a} := \eta$ and $\hat{c} := c \circ f^*(\text{‘}a)$, we get

$$\tilde{c}\text{‘} \circ \text{‘}\tilde{a} = (c \circ f^*(\text{‘}a))\text{‘} \circ \text{‘}\eta = (c \circ f^*(\text{‘}a))\text{‘} = c\text{‘} \circ \text{‘}a.$$

<u>If</u>: Let $\langle a,B,c \rangle$ be an interpolant such that for every interpolant $\langle \tilde{a},\tilde{B},\tilde{c} \rangle$ of the same arrow there is unique $\tilde{q} \in \mathcal{E}_K(B,\tilde{B})$ with $\tilde{a} = s^*(\tilde{q}) \circ a$. We must prove that $\tilde{c} \circ f^*(\tilde{q}) = c$.

Since $\langle \eta, s_!A, c \circ f^*(\text{‘}a) \rangle$ is an interpolant of the same arrow, there is $q_\eta \in \mathcal{E}_K(B, s_!A)$, such that $\eta = s^*(q_\eta) \circ a$. It is easy to see that

1) $q_\eta \circ \text{‘}a = id$ <u>and</u> $\text{‘}a \circ q_\eta = id$.

On the other hand, from $s^*(\tilde{q}) \circ a = \tilde{a} : A \longrightarrow s^*\tilde{B}$ follows $\tilde{q} \circ \text{‘}a = \text{‘}\tilde{a} : s_!A \longrightarrow \tilde{B}$. Using (1), we get

2) $\tilde{q} = \text{‘}\tilde{a} \circ q_\eta$.

Finally, the "left transpose" of $c\text{‘} \circ \text{‘}a$: $s_!A \longrightarrow f_*C$ is $c \circ f^*(\text{‘}a)$: $f^*s_!A \longrightarrow C$; the hypothesis $c\text{‘} \circ \text{‘}a = \tilde{c}\text{‘} \circ \text{‘}\tilde{a}$ <u>implies</u>

3) $c \circ f^*(\text{‘}a) = \tilde{c} \circ f^*(\text{‘}\tilde{a})$.

Now we can derive:

$$\tilde{c} \circ f^*(\tilde{q}) \overset{(2)}{=} \tilde{c} \circ f^*(\text{‘}\tilde{a} \circ q_\eta) \overset{(3)}{=} c \circ f^*(\text{‘}a \circ q_\eta) \overset{(1)}{=} c. \bullet$$

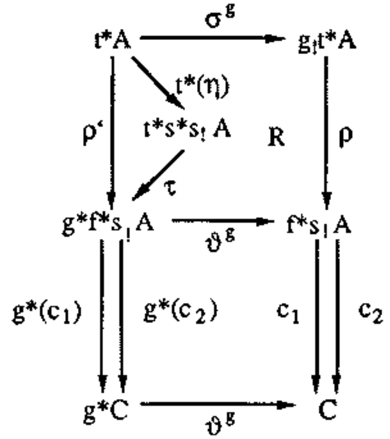
37. Theorem. A commutative square $Q = (f,g,s,t)$ in the base of a bifibration $E: \mathcal{E} \longrightarrow \mathbb{B}$ satisfies the uniform interpolation condition <u>iff</u> it satisfies the Beck-Chevalley condition.

• In section 2 we proved that the interpolation condition is satisfied <u>iff</u> $\rho: g_!t^*A \longrightarrow f^*s_!A$ is a split mono. It is now sufficient to show that the interpolation is uniform <u>iff</u> $\rho: g_!t^*A \longrightarrow f^*s_!A$ is an epi.

<u>If</u>: The <u>if</u>-part of the previous proof can be copied almost completely. The only difference is that equality (3) must be derived in a different way this time: namely, from lemma 26 and the fact that $\rho: g_!t^*A \longrightarrow f^*s_!A$ is an epi.

<u>Then</u>: For arbitrary arrows c_1, c_2: $f^*s_!A \longrightarrow C$, the equivalence

<u>Then</u>: For arbitrary arrows c_1, c_2: $f^*s_!A \longrightarrow C$, the diagram



shows that

$$c_1 \circ \rho = c_2 \circ \rho \Leftrightarrow g^*(c_1) \circ \tau \circ t^*(\eta) = g^*(c_2) \circ \tau \circ t^*(\eta).$$

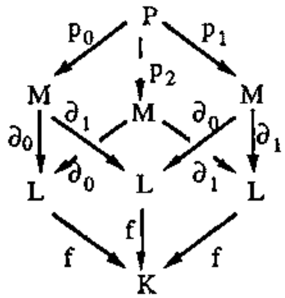
But this means that $c_1 \circ \rho = c_2 \circ \rho$ <u>implies</u> that $\langle \eta, s_!A, c_1 \rangle$ and $\langle \eta, s_!A, c_2 \rangle$ must be interpolants of the same arrow. The uniformity now <u>implies</u> that these interpolants must be initial; thus $c_1 = c_2$. So we derived that $c_1 \circ \rho = c_2 \circ \rho$ <u>implies</u> $c_1 = c_2$.•

4. Descent by interpolation

41. Geometric motivation. Let $\mathfrak{B}$ be a site (cf. Artin et al.), and D: $\mathfrak{D} \longrightarrow \mathfrak{B}$ a discrete fibration. If $f: L \longrightarrow K$ is a covering morphism in $\mathfrak{B}$, and $\partial_0, \partial_1: M \longrightarrow L$ its *kernel pair* – obtained by pulling back f along itself – the sheaf condition (ibid.) on D tells that for every $A \in \mathfrak{D}_L$, such that $\partial_0^*A = \partial_1^*A$, there must exist a unique $B \in \mathfrak{D}_K$ with $A = f^*B$. In other words, every "vertical arrow" $\partial_0^*A \longrightarrow \partial_1^*A$ (which is of course identity, since the fibration D is discrete) must have a unique interpolant over the kernel square $(f, \partial_0, f, \partial_1)$ of f. For arbitrary covering family $\{f_n: L_n \longrightarrow K \mid n \in N\}$, the sheaf condition can be expressed by saying that every family of "vertical arrows" $\{{}^n_m\partial_0^*A_n \longrightarrow {}^n_m\partial_1^*A_m \mid n,m \in N\}$ determines a unique common interpolant $B \in \mathfrak{D}_K$. The arrows ${}^n_m\partial_0: M_{nm} \longrightarrow L_n$ and ${}^n_m\partial_1: M_{nm} \longrightarrow L_m$ here are obtained in a pullback of f_n and f_m.

The notion of *descent* lifts the sheaf condition from the discrete fibrations to fibrations E: $\mathcal{E} \longrightarrow \mathfrak{B}$ in general. For simplicity, we shall consider covering by one arrow; the passage on covering *families* only requires some more involved formulations. The question will be: When can one *descend* along a morphism $f: L \to K$, and represent $\mathcal{E}_K$ in terms of $\mathcal{E}_L$? If this is possible, f is said to be an *effective descent* morphism.

42. Notation, terminology. To fix the notation, consider the following cube of pullback squares.



A *kernel square* of f is a pullback $(f) := (f, \partial_0, f, \partial_1)$. Note that the diagram above contains not only (f), but also (∂_0) and (∂_1). Three different pullback squares are obtained by pulling back (f) along f: (∂_0), (∂_1) and $S := (\partial_0, p_1, \partial_1, p_0)$.

By $\eta: L \to M$ we shall now denote the unique arrow such that $\partial_0 \circ \eta = \partial_1 \circ \eta = \mathrm{id}_L$. For cartesian liftings of ∂_0 we shall use $\vartheta^0: \partial_0{}^*A \to A$ (instead of ϑ^{∂_0}). Let $\upsilon^0: A \to \partial_0{}^*A$ be the unique splitting of ϑ^0 over η (i.e., $\vartheta^0 \circ \upsilon^0 = \mathrm{id}$ and $E\upsilon^0 = \eta$). It is straightforward to show that υ^0 must be cartesian. – Idem for the splitting υ^1 over η of the cartesian lifting ϑ^1 of ∂_1.

Given a fibration $E: \mathcal{E} \to \mathcal{B}$ and an (f)-interpolant $\langle a,B,c \rangle$ of $d \in \mathcal{E}_M(\partial_0{}^*A, \partial_1{}^*C)$, the triple

$$f^*\langle a,B,c \rangle := \langle \tau \circ \partial_0{}^*(a),\ f^*B,\ \partial_1{}^*(c) \circ \tau \rangle, \text{ where}$$
$$\tau \circ \partial_0{}^*(a): \partial_0{}^*A \to \partial_0{}^*f^*B \to \partial_1{}^*f^*B,$$
$$\partial_1{}^*(c) \circ \tau: \partial_0{}^*f^*B \to \partial_1{}^*f^*B \to \partial_1{}^*C,$$

is clearly an S-interpolant of

$$d^S := \tau \circ p_2{}^*(d) \circ \tau: p_0{}^*\partial_0{}^*A \to p_2{}^*\partial_0{}^*A \to p_2{}^*\partial_1{}^*C \to p_1{}^*\partial_1{}^*C.$$

We say that the interpolant $\langle a,B,c \rangle$ is f-*simple* if

$$\langle \mathrm{id}, \mathrm{id}, \mathrm{id} \rangle: f^*\langle a,B,c \rangle \to f^*\langle a,B,c \rangle$$

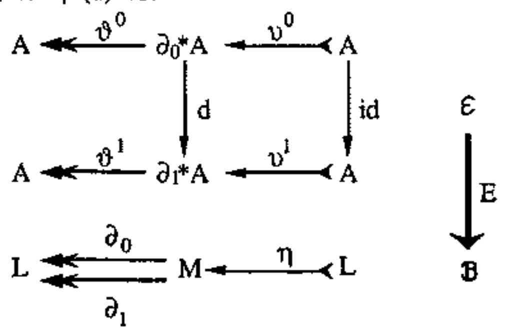
is the only arrow in the category of S-interpolants. In other words, the object $f^*\langle a,B,c \rangle$ has just one endomorphism in its R-fibre.

(In general, every pullback square Q and arrow q with the same target span a cube of pullbacks as above; and every Q-interpolant induces an interpolant over the square opposite to Q in this cube. So we can speak of q-simple interpolants in this general situation. For Q:=(f) and q:=f, we should actually consider two more interpolants induced by $\langle a,B,c \rangle$: namely, those over (∂_0)

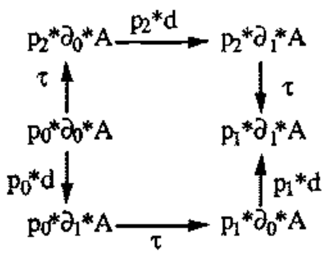
and (∂_1) – and include them in the definition of f-simplicity. This would, however, only add a couple of inessential sentences to the proofs below.)

43. Definition. Let a fibration E: $\mathcal{E} \longrightarrow \mathcal{B}$ and an arrow $f \in \mathcal{B}(L,K)$ be given. An f-*descent data* (for E) is a pair $\langle A,d \rangle$, $A \in |\mathcal{E}_L|$, $d \in \mathcal{E}_M(\partial_0{}^*A, \partial_1{}^*A)$, such that

RE) $\vartheta^1 \circ d \circ \upsilon^0 = id$; i.e. $\eta^*(d) = id$:



TR) $\langle d,A,d \rangle$ is an S-interpolant of $d' := \tau \circ p_2{}^*(d) \circ \tau$: $p_0{}^*\partial_0{}^*A \longrightarrow p_1{}^*\partial_1{}^*A$; i.e., the following diagram commutes



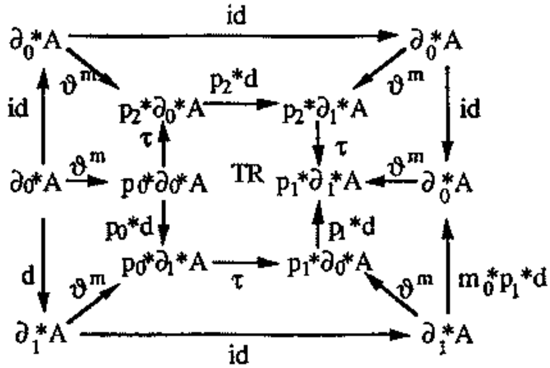
The descent data constitute a category, denoted $\underline{Des}_E(f)$, or just $\underline{Des}(f)$. A morphism $\langle A,d \rangle \longrightarrow \langle \tilde{A},\tilde{d} \rangle$ in this category is an arrow $h \in \mathcal{E}_L(A,\tilde{A})$, such that $\partial_1{}^*(h) \circ d = \tilde{d} \circ \partial_0{}^*(h)$.

44. Remarks. The arrow d occuring as a descent data must be an isomorphism. • Since $\partial_0 = \partial_0 \circ \eta \circ \partial_0$, and the square $(\partial_0, p_0, \partial_0, p_2)$ is a pullback, there is a unique arrow $m_0: M \longrightarrow P$ in $\mathcal{B}$, such that $p_0 \circ m_0 = id$ <u>and</u> $p_2 \circ m_0 = \eta \circ \partial_0$. It is routine to show that

$$\partial_0 \circ p_2 \circ m_0 = \partial_1 \circ p_2 \circ m_0 = \partial_0 \circ p_0 \circ m_0 = \partial_1 \circ p_1 \circ m_0 = \partial_0, \text{ and}$$

$$\partial_1 \circ p_0 \circ m_0 = \partial_0 \circ p_1 \circ m_0 = \partial_1.$$

If the inverse images of objects along m_0 are appropriately chosen, one obtains the following diagram.



The commutativity of the part TR now <u>implies</u>

$$m_0{}^*p_1{}^*(d)\circ d = id.$$

The arrow d is thus a split mono. An analogous argument with the arrow $m_1: M \longrightarrow P$, such that $p_1 \circ m_1 = id$ <u>and</u> $p_2 \circ m_1 = \eta \circ \partial_1$, shows that d is split epi too.•

An f-descent data for E is, in a sense, an action in $\mathcal{E}$ of the equivalence relation (internal groupoid) $\partial_0, \partial_1: M \longrightarrow L$ induced by f. Namely, $\langle A,d \rangle$ can be viewed as an internal groupoid in $\mathcal{E}$, with $\vartheta^0, \vartheta^1 \circ d: \partial_0{}^*A \longrightarrow A$ as the domain and codomain arrows, and with $\upsilon^0: A \longrightarrow \partial_0{}^*A$ as the arrow of identities. The category <u>Des</u>(f) is equivalent with the category of internal groupoids in $\mathcal{E}$ which are built from cartesian liftings of ∂_0, ∂_1, η etc. (Cf. Pavlović 1990, III.1.2.)

Note that for every $B \in |\mathcal{E}_K|$, the pair $\langle f^*B, \tau_B \rangle$ is a descent data (where f^*B is any inverse image, while $\tau_B: \partial_0{}^*f^*B \longrightarrow \partial_1{}^*f^*B$ is the vertical isomorphism between two inverse images of B along $f \circ \partial_0 = f \circ \partial_1$). Every arrow $b \in \mathcal{E}_K(B,B')$, induces a unique morphism $f^*b: \langle f^*B, \tau_B \rangle \longrightarrow \langle f^*B', \tau_{B'} \rangle$ in <u>Des</u>(f). Every choice of inverse image functors f^*, $\partial_0{}^*$ and $\partial_1{}^*$ determines a functor

$$f^\# : \mathcal{E}_K \longrightarrow \underline{Des}(f) : B \longmapsto \langle f^*B, \tau_B \rangle.$$

45. Definition. (Grothendieck 1959) An f-descent data $\langle A,d \rangle$ (for E) is *effective* if it is isomorphic (in $\underline{Des}_E(f)$) with one in the form $\langle f^*B, \tau_B \rangle$. An arrow f is said to be *effective* if all the f-descent data are effective.

f is a *descent* morphism if for all B, $\tilde{B} \in |\mathcal{E}_K|$ each arrow $\langle f^*B, \tau_B \rangle \longrightarrow \langle f^*\tilde{B}, \tau_{\tilde{B}} \rangle$ in $\underline{Des}_E(f)$ is in the form f^*b for a unique $b \in \mathcal{E}_K(B, \tilde{B})$.

In other words, f is effective <u>iff</u> each $f^\#$ is essentially surjective; f is a descent morphism <u>iff</u> each $f^\#$ is full and faithful; f is an *effective descent* morphism <u>iff</u> each $f^\#$ is an equivalence of categories.

46. Terminology. An interpolant $\langle a,B,c\rangle$ of a descent data $d:\partial_0{}^*A \to \partial_1{}^*A$ is *natural* if $a: A \to f^*B$ and $c: f^*B \to A$ are morphisms of descent data, i.e. if they satisfy

$\partial_1{}^*(a)\circ d = \tau_B\circ\partial_0{}^*(a)$ <u>and</u> $\partial_1{}^*(c)\circ\tau_B = d\circ\partial_0{}^*(c)$.

An arrow φ from a fibred category $\mathcal{E}$ is an E-*coequalizer* of a pair δ_0,δ_1 of parallel arrows if

- $\varphi\circ\delta_0 = \varphi\circ\delta_1$ <u>and</u>
- for every χ in $\mathcal{E}$, $\chi\circ\delta_0 = \chi\circ\delta_1$ <u>and</u> $E\chi = E\varphi$ <u>imply</u> that there is a unique vertical arrow b with $\chi = b\circ\varphi$.

47. Theorem. Let $E: \mathcal{E} \to \mathcal{B}$ be a fibration and f an arrow in its base. (Notation as above!)

i) f is an effective morphism <u>iff</u> each f-descent data has a natural, f-simple interpolant.

ii) f is a descent morphism <u>iff</u> every cartesian lifting $\vartheta^f: f^*B \to B$ is an E-coequalizer of its kernel pair.

• **i)** <u>If</u>: Let $\langle a,B,c\rangle$ be a natural, f-simple interpolant of f-descent data $\langle A,d\rangle$. So we have $d:\partial_0{}^*A \to \partial_1{}^*A$, $a: A \to f^*B$ and $c: f^*B \to A$ such that

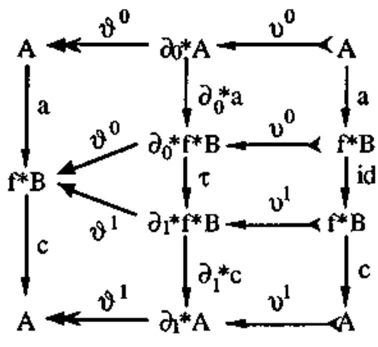
1) $d = \partial_1{}^*(c)\circ\tau\circ\partial_0{}^*(a)$,

2) $\partial_1{}^*(a)\circ d = \tau\circ\partial_0{}^*(a)$, and

3) $\partial_1{}^*(c)\circ\tau = d\circ\partial_0{}^*(c)$.

We shall prove $c\circ a = id_A$ <u>and</u> $a\circ c = id_{f^*B}$.

<u>$c\circ a = id$</u> : As before, choose the cartesian liftings υ^0 and υ^1 of η to be the splittings of ϑ^0 and ϑ^1 respectively.



We see that

$$\vartheta^1_A\circ\upsilon^1_A\circ c\circ a = \vartheta^1_A\circ\partial_1{}^*(c)\circ\tau\circ\partial_0{}^*(a)\circ\upsilon^0_A = \vartheta^1_A\circ d\circ\upsilon^0_A.$$

But now $\vartheta^1_A\circ\upsilon^1_A = id_A$ holds by the definition, while $\vartheta^1_A\circ d\circ\upsilon^0_A = id_A$ is just condition (RE).

<u>$a\circ c = id$</u> : By condition (TR), $\langle d,A,d\rangle$ is an interpolant of

$d' := \tau \circ p_2{}^*(d) \circ \tau : p_0{}^*\partial_0{}^*A \longrightarrow p_1{}^*\partial_1{}^*A.$

Of course, $f^*\langle a,B,c\rangle = \langle \tau\circ\partial_0{}^*(a), f^*B, \partial_1{}^*(c)\circ\tau\rangle$ is another interpolant of the same arrow. The equalities (2) and (1) mean that

$\langle id, a, id\rangle : \langle d,A,d\rangle \longrightarrow f^*\langle a,B,c\rangle.$

is a morphism of interpolants. On the other hand, (1) and (3) tell the same for

$\langle id, c, id\rangle : f^*\langle a,B,c\rangle \longrightarrow \langle d,A,d\rangle$

Hence, $\langle id, a\circ c, id\rangle$ is an arrow $f^*\langle a,B,c\rangle \longrightarrow f^*\langle a,B,c\rangle$. The hypothesis that $\langle a,B,c\rangle$ is f-simple now implies that

$a\circ c = id.$

Then: Suppose that $\langle A,d\rangle \cong \langle f^*B,\tau_B\rangle$ for some B; i.e., arrows $a: \langle A,d\rangle \longrightarrow \langle f^*B,\tau_B\rangle$ and $c: \langle f^*B,\tau_B\rangle \longrightarrow \langle A,d\rangle$ are given, such that $a\circ c = id$ and $c\circ a = id$. $\langle a, B, c\rangle$ is then an interpolant of d because

$d = \partial_1{}^*(c)\circ\partial_1{}^*(a)\circ d = \partial_1{}^*(c)\circ\tau\circ\partial_0{}^*(a).$

This interpolant is obviously natural. To show that it is f-simple, consider a morphism $h: f^*\langle a,B,c\rangle \longrightarrow f^*\langle a,B,c\rangle$. So we have an arrow $h\in \mathcal{E}_L(f^*B, f^*B)$, such that

$\tau\circ\partial_0{}^*(a) = \partial_1{}^*(h)\circ\tau\circ\partial_0{}^*(a)$ and $\partial_1{}^*(c)\circ\tau\circ\partial_0{}^*(h) = \partial_1{}^*(c)\circ\tau.$

Since c is an isomorphism, $\partial_1{}^*(c)$ is, and $\partial_0{}^*(h)$ must be an identity. Using this, we calculate:

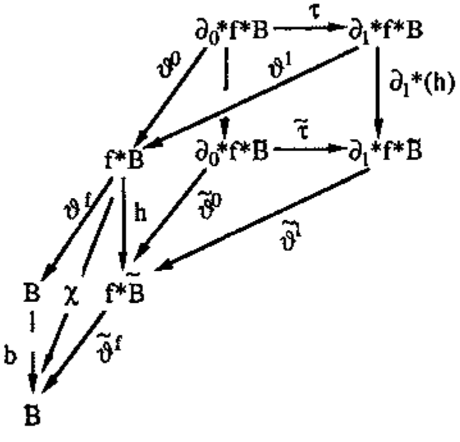
$\vartheta^f_B\circ h\circ a\circ\vartheta^0_A = \vartheta^f_B\circ\vartheta^0_{f^*B}\circ\partial_0{}^*(h)\circ\partial_0{}^*(a) = \vartheta^f_B\circ\vartheta^0_{f^*B}\circ\partial_0{}^*(a) = \vartheta^f_B\circ a\circ\vartheta^0_A,$

and conclude that $h\circ a = a$, because ϑ^0_A is an epimorphism (split by υ^0) and ϑ^f_B is cartesian, while $h\circ a$ and a are vertical arrows. Since a is an isomorphism, $h = id$.

ii) It follows from lemma 48 that $\vartheta^0, \vartheta^1\circ\tau : \partial_0{}^*f^*B \longrightarrow f^*B$ is a kernel pair of $\vartheta^f : f^*B \longrightarrow B$ (for every B).

Chasing the next diagram – where h is vertical, and $\chi = \tilde{\vartheta}^f\circ h$ – one easily proves that

$\partial_1{}^*(h)\circ\tau = \tilde{\tau}\circ\partial_0{}^*(h) \Leftrightarrow \chi\circ\vartheta^0 = \chi\circ\vartheta^1\circ\tau.$



Thus, if h is a morphism of descent data, the assumption that ϑ^f is an E-coequalizer of its kernel pair gives a unique arrow $b \in \mathcal{E}_K(B, \tilde{B})$, such that $\chi = b \circ \vartheta^f$. In other words, $h = f^*b$. Conversely, if an arrow $\chi: f^*B \longrightarrow \tilde{B}$ over f satisfies $\chi \circ \vartheta^0 = \chi \circ \vartheta^1 \circ \tau$, its vertical part h must be a morphism of descent data $\langle f^*B, \tau_B \rangle \longrightarrow \langle f^*\tilde{B}, \tau_{\tilde{B}} \rangle$. The hypothesis that f is a descent morphism means that there is a unique arrow $b \in \mathcal{E}_K(B, \tilde{B})$, such that $h = f^*b$. Thus, $\chi = b \circ \vartheta^f$.

48. Lemma. Let Q = (f,g,s,t) be a pullback square in $\mathcal{B}$ and $\Theta = (\varphi, \gamma, \sigma, \vartheta)$ a commutative square in $\mathcal{E}$ over Q (i.e., $\varphi \circ \gamma = \sigma \circ \vartheta$, $E\varphi = f$, $E\gamma = g$, $E\sigma = s$, $E\vartheta = t$). If φ and ϑ are cartesian, then Θ is a pullback square.

49. Comment. This descent theorem is a far descendant of the method which Joyal and Tierney (1984) used to prove that open surjections of toposes are effective descent morphisms – *in absence* of the Beck-Chevalley property. More recently, Moerdijk (1989) observed that an appropriately saturated class $\mathcal{O}$ of arrows in $\mathcal{B}$ must consist of effective descent morphisms with respect to the fibration Cod: $\mathcal{B}/\mathcal{B} \longrightarrow \mathcal{B}$ if it satisfies the following axioms:

i) A coequalizer of every parallel pair of arrows from $\mathcal{O}$ exists, and it is stable under pullbacks;

ii) Each arrow belonging to $\mathcal{O}$ is a coequalizer of its kernel pair.

The two parts of theorem 47 clearly correspond to these axioms.

On the other hand, it is perhaps interesting to put theorems 37 and 47 together, aligning the Beck-Chevalley property and descent – on the common ground of interpolation. This way, one can analyze how Bénabou-Roubaud-Beck's theorem (cf. Hyland-Moerdijk 1990) provides some sufficient conditions for descent *in presence* of the Beck-Chevalley property.

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This paper is in final form and will not be published elsewhere.