Notes on Serre fibrations

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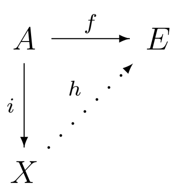
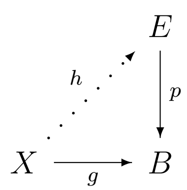
1 Introduction

Many problems in topology can be formulated abstractly as *extension problems*

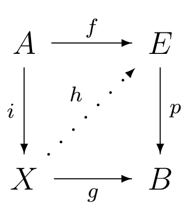


or *lifting problems*



Here the solid arrows represent maps that are given, and the problem is to find a map h commuting in the diagram. Usually the map i is an inclusion, and p is some kind of "bundle map" such as a local product. Note the special cases: (i) if i is an inclusion, $E = A$, and $f = 1_A$, then the extension problem asks whether A is a retract of X; and (ii) if p is surjective, X=B, and $g = 1_B$, the lifting problem asks whether p admits a section.

More generally, one can combine the two diagrams into one:



Here the problem is to find a map h such that both triangles commute. This situation arises, for example, when a section of a bundle is given on a subspace and we attempt to extend it to a global section. Note that we recover the extension problem by taking B to be a point, and the lifting problem by taking A to be the empty set. The map h (if it exists) is often called a *filler* or *solution* to the diagram. It is a map that simultaneously extends f and lifts g.

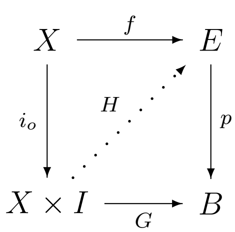
Now from a homotopy-theoretic perspective, all of these problems are ill-posed. For example, suppose that in the extension problem we have a map $f' : A \longrightarrow E$ homotopic to f, and f' extends. Then it does not follow that f extends. Similarly, if the map g in the lifting problem is homotopic to a map that lifts, it does not follow that g itself lifts. The reader can easily supply counterexamples in both cases. One motivation for *fibrations* and *cofibrations* is simply this: If i is a cofibration and f is homotopic to a map that extends, then f extends; if p is a fibration and g is homotopic to a map that lifts, then g lifts.

The dual concepts *cofibration* and *fibration* reflect a more general duality that is pervasive in homotopy theory. We will not attempt to formulate this duality precisely. Thus the word "dual", as used in these notes, has no technical meaning and should be implicitly placed in quotation marks. On the other hand, the duality is extremely useful for intuitive purposes, and the reader is urged to become familiar with it.

Note: Examples can be found in section 6, which should be read simultaneously with the previous sections.

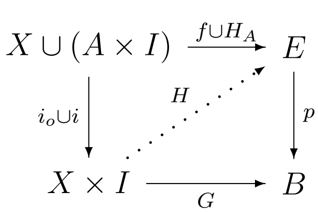
2 Definitions and basic properties

Let $p : E \longrightarrow B$ be a map of spaces. We say that p has the *homotopy lifting property* with respect to a space X if for all commutative diagrams



the filler H exists. Here I is the unit interval and i_0 is the inclusion $x \mapsto (x, 0)$. We are given a homotopy G into B and an initial map f into E; the homotopy lifting property says that we can always lift G to a homotopy H that starts with f. Note that replacing i_0 by i_1 would not change the definition.

The map p has the *relative* homotopy lifting property with respect to a pair of spaces (X, A), if, whenever we are given a diagram as above and a lift H_A already defined on $A \times I$, the lifted homotopy H can be taken to agree with H_A on $A \times I$. In other words, there is a filler H in the diagram



The map p is a *Hurewicz fibration* if it has the homotopy lifting property with respect to all spaces X, and is a *Serre fibration* if it has the homotopy lifting property with respect to all CW-complexes X. Note that any global product $X \times F \longrightarrow X$ is a Hurewicz fibration and hence also a Serre fibration.

Theorem 2.1 *Let $E \longrightarrow B$ be a local product, and suppose B is paracompact Hausdorff. Then E is a Hurewicz fibration.*

For the proof see [May], p. 49. Much easier is to show that any local product is a Serre fibration; see below.

The following properties are easily verified:

Proposition 2.2 *Any composition of Hurewicz fibrations [Serre fibrations] is a Hurewicz fibration [Serre fibration]. Any pullback of a Hurewicz fibration [Serre fibration] is a Hurewicz fibration [Serre fibration].*

Theorem 2.3 *Let $p : E \longrightarrow B$ be a map. Then the following are equivalent:*

a) p is a Serre fibration;

b) p has the homotopy lifting property with respect to all n-discs D^n;

c) p has the relative homotopy lifting property with respect to all pairs (D^n, S^{n-1});

d) p has the relative homotopy lifting property with respect to all CW-pairs (X, A).

Proof: a) $\Rightarrow$ b): This is immediate from the definitions.

b) $\Rightarrow$ c): It is visually obvious, and not hard to prove, that the pair $(D^n \times I, D^n \times 0 \cup S^{n-1} \times I)$ is homeomorphic to the pair $(D^n \times I, D^n \times 0)$. The desired implication follows easily from this.

c) $\Rightarrow$ d): Suppose that a lift H' is already given on $A \times I$. We extend H' over $X^n \times I \cup A \times I$ by induction on n. At the inductive step, we reduce to constructing a filler in a diagram of the form

$$\begin{array}{ccc} D^n \times 0 \cup S^{n-1} \times I & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ D^n \times I & \longrightarrow & B \end{array}$$

Such a filler exists by assumption (c).

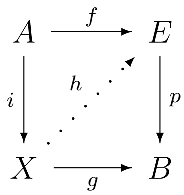
d) $\Rightarrow$ a): This is immediate, taking $A = \emptyset$.

Taking B to be a point in (d), we have incidentally proved:

Corollary 2.4 *Any CW-pair (X, A) has the homotopy extension property.*

Remark: Note that the proof of the theorem actually proves slightly more: Call $p : E \longrightarrow B$ an m-Serre fibration if p has the homotopy lifting property with respect to all CW-complexes of dimension at most m. Define relative m-Serre fibrations in the analogous way. Then the theorem remains valid when all four conditions are replaced by their evident m-analogues.

Theorem 2.5 *Consider a diagram*



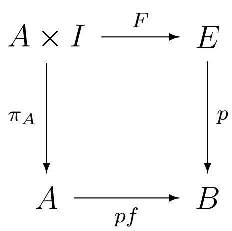
in which (X, A) is a CW-pair and p is a Serre fibration. Then

a) If g is homotopic rel A to a map g' such that a filler h' exists for g', then a filler h exists in the original diagram;

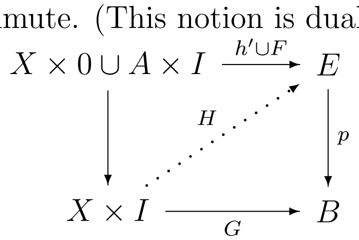
b) If f is fibrewise homotopic to a map f' such that a filler h' exists for f', then a filler h exists in the original diagram.

Proof: a) The assumption is that (i) there are maps g', h' such that $f = ih'$ and $ph' = g'$; and (ii) there is homotopy $G : X \times I \longrightarrow B$ such that $G_0 = g'$, $G_1 = g$, and $G(i(a), t) = gi(a)$ for all $a \in A$ and all t. Since G is constant on A, it can be lifted on A to the constant homotopy $H_A(i(a), t) = f(a)$. Now extend H_A to a lift H, as in part (d) of Theorem 2.3. Then $h = H_1$ is the desired filler.

b) By a *fibrewise homotopy* we mean a homotopy F that is only allowed to move f around within its fibre $p^{-1}p(f(a))$. More precisely, we require that the diagram



commute. (This notion is dual to relative homotopy.) Now consider the diagram



where F is a fibrewise homotopy that starts with h' and ends with f, and $G(x, t) = f(x)$. Note that the square commutes because F is a fibrewise homotopy. Then the filler H exists, and $h = H_1$ is the desired filler for the original diagram.

Note the following special cases:

(i) Take $A = \emptyset$ in (a). Then if g is homotopic to a map that lifts, g itself lifts.

(ii) Take B=point in (b). Then if f is homotopic to a map that extends, f itself extends. This is in fact true with i replaced by any cofibration.

In case (ii) the fibrewise homotopies of the theorem are just ordinary homotopies, so this is really a result about cofibrations. While we're on the subject, here is another important result on CW-pairs:

Theorem 2.6 *Let (X, A) be a CW-pair, with inclusion map $i : A \longrightarrow X$. Then*
a) If A is a homotopy retract of X, then A is a retract of X.
b) If i is a weak equivalence, then A is a deformation retract of X.

Proof: a) This part holds for any pair (X, A) having the homotopy extension property. By "homotopy retract" we mean that there is a map $r : X \longrightarrow A$ such that ri is homotopic to the identity. Now apply Theorem 2.5, part (b), with B=point, $f = 1_A$, and $f' = ri$.

b) By Whitehead's theorem, i is a homotopy equivalence. In particular, A is a homotopy retract of X, and so is an actual retract by part (a). Furthermore, if r is the retraction, then ir is homotopic to the identity of X. The remaining problem is to show that there is a homotopy rel A. For a proof see [Spanier], p. 31, Theorem 11 (note that his "strong deformation retract" is my "deformation retract"), or prove it yourself. (Hint: Start from a homotopy $ir \sim 1_X$ that may not be a homotopy rel A, and use the homotopy extension property to construct a homotopy of homotopies ending with the desired homotopy rel A. Here you only need the homotopy extension property for the pair $(X \times I, X \times 0 \cup A \times I \cup X \times 1)$; this is automatic for in the case of a CW-pair.)

3 Remarks on path-components

If X is an arbitrary topological space, with path-components X_α, then the natural map $f : \coprod X_\alpha \longrightarrow X$ is a continuous bijection, but not in general a homeomorphism. For example, if X is totally disconnected (e.g., the Cantor set), then $\coprod X_\alpha$ is just X with the discrete topology. To get a homeomorphism we would need the path-components of X to be open sets; this is true notably when X is locally path-connected. On the other hand, it is clear that f is always a weak equivalence.

Now suppose that $p : E \longrightarrow B$ is a Serre fibration. Let $\{B_\alpha\}$ denote the path-components of B, and let $\{E_{\alpha\beta}\}$ denote the path-components of $p^{-1}B_\alpha$ for each fixed α. Then there is a pullback diagram

$$\begin{array}{ccc} \coprod_{\alpha,\beta} E_{\alpha\beta} & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \coprod_{\alpha} B_\alpha & \longrightarrow & B \end{array}$$

in which the horizontal maps are weak equivalences (homeomorphisms if E and B are locally path-connected) and each individual map $E_{\alpha\beta} \longrightarrow B_\alpha$ is a Serre fibration. In this way, most questions about Serre fibrations are easily reduced to the case of a path-connected base and path-connected total space. We cannot assume the fibres are connected, however—think of a covering map, for example.

Note also that for any space B, the unique map from the empty set to B is a Hurewicz fibration and hence also a Serre fibration. This shows that fibrations need not be surjective. But if p is a Serre fibration and b is in the image of p, then the entire path-component of b must be in the image; this follows immediately from the definition, interpreting paths as homotopies of a point.

4 The exact homotopy sequence of a Serre fibration

A *pointed Serre fibration* is a Serre fibration $p : E \longrightarrow B$ equipped with basepoints $e_0 \in E$, $b_0 \in B$ such that $p(e_0) = b_0$. In this case we refer to $F = p^{-1}b_0$ as *the* fibre of p, and write $i : F \longrightarrow E$ for the inclusion.

Theorem 4.1 *Let $p : E \longrightarrow B$ be a pointed Serre fibration with fibre F. Then there is a natural long exact sequence*

$$\longrightarrow \pi_n F \xrightarrow{i} \pi_n E \xrightarrow{p} \pi_n B \xrightarrow{\partial} \pi_{n-1} F \longrightarrow$$

Remarks: a) This is a long exact sequence of groups as far as $\pi_1 E$. The sequence ends with

$$\pi_1 E \longrightarrow \pi_1 B \longrightarrow \pi_0 F \longrightarrow \pi_0 E \longrightarrow \pi_0 B$$

which is exact as pointed sets. Note that the last map need not be onto, since B could have path-components B_α such that $p^{-1}B_\alpha$ is empty. Frequently, however, E and B are path-connected, or at least we can easily reduce to that case by considering one component at a time. Hence the sequence typically ends with $\pi_1 B \longrightarrow \pi_0 F \longrightarrow 0$. For $n \geq 1$, only the basepoint components of B and E are relevant.

b) The sequence is natural in the sense that a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \downarrow & & \downarrow \\ E' & \xrightarrow[p']{} & B' \end{array}$$

leads to a commutative ladder of long exact sequences.

We will derive this sequence from the long exact sequence of a pair, so we begin by constructing the latter. Let I^n denote the n-cube, and for $n \geq 2$ let $bI^n \subset \partial I^n$ denote $\partial I^n - U$, where U is the interior of the "bottom" face $x_n = 0$. In other words, bI^n is like a cardboard box with the bottom cut out but with top and sides left intact. For $n = 1$ we set $bI^1 = \{0\}$. Now let (X, A) be a pointed pair. This means that X is pointed, A is a subspace of X, and the basepoint lies in A. The basepoint is denoted $*$, although it will be omitted from the notation entirely when no confusion can arise.

Now define the relative homotopy set $\pi_n(X, A)$ for $n \geq 1$ by

$$\pi_n(X,A) = [(I^n, \partial I^n, bI^n), (X, A, *)]$$

In other words, we take homotopy classes of maps of triples $(I^n, \partial I^n, bI^n) \longrightarrow (X, A, *)$, where the homotopies are required to keep ∂I^n in A and bI^n at the basepoint. For $n = 0$ we define $\pi_0(X,A) = \pi_0 X/\pi_0 A$. Observe that $\pi_n(X,*)$ is just the usual $\pi_n(X,*)$.

Note that $\pi_1(X,A)$ consists of homotopy classes of paths that start at $*$ and end in A. The homotopies must keep the initial point of the path at $*$ at each stage, but the endpoint is free to move around inside A. It should be clear that there is no reasonable way to concatenate such paths; thus $\pi_1(X,A)$ is only a pointed set, not a group.

If $n \geq 2$, on the other hand, we can define a product structure by the usual formula

$$(f * g)(x_1, ..., x_n) = \begin{cases} f(2x_1, ..., x_n) & x_1 \leq 1/2 \\ g(2x_1 - 1, ..., x_n) & x_1 \geq 1/2 \end{cases}$$

It is easy to check that this formula gives a map of triples as required. The reader should also check to see why it doesn't work for $n = 1$.

Proposition 4.2 *The product $f * g$ is well-defined on homotopy classes in $\pi_n(X,A)$ for $n \geq 2$, and gives $\pi_n(X,A)$ a group structure with identity element the constant map. This group structure is abelian for $n \geq 3$.*

Proof: The proof of the first statement is identical to the corresponding proof for the fundamental group, since all the action takes place in the first coordinate. The proof of the second statement is identical to the proof that $\pi_n(X,*)$ is abelian for $n \geq 2$. Note that the latter argument requires two degrees of freedom; we need $n \geq 3$ in the relative case because x_n is tied down by its special role in the definition of $\pi_n(X,A)$.

It is clear that $\pi_n(-,-)$ defines a functor from pointed pairs to sets (if $n \geq 0$), to groups (if $n \geq 2$), and to abelian groups (if $n \geq 3$). Note in particular that we have maps of pairs $i : (A,*) \longrightarrow (X,*)$ and $j : (X,*) \longrightarrow (X,A)$.

Given a map of triples $f : (I^n, \partial I^n, bI^n) \longrightarrow (X, A, *)$, the restriction of f to the bottom face $x_n = 0$ is a map of pairs $(I^{n-1}, \partial I^{n-1}) \longrightarrow (A, *)$. We denote this map by ∂f. It is immediate on inspection that the assignment $f \mapsto \partial f$ is well-defined on homotopy classes, yielding a map $\pi_n(X,A) \longrightarrow \pi_{n-1}A$ for which the same notation will be used. Furthermore, it is immediate that for $n \geq 3$, $\partial(f * g) = \partial f * \partial g$ on the nose—that is, no homotopies are required. Thus ∂ is a group homomorphism for $n \geq 3$.

Theorem 4.3 *There is a long exact sequence of groups (pointed sets for $n \leq 1$)*

$$\longrightarrow \pi_n A \xrightarrow{i} \pi_n X \xrightarrow{j} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}A \longrightarrow$$

This sequence is natural with respect to maps of pointed pairs.

Proof: The maps in the sequence have already been defined, and it is obvious that ∂ is natural with respect to maps of pointed pairs. What remains to be shown is the exactness. This is not hard, but somewhat tedious since there are so many things to check. We will sketch what is needed and leave further details to the reader. We recommend that the reader try to prove the theorem herself before even looking at the sketch.

Consider first the exactness of

$$\pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}A \xrightarrow{i} \pi_{n-1}X$$

A map of triples $(I^n, \partial I^n, bI^n) \longrightarrow (X, A, *)$ is the same thing as a map of pairs $g : (I^{n-1}, \partial I^{n-1}) \longrightarrow (A, *)$ together with a nullhomotopy of ig. Hence the exactness at $\pi_{n-1}A$ is immediate from the definitions.

Next, consider the segment

$$\pi_n X \xrightarrow{j} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}A$$

Then for any f we have $(\partial j)f = *$ on the nose; no homotopies are required. Now suppose $f : (I^n, \partial I^n, bI^n)(X, A, *)$ and ∂f is nullhomotopic. Let F be a nullhomotopy of ∂f. By the homotopy extension property, we can extend F to a homotopy $G : I^n \times I$ such that $G \equiv *$ on $bI^n \times I$. Hence $[f] = [G_1]$ in $\pi_n(X,A)$. Since $G_1(\partial I^n) = *$, this shows that $Ker\,\partial \subset Im\,j$.

Finally, consider the segment

$$\pi_n A \xrightarrow{i} \pi_n X \xrightarrow{j} \pi_n(X,A)$$

In this case we will need a simple lemma. Consider the $(n+1)$-cube $I^n \times I$ and single out the three faces A_0, A_1, B defined by $x_{n+1} = 0$, $x_{n+1} = 1$, and $x_n = 0$ respectively. If one pictures the case $n = 3$ in xyz-space, with the usual orientation of the axes and viewed from the positive x-axis, these are respectively the bottom, top and back faces of the cube. Thus a homotopy between two maps $f, g : (I^n, \partial I^n) \longrightarrow (X, *)$ is a map of the $(n+1)$-cube that has f on the bottom face, g on the top face, and $*$ on all remaining faces.

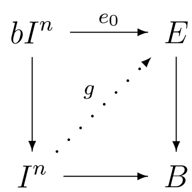
Lemma 4.4 *f and g are homotopic if and only if there is a map $F : I^n \times I \longrightarrow X$ that has f on the back face, g on the bottom face, and $*$ on all remaining faces.*

This is visually obvious, and not hard to prove rigorously. Now suppose given a map $f : (I^n, \partial I^n) \longrightarrow (A, *)$. Since f is homotopic to itself we can find F as in the lemma, with $g = f$. Then F is a nullhomotopy of $ji(f)$, proving that $ji = 0$. Conversely, suppose $f : (I^n, \partial I^n) \longrightarrow (A, *)$ and we are given a nullhomotopy F of jf. On the back face F maps $(I^n, \partial I^n) \longrightarrow (A, *)$. Hence by the lemma, f is homotopic to a map into A. In other words, $Ker\,j \subset Im\,i$. This completes the proof of the theorem.

We now turn to the proof of Theorem 4.1. Modulo noise in low degrees, this theorem follows from Theorem 4.3 and the following lemma:

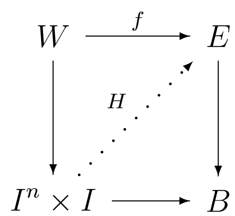
Lemma 4.5 *For $n \geq 1$, the natural map $p_* : \pi_n(E, F) \longrightarrow \pi_n B$ is an isomorphism.*

Proof: Suppose $\alpha \in \pi_n B$, and represent α by a map of pairs $f : (I^n, \partial I^n) \longrightarrow (B, b_0)$. Then there is a filler g in the diagram



where the top map is the constant map. Since g maps the bottom face into F by construction, g defines an element of $\pi_n(E, F)$ that maps to α. Hence p_* is onto.

Now suppose given $\beta_0, \beta_1 \in \pi_n(E, F)$ with $p_*\beta_0 = p_*\beta_1$. Represent β_i by a map of triples $h_i : (I^n, \partial I^n, bI^n) \longrightarrow (E, F, e_0)$. Then by assumption there is a homotopy $G : I^n \times I \longrightarrow B$ from $p\beta_0$ to $p\beta_1$, keeping $\partial I^n \times I$ at the basepoint. Choose a filler H in the diagram



where $W = I^n \times 0 \cup I^n \times 1 \cup bI^n \times I$, $f = h_i$ on $I^n \times i$, and f is constant on $bI^n \times I$. Then H is a homotopy showing $\beta_0 = \beta_1$. Hence p_* is one-to-one. This completes the proof of the lemma.

In view of Theorem 4.3, this yields the exact sequence of Theorem 4.1 except for the segment

$$\pi_0 F \longrightarrow \pi_0 E \longrightarrow \pi_0 B.$$

This case is clear because if $x \in E$ and $p(x)$ can be joined by a path to b_0, then a lift of this path with initial point x gives a path from x to some point of the fibre. (Note, however, that $\pi_0(E, F)$ need not biject to $\pi_0 B$; thus the last map may not be onto.)

We next take a closer look at the boundary map $\pi_1 B \longrightarrow \pi_0 F$. Note that the image of $\pi_1 E$ in $\pi_1 B$ need not be a normal subgroup, and that $\pi_0 F$ is only a set.

Proposition 4.6 *Suppose E and B are path-connected. Then the boundary map $\partial : \pi_1 B \longrightarrow \pi_0 F$ induces a bijection $\pi_1 B / p_*\pi_1 E \cong \pi_0 F$. (Here $\pi_1 B / p_*\pi_1 E$ denotes the set of cosets.)*

Proof: Unravelling the definitions, we find that ∂ is defined as follows: Given $\alpha \in \pi_1 B$, choose a loop $\lambda : I \longrightarrow B$ representing it. By the homotopy lifting property there is a lift $\tilde{\lambda} : I \longrightarrow E$ starting at the basepoint. Then $\partial\alpha$ is the path-component of $\tilde{\lambda}(1)$.

Notice that this is exactly how one defines the action of π_1 on a fibre in covering space theory. The only difference is that in covering space theory the lift $\tilde{\lambda}$ is unique; here it is not. In any event, the rest of the proof also resembles covering space theory, and is left to the reader.

As an application we prove:

Proposition 4.7 *Suppose given a commutative diagram of pointed spaces*

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

with all four spaces path-connected and p, p' Serre fibrations. Let $h : F \longrightarrow F'$ denote the induced map on fibres. Then if any two of f, g, h are weak equivalences, so is the third.

Proof: (If the fibres are also path-connected, this follows immediately from the long exact homotopy sequence and the 5-lemma. In the general case, more care is needed.)

Suppose that f and g are weak equivalences. Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_1 E & \longrightarrow & \pi_1 B & \xrightarrow{\partial} & \pi_0 F & \longrightarrow & * \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle \pi_0 h}\downarrow & & \\ \pi_1 E' & \longrightarrow & \pi_1 B' & \xrightarrow{\partial} & \pi_0 F' & \longrightarrow & * \end{array}$$

Note that the maps ∂ and $\pi_0 h$ are only maps of sets, so we must be careful about applying the 5-lemma. It is clear that $\pi_0 h$ is onto, but without further structure there is no reason that $\pi_0 h$ should be one-to-one. (The problem can be traced to the following simple fact: If a group homomorphism has trivial kernel, then it is one-to-one, but this is false for maps of pointed sets.) Fortunately, however, we do have the further structure provided by Proposition 4.6. It follows that $\pi_0 h$ is bijective. The 5-lemma then shows that with any choice of basepoints, $\pi_n h$ is an isomorphism for all $n \geq 1$. Thus h is a weak equivalence.

The other two cases of the proposition are left to the reader (use the 5-lemma, but with caution).

Theorem 4.8 *Let $p : E \longrightarrow B$ be a map, and suppose B path-connected. Then*

a) If p is a local product and B is locally path-connected, then any two fibres of p are homeomorphic;

b) If p is a Hurewicz fibration, any two fibres of p are homotopy equivalent;

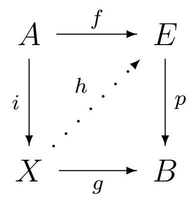
c) If p is a Serre fibration, any two fibres of p are weakly equivalent.

The proof of (a) is an easy exercise. For (b), see [Spanier], p. 101, Corollary 13. For (c), note that by pulling back over a path $I \longrightarrow B$, we reduce at once to the case B contractible. Then the long exact homotopy sequence shows that for any fibre $p^{-1}b$, the inclusion $p^{-1}b \subset E$ is a weak equivalence. Hence any two fibres are weakly equivalent.

5 The main lifting theorem

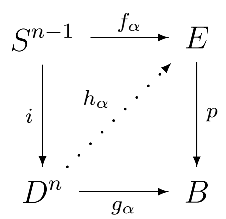
We now come to a particularly elegant lifting/extension theorem. By a *subcomplex inclusion* we mean the inclusion map of a subcomplex of a CW-complex.

Theorem 5.1 *In the diagram*



suppose that p is Serre fibration and i is a subcomplex inclusion. Then if either i or p is a weak equivalence, the filler h exists.

Proof: Suppose first that p is a weak equivalence. We will construct h inductively over $X^n \cup A$. The case $n = 0$ is easy, since X^0 is discrete. At the inductive step, we reduce to the special case



Here $g_\alpha = g \circ \phi_\alpha$ and $f_\alpha = h^{n-1} \circ \psi_\alpha$, where ϕ_α, ψ_α are respectively the characteristic map and attaching map for a typical n-cell e^n_α.

Now any map $D^n \longrightarrow B$ is homotopic rel S^{n-1} to a map that is constant on $D^n(1/2)$, the disc of radius 1/2. In view of Proposition 2.5, we may therefore assume that $g_\alpha(D^n(1/2)) \equiv b_0$ for some $b_0 \in B$. Let W denote the annulus consisting of $\{x \in D^n : 1/2 \leq| x |\leq 1\}$. Then W is homeomorphism to $S^{n-1} \times I$. Since p is a Serre fibration, there is a lift h'_α defined on W. Now observe that h'_α maps the sphere of radius 1/2 into the fibre $p^{-1}b_0$. Since p is a weak equivalence, the long exact homotopy sequence shows that this fibre is weakly contractible. Hence h'_α extends to a map $h_\alpha : D^n \longrightarrow E$, and by construction it lifts g_α. This completes the proof in the case p is a weak equivalence.

Now suppose i is a weak equivalence. Then by Theorem 2.6, A is a deformation retract of X. Let r denote the retraction. Since $ir \sim 1_X$ rel A, $gir \sim g$ rel A. But gir clearly admits a lift in the diagram—namely, fr—and hence g lifts by Proposition 2.5.

Corollary 5.2 *Let $p : E \longrightarrow B$ be a nonempty Serre fibration, with base space B a contractible CW-complex. Then p admits a section.*

Proof: Take $X = B$, $g = 1_B$, A a point of B, and f any map. Then h is the desired section.

Remark: As one might expect, a much stronger statement holds: If the base is contractible then the fibration itself is fibre-homotopy equivalent to the trivial fibration $B \times F \longrightarrow B$. See [Spanier], p. 102, Corollary 15.

Corollary 5.3 *Let $p : E \longrightarrow B$ be a Serre fibration with base space B a CW-complex, and suppose the fibre F is weakly contractible. Then p admits a section s. Furthermore, if A is a subcomplex of B and a section s_A is already given on A, s can be chosen to extend s_A.*

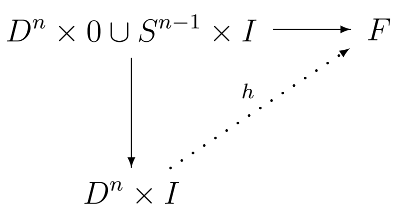
Proof: Since F is weakly contractible, the long exact homotopy sequence shows that p is a weak equivalence. Now take $X = B$ and $g = 1_B$.

Note that Theorem 5.1 also contains the very definition of a Serre fibration, by taking the left-hand vertical map to be the inclusion $i_0 : X \longrightarrow X \times I$. Further exploration of special cases is left to the reader.

6 Examples

Theorem 6.1 *A local product is a Serre fibration.*

Proof: We sketch the proof and let the reader supply the details. Suppose $p : E \longrightarrow B$ is a local product. By Theorem 2.3, it is enough to show that p has the homotopy lifting property with respect to all n-discs. We proceed by induction on n. At the inductive step we can assume (see the Remark following Theorem 2.3) that p has the homotopy lifting property with respect to all CW-complexes of dimension less than n. Using this together with the Lebesgue covering lemma, one can reduce to showing that a global product $U \times F \longrightarrow U$ has the relative homotopy lifting property with respect to the pair (D^n, S^{n-1}). This in turn amounts to showing that the extension h always exists in the diagram



But clearly $D^n \times 0 \cup S^{n-1} \times I$ is a retract of $D^n \times I$ (hang a lightbulb above the center of the cylinder and follow its rays). Hence the extension h exists, completing the proof.

One can also produce fibrations by starting from a cofibration and taking function spaces:

Proposition 6.2 *Suppose X is a locally compact Hausdorff space and $A \subset X$ is a cofibration. Then for any space Y, the restriction map $F(X, Y) \longrightarrow F(A, Y)$ on function spaces is a fibration.*

Proof: Exercise. The compact-open topology is compatible with precomposition in the first variable (and postcomposition in the second; the proof is easy); in particular the restriction map is continuous. Now note that A is necessarily closed (cf. [Hatcher], p. 14) and hence locally compact Hausdorff. The homotopy lifting property can then be deduced directly from the homotopy extension property, using Proposition A.14b from Hatcher.

The next general example does not arise as a local product. Let Y^I denote the path-space of the space Y; that is, the set of all continuous maps $I \longrightarrow Y$, equipped with the compact-open topology. Given a map $f : X \longrightarrow Y$, define N_f by the pullback diagram

$$\begin{array}{ccc} N_f & \longrightarrow & Y^I \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle e_0} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

where e_0 denotes evaluation at 0. Thus N_f is the space of pairs (x, λ) with $x \in X$ and λ a path in Y that starts at $f(x)$; this construction is dual to the mapping cylinder construction. Now define a map $p : N_f \longrightarrow Y$ by $p(x, \lambda) = \lambda(1)$.

Proposition 6.3 *$p : N_f \longrightarrow Y$ is a Hurewicz fibration.*

Proof: There is a pullback diagram

$$\begin{array}{ccc} N_f & \longrightarrow & Y^I \\ {\scriptstyle (q,\pi_Y)}\downarrow & & \downarrow{\scriptstyle (e_0,e_1)} \\ X \times Y & \underset{f\times Id}{\longrightarrow} & Y \times Y \end{array}$$

The map (e_0, e_1) is a fibration by Proposition 6.2, since it is just restriction to $\{0, 1\} \subset I$. Hence (q, π_Y) is a fibration, and therefore so is p since it is the composition $N_f \longrightarrow X \times Y \longrightarrow Y$.

As an important special casse, note that the fibre over a chosen basepoint y_0 is the space of pairs (x, λ) that start at $f(x)$ and end at y_0. In particular, taking X to be a point and f the inclusion of the basepoint, we obtain the *path-loop fibration* $PY \longrightarrow Y$. In this case, the fibre over the basepoint is the *loop-space of* Y, denote ΩY.

The dual construction starts from a map $g : X \longrightarrow Y$ and forms the cofibration $X \longrightarrow M_g$, where M_g is the reduced mapping cylinder. The cofibre is then the reduced mapping cone. Taking Y to be a point, the dual of the path-space is the reduced cone on X, and the dual of the loop space is the reduced suspension of X.

Proposition 6.4 *The projection map $q : N_f \longrightarrow X$ is a pointed homotopy equivalence. A homotopy inverse $s : X \longrightarrow N_f$ is given by $s(x) = (x, \alpha(x))$, where $\alpha(x)$ is the constant path at $f(x)$.*

The proof is an easy exercise: Clearly qs is actually equal to the identity, while a homotopy from sq to the identity is obtained by following each λ out to time t.

Corollary 6.5 *Any map $f : X \longrightarrow Y$ can be factored in the form*

$$X \xrightarrow{j} X' \xrightarrow{p} Y$$

with j a homotopy equivalence and p a Hurewicz fibration.

Proof: Take $X' = N_f$, $j = s$, p as above.

We recall here that there is a result dual to the last corollary:

Proposition 6.6 *Any map $f : X \longrightarrow Y$ can be factored in the form*

$$X \xrightarrow{i} Y' \xrightarrow{r} Y$$

with i a cofibration and r a homotopy equivalence.

Here one takes Y' the mapping cylinder of f, i the obvious inclusion at one end of the cylinder, and r the obvious deformation retraction onto Y.

Remark: In the late 60's Quillen introduced an axiomatic approach to homotopy theory. The idea is to start with a category $\mathcal{C}$ equipped with three distinguished classes of morphisms, called weak equivalences, fibrations and cofibrations. These classes are subject to certain axioms, the most significant of these being modeled on Theorem 5.1 (with i replaced by any distinguished cofibration), Corollary 6.5, and Proposition 6.6. The category $\mathcal{C}$, together with the three classes as morphisms,constitutes a *model category.* In the category of spaces there are several different interesting model category structures. For example, one can take the weak equivalences to be the homotopy equivalences, the fibrations to be the Hurewicz fibrations, and the cofibrations to be the closed maps that are cofibrations in the usual sense. Another model category structure on spaces, more relevant for our purposes, takes the weak equivalences to be the weak equivalences in the usual sense, the fibrations to be the Serre fibrations, and the cofibrations to be all maps that are retracts of cellular cofibrations. Without defining the latter class precisely, we remark that it includes all subcomplex inclusions, and is contained in the class of all ordinary cofibrations. Theorem 5.1 remains valid if i is replaced by one of these more general cofibrations.

This axiomatization has been applied applied to other categories having nothing to do with topological spaces *per se.* For example, there is a model category structure on the category of chain complexes in which the weak equivalences are the so-called quasi-isomorphisms; that is, the maps inducing an isomorphism on homology.

For a nice introduction to model categories, see [Dwyer-Spalinski].

7 Homotopy-fibres

All spaces, maps and homotopies in this section are pointed.

Let $f : X \longrightarrow Y$ be an arbitrary pointed map. Then the fibre $f^{-1}y_0$ is somewhat irrelevant from a homotopy-theoretic standpoint, because it is not homotopy-invariant. More precisely, suppose we are given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

with g and h homotopy-equivalences. Then it does not follow that the induced map on fibres $f^{-1}y_0 \longrightarrow f^{-1}y_0'$ is a homotopy equivalence, or even a weak equivalence. A dramatic illustration is provided by

$$\begin{array}{ccc} * & \longrightarrow & X \\ \big\downarrow & & \big\downarrow = \\ PX & \longrightarrow & X \end{array}$$

Here the fibre of the top map is just a point, while the fibre of the bottom map is the loop space ΩX. To make matters worse, if the given diagram is only homotopy-commutative then there is no induced map on fibres at all. What we need here is a homotopy-invariant replacement for the *geometric fibre* $p^{-1}y_0$. In fact such a replacement is already at hand. Given $f : X \longrightarrow Y$, define the *homotopy-fibre* L_f as the geometric fibre of the Hurewicz fibration $N_f \longrightarrow Y$. This definition is motivated by two key points: (i) N_f is homotopy-equivalent to X, and (ii) as already shown in Proposition 4.7, geometric fibres of fibrations are homotopically well-behaved.

For example, the homotopy-fibre of $* \longrightarrow Y$ is ΩY. In general, L_f is the space of pairs $(x, \lambda) \in X \times PY$ such that λ starts at $f(x)$ and ends at the basepoint y_0. It follows that for any space W, a map $W \longrightarrow L_f$ is the same thing as a map $\phi : W \longrightarrow X$ together with a nullhomotopy of $f \circ \phi$.

Theorem 7.1 *Suppose given a homotopy-commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\big\downarrow & & \big\downarrow h \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

Then there exists a map $\alpha : L_f \longrightarrow L_{f'}$ such that the diagram

$$\begin{array}{ccc} L_f & \xrightarrow{q} & X \\ \alpha\big\downarrow & & \big\downarrow g \\ L_{f'} & \xrightarrow[q']{} & X' \end{array}$$

is homotopy commutative. Furthermore

(a) If g, h are homotopy equivalences, then α is a homotopy equivalence;

(b) If g, h are weak equivalences, then α is a weak equivalence.

Proof: Form the diagram

$$\begin{array}{ccc} N_f & \xrightarrow{p} & Y \\ \Big\downarrow{\scriptstyle \bar{g}} & & \Big\downarrow{\scriptstyle h} \\ N_{f'} & \xrightarrow[p']{} & Y' \end{array}$$

where $\bar{g}$ is the composite

$$N_f \xrightarrow{q} X \xrightarrow{g} X' \xrightarrow{s'} N_{f'}.$$

It follows from the hypothesis and the definitions that this diagram is also homotopy-commutative. Since p' is a Hurewicz fibration, we can replace $\bar{g}$ by a homotopic map $\tilde{g}$ such that the new diagram

$$\begin{array}{ccc} N_f & \xrightarrow{p} & Y \\ \Big\downarrow{\scriptstyle \tilde{g}} & & \Big\downarrow{\scriptstyle h} \\ N_{f'} & \xrightarrow[p']{} & Y' \end{array}$$

is strictly commutative. We define $\alpha : L_f \longrightarrow L_{f'}$ to be the induced map on geometric fibres. Now consider the diagram

$$\begin{array}{ccccc} L_f & \longrightarrow & N_f & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{g}} & & \Big\downarrow{\scriptstyle g} \\ L_{f'} & \longrightarrow & N_{f'} & \longrightarrow & X' \end{array}$$

The first square strictly commutes by definition. The second square is homotopy-commutative because it is strictly commutative when $\tilde{g}$ is replaced by $\bar{g}$. Hence the outer rectangle is homotopy-commutative, as desired.

We omit the proof of (a), since we are in any case willing to work up to weak equivalence. Part (b) follows from Proposition 4.7.

Example 1. In a few cases L_f can be identified (up to weak equivalence at least) in more geometric terms. A striking example is the map of classifying spaces $BH \longrightarrow BG$ associated to a subgroup $H \subset G$. If H is sufficiently nice subgroup (e.g., a closed subgroup of a Lie group G), one can show that L_f is weak equivalent to the homogeneous space G/H. This is very useful for various purposes; e.g., (i) we get long exact sequence on homotopy groups with $\pi_* G/H$ in the fibre slot; and (ii) we can use spectral sequences to relate the cohomology of G/H, BH, and BG. For details and specific examples, see *Notes on principal bundles.*

Suppose given spaces X, Y, Z and maps $X \longrightarrow Y \longrightarrow Z$. Two such gadgets are said to be *weakly equivalent* if they are equivalent under the equivalence relation generated by commutative diagrams

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

Call $X \longrightarrow Y \longrightarrow Z$ a *fibre sequence* if it is weakly equivalent to some $F \longrightarrow E \xrightarrow{p} B$ with p a pointed Serre fibration and F the fibre of p. More generally, any sequence of maps $... \longrightarrow X_1 \longrightarrow X_2 \longrightarrow ... \longrightarrow X_n \longrightarrow ...$ (with or without initial/end terms) is a *fibre sequence* if any three consecutive terms give a fibre sequence in this sense. Note that if $X \longrightarrow Y \longrightarrow Z$ is a fibre sequence, then we get a long exact sequence

$$... \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \pi_n Z \longrightarrow \pi_{n-1} X \longrightarrow ...$$

To be continued...

8 Appendix: Whitehead's theorem

In this section we use Theorem 5.1 to give a very slick proof of Whitehead's theorem (I believe this proof is due to Quillen). We need to be careful to avoid a circular argument here, because we used Whitehead's theorem to prove half of Theorem 5.1—the case when i is a weak equivalence. But in the proof below we only use the other case, when p is a weak equivalence. The proof of this case did not involve Whitehead's theorem.

Theorem 8.1 *Suppose Y and Z are CW-complexes, and $f : Y \longrightarrow Z$ is a weak equivalence. Then f is a homotopy equivalence.*

This theorem follows from (and in fact is equivalent to):

Theorem 8.2 *Suppose Y and Z are arbitrary spaces, and $f : Y \longrightarrow Z$ is a weak equivalence. Then if X is any CW-complex, f induces a bijection $[X, Y] \xrightarrow{\cong} [X, Z]$*

The deduction of Theorem 8.1 from Theorem 8.2 is pure (and trivial) category theory. For in any category $\mathcal{C}$, a morphism $f : Y \longrightarrow Z$ is an isomorphism if and only if for all objects X, f induces a bijection $Hom_{\mathcal{C}}(X, Y) \xrightarrow{\cong} Hom_{\mathcal{C}}(X, Z)$. Here we take $\mathcal{C}$ to be the homotopy category of CW-complexes.

Proof of Theorem 8.2: By Proposition 6.5 we can factor f as a homotopy equivalence followed by a Hurewicz fibration: $Y \longrightarrow Y' \longrightarrow Z$. We therefore reduce at once to the case that f is both a weak equivalence and a Hurewicz fibration (hence also a Serre fibration). Applying Theorem 5.1 to the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Z \end{array}$$

then shows that $[X, Y] \longrightarrow [X, Z]$ is surjective. Now suppose $g, h : X \longrightarrow Y$ and there is a homotopy F from fg to fh. Applying Theorem 5.1 to the diagram

$$\begin{array}{ccc} X \times \{0\} \coprod X \times \{1\} & \xrightarrow{g \coprod h} & Y \\ \downarrow & \nearrow & \downarrow \\ X \times I & \xrightarrow[F]{} & Z \end{array}$$

shows that g is homotopic to h. Hence $[X, Y] \longrightarrow [X, Z]$ is also injective.

9 References

[Dwyer-Spalinski], Homotopy theory and model categories. I think this appeared in the *Handbook of Algebraic Topology*, edited by Ioan James.
[Maunder], *Algebraic Topology.*
[May], *A Concise Introduction to Algebraic Topology.*
[Spanier], *Algebraic Topology.*