Invariant Commutation and Propagation Functions

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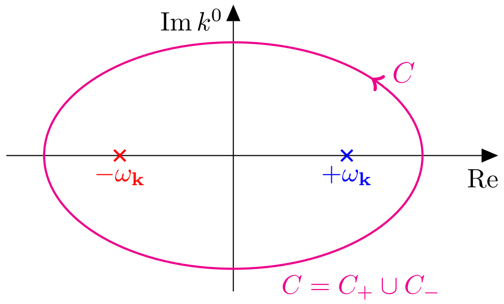
Commutation Functions (closed contours)

Pauli-Jordan-Schwinger function

$$\Delta(x-y) = \int_C \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,\frac{-1}{k^2-m^2}$$

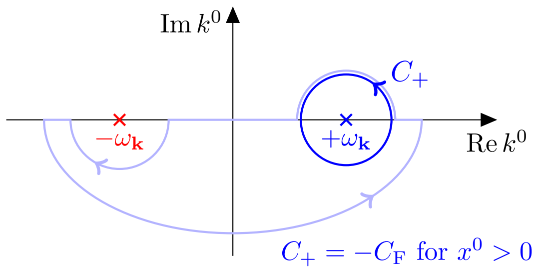
$$\mathrm{i}\Delta(x-y) = [\,\phi(x),\,\phi(y)\,]$$



Positive frequency commutation function

$$\Delta^+(x-y) = \int_{C^+} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,\frac{-1}{k^2-m^2}$$

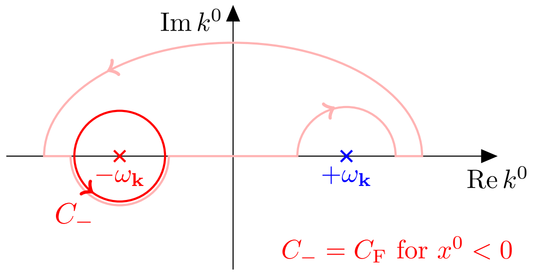
$$\mathrm{i}\Delta^+(x-y) = [\,\phi^+(x),\,\phi^-(y)\,]$$



Negative frequency commutation function

$$\Delta^-(x-y) = \int_{C^-} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,\frac{-1}{k^2-m^2}$$

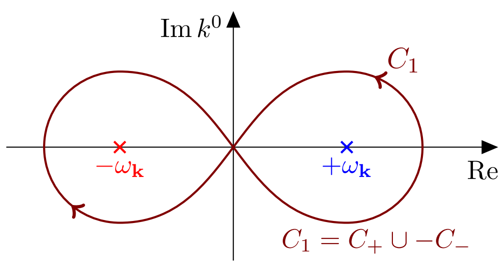
$$\mathrm{i}\Delta^-(x) = [\,\phi^-(x),\,\phi^+(y)\,]$$



Anticommutation (or auxiliary) function

$$\Delta_1(x-y) = \int_{C_1} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,\frac{-1}{k^2-m^2}$$

$$\mathrm{i}\Delta_1(x-y) = \{\,\phi(x),\,\phi(y)\,\}$$



Main properties of the commutation functions:

1° The commutation functions are Lorentz invariant.

$$\mathrm{i}\Delta(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \mathrm{d}k^0\;\mathrm{e}^{-\mathrm{i}k\cdot x}\,\delta(k^2-m^2)\;\mathrm{sgn}(k^0) \tag{1}$$

$$\mathrm{i}\Delta_1(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \mathrm{d}k^0\;\mathrm{e}^{-\mathrm{i}k\cdot x}\,\delta(k^2-m^2) \tag{2}$$

$$\mathrm{i}\Delta^{\pm}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \mathrm{d}k^0\;\mathrm{e}^{-\mathrm{i}k\cdot x}\,\delta(k^2-m^2)\;(\pm\theta(\pm k^0)) \tag{3}$$

2° The following *initial conditions at vanishing time difference* hold:

$$\Delta(0,\mathbf{x}) = 0, \qquad \partial_0\Delta(x^0,\mathbf{x})|_{x^0=0} = -\delta^3(\mathbf{x})\,. \tag{4}$$

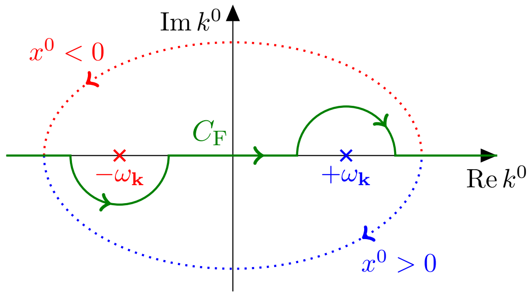
As a corrolary, the ETCR are regained.

3° From 2°, it also follows the condition of *microcausality*: $\Delta(x-y) = 0$ for $(x-y)^2 < 0$, i.e. $\Delta(x-y)$ vanishes if the argument is spacelike. (This is a very fundamental property!)

4° All the commutation functions satisfy EoM; e.g. $-(\Box_x + m^2)\,\Delta(x-y) = 0$.

Note that 2° and 4° are sufficient to provide a unique definition of the function $\Delta(x)$. (EoM is a hyperbolic PDE with the initial data (4) on a space-like hypersurface $x^0 = 0$). Also, all the Δ-functions show singular behavior on the light cone.

Propagation Functions (open contours that extend to infinity)

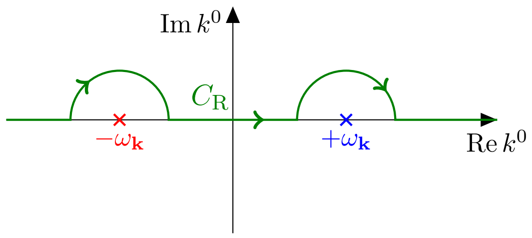


Feynman propagator
(causal propagator or causal Green's function)

$$\Delta_{\mathrm{F}}(x-y) = \int_{C_{\mathrm{F}}} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\,\frac{1}{k^2-m^2}$$

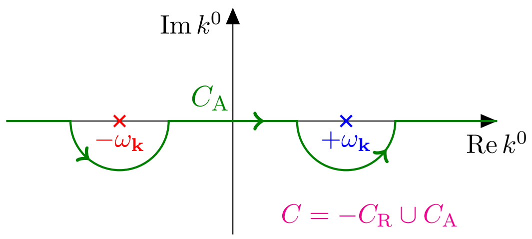
$$\mathrm{i}\Delta_{\mathrm{F}}(x-y) = \langle 0|\mathrm{T}\{\,\phi(x),\,\phi(y)\,\}|0\rangle$$

$$\begin{aligned}\Delta_{\mathrm{F}}(x) &= \theta(x^0)\,\Delta^+(x) - \theta(-x^0)\,\Delta^-(x)\\ &= \tfrac{1}{2}\left(\mathrm{sgn}(x^0)\,\Delta(x) + \Delta_1(x)\right)\end{aligned}$$



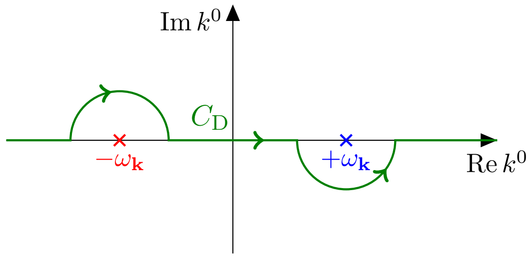
Causal retarded propagator

$$\Delta_{\mathrm{R}}(x) = -\theta(x^0)\,\Delta(x)$$



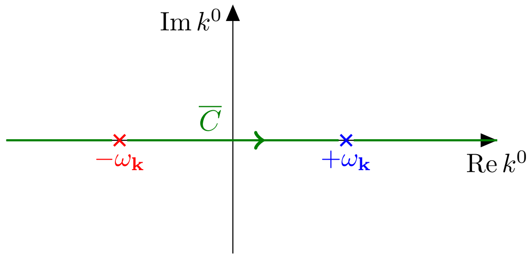
Causal advanced propagator

$$\Delta_{\mathrm{A}}(x) = \theta(-x^0)\,\Delta(x)$$



Dyson propagator
(anticausal propagator)

$$\begin{aligned}\Delta_{\mathrm{D}}(x) &= \theta(x^0)\,\Delta^-(x) - \theta(-x^0)\,\Delta^+(x)\\ &= \tfrac{1}{2}\left(\mathrm{sgn}(x^0)\,\Delta(x) - \Delta_1(x)\right)\end{aligned}$$



Principal-part propagator

$$\begin{aligned}\overline{\Delta}(x) &= \tfrac{1}{2}\,\mathrm{sgn}(x^0)\,\Delta(x)\\ &= \tfrac{1}{2}\left(\Delta_{\mathrm{R}}(x) + \Delta_{\mathrm{A}}(x)\right)\end{aligned}$$

Relations between Δ-functions:

$$\begin{aligned}
\Delta^{\pm}(-x) &= -\Delta^{\mp}(x)\\
\Delta(-x) &= -\Delta(x)\\
\Delta_1(-x) &= \Delta_1(x)\\
\Delta(x) &= \Delta^+(x) + \Delta^-(x)\\
\Delta_1(x) &= \Delta^+(x) - \Delta^-(x)\\
\Delta^+(x) &= \tfrac{1}{2}\left(\Delta(x) + \Delta_1(x)\right)\\
\Delta^-(x) &= \tfrac{1}{2}\left(\Delta(x) - \Delta_1(x)\right)\\
\Delta_1(x) &= \Delta_{\mathrm{F}}(x) - \Delta_{\mathrm{D}}(x)\\
\Delta(x) &= \Delta_{\mathrm{R}}(x) - \Delta_{\mathrm{A}}(x)
\end{aligned}$$

Real scalar field:

$$\mathcal{L}[\phi,\partial_\mu\phi] = -\tfrac{1}{2}\phi(x)\,(\Box + m^2)\,\phi(x)\,,$$

$$\text{EoM:}\quad \frac{\delta\mathcal{L}}{\delta\phi(x)} = -(\Box + m^2)\,\phi(x) = 0\,.$$

The complete set of plane wave states:

$$\phi(x) = \phi^+(x) + \phi^-(x)\,,$$

$$\phi^+(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,a(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}k\cdot x}\,,$$

$$\phi^-(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,a^\dagger(\mathbf{k})\,\mathrm{e}^{\mathrm{i}k\cdot x}\,.$$

Contour integrals:

$$\oint_\gamma \frac{f(z)\,\mathrm{d}z}{z - z_0} = \begin{cases} 2\pi\mathrm{i}\,f(z_0), & z_0 \in \mathrm{int}\,\gamma,\\ 0, & z_0 \notin \mathrm{int}\,\gamma;\end{cases}$$

$$I_\gamma(x) = \oint_\gamma \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}k^0}{k^2 - m^2} = \oint_\gamma \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}k^0}{(k^0-\omega_{\mathbf{k}})(k^0+\omega_{\mathbf{k}})}\,.$$

Positive and negative frequency poles:

$$\begin{aligned}
I_\gamma(x) &= +2\pi\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{2\omega_{\mathbf{k}}}, \quad k^0 = +\omega_{\mathbf{k}} \in \gamma = C^+,\\
I_\gamma(x) &= -2\pi\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{2\omega_{\mathbf{k}}}, \quad k^0 = -\omega_{\mathbf{k}} \in \gamma = C^-,\\
I_\pm(x) &= \pm 2\pi\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{2\omega_{\mathbf{k}}}\delta(k^0 \mp \omega_{\mathbf{k}}).
\end{aligned}$$

Hence,

$$\Delta_\gamma(x) = \int_\gamma \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k\cdot x}\,\frac{-1}{k^2-m^2} = -\int \frac{\mathrm{d}^3k}{(2\pi)^4} I_\gamma\,,$$

$$\mathrm{i}\Delta^{\pm}(x) = \pm\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\cdot x}\;\delta(k^0 \mp \omega_{\mathbf{k}})\frac{1}{2\omega_{\mathbf{k}}},$$

where, from the properties of the δ-function,

$$\delta(k^2 - m^2)\,\theta(\pm k^0) = \delta(k^0 \mp \omega_{\mathbf{k}})\frac{1}{2\omega_{\mathbf{k}}}\,,$$

$$\pm\int \mathrm{d}k^0\,\delta(k^2-m^2)\,\theta(\pm k^0) = \frac{1}{2\omega_{\mathbf{k}}}\,.$$

All the propagation functions are Green's functions as they yield the δ-function when the Klein-Gordon operator is applied. Namely, they contain a product of the function $\Delta(x)$ with a unit step function in time, either $\theta(x^0)$ or $\frac{1}{2}\,\mathrm{sgn}(x^0)$, of which derivativs are the δ-functions.

For example, for $(\Box + m^2)\Delta_{\mathrm{F}}(x)$ we have:

$$\begin{aligned}(\Box + m^2)\left(\tfrac{1}{2}\left(\mathrm{sgn}(x^0)\,\Delta(x) + \Delta_1(x)\right)\right)\,&(\delta_0^2 - \boldsymbol{\nabla}^2 + m^2)\left(\tfrac{1}{2}\,\mathrm{sgn}(x_0)\Delta(x)\right) =\\ &= (\partial_0\delta(x))\,\Delta(x) + 2\delta(x^0)\,(\partial_0\Delta(x)) + \tfrac{1}{2}\,\mathrm{sgn}(x_0)(\Box + m^2)\Delta(x)\,.\end{aligned}$$

The last term vanishes because $\Delta(x)$ solves EoM, and the first term is equivalent to $-\delta(x^0)(\partial_0\Delta(x))$. Using a test function $f(x^0)$, an integration by parts yields,

$$\int \mathrm{d}x^0\,(\partial_0\delta(x))\,\Delta(x)f(x^0) = -\partial_0\Delta(x)\,|_{x^0=0}\,f - \Delta(0,\mathbf{x})\,\partial_0\,f\,|_{x^0=0} = -\partial_0\Delta(x)\,|_{x^0=0}\,f = -\delta^3(\mathbf{x})\,f\,.$$

Here, we applied the initial conditions (4) from Property 2°. Hence,

$$(\Box + m^2)\Delta_{\mathrm{F}}(x) = -\delta(x^0)\,(\partial_0\Delta(x)) + 2\delta(x^0)\,(\partial_0\Delta(x)) = \delta(x^0)\partial_0\Delta(x) = -\delta(x^0)\delta^3(\mathbf{x}) = -\delta^4(x)\,.$$