Fields Institute Communications
Volume **17**, 1997

A Closed Model Structure for Differential Graded Algebras

J. F. Jardine
Mathematics Department
University of Western Ontario
London, Ontario, Canada N6A 5B7
jardine@uwo.ca

Abstract. We derive a closed model structure for the category of noncommutative differential graded algebras over an arbitrary commutative ring with unit.

This short note constructs a closed model structure, and hence a homotopy theory, for the category of differential graded algebras over an arbitrary commutative unitary ring. Such differential graded algebras are not assumed to be commutative.

This structure was obtained in the context of a joint project between the author and Paul Goerss to construct explicit algebraic models for mod p homotopy theory. An initial hope, which was inspired by Karoubi's result [**4**] that the Steenrod operations could be modelled in the category of noncommutative DGAs over $\mathbb{F}_p$, was that non-commutative differential graded algebras might be an adequate setting in which to realize a model for Bousfield's $H_*(\ ,\mathbb{Z}/p)$-local homotopy theory [**1**]. This idea did not work — the problem of giving a computable algebraic model for mod p homotopy theory, which was formulated one way or another in the late 1970's, remains one of the difficult problems of homotopy theory (see also [**3**]) — but the closed model structure for non-commutative DGAs survives.

This structure is apparently of some interest within the cyclic homology community, so the result is presented here: it is Theorem 5 below. Subject to a proper understanding of coproducts in the category of non-commutative DGAs, the proof of Theorem 5 is analogous to that of the corresponding result given by Bousfield and Gugenheim [**2**] for commutative differential graded algebras over fields of characteristic zero.

Suppose that k is a commutative ring with 1. The category of differential graded algebras over k will be denoted by DGA_k. An object of DGA_k is a graded k-algebra $A = \{A^0, A^1, A^2, \dots\}$ which is not necessarily commutative, and is equipped with

1991 *Mathematics Subject Classification.* Primary 55U15; Secondary 16W50, 18G55.
This research was supported by NSERC.

a degree +1 differential $d : A^n \to A^{n+1}$ satisfying the Leibnitz formula

$$d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y), \tag{1}$$

where $|x|$ denotes the degree of x, and $x \cdot y$ is the product of x and y with respect to the algebra structure on A. We shall also require that

$$d(1) = 0 \tag{2}$$

for $1 \in A^0$, and that k is in the centre of A. It follows that the differential d is a k-module map. I do not require my differential graded algebras A to be augmented.

The category DGA_k obviously has all inverse limits (with terminal object 0), all filtered colimits and all coequalizers. The coproduct $A *_k B$ of the differential graded k-algebras A and B is formed by setting

$$A *_k B = T(A \otimes_k B)/I,$$

where $T(A \otimes_k B)$ is the tensor algebra

$$T(A \otimes_k B) = \bigoplus_{n \geq 0} (A \otimes_k B)^{\otimes n}$$

for the k-chain complex $A \otimes_k B$, and I is the ideal which is (multiplicatively) generated by elements of the form

$$\begin{cases} (a_1 \otimes b_1) \otimes (1 \otimes b_2) - a_1 \otimes b_1 b_2 & \text{and} \\ (a_1 \otimes 1) \otimes (a_2 \otimes b_2) - a_1 a_2 \otimes b_2. \end{cases}$$

Note that I is a differential ideal in the sense that it is preserved by the differential. The category DGA_k has all finite coproducts (with initial object k, concentrated in degree 0), and all filtered colimits, so DGA_k has all coproducts, and is therefore complete and cocomplete.

Suppose that x is a variable of degree n. The differential graded k-algebra $S(x)$ is defined to be the free graded k-algebra $k\{x\}$ on x, equipped with a differential d which is uniquely specified by $d(x) = 0$. If A is a differential graded k-algebra, and $a \in A$ is an element such that $|a| = n$ and $d(a) = 0$, then there is a unique map $\iota_a : S(x) \to A$ of differential graded k-algebras such that $\iota_a(x) = a$.

The differential graded k-algebra $T(x)$ is the k-algebra $k\{x, dx\}$ which is freely generated by x and dx (with $|dx| = n+1$), and with differential d uniquely specified by $d(x) = dx$ and $d(dx) = 0$. If B is a differential graded k-algebra and $b \in B$ is an element such that $|b| = n$, then there is a unique map of differential graded k-algebras $\iota_b : T(x) \to B$ such that $\iota_b(x) = b$.

In other words, $T(x)$ is the free differential graded k-algebra on a generator x of degree n. The free k-cochain complex $C(x)$, a generator x of degree n, has the form

$$C(x)^i = \begin{cases} 0 & \text{if } i \neq n, n+1, \text{ and} \\ k & \text{if } i = n, n+1. \end{cases}$$

The differential $d : C(x)^n \to C(x)^{n+1}$ is the identity map on k. We shall identify x with the k-generator $1 \in k = C(x)^n$. Note that $C(x)$ is acyclic. Observe as well that the free object $T(x)$ can be identified up to isomorphism with the tensor algebra

$$T(C(x)) = \bigoplus_{n \geq 0} C(x)^{\otimes n}$$

on the free cochain complex $C(x)$. It follows that there are isomorphisms in cohomology of the form

$$H^iT(x) = \begin{cases} k & \text{if } i = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

More generally, if A is a differential graded k-algebra and C is a k-cochain complex, then there is an object $A[C] \in DGA_k$ with

$$A[C] = A \oplus (A \otimes C \otimes A) \oplus (A \otimes C \otimes A \otimes C \otimes A) \oplus \ldots$$

and with multiplication specified by

$$\begin{aligned}(a_1 \otimes b_1 \otimes \cdots \otimes b_k \otimes a_{k+1}) \cdot (a_1' \otimes b_1' \otimes \cdots \otimes b_l' \otimes a_{l+1}') \\ = a_1 \otimes b_1 \otimes \ldots b_k \otimes a_{k+1}a_1' \otimes b_1' \otimes \cdots \otimes b_l' \otimes a_{l+1}'.\end{aligned}$$

This construction can be thought of (abusively, since A is not central) as the tensor algebra over A for the free differential A-bimodule $A \otimes C \otimes A$ on the cochain complex C.

Any map $f : A[C] \to B$ in DGA_k is uniquely determined by its restriction to A and the chain map given by the composite

$$C \xrightarrow{in} A \otimes C \otimes A \subset A[C] \xrightarrow{f} B,$$

where the map in is defined by $c \mapsto 1 \otimes c \otimes 1$. It follows in particular that there is an isomorphism

$$A *_k T(x) \cong A[C(x)],$$

and that the canonical map $A \to A *_k T(x)$ is a cohomology isomorphism.

A map $f : A \to B$ in DGA_k is said to be:

(a) a *weak equivalence* if f is a cohomology isomorphism,

(b) a *fibration* if f is surjective in all degrees, or

(c) a *cofibration* if f has the left lifting property with respect to all maps which are fibrations and weak equivalences (aka. trivial fibrations).

These definitions (and the arguments which follow below) are direct analogues of ideas appearing in a paper of Bousfield and Gugenheim [**2**]. In particular, one can make the following observations:

Lemma 1

(1) *The canonical maps $k \to T(x)$, $k \to S(x)$ and $S(dx) \to T(x)$ are all cofibrations.*

(2) *A map $f : A \to B$ is a trivial fibration if and only if f is a fibration and has the right lifting property with respect to all maps of the form $k \to S(x)$ and $S(dx) \to T(x)$.*

Lemma 2 *Any map $f : A \to B$ in DGA_k can be factored $f = q \cdot j$, where j has the left lifting property with respect to all fibrations and is a weak equivalence, and q is a fibration.*

Proof Form the factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "f"'] & A * (*_{b \in B} T(b)) \arrow[d, "q"] \\ & B, \end{tikzcd}$$

where j is the canonical map, and q is the A-algebra map which sends the generator $b \in T(b)$ to $b \in B$. q is obviously a fibration, and j is a filtered colimit of maps of the form $A \to T(b_1) * \cdots * T(b_n) * A$, each of which is a trivial cofibration by successive applications of the construction $T(b_i) * A \cong A[C(b_i)]$ given above. □

Lemma 3 *Any map $f : A \to B$ in DGA_k may be factored $f = p \cdot i$, where p is a trivial fibration and the map i is a cofibration.*

Proof It suffices, by Lemma 2, to presume that f is a fibration. But then the result follows from a small object argument, based on Lemma 1 above. □

Lemma 4 *Suppose that the map $i : A \to B$ in DGA_k is a cofibration and a weak equivalence. Then i has the left lifting property with respect to all fibrations.*

Proof This is a standard consequence of Lemma 2. Find a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & \overline{B} \arrow[d, "q"] \\ & B \end{tikzcd}$$

as in Lemma 2, so that j has the left lifting property with respect to all fibrations and is a weak equivalence, and q is a fibration. Then q is a trivial fibration, so the dotted arrow exists making the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & \overline{B} \arrow[d, "q"] \\ B \arrow[r, "1_B"'] \arrow[ur, dotted] & B \end{tikzcd}$$

commute. The map i is therefore a retract of j, so i has the desired lifting property. □

Theorem 5 *Subject to the definitions given above, the category DGA_k and the classes of cofibrations, fibrations and weak equivalences satisfy the axioms for a closed model category.*

Proof CM1 is a consequence of the completeness and cocompleteness of the category DGA_k. The weak equivalence axiom **CM2** and the retraction axiom **CM3** are both trivial. The factorization axiom **CM5** is a consequence of Lemma 2 and Lemma 3, and **CM4** follows from Lemma 4. □

References

[1] Bousfield, A. K. [1975], *The localization of spaces with respect to homology*, Topology **14**, 133–150.

[2] Bousfield, A. K. and Gugenheim, V. K. A. M. [1976], *On PL De Rham theory and rational homotopy type*, Memoirs A.M.S. **179**.

[3] Goerss, P. G. [1995], *Simplicial chains over a field and p-local homotopy theory*, Math. Z. **220**, 523–544.

[4] Karoubi, M. [1995], *Formes differentielles non commutatives et opérations de Steenrod*, Topology **34**, 699–715.