Bundle Theory for Categories

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Joint work with Steve Lack.

Outline

1. Bundles of finite sets
2. Bundles of categories
3. Bundles of monoidal categories
4. Bundles of bicategories

Global and local descriptions

A bundle of finite sets is a set map

$$p : E \to B,$$

such that $p^{-1}(b)$ is a finite set for all $b \in B$.

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where FinSet is the set of finite sets.

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where FinSet is the set of finite sets.

Any set map $\Phi : B \to \mathrm{FinSet}$ can be globalized: let

$$E_\Phi = \{(b, x) \mid x \in \Phi(b), b \in B\}$$

and

$$p_\Phi : E_\Phi \to B : (b, x) \mapsto b.$$

The tautological bundle

Let

$$\mathrm{FinSet}_* = \{(X, x) \mid X \in \mathrm{FinSet}, x \in X\}.$$

The tautological bundle of finite sets is the map

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Observe that $\tau_{set}^{-1}(X) = X$ for all $X \in \mathrm{FinSet}$.

Classification

Proposition

The bundle τ_{set} classifies bundles of finite sets.

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Proof.

The globalization $p_\Phi : E_\Phi \to B$ of $\Phi : B \to \mathrm{FinSet}$ fits into a pullback square

$$\begin{array}{ccc} E_\Phi & \longrightarrow & \mathrm{FinSet}_* \\ {\scriptstyle p_\Phi}\downarrow & & \downarrow{\scriptstyle \tau_{set}} \\ B & \xrightarrow{\Phi} & \mathrm{FinSet} \end{array} .$$

Moreover, it is obvious that

$$p_{\Phi_p} = p \quad \text{and} \quad \Phi_{p_\Phi} = \Phi$$

for all $p : E \to B$ and for all $\Phi : B \to \mathrm{FinSet}$. □

The local description

Let **Cat** denote the category of small categories

Let **B** denote any category. Local category bundle data over **B** is a functor

$$\Phi : \mathbf{B} \longrightarrow \mathbf{Cat}.$$

The global description

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The global description

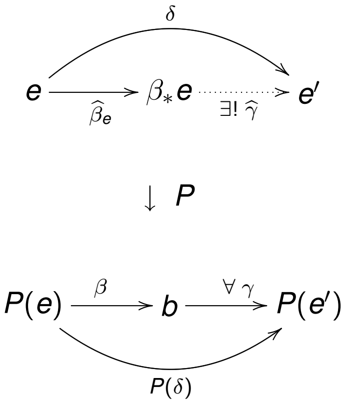
A functor $P : \mathbf{E} \to \mathbf{B}$ is a bundle of categories if it is a split opfibration with small fibers , i.e.:

- (Existence of a lift)

$$\begin{array}{ccccc} e & \overset{\exists\, \widehat{\beta}_e}{\dashrightarrow} & \beta_* e & & \in \mathbf{E} \\ & \downarrow P & & & \downarrow P \\ P(e) & \xrightarrow{\forall\, \beta} & b & & \in \mathbf{B}. \end{array}$$

The global description

- (Universal property of the lift)



The global description

- $$\begin{array}{ccccc} & & & & (\gamma\beta)_*(e) \\ & & & \nearrow^{\widehat{\gamma\beta}_e} & \| \\ e & \xrightarrow[\widehat{\beta}_e]{} & \beta_* e & \xrightarrow[\widehat{\gamma}_{\beta_* e}]{} & \gamma_*\beta_* e \end{array}$$

  $$\downarrow \ P$$

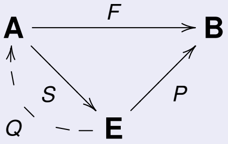
  $$P(e) \xrightarrow{\forall\ \beta} b \xrightarrow{\forall\ \gamma} c$$

  and $\widehat{Id_{P(e)}}_e = Id_e$ for all e.
- Each $P^{-1}(b)$ is a small category.

Associated bundles

Proposition

For any $F : \mathbf{A} \to \mathbf{B}$*, there is a natural factorization*



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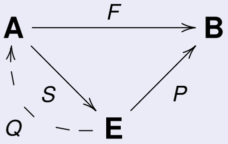

such that P *is bundle of categories,* $QS = Id_{\mathbf{A}}$ *and there is a natural transformation* $SQ \Rightarrow Id_{\mathbf{E}}$.

The proof is highly analogous to the usual proof that any continuous map can be factored as a homotopy equivalence followed by a fibration.

Invariance under pullback

Proposition

If $P : \mathbf{E} \to \mathbf{B}$ is a bundle of categories and $F : \mathbf{A} \to \mathbf{B}$ is any functor, then the pullback

$$F^*P : \mathbf{E} \underset{\mathbf{B}}{\times} \mathbf{A} \to \mathbf{A}$$

of P along F is also a bundle of categories.

The proof is very straightforward.

The tautological bundle

Let $\mathbf{Cat}_*$ be the category of pointed, small categories:

$$\mathrm{Ob}\,\mathbf{Cat}_* = \{(\mathbf{A}, a) \mid \mathbf{A} \in \mathrm{Ob}\,\mathbf{Cat}, a \in \mathrm{Ob}\,\mathbf{A}\};$$

$$\mathbf{Cat}_*\big((\mathbf{A}, a), (\mathbf{B}, b)\big) = \{(F, f) \mid F : \mathbf{A} \to \mathbf{B}, f : F(a) \to b\}.$$

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The tautological bundle of categories is the functor

$$\tau_{cat} : \mathbf{Cat}_* \to \mathbf{Cat} : \begin{cases} (\mathbf{A}, a) \mapsto \mathbf{A} \\ (F, f) \mapsto F. \end{cases}$$

Observe that $\tau_{cat}^{-1}(\mathbf{A}) \cong \mathbf{A}$ for all small categories **A.**

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Proof.

- Local to global: Given local category bundle data $\Phi : \mathbf{B} \to \mathbf{Cat}$, consider the pullback

$$\begin{array}{ccc} \mathbf{E}_\Phi & \longrightarrow & \mathbf{Cat}_* \\ {\scriptstyle P_\Phi}\downarrow & & \downarrow{\scriptstyle \tau_{cat}} \\ \mathbf{B} & \xrightarrow{\Phi} & \mathbf{Cat}. \end{array}$$

  Since τ_{cat} is a bundle of categories, P_Φ is also a bundle of categories.

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□

Classification

Proof.

- Global to local: Given a bundle of categories $P : \mathbf{E} \to \mathbf{B}$, define $\Phi_P : \mathbf{B} \to \mathbf{Cat}$ by $\Phi_P(b) = P^{-1}(b)$ and

$$
\begin{array}{ccc}
e_1 & \xrightarrow{\widehat{\beta}_{e_1}} & \beta_* e_1 \\
\uparrow{\scriptstyle \xi} & & \vdots \uparrow {\scriptstyle \widehat{Id_{b'}} = \Phi(\beta)(\xi)} \\
e_0 & \xrightarrow{\widehat{\beta}_{e_0}} & \beta_* e_0
\end{array}
$$

$$\downarrow \ P$$

$$b \underset{\beta}{\overset{P(\widehat{\beta}_{e_1} \circ \xi)}{\rightrightarrows}} b'.$$

□

The “geometric” viewpoint

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The functor $\Phi_P(\beta) : P^{-1}(b) \to P^{-1}(b')$ can therefore be seen as “parallel transport” along the path β from the fiber over b to the fiber over b'.

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Thus: bundles of categories can be thought of as functors endowed with a flat connection.

(The nonflat case corresponds to considering *pseudofunctors* **B** $\to$ **CAT**.)

Example: covering spaces

For any topological space X, let $\Pi(X)$ denote its fundamental groupoid, i.e., $\text{Ob}\,\Pi(X) = X$ and $\Pi(X)(x, x')$ is the set of based homotopy classes of paths from x to x'.

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If $p : E \to B$ is a covering map of topological spaces, then $\Pi(p) : \Pi(E) \to \Pi(B)$ is bundle of categories.

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If $p : E \to B$ is a covering map of topological spaces, then $\Pi(p) : \Pi(E) \to \Pi(B)$ is bundle of categories.

The corresponding local data $\Phi_p : \Pi(B) \to \mathbf{Cat}$ is such that $\Phi_p(b)$ is the fundamental groupoid of $p^-1(b)$.

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- (Flores) Classifying spaces for families of subgroups: to a discrete group G and a family $\mathcal{F}$ of subgroups of G, there is associated a functor

$$R : \mathbf{O}_{\mathcal{F}} \to \mathbf{Cat} : G/H \to G/H.$$

The nerve of $\mathbf{E}_R$ is then a model for $E_{\mathcal{F}}G$: it is a G-CW-complex such that every isotropy group belongs to $\mathcal{F}$ and the fixed-point space with respect to any element of $\mathcal{F}$ is contractible.

The local description

Recall that **Cat** admits a monoidal structure, given by cartesian product.

Let **B** denote any monoidal category.

Local monoidal bundle data over **B** is a monoidal functor

$$\Phi : \mathbf{B} \to \mathbf{Cat}.$$

The global description

Let **B** and **E** be monoidal categories.

A strict monoidal functor $P : \mathbf{E} \to \mathbf{B}$ that is a bundle of categories is a bundle of monoidal categories if the lifts satisfy:

$$\widehat{\beta}_e \otimes \widehat{\beta'}_{e'} = \widehat{\beta \otimes \beta'}_{e \otimes e'},$$

for all $\beta : P(e) \to b$ and $\beta' : P(e') \to b'$ in **B**.

The tautological bundle

The tautological bundle of categories

$$\tau_{cat} : \mathbf{Cat}_* \longrightarrow \mathbf{Cat}$$

is a bundle of monoidal categories.

Classification

Theorem

The tautological bundle τ_{cat} classifies bundles of monoidal categories.

Proof.

Using the constructions of the previous classification theorem, we see that

$$\Phi : \mathbf{B} \to \mathbf{Cat} \text{ monoidal } \Rightarrow$$

$$P_\Phi : \mathbf{E}_\Phi \to \mathbf{B} \text{ bundle of monoidal categories}$$

and

$$P : \mathbf{E} \to \mathbf{B} \text{ bundle of monoidal categories} \Rightarrow$$

$$\Phi_P : \mathbf{B} \to \mathbf{Cat} \text{ monoidal }.$$

□

The "geometric" viewpoint

If $P : \mathbf{E} \to \mathbf{B}$ is a bundle of monoidal categories, then the associated "parallel transport" is such that

$$\begin{array}{ccc} P^{-1}(b_0) \times P^{-1}(b_0') & \xrightarrow{\mu} & P^{-1}(b_0 \otimes b_0') \\ {\scriptstyle \Phi_P(\beta)\times\Phi_P(\beta')}\downarrow & & \downarrow{\scriptstyle \Phi_P(\beta\otimes\beta')} \\ P^{-1}(b_1) \times P^{-1}(b_1') & \xrightarrow{\mu} & P^{-1}(b_1 \otimes b_1') \end{array}$$

commutes for all "paths" $\beta : b_0 \to b_1$ and $\beta' : b_0' \to b_1'$.

Example: modules over a fixed ring

Let R be a ring, and let X be a left R-module.

Let $(\mathbf{R}, \otimes, I)$ be the monoidal category where

- $\operatorname{Ob}\mathbf{R} = \mathbb{N}$;
- $\mathbf{R}(m,n) = \mathfrak{M}_{nm}(R)$, the set of $(n \times m)$-matrices with coefficients in R;
- composition is given by matrix multiplication;
- $m \otimes m' := m + m'$, $I := 0$ and for all $M \in \mathfrak{M}_{nm}(R)$, $M' \in \mathfrak{M}_{n'm'}(R)$

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Let $\mathbf{X}$ be the category with one object $*$ and with morphism set equal to X.

Example: modules over a fixed ring

Let $\Phi_X : \mathbf{R} \to \mathbf{Cat}$ denote the functor given by

$$\Phi_X(m) = \mathbf{X}^{\times m}$$

and

$$\Phi_X(M) : \mathbf{X}^{\times m} \to \mathbf{X}^{\times n} : \begin{cases} * \mapsto * \\ \vec{x} \mapsto M\vec{x} \end{cases}$$

for all $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in X^{\times n}$.

It is easy to see that Φ_X is monoidal and therefore gives rise to a bundle of monoidal categories

$$P_X : \mathbf{E}_X \to \mathbf{R}.$$

Example: modules over a fixed ring

Proposition

The categories of left and of right modules over a fixed ring R embed into the category of bundles of monoidal categories over $\mathbf{R}$.

The matrix bicategory

$\mathcal{MAT}$ is specified by

- $\mathcal{MAT}_0 = \mathrm{Ob}\,\mathbf{Set}$ and
- for all $U, V \in \mathcal{MAT}_0$,

$$\mathcal{MAT}(U, V) = \mathbf{Cat}^{U \times V},$$

where U and V are seen as discrete categories.

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where U and V are seen as discrete categories.

Horizontal composition

$$\mathcal{MAT}(U, V)\times\mathcal{MAT}(V, W) \longrightarrow \mathcal{MAT}(U, W) : (A, B) \mapsto A*B$$

is given by matrix multiplication, i.e.,

$$(A * B)(u, w) = \coprod_{v\in V} A(u, v) \times B(v, w)$$

for all $u \in U$ and $w \in W$.

The local description

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Local bicategory bundle data over $\mathcal{B}$ consist of a lax functor $\Phi : \mathcal{B} \to \mathcal{MAT}$.

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Local bicategory bundle data over $\mathcal{B}$ consist of a lax functor $\Phi : \mathcal{B} \to \mathcal{MAT}$.

This is a sort of "parametrized" version of the local data for a bundle of monoidal categories. In particular, local bicategory bundle data is obtained when local data for a bundle of monoidal categories is "suspended."

The global description

A strict homomorphism of bicategories $\Pi : \mathcal{E} \to \mathcal{B}$ is a bundle of bicategories if

- The induced functor on hom-categories $\Pi : \mathcal{E}(e, e') \to \mathcal{B}\big(\Pi(e), \Pi(e')\big)$ is a bundle of categories for all 0-cells e, e' in $\mathcal{E}$.

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- The composition functors

$$\begin{array}{ccc} \mathcal{E}(e,e') \times \mathcal{E}(e',e'') & \xrightarrow{\quad c \quad} & \mathcal{E}(e,e'') \\ \Big\downarrow{\scriptstyle \Pi\times\Pi} & & \Big\downarrow{\scriptstyle \Pi} \\ \mathcal{B}\big(\Pi(e),\Pi(e')\big) \times \mathcal{B}\big(\Pi(e'),\Pi(e'')\big) & \xrightarrow{\ c\ } & \mathcal{B}\big(\Pi(e),\Pi(e'')\big) \end{array}$$

  are morphisms of bundles of categories.

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  are morphisms of bundles of categories.
- The associator and the unitors in $\mathcal{B}$ lift to the associator and the unitors in $\mathcal{E}$.

The pointed matrix bicategory

$\mathcal{MAT}_*$ is specified by

- $(\mathcal{MAT}_*)_0 = \mathbf{Set}_*$ and
- for all $(U, u), (V, v) \in (\mathcal{MAT}_*)_0$,

$$\mathcal{MAT}_*\big((U, u), (V, v)\big) = \mathbf{Cat}_*^{\big(U \times V, (u,v)\big)},$$

where $\big(U \times V, (u, v)\big)$ is seen as a discrete, pointed category.

Horizontal composition is again given by matrix multiplication.

The tautological bundle

The tautological bundle of bicategories is the strict homomorphism

$$\tau_{bicat} : \mathcal{MAT}_* \to \mathcal{MAT}$$

given by forgetting basepoints.

Classification

Theorem

The bundle τ_{bicat} classifies bundles of bicategories.

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Proof.

- Local to global: Given local bicategory bundle data $\Phi : \mathcal{B} \to \mathcal{MAT}$, consider the pullback

$$\begin{array}{ccc} \mathcal{E}_\Phi & \longrightarrow & \mathcal{MAT}_* \\ \Pi_\Phi \downarrow & & \downarrow \tau_{bicat} \\ \mathcal{B} & \xrightarrow{\Phi} & \mathcal{MAT}. \end{array}$$

  Then Π_Φ is a bundle of bicategories, a sort of parametrized Grothendieck construction on Φ.

□

The “geometric” viewpoint: fibers over 1-cells

Let $\Pi : \mathcal{E} \to \mathcal{B}$ be a bundle of bicategories.

Let $f : b \to b'$ be a 1-cell in $\mathcal{B}$. Let e, e' be 0-cells of $\mathcal{E}$ such that $\Pi(e) = b$, $\Pi(e') = b'$.

The fiber category $\mathbf{Fib}^f_{e,e'}$ over f with respect to (e, e'):

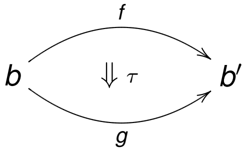
$$\hat{f} \in \operatorname{Ob} \mathbf{Fib}^f_{e,e'} \Rightarrow \hat{f} : e \to e' \quad \text{and} \quad \Pi(\hat{f}) = f$$

and

$$\alpha \in \mathbf{Fib}^f_{e,e'}(\hat{f}, \hat{f}') \Rightarrow \alpha : \hat{f} \to \hat{f}' \quad \text{and} \quad \Pi(\alpha) = Id_f.$$

The “geometric” viewpoint: parallel transport along 2-cells

Since $\Pi : \mathcal{E}(e, e') \to \mathcal{B}\big(\Pi(e), \Pi(e')\big)$ is a bundle of categories, for each 2-cell

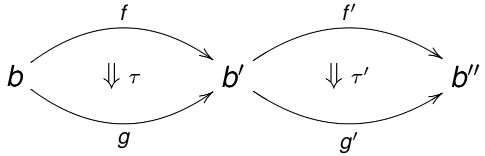


there is a functor

$$\nabla^{\tau}_{e,e'} : \mathbf{Fib}^{f}_{e,e'} \longrightarrow \mathbf{Fib}^{g}_{e,e'}.$$

The "geometric" viewpoint: parallel transport and composition

Furthermore, for all



in $\mathcal{B}$,

$$\begin{array}{ccc}
\mathbf{Fib}^{f}_{e,e'} \times \mathbf{Fib}^{f'}_{e',e''} & \xrightarrow{\;C\;} & \mathbf{Fib}^{f'f}_{e,e''} \\
\Big\downarrow{\scriptstyle \nabla^{\tau}_{e,e'} \times \nabla^{\tau'}_{e',e''}} & & \Big\downarrow{\scriptstyle \nabla^{\tau'\tau}_{e,e''}} \\
\mathbf{Fib}^{g}_{e,e'} \times \mathbf{Fib}^{g'}_{e',e''} & \xrightarrow{\;C\;} & \mathbf{Fib}^{g'g}_{e,e''}
\end{array}$$

commutes.

Examples

- Charted bundles with coefficients in a topological bicategory (cf., Baas-Dundas-Rognes or Baas-Bökstedt-Kro) naturally give rise to bundles of bicategories.
- Parametrized Kleisli constructions.
- The domain projection from the Bénabou bicategory of cylinders in a fixed bicategory $\mathcal{B}$ down to $\mathcal{B}$ is a bundle of bicategories.

Work in progress

- *K-theory:* All these categories of bundles admit "Whitney sum" and "tensor product"-type operations. What information does the associated "bundle K-theory" carry? Should englobe both topological and algebraic K-theory.

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- *K-theory:* All these categories of bundles admit "Whitney sum" and "tensor product"-type operations. What information does the associated "bundle K-theory" carry? Should englobe both topological and algebraic K-theory.
- *Homotopy theory:* How do these bundle notions interact with the homotopy theory of **Cat** and of **Bicat**?