Cohomotopy groups capture robust Properties of Zero Sets via Homotopy Theory

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- In many cases, the preimage of zero (or any single point in $\mathbb{R}^n$) plays a crucial role.
- Often we have access only to an approximation of the actual map.

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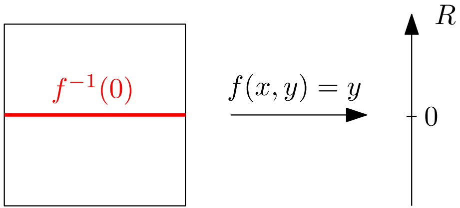

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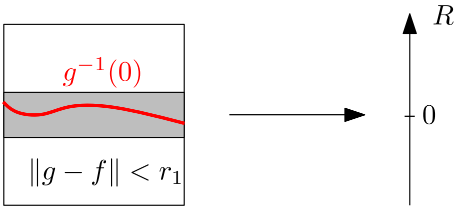

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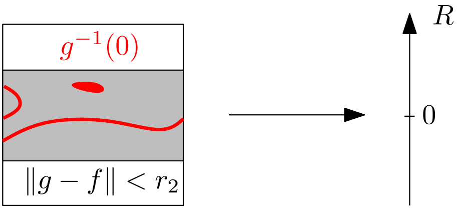

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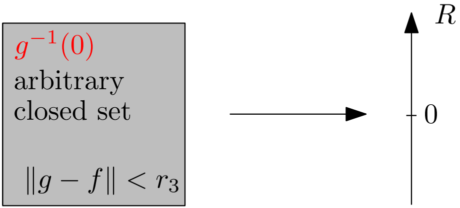

Formalization

For $f : X \to \mathbb{R}^n$ and $r > 0$, let

$$Z_r(f) := \{g^{-1}(0) : \ g : X \to \mathbb{R}^n \text{ s.t. } \|g - f\| < r\}$$

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- Robust volume: $\inf_{Z \in Z_r(f)} \mathcal{H}^{m-n}(Z)$ where $m = \dim X$

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not to study f where it is small
but rather where it is big
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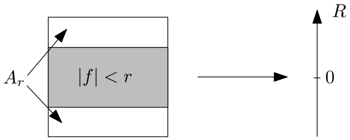

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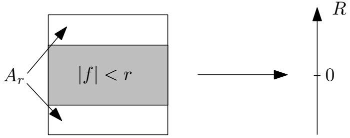

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Let $f : X \to \mathbb{R}^n$ and X be compact. If $A_r := \{x : |f(x)| \geq r\}$ is given, then $Z_r(f)$ is determined by the homotopy class of $f/|f| : A_r \to S^{n-1}$.

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- $\supseteq$ ($e \rightsquigarrow g$):
  multiply e by a scalar function that is 1 of A_r and goes quickly to 0 elsewhere.

□

Robust nonemptiness

Immediate consequence:

$$\emptyset \notin Z_r(f) \Leftrightarrow f/|f| \colon A_r \to S^{n-1} \text{ can be extended to } X \to S^{n-1}$$

The extendability problem is in decidable when $\dim X \leq 2n-3$ (or $n = 1, 2$ or n even) and is undecidable otherwise.

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Moreover, if $A_r \subseteq X$ are CW complexes and $\dim X \leq 2n-3$, then $Z_r(f)$ is determined by the δ-image of the above homotopy class, where δ is the "connecting homomorphism" in the sequence

$$\begin{array}{ccccccc} \dots & \to & [X, S^{n-1}] & \xrightarrow{i^*} & [A_r, S^{n-1}] & \xrightarrow{\delta} & [X/A_r, S^n] \\ & & & & \Cup & & \Cup \\ & & & & [f/|f|] & \mapsto & [f_{/A_r}] \end{array}$$

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- We denote $\mathrm{Im}\,\delta$ by π_r (group of all descriptors)
- The theorem does not give recipes for how to decode particular robust features from the homotopy class... but it yields a persistence-like tool for distinguishing

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- Similarly, there is a map $\pi_r \to \pi_s$ that takes $[f_{/A_r}]$ to $[f_{/A_s}]$

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After tensoring with a field, Π_f may be represented via a *pointed barcode*

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Let X be a finite simplicial complex, $f : X \to \mathbb{R}^n$ be simplexwise linear with rational values on vertices and assume $\dim X \leq 2n - 3$. Then the isomorphism type of Π_f as well as barcode of $\Pi_f \otimes \mathbb{F}$ for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}$ finite can be computed.

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- Computability of cohomotopy groups $[Y, S^{n-1}]$ in the dimension range $\dim Y \leq 2n - 4$.
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- Approximation of A_r (up to homotopy equivalence) by simplicial subcomplex $A_r^{\Delta} \subseteq X$.
- Simplicial approximation of $f/|f| : A_r^{\Delta} \to (S^{n-1})^{\Delta}$.

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- $n = \dim X = 3$ already satisfies $\dim X \leq 2n - 3$. However, if $\dim X = 4$ and $n = 3$ then $\emptyset \in Z_r(f)$ is undecidable!
- $n = 4$ is nice: $f/|f| \in [A_r, S^3]$ and $[Y, S^3]$ is a group for any Y due to quaternion multiplications – computability of $[Y, S^3]$ is work in progress.

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- **Cap image groups:** computable replacement of well groups
  - subgroups of well groups
  - based on primary obstruction to extending $f/|f|\colon A_r \to S^{n-1}$ the primary obstruction is the "first component" of $[f_{/A_r}]$

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- **Well groups:** capture homological properties common to all $Z \in Z_r(f)$ (informally)
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  - undecidability for $\dim X = 2n - 2$
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Our coding effort: compute the secondary (terciary) obstructions and see how much they matter.
Cap image groups can be used to study preimages of all points in $\mathbb{R}^n$ simultaneously in some sense: provide an alternative to multidimensional persistence.

Optimality of Π_f

Still, the homotopy class $[f_{/A_r}]$ carries more information than needed to encode $Z_r(f)$. If A_r is given, then different elements of π_r may determine the same family of zero sets.

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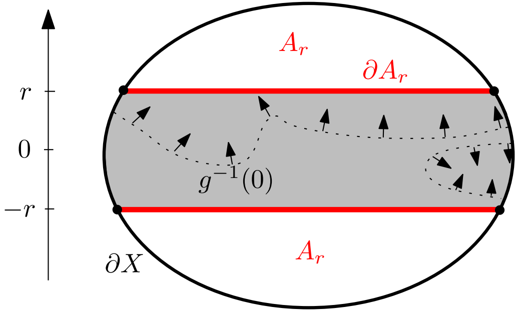
Elements of Z_r^{fr} are framed dim $X - n$ dimensional submanifolds of X, contained in the complement of A_r (trivialization of the normal bundle).



Optimality of Π_f

Theorem

Assume that X is a smooth compact m-manifolds, $r > 0$, $A_r = h^{-1}[0, \infty)$ for some regular h, and $m \leq 2n - 3$. Then there is a bijection

$$\{Z_r^{\mathrm{fr}}(f) \mid f : X \to \mathbb{R}^n \text{ such that } A_r = |f|^{-1}[r, \infty)\} \longleftrightarrow \pi_r$$

satisfying that each $Z_r^{\mathrm{fr}}(f)$ is mapped to $[f_{/A}]$.

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Moreover, each element of $Z_r^{\mathrm{fr}}(f)$ determines $[f_{/A}]$.

- So, $[f_{/A_r}]$ is an invariant of $Z_r^{\mathrm{fr}}(f)$
- The additional information in $[f_{/A_r}] \in \pi_r$ encodes the infinitesimal behaviour of perturbation(s) around their zero sets.

Key idea of the proof

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- Again, regular perturbations can be replaced by "regular homotopy perturbations."
- From a regular homotopy we get a framed cobordism easily.
- The other direction is more difficult.

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- **Follow the approach of cap image groups.**
  We can construct $\Pi_f(c)$ for any $c \in \mathbb{R}^n$, not just $c = 0$. Can we compute some data structure built from $\Pi_f(c)$, $c \in \mathbb{R}^n$, that robustly describes f itself (not just the zero set)?

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- Practical implementation