24. November 2010

# POVMs and superoperators

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In this talk, it was described how positive operator valued measures (POVMs) can be used to generalize our understanding of what a measurement actually is. The connection between ordinary POVMs and the more special, but also more familiar projection valued measures (PVMs) was discussed, and the important Neumark's theorem was stated. Later, we discussed how POVMs (under the name of Kraus operators) are connected to the time evolution of some subsystem of a larger Hilbert space. This helped us to refine our understanding of the Schrödinger's cat paradox.

## 1 Introduction

The kind of measurement that we are used to in the mathematical description of qantum mechanics is represented by a hermitian operator which is applied to quantum states. Via the spectral decomposition, such an operator can be described as a sum of mutually commuting projection operators, weighted with the possible outcomes of the measurement. Two important questions which arise in the study of such measurements will be answered in this talk, and the answer will in both cases be: positive operator valued measures, or a little bit shorter: POVMs!

The first question is: Can we generalise our understanding of measurements? In particular, if we perform an ordinary measurement on a given Hilbert space, how can we describe the action of the measurement on a subsystem without referring to the whole Hilbert space? The answer to this question will be given in section 3 by *Neumark's theorem* which will tell us that on the subsystem, the action of a projective measurement will always be described by a POVM (which we will define in section 2).

The second question is: If the Schrödinger equation tells us that the evolution of a closed

quantum system is governed by a unitary transformation, how can we describe the evolution of a subsystem? We will see in the sections 4 and 5 that the answer to this is that the evolution of the subsystem is described by so called *Kraus operators* which by definition also form a POVM.

# 2 Positive operator valued measures (POVMs)

## 2.1 Definition

On a finite dimensional Hilbert space $\mathcal{H}$ (which will be sufficient for this talk), a POVM is a set of operators $\{E_m\}$ $(m = 1, 2, ...)$ with the following properties [1]:

$$E_m^\dagger = E_m \text{ for all } m$$

$$\langle\psi| E_m |\psi\rangle \geq 0 \text{ for all } m \text{ and all } |\psi\rangle \in \mathcal{H}$$

We say the $E_m$ are *positive operators*

$$\sum_m E_m = \mathbf{1} \quad \text{This is called a } \textit{partition of unity}$$

The underlying philosophy is the following: Imagine we have a given measurement-apparatus (which shall be represented by the POVM), with a set of possible measurement-results. Then we assign a *POVM element* $E_m$ to every possible outcome, and we choose the $E_m$ so that the probability of the corresponding outcome for a state $|\psi\rangle$ is given by the expectation value $\text{Prob}(m) = \langle\psi| E_m |\psi\rangle$. The requirements that we imposed upon the POVM elements ensure that all these probabilities are positive and add up to one [1]. Note that the condition of hermiticity is redundant if the condition of positivity is given, but above it was nonetheless seperately stated for the sake of clarity. Before looking at an application-example, we will clarify some more mathematical details. Afterwards, in section 3 we will discuss Neumark's theorem and hence get a better understanding of the meaning of POVMs. For a detailed description of how any possible POVM on the 2-dimensional Hilbert space of photon-polarization can be implemented in an actual measurement, see [2]. For the definition of POVMs in the case of infinite-dimensional Hilbert spaces see [3].

## 2.2 Measurement operators

There always exists a set of *measurement operators* $\{M_m\}$ which describes the POVM $\{E_m\}$ via

$$E_m = M_m^\dagger M_m$$

We write $M_m = \sqrt{E_m}$. Such a set need not be unique, but for every POVM it is possible to find one, due to the positivity of the operators $E_m$. [1]
As we know, the probabilities for the different possible outcomes of the measurement are already given by the POVM elements, the measurement operators are only needed if we are interested in the new state after the measurement, which in the case of the result corresponding to $E_m$ is given by ([1])

$$|\psi_m\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}$$

So obviously the state after the measurement depends on the result of the measurement, this corresponds to the *collapse of the wave function* in the Copenhagen interpretation of quantum mechanics. Please note that in the case of POVMs the word "collapse" may not be fully adequate, as the repeated application of a POVM-measurement on a certain quantum system will not always yield the same result again, in contrast to the projection valued measures discussed in the next section. Nevertheless, such a kind of (time) evolution of a quantum system is not unitary and not time-invertible, in contrast to the time evolution described by the Schrödinger equation.
In general, the initial state may be described by a density matrix. If we now imagine that a measurement has been performed, but we have forgotten the result, then the state after the measurement has again to be described by a density matrix ([4])

$$\rho' = \sum_m M_m \rho M_m^\dagger$$

We will recover exactly the same equation later as (1) in a completely different context. Note that:

$$\mathrm{tr}[\rho'] = \mathrm{tr}[\sum_m M_m \rho M_m^\dagger] = \sum_m \mathrm{tr}[M_m^\dagger M_m \rho] = \mathrm{tr}[\sum_m E_m \rho] = \mathrm{tr}[\rho]$$

## 2.3 Projection valued measures (PVMs)

Usually, an observable is described by a hermitian operator $A$ with decomposition

$$A = \sum_m A_m P_m$$

in which $A_m$ is the $m$-th eigenvalue, and $P_m$ is the projection operator on the corresponding eigenspace of $A$, with the Properties $P_m^\dagger = P_m$, $P_m P_n = \delta_{m,n} P_m$ and $\sum_m P_m = \mathbf{1}$. Such a PVM is a special case of a POVM with $M_m = E_m = P_m$. Note the following differences between POVMs and PVMs:
The operators $P_m$ are mutually orthogonal, while this need not be the case for the $E_m$. Therefore, the number of projection operators $P_m$ in the decomposition of $A$ is limited by the dimension of $\mathcal{H}$, while the number of $E_m$ is unlimited. Also, if a PVM is applied to a certain quantum state repeatedly, it will always yield the same result, while this need not be true for a general POVM.[1]

## 2.4 Application example

Suppose Alice sends Bob Q-bits in one of the two states $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ with the same probability of $1/2$. We know that Bob is principially not able to distinguish between $|\psi_1\rangle$ and $|\psi_2\rangle$ perfectly, as they are not orthogonal. But Bob can use a POVM of the form

$$\begin{aligned} E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1| \\ E_2 &= \frac{\sqrt{2}}{(1+\sqrt{2})2} (|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\ E_3 &= \mathbf{1} - E_1 - E_2 \end{aligned}$$

So now, if Bob gets the result corresponding to $E_1$ ($E_2$) he can be sure that he got the state $|\psi_2\rangle$ ($|\psi_1\rangle$) from Alice because $E_1 |\psi_1\rangle = E_2 |\psi_2\rangle = 0$. If he gets the result $E_3$, the measurement has to be discarded, but in contrast to any possible PVM (like for example $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$), Bob exactly kows when he has to ignore the outcome of a measurement, and therefore never makes a mistake in distinguishing between the two possible states. [1]
How can this example be related to the usual projection valued measures that we are familiar with? It turns out that, by adding one more dimension to the Hilbert space, for example by coupling the measured state to a known *ancilla* state, our given POVM

can be described as a PVM in the larger dimensional Hilbert space [4]. This statement can be generalized and is related to Neumark's theorem which will be dealt with in the next section.

In this particular example, we first need to calculate the spectral decompositions of the POVM elements [4]. It turns out that all of them have one eigenvalue = 0 and can therefore be written in the form $E_i = \left|\tilde{\psi}_i\right\rangle \left\langle\tilde{\psi}_i\right|$ with non normalized states $\left|\tilde{\psi}_i\right\rangle$. For $i = 1, 2$ this is just the form in which they are given, in the case $i = 3$ the fact that one eigenvalue is zero is noteworthy. The prefactors in the defenitions of $E_1$ and $E_2$ have to be chosen as large as possible, in order to minnimize the expectation value of $E_3$, and therefore also the probability to get an inconclusive result. But also, we have to ensure that $E_3$ is a positive operator, in order to be a POVM element. As it turns out, one eigenvalue of $E_3$ is zero because the prefactors of $E_1$ and $E_2$ are chosen optimally. If they were larger, the second eigenvalue of $E_3$ would be negative and the POVM-conditions would be violated.

Using the basis $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we find:

$$\left|\tilde{\psi}_1\right\rangle = \sqrt{\frac{\sqrt{2}}{1+\sqrt{2}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left|\tilde{\psi}_2\right\rangle = \sqrt{\frac{\sqrt{2}}{2\left(1+\sqrt{2}\right)}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left|\tilde{\psi}_3\right\rangle = \sqrt{\frac{1}{\sqrt{2}}} \begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$$

If we add one dimension to our Hilbert space, then we have to extend $E_1$, $E_2$ and $E_3$ to a set of 3-dimensional projection operators $P_1$, $P_2$ and $P_3$ which project onto mutually orthogonal subspaces of the Hilbert space, i.e. which mutually commute and therefore can be composed to an ordinary observable $A = \sum_m A_m P_m$. With the definition $P_i = |\phi_i\rangle \langle\phi_i|$, we now have to search for an orthonormal set of 3-dimensional vectors $\phi_i = \begin{pmatrix} \tilde{\psi}_i \\ ?_i \end{pmatrix}$ with a yet to be determined third component (we abandon the bra-cet-notation for a while). We know that if the rows of a square matrix form an orthonormal basis (ONB), then the columns do so as well. So we can take the rows of the $3 \times 3$-matrix $(\phi_1, \phi_2, \phi_3)$ to be a set of two known vectors $v_1$ and $v_2$ and a completely unknown vector $v_3 = (?_1, ?_2, ?_3)^T$. Indeed, the two vectors $v_1$ and $v_2$ are already orthonormal with respect to each other. This may seem surprising, but indeed it is a result of the requirement

that our POVM has to be a partition of unity [4]. The construction of $v_3$ is now quite easy: we just take an arbitrary vector which is not in span $(v_1, v_2)$ (like $(1,0,0)^T$) and then apply the Gram-Schmidt orthonormalisation-procedure. In our case, this yields $v_3 = \left((1+\sqrt{2})^{-1/2}, (1+\sqrt{2})^{-1/2}, 1-\sqrt{2}\right)^T$ and therefore we find:

$$|\phi_1\rangle = \begin{pmatrix} \sqrt{\frac{\sqrt{2}}{1+\sqrt{2}}} \\ 0 \\ \frac{1}{\sqrt{1+\sqrt{2}}} \end{pmatrix} \quad |\phi_2\rangle = \begin{pmatrix} \frac{-1}{\sqrt{2+\sqrt{2}}} \\ \frac{1}{\sqrt{2+\sqrt{2}}} \\ \frac{1}{\sqrt{1+\sqrt{2}}} \end{pmatrix} \quad |\phi_3\rangle = \begin{pmatrix} \frac{\sqrt{2}-1}{\sqrt[4]{2}} \\ \frac{1}{\sqrt[4]{2}} \\ 1-\sqrt{2} \end{pmatrix}$$

This finally yields

$$P_1 = \begin{pmatrix} \frac{\sqrt{2}}{1+\sqrt{2}} & 0 & \frac{\sqrt[4]{2}}{1+\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{\sqrt[4]{2}}{1+\sqrt{2}} & 0 & \frac{1}{1+\sqrt{2}} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \frac{1}{2+\sqrt{2}} & \frac{-1}{2+\sqrt{2}} & \frac{-1}{\sqrt[4]{2}(1+\sqrt{2})} \\ \frac{-1}{2+\sqrt{2}} & \frac{1}{2+\sqrt{2}} & \frac{1}{\sqrt[4]{2}(1+\sqrt{2})} \\ \frac{-1}{\sqrt[4]{2}(1+\sqrt{2})} & \frac{1}{\sqrt[4]{2}(1+\sqrt{2})} & \frac{1}{1+\sqrt{2}} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} \frac{3-2\sqrt{2}}{\sqrt{2}} & \frac{\sqrt{2}-1}{\sqrt{2}} & \frac{2\sqrt{2}-3}{\sqrt[4]{2}} \\ \frac{\sqrt{2}-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1-\sqrt{2}}{\sqrt[4]{2}} \\ \frac{2\sqrt{2}-3}{\sqrt[4]{2}} & \frac{1-\sqrt{2}}{\sqrt[4]{2}} & 3-2\sqrt{2} \end{pmatrix}$$

and the $2\times 2$ parts in the upper left corners are exactly the representations of the $E_i$. The measurement of one of the states $|\psi_1\rangle = (0,1)^T$ or $|\psi_2\rangle = 1/\sqrt{2}\,(1,1)^T$ with the given POVM in the original Hilbert space turns now out to be analogous to the measurement of the states $(0,1,0)^T$ or $1/\sqrt{2}\,(1,1,0)^T$ with the PVM given above in the extended 3-dimensional Hilbert space.

## 3 Neumark's theorem

*Theorem:* Any POVM on a Hilbert space $\mathcal{H}_A$ can be realized by a PVM in a larger Hilbert space $\mathcal{H}$ containing $\mathcal{H}_A$.
*Please note:* As mentioned in the introduction, the reverse is also true: Every PVM on a Hilbert space $\mathcal{H}$ leads to a POVM on any subspace $\mathcal{H}_A$. [1][4]
Therefore, if we want to understand the effect of measurements on a subsystem without referring to the whole Hilbert space, POVMs appear to be a necessarry and indeed very

useful generalisation of PVMs.

### 3.1 Consider the case $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Suppose that we have given an initial state $\rho_{AB} = \rho_A \otimes \rho_B$ and a PVM described by the set of operators $P_m$ acting on $\mathcal{H}$. It is our aim to describe the probability to get the result corresponding to index $m$ only by making use of operators and states on $\mathcal{H}_A$. We therefore define a set of operators $F_m$ on $\mathcal{H}_A$ such that [4]:

$$\text{Prob}(m) = \text{tr}_{AB}\left[P_m \rho_{AB}\right] = \text{tr}_A\left[\text{tr}_B\left[P_m \rho_{AB}\right]\right] \equiv \text{tr}_A\left[F_m \rho_A\right]$$

It is easy to find such $F_m$ by writing the above equation in terms of components:

$$\begin{aligned}\sum_{i,j}(F_m)_{ji}(\rho_A)_{ij} &= \sum_{i,j,\mu,\nu}(P_m)_{ji,\nu\mu}(\rho_B)_{\mu\nu}(\rho_A)_{ij} \\ \Rightarrow (F_m)_{ji} &= \sum_{\mu,\nu}(P_m)_{ji,\nu\mu}(\rho_B)_{\mu\nu}\end{aligned}$$

Here the indices $i, j$ $(\mu, \nu)$ denote components corresponding to basevectors in $\mathcal{H}_A$ $(\mathcal{H}_B)$. Based on this definition of the $F_m$, it is easy to compute the following properties:
- $F_m = F_m^\dagger$
- $F_m$ is a positive operater for all $m$
- $\sum_m F_m = \mathbf{1}_A$

Proof 1: $(F_m)^*_{ij} = \sum_{\mu,\nu}(P_m)^*_{ij,\nu\mu}(\rho_B)^*_{\mu\nu} = \sum_{\mu,\nu}(P_m)_{ji,\mu\nu}(\rho_B)_{\nu\mu} = (F_m)_{ji}$ $\square$
Proof 2: We set the basis such that $\rho_B = \sum_\mu p_\mu \left|\mu_B\right\rangle \left\langle\mu_B\right|$. Then:
$\left\langle\psi_A\right| F_m \left|\psi_A\right\rangle = \sum_\mu p_\mu(\left\langle\psi_A\right| \otimes \left\langle\mu_B\right|)P_m(\left|\psi_A\right\rangle \otimes \left|\mu_B\right\rangle) \geq 0 \ \forall \psi_A \in \mathcal{H}_A$ $\square$
Proof 3: $(\sum_m F_m)_{ij} = \sum_{\mu,\nu}\sum_m (P_m)_{ij,\nu\mu}(\rho_B)_{\mu\nu} = \sum_{\mu,\nu}(\mathbf{1}_{AB})_{ij,\nu\mu}(\rho_B)_{\mu\nu} = (\text{tr}_B[\mathbf{1}_{AB}\rho_B])_{ij} = (\mathbf{1}_A)_{ij}$ $\square$
Therefore, the set $F_m$ indeed forms a POVM. In the proofs we used the hermiticity of the $P_m$ and $\rho_B$ as well as the positivity of $\rho_B$ and the fact that $\text{tr}_B[\rho_B] = 1$. [4]

## 4 Superoperators and unitary evolution

*Postulate: The time evolution of a closed quantum system is governed by unitary transformations. [1]*

But this need not be true in a subsystem $A$ of a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$! Suppose

our initial state is given by a density matrix

$$\rho_{AB} = \rho_A \otimes |0\rangle \langle 0|_B$$

After the unitary evolution defined by the unitary operator $U_{AB}$ on $\mathcal{H}$, the new state in subsystem $A$ is given by

$$\rho'_A = \mathrm{tr}_B \left( U_{AB} \rho_{AB} U_{AB}^\dagger \right) = \sum_\mu \underbrace{\langle \mu_B | \, U_{AB} \, | 0 \rangle}_{\equiv M_\mu} \rho_A \underbrace{\langle 0 | \, U_{AB}^\dagger \, | \mu_B \rangle}_{= M_\mu^\dagger}$$

where the vectors $|\mu_B\rangle$ form an orthonormal basis in $\mathcal{H}_B$. Now we define a *superoperator* $\$(\rho_A) \equiv \rho'_A$ describing the time evolution of $\rho_A$ entirely in terms of operators acting on $\mathcal{H}_A$ by

$$\$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger \tag{1}$$

in this example. The $M_\mu$ are called *Kraus operators* and satisfy

$$\left( M_\mu^\dagger M_\mu \right) \text{ is hermitian and positive for all } \mu$$
$$\sum_\mu M_\mu^\dagger M_\mu = \mathbf{1}_A$$

therefore they are the measurement operators of a POVM [4]. In order to describe the evolution of the density matrix $\rho_A$ on $\mathcal{H}_A$ in a way that physically makes sense, i.e. in order for $\$ to map a density matrix $\rho_A$ on another density matrix $\$(\rho_A)$, $\$ has to (and does) satisfy the following properties:
0) $\$(\rho_A)$ is linear
1) $\rho_A$ is hermitian $\Rightarrow \$(\rho_A)$ is hermitian
2) $\mathrm{tr}_A(\rho_A) = 1 \Rightarrow \mathrm{tr}_A(\$(\rho_A)) = 1$
3a) $\rho_A$ is positive $\Rightarrow \$(\rho_A)$ is positive
[4][5]. It may not immediately be clear that condition 3a) is actually not enough, as we can imagine system $A$ to be extended by any possible Hilbert space $\mathcal{H}_C$, we must extend 3a) to a condition that is called *complete positivity*.
3b) $\$(\rho_A) \otimes \mathbf{1}_C$ is positive for any extension $\mathcal{H}_C$
There are in fact examples of operators which satisfy 3a) but not 3b), as for example $\$(\rho) = \rho^T$[4]. Also, whether 0) is a necessarry condition or not is a much more subtle question than it may have become clear up to now, for a detailed discussion see [4].

*Kraus representation theorem:* Every operator \$ satisfying 0), 1), 2) and 3b) (called a *superoperator*) has an operator-sum representation (1) and therefore also a unitary representation on a larger Hilbert space $\mathcal{H}$ [4][5].
Proof:
In order to prove this important result, we first introduce the so called *relative state method* [4]: We suppose to have given a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_\mathcal{B}) \geq \dim(\mathcal{H}_\mathcal{A})$ and a state $|\psi\rangle_{AB} = \sum_i |i_A\rangle \otimes |i'_B\rangle \in \mathcal{H}$, where the vectors $|i_A\rangle$ and $|i'_B\rangle$ are the first $\dim(\mathcal{H}_\mathcal{A})$ base-vectors in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. In order to express $|\psi\rangle_{AB}$ in this simple way (with a single sum $\sum_i$ instead of a double sum) we have to choose the orthonormal basis $|i'_B\rangle$ of $\mathcal{H}_B$ appropriately, the possibility to do this is ensured by the *Schmidt decomposition* theorem.
Now if $|\phi_A\rangle = \sum_i a_i |i_A\rangle \in \mathcal{H}_A$, we define the *relative state* $|\phi *_B\rangle = \sum_i a_i^* |i'_B\rangle \in \mathcal{H}_B$ such that $|\phi_A\rangle = \langle \phi *_B |\psi_{AB}\rangle$. This is called a partial scalar product. So for

$$(O_A \otimes \mathbf{1}_B) |\psi_{AB}\rangle = \sum_i O_A |i_A\rangle \otimes |i'_B\rangle$$

we find by applying the relative state method:

$$O_A |\phi_A\rangle = \langle \phi *_B| (O_A \otimes \mathbf{1}_B) |\psi_{AB}\rangle$$

Now we apply this method in the proof of the Kraus-representation-theorem. Our superoperator \$ acts in the following way:

$$(\$_A \otimes \mathbf{1}_B) \underbrace{\rho_{AB}}_{\equiv |\psi_{AB}\rangle\langle\psi_{AB}|} = \underbrace{\sum_\mu q_\mu |\Phi_{\mu AB}\rangle \langle\Phi_{\mu AB}|}_{=\rho'_{AB}}$$

Note that the pure state form $\rho_{AB} = |\psi_{AB}\rangle \langle\psi_{AB}|$ can, for a given density matrix in subspace $\mathcal{H}_A$, always be achieved by choosing the dimension of $\mathcal{H}_B$ large enough. Via the relative state method, we optain

$$\$_A(|\phi_A\rangle \langle\phi_A|) = \langle\phi_B^*| (\$_A \otimes \mathbf{1}_B)\rho_{AB} |\phi_B^*\rangle = \sum_\mu \underbrace{\sqrt{q_\mu} \langle\phi_B^*|\Phi_{\mu AB}\rangle}_{\equiv M_\mu |\phi_A\rangle} \underbrace{\langle\Phi_{\mu AB}|\phi_B^*\rangle \sqrt{q_\mu}}_{=\langle\phi_A| M_\mu^\dagger}$$

These $M_\mu$ have the following properties:
- they define a linear map $|\phi_A\rangle \rightarrow \sqrt{q_\mu} \langle\phi_B^*|\Phi_{\mu AB}\rangle$
- $\$(|\phi_A\rangle \langle\phi_A|) = \sum_\mu M_\mu |\phi_A\rangle \langle\phi_A| M_\mu^\dagger \; \forall |\phi_A\rangle \in \mathcal{H}$

$\Rightarrow$ $\$_A(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger$ if \$ is linear.
- $\sum_\mu M_\mu M_\mu^\dagger = \mathbf{1}_A$, because $\$_A$ is trace preserving for any $\rho_A$
$\Rightarrow$ $\$_A$ has an operator-sum- respectively Kraus-representation (1)[4].
Now we still have to prove that from a operator-sum representation there follows a unitary representation on a larger Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_C$ such that $M_\mu = \langle \mu_C | U_{AC} | 0_C \rangle$. Note that the Hilbert space $\mathcal{H}_C$ with which we extend $\mathcal{H}_A$ need not be equal to the Hilbert space $\mathcal{H}_B$ which we used in the proof of the Kraus-representation. We define a operator $U_{AC}$ such that

$$U_{AC}\left(|\phi_A\rangle \otimes |0_C\rangle\right) \equiv \sum_\mu M_\mu |\phi_A\rangle \otimes |\mu_C\rangle$$

with some normalized state $|0_C\rangle \in \mathcal{H}_C$. It can easily be checked that the operator $U_{AC}$ that is defined via this equation preserves the inner product:

$$(\langle 0_C| \otimes \langle \phi_{2A}| U_{AC}^\dagger)(U_{AC} |\phi_{1A}\rangle \otimes |0_C\rangle) = ... = \langle \phi_{2A} | \phi_{1A} \rangle$$

From this it follows that $U_{AC}$ can be extended to be a unitary operator on the whole Hilbert space, and indeed from the so defined operator we can obtain back the Kraus operators $M_\mu$ by performing the same steps as above. [4] $\square$

So, in this section we found that while the evolution of a closed quantum system is governed by unitary transformations (which can always be inverted, such that the evolution is reversed), the evolution of a subsystem is governed by much more general superoperators which are closely related to the formalism of POVMs. One important fact that has not yet been explicitly stated is that in general the inverse of a superoperator \$ need not exist, or be again a superoperator itself. Therefore, it may not be possible to reverse the time evolution in a subsystem of a larger quantum system, so to say the subsystem "forgets" its past. This resolves some philosophical problems of quantum mechanics, as a unitary time evolution can be reversed, but the often postulated collapse of the wavefunction is an irriversible process. [4]
Also, it should be pointed out that the description of the time evolution of a subsystem discussed above leads to a large variety of possible transformations that can be expressed via superoperators, unitary as well as non-unitary ones. But of course there cannot be superoperators describing a time evolution which violates a basic quantum mechanical principle like for example the *no-cloning theorem*.
So far, we have only used superoperators mapping density matrices of a (sub)space $\mathcal{H}_A$

on density matrices of the same Hilbert space $\mathcal{H}_A$. But this is not the most general case possible. If we have a given Hilbert space $\mathcal{H}$ with two distinct subspaces $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, then it is possible (if a unitary transformation on $\mathcal{H}$ is given) to find a corresponding superoperator \$ which maps a density matrix $\rho_A$ (defined on $\mathcal{H}_A$) on a density matrix $\rho_{A'}$ (defined on $\mathcal{H}_{A'}$). Imagine for example a large laboratory (mathematically described by $\mathcal{H}$) in which the state of a single Q-bit ($\mathcal{H}_A$) at the beginning of an experiment determines the state of some apparatus ($\mathcal{H}_{A'}$) after the experiment. Then there exists a superoperator \$ mapping the initial state in $\mathcal{H}_A$ on the final state in $\mathcal{H}_{A'}$. In the following section, when we will have a look on Schrödinger's cat, we will consider the cat *together* with the nucleus as *one* subsystem, but as shown above we could as well consider them to be two distinct subsystems.

## 5 Schrödinger's cat's revival

I assume the reader to be familiar with the basic concept of the paradox known as "Schrödinger's cat". The cat together with the nucleus can be described roughly by a two dimensional Hilbert space with base-vectors $|\uparrow\rangle$ which means the nucleus is not decayed, the cat is still alive, and the vector $|\downarrow\rangle$ which corresponds to a decayed atom and a dead cat. Now we assume this subsystem to be in contact with the environment which acts as some kind of thermal bath. We now have to come up with a reasonable model of the (unitary) evolution of the whole system. For this purpose, we will describe the environment via a 3-dimensional Hilbert space with basevectors $|0\rangle_E$, $|1\rangle_E$, $|2\rangle_E$. With a high probability (for sufficiently small timesteps), no molecule is scattered of the cat, so that the whole system stays in the same state, but with a small probability $p$ a molecule is scattered of the cat and the environment switches to another state, which depends on the state of the cat. Also, the state of the cat will never be changed by interaction with the environment. For example, if cats could be killed by the hit of an air-molecule, this would be a severe shortcoming in Darwinian evolution. In formulas, this model reads:

$$|\uparrow\rangle |0\rangle_E \to \sqrt{1-p}\,|\uparrow\rangle |0\rangle_E + \sqrt{p}\,|\uparrow\rangle |1\rangle_E$$
$$|\downarrow\rangle |0\rangle_E \to \sqrt{1-p}\,|\downarrow\rangle |0\rangle_E + \sqrt{p}\,|\downarrow\rangle |2\rangle_E$$

As can easily be checked, the evolution described by this rules is indeed inner product preserving and can therefore be extended to a unitary transformation. Via the usual

steps, we find the Kraus operators on the 2-dimensional Hilbert space of the cat

$$M_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

They are indeed the measurement operators of a POVM, as again can easily be checked. Suppose the initial (pure!) state of our feline subsystem is

$$\rho_A = 1/2(|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Now via the Kraus-representation, we find that after one timestep the new density matrix is

$$\$(\rho_A) = ... = \frac{1}{2}\begin{pmatrix} 1 & -(1-p) \\ -(1-p) & 1 \end{pmatrix}$$

This looks like a promising beginning! Now, we define a scattering-rate $\Gamma$ and a timestep $\Delta t \ll 1$ such that $p = \Gamma\Delta t$. Then after a finite time $t = n\Delta t$ the cat has aquired the state

$$\rho'_A = \$^n(\rho_A) = \frac{1}{2}\begin{pmatrix} 1 & -\left(1-\frac{\Gamma t}{n}\right)^n \\ -\left(1-\frac{\Gamma t}{n}\right)^n & 1 \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So obviously, via interaction with the environment, the initial *pure state* (which describes the seemingly paradoxical quantum-superposition of life and death of a macroscopic object) has become a *mixed state*! Of course, we still don't know whether the cat will be alive or dead before we have opened the box, but the kind of uncertainty expressed by a mixed state density matrix is classical rather than the kind of uncertainty expressed by a pure state density matrix. [4]

## References


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