CHAPTER 13

UNITARY REPRESENTATIONS OF LOCALLY COMPACT GROUPS

13.1. Elementary definitions concerning representations

13.1.1. DEFINITION. Let G be a topological group and H a Hilbert space. A *continuous unitary representation* of G in H is a morphism of the group G into the unitary group of H which is continuous for the strong topology.

In other words, a continuous unitary representation of G in H is a mapping π of G into the unitary group of H such that $\pi(st) = \pi(s)\pi(t)$ and such that, for every $\xi \in H$, the mapping $s \to \pi(s)\xi$ of G into H is continuous (for the norm topology of H). The condition $\pi(st) = \pi(s)\pi(t)$ implies that $\pi(e) = 1$ (the neutral element of the group being studied will always be denoted by e) and $\pi(s^{-1}) = \pi(s)^{-1} = \pi(s)^*$.

The functions $s \to \varphi_{\xi,\eta}(s) = (\pi(s)\xi \mid \eta)$ on G (ξ, η being fixed elements of H) are called the coefficients of π.

We have $\varphi_{\xi,\eta}(s) = \bar{\varphi}_{\eta,\xi}(s^{-1})$.

13.1.2. The strong and weak topologies coincide on the unitary group of H. In the above, we can therefore replace the strong topology by the weak topology throughout. On the other hand, if we were to stipulate continuity for the norm topology, we would obtain a more restrictive definition, of no interest for what follows. (Of course, if $\dim H < +\infty$, all these notions of continuity are the same).

13.1.3. The Hilbert dimension of H is called the dimension of π and is denoted by $\dim \pi$. The space H is called the space of π, and is denoted by H_π.

By analogy with the case of involutive algebras, the following notions are also easily defined: equivalent representations, class of representations, intertwining operator, intertwining number, direct sum of representations, multiple of a representation, subrepresentation, representation contained in a representation, cyclic vector (ξ is said to be a cyclic vector for π if $\pi(G)\xi$ is total in H_π). We employ the same notation as in the algebra case, and we have the same elementary properties; we leave it to the reader to convince himself of this.

There is no need to define here the essential subspace of a representation. Everything proceeds as if continuous unitary representations were automatically non-degenerate.

13.1.4. Let π be a continuous unitary representation of G. The following conditions are equivalent: (i) the only closed subspaces of H_π invariant under $\pi(G)$ are 0 and H; (ii) the commutant of $\pi(G)$ in $\mathscr{L}(H_\pi)$ is just the scalar operators; (iii) every non-zero vector of H_π is a cyclic vector for π. If these conditions are satisfied, and, in addition, $H_\pi \neq 0$, then π is said to be *topologically irreducible* or simply *irreducible*. (We will never encounter the notion of algebraic irreducibility, except when $\dim \pi < +\infty$, in which case it is equivalent to topological irreducibility). We denote by $\hat{G}$ the set of classes of irreducible representations of G.

Just as in Chapter 5, we define the terms disjoint, factor, quasi-equivalent, of type I, II, ..., multiplicity-free, of multiplicity n as applied to representations. All the arguments and all the results of Chapter 5 extend immediately to group representations. (We can also make use of the correspondence between representations of G and representations of $L^1(G)$ which will be established in 13.3, at least if g is locally compact).

13.1.5. We are now going to define, for group representations, operations which have no meaning for representations of involutive algebras.

Let π, π' be continuous unitary representations of G. For every $s \in G$, we form, in the Hilbert tensor product $H_\pi \otimes H_{\pi'}$, the (unitary) operator $\pi(s) \otimes \pi'(s)$. It is immediate that $s \to \pi(s) \otimes \pi'(s)$ is a continuous unitary representation of G in $H_\pi \otimes H_{\pi'}$, called the tensor product of π and π', and denoted by $\pi \otimes \pi'$.

Let π be a continuous unitary representation of G in H, and let $\bar{H}$ be the Hilbert space conjugate to H. Each $\pi(s)$ is a unitary operator in $\bar{H}$, and it is immediate that $s \to \pi(s)$ is also a continuous unitary representation of G in $\bar{H}$, called the conjugate representation of π, and denoted by $\bar{\pi}$.

13.1.6. Let G be a locally compact group endowed with a left Haar measure ds. For every $s \in G$, let $\lambda(s)$ be the operator in $L^2(G)$ defined by

$$(\lambda(s)f)(x) = f(s^{-1}x) \quad (f \in L^2(G), x \in G).$$

We immediately verify that λ is a continuous unitary representation of G in $L^2(G)$, called the left regular representation.

Let $\Delta : G \to \mathbf{R}$ be the modular function of G. For every $s \in G$, let $\rho(s)$ be the operator in $L^2(G)$ defined by

$$(\rho(s)f)(x) = \Delta(s)^{1/2} f(xs) \quad (f \in L^2(G), x \in G).$$

We immediately verify that ρ is a continuous unitary representation of G in $L^2(G)$, called the right regular representation.

The representations λ and ρ are injective.

For every $f \in L^2(G)$, define $f' \in L^2(G)$ by $f'(x) = \Delta(x)^{-1/2} f(x^{-1})$. Then $f \to f'$ is an isomorphism of the Hilbert space $L^2(G)$ onto itself, and we have, for every $s \in G$,

$$(\lambda(s)f)'(x) = \Delta(x)^{-1/2} f(s^{-1}x^{-1}) = \Delta(s)^{1/2} \Delta(xs)^{-1/2} f((xs)^{-1}) = (\rho(s)f')(x).$$

Hence the isomorphism $f \to f'$ transforms λ into ρ, from which it follows that $\lambda \simeq \rho$. We thus speak sometimes of the "regular representation" of G, without specifying left or right.

For every $f \in L^2(G)$, let $\bar{f} \in L^2(G)$ be the complex conjugate function. Then $f \to \bar{f}$ is an isomorphism of the Hilbert space $L^2(G)$ onto the conjugate Hilbert space which transforms λ into $\bar{\lambda}$. Hence $\lambda \simeq \bar{\lambda}$ and similarly $\rho \simeq \bar{\rho}$.

13.1.7. Let a be a cardinal and H a Hilbert space of (Hilbert) dimension a. The representation $s \to 1$ of G in H is called the trivial representation of G in H, or the trivial representation of G of dimension a. It is denoted by 1_H, or just by 1 when there is no uncertainty as to what H is.

13.1.8. Proposition. *Let G_1, G_2 be topological groups, $G = G_1 \times G_2$, and σ an irreducible continuous unitary representation of G. Suppose that every continuous unitary factor representation of G_1 is of type I. Then there exist, for $i = 1, 2$, an irreducible continuous unitary representation σ_i of G_i such that σ is equivalent to the representation*

$$(s_1, s_2) \to \sigma_1(s_1) \otimes \sigma_2(s_2)$$

in the Hilbert space $H_{\sigma_1} \otimes H_{\sigma_2}$.

Let $\mathscr{A}_i$ be the von Neumann algebra generated by $\sigma(G_i)$. Then $\mathscr{A}_1, \mathscr{A}_2$ commute with each other and generate the von Neumann algebra $\mathscr{L}(H_\sigma)$. Every element of the centre of $\mathscr{A}_1$ commutes with $\mathscr{A}_1$ and with $\mathscr{A}_2$, hence with $\mathscr{L}(H_\sigma)$, and is therefore a scalar operator. Hence $\mathscr{A}_1$ is a factor, of type I by the hypothesis made about G_1. There then exist Hilbert spaces

H_1, H_2 such that H_σ may be identified with $H_1 \otimes H_2$ and $\mathscr{A}_1$ with $\mathscr{L}(H_1) \otimes \mathbf{C}$ (A 36). For every $s_1 \in G_1$, $\sigma(s_1)$ may be written $\sigma_1(s_1) \otimes 1$, where $\sigma_1(s_1)$ is a unitary operator on H_1, and σ_1 is a continuous unitary representation of G_1 in H_1. The $\sigma_1(s_1)$ generate the von Neumann algebra $\mathscr{L}(H_1)$, and so σ_1 is irreducible. We have $\mathscr{A}_2 \subseteq \mathbf{C} \otimes \mathscr{L}(H_2)$ (A 18), hence, for every $s_2 \in G_2$, $\sigma(s_2)$ may be written $1 \otimes \sigma_2(s_2)$, where σ_2 is a continuous unitary representation of G_2 in H_2. If $T \in \mathscr{L}(H_2)$ commutes with the $\sigma_2(s_2)$, then $1 \otimes T$ commutes with $\mathscr{A}_1$ and $\mathscr{A}_2$, and is therefore a scalar operator; hence T is a scalar operator and σ_2 is irreducible.

References: [995], [1108].

13.2. The involutive algebra $L^1(G)$

13.2.1. Let G be a locally compact group and $M^1(G)$ the algebra (under convolution) of bounded complex measures on G. This is a Banach algebra, having as identity the Dirac measure ϵ_e at the point e. If, for every $\mu \in M^1(G)$, we define μ^* by the relation $d\mu^*(s) = \overline{d\mu(s^{-1})}$, it can be checked that $M^1(G)$ becomes an involutive algebra, and that $\|\mu^*\| = \|\mu\|$. Hence $M^1(G)$ is an involutive Banach algebra.

13.2.2. Choose, once and for all, a left Haar measure ds on G. If, with every $f \in L^1(G)$, we associate the measure $d\mu(s) = f(s)\,ds \in M^1(G)$, we obtain an isometric morphism Φ of the Banach algebra $L^1(G)$ into the Banach algebra $M^1(G)$. For $g \in \mathscr{K}(G)$ (the set of continuous complex-valued functions on G with compact support), we have, denoting the modular function of G by Δ,

$$\int g(s)\,d\mu^*(s) = \overline{\int \bar{g}(s^{-1})\,d\mu(s)} = \int g(s^{-1})\overline{f(s)}\,ds = \int g(s)\bar{f}(s^{-1})\Delta(s^{-1})\,ds.$$

For every complex-valued function f on G, define f^* by

$$f^*(s) = \bar{f}(s^{-1})\Delta(s^{-1}).$$

The above then shows that $d\mu^*(s) = f^*(s)\,ds$. Hence $f \to f^*$ is an isometric involution on $L^1(G)$, and Φ is a morphism of the involutive algebra $L^1(G)$ into the involutive algebra $M^1(G)$. We identify $L^1(G)$ with $\Phi(L^1(G))$. In general, $L^1(G)$ is not a C^*-algebra.

13.2.3. For every complex-valued function f on G, we define $\check{f}$ and $\tilde{f}$ by

$\check{f}(s) = f(s^{-1})$, $\tilde{f}(s) = \bar{f}(s^{-1})$. We thus have $f^* = \tilde{f}\Delta^{-1}$. If $a \in G$, put

$$ {}_af(s) = f(as), \qquad f_a(s) = f(sa). $$

We have (denoting the Dirac measure at the point a by ϵ_a):

$$ \epsilon_a * f = {}_{a^{-1}}f, \qquad f * \epsilon_a = \Delta(a^{-1})f_{a^{-1}}. $$

The formula $(\epsilon_a * f)^* = f^* * \epsilon_a^* = f^* * \epsilon_{a^{-1}}$ then gives

$$ ({}_{a^{-1}}f)^* = \Delta(a)(f^*)_a $$

13.2.4. If G is discrete, then $L^1(G)$ has an identity. If G is separable, then $L^1(G)$ is separable.

13.2.5. Let $s \in G$. Let I be the family of neighbourhoods of s, ordered by reverse inclusion. For every $i \in I$, let u_i be a positive function of $L^1(G)$, vanishing on $G\backslash i$, with integral equal to 1. We have, for every $f \in L^1(G)$,

$$ \|u_i * f - \epsilon_s * f\|_1 \to 0, \qquad \|f * u_i - f * \epsilon_s\|_1 \to 0. $$

[It is enough to verify this for $f \in \mathscr{K}(G)$, and this is then immediate by uniform continuity.] Applying this to $s = e$, we see that $L^1(G)$ possesses an approximate identity.

References: [995], [1108].

13.3. Representations of G and representations of $L^1(G)$.

13.3.1. PROPOSITION. *Let π be a continuous unitary representation of G. For every $\mu \in M^1(G)$, put $\pi(\mu) = \int \pi(s)\, d\mu(s) \in \mathscr{L}(H_\pi)$. Then, $\mu \to \pi(\mu)$ is a representation of the involutive algebra $M^1(G)$ in H_π, whose restriction to $L^1(G)$ is non-degenerate.*

The fact that $\mu \to \pi(\mu)$ is a representation of the involutive algebra $M^1(G)$ follows from easy calculations; for example, for $\mu \in M^1(G)$, $\xi \in H_\pi$, $\eta \in H_\pi$, we have

$$ (\eta \mid \pi(\mu^*)\xi) = \overline{(\pi(\mu^*)\xi \mid \eta)} = \int \overline{(\pi(s^{-1})\xi \mid \eta)}\, d\mu(s) $$
$$ = \int (\pi(s)\eta \mid \xi)\, d\mu(s) = (\pi(\mu)\eta \mid \xi) = (\eta \mid \pi(\mu)^*\xi), $$

whence $\pi(\mu^*) = \pi(\mu)^*$. Moreover, let $s \in G$, $\xi \in H_\pi$ and $\epsilon > 0$. With the notation of 13.2.5, there exists an $i \in I$ such that $\|\pi(s')\xi - \pi(s)\xi\| \leq \epsilon$ for $s' \in i$, whence $\|\pi(u_i)\xi - \pi(s)\xi\| \leq \epsilon$. Hence

$$\pi(u_i) \to \pi(s) \tag{1}$$

for the strong operator topology. Applying this to $s = e$, we see that the representation π of $L^1(G)$ is non-degenerate.

13.3.2. The representations of $M^1(G)$ and $L^1(G)$ obtained above are said to be associated with the given representation of G.

13.3.3. For $s \in G$, we have $\pi(\epsilon_s) = \pi(s)$. Hence, if $f \in L^1(G)$,

$$\pi({}_a f) = \pi(\epsilon_{a^{-1}} * f) = \pi(a^{-1})\pi(f),$$
$$\pi(f_a) = \pi(\Delta(a^{-1})f * \epsilon_{a^{-1}}) = \Delta(a^{-1})\pi(f)\pi(a^{-1}).$$

13.3.4. PROPOSITION. *Let H be a Hilbert space, and π a non-degenerate representation of the involutive algebra $L^1(G)$ in H. Then π is associated with exactly one continuous unitary representation of G.*

The uniqueness follows from formula (1) of 13.3.1. We prove the existence. Let H' be the subspace of H generated by the $\pi(f)H$. By hypothesis, H' is dense in H. Let $s \in G$. With the notation of 13.2.5, we have

$$\|u_i * f - \epsilon_s * f\| \to 0, \quad \text{hence } \|\pi(u_i)\pi(f) - \pi(\epsilon_s * f)\| \to 0.$$

Hence $\pi(u_i)$ converges, for the topology of pointwise convergence over H', to an operator $\pi(s)$ on H', such that

$$\pi(s)\pi(f) = \pi(\epsilon_s * f). \tag{1}$$

Since $\|\pi(u_i)\| \leq \|u_i\| = 1$, $\pi(s)$ has a unique extension to a continuous linear operator on H, which we again denote by $\pi(s)$, such that $\|\pi(s)\| \leq 1$. For $s, t \in G$ and $f \in L^1(G)$ we have, by (1),

$$\begin{aligned}\pi(st)\pi(f) &= \pi(\epsilon_{st} * f) \\ &= \pi(\epsilon_s * \epsilon_t * f) = \pi(s)\pi(\epsilon_t * f) = \pi(s)\pi(t)\pi(f)\end{aligned}$$

and so $\pi(st) = \pi(s)\pi(t)$ on H' and consequently on H. Equality (1) shows that $\pi(e) = 1$, and that $\pi(s)\xi$ depends continuously on s for $\xi \in H'$, and therefore for $\xi \in H$. Finally, since $\pi(s)$ and $\pi(s^{-1}) = \pi(s)^{-1}$ are both norm-reducing, $\pi(s)$ is unitary, which proves that π is a continuous unitary representation of G.

We now show that the associated representation of $L^1(G)$ is just the original representation with which we started. Let $f, g \in L^1(G)$. Since

$$f * g = \int f(s)(\epsilon_s * g)\, ds$$

in $L^1(G)$, we have

$$\pi(f)\pi(g) = \pi(f * g) = \int f(s)\pi(\epsilon_s * g)\, ds$$

$$= \int f(s)\pi(s)\pi(g)\, ds = \left(\int f(s)\pi(s)\, ds\right)\pi(g),$$

hence $\pi(f) = \int f(s)\pi(s)\, ds$.

13.3.5. Propositions 13.3.1 and 13.3.4 establish a bijective correspondence between the continuous unitary representations of G and the non-degenerate representations of $L^1(G)$. Let π be a continuous unitary representation of G and π' the associated representation of $L^1(G)$. The closed subspaces invariant under $\pi(G)$ and $\pi'(L^1(G))$ are the same and the cyclic vectors for $\pi(G)$ and $\pi'(L^1(G))$ are the same (in fact, by formula (1) of 13.3.1). Hence $\pi(G)$ and $\pi'(L^1(G))$ have the same commutant and generate the same von Neumann algebra. In particular, the topological irreducibility of π' is equivalent to that of π. We have thus established a bijective correspondence between the irreducible continuous unitary representations of G and the topologically irreducible representations of $L^1(G)$. Similarly, to say that π is a factor representation, is of type I,... is equivalent to saying that π' is a factor representation, is of type I,.... If π_1, π_2 are continuous unitary representations of G and π_1', π_2' the associated representations of $L^1(G)$, then the intertwining operators for π_1 and π_2 are the same as those for π_1' and π_2', we have $\pi_1' \ominus \pi_2' = (\pi_1 \oplus \pi_2)'$ etc. Henceforth, we will denote by the same letter a continuous unitary representation of G and the associated representation of $L^1(G)$.

13.3.6. Let λ be the left regular representation of G in $L^2(G)$. If $f \in L^1(G)$, then $\lambda(f)$ is the operator $g \to f * g$ in $L^2(G)$. This representation λ of $L^1(G)$ is called the left regular representation of $L^1(G)$ in $L^2(G)$. If $\lambda(f) = 0$, we have $f * g = 0$ for every $g \in \mathcal{K}(G)$, and hence $f = 0$ by 13.2.5. Hence the left regular representation of $L^1(G)$ in $L^2(G)$ is injective.

If G is commutative, and if $\hat{G}$ denotes the dual group of G, the Plancherel isomorphism of $L^2(G)$ onto $L^2(\hat{G})$ transforms $\lambda(f)$ into the operator of multiplication by $\mathscr{F}f$ (the Fourier transform of f) in $L^2(\hat{G})$. We therefore have

$$\|\lambda(f)\| = \sup_{t\in\hat{G}} |(\mathscr{F}f)(t)|.$$

In general, $\int |f(s)|\,ds \neq \sup|\mathscr{F}f|$, and so λ is not isometric (from which it follows that $L^1(G)$ is not a C^*-algebra).

References: [995], [1108].

13.4. Positive forms on $L^1(G)$ and positive-definite functions

13.4.1. Section 13.3 leads us to the study of the representations of $L^1(G)$, or, which amounts to the same thing, the continuous positive forms on $L^1(G)$. Now a continuous linear form on $L^1(G)$ is defined by an element of $L^\infty(G)$. This leads to the introduction of certain bounded functions on G.

Definition. A continuous complex-valued function φ on G is said to be *positive-definite* if, for any elements $s_1, \ldots, s_n$ of G, the matrix $(\varphi(s_i^{-1}s_j))_{1\leq i,j\leq n}$ is positive hermitian.

In other words, for any $s_1, \ldots, s_n \in G$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$, we have

$$\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j\varphi(s_i^{-1}s_j) \geq 0. \tag{1}$$

13.4.2. The sum of two positive-definite continuous functions is positive-definite. If φ is positive-definite continuous and if $\lambda \geq 0$, then $\lambda\varphi$ is positive-definite. For examples of positive-definite continuous functions, cf. 13.6.3.

13.4.3. In (1), put $n = 2$, $s_1 = e$, $s_2 = s \in G$. The matrix

$$\begin{pmatrix} \varphi(e) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(e) \end{pmatrix}$$

must be positive hermitian. This implies that

$$\varphi(s^{-1}) = \overline{\varphi(s)}$$
$$|\varphi(s)| \leq \varphi(e)$$

for every $s \in G$. In particular, φ is bounded, and $\|\varphi\|_\infty = \varphi(e)$.

13.4.4. Proposition. *Let φ be a continuous complex-valued function on G. Then the following conditions are equivalent:*

(i) *φ is positive-definite;*

(ii) *φ is bounded and, for every bounded complex-valued measure μ on G, we have $\langle \varphi, \mu^* * \mu \rangle \geq 0$, in other words*

$$\int\int \varphi(y^{-1}z)\,\overline{d\mu(y)}\,d\mu(z) \geq 0;$$

(iii) *for every $f \in \mathcal{K}(G)$, we have $\langle \varphi, f^* * f \rangle \geq 0$, in other words*

$$\int\int \varphi(y^{-1}z)\bar{f}(y)\bar{f}(z)\,dy\,dz \geq 0.$$

(ii) $\Rightarrow$ (i): Suppose condition (ii) is satisfied. Let $s_1, \dots, s_n \in G$, $\alpha_1, \dots, \alpha_n \in \mathbf{C}$, and μ be the measure on G defined by masses α_i at the points s_i. We have

$$0 \leq \int\int \varphi(y^{-1}z)\,\overline{d\mu(y)}\,d\mu(z) = \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j \varphi(s_i^{-1}s_j),$$

and so φ is positive-definite.

(i) $\Rightarrow$ (iii): Suppose φ is positive definite. Let $f \in \mathcal{K}(G)$. The function $(y, z) \to \varphi(y^{-1}z)\bar{f}(y)f(z)$ on $G \times G$ is continuous and of compact support, S; S is contained in a set $K \times K$, for K some compact subset of G. The measure induced on K by the Haar measure is the weak*-limit of positive measures ν_i of finite support which are norm-bounded; hence the measure induced on $K \times K$ by the Haar measure of $G \times G$ is the weak*-limit of the $\nu_i \otimes \nu_i$. Now, if ν_i is defined by masses $\beta_1, \dots, \beta_n$ at the points $s_1, \dots, s_n$ we have

$$\int\int \varphi(y^{-1}z)\bar{f}(y)f(z)\,d\nu_i(y)\,d\nu_i(z) = \sum_{i,j} \varphi(s_i^{-1}s_j)\bar{f}(s_i)f(s_j)\beta_i\beta_j \geq 0$$

and, in the limit, $\int\int \varphi(y^{-1}z)\bar{f}(y)f(z)\,dy\,dz \geq 0$.

(iii) $\Rightarrow$ (ii): Suppose condition (iii) is satisfied. Let μ be a complex-valued measure on G with compact support. For every $f \in \mathcal{K}(G)$, $\mu * f$ is an element of $\mathcal{K}(G)$, hence

$$\begin{aligned} 0 &\leq \langle \varphi, f^* * \mu^* * \mu * f \rangle \\ &= \int\int\int\int \varphi(xyzt)f^*(x)f(t)\,dx\,dt\,d\mu^*(y)\,d\mu(z) \\ &= \int\int d\mu^*(y)\,d\mu(z) \int\int \varphi(xyzt)f^*(x)f(t)\,dx\,dt. \end{aligned}$$

Taking $f \geqslant 0$, with integral equal to 1, and with support contained in a smaller and smaller neighbourhood of e, $\int\int \varphi(xyzt)f^*(x)f(t)\,dx\,dt$ converges to $\varphi(yz)$ uniformly over every compact subset of $G \times G$. We thus find that, in the limit, $\langle \varphi, \mu^* * \mu \rangle \geqslant 0$. Reasoning as in (ii) $\Rightarrow$ (i), we at once see that φ is positive-definite, and therefore bounded. Now let μ be an arbitrary bounded measure on G. Then μ is the norm-limit of measures with compact support ν. Since φ is bounded, $\langle \varphi, \mu^* * \mu \rangle$ is the limit of $\langle \varphi, \nu^* * \nu \rangle$, and is therefore $\geqslant 0$, by the above.

13.4.5. THEOREM. (i) *Let $\varphi \in L^\infty(G)$, and ω be the continuous linear form on $L^1(G)$ defined by φ. Then ω is positive if and only if φ is equal, locally almost everywhere, to a continuous positive-definite function.*

(ii) *A complex-valued function ψ on G is continuous positive-definite if and only if there exists a continuous unitary representation π of G and an $\xi \in H_\pi$ (which we can suppose is a cyclic vector for π) such that*

$$\psi(s) = (\pi(s)\xi \mid \xi).$$

We then have $\|\varphi\|_\infty = (\xi \mid \xi)$.

(iii) *Let π and π' be continuous unitary representations of G, and ξ (resp. ξ') a cyclic vector for π (resp. π'). If $(\pi(s)\xi \mid \xi) = (\pi'(s)\xi' \mid \xi')$ for any $s \in G$, there exists an isomorphism of H_π onto $H_{\pi'}$ which transforms π into π' and ξ into ξ'.*

Suppose that φ is equal, locally almost everywhere, to a continuous positive-definite function. Then, for every $f \in L^1(G)$, we have

$$\int \varphi(s)(f^* * f)(s)\,ds \geqslant 0$$

by 13.4.4. Hence ω is a positive form on $L^1(G)$. Conversely, suppose that $\omega \geqslant 0$. Form the representation π_ω of $L^1(G)$ and the vector ξ_ω. By 13.3.4, π_ω is associated with a continuous unitary representation of G in the space of π_ω, which we again denote by π_ω. Then, for any $f \in L^1(G)$, we have

$$\int \varphi(s)f(s)\,ds = \omega(f) = (\pi_\omega(f)\xi_\omega \mid \xi_\omega) = \int (\pi_\omega(s)\xi_\omega \mid \xi_\omega)f(s)\,ds,$$

and so

$$\varphi(s) = (\pi_\omega(s)\xi_\omega \mid \xi_\omega) \tag{1}$$

locally almost everywhere. If φ is continuous, equality (1) holds everywhere, since the right-hand side is a continuous function of s. To accomplish the proof of (i) and (ii), it is enough to prove that, if π is a continuous unitary representation of G, and if $\xi \in H_\pi$, then $s \to (\pi(s)\xi \mid \xi)$ is positive-definite. Now, if $\mu \in M^1(G)$, we have

$$0 \leq \|\pi(\mu)\xi\|^2 = (\pi(\mu^* * \mu)\xi \mid \xi) = \int (\pi(s)\xi \mid \xi)\, d(\mu^* * \mu)(s),$$

whence our assertion, by 13.4.4.

We now adopt the hypothesis and notation of (iii). We have

$$(\pi(f)\xi \mid \xi) = (\pi'(f)\xi \mid \xi) \quad \text{for every } f \in L^1(G).$$

Hence there exists an isomorphism Φ of H_π onto $H_{\pi'}$ which transforms ξ into ξ', and $\pi(f)$ into $\pi'(f)$ for every $f \in L^1(G)$ (2.4.1 (ii)). By 13.3.4, Φ transforms $\pi(s)$ into $\pi'(s)$ for every $s \in G$.

13.4.6. Let π be a continuous unitary representation of G, and let $\xi \in H_\pi$. We say that the function $s \to (\pi(s)\xi \mid \xi)$ is the positive-definite function *defined by π and ξ*. For π fixed and ξ varying in H_π, we obtain the positive-definite functions *associated with π*. If S is a set of representations of G, the positive-definite functions associated with S are just the positive-definite functions associated with the various elements of S.

Let φ be a continuous positive-definite function on G. It defines a positive form ω on $L^1(G)$, and therefore a pair (π_ω, ξ_ω), where π_ω can be regarded as a continuous unitary representation of G. This pair is also denoted by $(\pi_\varphi, \xi_\varphi)$ and is said to be *defined by φ*. It is characterised, up to isomorphism, by the fact that $\varphi(s) = (\pi_\varphi(s)\xi_\varphi \mid \xi_\varphi)$ for every $s \in G$ and that ξ_φ is a cyclic vector for π_φ.

13.4.7. PROPOSITION. *Let φ be a continuous positive-definite function on G. If $s, t \in G$, we have*

$$|\varphi(s) - \varphi(t)|^2 \leq 2\varphi(e)(\varphi(e) - \operatorname{Re} \varphi(s^{-1}t)).$$

In fact, let $\pi = \pi_\varphi$, $\xi = \xi_\varphi$. We have

$$\begin{aligned}|\varphi(s) - \varphi(t)|^2 &= |((\pi(s) - \pi(t))\xi \mid \xi)|^2 \leq \|\xi\|^2\|\pi(s)\xi - \pi(t)\xi\|^2 \\ &= \varphi(e)(\|\pi(s)\xi\|^2 + \|\pi(t)\xi\|^2 - 2\operatorname{Re}(\pi(s)\xi \mid \pi(t)\xi)) \\ &= \varphi(e)(2\|\xi\|^2 - 2\operatorname{Re}(\pi(s^{-1}t)\xi \mid \xi)) = 2\varphi(e)(\varphi(e) - \operatorname{Re}\varphi(s^{-1}t)).\end{aligned}$$

13.4.8. PROPOSITION. *Every translate of a continuous positive-definite function is a linear combination of four continuous positive-definite functions.*

Let φ be a continuous positive-definite function on G, $\pi = \pi_\varphi$, $\xi = \xi_\varphi$, $a, b \in G$. Put $\pi(a)\xi = \eta$, $\pi(b)^{-1}\xi = \zeta$. Then

$$\begin{aligned} 4\varphi(bsa) &= 4(\pi(bsa)\xi \mid \xi) = 4(\pi(s)\eta \mid \zeta) \\ &= (\pi(s)(\eta+\zeta) \mid \eta+\zeta) - (\pi(s)(\eta-\zeta) \mid \eta-\zeta) \\ &\quad + \mathrm{i}(\pi(s)(\eta+\mathrm{i}\zeta) \mid \eta+\mathrm{i}\zeta) - \mathrm{i}(\pi(s)(\eta-\mathrm{i}\zeta) \mid \eta-\mathrm{i}\zeta). \end{aligned}$$

13.4.9. PROPOSITION. *Let φ, φ' be two continuous positive-definite functions on G. Then $\varphi\varphi'$ is positive-definite. More precisely, let $\pi = \pi_\varphi$, $\xi = \xi_\varphi$, $\pi' = \pi_{\varphi'}$, $\xi' = \xi_{\varphi'}$. Then*

$$\varphi(s)\varphi'(s) = ((\pi \otimes \pi')(s)(\xi \otimes \xi') \mid \xi \otimes \xi').$$

In fact,

$$\begin{aligned} ((\pi \otimes \pi')(s)(\xi \otimes \xi') \mid \xi \otimes \xi') &= (\pi(s)\xi \otimes \pi'(s)\xi' \mid \xi \otimes \xi') \\ &= (\pi(s)\xi \mid \xi)(\pi'(s)\xi' \mid \xi') = \varphi(s)\varphi'(s). \end{aligned}$$

13.4.10. PROPOSITION. *Let φ be a continuous positive-definite function on G, and $\pi = \pi_\varphi$. Then every continuous positive-definite function associated with π is the uniform limit over G of functions of the form*

$$s \to \sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \varphi(s_j^{-1} s s_i), \quad \textit{where, } s_1, \ldots, s_n \in G,\ \lambda_1, \ldots, \lambda_n \in \mathbf{C}.$$

Let $\xi = \xi_\varphi$, $\eta \in H_\pi$, and $\epsilon > 0$. There exist $s_1, \ldots, s_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ such that

$$\left\| \eta - \sum_{i=1}^{n} \lambda_i \pi(s_i)\xi \right\| \leq \epsilon.$$

Then, for every $s \in G$, we have

$$\left| (\pi(s)\eta \mid \eta) - \left(\pi(s)\left(\sum \lambda_i \pi(s_i)\xi \right) \middle| \sum \lambda_i \pi(s_i)\xi \right) \right| \leq \epsilon\|\eta\| + \epsilon(\|\eta\| + \epsilon).$$

Now the left-hand side may be written

$$\left| (\pi(s)\eta \mid \eta) - \sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \varphi(s_j^{-1} s s_i) \right|.$$

13.4.11. Let $f \in L^2(G)$. If λ denotes the left regular representation of G, we have

$$\overline{(\lambda(s)f \mid f)} = \int \bar{f}(s^{-1}t)f(t)\,dt = \int f(t)\tilde{f}(t^{-1}s)\,dt = (f * \tilde{f})(s).$$

Hence $f * \tilde{f}$ is a continuous positive-definite function associated with λ. If $f \in \mathscr{K}(G)$, then $f * \tilde{f} \in \mathscr{K}(G)$; passing to the limit, if $f \in L^2(G)$, then $f * \tilde{f}$ is the uniform limit of functions of $\mathscr{K}(G)$, i.e. it vanishes at infinity.

References: [620], [635], [818], [1101], [1454], [1455].

13.5. Weak*-convergence and compact convergence of continuous positive-definite functions

13.5.1. LEMMA. *Let A be a bounded set in $L^\infty(G)$. Let $f \in L^1(G)$. If $\varphi \in A$ weak*-converges to $\varphi_0 \in A$, then $f * \varphi$ converges to $f * \varphi_0$ for the topology of compact convergence.*

In fact,

$$(f * \varphi)(s) = \int f(t)\varphi(t^{-1}s)\,dt = \int f(st)\varphi(t^{-1})\,dt = \langle \check{\varphi}, {}_s f\rangle.$$

When φ weak*-converges to φ_0 while remaining in A, $\langle \check{\varphi}, g\rangle$ converges to $\langle \check{\varphi}_0, g\rangle$ uniformly over every compact subset of $L^1(G)$. Now, when s runs through a compact subset of G, the set of the ${}_s f$ is norm-compact in $L^1(G)$, since the mapping $s \to {}_s f$ of G into $L^1(G)$ is norm-continuous.

13.5.2. THEOREM. *Let P_1 be the set of continuous positive-definite functions φ on G such that $\varphi(e) = 1$. On P_1, the weak*-topology $\sigma(L^\infty(G), L^1(G))$ coincides with the topology of compact convergence.*

Since $\|\varphi\|_\infty = 1$ for every $\varphi \in P_1$, it is clear that, on P_1, the topology of compact convergence is finer than the weak*-topology.

Now let $\varphi_0 \in P_1$, K be a compact subset of G, and $\epsilon > 0$. We are going to prove that, if $\varphi \in P_1$ is in a suitable weak*-neighbourhood of φ_0, we have $|\varphi(s) - \varphi_0(s)| \leq \epsilon + 4\sqrt{\epsilon}$ for every $s \in K$. This will finish off the proof.

There exists a compact neighbourhood V of e in G such that

$$|\varphi_0(s) - 1| = |\varphi_0(s) - \varphi_0(e)| \leq \epsilon \quad \text{for every } s \in V.$$

Let χ be the characteristic function of V, and $a > 0$ the measure of V. Let $\mathscr{V}$ be the weak*-neighbourhood of φ_0 in P_1 defined by the condition

$|\langle\varphi - \varphi_0, \chi\rangle| \leq \epsilon a$, i.e. by the condition

$$\left|\int_V (\varphi(s) - \varphi_0(s))\, ds\right| \leq \epsilon a.$$

For $\varphi \in \mathcal{V}$, we have

$$\left|\int_V (1 - \varphi(s))\, ds\right| \leq \left|\int_V (1 - \varphi_0(s))\, ds\right| + \left|\int_V (\varphi(s) - \varphi_0(s))\, ds\right| \leq 2\epsilon a.$$

Moreover, for $\varphi \in \mathcal{V}$ and $s \in G$, we have

$$\begin{aligned}|(a^{-1}\chi * \varphi)(s) - \varphi(s)| &= \left|a^{-1}\int \chi(t)\varphi(t^{-1}s)\, dt - \varphi(s)\right| \\ &= \left|a^{-1}\int_V \varphi(t^{-1}s)\, dt - a^{-1}\int_V \varphi(s)\, dt\right| \\ &\leq a^{-1}\int_V |\varphi(t^{-1}s) - \varphi(s)|\, dt.\end{aligned}$$

In view of 13.4.7, this is dominated by

$$\begin{aligned}a^{-1}\int_V \sqrt{2}(1 - \operatorname{Re}\varphi(t))^{1/2}\, dt &\leq \sqrt{2}a^{-1}\left(\int_V (1 - \operatorname{Re}\varphi(t))\, dt\right)^{1/2}\left(\int_V 1 \cdot dt\right)^{1/2} \\ &\leq \sqrt{2}a^{-1}\sqrt{(2\epsilon a)}\sqrt{a} = 2\sqrt{\epsilon}.\end{aligned}$$

Now, there exists (13.5.1) a weak*-neighbourhood $\mathcal{V}'$ of φ_0 in P_1 such that $\varphi \in \mathcal{V}'$ implies

$$|(a^{-1}\chi * \varphi_0)(s) - (a^{-1}\chi * \varphi)(s)| \leq \epsilon \quad \text{for every } s \in K.$$

Then, for $\varphi \in \mathcal{V} \cap \mathcal{V}'$, we have

$$|\varphi(s) - \varphi_0(s)| \leq \epsilon + 4\sqrt{\epsilon} \quad \text{for every } s \in K.$$

References: [1309], [1867].

13.6. Pure positive-definite functions

13.6.1. Definition. A continuous positive-definite function φ on G is said to be *pure* if π_φ is irreducible.

13.6.2. This amounts to saying that the positive form defined on $L^1(G)$ by φ is pure (2.5.4). In view of 13.4.5, this again amounts to saying that, in every decomposition $\varphi = \varphi_1 + \varphi_2$ of φ into a sum of two continuous positive-definite functions, φ_1 and φ_2 are proportional to φ.

13.6.3. If G is commutative, the irreducible continuous unitary representations of G are the characters of G. The pure continuous positive-definite functions on G are therefore the functions of the form $\lambda\chi$ ($\lambda \geqslant 0, \chi$ a character of G).

13.6.4. THEOREM. *Let φ be a continuous positive-definite function on G such that $\varphi(e) = 1$. Then φ is the limit, for the topology of compact convergence, of functions of the form $\lambda_1\phi_1 + \cdots + \lambda_n\phi_n$, where $\varphi_1, \ldots, \varphi_n$ are pure continuous positive-definite functions equal to* 1 *at e, and where $\lambda_1, \ldots, \lambda_n$ are non-negative numbers such that $\lambda_1 + \cdots + \lambda_n = 1$.*

The positive form ψ on $L^1(G)$ defined by φ is the weak*-limit of convex combinations ψ_i of pure states of A and of 0 (2.5.5). Since $\underline{\lim}\, \|\psi_i\| \geqslant \|\psi\| = 1$ and $\|\psi_i\| \leqslant 1$, we can even suppose, multiplying the ψ_i by suitable scalars, that $\|\psi_i\| = 1$. Then ψ_i is the positive form on $L^1(G)$ defined by a function $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$ ($\varphi_1, \ldots, \varphi_n$ pure continuous positive-definite functions, $\varphi_1(e) = \cdots = \varphi_n(e) = 1$, $\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0$, $\lambda_1 + \cdots + \lambda_n = 1$). It is now enough to apply 13.5.2.

13.6.5. COROLLARY. *Every continuous complex-valued function on G is the limit, for the topology of compact convergence, of linear combinations of pure positive-definite functions.*

It is clearly enough to prove the corollary for a function f of $\mathcal{K}(G)$. Now such a function is the uniform limit over G of functions $f * g$, with $g \in \mathcal{K}(G)$. Moreover,

$$4f * g = (f+g) * (f+g)^{\sim} - (f-g) * (f-g)^{\sim}$$
$$+ \mathrm{i}(f+\mathrm{i}g) * (f+\mathrm{i}g)^{\sim} - \mathrm{i}(f-\mathrm{i}g) * (f-\mathrm{i}g)^{\sim}.$$

Lastly, each function of the form $h * h^{\sim}$, where $h \in \mathcal{K}(G)$, is continuous positive-definite (13.4.11), and 13.6.4 can be applied to it.

13.6.6. COROLLARY. *For every $s \in G$ different from e, there exists an irreducible continuous unitary representation π of G such that $\pi(s) \neq 1$.*

There exists a continuous complex-valued function on G taking different values at s and at e, hence (13.6.5) a pure continuous positive-definite function φ on G taking different values at s and at e. We have

$$(\pi_\varphi(s)\xi_\varphi \mid \xi_\varphi) = \varphi(s) \neq \varphi(e) = (\xi_\varphi \mid \xi_\varphi),$$

hence $\pi_\varphi(s) \neq 1$, and π_φ is irreducible.

13.6.7. Corollary 13.6.6, due to Gelfand and Raikov, expresses the fact that a locally compact group admits "enough" irreducible continuous unitary representations. This corollary would have been false, had we been limited to finite-dimensional continuous unitary representations. This justifies the study of infinite-dimensional continuous unitary representations.

13.6.8. We saw in 13.6.4 how to recover all continuous positive-definite functions using the pure continuous positive-definite functions. We are going to give a result in the same spirit, but in an integral form.

Suppose that G is separable. Let B be the convex set of continuous positive-definite functions on G whose value at e is ≤ 1. This is a compact set for the weak*-topology, separable, because $L^1(G)$ is separable (B 7). For every $s \in G$, the function $\varphi \to \varphi(s)$ on B, which is bounded in absolute value by 1, is Borel, being the limit of a sequence of continuous functions $\varphi \to \int \varphi(t)f_n(t)\,dt$ (take the f_n in $\mathcal{K}(G)$, ≥ 0, with integral equal to 1, and with supports lying within smaller and smaller neighbourhoods of s). Moreover, the set P of pure continuous positive-definite functions equal to 1 at e is the set of extreme points of B, less 0 (2.5.5), and is therefore a G_δ set in B (B 13). This established, we have:

PROPOSITION. *Let φ be a continuous positive-definite function on G such that $\varphi(e) = 1$. Then there exist a positive measure of norm 1 on B, concentrated on P, such that $\varphi(s) = \int_P \zeta(s)\,d\mu(\zeta)$ for every $s \in G$.*

There exists a positive measure μ of norm 1 on B, concentrated on $P \cup \{0\}$, such that $\varphi = \int_{P\cup\{0\}} \zeta\,d\mu(\zeta)$, the integral being taken in the weak sense (B 13). We can clearly suppose that $\mu(\{0\}) = 0$, hence that μ is concentrated on P. Let $f \in L^1(G)$. We have

$$\int_G \varphi(s)f(s)\,ds = \int_B d\mu(\zeta) \int_G \zeta(s)f(s)\,ds. \tag{1}$$

The function $(\zeta, s) \to \zeta(s)$ is continuous on $P \times G$ by 13.5.2, and is therefore measurable on $B \times G$ for the product measure $d\mu(\zeta)\,ds$; it is, on the other hand, bounded in absolute value by 1. Hence $(\zeta, s) \to$

$\zeta(s)f(s)$ is integrable for $d\mu(\zeta)\,ds$ and (1) can be written

$$\int_G \varphi(s)f(s)\,ds = \int_G f(s)\,ds \int_B \zeta(s)\,d\mu(\zeta).$$

Hence

$$\varphi(s) = \int_B \zeta(s)\,d\mu(\zeta) \tag{2}$$

almost everywhere on G. Moreover, if $(s_1, s_2, \ldots)$ is a sequence of elements of G converging to s, we have $\zeta(s_n) \to \zeta(s)$ for every $\zeta \in B$, and so

$$\int_B \zeta(s_n)\,d\mu(\zeta) \to \int_B \zeta(s)\,d\mu(\zeta),$$

by Lebesgue's theorem. Hence $\int_B \zeta(s)\,d\mu(\zeta)$ depends continuously on s, from which it follows that (2) holds everywhere on G.

References: [620], [635], [638], [1101], [1455], [1868].

13.7. Positive-definite measures

13.7.1. Definition. A complex-valued measure μ on G is said to be *positive-definite* if

$$\langle \mu, f * \tilde{f} \rangle \geq 0 \tag{1}$$

for every $f \in \mathcal{K}(G)$. We then write $\mu \gg 0$.

13.7.2. If $\mu \gg 0$, we have $\mu^* = \mu$. In fact,

$$\langle \mu^*, f * \tilde{f} \rangle = \overline{\langle \mu, (f * \tilde{f})^{\sim} \rangle} = \overline{\langle \mu, f * \tilde{f} \rangle} = \langle \mu, f * \tilde{f} \rangle$$

for every $f \in \mathcal{K}(G)$; hence, by polarisation, $\langle \mu^*, f * \tilde{g} \rangle = \langle \mu, f * \tilde{g} \rangle$ for any $f, g \in \mathcal{K}(G)$; hence, passing to the limit, $\langle \mu^*, h \rangle = \langle \mu, h \rangle$ for any $h \in \mathcal{K}(G)$.

13.7.3. Relation (1) of 13.7.1 may be written

$$\iint f(s)\bar{f}(t^{-1}s)\,ds\,d\mu(t) \geq 0$$

for every $f \in \mathcal{K}(G)$, or again, changing f into $\bar{f}$,

$$\int (\mu * f)(s)\bar{f}(s)\, ds \geq 0$$

for every $f \in \mathcal{K}(G)$.

13.7.4. Suppose that μ is bounded. If λ is the left regular representation of G, we can form $\lambda(\mu)$. By 13.7.3, to say that $\mu \gg 0$ means that $(\lambda(\mu)f \mid f) \geq 0$ for every $f \in \mathcal{K}(G)$. Since $\mathcal{K}(G)$ is dense in $L^2(G)$, we have

$$\mu \gg 0 \Leftrightarrow \lambda(\mu) \geq 0.$$

13.7.5. Let Δ be the modular function of G, and $f \in \mathcal{K}(G)$. We have

$$(f * f^*)(s)\Delta(s)^{1/2} = \int f(t)\bar{f}(s^{-1}t)\Delta(s^{-1}t)\Delta(s)^{1/2}\, dt$$

$$= \int f(t)\Delta(t)^{1/2}\bar{f}(s^{-1}t)\Delta(s^{-1}t)^{1/2}\, dt = ((f\Delta^{1/2}) * (f\Delta^{1/2})^{\sim})(s)$$

Condition (1) of 13.7.1 can thus also be expressed, changing f into $f\Delta^{1/2}$, by

$$\langle \Delta^{1/2}\mu, f * f^*\rangle \geq 0 \quad \text{for every } f \in \mathcal{K}(G).$$

13.7.6. DEFINITION. A locally integrable function φ on G is said to be *positive-definite* if the measure $\Delta^{-1/2}(s)\varphi(s)\, ds$ on G is positive-definite. We then write $\varphi \gg 0$.

By 13.7.5, this may be expressed by

$$\langle \varphi, f * f^*\rangle \geq 0 \quad \text{for every } f \in \mathcal{K}(G).$$

For continuous φ, we just recover the previous notion (13.4.4).

If φ and ψ are two locally integrable functions on G such that $\psi - \varphi \gg 0$, we write $\psi \gg \varphi$.

13.7.7. If $\varphi \gg 0$, we have the following equality of measures:

$$\Delta^{-1/2}(s)\varphi(s)\, ds = (\Delta^{-1/2}(s)\varphi(s)\, ds)^* = \Delta^{1/2}(s)\bar{\varphi}(s^{-1})\, d(s^{-1})$$
$$= \Delta^{-1/2}(s)\bar{\varphi}(s^{-1})\, ds,$$

and so

$$\varphi = \tilde{\varphi} \quad \text{locally almost everywhere.}$$

13.7.8. Let φ be a complex-valued function on G such that $\Delta^{-1/2}\varphi \in$

$L^1(G)$. By 13.7.4, $\varphi \gg 0$ if and only if $\lambda(\Delta^{-1/2}\varphi) \geqslant 0$, where λ denotes the left regular representation of G.

13.7.9. Let μ be a positive-definite measure on G. For $f, g \in \mathcal{K}(G)$, put

$$(f \mid g)_\mu = \langle \mu, \tilde{g} * f \rangle = \iint f(t^{-1}s)\bar{g}(t^{-1})\, dt\, d\mu(s).$$

$\mathcal{K}(G)$ then becomes a pre-Hilbert space. Let H_μ be the (Hausdorff) Hilbert space which is the completion of this pre-Hilbert space. For $s \in G$, $f \in \mathcal{K}(G)$, put

$$\sigma(s)f = (\epsilon_s * f)\Delta^{1/2}(s),$$

i.e.

$$(\sigma(s)f)(x) = f(s^{-1}x)\Delta^{1/2}(s).$$

We have

$$\begin{aligned}
\|\sigma(s)f\|_\mu^2 &= \iint \overline{(\sigma(s)f)}(t^{-1})(\sigma(s)f)(t^{-1}u)\, dt\, d\mu(u) \\
&= \iint \bar{f}(s^{-1}t^{-1})\Delta^{1/2}(s)f(s^{-1}t^{-1}u)\Delta^{1/2}(s)\, dt\, d\mu(u) \\
&= \iint \bar{f}(t^{-1})f(t^{-1}u)\Delta(s)\Delta(s)^{-1}\, dt\, d\mu(u) \\
&= \iint \bar{f}(t^{-1})f(t^{-1}u)\, dt\, d\mu(u) = \|f\|_\mu^2.
\end{aligned}$$

Hence $\sigma(s)$ defines a unitary operator on H_μ, that we again denote by $\sigma(s)$. It is clear that σ is a unitary representation of G in H_μ. Moreover, when $s \to s_0$, $\sigma(s)f$ converges uniformly to $\sigma(s_0)f$, and its support remains within a fixed compact set, from which we easily deduce that the unitary representation σ is continuous.

13.7.10. PROPOSITION. *Let V be a neighbourhood of e, and μ a positive-definite measure on G such that*

$$\langle \mu, \tilde{f} * f \rangle \leqslant K\left(\int f(x)\, dx\right)^2 \tag{1}$$

for all non-negative f in $\mathcal{K}(G)$ which vanish outside V (K denotes a number that does not depend on f). Then $d\mu(s) = \varphi(s)\Delta^{-1/2}(s)\, ds$, where φ is a continuous positive-definite function.

There exists a net (f_i) of non-negative functions of $\mathscr{K}(G)$ vanishing outside V such that $\int f_i(x)\,dx = 1$, and such that

$$\int f_i(x)g(x)\,dx \to g(e) \quad \text{for every } g \in \mathscr{K}(G).$$

By (1), we have $\|f_i\|_\mu^2 \leqslant K$ for every i. Moreover, for every $g \in \mathscr{K}(G)$, we have

$$\begin{aligned}(f_i \mid g)_\mu &= \iint \bar{g}(t^{-1})f_i(t^{-1}s)\,dt\,d\mu(s)\\ &= \int d\mu(s)\int \bar{g}(t^{-1}s^{-1})f_i(t^{-1})\,dt \to \int \bar{g}(s^{-1})\,d\mu(s).\end{aligned}$$

Hence, if Λ denotes the canonical mapping of $\mathscr{K}(G)$ into H_μ, Λf_i converges weakly to an element ϵ of H_μ, such that

$$(\epsilon \mid \Lambda g) = \int \bar{g}(s^{-1})\,d\mu(s) \tag{2}$$

for every $g \in \mathscr{K}(G)$. Put $\varphi(s) = (\sigma(s)\epsilon \mid \epsilon)$ for every $s \in G$. Then φ is a continuous positive-definite function on G (13.4.5). Moreover, (2) implies, for every $t \in G$, that

$$\begin{aligned}(\sigma(t)\epsilon \mid \Lambda g) &= (\epsilon \mid \sigma(t^{-1})\Lambda g)\\ &= \int \overline{(\sigma(t^{-1})g)}(s^{-1})\,d\mu(s) = \int \bar{g}(ts^{-1})\Delta^{-1/2}(t)\,d\mu(s)\end{aligned}$$

hence, for every $f \in \mathscr{K}(G)$,

$$\begin{aligned}\int f(t)(\sigma(t)\epsilon \mid \Lambda g)\,dt &= \iint f(t)\bar{g}(ts^{-1})\Delta^{-1/2}(t)\,dt\,d\mu(s)\\ &= \iint f(t^{-1})\bar{g}(t^{-1}s^{-1})\Delta^{-1/2}(t)\,dt\,d\mu(s)\\ &= \iint \bar{g}(t^{-1})f(t^{-1}s)\Delta^{-1/2}(s^{-1}t)\,dt\,d\mu(s)\\ &= \langle \mu, \bar{g} * (\Delta^{1/2}f)\rangle = (\Delta^{1/2}f \mid g)_\mu.\end{aligned}$$

From this we deduce that

$$\int f(t)(\sigma(t)\epsilon \mid \epsilon)\,dt = (\Delta^{1/2}f \mid \epsilon),$$

i.e., in view of (2) and the fact that $\mu^* = \mu$,

$$\int f(t)\varphi(t)\,dt = \int \Delta^{1/2}(t^{-1})f(t^{-1})\,d\check{\mu}(t) = \int \Delta^{1/2}(t)f(t)\,d\mu(t).$$

Since this holds for every $f \in \mathcal{K}(G)$, we conclude that

$$d\mu(t) = \Delta^{-1/2}(t)\varphi(t)\,dt.$$

13.7.11. COROLLARY. *Let φ be a continuous positive-definite function, and ψ a locally integrable positive-definite function such that $\varphi \gg \psi$. Then ψ is equal locally almost everywhere to a continuous positive-definite function.*

In fact, for every $f \in \mathcal{K}(G)$, we have

$$\int \psi(s)\Delta^{-1/2}(s)(\tilde{f} * f)(s)\,ds \leq \int \varphi(s)\Delta^{-1/2}(s)(\tilde{f} * f)(s)\,ds.$$

If $f \in \mathcal{K}(G)$ is non-negative and vanishes outside a compact symmetric neighbourhood V of e, the right-hand side is dominated by

$$\sup_{s\in V^2} |\varphi(s)\Delta^{-1/2}(s)| \iint f(t^{-1})f(t^{-1}s)\,dt\,ds$$
$$= \sup_{s\in V^2} |\varphi(s)\Delta^{-1/2}(s)| \sup_{t\in V} \Delta(t)\left(\int f(t)\,dt\right)^2$$

and it suffices to apply 13.7.10.

References: [620], [635].

13.8. Square-integrable positive-definite functions

13.8.1. Throughout 13.8, we will denote by λ the left regular representation of G. Let $f \in L^2(G)$. For every $g \in \mathcal{K}(G)$, we have $\lambda(g)f \in L^2(G)$. If there exists a finite constant M such that $\|\lambda(g)f\|_2 \leq M\|g\|_2$ for every $g \in \mathcal{K}(G)$, we will say that f is a *moderated* element of $L^2(G)$. The mapping $g \to \lambda(g)f$ then extends uniquely to a continuous linear operator on $L^2(G)$, which we will denote by $\rho(f)$. For every $g \in \mathscr{K}(G)$, we have

$$(\rho(f)\bar{g} \mid \bar{g}) = (\lambda(\bar{g})f \mid \bar{g}) = \int (\bar{g} * f)(s)g(s)\,ds$$

$$= \iint \bar{g}(t)f(t^{-1}s)g(s)\,ds\,dt = \langle f, g^* * g\rangle.$$

Hence $f \gg 0$ if and only if $\rho(f) \geqslant 0$ (13.7.6).

13.8.2. Now let f be an element of $L^2(G)$, positive-definite but not necessarily moderated. The operator $g \to \lambda(g)f$, defined on $\mathscr{K}(G)$ and with values in $L^2(G)$, is $\geqslant 0$ by the same calculation as in 13.8.1. We will denote by $\rho(f)$ its Friedrichs extension (B 23), which is self-adjoint and $\geqslant 0$. When f is moderated, we recover the operator $\rho(f)$ defined in 13.8.1.

13.8.3. LEMMA. *Let f be a positive-definite element of $L^2(G)$.*

(i) *For every $s \in G$, $\rho(f)$ commutes with $\lambda(s)$.*

(ii) *If h belongs to the domain of definition D of $\rho(f)$, we have $\rho(f)h = h * f$.*

For $g \in \mathscr{K}(G)$ and $s \in G$, we have

$$\epsilon_s * g \in \mathscr{K}(G) \quad \text{and} \quad \epsilon_s * (g * f) = (\epsilon_s * g) * f,$$

in other words $\lambda(s)\rho(f)g = \rho(f)\lambda(s)g$. Hence

$$\rho(f) \mid \mathscr{K}(G) = \lambda(s)\rho(f)\lambda(s)^{-1} \mid \mathscr{K}(G)$$

and, consequently, $\rho(f) = \lambda(s)\rho(f)\lambda(s)^{-1}$ (B 23).

Moreover, $\rho(f)$ coincides on D with the adjoint of $\rho(f) \mid \mathscr{K}(G)$ (B 23). Hence, if $h \in D$ and $g \in \mathscr{K}(G)$, we have

$$(1) \qquad (\rho(f)h \mid g) = (h \mid \rho(f)g) = \int h(s)\,ds \int \bar{g}(t)\bar{f}(t^{-1}s)\,dt.$$

Now $h(s)\bar{g}(t)f(t^{-1}s)$ is $ds\,dt$-measurable on $G \times G$, and zero outside a countable union of $ds\,dt$-integrable sets; moreover,

$$\int^* ds \int^* |h(s)g(t)f(t^{-1}s)|\,dt \leqslant \int^* |h(s)|(|g| * |f|(s))\,ds < +\infty,$$

because $|h| \in L^2(G)$ and $|g| * |f| \in L^2(G)$. Hence $h(s)\bar{g}(t)\bar{f}(t^{-1}s)$ is $ds\,dt$-

integrable, and (1) may be written

$$(\rho(f)h \mid g) = \int \left[\int h(s)f(s^{-1}t)\,ds\right] \bar{g}(t)\,dt = \int (h * f)(t)\bar{g}(t)\,dt,$$

whence $\rho(f)h = h * f$ almost everywhere.

13.8.4. LEMMA. *Let a, b be two moderated positive-definite elements of $L^2(G)$ such that $\rho(a)$ and $\rho(b)$ commute. Then $a * b$ is a continuous positive-definite function, and is a moderated element of $L^2(G)$. We have $\rho(a * b) = \rho(a)\rho(b)$, and $(a \mid b) \geqslant 0$. If, further, $a \ll b$, we have*

$$\|b - a\|_2^2 \leqslant \|b\|_2^2 - \|a\|_2^2.$$

The convolution product of two functions of $L^2(G)$ is a continuous function on G. Let (a_n) be a sequence of functions of $\mathcal{K}(G)$ converging to a in $L^2(G)$. Then $a_n * b = \rho(b)a_n$ converges to $\rho(b)a$ in $L^2(G)$. Moreover,

$$\|a_n * b - a * b\|_\infty \to 0, \quad \text{and so} \quad a * b = \rho(b)a \in L^2(G).$$

Let $f \in \mathcal{K}(G)$. We have

$$f * a_n \in \mathcal{K}(G),$$

and $f * a_n$ converges to $f * a$ in $L^2(G)$, hence $(f * a_n) * b = \rho(b)(f * a_n)$ converges to $\rho(b)(f * a) = \rho(b)\rho(a)f$ in $L^2(G)$. Moreover, $f * (a_n * b)$ converges to $f * (a * b)$ in $L^2(G)$. Hence

$$f * (a * b) = \rho(b)\rho(a)f = \rho(a)\rho(b)f \quad \text{for every } f \in \mathcal{K}(G),$$

which proves that $a * b$ is moderated and that $\rho(a * b) = \rho(a)\rho(b)$. Since $\rho(a), \rho(b)$ are positive and commute, $\rho(a)\rho(b)$ is positive, and so $a * b$ is positive-definite. We have

$$(a \mid b) = \int a(s)b(s^{-1})\,ds = (a * b)(e) \geqslant 0.$$

Finally, if $a \ll b$, let $c = b - a \gg 0$; c is moderated, and, by the above

$$(a \mid a) \leqslant (a \mid a) + (a \mid c) = (a \mid b),$$

hence

$$\|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2(a \mid b) \leqslant \|b\|^2 - \|a\|^2.$$

13.8.5. LEMMA. *Let $a_1, a_2, \ldots$ be moderated positive-definite elements of $L^2(G)$, such that $a_1 \ll a_2 \ll \cdots$, and the $\rho(a_i)$ commute pairwise. If $\sup \|a_i\|_2 < +\infty$, the a_i have a norm-limit in $L^2(G)$.*

By 13.8.4, the $\|a_i\|_2$ form an increasing, and therefore convergent, sequence. Applying 13.8.4 again, we then see that the a_i form a Cauchy sequence.

13.8.6. THEOREM. *Let φ be a square-integrable continuous positive-definite function on G. Then there exists a square-integrable positive-definite function ψ on G such that $\varphi = \psi * \psi = \psi * \tilde{\psi}$.*

Suppose, to begin with, that φ is moderated. We can suppose, multiplying φ by a constant >0, that $0 \leq \rho(\varphi) \leq 1$. Let $(p_1(t), p_2(t), \ldots)$ be an increasing sequence of non-negative polynomials on $[0, 1]$, vanishing at 0, which converges uniformly to $\sqrt{t}$ over $[0, 1]$. Thanks to 13.8.4, we can form $\psi_1 = p_1(\varphi)$, $\psi_2 = p_2(\varphi), \ldots$ (using convolution for the multiplication); these elements are in $L^2(G)$ and are moderated, and we have

$$\rho(\psi_1) = p_1(\rho(\varphi)), \qquad \rho(\psi_2) = p_2(\rho(\varphi)), \ldots .$$

We see that $0 \leq \rho(\psi_1) \leq \rho(\psi_2) \leq \cdots$, hence $0 \ll \psi_1 \ll \psi_2 \ll \cdots$; also, the $\rho(\psi_n)$ commute pairwise. Since $p_n^2(t) \leq t$ on $[0, 1]$, we have $\rho(\psi_n)^2 \leq \rho(\varphi)$, hence $\psi_n * \psi_n \ll \varphi$, and so $(\psi_n * \psi_n)(e) = \|\psi_n\|^2 \leq \varphi(e)$. Hence the ψ_n converge in norm to an element ψ of $L^2(G)$ (13.8.5). At the same time, $\rho(\psi_n) = p_n(\rho(\varphi))$ converges in norm to $\rho(\varphi)^{1/2}$. For $f \in \mathcal{K}(G)$, $f * \psi_n = \rho(\psi_n)f$ converges in $L^2(G)$ to $f * \psi$ on the one hand, and to $\rho(\varphi)^{1/2}f$ on the other hand. Hence ψ is moderated and positive-definite, and $\rho(\psi) = \rho(\varphi)^{1/2}$. This implies that $\rho(\varphi) = \rho(\psi)^2 = \rho(\psi * \psi)$, whence $\varphi = \psi * \psi$.

We now pass to the general case. Let $\rho(\varphi) = \int_0^\infty \zeta \, dE_\zeta$ be the spectral decomposition of $\rho(\varphi)$. The projections E_ζ commute with the $\lambda(s)(s \in G)$ by 13.8.3 (i). Let $\varphi_\zeta = E_\zeta \varphi$. For every $g \in \mathcal{K}(G)$, we have

$$\begin{aligned}(g * \tilde{\varphi}_\zeta)(t) &= \int g(s)\tilde{\varphi}_\zeta(s^{-1}t)\, ds = \int g(s)\bar{\varphi}_\zeta(t^{-1}s)\, ds \\ &= (g \mid \lambda(t)E_\zeta\varphi) = (E_\zeta g \mid \lambda(t)\varphi) = \int (E_\zeta g)(s)\bar{\varphi}(t^{-1}s)\, ds \\ &= \int (E_\zeta g)(s)\varphi(s^{-1}t)\, ds = (E_\zeta g * \varphi)(t)\end{aligned}$$

and this is equal to $(\rho(\varphi)E_\zeta g)(t)$ by 13.8.3 (ii). Hence $\tilde{\varphi}_\zeta$ is moderated and $\rho(\tilde{\varphi}_\zeta) = \rho(\varphi)E_\zeta \geq 0$, hence $\tilde{\varphi}_\zeta = \varphi_\zeta \gg 0$. Besides, $\rho(\varphi_\zeta) \leq \rho(\varphi)$, and so $\varphi_\zeta \ll \varphi$, hence φ_ζ is continuous (13.7.11). By the first part of the proof, there exists a moderated positive-definite element ω_ζ such that $\varphi_\zeta = \omega_\zeta * \omega_\zeta$. Take, in particular, $\zeta = 1, 2, \ldots, n, \ldots$. The $\rho(\varphi_n)$ commute pairwise and increase with n, and hence the same is true of the $\rho(\omega_n) = \rho(\varphi_n)^{1/2}$. Hence $\omega_1 \leqslant \omega_2 \leqslant \cdots$. Moreover $\|\omega_n\|^2 = \varphi_n(e) \leqslant \varphi(e)$, and hence the ω_n have a norm-limit $\omega \geqslant 0$ (13.8.5). Then φ_n, i.e. $\omega_n * \omega_n$, converges uniformly on G to $\omega * \omega$; on the other hand, $\varphi_n = E_n\varphi$ converges in norm to φ in $L^2(G)$. Hence $\varphi = \omega * \omega$.

Reference: [635].

13.9. The C^*-algebra of a locally compact group

13.9.1. Since $L^1(G)$ is an involutive Banach algebra with an approximate identity, we can form its enveloping C^*-algebra (2.7.2). This C^*-algebra is called the C^*-algebra of G and is denoted by $C^*(G)$.

For $f \in L^1(G)$, put $\|f\|' = \sup \|\pi(f)\| \leqslant \|f\|_1$, where π runs through the set of non-degenerate representations of $L^1(G)$, or, which amounts to the same thing, the set of continuous unitary representations of G. Then $f \to \|f\|'$ is a seminorm on $L^1(G)$ (2.7.1), and, indeed, a norm, since $L^1(G)$ admits an injective representation (13.3.6). The C^*-algebra of G is just the completion of $L^1(G)$ for this norm.

13.9.2. If G is discrete, $C^*(G)$ admits an identity element. If G is separable, $C^*(G)$ is separable (13.2.4).

13.9.3. By 2.7.4 and 13.3.5, there exists a bijective correspondence between continuous unitary representations of G and non-degenerate representations of $C^*(G)$. Everything that was said in 13.3.5 is still valid when $L^1(G)$ is replaced by $C^*(G)$.

To the left regular representation of G, corresponds a representation of $C^*(G)$ called the left regular representation of $C^*(G)$ in $L^2(G)$.

13.9.4. The group G is said to be liminal, postliminal, antiliminal, of type I, if $C^*(G)$ is liminal, postliminal, antiliminal, of type I.

G is liminal if and only if, for every irreducible continuous unitary representation π of G and every $f \in L^1(G)$, $\pi(f)$ is compact.

Suppose that G is postliminal; then, for every irreducible continuous unitary representation π of G, the norm-closure of $\pi(L^1(G))$ contains $\mathscr{LC}(H_\pi)$ (4.3.7). The converse is true if G is separable (9.1) (and even in general: cf. 9.5.9).

G is of type I if and only if, for every continuous unitary representation of G, the von Neumann algebra generated by $\pi(G)$ is of type I. If G is postliminal, G is of type I (5.5.2). If G is separable (and even in general: cf. 9.5.9), the following conditions are equivalent: (1) G is of type I; (2) for every continuous unitary factor representation π of G, the factor generated by $\pi(G)$ is of type I; (3) G is postliminal (9.1).

References: [582], [896].

13.10. The Hilbert algebra of a unimodular locally compact group

13.10.1. For the convolution product and the involution $f \to f^*$, $\mathcal{K}(G)$† is an involutive algebra. We endow $\mathcal{K}(G)$ with the scalar product $(f \mid g) = \int f(s)\bar{g}(s)\,ds$. It is easy to see that $\mathcal{K}(G)$ then becomes a Hilbert algebra; the completed Hilbert space is just $L^2(G)$. The full Hilbert algebra A of bounded elements (A 57) is called the *Hilbert algebra of* G; we have $\mathcal{K}(G) \subseteq A \subseteq L^2(G)$. It is clear that A is closed under the mapping $f \to \bar{f}$, and also, therefore, for the mapping $f \to \check{f} = (\bar{f})^*$.

13.10.2. A continuous linear operator on $L^2(G)$ commutes with the left-translation operators if and only if it commutes with the operators of left-convolution by the elements of $\mathcal{K}(G)$ (this follows, for example, from 13.3.5 applied to the left regular representation of G). Hence $\mathcal{U}(A)$ is the von Neumann algebra generated by the left-translation operators on $L^2(G)$. Similarly, $\mathcal{V}(A)$ is the von Neumann algebra generated by the right-translation operators on $L^2(G)$.

13.10.3. If $f \in A$, recall that U_f, V_f denote the continuous linear operators on $L^2(G)$ which extend left- and right-multiplication by f in A. If $f \in A$ and $g \in L^2(G)$, we have $U_f g = f * g$ (and similarly $V_f g = g * f$). In fact, suppose to start with that $g \in \mathcal{K}(G)$; let (f_n) be a sequence of elements of $\mathcal{K}(G)$ converging to f in the L^2-sense; then $V_g f_n = f_n * g$ converges to $V_g f = U_f g$ in the L^2-sense, and also to $f * g$, so that $U_f g = f * g$. In the general case, let (g_n) be a sequence of elements of $\mathcal{K}(G)$ converging to g in the L^2-sense; then $U_f g_n$ converges to $U_f g$ in the L^2-sense, and $f * g_n$ converges to $f * g$ uniformly over G; since $U_f g_n = f * g_n$ by the first part of the proof, we have $U_f g = f * g$.

13.10.4. Recall that the mapping $f \to f^*$ commutes with every hermitian element of $\mathcal{U}(A) \cap \mathcal{V}(A)$ (the common centre of $\mathcal{U}(A)$ and $\mathcal{V}(A)$) (A 54). Recall also that each of $\mathcal{U}(A)$, $\mathcal{V}(A)$ is the commutant of the other, and that each is a semi-finite von Neumann algebra (A 60).

†Throughout this section, G denotes a unimodular locally compact group.

13.10.5. PROPOSITION. *Let G be a unimodular locally compact group, and A its Hilbert algebra. Then the following conditions are equivalent:*

(i) (*resp.* (i)′) *The von Neumann algebra $\mathscr{U}(A)$ (resp. $\mathscr{V}(A)$) is a finite von Neumann algebra;*

(ii) *There exists in G a base of compact neighbourhoods of e invariant under the inner automorphisms of G.*

Conditions (i) and (i′) are equivalent for every Hilbert algebra (A 63).

Suppose that there exists a base of compact neighbourhoods $(V_i)_{i\in I}$ of e invariant under the inner automorphisms of G. Let f_i be the characteristic function of V_i. The f_i are central elements of A (they belong to the centre of $L^1(G)$). Moreover, every $f \in L^2(G)$ is in the norm-closure of the set of the $f_i * f$. Hence the characteristic projection of A is 1 (A 62). Hence $\mathscr{U}(A)$ and $\mathscr{V}(A)$ are finite von Neumann algebras (A 63).

Suppose that $\mathscr{U}(A)$ and $\mathscr{V}(A)$ are finite von Neumann algebras. The characteristic projection of A is 1 (A 63). Let s be an element of G different from e. The operator $f \to {}_sf$ on $L^2(G)$ is not the identity; now it commutes with right-translations, and the set of right-translations by the elements of $L^2(G)$ central relative to A is total in $L^2(G)$ (A 62). Hence there exists an $f \in L^2(G)$ central relative to A and such that ${}_sf$ is distinct from f. Since f is central, we have ${}_tf = f_t$ for every $t \in G$ (A 62), and so f is invariant under the inner automorphisms of G. Consider the function $t \to g(t) = ({}_tf \mid f)$ on G. It is continuous, invariant under inner automorphisms, vanishes at infinity, and $g(s) \neq g(e)$. If W is a compact neighbourhood of $g(e)$ in $\mathbf{C}$ containing neither 0 nor $g(s)$, the relation $g(t) \in W$ defines a compact neighbhourhood of e, invariant under inner automorphisms, and not containing s. Hence the intersection of all the compact neighbourhoods of e invariant under inner automorphisms is just $\{e\}$. Every neighbourhood of e therefore contains a compact neighbourhood invariant under inner automorphisms.

References: [639], [641], [1036], [1037], [1457], [1458].

13.11. Addenda

13.11.1. Let G be a locally compact group, and $f \in L^1(G)$. If $f(s) \geqslant 0$ for every $s \in G$, the norm of f in $L^1(G)$ and in $C^*(G)$ is the same. (Consider the trivial 1-dimensional representation of G). [582].

13.11.2. Let G be a locally compact group, π a continuous unitary representation of G, S a set of continuous unitary representations of G, and K the set of $\xi \in H_\pi$ such that the function $s \to (\pi(s)\xi \mid \xi)$ is the uniform limit over every compact set of sums of positive-definite functions associated with S. Then K is a closed subspace of H_π invariant under $\pi(G)$. [587].

13.11.3. Let G be a locally compact group, λ the left regular representation of G, and π a continuous unitary representation of G. Then $\lambda \otimes \pi \simeq (\dim \pi) \cdot \lambda$. (Consider the isomorphism of $L^2(G) \otimes H_\pi$ onto $L^2(G) \otimes H_\pi = L^2_{H_\pi}(G)$ which transforms $f \otimes \xi$ into the function $s \to f(s)\pi(s^{-1})\xi$). [585].

13.11.4. Let G be the locally compact group of affine transformations of $\mathbf{R}$, G' the ← normal closed subgroup of translations. If π is a continuous unitary representation of G and if χ is a non-trivial character of G', then $\pi \mid G'$ does not contain χ. (If $\pi \mid G'$ contained χ, then $\pi \mid G'$ would contain all the non-trivial characters of G' by the action in G' of the inner automorphisms of G; the corresponding subspaces of H_π would be mutually orthogonal). Deduce from this that a non-trivial character of G' cannot be extended to a continuous positive-definite function on G. (Douady, unpublished.)

13.11.5. Let G be a locally compact group, and Δ its modular function. If a continuous positive-definite function on G is integrable for $\Delta(s)^{-1/2}\,ds$, it is square-integrable for ds. [635].

*13.11.6. Let G be a locally compact group, and E the set of linear combinations of continuous positive-definite functions. For $\varphi \in E$, let K^l_φ(resp. K^r_φ) be the closed convex hull of the set of left- (resp. right-) translates of φ in the space of continuous complex-valued functions on G endowed with the uniform norm. Then K^l_φ and K^r_φ both contain the same single constant $M(\varphi)$. $M(\varphi\bar{\varphi}) = 0$ for a continuous positive-definite function φ if and only if π_φ does not admit any non-zero finite-dimensional subrepresentation. [635].

13.11.7. Let G_1, G_2 be two topological groups.

(a) Let π_1 be an irreducible continuous unitary representation of G_1, and π_2 and π_2' continuous unitary representations of G_2. If the representations

$$(s_1, s_2) \to \pi_1(s_1) \otimes \pi_2(s_2) \quad \text{and} \quad (s_1, s_2) \to \pi_1(s_1) \otimes \pi_2'(s_2).$$

are equivalent, then π_2 and π_2' are equivalent.

(b) Let π be a continuous unitary representation of $G_1 \times G_2$. If π is a factor representation, then $\pi \mid G_1$ and $\pi \mid G_2$ are factor representations.

(c) Let π be a continuous unitary representation of $G_1 \times G_2$, $\pi_1 = \pi \mid G_1$ and $\pi_2 = \pi \mid G_2$. Suppose that π_1 is a factor representation of type I. Then there exist $\pi_1' \approx \pi_1$, $\pi_2' \approx \pi_2$ such that π is equivalent to the representation $(s_1, s_2) \to \pi_1'(s_1) \otimes \pi_2'(s_2)$.

(d) If G_1 and G_2 are of type I, then $G_1 \times G_2$ is of type I. [1005].

13.11.8. Let G be a locally compact group, and π a unitary representation of G. Suppose that, for all ξ, $\eta \in H = H_\pi$, the function $s \to (\pi(s)\xi \mid \eta)$ is measurable for the Haar measure on G. We can define $\pi(f) = \int f(s)\pi(s)\, ds$ for every $f \in L^1(G)$. Let $K \subseteq H$ be the essential subspace for the representation π of $L^1(G)$. Then K and $H \ominus K$ are invariant under $\pi(G)$. The representation $s \to \pi(s) \mid K$ of G is continuous. For ξ, $\eta \in H \ominus K$, we have $(\pi(s)\xi \mid \eta) = 0$ locally almost everywhere on G. The space $H \ominus K$ is either zero or non-separable. It can happen that it is non-zero. [1473].

13.11.9. Let G be a topological group. Let π be a continuous unitary representation of G. In $H_\pi \otimes H_\pi \otimes \cdots \otimes H_\pi$ (n factors), the symmetric tensors generate a closed subspace K invariant under $\rho = \pi \otimes \pi \otimes \cdots \otimes \pi$. The subrepresentation ρ_K of ρ is called the n^{th} symmetric tensor power of π. The antisymmetric tensor powers are defined analogously.

*13.11.10. Let G be a connected real Lie group, and π a unitary representation of G, continuous for the norm topology of operators and irreducible. Then π is finite-dimensional. [1484].

13.11.11. Let G be a topological group. A unitary representation π of G is said to be real if there exists a closed real subspace K of H_π such that H_π is the direct sum of K and of iK, the scalar product is real on K, and $\pi(G)$ leaves K invariant (in other words, H_π is the Hilbert space which is the complexification of a real Hilbert space K, and π is the complexification of a representation of G in K). It comes to the same thing to say that there exists an involution J of H commuting with $\pi(G)$. If π is real, we have $\pi \approx \bar{\pi}$, but the converse is not true.

*13.11.12. Semisimple connected real Lie groups and nilpotent connected real Lie groups are liminal (cf. 15.5.6). A real algebraic linear group is postliminal. Large classes of postliminal solvable Lie groups are known, but Mautner has given an example of a non-postliminal solvable Lie group. The non-commutative, 2-dimensional, solvable, connected, real Lie group is postliminal but not liminal. A countable discrete group is of type I if and only if it is the extension of a finite group by a commutative group. Problems: is a p-adic algebraic group postliminal? Does the C^*-algebra of a liminal real Lie group have generalised continuous trace? [439], [441], [448], [586], [758], [903], [922], [1507], [1608], [1774].

*13.11.13. Let G be a locally compact group, and π an irreducible continuous unitary representation of G. Then $\pi(L^1(G))$ is not algebraically irreducible in general.

13.11.14. (a) Let G be a discrete group, A its Hilbert algebra, $\mathcal{U} = \mathcal{U}(A)$, which is a finite von Neumann algebra, and f the natural trace on $\mathcal{U}^+$ defined by A. Since the operator $1 \in \mathcal{U}$ corresponds to the characteristic function of e, we have $\mathfrak{m}_f = \mathfrak{n}_f = \mathcal{U}$, and every element of $\mathcal{U}$ is of the form U_x, where $x \in L^2(G)$. We will still denote by f the linear extension of f to $\mathcal{U}$. If $U_x \in \mathcal{U}$ with $x \in L^2(G)$, we have $f(U_x) = x(e)$.

(b) Let E_{I_i} (resp. E_{II}) be the greatest projection of the centre of $\mathcal{U}$ such that the corresponding algebra induced by $\mathcal{U}$ is of type I_i (resp. of type II) (i takes the values 1, 2, ...). We have $E_{II} + \Sigma E_{I_i} = 1$. Put $f(E_{II}) = r$, $f(E_{I_i}) = r_i$, whence $r + \Sigma r_i = 1$.

(c) Let C be the group of commutators of G. Let φ_C be the characteristic function of C. If C is infinite, then $r_1 = 0$. If C is finite, then $r_1 = (\mathrm{Card}\, C)^{-1}$, and e_1 is the element of $\mathcal{U}$ defined by the function $(\mathrm{Card}\, C)^{-1}\varphi_C$ on G.

(d) If G is infinite, and if C coincides with the centre of G and is of prime order p, then $\mathcal{U}$ is the product of a commutative von Neumann algebra and $p-1$ factors of type II_1. We have $r_1 = 1/p$, $r = (p-1)/p$.

(e) Let (G_i) be an infinite family of non-commutative finite groups. Let G be the set of elements of $\Pi\, G_i$ all but a finite number of whose components are equal to e. Regard G as a discrete group. Then $\mathcal{U}$ is of type II_1.

(f) Let G be a discrete group, and G_0 the subgroup of G which is the union of the finite classes of G. If G/G_0 is infinite, then $\mathcal{U}$ is of type II_1. If $G_0 = \{e\}$, and G is infinite, then $\mathcal{U}$ is a factor of type II_1. [897], [1006], [1035], [1039].