Appendix to §1: Graded super vector spaces

Some classical constructions give objects which are graded and are subject to Koszul's sign rule. Examples: cohomology groups of all kinds (H^*, Tor_*, $\mathrm{Ext}^*\dots$), the de Rham complex and other standard complexes.

To handle their analogue in the super world, two points of view have been used. As we will see, they are basically equivalent. However, they lead to different sign conventions.

Point of View I

One considers such objects as graded objects in the super category, i.e., graded mod 2 graded objects. The grading which is the analogue of the grading in the classical case will be called the cohomological grading. The commutativity isomorphism for the tensor product of graded mod 2 graded vector spaces is defined as follows: for v of bidegree (p,n), viz. parity p and cohomological degree n, and w of bidegree (q,m),

$$c^{\mathbf{I}}_{V,W}\colon V\otimes W\to W\otimes V \quad \text{is} \quad v\otimes w\longmapsto(-1)^{pq+nm}w\otimes v.$$

This is the point of view adopted in §1.3.6.

Point of View II

One considers that the classical construction already lives in the super world, with parity being cohomological degree modulo 2. In the super world, one continues to obtain super objects; they also have a cohomological degree, with no influence on signs: the commutativity isomorphism of graded mod 2 graded vector spaces is defined as

$$c^{\mathbf{II}}_{V,W}\colon V\otimes W\to W\otimes V\colon v\otimes w\to(-1)^{pq}w\otimes v$$

for v of parity p and w of parity q. Of course, one could consider modules instead of vector spaces.

Comparison of I and II

In his lectures, Bernstein used the point of view **II**, which he prefers. In our rendition of it here, we have used the point of view **I**. Here are our reasons for doing so.

(A) In classical occurrences of super objects, the sign rule is violated by some of the standard conventions. For instance, for V and $V^\vee$ vector spaces in duality it is usual to define the duality between $\wedge^p V$ and $\wedge^p V^\vee$ by

$$\langle\omega_1\wedge\dots\wedge\omega_p,\ v_1\wedge\dots\wedge v_p\rangle=\det\left(\langle\omega_i,v_j\rangle\right)\ .$$

In the determinant, the diagonal term is $\langle\omega_1,v_1\rangle\cdots\langle\omega_p,v_p\rangle$, with no sign, despite the fact that v_i passed over ω_j for $i<j$. Using the point of view **I** allows us to keep the inherited classical conventions, while consistently using the sign rule as far as parity is concerned.

(B) The point of view **I** is forced on us by the categorical method 1.2 to hide signs. Indeed, when mimicking in a tensor category $\mathcal{T}$ a classical construction which could be construed as graded (or super), we are led to consider the category of graded (or just mod 2 graded) objects of $\mathcal{T}$, with the commutativity of tensor product being given by modifying the one induced from $\mathcal{T}$ by the Koszul sign rule. When $\mathcal{T}$ is the category of super vector spaces, this gives $c^{\mathbf{I}}$ of **I**.

(C) One does not have to decide early on whether an object should be seen as having a cohomological degree.

Example. Let A be a commutative super k-algebra. The point of view **I** suggests defining the module Ω^1_A of Kähler differentials as being an A-module (hence bimodule) Ω, provided with a morphism of super k-vector spaces $d\colon A \to \Omega$ such that

$$d(ab) = a \cdot db + da \cdot b, \tag{C.1}$$

which is universal. Later, when considering the de Rham complex, one may decide that Ω^1_A is of cohomological degree 1. In the point of view **II**, deciding that in the classical case (purely even A) Ω^1_A is odd requires d to be an odd map, and that (C.1) be replaced by

$$d(ab) = (-1)^{p(a)} a \cdot db + da \cdot b \tag{C.2}$$

(D) The point of view **I** minimizes the use of the parity change functor Π. It is replaced by the imposition of an odd cohomological degree. Using the functor Π leads to nightmares of signs, for the following reasons.

(i) Let $\Pi\, k$ be k, viewed as an odd k-vector space. The functor Π is best viewed as being the tensor product with $\Pi\, k$. One has to decide whether it is $V \mapsto \Pi\, k \otimes V$ or $V \mapsto V \otimes \Pi\, k$. The two are canonically isomorphic, but lead to different sign conventions.

(ii) One has natural isomorphisms $(\Pi V) \otimes W \xrightarrow{\sim} \Pi(V \otimes W)$ and $(V \otimes \Pi W) \xrightarrow{\sim} \Pi(V \otimes W)$, exchanged by the commutativity of $\otimes$. The diagram

$$\begin{array}{ccc} \Pi V \otimes \Pi W & \longrightarrow & \Pi(V \otimes \Pi W) \\ \downarrow & & \downarrow \\ \Pi(\Pi V \otimes W) & \longrightarrow & \Pi\,\Pi(V \otimes W) \end{array}$$

is anticommutative, rather than commutative.

The point of view **II** has advantages too: one has only one parity to consider, for applying the sign rule, and some constructions are more natural. For example, if D^- is the standard odd line (coordinate ring $k[\theta]$, θ odd, $\theta^2 = 0$), the de Rham complex of a super manifold M is (up to a completion) the space of functions on the super manifold

$$\underline{\mathrm{Hom}}(D^-, M)\ ,$$

where the $\underline{\mathrm{Hom}}$ is defined by $\mathrm{Hom}(S, \underline{\mathrm{Hom}}(D^-, M)) = \mathrm{Hom}(S \times D^-, M)$ functorially in S. In the point of view **I**, one has to apply to the de Rham complex the functor "associated simply graded object" explained below.

The points of view **I** and **II** are equivalent, in the sense that the tensor categories $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{II}}$ of graded mod 2 graded vector spaces introduced in **I** and **II** are equivalent. That is to say, there is an equivalence of categories

$$\widetilde{\mathrm{s}}\colon \mathcal{T}_{\mathbf{I}} \longrightarrow \mathcal{T}_{\mathbf{II}}$$

and an isomorphism of functors

$$\alpha\colon \widetilde{\mathrm{s}}(V \otimes_{\mathbf{I}} W) \xrightarrow{\sim} \widetilde{\mathrm{s}}(V) \otimes_{\mathbf{II}} \widetilde{\mathrm{s}}(W)$$

compatible with the associativity and commutativity isomorphisms for $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$.

The functor $\widetilde{\mathrm{s}}$ is a regrading functor: $\widetilde{\mathrm{s}}(V)$ has the same underlying vector space as V, with v of parity p and cohomological degree n acquiring the parity $p+n$, and keeping its cohomological degree.

On the underlying vector space, $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$ are both the usual tensor product. For v in $V^{p,n}$ and w in $W^{q,m}$, α is defined as

$$\alpha\colon v \otimes w \longmapsto (-1)^{nq} v \otimes w.$$

The compatibility with the commutativity isomorphisms is the commutativity of the diagram

$$\begin{array}{ccc}
V^{p,n} \otimes W^{q,m} & \xrightarrow{(-1)^{nq}} & V^{p,n} \otimes W^{q,m} \\
{\scriptstyle (-1)^{pq+nm}}\downarrow & & \downarrow{\scriptstyle (-1)^{(p+n)(q+m)}} \\
W^{q,m} \otimes V^{p,n} & \xrightarrow{(-1)^{mp}} & W^{q,m} \otimes V^{p,n}
\end{array}\quad .$$

The compatibility with the associativity isomorphisms comes from the identity

$$n_1(p_2+p_3) + n_2p_3 = n_1p_2 + (n_1+n_2)p_3 \ .$$

The functor "*associated simply* (mod 2) *graded*" is the composite of $\widetilde{\mathrm{s}}$ with the functor "forgetting the cohomological degree" to super vector spaces. We denote this composite by s.

Example. Let A be a graded mod 2 graded algebra (point of view **I**). It is given by a multiplication

$$\cdot : A \otimes A \to A.$$

Applying s, we obtain a super algebra, with a of parity p and cohomological degree n becoming of parity $p+n$. The new product

$$*\colon \mathrm{s}(A) \otimes \mathrm{s}(A) \xleftarrow{\sim} \mathrm{s}(A \otimes A) \xrightarrow{\mathrm{s}(\cdot)} \mathrm{s}(A)$$

is

$$x * y = (-1)^{nq} x \cdot y$$

for x in $A^{p,n}$ and y in $A^{q,m}$. If A is commutative, meaning that for x of bidegree (p,n) and y of bidegree (q,m) one has $xy = (-1)^{pq+nm} yx$, then the superalgebra $(\mathrm{s}(A), *)$ is a commutative super algebra.