DIFFERENTIABLE MANIFOLDS

DIFFERENTIAL GEOMETRY OF FIBER BUNDLES

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The aim of this lecture is to give a discussion of the main results and ideas concerning a certain aspect of the so-called differential geometry in the large which has made some progress in recent years. Differential geometry in the large in its vaguest sense is concerned with relations between global and local properties of a differential-geometric object. In order that the methods of differential calculus may be applicable, the spaces under consideration are not only topological spaces but are differentiable manifolds. The existence of such a differentiable structure allows the introduction of notions as tangent vector, tangent space, differential forms, etc. In problems of differential geometry there is usually an additional structure such as: (1) a Riemann metric, that is, a positive definite symmetric covariant tensor field of the second order; (2) a system of paths with the property that through every point and tangent to every direction through the point there passes exactly one path of the system; (3) a system of cones of directions, one through each point, which correspond to the light cones in general relativity theory, etc. Among such so-called geometric objects the Riemann metric is perhaps the most important, both in view of its rôle in problems of analysis, mechanics, and geometry, and its richness in results. In 1917 Levi-Civita discovered his celebrated parallelism which is an infinitesimal transportation of tangent vectors preserving the scalar product and is the first example of a connection. The salient fact about the Levi-Civita parallelism is the result that it is the parallelism, and not the Riemann metric, which accounts for most of the properties concerning curvature.

The Levi-Civita parallelism can be regarded as an infinitesimal motion between two infinitely near tangent spaces of the Riemann manifold. It was Elie Cartan who recognized that this notion admits an important generalization, that the spaces for which the infinitesimal motion is defined need not be the tangent spaces of a Riemann manifold, and that the group which operates in the space plays a dominant rôle. In his theory of generalized spaces (*Espaces généralisés*) Cartan carried out in all essential aspects the local theory of what we shall call connections [1; 2]. With the development of the theory of fiber bundles in topology, begun by Whitney for the case of sphere bundles and developed by Ehresmann, Steenrod, Pontrjagin, and others, [8; 19], it is now possible to give a modern version of Cartan's theory of connections, as was first carried out by Ehresmann and Weil [7; 22].

Let F be a space acted on by a topological group G of homeomorphisms. A fiber bundle with the director space F and structural group G consists of topological spaces B, X and a mapping ψ of B onto X, together with the following: (1) X is covered by a family of neighborhoods $\{U_\alpha\}$, called the coordinate neighborhoods, and to each U_α there is a homeomorphism (a coordinate function) φ_α: $U_\alpha \times F \to \psi^{-1}(U_\alpha)$, with $\psi\varphi_\alpha(x, y) = x$, $x \in U_\alpha$, $y \in F$.

(2) As a consequence of (1), a point of $\psi^{-1}(U_\alpha)$ has the coordinates (x, y), and a point of $\psi^{-1}(U_\alpha \cap U_\beta)$ has two sets of coordinates (x, y) and (x, y'), satisfying $\varphi_\alpha(x, y) = \varphi_\beta(x, y')$. It is required that $g_{\alpha\beta}(x)$: $y' \to y$ be a continuous mapping of $U_\alpha \cap U_\beta$ into G.

The spaces X and B are called the base space and the bundle respectively. Each subset $\psi^{-1}(x) \subset B$ is called a fiber.

This definition of a fiber bundle is too narrow in the sense that the coordinate neighborhoods and coordinate functions form a part of the definition. An equivalence relation has thus to be introduced. Two bundles (B, X), (B', X) with the same base space X and the same F, G are called equivalent if, $\{U_\alpha, \varphi_\alpha\}$, $\{V_\beta, \theta_\beta\}$ being respectively their coordinate neighborhoods and coordinate functions, there is a fiber-preserving homeomorphism T: $B \to B'$ such that the mapping $h_{\alpha\beta}(x)$: $y \to y'$ defined by $\theta_\beta(x, y) = T\varphi_\alpha(x, y')$ is a continuous mapping of $U_\alpha \cap V_\beta$ into G.

An important operation on fiber bundles is the construction from a given bundle of other bundles with the same structural group, in particular, the principal fiber bundle which has G as director space acted upon by G itself as the group of left translations. The notion of the principal fiber bundle has been at the core of Cartan's method of moving frames, although its modern version was first introduced by Ehresmann. It can be defined as follows: For $x \in X$, let G_x be the totality of all maps $\varphi_{\alpha,g}(x)$: $F \to \psi^{-1}(x)$ defined by $y \to \varphi_\alpha(x, g(y))$, $y \in F$, $g \in G$, relative to a coordinate neighborhood U_α containing x. G_x depends only on x. Let $B^* = \bigcup_{x \in X} G_x$ and define the mapping ψ^*: $B^* \to X$ by $\psi^*(G_x) = x$ and the coordinate functions $\varphi_\alpha^*(x, g) = \varphi_{\alpha,g}(x)$. Topologize B^* such that the φ_α^*'s define homeomorphisms of $U_\alpha \times G$ into B^*. The bundle (B^*, X) so obtained is called a principal fiber bundle. This construction is an operation on the equivalence classes of bundles in the sense that two fiber bundles are equivalent if and only if their principal fiber bundles are equivalent. Similarly, an inverse operation can be defined, which will permit us to construct bundles with a given principal bundle and having as director space a given space acted upon by the structural group G. An important property of the principal fiber bundle is that B^* is acted upon by G as right translations.

For the purpose of differential geometry we shall assume that all spaces under consideration are differentiable manifolds and that our mappings are differentiable with Jacobian matrices of the highest rank everywhere. In particular, the structural group G will be assumed to be a connected Lie group. For simplicity we suppose our base space X to be compact, although a large part of our discussions holds without this assumption.

The implications of these assumptions are very far-reaching indeed. First of all we can draw into consideration the Lie algebra $L(G)$ of G. $L(G)$ is invariant under the left translations of G, while the right translations and the inner automorphisms of G induce on $L(G)$ a group of linear endomorphisms $\mathrm{ad}(G)$, called the adjoint group of G. Relative to a base of $L(G)$ there are the left-invariant linear differential forms ω^i and the right-invariant linear differential forms π^i, each set consisting of linearly independent forms whose number is equal to the dimension of G. A fundamental theorem on Lie groups asserts that their exterior derivatives are given by

$$(1) \qquad \begin{aligned} d\omega^i &= -\frac{1}{2}\sum_{j,k} c^i_{jk}\omega^j \wedge \omega^k, \\ d\pi^i &= +\frac{1}{2}\sum_{j,k} c^i_{jk}\pi^j \wedge \pi^k, \qquad i, j, k = 1, \cdots, \dim G, \end{aligned}$$

where c^i_{jk} are the so-called constants of structure which are antisymmetric in the lower indices and which satisfy the well-known Jacobi relations.

Returning to our fiber bundle, the dual mapping of the mapping $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to G$ carries ω^i and π^i into linear differential forms in $U_\alpha \cap U_\beta$, which we shall denote by $\omega^i_{\alpha\beta}$ and $\pi^i_{\alpha\beta}$ respectively. Since $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$(2) \qquad \omega^i_{\alpha\gamma} = \sum_j \operatorname{ad}(g_{\beta\gamma})^i_j\omega^j_{\alpha\beta} + \omega^i_{\beta\gamma}.$$

We can also interpret $\omega^i_{\alpha\beta}$ as a vector-valued linear differential form in $U_\alpha \cap U_\beta$, with values in $L(G)$, and shall denote it simply by $\omega_{\alpha\beta}$ when so interpreted.

The generalization of the notion of a tensor field in classical differential geometry leads to the following situation: Let E be a vector space acted on by a representation $M(G)$ of G. A tensorial differential form of degree r and type $M(G)$ is an exterior differential form u_α of degree r in each coordinate neighborhood U_α, with values in E, such that, in $U_\alpha \cap U_\beta$, $u_\alpha = M(g_{\alpha\beta})u_\beta$. The exterior derivative du_α of u_α is in general not a tensorial differential form. It is in order to preserve the tensorial character of the derivative that an additional structure, a connection, is introduced into the fiber bundle.

A connection in the fiber bundle is a set of linear differential forms θ_α in U_α, with values in $L(G)$, such that

$$(3) \qquad \omega_{\alpha\beta} = -\operatorname{ad}(g_{\alpha\beta})\theta_\alpha + \theta_\beta, \qquad \text{in } U_\alpha \cap U_\beta.$$

It follows from (2) that such relations are consistent in $U_\alpha \cap U_\beta \cap U_\gamma$. As can be verified without difficulty, a connection defines in the principal fiber bundle a field of tangent subspaces transversal to the fibers, that is, tangent subspaces which, together with the tangent space of the fiber, span at every point the tangent space of the principal bundle. It follows from elementary extension theorems that in every fiber bundle there can be defined a connection. As there is great freedom in the choice of the connection, the question of deciding the relationship between the properties of the bundle and those of the connection will be our main concern in this paper.

Let us first define the process of so-called absolute differentiation. Let $\bar{M}(X)$, $X \in L(G)$, be the representation of the Lie algebra $L(G)$ induced by the representation $M(G)$ of G. Then we have

$$dM(g_{\alpha\beta}) = M(g_{\alpha\beta})\bar{M}(\theta_\beta) - \bar{M}(\theta_\alpha)M(g_{\alpha\beta}). \tag{4}$$

It follows that if we put for our tensorial differential form u_α of degree r and type $M(G)$

$$Du_\alpha = du_\alpha + \bar{M}(\theta_\alpha) \wedge u_\alpha, \tag{5}$$

the form Du_α will be a tensorial differential form of degree $r + 1$ and the same type $M(G)$.

To study the local properties of the connection we again make use of a base of the Lie algebra, relative to which the form θ_α has the components θ^i_α. We put

$$\Theta^i_\alpha = d\theta^i_\alpha + \frac{1}{2}\sum_{j,k} c^i_{jk}\theta^j_\alpha \wedge \theta^k_\alpha, \qquad \text{in } U_\alpha. \tag{6}$$

The form Θ_α, whose components relative to the base are Θ^i_α, is then an exterior quadratic differential form of degree 2, with values in $L(G)$. It is easy to verify that $\Theta_\alpha = \text{ad}(g_{\alpha\beta})\Theta_\beta$ in $U_\alpha \cap U_\beta$. The Θ_α's therefore define a tensorial differential form of degree 2 and type $\text{ad}(G)$, called the curvature tensor of the connection. In a manner which we shall not attempt to describe here, the curvature tensor and tensors obtained from it by successive absolute differentiations give all the local properties of the connection. In particular, the condition $\Theta_\alpha = 0$ is a necessary and sufficient condition for the connection to be flat, that is, to be such that $\theta_\alpha = 0$ by a proper choice of the coordinate functions.

The following formulas for absolute differentiation can easily be verified:

$$\begin{aligned} \bar{M}(\Theta_\alpha) &= d\bar{M}(\theta_\alpha) + \bar{M}(\theta_\alpha)^2, \\ D\Theta_\alpha &= 0, \\ D^2u_\alpha &= \bar{M}(\Theta_\alpha)u_\alpha. \end{aligned} \tag{7}$$

Such relations are known in classical cases, the second as the Bianchi identity.

We now consider real-valued symmetric multilinear functions $P(Y_1, \cdots, Y_k)$, $Y_i \in L(G)$, $i = 1, \cdots, k$, which are invariant, that is, which are such that $P(\text{ad}(a)Y_1, \cdots, \text{ad}(a)Y_k) = P(Y_1, \cdots, Y_k)$ for all $a \in G$. For simplicity we shall call such a function an invariant polynomial, k being its degree. By the definition of addition,

$$(P + Q)(Y_1, \cdots, Y_k) = P(Y_1, \cdots, Y_k) + Q(Y_1, \cdots, Y_k), \tag{8}$$

all invariant polynomials of degree k form an abelian group. Let $I(G)$ be the direct sum of these abelian groups for all $k \geqq 0$. If P and Q are invariant polynomials of degrees k and l respectively, we define their product PQ to be an invariant polynomial of degree $k + l$ given by

$$(PQ)(Y_1, \cdots, Y_{k+l}) = \frac{1}{N}\sum P(Y_{i_1}, \cdots, Y_{i_k})Q(Y_{i_{k+1}}, \cdots, Y_{i_{k+l}}), \tag{9}$$

where the summation is extended over all permutations of the vectors Y_i, and N is the number of such permutations. This definition of multiplication, together with the distributive law, makes $I(G)$ into a commutative ring, the ring of invariant polynomials of G.

Let $P \in I(G)$, with degree k. For Y_i we substitute the curvature tensor Θ. Then $P(\Theta) = P(\Theta, \cdots, \Theta)$ is an exterior differential form of degree $2k$, which, because of the invariance property of P, is defined everywhere in the base space X. From the Bianchi identity (7_2) it follows that $P(\Theta)$ is closed. Therefore, by the de Rham theory, $P(\Theta)$ determines an element of the cohomology ring $H(X)$ of X having as coefficient ring the field of real numbers. This mapping is a ring homomorphism

$$h: I(G) \rightarrow H(X) \tag{10}$$

of the ring of invariant polynomials of G into the cohomology ring of X. It is defined with the help of a connection in the bundle.

Our first main result is the following theorem of Weil: h is independent of the choice of the connection [22]. In other words, two different connections in the fiber bundle give rise to the same homomorphism h. To prove this we notice that if θ_α and θ'_α are the linear differential forms defining these connections, their difference $u_\alpha = \theta'_\alpha - \theta_\alpha$ is a linear differential form of type $\mathrm{ad}(G)$, with values in $L(G)$. With the help of u_α Weil constructs a differential form whose exterior derivative is equal to the difference $P(\Theta') - P(\Theta)$, for a given invariant polynomial P. Another proof has been given recently by H. Cartan, by means of an invariant definition of the homomorphism h.

Our next step consists in setting up a relationship between this homomorphism h and a homomorphism which is defined in a purely topological manner. This requires the concepts of an induced fiber bundle and a universal fiber bundle.

Let a mapping $f: Y \rightarrow X$ be given. The neighborhoods $\{f^{-1}(U_\alpha)\}$ then form a covering of Y and coordinate functions $\varphi'_\alpha: f^{-1}(U_\alpha) \times F \rightarrow f^{-1}(U_\alpha) \times \psi^{-1}(U_\alpha)$ can be defined by $\varphi'_\alpha(\eta, y) = \eta \times \varphi_\alpha(f(\eta), y)$. This defines a fiber bundle $Y \times \psi^{-1}(f(Y))$ over Y, with the same director space F and the same group G. The new bundle is said to be induced by the mapping f. If the original bundle has a connection given by the differential form θ_α in U_α, the dual mapping f^* of f carries θ_α into $f^*\theta_\alpha$ in $f^{-1}(U_\alpha)$ for which the relation corresponding to (3) is valid. The forms $f^*\theta_\alpha$ therefore define an induced connection in the induced bundle.

This method of generating new fiber bundles from a given bundle is very useful. Its value is based on the fact that it provides a way for the enumeration of fiber bundles. In fact, let the director space and the structural group G be given and fixed for our present considerations. A bundle with the base space X_0 is called universal relative to a space X if every bundle over X is equivalent to a bundle induced by a mapping $X \rightarrow X_0$ and if two such induced bundles are equivalent when and only when the mappings are homotopic. If, for a space X, there exists a universal bundle with the base space X_0, then the classes of bundles over X are in one-one correspondence with the homotopy classes of mappings $X \rightarrow X_0$, so that the enumeration of the bundles over X reduces to a homotopy classification problem.

It is therefore of interest to know the circumstances under which a universal bundle exists. A sufficient condition for the bundle over X_0 to be universal for all compact spaces X of dimension less than or equal to n is that the bundle B_0 of its principal fiber bundle have vanishing homotopy groups up to dimension n inclusive: $\pi_i(B_0) = 0$, $0 \leqq i \leqq n$, where the condition $\pi_0 = 0$ means connectedness.

Under our assumptions that X is compact and that G is a connected Lie group, bundles can be found such that these conditions are fulfilled. First of all, according to a theorem due to E. Cartan, Malcev, Iwasawa, and Mostow, [12; 14; 15], G contains a maximal compact subgroup G_1, and the homogeneous space G/G_1 is homeomorphic to a Euclidean space. This makes it possible to reduce problems of equivalence, classification, etc. of bundles with the group G to the corresponding problems for G_1. Since G_1 is a compact Lie group, it has a faithful orthogonal representation and can be considered as a subgroup of the rotation group $R(m)$ operating in an m-dimensional Euclidean space E^m. Imbed E^m in an $(m + n + 1)$-dimensional Euclidean space E^{m+n+1} and consider the homogeneous space $\tilde{B} = R(m + n + 1)/(I_m \times R(n + 1))$ as a bundle over $X_0 = R(m + n + 1)/(G_1 \times R(n + 1))$, where I_m is the identical automorphism of E^m, and $R(n + 1)$ is the rotation group of the space E^{n+1} perpendicular to E^m in E^{m+n+1}. This is a principal bundle with G_1 as its structural group. By the covering homotopy theorem we can prove that $\pi_i(\tilde{B}) = 0$, $0 \leqq i \leqq n$. In this way the existence of a universal bundle is proved by an explicit construction.

Suppose that a universal bundle exists, with the base space X_0. Let $H(X, R)$ be the cohomology ring of X, relative to the coefficient ring R. Since the classes of bundles over X are in one-one correspondence with the homotopy classes of mappings $X \rightarrow X_0$, the homomorphism $h'\colon H(X_0, R) \rightarrow H(X, R)$ is completely determined by the bundle. h' will be called the characteristic homomorphism, its image $h'(H(X_0, R)) \subset H(X, R)$ the characteristic ring, and an element of the characteristic ring a characteristic cohomology class. It will be understood that the coefficient ring R will be the field of real numbers whenever it is dropped in the notation.

The universal bundle is of course not unique. However, given any two bundles which are universal for compact base-spaces of dimension less than or equal to n, it is possible to establish between their base spaces X_0 and X_0' a chain transformation of the singular chains of dimension less than or equal to n which gives rise to a chain equivalence. From this it follows that up to the dimension n inclusive, the cohomology rings of X_0 and X_0' are in a natural isomorphism. The characteristic homomorphism is therefore independent of the choice of the universal bundle. Although this conclusion serves our purpose, it may be remarked that, in terms of homotopy theory, a stronger result holds between X_0 and X_0', namely, they have the same homotopy-n-type. From this the above assertion follows as a consequence.

A knowledge of $H(X_0, R)$ would be necessary for the description of the characteristic homomorphism. Since elements of dimension greater than n ($=\dim X$) of $H(X_0, R)$ are mapped into zero by dimensional considerations, $H(X_0, R)$ can be replaced by any ring which is isomorphic to it up to dimension n inclusive. On the other hand, it follows from the discussions of the last section that the choice of the universal bundle is immaterial, so that we can take the one whose base space is $X_0 = R(m+n+1)/(G_1 \times R(n+1))$. Using a connection in this universal bundle, we can, according to a process given above, define a homomorphism $h_0 = I(G_1) \to H(X_0)$ of the ring of invariant polynomials of G_1 into $H(X_0)$. X_0 being a homogeneous space, its cohomology ring $H(X_0)$ with real coefficients can be studied algebraically by methods initiated by E. Cartan and recently developed with success by H. Cartan, Chevalley, Kozsul, Leray, and Weil [13]. Thus it has been shown that, up to dimension n, h_0 is a one-one isomorphism. We may therefore replace $H(X_0)$ by $I(G_1)$ in the homomorphism h' and write the characteristic homomorphism as

$$h'\colon I(G_1) \to H(X). \tag{11}$$

This homomorphism h' is defined by the topological properties of the fiber bundle.

On the other hand, the homomorphism $h\colon I(G) \to H(X)$ defined above can be split into a product of two homomorphisms. Since an invariant polynomial under G is an invariant polynomial under G_1, there is a natural homomorphism

$$\sigma\colon I(G) \to I(G_1). \tag{12}$$

Since G_1 can be taken to be the structural group, the homomorphism

$$h_1\colon I(G_1) \to H(X) \tag{13}$$

is defined. Now, a connection with the group G_1 can be considered as a connection with the group G. Using such a connection, we can easily prove

$$h = h_1\sigma. \tag{14}$$

Our main result which seems to include practically all our present knowledge on the subject consists in the statement:

$$h' = h_1. \tag{15}$$

Notice that h' is defined by the topological properties of the bundle and h_1 by the help of a connection, so that our theorem gives a relationship between a bundle and a connection defined in it, which is restrictive in one way or the other. In particular, when the structural group G is compact, we have $G_1 = G$ and σ is the identity, and the characteristic homomorphism is in a sense determined by the connection. For instance, it follows that the characteristic ring of the bundle has to be zero when a connection can be defined such that $h(I(G)) = 0$.

A proof for this theorem is obtained by first establishing it for the universal bundle. Under the mapping $f: X \to X_0$ it is then true for the induced bundle and the induced connection. Using the theorem of Weil that h is independent of the choice of the connection, we see that the relation is true for any connection in the bundle.

A great deal can be said about the rings of invariant polynomials $I(G)$, $I(G_1)$ and the homomorphism σ. When the structural group is compact, such statements can usually be proved more simply by topological considerations. In the other case we have to make use of the cohomology theory of Lie algebras. As we do not wish to discuss this, we shall restrict ourselves to the explanation of the corresponding topological notions. For this purpose we shall first discuss compact groups, that is, we begin by confining our attention to G_1.

We first recall some results on compact group manifolds. All the maximal abelian subgroups are conjugate and are isomorphic to a torus whose dimension is called the rank of the group. By an idea due essentially to Pontrjagin [16] we can define an operation of the homology classes of G_1 on the cohomology classes of G_1. In fact, $m: G_1 \times G_1 \to G_1$ being defined by the group multiplication, the image $m^*\gamma^k$ of a cohomology class of dimension k of G under the dual homomorphism m^* can be written $m^*\gamma^k = \sum u_i^r \times v_i^{k-r}$. The operation of a homology class c^s of dimension $s \leqq k$ on γ^k is then defined as $i(c^s)\gamma^k = \sum_i KI(c^s, u_i^s)v_i^{k-s}$. We call this operation an interior product. A cohomology class γ^k of G_1 is called primitive if its interior product by any homology class of dimension s, $1 \leqq s \leqq k - 1$, is zero. The homology structure of compact group manifolds (with real coefficients) has a description given by the following theorem of Hopf and Samelson [11; 18]: (1) all primitive cohomology classes are of odd dimension; (2) the vector space of the primitive classes has as dimension the rank of G; (3) the cohomology ring of G_1 is isomorphic to the Grassmann algebra of the space of primitive classes.

The primitive classes play a rôle in the study of the universal principal fiber bundle $\psi: B_0 \to X_0$. Identify a fiber $\psi^{-1}(x)$ ($x \in X_0$) with G_1, and let i be the inclusion mapping of G_1 into B_0. If γ^k is a cocycle of X_0, $\psi^*\gamma^k$ is a cocycle of B_0. Since B_0 is homologically trivial, there exists a cochain β^{k-1} having $\psi^*\gamma^k$ as coboundary. Then $i^*\beta^{k-1}$ is a cocycle in G_1 whose cohomology class depends only on that of γ^k. The resulting mapping of the cohomology classes is called a transgression. It is an additive homomorphism of the ring of invariant polynomials of G_1 into the cohomology ring of G_1 and it carries an invariant polynomial of degree k into a cohomology class of dimension $2k - 1$. Chevalley and Weil proved that the image is precisely the space of the primitive classes.

When the group G is noncompact, the consideration of its Lie algebra allows us to generalize the above notions, at least under the assumption that G is semisimple. H. Cartan, Chevalley, and Koszul have developed a very comprehensive theory dealing with the situation, which can be considered in a sense as the algebraic counterpart of the above treatment. Among their consequences we mention the following which is interesting for our present purpose: The ring of invariant polynomials under G has a set of generators equal to the rank of G; these can be so chosen that their images under transgression span the space of primitive classes of G.

Using the fact that the cohomology theory of Lie algebras and transgression can be defined algebraically, and therefore for G, we have the following diagram

$$\begin{array}{ccc} H(G) & \xrightarrow{i^*} & H(G_1) \\ t\uparrow & & \uparrow t_1 \\ I(G) & \xrightarrow{\sigma} & I(G_1). \end{array}$$

It is not difficult to prove that commutativity holds in this diagram. Hence the image under σ depends on the image under i^* of $H(G)$, that is, on the "homological position" of G_1 in G. In general, $\sigma[I(G)] \neq I(G_1)$.

There are relations between the characteristic cohomology classes in our definition and the classes carrying the same name in the topological method of obstructions but we cannot discuss them in detail. The latter come into being when one attempts to define a cross-section in the fiber bundle (that is, a mapping f of X into B, such that ψf is the identity) by extension over the successive skeletons; they are cohomology classes over groups of coefficients which are the homotopy groups of the director space. As we shall see from examples, it is sometimes possible to identify them by identifying the coefficient groups. In general, however, our characteristic classes are based on homological considerations, while those of obstruction theory are based on homotopy considerations. Their rôles are complementary.

We shall devote the rest of this lecture to the consideration of examples. Although the main results will follow from the general theorems, special problems arise in individual cases which can be of considerable interest. To begin with, take for G the rotation group in m variables, and suppose that a connection is given in the bundle. This includes in particular the case of orientable Riemann manifolds with a positive definite metric, the bundle being the tangent bundle of the manifold and the connection being given by the parallelism of Levi-Civita; it also includes, among other things, the theory of orientable submanifolds imbedded in an orientable Riemann manifold.

By a proper choice of a base of the Lie algebra of $G = R(m)$, the space of the Lie algebra can be identified with the space of skew-symmetric matrices of order m. The connection can therefore be defined, in every coordinate neighborhood, by a skew-symmetric matrix of linear differential forms $\theta = (\theta_{ij})$, and its curvature tensor by a skew-symmetric matrix of quadratic differential forms $\Theta = (\Theta_{ij})$. The effect of the adjoint group is given by $\mathrm{ad}(a)\Theta = A\Theta\, {}^tA$, where A is a proper orthogonal matrix and ${}^tA$ is its transpose.

The first question is of course to determine a set of generators for the ring of invariant polynomials; using the fundamental theorem on invariants, it is easy to do this explicitly [23]. Instead of the invariant polynomials we write the corresponding differential forms:

$$\begin{aligned} &\Delta_s = \Theta_{i_1 i_2} \cdots \Theta_{i_s i_1}, \quad s = 2, 4, \cdots, m + 1, \quad && m \text{ odd} \\ (16) \quad &\Delta_s = \Theta_{i_1 i_2} \cdots \Theta_{i_s i_1}, \quad s = 2, 4, \cdots, m - 2, \quad && m \text{ even} \\ &\Delta_0 = \epsilon_{i_1 \cdots i_m} \Theta_{i_1 i_2} \cdots \Theta_{i_{m-1} i_m}, && m \text{ even,} \end{aligned}$$

where repeated indices imply summation and where $\epsilon_{i_1 \cdots i_m}$ is the Kronecker tensor, equal to $+1$ or -1 according as $i_1, \cdots, i_m$ form an even or odd permutation of $1, \cdots, m$ and otherwise to 0. Since the rank of $R(m)$ is $(m + 1)/2$ or $m/2$ according as m is odd or even, we verify here that the number of the above generators is equal to the rank. They form a complete set of generators, because they are obviously independent.

It follows that the cohomology classes determined by these differential forms or by polynomials in these differential forms depend only on the bundle and not on the connection. As a consequence, if all these differential forms are zero, the characteristic ring is trivial. The differential forms in (16) were first given by Pontrjagin [17].

For geometric applications it is useful to have a more explicit description of the base space of a universal bundle. This is all the more significant, since it would then allow us to study the characteristic homomorphisms with coefficient rings other than the field of real numbers. Our general theory gives as such a base space the Grassmann manifold

$$X_0 = R(m + n + 1)/(R(m) \times R(n + 1)),$$

which can be identified with the space of all oriented m-dimensional linear spaces through a point 0 of an $(m + n + 1)$-dimensional Euclidean space E^{m+n+1}.

The homology structure of Grassmann manifolds has been studied by Ehresmann [9, 10]. A cellular decomposition can be constructed by the following process: Take a sequence of linear spaces

$$0 \subset E^1 \subset E^2 \subset \cdots \subset E^{m+n} \subset E^{m+n+1}.$$

Corresponding to a set of integers

$$0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_m \leqq n + 1,$$

denote by $(a_1 \cdots a_m)$ the set of all m-dimensional linear spaces $\xi \in X_0$ such that

$$\dim (\xi \cap E^{a_i+i}) \geqq i, \qquad i = 1, \cdots, m.$$

The interior points of $(a_1 \cdots a_m)$ form two open cells of dimension $a_1 + \cdots + a_m$. These open cells constitute a cellular decomposition of X_0, whose incidence relations can be determined. From this we can determine the homology and cohomology groups of X_0. In particular, it follows that the symbol $(a_1 \cdots a_m)^{\pm}$ can be used to denote a cochain, namely, the one which has the value $+1$ for the corresponding open cells and has otherwise the value zero. The characteristic homomorphism can then be described as a homomorphism of combinations of such symbols into the cohomology ring $H(X, R)$ of X. When R is the field of real numbers, the result is particularly simple. In fact, a base for the cohomology groups of dimensions less than or equal to n consists of cocycles having as symbols those for which all a_i are even, together with the cocycle $(1 \cdots 1)$ when m is even.

This new description of the characteristic homomorphism allows us to give a geometric meaning to individual characteristic classes. In this respect the class $h'((1 \cdots 1))$, which exists only when m is even, deserves special attention. In fact, the bundle with the director space $S^{m-1} = R(m)/R(m-1)$ constructed from the principal bundle is a bundle of $(m-1)$-spheres in the sense of Whitney. For such a sphere bundle, Whitney introduced a characteristic cohomology class W^m with integer coefficients. It can be proved that W^m, when reduced to real coefficients, is precisely the class $h'((1 \cdots 1))$. On the other hand, the latter can be identified on the universal bundle with a numerical multiple of the class defined by the differential form Δ_0. Taking the values of these classes for the fundamental cycle of the base manifold, we can write the result in an integral formula

$$W^m \cdot X = c \int_X \Delta_0, \tag{17}$$

where c is a numerical factor and X denotes a fundamental cycle of the base manifold. For a Riemann manifold, $W^m \cdot X$ is equal to the Euler-Poincaré characteristic of X and our formula reduces to the Gauss-Bonnet formula [3].

We introduce the notations

$$\begin{aligned} P^{4k} &= h'(0 \cdots 0\underbrace{2 \cdots 2}_{2k \text{ times}}) \\ \bar{P}^{4k} &= h'(0 \cdots 0\ 2k\ 2k) \\ \chi^m &= h'(1 \cdots 1), \qquad m \text{ even}, \end{aligned} \tag{18}$$

where the symbols denote also the cohomology classes to which the respective cocycles belong. By studying the multiplicative structure of the cohomology ring of X_0, we can prove that the characteristic homomorphism is determined by the classes P^{4k}, χ^m, $4k \leqq \dim X$ or the classes $\bar{P}^{4k}$, χ^m, $4k \leqq \dim X$.

We shall mention an application of the classes $\bar{P}^{4k}$. Restricting ourselves for simplicity to the tangent bundle of a compact differentiable manifold, the conditions $\bar{P}^{4k} = 0$, $2k \geqq n + 2$, are necessary for the manifold to be imbeddable into a Euclidean space of dimension $m + n + 1$. We get thus criteria on the impossibility of imbedding which can be expressed in terms of the curvature tensor of a Riemann metric on the manifold.

The second example we shall take up is the case that G is the unitary group. Such bundles occur as tangent bundles of complex analytic manifolds, and the introduction of an Hermitian metric in the manifold would give rise to a connection in the bundle.

The space of the Lie algebra of the unitary group $U(m)$ in m variables can be identified with the space of $m \times m$ Hermitian matrices A (${}^t\bar{A} = A$). A connection is therefore defined in each coordinate neighborhood by an Hermitian matrix of linear differential forms $\theta = (\theta_{ij})$ and its curvature tensor by an Hermitian matrix of quadratic differential forms $\Theta = (\Theta_{ij})$. Under the adjoint group the curvature tensor is transformed according to $\mathrm{ad}(a)\Theta = A\Theta{}^t\bar{A}$, A being a unitary matrix. Using this representation of the adjoint group, a set of invariant polynomials can be easily exhibited. We give their corresponding differential forms as

$$\Lambda_k = \Theta_{i_1 i_2} \cdots \Theta_{i_k i_1}, \qquad k = 1, \cdots, m. \tag{19}$$

Since they are clearly independent and their number is equal to the rank m of $U(m)$, they form a complete set of generators in the ring of invariant polynomials.

As in the case of the rotation group the complex Grassmann manifold $X_0 = U(m+n)/(U(m) \times U(n))$ is the base space of a universal bundle, whose study would be useful for some geometric problems. The results are simpler than the real case, but we shall not describe them here. A distinctive feature of the complex case is that a set of generators can be chosen in the ring of invariant polynomials whose corresponding differential forms are

$$\Psi_r = \frac{1}{(2\pi(-1)^{1/2})^{m-r+1}(m-r+1)!} \sum \delta(i_1 \cdots i_{m-r+1}; j_1 \cdots j_{m-r+1}) \cdot \Theta_{i_1 j_1} \cdots \Theta_{i_{m-r+1} j_{m-r+1}}, \qquad r = 1, \cdots, m, \tag{20}$$

where $\delta(i_1 \cdots i_{m-r+1}; j_1 \cdots j_{m-r+1})$ is zero except when $j_1, \cdots, j_{m-r+1}$ form a permutation of $i_1, \cdots, i_{m-r+1}$, in which case it is $+1$ or -1 according as the permutation is even or odd, and where the summation is extended over all indices $i_1, \cdots, i_{m-r+1}$ from 1 to m. This set of generators has the advantage that the cohomology classes determined by the differential forms have a simple geometrical meaning. In fact, they are the classes, analogous to the Stiefel-Whitney classes, for the bundle with the director space $U(m)/U(m-r)$. As such they are primary obstructions to the definition of a cross-section and are therefore more easily dealt with [4]. Substantially the same classes have been introduced by M. Eger and J. A. Todd in algebraic geometry, even before they first made their appearance in differential geometry [6; 20].

The situation is different for bundles with the rotation group, since the Stiefel-Whitney classes, except the highest-dimensional one, are essentially classes mod 2 and therefore do not enter into our picture. However, there is a close relationship between bundles with the group $R(m)$ and bundles with the group $U(m)$. In fact, given a bundle with the group $R(m)$, we can take its Whitney product with itself, which is a bundle with the same base space and the group $R(m) \times R(m)$. The latter can be imbedded into $U(m)$, so that we get a bundle with the group $U(m)$. Such a process is frequently useful in reducing problems on bundles with the rotation group to those on bundles with the unitary group.

We shall take as last example the case that the group is the component of the identity of the general linear group $GL(m)$ in m variables. A connection in the bundle is called an affine connection. An essential difference from the two previous examples is that the group is here noncompact.

The Lie algebra of the group $GL(m)$ can be identified with the space of all m-rowed square matrices, so that the curvature tensor in each coordinate neighborhood is given by such a matrix of exterior quadratic differential forms: $\Theta = (\Theta_i^j)$. The effect of the adjoint group being defined by $\mathrm{ad}(a)\Theta = A\ \Theta\ A^{-1}$, $a \in GL(m)$, it is easily seen that a set of generators of the ring of invariant polynomials can be so chosen that the corresponding differential forms are

$$M_s = \Theta_{i_1}^{i_2} \cdots \Theta_{i_s}^{i_1}, \qquad s = 1, \cdots, m-1. \tag{21}$$

According to the general theory it remains to determine the homomorphism of the ring of invariant polynomials under $GL(m)$ into the ring of invariant polynomials under its maximal compact subgroup, which is in this case the rotation group $R(m)$. It is seen that M_s, for even s, is mapped into Δ_s, and, for odd s, is mapped into zero. The class defined by Δ_0 does not belong to the image of the homomorphism. This fact leads to the interesting explanation that a formula analogous to the Gauss-Bonnet formula does not exist for an affine connection.

Perhaps the most important of the bundles is the tangent bundle of a differentiable manifold. We mentioned above the identification of a certain characteristic class with the Euler-Poincaré characteristic of the manifold, at least for the case that the manifold is orientable and of even dimension. Beyond this very little is known on the relations between topological invariants of the manifold and the characteristic homomorphism of its tangent bundle. Recently, contributions have been made by Thom and Wu which bear on this question [21; 25]. Although it is not known whether a topological manifold always has a differentiable structure, nor whether it can have two essentially different differentiable structures, Thom and Wu proved that the characteristic homomorphisms of the tangent bundle, with coefficients mod 2 and with coefficients mod 3, are independent of the choice of the differentiable structure, provided one exists. Briefly speaking, this means that such characteristic homomorphisms are topological invariants of differentiable manifolds. The proof for coefficients mod 3 is considerably more difficult than the case mod 2.

For bundles with other groups such questions have scarcely been asked. The next case of interest is perhaps the theory of projective connections derived from the geometry of paths. In this case the bundle with the projective group depends both on the tangent bundle and the family of paths. It would be of interest to know whether or what part of the characteristic homomorphism is a topological invariant of the manifold.

Before concluding we shall mention a concept which has no close relation with the above discussion, but which should be of importance in the theory of connections, namely, the notion of the group of holonomy. It can be defined as follows: if ω is the left-invariant differential form in G, with values in $L(G)$, and if θ_α defines a connection, the equation

$$\theta_\alpha + \omega = 0 \tag{22}$$

is independent of the coordinate neighborhood. When a parametrized curve is given in the base manifold, this differential equation defines a family of integral curves in G invariant under left translations of the group. Let $x \in X$ and consider all closed parametrized curves in X having x as the initial point. To every such curve C let $a(C)$ be the endpoint of the integral curve which begins at the unit element e of G. All such points $a(C)$ form a subgroup H of G, the group of holonomy of the connection.

Added in proof: The details of some of the discussions in this article can be found in mimeographed notes of the author, *Topics in differential geometry*, Institute for Advanced Study, Princeton, 1951.

BIBLIOGRAPHY

1. E. CARTAN, *Les groupes d'holonomie des espaces généralisés*, Acta Math. vol. 48 (1926) pp. 1–42.

2. ———, *L'extension du calcul tensoriel aux géométries non-affines*, Ann. of Math. (2) vol. 38 (1947) pp. 1–13.

3. S. S. CHERN, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. (2) vol. 45 (1944) pp. 747–752.

4. ———, *Characteristic classes of Hermitian manifolds*, Ann. of Math. (2) vol. 47 (1946) pp. 85–121.

5. ———, *Some new viewpoints in differential geometry in the large*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1–30.

6. M. EGER, *Sur les systèmes canoniques d'une variété algébrique*, C. R. Acad. Sci. Paris vol. 204 (1937) pp. 92–94, 217–219.

7. C. EHRESMANN, *Les connexions infinitésimales dans un espace fibré différentiable*, Seminaire Bourbaki, March, 1950. This paper contains references to earlier notes of the author on the subject in C. R. Acad. Sci. Paris.

8. ———, *Sur la théorie des espaces fibrés*, Colloque de Topologie Algébrique, Paris, 1947, pp. 3–15.

9. ———, *Sur la topologie de certains espaces homogènes*, Ann. of Math. (2) vol. 35 (1934) pp. 396–443.

10. ———, *Sur la topologie des certaines variétés algébriques réelles*, Journal de Mathématiques vol. 104 (1939) pp. 69–100.

11. H. HOPF, *Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. (2) vol. 42 (1941) pp. 22–52.

12. K. IWASAWA, *On some types of topological groups*, Ann. of Math. (2) vol. 50 (1949) pp. 507–558.

13. J. L. KOSZUL, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France vol. 78 (1950) pp. 65–127.

14. A. MALCEV, *On the theory of Lie groups in the large*, Rec. Math. (Mat. Sbornik) N. S. vol. 16 (1945) pp. 163–189.

15. G. D. MOSTOW, *A new proof of E. Cartan's theorem on the topology of semi-simple groups*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 969–980.

16. L. PONTRJAGIN, *Homologies in compact Lie groups*, Rec. Math. (Mat. Sbornik) N. S. vol. 48 (1939) pp. 389–422.

17. ———, *On some topological invariants of Riemannian manifolds*, C. R. (Doklady) Acad. Sci. URSS vol. 43 (1944) pp. 91–94.

18. H. SAMELSON, *Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten*, Ann. of Math. (2) vol. 42 (1941) pp. 1091–1137.

19. N. E. STEENROD, *Topology of fiber bundles*, Princeton, 1951. With few exceptions this will be our standard reference on fiber bundles.

20. J. A. TODD, *The geometrical invariants of algebraic loci*, Proc. London Math. Soc. (2) vol. 43 (1937) pp. 127–138.

21. R. THOM, *Variétés plongées et i-carrés*, C. R. Acad. Sci. Paris vol. 230 (1950) pp. 507–508.

22. A. WEIL, *Géométrie différentielle des espaces fibrés*, unpublished.

23. H. WEYL, *Classical groups*, Princeton, 1939.

24. H. WHITNEY, *On the topology of differentiable manifolds*, Lectures on Topology, Michigan, 1941, pp. 101–141.

25. WEN-TSÜN WU, *Topological invariance of Pontrjagin classes of differentiable manifolds*, to be published.

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