CHAPTER II.2

SUPER LIE ALGEBRAS, SUPERMANIFOLDS AND SUPERGROUPS

II.2.1 - The definition of superalgebras and the example of N-extended super Poincare algebra

We begin with the definition. A super Lie algebra is a vector space $\mathbb{A}$ over the field of complex or real numbers which splits into two subspaces $\mathbb{G}$ and $\mathbb{U}$, called respectively the even and odd subspace

$$\mathbb{A} = \mathbb{G} \oplus \mathbb{U} \qquad (II.2.1)$$

Besides the operations of a vector space (sum and multiplication by a scalar), in order to turn $\mathbb{A}$ into an algebra one must define a further product operation which we shall call the Lie bracket and denote by $[\ \ ,\ \ \}$.

The following are the defining properties of the Lie bracket:

i) If $X \in \mathbb{G}$, $Y \in \mathbb{G}$ are two elements of the even subspace then their Lie bracket belongs to the same subspace and it is antisymmetric:

$$[X,Y\} \in \mathbb{G} \quad ; \qquad [X,Y\} = -\,[Y,X\} \qquad (II.2.2)$$

Furthermore if $X, Y, Z \in \mathbb{G}$ are three elements of the even subspace, then the Jacobi identity is satisfied:

$$[X,[Y,Z\}\} + [Y,[Z,X\}\} + [Z,[X,Y\}\} = 0 \qquad (II.2.3a)$$

ii) If $X \in \mathbb{G}$ and $\Psi \in \mathbb{U}$, the Lie bracket of these two elements lies in the odd subspace and it is antisymmetric:

$$[X,\Psi\} \in \mathbb{U} \quad ; \qquad [X,\Psi\} = -\,[\Psi,X\} \qquad (II.2.3b)$$

Furthermore if $X, Y \in \mathbb{G}$ and $\Psi \in \mathbb{U}$ we demand that the following identity be satisfied:

$$[X,[Y,\Psi\}\} + [Y,[\Psi,X\}\} + [\Psi,[X,Y\}\} = 0 \qquad (II.2.4)$$

Eq. (II.2.4) can also be rewritten as follows:

$$[X,[Y,\Psi\}\} - [Y,[X,\Psi\}\} = [[X,Y\},\Psi\} \qquad (II.2.5)$$

iii) If $\Psi \in \mathbb{U}$, $\Xi \in \mathbb{U}$ are two elements of the odd subspace then their Lie bracket is symmetric and lies in the even subspace:

$$[\Psi,\Xi\} \in \mathbb{G} \quad ; \qquad [\Psi,\Xi\} = [\Xi,\Psi\} \qquad \text{(II.2.6)}$$

Moreover if $\psi, \Xi, \Lambda \in \mathbb{U}$ are all odd the following is an identity:

$$[\Psi,[\Xi,\Lambda\}\} + [\Lambda,[\Psi,\Xi\}\} + [\Xi,[\Lambda,\Psi\}\} = 0 \qquad \text{(II.2.7)}$$

while if $\psi, \Xi \in \mathbb{U}$ are odd and $X \in \mathbb{G}$ is even we have:

$$[X,[\Psi,\Xi\}\} - [\Xi,[X,\Psi\}\} + [\Psi,[\Xi,X\}\} = 0 \qquad \text{(II.2.8)}$$

iv) Finally the Lie bracket is distributive with respect to the vector space operations, namely if $\alpha, \beta \in \mathbb{C}$ (or $\mathbb{R}$) and $A, B, C \in \mathbb{A}$ then we have:

$$[\alpha A + \beta B, C\} = \alpha[A,C\} + \beta[B,C\} \qquad \text{(II.2.9)}$$

Let us discuss the meaning of these properties. Eqs. (II.2.2) and (II.2.3) are equivalent to stating that $\mathbb{G}$ is closed under the Lie bracket, namely it is a subalgebra. Not only. On this subspace the properties of our Lie bracket are the same as the properties of the Lie bracket of an ordinary Lie algebra. Hence the even subspace $\mathbb{G}$ is an ordinary Lie algebra. Let us now consider Eqs. (II.2.3b) and (II.2.4). They state that the odd subspace $\mathbb{U}$ is a carrier space for a representation of the Lie algebra $\mathbb{G}$, the Lie bracket $[\ ,\ \}$ defining the action of $\mathbb{G}$ on $\mathbb{U}$. Indeed Eq. (II.2.4), once rewritten in the form (II.2.5), is the statement that the action of elements of $\mathbb{G}$ is consistent with the Lie bracket defined over $\mathbb{G}$.

Eqs. (II.2.6) and (II.2.8) are really novel. They introduce a symmetric Lie bracket, that is an anticommutator, over the odd-subspace $\mathbb{U}$ and they state that the anticommutator of two odd elements is an even one. In other words the odd elements are the square-roots of the ordinary Lie algebra $\mathbb{G}$.

We can now condense all the Eqs. (II.2.2 - II.2.8) in a much more compact notation if we introduce the concept of grading. Let $\mathbb{Z}_2$ be the set of integer numbers mod 2; representatives of the two equivalence classes are 0 and 1. To each element $A \in \mathbb{A}$ we associate a grading a which is an element of $\mathbb{Z}_2$:

$$\forall A \in \mathbb{A} \qquad a = \text{grad } A \in \mathbb{Z}_2 \qquad \text{(II.2.10)}$$

a is 1 if A lies in the odd subspace, while it is zero if A lies in the even subspace:

$$A \in \mathbb{U} \quad \Rightarrow \quad a = 1 \ (\text{mod}2) \qquad \text{(II.2.11a)}$$

$$A \in \mathbb{G} \quad \Rightarrow \quad a = 0 \ (\text{mod}2) \qquad \text{(II.2.11b)}$$

Using this notation we can rewrite the defining properties of the Lie bracket in the following way. First we note that, utilizing the distributive property (II.2.9), the Lie bracket of two arbitrary elements of the superalgebra, which in general do not have a definite grading since they are the sum of an even and an odd part, can be decomposed into a sum of terms which are Lie brackets of elements possessing a definite grading. Then if A, B, C are elements of $\mathbb{A}$ endowed with a definite grading, we can write:

$$[A,B\} = (-)^{1+ab}[B,A\} \qquad \text{(II.2.12a)}$$

$$[A,[B,C\}\} + (-)^{a(b+c)}[B,[C,A\}\} + (-)^{b(a+c)}[C,[A,B\}\} = 0 \qquad \text{(II.2.12b)}$$

which summarizes Eqs. (II.2.2 - II.2.8).

As it happens for ordinary Lie algebras, superalgebras are most conveniently described in terms of a basis of the vector space $\mathbb{A}$. Let $\{T_A\}$ be such a basis ($A=1,\dots,d$) where $d=\dim \mathbb{A}$. Since $\mathbb{A}$ is the direct sum of $\mathbb{G}$ and $\mathbb{U}$, the basis $\{T_A\}$ can be chosen in such a way that it is the union of a basis for $\mathbb{G}$ and a basis for $\mathbb{U}$: in other words the basis elements T_A have a definite grading. The super algebra is completely specified if we give the Lie bracket of any two basis elements, from now on referred to as generators:

$$[T_A, T_B\} = C_{AB}^{\cdot\cdot F} T_F \qquad \text{(II.2.13)}$$

In Eq. (II.2.13) the summation convention on the index F is adopted and $C_{AB}^{\cdot\cdot F}$ are constants. They are the graded structure constants of the superalgebra and satisfy the following properties, inherited from eq.s (II.2.12):

$$C_{AB}^{\cdot\cdot F} = (-)^{1+ab} C_{BA}^{\cdot\cdot F} \qquad \text{(II.2.14a)}$$

$$C_{AL}^{\cdot\cdot M} C_{BC}^{\cdot\cdot L} + (-)^{a(b+c)} C_{BL}^{\cdot\cdot M} C_{CA}^{\cdot\cdot L} + (-)^{b(a+c)} C_{CL}^{\cdot\cdot M} C_{AB}^{\cdot\cdot L} = 0 \qquad \text{(II.2.14b)}$$

Furthermore if we adopt the convention that the capital latin index A is replaced by a lower case latin index when the grading is even and by a lower case Greek index when the grading is odd we have:

$$C_{ab}^{\gamma} = C_{a\beta}^{b} = C_{\alpha\beta}^{\gamma} = 0 \qquad \text{(II.2.15)}$$

The only structure constants which can be different from zero are

$$C_{ab}^{\cdot\cdot c} \quad ; \quad C_{a\beta}^{\cdot\cdot\gamma} \quad ; \quad C_{\alpha\beta}^{\cdot\cdot a} = C_{\beta\alpha}^{\cdot\cdot a} \qquad \text{(II.2.16)}$$

where:

i) $C_{ab}^{\cdot\cdot c}$ are the structure constants of the Lie algebra $\mathbb{G}$

ii) $C_{a\beta}^{\cdot\cdot\gamma}$ are $u \times u$ matrices (u being the dimension of $\mathbb{U}$) which satisfy the Lie algebra and generate one of its representations

iii) $C_{\alpha\beta}^{\cdot\cdot a}$ are the symmetric structure constants to whose existence the existence of the entire superalgebra is due.

Before proceeding to the classification of the superalgebras we want to show that the concept is not empty. To this effect we introduce an example which is of the highest relevance, namely the super Poincaré algebra. As the word suggests, this is a superalgebra $\mathbb{A}$ where the ordinary subalgebra $\mathbb{G}$ is the familiar Lie algebra of the Poincaré group. This latter is ten dimensional and its natural basis is provided by the 4 translation generators P_a plus the six Lorentz generators $M_{ab}=-M_{ba}$, the index a running from 0 to 3:

$$a = 0,\ 1,\ 2,\ 3 \qquad \text{(II.2.17)}$$

Utilizing the Minkowskian metric

$$\eta_{ab} = \begin{pmatrix} 1, & 0 & 0 & 0 \\ 0, & -1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix} \qquad \text{(II.2.18)}$$

to rise and lower the indices we can write the Poincaré Lie algebra in the following way:

$$[M_{ab},M_{cd}] = \tfrac{1}{2}(\eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{bd}M_{ac} - \eta_{ac}M_{bd}) \qquad \text{(II.2.19a)}$$

$$[P_a,P_b] = 0 \qquad \text{(II.2.19b)}$$

$$[M_{ab},P_c] = -\tfrac{1}{2}(\eta_{ca}P_b - \eta_{cb}P_a) \qquad \text{(II.2.19c)}$$

This algebra is not semisimple and it is obtained as the semidirect product of the simple Lorentz algebra SO(1,3) with its 4-dimensional vector representation. To extend it to a superalgebra we need one of its representations: we choose the 4-dimensional spinor representation.

Following a standard procedure we introduce the four gamma matrices γ_a satisfying the Clifford algebra:

$$\{\gamma_a,\gamma_b\} = 2\eta_{ab} \qquad \text{(II.2.20)}$$

and we define the matrices (see Chapter II.7 for further details on the conventions)

$$S_{ab} = \frac{1}{8}\,[\gamma_a,\gamma_b] = \frac{1}{4}\,\gamma_{ab} \qquad \text{(II.2.21)}$$

which satisfy the Lorentz algebra in the form (II.2.19a):

$$[S_{ab},S_{cd}] = \tfrac{1}{2}(\eta_{bc}S_{ad} + \eta_{ad}S_{bc} - \eta_{bd}S_{ac} - \eta_{ac}S_{bd}) \qquad \text{(II.2.22)}$$

(II.2.22)

An element of the carrier space for the spinor representation (II.2.21) is a 4-component spinor. Hence we introduce new generators, called $\bar{Q}_\alpha$ (α = 1,2,3,4) which transform as barred spinors under the Lorentz algebra

$$[M_{ab},\bar{Q}_\beta] = \bar{Q}_\alpha (S_{ab})_{\alpha\beta} = \frac{1}{4}\,\bar{Q}_\alpha\,(\gamma_{ab})_{\alpha\beta} \qquad \text{(II.2.23)}$$

and we declare that the action of the translation P_a on $\bar{Q}_\alpha$ is null

$$[P_a,\bar{Q}_\alpha] = 0 \qquad \text{(II.2.24)}$$

In this way $\bar{Q}_\alpha$ carries a representation of the full Poincaré algebra and the structure constants $C_{A\beta}^{\cdot\cdot\gamma}$ have been identified. (A runs on the Poincaré adjoint: A = a,(ab).)

$$C_{(ab)\beta}^{\cdot\ \cdot\alpha} = \frac{1}{4}\,(\gamma_{ab})_{\alpha\beta} \qquad \text{(II.2.25a)}$$

$$C_{a\beta}^{\cdot\cdot\gamma} = 0 \qquad \text{(II.2.25b)}$$

It remains to be checked whether we can construct the structure constants $C_{\alpha\beta}^{\cdot\cdot A}$. To this effect we first recall that the Poincaré Lie algebra (II.2.19) is constructed over the field of real numbers and the generators P_a, M_{ab} are antihermitean

$$P_a^\dagger = -P_a \quad ; \quad M_{ab}^\dagger = -M_{ab} \qquad \text{(II.2.26)}$$

To extend it consistently, we must impose suitable reality conditions also on the extra spinorial generators Q_α. We do this by requiring Q_α (the spinor of which $\bar{Q}_\alpha$ is the Dirac conjugate) to be a Majorana spinor. Hence we write:

$$Q = C\,\bar{Q}^T \quad ; \quad Q \equiv \gamma_0\bar{Q}^\dagger \tag{II.2.27}$$

where C is the charge conjugation matrix (see Chapter II.7 again). Equation (II.2.27) can be rewritten as follows

$$\bar{Q}_\alpha = Q^*_\beta(\gamma_0)_{\beta\alpha} = Q_\beta C_{\beta\alpha} = Q^T C \tag{II.2.28}$$

The generators S_{ab} not only satisfy the Lorentz algebra but fulfill the extra property of transforming Majorana spinors into Majorana spinors. Indeed we have:

$$\begin{aligned}(\overline{\gamma_{ab}Q}) &= Q^\dagger\gamma^\dagger_{ab}\gamma_0 = \bar{Q}\,\gamma_0\gamma^\dagger_{ab}\gamma_0 = \\ &= -\,Q^T C\,\gamma_{ab} = Q^T\gamma^T_{ab}C = (\gamma_{ab}Q)^T C\end{aligned} \tag{II.2.29}$$

which follows from the two identities:

$$\gamma_0\gamma^\dagger_{ab}\gamma_0 = -\,\gamma_{ab} \quad ; \quad C\,\gamma_{ab}C^{-1} = -\,\gamma^T_{ab} \tag{II.2.30}$$

This is essential for equation (II.2.23) to be consistent.

We close the superalgebra by writing the anticommutator of two spinorial generators:

$$\{\bar{Q}_\alpha,\bar{Q}_\beta\} = i(C\gamma^a)_{\alpha\beta}P_a \tag{II.2.31}$$

In this way the structure constants $C^{\cdot\cdot A}_{\alpha\beta}$ are identified:

$$C^{\cdot\cdot a}_{\alpha\beta} = i(C\gamma^a)_{\alpha\beta} = i(C\gamma^a)_{\beta\alpha} \tag{II.2.32a}$$

$$C^{\cdot\cdot ab}_{\alpha\beta} = 0 \tag{II.2.32b}$$

The fulfilment of the Jacobi identities (II.2.7) and (II.2.8) is easily checked. We have

$$\begin{aligned}&[\bar{Q}_\alpha,\{\bar{Q}_\beta,\bar{Q}_\gamma\}] + [\bar{Q}_\beta,\{\bar{Q}_\gamma,\bar{Q}_\alpha\}] + \\ &\qquad + [\bar{Q}_\gamma,\{\bar{Q}_\alpha,\bar{Q}_\beta\}] = 0\end{aligned} \tag{II.2.33}$$

since $\{\bar{Q},\bar{Q}\} = P$ and P_a commutes with $\bar{Q}_\alpha$ (Eq. (II.2.24)). Similarly we get:

$$[P_a,\{\bar{Q}_\alpha,\bar{Q}_\beta\}] - [\bar{Q}_\alpha,\{\bar{Q}_\beta,P_a\}] + \{\bar{Q}_\beta,[P_a,\bar{Q}_\alpha]\} = 0 \tag{II.2.34}$$

since P_a commutes with P_b and $\bar{Q}_\alpha$. Finally we should have

$$[M_{ab},\{\bar{Q}_\alpha,\bar{Q}_\beta\}] = \{\bar{Q}_\alpha,[M_{ab},\bar{Q}_\beta]\} + \{\bar{Q}_\beta,[M_{ab},\bar{Q}_\alpha]\} \tag{II.2.35}$$

To check the validity of (II.2.35) we just substitute Eqs. (II.2.23) and (II.2.31). We get

$$\begin{aligned}&-\frac{i}{2}\,(C\gamma^c)_{\alpha\beta}(\eta_{ca}P_b - \eta_{cb}P_a) \overset{?}{=} \\ &i(C\gamma^c)_{\alpha\gamma}(S^T_{ab})_{\beta\gamma}P_c + i(C\gamma^c)_{\beta\gamma}(S^T_{ab})_{\alpha\gamma}P_c\end{aligned} \tag{II.2.36}$$

which is fulfilled if the following matrix identity holds

$$-\frac{1}{2}\,(C\gamma_a\delta_{bm}-C\gamma_b\delta_{am})=S^T_{ab}(C\gamma^c)^T+C\gamma^c S_{ab} \qquad \text{(II.2.37)}$$

Multiplying by C^{-1} and using Eqs. (II.2.30) plus the definition of the matrix C:

$$C\gamma_a C^{-1}=-\gamma^T_a \qquad \text{(II.2.38)}$$

we find that equation (II.2.37) is the same as

$$-\frac{1}{2}\,(\gamma_a\delta_{bm}-\gamma_b\delta_{am})=\left[\gamma_m,S_{ab}\right]=\frac{1}{4}\,\left[\gamma_m,\gamma_{ab}\right] \qquad \text{(II.2.39)}$$

which is obviously true.

This check shows that when we decided that $\bar{Q}$ was a barred spinor and we wrote the action of S_{ab} on the right rather than on the left of $\bar{Q}$ we did not make an arbitrary choice but we just made the only one consistent with equation (II.2.31). If we wanted we could use Q instead of $\bar{Q}$ but then Eq. (II.2.31) would be replaced by:

$$\{Q_\alpha,Q_\beta\}=i(\gamma_a C)_{\alpha\beta}P_a \qquad \text{(II.2.40)}$$

The reason why we chose to work with $\bar{Q}_\alpha$ rather than Q_α is related to the privileged role we want to give to the dual formulation of the superalgebra. In Chapter II.3 we shall rewrite the super Poincaré algebra in terms of forms and the $\bar{Q}_\alpha$ generator will be identified with the dual tangent vector to a 1-form ψ^β which is an unbarred Majorana spinor:

$$\psi^\alpha(\bar{Q}_\beta)=\delta^\alpha_\beta \qquad \text{(II.2.41)}$$

Since from our point of view the fundamental object is ψ rather than $\bar{Q}$, we choose the first to be unbarred.

The superalgebra we have discussed is the N=1 Poincaré super algebra in 4-dimensions. We call it N=1 because it contains only one spinorial generator and we say that it is 4-dimensional because the vector indices a,b run from 0 to 3 while the spinor indices span the 4-dimensional spinor space. The generalization to other dimensions is not automatic as it is for the pure Poincaré algebra: indeed in order to proceed one must make sure that in the chosen dimension D the following properties hold true:

i) A charge conjugation matrix defined by Eq. (II.2.38) exists
ii) Majorana spinors, defined by Eq. (II.2.27) exist
iii) The matrix $C\gamma^a$ is symmetric.

This does not happen in all dimensions. In those dimensions where it happens we have N=1 extensions of the Poincaré group; in the others superextensions may still exist but they are more complicated: in any case, as we shall see, in every dimension D>4 the algebraic structure underlying supergravity theories is wider and more complicated than the one presented here. Hence the only dimension where the simple N=1 extension of the Poincaré group leads to interesting physical theories is precisely D=4 and this is the reason why we chose it.

Remaining in D=4 we could ask what happens if, instead of one spinorial generator $\bar{Q}_\alpha$ we introduced several, labeled by an additional index A which runs from 1 to N:

$$\bar{Q}^A_\alpha \qquad (A=1,\,2,\ldots,N) \qquad \text{(II.2.42)}$$

In this case we would still set

$$[M_{ab},\bar{Q}^A_\beta] = \frac{1}{4}\,\bar{Q}^A_\alpha(\gamma_{ab})_{\alpha\beta} \qquad \text{(II.2.43a)}$$

$$[P_a,\bar{Q}^A_\beta] = 0 \qquad \text{(II.2.43b)}$$

but we could replace Eq. (II.2.31) by:

$$\{\bar{Q}^A_\alpha,\bar{Q}^B_\beta\} = i(C\gamma^a)_{\alpha\beta}P_a\delta^{AB} + C_{\alpha\beta}Z^{AB}_{(+)} + i(C\gamma_5)_{\alpha\beta}Z^{AB}_{(-)} \qquad \text{(II.2.44)}$$

where $Z_{(+)}{}^{AB} = -Z_{(+)}{}^{BA}$ and $Z_{(-)}{}^{AB} = -Z_{(-)}{}^{BA}$ are new even generators having the property of commuting with everything else:

$$\begin{aligned}
[Z^{AB}_{(+)},Z^{CD}_{(+)}] &= [Z^{AB}_{(+)},Z^{CD}_{(-)}] = [Z^{AB}_{(-)},Z^{CD}_{(-)}] = \\
&= [Z^{AB}_{(+)},P_a] = [Z^{AB}_{(+)},M_{ab}] = [Z^{AB}_{(+)},\bar{Q}^C_\alpha] = \\
&= [Z^{AB}_{(-)},P_a] = [Z^{AB}_{(-)},M_{ab}] = [Z^{AB}_{(-)},\bar{Q}^C_\alpha] = 0 \qquad \text{(II.2.45)}
\end{aligned}$$

For this reason they are called central charges. From the algebraic point of view they are optional; we can either introduce Z_+ or Z_- or both or none of the two: the algebra closes in any case. When we shall consider the multiplets, namely the representations of the above algebras we shall see that the massless representation beginning at spin 2 (the supergravity multiplet) is consistent only if we include the spin 1 gauge field of $Z_{(+)}{}^{AB}$ but not of $Z_{(-)}{}^{AB}$; hence the algebra which leads to a supergravity theory is selected. This will be further clarified when the algebra (II.2.44) will be obtained as Inonu Wigner contraction of a simple algebra.

Eqs. (II.2.43) and (II.2.44), together with Eqs. (II.2.19) define the N-extended Poincaré superalgebras.

II.2.2 - Classification of the simple superalgebras whose Lie algebra is reductive

In this subsection we give the list of all the possible superalgebras which have the following property:

a) $\mathbb{A}$ is simple in the sense that it contains no non trivial ideals $\mathbb{J}$.

b) The ordinary subalgebra $G \subset \mathbb{A}$ is reductive in the sense that

$$G = G_1 \otimes G_2 \qquad \text{(II.2.46)}$$

where G_1 is a semisimple Lie algebra and G_2 is an abelian one. In other words G is the tensor product of some simple factors times a certain number of U(1) factors.

For the reader's convenience we recall that an ideal $\mathbb{J}$ is a subalgebra with the further property that the Lie bracket of any element $X \in \mathbb{A}$ with any element $Z \in \mathbb{J}$ is still an element of $\mathbb{J}$.

In full analogy with the treatment of ordinary Lie algebras the algebras classified here are taken over the field of complex numbers: it is then our privilege to choose a real form for them by introducing suitable reality conditions. The classification theorem is due to Scheunert, Nahm and Rittenberg: its proof being quite technical and complicated we restrict ourselves to enunciating the thesis: the interested reader is referred to the original article.

Theorem: The simple superalgebras whose Lie algebra is reductive are the following ones:

A) The infinite series of orthosymplectic algebras Osp(2p/N).

B) The infinite series of superunitary algebras SU(m/N).

C) The infinite series of P(n) and Q(n) algebras.

D) The three exceptional algebras D(2,1,α), G(3) and F(4).

Let us describe these algebras one by one emphasizing once more that they are classified as complex algebras although in case a) and b) their name is taken from the name of their most important real form.

The basic idea for the construction is that of considering complex matrices in dimension

$$d = m + N \tag{II.2.47}$$

where m and N are two integer numbers. Any d x d matrix can be written in block form as follows

$$Q = \left(\begin{array}{c||c} A & B \\ \hline\hline \underbrace{C}_{m} & \underbrace{D}_{N} \end{array}\right)\begin{array}{l} \}m \\ \}N \end{array} \tag{II.2.48}$$

where A is m x m, D is N x N, and B and C are m x N and N x m respectively. The space of d x d matrices is a d^2-dimensional vector space which can be split, according to (II.2.1), into an even and odd subspace by defining:

$$Q \in \mathbb{G} \Longleftrightarrow B = C = 0 \Longrightarrow Q = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \tag{II.2.49a}$$

$$Q \in \mathbb{U} \Longleftrightarrow A = D = 0 \Longrightarrow Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{II.2.49b}$$

The Lie bracket can now be introduced, consistently with the grading (II.2.49) and with all the axioms (II.2.2-II.2.9) of a superalgebra, by means of the following recipe:

$$\text{if} \quad Q_1,Q_2 \in \mathbb{G} : [Q_1,Q_2\} = [Q_1,Q_2] \tag{II.2.50a}$$

$$\text{if} \quad Q_1 \in \mathbb{G}\,,\ Q_2 \in \mathbb{U} : [Q_1,Q_2\} = [Q_1,Q_2] \tag{II.2.50b}$$

$$\text{if} \quad Q_1,\ Q_2 \in \mathbb{U} : [Q_1,Q_2\} = \{Q_1,Q_2\} \tag{II.2.50c}$$

where [,], { , } denote, respectively, the ordinary commutator and anticommutator of matrices:

$$[Q_1,Q_2] = Q_1Q_2 - Q_2Q_1 \tag{II.2.51a}$$

$$\{Q_1,Q_2\} = Q_1Q_2 + Q_2Q_1 \tag{II.2.51b}$$

Eqs. (II.2.50) can be summarized by stating that the Lie bracket of any two matrices Q_1 and Q_2 of type (II.2.48) is a new matrix Q_3 of the same type:

$$[Q_1,Q_2\} = Q_3 = \left(\begin{array}{c|c} A_3 & B_3 \\ \hline C_3 & D_3 \end{array}\right) \tag{II.2.52}$$

where:

$$A_3 = [A_1,A_2] + B_1C_2 + B_2C_1 \qquad (II.2.53a)$$

$$D_3 = [D_1,D_2] + C_1B_2 + C_2B_1 \qquad (II.2.53b)$$

$$B_3 = A_1B_2 - B_2D_1 - A_2B_1 + B_1D_2 \qquad (II.2.53c)$$

$$C_3 = D_1C_2 - C_2A_1 - D_2C_1 + C_1A_2 \qquad (II.2.53d)$$

The superalgebra obtained in this way is called the general graded Lie algebra GL(m/N): it is not simple. The simple algebras Osp(2p/N), SU(m/N), P(n), Q(n) are obtained as subalgebras of GL(m/N) by imposing further conditions on the block matrices Q .

A) The orthosymplectic algebras Osp(2p/N)

An element of Osp(2p/N), which exists only when m=2p is even, is a matrix $Q \in GL\ (m{=}2p/N)$ characterized by the following conditions:

$$A^T\,\Omega_{(2p)} + \Omega_{(2p)}A = 0 \qquad (II.2.54a)$$

$$D^T\,\Omega_{(N)} + \Omega_{(N)}D = 0 \qquad (II.2.54b)$$

$$C = \Omega_{(N)}\,B^T\,\Omega_{(2p)} \qquad (II.2.54c)$$

where the two matrices $\Omega_{(2p)}$ and $\Omega_{(N)}$ have the following properties:

$$\Omega^2_{(2p)} = -\mathbb{1} \quad ; \quad \Omega^T_{(2p)} = -\Omega_{(2p)} \qquad (II.2.55a)$$

$$\Omega^T_{(N)} = \Omega_{(N)} \qquad (II.2.55b)$$

From (II.2.55a) we see that since $\Omega_{(2p)}$ is antisymmetric the matrices A span a symplectic subalgebra Sp(2p,C) of Osp(2p/N). On the other hand $\Omega_{(N)}$, being symmetric, is an orthogonal metric and the submatrices D span an orthogonal subalgebra O(N,C) of Osp(2p/N).

The ordinary Lie subalgebra of Osp(2p/N) is therefore

$$G = Sp(2p) \otimes O(N) \qquad (II.2.56)$$

which explains the name chosen for the superalgebra.

The off-diagonal matrices B and C which are related to each other by Eq. (II.2.54) are acted on by the symplectic and orthogonal algebra transforming respectively in the defining representations of Sp(2p) and O(N). Of particular interest to us will be the algebras Osp(4/N) with $1 \le N \le 8$. In this case one exploits the Lie algebra isomorphism

$$Sp(4,C) \sim O(5,C) \qquad (II.2.57)$$

and imposing suitable reality conditions one obtains a real superalgebra Osp(4/N) whose Lie algebra is SO(2,3) x SO(N).

SO(2,3) is the group of motions of anti de Sitter space (the anti de Sitter group) containing the Lorentz generators M_{ab} and the non commuting anti de Sitter translations P_a: the off-diagonal generators transform as vectors under SO(N) and as spinors under SO(2,3) playing a role analogue to the role of the supersymmetry generators in the N-extended Poincaré superalgebra. Actually, as we shall see, by an Inonü-Wigner contraction which sends the cosmological constant to zero the Lie algebra Osp(4/N) reduces to the N-extended Poincaré superalgebra, the central charges $Z_{(+)}^{AB}$ being the limit of the O(N) generators.

B) The superunitary algebras SU(p,q/N)

As we already mentioned the algebras we define here are complex algebras: hence we are allowed to use, for their definition, only those properties which do not distinguish between real and complex numbers. Such a property in the case of the SU(m/N) family is the following

$$\mathrm{Tr}\, A = \mathrm{Tr}\, D \qquad \text{(II.2.58)}$$

which is conserved by the Lie bracket product (II.2.52) and (II.2.53). The elements of GL(m/N) satisfying (II.2.58) span a simple superalgebra whose Lie algebra is immediately seen to be:

$$\mathbb{G} = SL(m,C) \otimes SL(N,C) \otimes GL(1,C) \qquad \text{(II.2.59)}$$

In view of this the most appropriate name for this complex superalgebra would be SL(m/N); the name superunitary SU(p,q/N) originates from the possibility of imposing the following reality conditions

$$H_{(m)}\, A\, H^{-1}_{(m)} = -\, A^{\dagger} \qquad \text{(II.2.60a)}$$

$$H_{(N)}\, D\, H^{-1}_{(N)} = -\, D^{\dagger} \qquad \text{(II.2.60b)}$$

$$H_{(m)}\, B\, H^{-1}_{(N)} = -\, C^{\dagger} \qquad \text{(II.2.60c)}$$

where $H_{(m)}$ and $H_{(N)}$ are hermitean

$$H^{\dagger}_{(m)} = H_{(m)} \quad ; \quad H^{\dagger}_{(N)} = H_{(N)} \qquad \text{(II.2.61)}$$

and we admit p positive and q negative eigenvalues for $H_{(m)}$ while we choose all positive eigenvalues for $H_{(N)}$.

Conditions (II.2.60) are preserved by the Lie bracket (II.2.52-II.2.53) and imply that A and D span a U(p,q) (p+q=m) and U(N) Lie algebra respectively. Condition (II.2.58) then tells us that the full Lie algebra of our real superalgebra is

$$\mathbb{G} = SU(p,q) \otimes SU(N) \otimes U(1) \qquad \text{(II.2.62)}$$

This property of the real form (II.2.60) justifies the name SU(p,q/N) given to the whole family of complex algebras.

Of special interest to us will be the case m=4.

Utilizing the isomorphism:

$$SU(2,2) \sim SO(2,4) \qquad \text{(II.2.63)}$$

we can reinterpret the Lie algebra sector SU(2,2) x SU(N) x U(1) of SU(2,2/N) as the anti de Sitter algebra in five space time dimensions times an internal symmetry SU(N) x U(1): the off-diagonal generators transforming as spinors under SO(2,4) will be the supersymmetry generators. Hence SU(2,2/N) is the superalgebra of 5-dimensional supergravity. We mention that SU(2,2/N) can also be interpreted as N-extended superconformal algebra in D=4 and it is the basis of conformal supergravity: however since this theory and its applications are out of the scope and of the philosophy of this book we will not discuss this point further.

Before leaving the SU(p,q/N) algebras we note that if p+q=m=N then condition (II.2.58) does not define a simple algebra. Indeed in this case the matrices of the type

$$Q = \left(\begin{array}{c|c} \lambda \mathbb{1}_{(N)} & 0 \\ \hline 0 & \lambda \mathbb{1}_{(N)} \end{array} \right) \tag{II.2.64}$$

span an abelian ideal $Z_{(N)}$ of SL(N/N). To obtain a simple algebra we take the quotient

$$SL(N/N)/Z_{(N)} \Rightarrow \mathrm{Tr}\ A = 0 \tag{II.2.65}$$

and we obtain a superalgebra whose Lie algebra is

$$G = SL(N,C) \otimes SL(N,C) \tag{II.2.66}$$

with the GL(1,C) factor omitted. Correspondingly the real forms SU(p,q/p+q) have, as Lie algebra

$$G = SU(p,q) \otimes SU(p+q) \tag{II.2.67}$$

with the U(1)-factor omitted.

C) The superalgebras P(n) and Q(n)

The superalgebras P(n) are defined for m=n ≥ 3 as the set of matrices (II.2.48) fulfilling the conditions:

$$A^T + D = 0 \qquad \mathrm{Tr}\ A = 0 \tag{II.2.68a}$$

$$B^T = B \quad ; \quad C^T = - C \tag{II.2.68b}$$

while the Q(n) algebra are defined, under the same restrictions on m and n by

$$A = D \quad ; \quad \mathrm{Tr}\ A = 0 \tag{II.2.69a}$$

$$B = C \quad ; \quad \mathrm{Tr}\ B = 0 \tag{II.2.69b}$$

In both cases the Lie algebra is

$$G = SL(n,C) \tag{II.2.70}$$

No physical application of these algebras has been found so far.

D) The exceptional superalgebras D(2,1,α), G(3), F(4)

These exceptional superalgebras have so far found no interesting applications and, therefore will be described very briefly.

i) $\underline{D(2,1,\alpha)}$: The Lie algebra is

$$G = SL(2,C) \otimes SL(2,C) \otimes SL(2,C) \qquad (II.2.71)$$

and the odd-generators, which are 8, transform as the tensor product of the fundamental 2-dimensional representations of the three SL(2,C). Since in the anticommutator of two odd-generators there is a number α which can be arbitrarily chosen, $D(2,1,\alpha)$ is actually a one-parameter family of 17-dimensional simple superalgebras.

ii) $\underline{G(3)}$: This is a superalgebra whose Lie algebra is

$$G = SL(2,C) \otimes G_2 \qquad (II.2.72)$$

The 14 odd generators transform in the $\underline{2}$ of SL(2,C) and in the $\underline{7}$ of G_2. Altogether the superalgebra has 3+14+14=31 generators.

iii) $\underline{F(4)}$: The superalgebra is

$$G = SL(2,C) \otimes SO(7,C) \qquad (II.2.73)$$

The 16 odd generators transform in the $\underline{2}$ of SL(2,C) and in the $\underline{8}$-spinorial representation of SO(7,C).

The superalgebra is therefore 40-dimensional.

II.2.3 - Grassmann algebras

In order to exponentiate our superalgebras and obtain the corresponding supergroups it is convenient to introduce the concept of Grassmann algebra whose elements will be the parameters of the supergroups.

A Grassmann algebra GA_n is an extension of the field of complex numbers defined through the following construction.

Let

$$\pi_i \qquad i = 1, 2, 3, \ldots, n \qquad (II.2.74)$$

be n-objects, called generators of the Grassman algebra, which satisfy the following anticommutation relations:

$$\pi_i \pi_j = -\pi_j \pi_i \Rightarrow \pi_i^2 = 0 \qquad (II.2.75)$$

and let us consider all the possible monomials $\pi_{i_1} \ldots \pi_{i_k}$.

The number N_k of different k-monomials is

$$N_k = \binom{n}{k} \qquad (II.2.76)$$

and the total number of monomials is

$$N_{monomials} = \sum_{s=0}^{n} \binom{n}{s} = 2^m \qquad (II.2.77)$$

The Grassman algebra GA_n generated by $\{\pi_i\}$ is the 2^n-dimensional complex vector space spanned by all the linear combinations of the 2^n-monomials $\pi_{i_1} \ldots \pi_{i_k}$. Note that we have included the k=0 monomial which is by definition the complex number 1.

An element $\alpha \epsilon GA_n$ of the Grassmann algebra is therefore written as follows

$$\alpha = z + \alpha_i \pi^i + \alpha_{ij} \pi^i \pi^j + \alpha_{ijk} \pi^i \pi^j \pi^k \qquad (II.2.78)$$

where $z, \alpha_i, \alpha_{ij}, \alpha_{ijk}, \ldots$ are complex numbers. In particular if $\alpha_i = \alpha_{ij} = \alpha_{ijk} = \ldots = 0$ α is an ordinary complex number. Note moreover that $\alpha_{i_1 \ldots i_k}$ is by definition an antisymmetric tensor.

GA_n is an algebra because the product of the generators induces, canonically, a product operation of the elements of GA_n. Explicitly we have

$$\alpha^{(1)} \cdot \alpha^{(2)} = \alpha^{(3)} \qquad (II.2.79)$$

where

$$z^{(3)} = z^{(1)} z^{(2)} \qquad (II.2.80a)$$

$$\alpha_i^{(3)} = z^{(1)} \alpha_i^{(2)} + z^2 \alpha_i^{(1)} \qquad (II.2.80b)$$

$$\alpha_{ij}^{(3)} = \alpha_i^{(1)} \alpha_j^{(2)} - \alpha_i^{(2)} \alpha_j^{(1)} + z^{(1)} \alpha_{ij}^{(2)} + z^{(2)} \alpha_{ij}^{(1)} \qquad (II.2.80c)$$

$$\alpha_{(ijk)}^{(3)} = \ldots\ldots\ldots\ldots\ldots\ldots \qquad (II.2.80d)$$

The product operation in $GA_{(n)}$ is associative and distributive but it is not commutative. Every even monomial (k=2p) commutes with any other monomial, odd or even (k=2p or k=2p+1). This suggests that every element of $GA_{(n)}$ should be split into an even and an odd part:

$$\alpha = \alpha^{(+)} + \alpha^{(-)} \Rightarrow GA_{(n)} = GA_{(m)}^{(+)} \oplus GA_{(n)}^{(-)} \qquad (II.2.81)$$

where the even part is a linear combination of the even monomials and the odd part a linear combination of the odd ones. This operation induces a $\mathbb{Z}_2$-grading of the Grassmann algebra. Indeed if $GA_{(n)}^{(+)}$ and $GA_{(n)}^{(-)}$ are the even and odd subspaces of GA_n the following properties are easily checked

$$GA_{(n)}^{(+)} \cdot GA_{(n)}^{(+)} \subset GA_{(n)}^{(+)} \qquad (II.2.82a)$$

$$GA_{(n)}^{(+)} \cdot GA_{(n)}^{(-)} \subset GA_{(n)}^{(-)} \qquad (II.2.82b)$$

$$GA_{(n)}^{(-)} \cdot GA_{(n)}^{(-)} \subset GA_{(n)}^{(+)} \qquad (II.2.82c)$$

Furthermore while an even element $\alpha^+ \in GA^+$ commutes with any element $\alpha \in GA_{(n)}$ the product of two odd elements α^-, $\beta^- \in GA_{(n)}^{(-)}$ is anticommutative:

$$\alpha^- \beta^- = - \beta^- \alpha^- \qquad (II.2.83)$$

Defining the grading a of an element $\alpha \in GA_{(n)}$ to be zero if it is even and to be one if it is odd we can write

$$\alpha\beta = (-)^{ab}\beta\alpha \qquad \text{(II.2.83)}$$

Equation (II.2.83) makes it clear that if we decide that the parameters multiplying odd elements of a given superalgebra are odd elements of a Grassmann algebra while those multiplying the even elements of the superalgebra are even elements of the same Grassmann algebra then all signs will be automatically taken care of and all factors $(-)^{ab}$ will disappear. However before showing how this happens we want to discuss some more properties of the Grassmann algebras: in particular we want to define complex conjugation. Let n=2p and let us label the generators π_i in the following way

$$\pi_{\alpha} \quad (\alpha = 1,2,\dots,p) \quad ; \quad \pi_{\bullet\alpha} \quad (\bullet\alpha = p+1,\dots,2p) \qquad \text{(II.2.84)}$$

We define a mapping * which acts on the generators π in the following way:

$$(\pi_{\alpha})^* = \pi_{\bullet\alpha} \qquad \text{(II.2.85a)}$$

$$(\pi_{\bullet\alpha})^* = \pi_{\alpha} \qquad \text{(II.2.85b)}$$

$$(\pi_i\pi_j)^* = (\pi_j)^*(\pi_i)^* \qquad \text{(II.2.85c)}$$

$$(a\pi_i)^* = a^*(\pi_i)^* \qquad \text{(II.2.85d)}$$

where a^* is the complex conjugate of the complex number a.

The mapping * extends canonically to all the elements of the Grassmann algebra. If $\alpha \in GA_{(2p)}$ is given by (II.2.78) we have:

$$\alpha^* = z^* + \alpha_i^*(\pi^i)^* + \alpha_{ij}^*(\pi^j)^*(\pi^i)^* + \dots \qquad \text{(II.2.86)}$$

The operation *, which is called the complex conjugation in the even Grassmann algebra $GA_{(2p)}$, has the following formal properties:

$$\forall \alpha \in GA_{(2p)} : (\alpha^*)^* = \alpha \qquad \text{(II.2.87a)}$$

$$\forall \alpha_1,\alpha_2 \in GA_{(2p)} : (\alpha_1\alpha_2)^* = \alpha_2^*\alpha_1^* \qquad \text{(II.2.87b)}$$

$$\forall a \in \mathbb{C},\ \forall \alpha \in GA_{(2p)} : (a\alpha)^* = a^*\alpha^* \qquad \text{(II.2.87c)}$$

Eq.s (II.2.87) could also be regarded as the defining axioms.

Given the complex conjugation, the notions of reality and of norm are defined in the same way as for complex numbers

$$\alpha = \text{real} \Rightarrow \alpha^* = \alpha \qquad \text{(II.2.88a)}$$

$$\| \alpha \|^2 = \alpha^*\alpha \qquad \text{(II.2.88b)}$$

It is important, however, to keep in mind that $||\alpha||^2$ is an element of the Grassmann algebra and it is not positive definite. For instance the norm of an imaginary odd element is always zero

$$\alpha = -\alpha \quad , \quad \alpha \in GA^{(-)}_{(2p)} \Rightarrow \| \alpha \|^2 = 0 \qquad \text{(II.2.89)}$$

Let now $GA_{(n)}$ be a Grassmann algebra with n generators: an analytic function mapping $GA_{(n)}$ into itself:

$$f : GA_{(n)} \rightarrow GA_{(n)} \qquad \text{(II.2.90)}$$

can be defined via a power series expansion:

$$\forall\alpha \in GA_{(n)} : f(\alpha) = \sum_{m=0}^{\infty} f_m(\alpha)^m \in GA_{(n)} \qquad \text{(II.2.91)}$$

where f_m are the coefficients of a series with finite convergence radius. If α is an even element the series (II.2.91) may extend to infinity (we say may because α, although even, may be nilpotent ($\alpha^m = 0$ for some $0< m <\infty$)); however if α is odd the series necessarily stops after the first element since $\alpha^2=0$.

A function of several variables mapping the tensor product $GA_{(n)}$ x ... x $GA_{(n)}$ into $GA_{(n)}$ can also be defined via a power series. In this case if the arguments of the function are odd the series does not stop at the first term but it degenerates into a polynomial of finite degree equal to the number of arguments of the function.

II.2.4 - Supermanifolds

Equipped with the notion of Grassmann algebras one can introduce the concept of supermanifold. Without entering the endless discussions and subtleties which this concept has stirred in the mathematical and physical-mathematical literature one can take the simple minded point of view that a supermanifold is a smooth space whose points are labeled by two sets of coordinates: the bosonic and the fermionic ones.

The bosonic coordinates are even elements of a Grassmann algebra while the fermionic coordinates are odd elements of the same algebra GA_∞, which, to avoid pitfalls, should be chosen to have an infinite number of generators π_i. Since the concept of function on a Grassmann algebra is well-defined, we can introduce charts, atlases and transition functions: in a word the whole machinery of differential geometry.

Therefore by $\mathscr{M}^{p/q}$ we shall denote a supermanifold with p bosonic dimensions and q-fermionic ones. The coordinates of a point $p \in \mathscr{M}^{p/q}$ will be denoted by:

$$p \Longrightarrow \{x^a, \theta^\alpha\} \qquad \text{(II.2.92)}$$

where x^a $(a=1,2,\ldots,p)$ are bosonic and θ^α $(\alpha = 1,\ldots,q)$ are fermionic.

The functions of several variables which map $\mathscr{M}^{p/q}$ into GA_∞ :

$$\Phi : \mathscr{M}^{p/q} \Longrightarrow GA_\infty \qquad \text{(II.2.93)}$$

are called the superfields. Utilizing the nilpotency of θ^α the superfield Φ (x, Θ) can be written as a polynomial in θ^α, whose coefficients are functions of the bosonic coordinates only:

$$\Phi(x,\Theta) = \varphi(x) + \varphi_\alpha(x)\theta^\alpha + \varphi_{\alpha_1\alpha_2}(x)\theta^{\alpha_1}\theta^{\alpha_2}$$
$$+ \ldots + \varphi_{\alpha_1\ldots\alpha_q}(x)\theta^{\alpha_1}\ldots\theta^{\alpha_q} \qquad \text{(II.2.94)}$$

As one sees, a superfield is just a bookkeeping device for a collection of ordinary fields with different tensor structures. (We emphasize that all the $\varphi_{\alpha_1..\alpha_m}(x)$ are completely antisymmetric in their indices because of the anticommutativity of the Θ.s). In supersymmetric theories where the fermionic coordinates Θ^α are spinors, the fields in the collection have different spins, bosons and fermions necessarily coexisting in the same superfield.

The space of superfields, named $C(\mathcal{M}^{p/q})$ is acted on by differential operators which are linear combinations of the fundamental derivatives:

$$\vec{\partial}_a = \frac{\partial}{\partial x^a} \tag{II.2.95a}$$

$$\vec{\partial}_\alpha = \frac{\partial}{\partial \Theta^\alpha} \tag{II.2.95b}$$

The action of $\vec{\partial}_a$ and $\vec{\partial}_\alpha$ is defined through the following formulae

$$\vec{\partial}_a \Phi(x,\Theta) = \partial_a \varphi(x) + \partial_a \varphi_\alpha(x)\Theta^\alpha + \ldots \tag{II.2.96a}$$

$$\partial_\alpha \Phi(x,\Theta) = \varphi_\alpha(x) + 2\varphi_{\alpha\beta}(x)\Theta^\beta + 3\varphi_{\alpha\beta\gamma}(x)\Theta^\beta\Theta^\gamma + \ldots \tag{II.2.96b}$$

and the following formal properties are easily verified:

$$\vec{\partial}_a\vec{\partial}_b - \vec{\partial}_b\vec{\partial}_a = 0 \tag{II.2.97a}$$

$$\vec{\partial}_a\vec{\partial}_\beta - \vec{\partial}_\beta\vec{\partial}_a = 0 \tag{II.2.97b}$$

$$\vec{\partial}_\alpha\vec{\partial}_\beta + \vec{\partial}_\beta\vec{\partial}_\alpha = 0 \tag{II.2.97c}$$

$$\vec{\partial}_\alpha(\Theta^\beta \phi(x,\Theta)) = \delta^\beta_\alpha \Phi - \Theta^\beta \vec{\partial}_\alpha \Phi \tag{II.2.97d}$$

The differential operators

$$\vec{t} = t^a(x,\Theta)\vec{\partial}_a + t^\alpha(x,\Theta)\vec{\partial}_\alpha \tag{II.2.98}$$

where t^a and t^α are respectively bosonic and fermionic superfields span the tangent space to $\mathcal{M}^{p/q}$ named $T(\mathcal{M}^{p/q})$.

At each point $p=\{x, \Theta\}$ $T(\mathcal{M}^{p/q})$ is a graded vector space with p-bosonic and q-fermionic dimensions.

A graded vector space $V(n/m)$ can be defined in the following way. Let $\{\vec{e}_a, \vec{e}_\alpha\}$ be a collection of n elements $\vec{e}_a$ $(a=1,\ldots,n)$ and m elements $\vec{e}_\alpha$ $(\alpha = 1,\ldots,m)$ respectively called the bosonic and fermionic fundamental vectors. An element $\nu \in V\,(n,m)$ is a linear combination

$$\nu = \nu^a\vec{e}_a + \nu^\alpha\vec{e}_\alpha \tag{II.2.99}$$

where

$$\nu^a \in GA_\infty^{(+)} \tag{II.2.100a}$$

$$\nu^\alpha \in GA_\infty^{(-)} \tag{II.2.100b}$$

In other words the ν^a components of a graded vector are even elements of an infinitely generated Grassmann algebra GA_∞, while the ν^a components of the same vector are odd elements of GA_∞.

In complete analogy to ordinary vector space theory one can introduce the notion of the dual space $V^*(n/m)$. It suffices to introduce a dual basis of linear functionals $\{e_*^a, e_*^\alpha\}$:

$$\forall w \in V(n/m) : (w,e_*) \in GA_\infty \tag{II.2.101a}$$

$$(\vec{e}_a, e_*^b) = \delta_a^b \quad ; \quad (\vec{e}_\alpha, e_*^b) = 0 \tag{II.2.101b}$$

$$(\vec{e}_a, e_*^\beta) = 0 \quad ; \quad (\vec{e}_\alpha, e_*^\beta) = \delta_\alpha^\beta \tag{II.2.101c}$$

and to define the elements of $V^*(n/m)$ as the linear combinations

$$\nu^* \in V^*(n/m) \qquad \nu^* = e_*^a \nu_a + e_*^\alpha \nu_\alpha \tag{II.2.102}$$

It follows that:

$$\forall \nu \in V(n/m) \quad , \quad \forall w^* \in V^*(n/m) :$$

$$(\nu, w^*) = (\nu^a w_a + \nu^\alpha w_\alpha) \in GA_\infty \tag{II.2.103}$$

Notice that to avoid ordering problems we have written the coefficients of the vectors on the left and the coefficients of the dual vectors on the right.

As usual we can define differential 1-forms on $\mathscr{M}^{p/q}$ as

elements of the dual vector space to $T(\mathscr{M}^{p/q})$. The dual basis to the derivatives $\vec{\partial}_a$ and $\vec{\partial}_\alpha$ is provided by the differentials dx^a and $d\Theta^\alpha$. Writing

$$(\vec{\partial}_a, dx^b) = \delta_a^b \quad ; \quad (\vec{\partial}_\alpha, dx^b) = 0 \tag{II.2.104a}$$

$$(\vec{\partial}_a, d\Theta^\beta) = 0 \quad ; \quad (\vec{\partial}_\alpha, d\Theta^\beta) = \delta_\alpha^\beta \tag{II.2.104b}$$

one can define a 1-form $\omega \in T^*(\mathscr{M}^{p/q})$ via the following equation

$$\omega = dx^a \omega_a(x,\Theta) + d\Theta^\alpha \omega_\alpha(x,\Theta) \tag{II.2.105}$$

where $\omega_a(x,\Theta)$, $\omega_\alpha(x,\Theta)$ are respectively bosonic and fermionic superfields.

p-forms can now be introduced as elements of the exterior product of p copies of the cotangent vector space $T^*(\mathscr{M}^{p/q})$.

Consistency is obtained if we set the following rules for the exterior product.

$$dx^a \wedge dx^b = - dx^b \wedge dx^a \tag{II.2.106a}$$

$$dx^a \wedge d\Theta^\beta = - d\Theta^\beta \wedge dx^a \tag{II.2.106b}$$

$$d\Theta^\alpha \wedge d\Theta^\beta = d\Theta^\beta \wedge d\Theta^\alpha \tag{II.2.106c}$$

and if we define

$$\omega^{(p)} = \omega_{a_1 \ldots a_p}(x,\Theta)\, dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p} +$$
$$+\, \omega_{\alpha_1 a_2 \ldots a_p}(x,\Theta)\, d\Theta^{\alpha_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p} +$$
$$+ \ldots + \omega_{\alpha_1 \ldots \alpha_p}\, d\Theta^{\alpha_1} \wedge \ldots \wedge d\Theta^{\alpha_p} \qquad \text{(II.2.107)}$$

where again $\omega_{\alpha_1 \cdots \alpha_m a_{m+1} \cdots a_p}(x,\Theta)$ are fermionic or bosonic superfields depending on whether the number of Greek indices is odd or even: in this way the usual grading of the exterior product of forms is respected:

$$\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq} \omega^{(q)} \wedge \omega^{(p)} \qquad \text{(II.2.108)}$$

Equation (II.2.106c), however, suggests that (II.2.108) can be generalized. Indeed the above choice of the fermionic or bosonic character of $\omega_{\alpha_1 \ldots \alpha_m a_{m+1} \ldots a_p}$ is the right one for a bosonic p-form $\omega^{(p)}$. It is perfectly legitimate, however, to consider fermionic p-forms: such are, for instance, the coordinate differentials $d\Theta^{\alpha}$ and, in general, all the p-forms carrying free fermionic indices in an odd number.

When the forms are fermionic they must commute with opposite sign with respect to the bosonic forms of the same degree. Hence we declare that each form ω has two gradings: a grading a=0,1 which tells you whether it is bosonic or fermionic and a grading p which is its degree. Equation (II.2.108) is then replaced by

$$\omega^{(p)}_{(a)} \wedge \omega^{(q)}_{(b)} = (-)^{ab+pq} \omega^{(q)}_{(b)} \wedge \omega^{(p)}_{(a)} \qquad \text{(II.2.109)}$$

This is as much as we need, for the moment, of supermanifolds: let us turn to the exponentiation of superalgebras.

II.2.5 - Supergroups and graded matrices

We come back to the definition of the simple superalgebras discussed in Section II.2.2. The two classical infinite families Osp(m/N) and SU(m/N), which are the most relevant to our purposes, are described in terms of ordinary matrices Q (see Eq. (II.2.48)) whose Lie bracket, however, is the unusual one defined by Eqs. (II.2.52) and (II.2.53). This Lie bracket can be understood if we perform the following construction. Consider GL(m/N), namely the algebra of (m+N) x (m+N) complex matrices, which is closed under (II.2.52), and let $\{t_a, t_\alpha\}$ be a basis of GL(m/N). $\{t_a\}$ $(a=1,2,\ldots, m^2+N^2)$ is a basis of the even subspace:

$$t_a = \left(\begin{array}{c|c} A_a & 0 \\ \hline 0 & D_a \end{array}\right) \qquad \text{(II.2.110)}$$

while $\{t_\alpha\}$ $(\alpha = 1,2,\ldots,2mN)$ is a basis of the odd subspace:

$$t_\alpha = \left(\begin{array}{c|c} 0 & B_\alpha \\ \hline C_\alpha & 0 \end{array}\right) \qquad \text{(II.2.111)}$$

Any matrix $Q \in GL(m/N)$ can be written as

$$Q = Q^a t_a + Q^\alpha t_\alpha \qquad \text{(II.2.112)}$$

where Q^a, $Q^\alpha \in \mathbb{C}$ are complex numbers and the Lie bracket of two matrices Q_1 and Q_2 is, according to (II.2.52-53), the following one:

$$[Q_1,Q_2\} = Q_1^aQ_2^b[t_a,t_b] + (Q_1^aQ_2^\beta - Q_1^\beta Q_2^a)[t_a,t_\beta] + Q_1^\alpha Q_2^\beta\{t_\alpha,t_\beta\} \qquad \text{(II.2.113)}$$

At this point notice that the right-hand side of Eq. (II.2.113) would be the ordinary commutator of Q_1 and Q_2

$$[Q_1,Q_2] = Q_1Q_2 - Q_2Q_1 \qquad \text{(II.2.114)}$$

if $Q^a\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ and $Q^\alpha\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$, instead of being complex numbers, were, respectively, even and odd elements of a Grassmann algebra GA_∞. In view of this, to every superalgebra and, in particular, to GL(m/N) we associate a graded vector space spanned by the linear combinations of the even generators with even elements of a GA_∞ and of the odd generators with odd elements of the same GA_∞. The ordinary commutator of elements of the associated graded vector space provides, in view of our previous observation, an isomorphic realization of the superalgebra.

The advantage of this point of view is that we are now able to define the supergroup corresponding to a given superalgebra as the exponentiation of the associated graded vector space. Formally we can write:

$$\mathbb{A} \Rightarrow \hat{\mathbb{A}} = \text{graded vector space where complex numbers are replaced by elements of the Grassmann algebra} \qquad \text{(II.2.115a)}$$

$$\mathcal{G} = \exp(\,\hat{\mathbb{A}}\,) = \text{supergroup associated to } \mathbb{A} \qquad \text{(II.2.115b)}$$

In the case of GL(m/N) our construction introduces the notion of a graded matrix: an element of the associated graded vector space is a matrix whose entries are elements of the Grassmann algebra: even in the diagonal blocks (A,D) odd in the off-diagonal ones (B,C). Such objects are worth considering for their own sake: indeed they can be viewed as GA_∞-linear operators on graded vector spaces and the supergroups Osp(m/N) and SU(m/N) can be viewed as groups of graded matrices. The product operation is the ordinary product of graded-matrices.

Let

$$Q = \left(\begin{array}{c|c} A & \Sigma \\ \hline \Pi & D \end{array}\right) \qquad \text{(II.2.116)}$$

be a graded matrix. A,D are m x m and N x N matrices, with commuting entries while Σ, Π are m x N and Nxm matrices, respectively, with anticommuting entries.

The product Q_1Q_2 is defined as for ordinary matrices. We have:

$$Q_1Q_2 = Q_3 = \left(\begin{array}{c|c} A_3 & \Sigma_3 \\ \hline \Pi_3 & D_3 \end{array}\right) \qquad \text{(II.2.117a)}$$

$$A_3 = A_1A_2 + \Sigma_1\Pi_2 \quad ; \quad D_3 = \Pi_1\Sigma_2 + D_1D_2 \qquad \text{(II.2.117b)}$$

$$\Sigma_3 = A_1\Sigma_2 + \Sigma_1D_2 \quad ; \quad \Pi_3 = \Pi_1A_2 + D_1\Pi_2 \qquad \text{(II.2.117c)}$$

The operations of transposition, hermitean conjugation plus the definition of supertrace and superdeterminant are given below:

$$Q^T = \begin{pmatrix} A^T & \Pi^T \\ -\Sigma^T & D^T \end{pmatrix} \quad ; \quad Q^\dagger = \begin{pmatrix} A^\dagger & \Pi^\dagger \\ \Sigma^\dagger & D^\dagger \end{pmatrix} \qquad \text{(II.2.118a)}$$

$$\text{Str } Q = \text{Tr } A - \text{Tr } D \qquad \text{(II.2.118b)}$$

$$\text{S det } Q = (\det A)(\det D') \qquad \text{(II.2.118c)}$$

where D'is defined by the following equations:

$$Q^{-1}Q = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad \text{(II.2.119a)}$$

$$Q^{-1} = \begin{pmatrix} A' & \Sigma' \\ \Pi' & D' \end{pmatrix} \qquad \text{(II.2.119b)}$$

These definitions are designed in such a way that the following properties valid for ordinary matrices hold true also for graded matrices:

$$(Q_1Q_2)^T = Q_2^TQ_1^T \qquad \text{(II.2.120a)}$$

$$(Q_1Q_2)^\dagger = Q_2^\dagger Q_1^\dagger \qquad \text{(II.2.120b)}$$

$$\text{Str}(Q_1Q_2) = \text{Str}(Q_2Q_1) \qquad \text{(II.2.120c)}$$

$$\text{Sdet}(Q_1Q_2) = (\text{Sdet}Q_1)(\text{Sdet}Q_2) \qquad \text{(II.2.120d)}$$

$$\text{Sdet}(\exp Q) = \exp(\text{Str } Q) \qquad \text{(II.2.120e)}$$

The proof is a straightforward exercise and it is left to the reader: we just point out that in checking Eq. (II.2.120b) one must utilize Eq. (II.2.85) to evaluate the complex conjugate of the product of two entries. In terms of graded matrices the supergroups Osp(m/N) and SU(m/N) have a simple interpretation.

Let

$$\hat{\Omega} = \left(\begin{array}{c|c} \Omega_{(m)} & 0 \\ \hline 0 & \Omega_{(N)} \end{array}\right) \qquad \text{(II.2.121a)}$$

$$\hat{H} = \left(\begin{array}{c|c} H_{(m)} & 0 \\ \hline 0 & H_{(N)} \end{array}\right) \qquad \text{(II.2.121b)}$$

be two graded matrices of even-type whose diagonal blocks are defined in Eqs. (II.2.55) and (II.2.61) respectively.

$\hat{\Omega}$ is called an orthosymplectic metric while $\hat{H}$ is named a superhermitean one. The reason is that they can be utilized to define two quadratic forms on a graded vector space V(m/N) of which the first is the generalization of a symplectic plus an orthogonal quadratic form while the second is the generalization of two hermitean forms.

Given any two elements ν, $w \in V(n/m)$ we set

$$\hat{\Omega}(\nu,w) = \nu^T \hat{\Omega}\, w = \nu^{\alpha} w^{\beta} \Omega_{(m)\alpha\beta} + \nu^a w^b \Omega_{(N)ab} \qquad \text{(II.2.122a)}$$

$$\hat{H}(\nu,w) = \nu^{\dagger} \hat{H}\, w = (\nu^{\alpha})^* w^{\beta} H_{(m)\alpha\beta} + (\nu^a)^* w^b H_{(N)ab} \qquad \text{(II.2.122b)}$$

and we can define the complex orthosymplectic group Osp(m/N;C) as the group of graded matrices $\mathcal{O}$ which preserve the orthosymplectic quadratic form Ω:

$$\hat{\Omega}(\mathcal{O}\nu,\mathcal{O}w) = \hat{\Omega}(\nu,w) \qquad \text{(II.2.123)}$$

Equation (II.2.123) implies

$$\mathcal{O}^T \hat{\Omega}\, \mathcal{O} = \hat{\Omega} \qquad \text{(II.2.124)}$$

which can be assumed as the defining property of the orthosymplectic graded matrices. Setting

$$\mathcal{O} = \exp(\Lambda) \qquad \text{(II.2.125)}$$

and considering Λ infinitesimal we see that (II.2.124) is equivalent to

$$\hat{\Omega}\, \Lambda\, \hat{\Omega}^{-1} = -\Lambda^T \qquad \text{(II.2.126)}$$

This can be taken as the defining property of the complex orthosymplectic algebra Osp(m/N;C). Condition (II.2.126), once written in explicit block form, coincides with Eqs. (II.2.54).

Similarly we can define the superunitary group SU(m/N) as the group of graded matrices which, besides having superdeterminant equal to 1

$$\mathrm{s\,det}\, \mathcal{U} = 1 \qquad \text{(II.2.127)}$$

have the property of preserving the superhermitean quadratic form H:

$$\hat{H}(\mathcal{U}\nu,\mathcal{U}w) = \hat{H}(\nu,w) \qquad \text{(II.2.128)}$$

Equation (II.2.128) implies

$$\mathcal{U}^{\dagger} \hat{H}\, \mathcal{U} = \hat{H} \qquad \text{(II.2.129)}$$

and setting

$$\mathcal{U} = \exp(\Sigma) \qquad \text{(II.2.130)}$$

at the infinitesimal level we get:

$$\hat{H}\, \Sigma\, \hat{H}^{-1} = -\Sigma^{\dagger} \qquad \text{(II.2.131a)}$$

$$\mathrm{Str}\, \Sigma = 0 \qquad \text{(II.2.131b)}$$

the last equation following from Eq. (II.2.127).

These conditions are the same as Eqs. (II.2.58) and (II.2.60). The real form of the complex orthosymplectic group Osp(4/N;C) which is relevant to the construction of supergravity theories, henceforth called Osp(4/N), is the intersection of Osp(4/N;C) with SU(2,2/N):

$$\mathrm{Osp}(4/N) = \mathrm{Osp}(4/N;C) \cap \mathrm{SU}(2,2/N) \tag{II.2.132}$$

In other words an element Λ of the Osp(4/N) algebra satisfies the following two conditions

$$\hat{\Omega}\Lambda\hat{\Omega}^{-1} = -\Lambda^T \tag{II.2.133a}$$

$$\hat{H}\Lambda\hat{H}^{-1} = -\Lambda^{\dagger} \tag{II.2.133b}$$

where $\hat{H}_{(4)}$ has two positive and two negative eigenvalues.

It is to a more detailed study of this supergroup that the last section of this chapter is devoted.

II.2.6 - Osp(4/N) as the N-extended supersymmetry algebra in anti de Sitter space

As we anticipated the simple superalgebra Osp(4/N) plays a special role in supergravity because it is the generalization to the case of an anti de Sitter space of the N-extended super Poincaré algebra (II.2.44). To obtain its explicit form we consider equations (II.2.133) and we make the following choice for the matrices $\hat{\Omega}$ and $\hat{H}$:

$$\hat{\Omega} = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & \mathbb{1}_{(N)} \end{array}\right) \quad ; \quad \hat{H} = \left(\begin{array}{c|c} \gamma_0 & 0 \\ \hline 0 & -\mathbb{1}_{(N)} \end{array}\right) \tag{II.2.134}$$

where C is the charge-conjugation matrix defined by Eq. (II.2.38), γ_0 is the gamma matrix in the time-direction and $\mathbb{1}_{(N)}$ is the unit-matrix in N-dimensions.

The most general graded matrix Λ which satisfies Eqs. (II.2.133) has the following form:

$$\Lambda = \left(\begin{array}{c|c} -\frac{1}{4}\varepsilon^{ab}\gamma_{ab} + \frac{i}{2}\varepsilon^a\gamma_a & \xi^B \\ \hline \bar{\xi}^A & \frac{1}{2}\varepsilon^{AB} \end{array}\right) \tag{II.2.135}$$

where $\varepsilon^{ab} = -\varepsilon^{ba}$ are the parameters of the Lorentz subalgebra and ε^a may be interpreted as the parameters of the anti de Sitter boosts. Indeed the 4x4 matrices

$$L = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} - \frac{i}{2}\varepsilon^a\gamma_a \tag{II.2.136}$$

generate the anti de Sitter group SO(2,3). Furthermore the antisymmetric parameters $\varepsilon_{AB} = -\varepsilon_{BA}$ correspond to the generators of SO(N) while the ξ_A are Majorana spinors

$$\xi_A = C\,\bar{\xi}_A^T \tag{II.2.137}$$

which play the role of supersymmetry parameters.

Writing Λ as a GA_∞-linear combination of matrices:

$$\Lambda = -\,(\varepsilon^{ab}M_{ab} + \varepsilon^{a}P_{a} + \varepsilon^{AB}T_{AB} + \bar{Q}_{A}\xi^{A}) \qquad \text{(II.2.138)}$$

we single out the definition of the generators of the superalgebra.

Calculating the commutator

$$[\Lambda_1,\Lambda_2] = \Lambda_3 \qquad \text{(II.2.139)}$$

and reexpanding the result along the generators M_{ab}, P_a, T_{AB} and Q_A we obtain the following commutation relations:

$$[M_{ab},M_{cd}] = \frac{1}{2}\,(\eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{bd}M_{ac} - \eta_{ac}M_{bd}) \qquad \text{(II.2.140a)}$$

$$[P_a,P_b] = -\,2M_{ab} \qquad \text{(II.2.140b)}$$

$$[M_{ab},P_c] = -\,\frac{1}{2}\,(\eta_{ac}P_b - \eta_{bc}P_a) \qquad \text{(II.2.140c)}$$

$$[M_{ab},\bar{Q}_{A\beta}] = \frac{1}{4}\,\bar{Q}_{A\alpha}(\gamma_{ab})_{\alpha\beta} \qquad \text{(II.2.140d)}$$

$$[P_a,\bar{Q}_{A\beta}] = -\,\frac{i}{2}\,\bar{Q}_{A\beta}(\gamma_a)_{\alpha\beta} \qquad \text{(II.2.140e)}$$

$$[T_{AB},\bar{Q}_{C\alpha}] = \frac{1}{4}\,(\delta_{CA}\bar{Q}_{B\alpha} - \delta_{CB}\bar{Q}_{A\alpha}) \qquad \text{(II.2.140f)}$$

$$\begin{aligned}\{Q_{A\alpha},Q_{B\beta}\} = {} & i(C\gamma_a)_{\alpha\beta}\delta_{AB}P_a - (C\gamma_{ab})_{\alpha\beta}\delta_{AB}M_{ab} \\ & -\,4\;C_{\alpha\beta}T_{AB}\end{aligned} \qquad \text{(II.2.140g)}$$

$$[T_{AB},T_{CD}] = \frac{1}{4}\,(\delta_{BC}T_{AD} + \delta_{AD}T_{BC} - \delta_{BD}T_{AC} - \delta_{AC}T_{BD}) \qquad \text{(II.2.140h)}$$

In Eqs. (II.2.140 a-b-c) we recognize the anti de Sitter algebra SO(2,3) of Eqs. (I.3.173): Eqs. (II.2.140 d-e-f) tell us that the supersymmetry generators $Q_{A\alpha}$ transform under SO(2,3) as a 4-dimensional spinor and under SO(N) as a vector. Finally Eq. (II.2.140g) shows that the Q.s are square roots of all the bosonic generators: translations, Lorentz rotations and SO(N) rotations.

The relation between the Osp(4/N) algebra (II.2.140) and the super Poincaré algebra (II.2.19), (II.2.43), (II.2.44), (II.2.45) can be obtained through the following rescaling procedure which amounts to an Inonü-Wigner contraction. Let us redefine our generators in the following way:

$$(M_{ab})^{\text{old}} = (M_{ab})^{\text{new}} \qquad \text{(II.2.141a)}$$

$$(T_{AB})^{\text{old}} = \frac{1}{2\bar{e}}\,(T_{AB})^{\text{new}} \qquad \text{(II.2.141b)}$$

$$(P_a)^{\text{old}} = \frac{1}{2\bar{e}}\,(P_a)^{\text{new}} \qquad \text{(II.2.141c)}$$

$$(\bar{Q}_{A\alpha})^{\text{old}} = \frac{1}{\sqrt{2\bar{e}}} (\bar{Q}_{A\alpha})^{\text{new}} \qquad \text{(II.2.141d)}$$

where $\bar{e}$ is a dimensionful parameter which we shall interpret as the inverse radius of the anti de Sitter space: if $\bar{e}\neq 0$, then equations (II.2.141) are simply a change of basis.

In the new basis the commutation relations (II.2.140) become:

$$[M_{ab},M_{cd}] = \frac{1}{2} (\eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{bd}M_{ac} - \eta_{bc}M_{bd}) \qquad \text{(II.2.142a)}$$

$$[P_a,P_b] = -8\,\bar{e}^2 M_{ab} \qquad \text{(II.2.142b)}$$

$$[M_{ab},P_c] = -\frac{1}{2} (\eta_{ac}P_b - \eta_{bc}P_a) \qquad \text{(II.2.142c)}$$

$$[M_{ab},\bar{Q}_{A\beta}] = \frac{1}{4} \bar{Q}_{A\alpha}(\gamma_{ab})_{\alpha\beta} \qquad \text{(II.2.142d)}$$

$$[P_a,\bar{Q}_{A\beta}] = -i\,\bar{e}\,\bar{Q}_{A\alpha}(\gamma_a)_{\alpha\beta} \qquad \text{(II.2.142e)}$$

$$[T_{AB},\bar{Q}_{C\alpha}] = \frac{1}{2}\,\bar{e}\,(\delta_{CA}\bar{Q}_{B\alpha} - \delta_{CB}\bar{Q}_{A\alpha}) \qquad \text{(II.2.142f)}$$

$$\{\bar{Q}_{A\alpha},Q_{B\beta}\} = i(C\gamma^a)_{\alpha\beta}\delta_{AB}P_a - 2\bar{e}(C\gamma_{ab})_{\alpha\beta}\delta_{AB}M^{ab} - 4\,C_{\alpha\beta}T_{AB} \qquad \text{(II.2.142g)}$$

$$[T_{AB},T_{CD}] = \frac{\bar{e}}{2} (\delta_{BC}T_{AD} + \delta_{AD}T_{BC} - \delta_{BD}T_{AC} - \delta_{AC}T_{BD}) \qquad \text{(II.2.142h)}$$

and are equivalent to the old ones. The limit $\bar{e}\to 0$, however is singular and it gives rise to a new non semisimple algebra. We easily see what happens. The translations P_a become abelian and commute with the spinorial charges $Q_{A\alpha}$; similarly the SO(N) generators become abelian central charges (they commute with everything else) and the Lorentz generator in the anticommutator of two Q.s drops out. The result is exactly the N-extended super Poincaré algebra with the identification:

$$Z_+^{AB} = -4 \lim_{\bar{e}\to 0} T_{AB} \qquad \text{(II.2.143a)}$$

$$Z_-^{AB} = 0 \qquad \text{(II.2.143b)}$$

In this way we obtain an algebraic justification for discarding the pseudoscalar central charges Z_-^{AB}. Since the N-extended super Poincaré group with scalar central charges (Z_+^{AB}) is the Inonü-Wigner contraction of Osp(4/N) we refer to it as the $\overline{\text{Osp}(4/N)}$ group (or algebra). For every $1 \le N \le 8$ we shall have two theories of supergravity depending on whether a certain parameter is zero or non zero. In the first case the vacuum state is Minkowski space and the theory can be regarded as the gauging of the $\overline{\text{Osp}(4/N)}$ algebra (Poincaré supergravity) in the second case the vacuum state is anti de Sitter space and the theory can be viewed as the gauging of the non contracted Osp(4/N) algebra (de Sitter supergravity). In the Poincaré case the N(N-1)/2 vectors associated to the charges Z^{AB} are abelian while in the anti de Sitter case they gauge the SO(N) subgroup. The most intriguing feature of the extended theories (N>1) is the rigid relation between the cosmological constant and the SO(N) gauge coupling constant:

$$g_{SO(N)} = \bar{e} \quad ; \quad \Lambda = -4\,\bar{e}^2 \qquad \text{(II.2.144)}$$

which is a direct consequence of the Osp(4/N) algebra (II.2.142).

Eq. (II.2.144) implies that a realistic coupling constant $g_{SO(N)} \sim 1$ yields a non-realistic curvature radius of the Universe ($R=10^{-33}$cm) and vice-versa: hence it is very embarassing. Unfortunately it seems also very central to the whole business since it reappears both in Kaluza-Klein theory where one tries to obtain the gauge group from the extra dimensions and in the partial breaking of supersymmetry in 4-dimensions where one tries to obtain an effective N=1 Lagrangian from a higher N Lagrangian. The only theory which is free from this disease is N=1 supergravity based on either Osp(4/1) or $\overline{\text{Osp}(4/1)}$.

Moreover this theory is the only one which admits chiral fermions. For these two facts it is the natural candidate for a description of particle phenomenology at low-energies and as such it has many advantages (see Part Four): it is unsatisfactory, however, because it provides a too low degree of unification (it has too many freedoms) and because it is not finite nor renormalizable. The central problem in the programme of superunification is therefore "how to obtain N=1 supergravity as a low-energy approximation of a more constrained theory". In this effort, higher supersymmetries, higher dimensions and recently strings have all come into play. The patient reader of this book will discover how and will see the

advantages and disadvantages of each structure. He will realize that at every stage a universal threat is waiting in ambush: anti de Sitter space.

Such universality means only two things: either there is a unique way out which selects Minkowski space and that is the truth precisely because it is unique (this may be the compactification of the heterotic string) or we are biased by a pseudo problem, anti de Sitter space being acceptable if suitably reinterpreted (see Part Five). In both cases, as a trap to avoid or as a path to follow the properties of supersymmetry in anti de Sitter space are an essential ingredient for the student of supergravity. More of it is going to come in the next chapters.