Equivariant K-theory, groupoids and proper actions

by

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B.Sc., Universidad de Malaga, 2004

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in

The Faculty of Graduate Studies

(Mathematics)

The University Of British Columbia

(Vancouver)

April 2009

Abstract

Equivariant K-theory for actions of groupoids is defined and shown to be a cohomology theory on the category of finite equivariant CW-complexes. Under some conditions, these theories are representable. We use this fact to define twisted equivariant K-theory for actions of groupoids. A classification of possible twistings is given. We also prove a completion theorem for twisted and untwisted equivariant K-theory. Finally, some applications to proper actions of Lie groups are discussed.

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Acknowledgements

I would like to express my utmost gratitude to my supervisor Alejandro Adem in the first place, for he has guided me through my Ph.D. program patiently, introduced me into the research world and given me many opportunities to advance professionally and personally. I owe special thanks to Antonio Viruel for helping me with the choice of school and advisor for my graduate studies. I am very grateful to some of my colleagues who were specially helpful with my research at some point, namely, Bernardo Uribe, Nitya Kitchloo, Jose Manuel Gomez and Ali Duman. Thanks to the members of my supervisory committee, Denis Sjerve and Dale Rolfsen, and the rest of the examining committee for their time.

On a personal level, I would like to thank my family and friends for their support and encouragement. My girlfriend Dacil Garcia deserves my special appreciation for her love and positive influence, which have been fundamental during this particularly stressful time. Finally, I would like to thank everybody who was important to the successful realization of this thesis and the completion of my Ph.D. program, and apologize that I could not mention personally all of them.

Chapter 1

Introduction

Symmetries have always played a very important role in mathematics. In algebraic topology, these are usually realized by actions of groups on topological spaces. A groupoid is a generalization of a group with the difference that multiplication is not globally defined. Actions of groupoids give rise to more general symmetries. Groupoids can also be seen as a generalizations of topological spaces. A particular kind of groupoids, orbifolds, have been extensively studied lately in algebraic topology, algebraic geometry and physics. When studying orbifolds, it is convenient to consider two orbifolds to be the same not when they are isomorphic, but when they satisfy a weaker condition, called Morita equivalence. In fact, a similar concept, that of weak equivalence of groupoids, helps us identify when two actions are equivalent in a sense.

The recent theorem of Freed, Hopkins and Teleman [17, 18, 19] relates the complex equivariant twisted K-theory of a simply-connected compact Lie group acting on itself by conjugation to the Verlinde algebra. This result links information about the conjugation action with the action of the loop group on its universal space for proper actions. If we use the language of groupoids, the two associated groupoids are Morita equivalent. The invariance of orbifold K-theory under Morita equivalence [1] also seems to suggest that the language of groupoids is an appropriate framework to work with proper actions. We introduce all the necessary background on groupoids in chapter 2, as well as new constructions that will allow us to construct equivariant K-theory for groupoid actions.

The complex representation ring of a compact Lie group G can be identified with the G-equivariant complex K-theory of a point. Equivariant complex K-theory is defined via equivariant complex bundles, but this procedure does not give a cohomology theory for proper actions of non-compact Lie groups in general, as shown in [35]. Phillips constructed an equivariant cohomology theory for any second countable locally compact group G on the category of proper locally compact G-spaces. This is done using infinite-dimensional complex G-Hilbert bundles. Sometimes it is enough to use finite-dimensional vector bundles, for example in the case of discrete groups [28]. In this paper we will construct complex equivariant K-theory for actions of a Lie groupoid by using extendable complex equivariant bundles, defined in section 2.7. These bundles are finite-dimensional, but are required to satisfy an additional condition which will make sure that we have a Mayer-Vietoris sequence.

The Grothendieck construction then gives a cohomology theory on the category of $\mathcal{G}$-spaces. For any $\mathcal{G}$-space X, $K^*_{\mathcal{G}}(X)$ is a module over $K^*_{\mathcal{G}}(G_0)$ and the latter can be identified with $K^*_{orb}(\mathcal{G})$ when $\mathcal{G}$ is an orbifold. But this theory does not satisfy Bott periodicity in general. In fact, it may fail to agree with classical equivariant K-theory when the action on the space is equivalent to the action of a compact Lie group. In order to solve this problem we introduce $\mathcal{G}$-cells, which are $\mathcal{G}$-spaces whose $\mathcal{G}$-action is equivalent to the action of a compact Lie group on a finite complex. The condition of Bredon-compatibility makes sure that $\mathcal{G}$-equivariant K-theory agrees with classical equivariant K-theory on the $\mathcal{G}$-cells. This condition also implies Bott periodicity for finite $\mathcal{G}$-CW-pairs. The following theorem is proved in chapter 3:

Theorem 1.0.1. *If $\mathcal{G}$ is a Bredon-compatible Lie groupoid, the groups $K^n_{\mathcal{G}}(X, A)$ define a $\mathbb{Z}/2$-graded multiplicative cohomology theory on the category of finite $\mathcal{G}$-CW-pairs.*

Atiyah and Segal twist equivariant K-theory for actions of a compact Lie group G using G-stable projective bundles [6]. Since stable projective bundles and sections behave well under weak equivalences, it seems natural to use $\mathcal{G}$-stable projective bundles, which can be defined in an similar way, to twist $\mathcal{G}$-equivariant K-theory.

G-equivariant K-theory can be represented by a space of Fredholm operators on a G-stable Hilbert space. This is used to construct twisted K-theory in [6]. For a G-stable projective bundle, we can consider a suitable bundle of Fredholm operators associated to it and define twisted K-theory with the sections of this bundle [6]. For actions of groupoids, we have a representability theorem if we consider not all maps into such a space of Fredholm operators, but only those which are extendable. The definition of extendable sections and the space $Fred'(H)$ can be found in section 3.4.

Theorem 1.0.2. *Let H be a stable representation of a Bredon-compatible finite Lie groupoid $\mathcal{G}$. Then:*

$$K_{\mathcal{G}}(X) = [X, Fred'(H)]^{ext}_{\mathcal{G}}$$

Choosing all sections of the Fredholm bundle corresponds to choosing all vector bundles in the untwisted case. To make these new theories an extension of untwisted K-theory, we need to consider extendable sections. Then we can define twisted $\mathcal{G}$-equivariant K-theory as the group of extendable homotopy classes of extendable sections of a suitable Fredholm bundle, that is, homotopy classes where the homotopies run over extendable sections. Extending it to all degrees as in [6], we obtain a cohomology theory:

Theorem 1.0.3. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, the groups ${}^{P}K^{n}_{\mathcal{G}}(X)$ define a $\mathbb{Z}/2$-graded cohomology theory on the category of finite $\mathcal{G}$-CW-complexes with $\mathcal{G}$-stable projective bundles, which is a module over untwisted $\mathcal{G}$-equivariant K-theory.*

The category of $\mathcal{G}$-orbits behaves similarly to the corresponding category for a compact Lie group. In particular, we are able to use some of these properties to prove an analogue of Elmendorf's construction [15]. This construction is the key to the classification of $\mathcal{G}$-stable projective bundles. In fact, we show that isomorphism classes of $\mathcal{G}$-stable projective bundles over X

are classified by $H^3_{\mathcal{G}}(X)$. This is done by constructing a particular model for the space that represents $H^3_{\mathcal{G}}(-)$ which admits a natural $\mathcal{G}$-stable projective bundle on it. Chapter 4 contains all the definitions and results concerning twisted K-theory.

In chapter 5, we prove corresponding completion theorems for twisted and untwisted K-theory. We introduce a universal $\mathcal{G}$ space $E\mathcal{G}$ as the limit of a sequence of free $\mathcal{G}$-spaces $E^n\mathcal{G}$ as in the case of compact Lie groups. The quotient of $E\mathcal{G}$ by the $\mathcal{G}$-action is $B\mathcal{G}$, the classifying space of $\mathcal{G}$. We can then form the fibered product $X \times_\pi E^n\mathcal{G}$ over G_0 and prove a generalization of the completion theorem of Atiyah and Segal [5] when $\mathcal{G}$ is finite, that is, when G_0 is a finite $\mathcal{G}$-CW-complex:

Theorem 1.0.4. *Let $\mathcal{G}$ be a Bredon-compatible, finite Lie groupoid and X a finite $\mathcal{G}$-CW-complex. Then we have an isomorphism of pro-rings*

$$\{K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}K^*_{\mathcal{G}}(X)\} \longrightarrow \{K^*(X \times_\pi E^n\mathcal{G}/\mathcal{G})\}$$

The recent completion theorem for twisted equivariant K-theory for actions of compact Lie groups in [26] provides the necessary results to use induction over cells. In the twisted case, however, the completion theorem will relate the completion of twisted $\mathcal{G}$-equivariant K-theory of X with respect to $I_{\mathcal{G}}$ to the twisted $\mathcal{G}$-equivariant K-theory of $X \times_\pi E\mathcal{G}$.

Theorem 1.0.5. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid, X a finite $\mathcal{G}$-CW-complex and P a $\mathcal{G}$-stable projective bundle on X. Then we have an isomorphism of $K^*_{\mathcal{G}}(G_0)$-modules:*

$${}^{P}K^n_{\mathcal{G}}(X)^{\wedge}_{I_{\mathcal{G}}} \longrightarrow {}^{P\times_\pi E\mathcal{G}}K^n_{\mathcal{G}}(X \times_\pi E\mathcal{G})$$

Some applications are discussed in chapter 6. When S is a Lie group, not necessarily compact, we can define twisted equivariant K-theory for proper actions of S using the groupoid $S \rtimes \underline{E}S$, where $\underline{E}S$ is the universal space for proper actions of S. These groupoids provide particular instances where these theorems apply and can be used to study the proper actions of these particular groups. The case of compact Lie groups and finite groups were studied by Atiyah and Segal [5]. Discrete groups are dealt with in [28]. Almost compact groups and matrix groups are studied in [34]. Proper actions of pro-discrete groups are shown to be Bredon-compatible in [38]. The results here provide a way to define twisted K-theory for such actions as well as completion theorems.

Chapter 2

Groupoids

2.1 Basic facts

In this section we review some basic facts about groupoids. All this material can be found in [1] and [32].

Definition 2.1.1. A topological groupoid $\mathcal{G}$ consists of a space G_0 of objects and a space G_1 of arrows, together with five continuous structure maps, listed below.

- The source map $s : G_1 \to G_0$ assigns to each arrow $g \in G_1$ its source $s(g)$.
- The target map $t : G_1 \to G_0$ assigns to each arrow $g \in G_1$ its target $t(g)$. For two objects x, $y \in G_0$, one writes $g : x \to y$ to indicate that $g \in G_1$ is an arrow with $s(g) = x$ and $t(g) = y$.
- If g and h are arrows with $s(h) = t(g)$, one can form their composition hg, with $s(hg) = s(g)$ and $t(hg) = t(h)$. The composition map $m : G_1 \times_{s,t} G_1 \to G_1$, defined by $m(h,g) = hg$, is thus defined on the fibered product

$$G_1 \times_{s,t} G_1 = \{(h,g) \in G_1 \times G_1 \mid s(h) = t(g)\}$$

  and is required to be associative.
- The unit map $u : G_0 \to G_1$ which is a two-sided unit for the composition. This means that $su(x) = x = tu(x)$, and that $gu(x) = g = u(y)g$ for all x, $y \in G_0$ and $g : x \to y$.

- An inverse map $i : G_1 \to G_1$, written $i(g) = g^{-1}$. Here, if $g : x \to y$, then $g^{-1} : y \to x$ is a two-sided inverse for the composition, which means that $g^{-1}g = u(x)$ and $gg^{-1} = u(y)$.

Definition 2.1.2. A Lie groupoid is a topological groupoid $\mathcal{G}$ for which G_0 and G_1 are smooth manifolds, and such that the structure maps are smooth. Furthermore, s and t are required to be submersions so that the domain $G_1 \times_{s,t} G_1$ of m is a smooth manifold.

Example 2.1.3. Suppose a Lie group K acts smoothly on a manifold M. One defines a Lie groupoid $K \rtimes M$ by $(K \rtimes M)_0 = M$ and $(K \rtimes M)_1 = K \times M$, with s the projection and t the action. Composition is defined from the multiplication in the group K. This groupoid is called the action groupoid.

Definition 2.1.4. Let $\mathcal{G}$ be a Lie groupoid. For a point $x \in G_0$, the set of all arrows from x to itself is a Lie group, denoted by $\mathcal{G}_x$ and called the isotropy group at x. The set $ts^{-1}(x)$ of targets of arrows out of x is called the orbit of x. The quotient $|\mathcal{G}|$ of G_0 consisting of all the orbits in $\mathcal{G}$ is called the orbit space. Conversely, we call $\mathcal{G}$ a groupoid presentation of $|\mathcal{G}|$.

Definition 2.1.5. A Lie groupoid $\mathcal{G}$ is proper if $(s,t) : G_1 \to G_0 \times G_0$ is a proper map. Note that in a proper Lie groupoid, every isotropy group is compact.

Definition 2.1.6. Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A strict homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ consists of two smooth maps $\phi : H_0 \to G_0$ and $\phi : H_1 \to G_1$ that commute with all the structure maps for the two groupoids.

Given a Lie groupoid $\mathcal{G}$, we can associate an important topological construction to it, namely its classifying space $B\mathcal{G}$. Moreover, this construction is well-behaved under Morita (weak) equivalence. For $n \geq 1$, let G_n be the iterated fibered product

$$G_n = \{(g_1, ..., g_n) \mid g_i \in G_1, s(g_i) = t(g_{i+1}), i = 1, ..., n-1\}$$

Together with the objects G_0, these G_n have the structure of a simplicial manifold called the nerve of $\mathcal{G}$. Here we are really just thinking of

$\mathcal{G}$ as a category. Following the usual convention, we define face operators $d_i : G_n \to G_{n-1}$ for $i = 0, ..., n$, given by

$$d_i(g_1, ..., g_n) = \begin{cases} (g_2, ..., g_n) & \text{if } i = 0 \\ (g_1, ..., g_{n-1}) & \text{if } i = n \\ (g_1, ..., g_i g_{i+1}, ... g_n) & \text{otherwise} \end{cases}$$

for $0 < i < n$ when $n > 1$. Similarly, we define $d_0(g) = s(g)$ and $d_1(g) = t(g)$ when $n = 1$.

For such a simplicial space, we can glue the disjoint union of the $G_n \times \Delta^n$ as follows, where Δ^n is the topological n-simplex. Let $\delta_i : \Delta^{n-1} \to \Delta^n$ be the linear embedding of Δ^{n-1} into Δ^n as the i-th face. We define the classifying space of $\mathcal{G}$ (the geometric realization of its nerve) as the identification space

$$B\mathcal{G} = \coprod_n (G_n \times \Delta^n)/(d_i(g), x) \sim (g, \delta_i(x))$$

This is usually called the fat realization of the nerve, meaning that we have chosen to leave out identifications involving degeneracies. The two definitions will produce homotopy equivalent spaces provided that the topological category has sufficiently nice properties. Another nice property of the fat realization is that if every G_n has the homotopy type of a CW-complex, then the fat realization will also have the homotopy type of a CW-complex.

Definition 2.1.7. A smooth left Haar system for a Lie groupoid $\mathcal{G}$ is a family $\{\lambda^a \mid a \in G_0\}$, where each λ^a is a positive, regular Borel measure on the manifold $t^{-1}(a)$ such that:

- If (V, ψ) is an open chart of G_1 satisfying $V \cong t(V) \times W$, and if λ_W is the Lebesgue measure on $\mathbb{R}^k$ restricted to W, then for each $a \in t(V)$, the measure $\lambda^a \circ \psi$ is equivalent to λ_W, and the map $(a, w) \longrightarrow d(\lambda^a \circ \psi_a)/d\lambda_W(w)$ belongs to $C^\infty(t(V) \times W)$ and is strictly positive.

- For any $x \in G_1$ and $f \in C_c^\infty(G_1)$, we have

$$\int_{t^{-1}(s(x))} f(xz)d\lambda^{s(x)}(z) = \int_{t^{-1}(t(x))} f(y)d\lambda^{t(x)}(y)$$

Proposition 2.1.8. *Every Lie groupoid admits a smooth left Haar system.*

Proof. The proof can be found in [33]. □

2.2 Equivalent groupoids

Definition 2.2.1. A strict homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ between Lie groupoids is called an equivalence if:

- The map

  $$t\pi_1 : G_1 \times_{s,\phi} H_0 \to G_0$$

  is a surjective submersion, where the fibered product of manifolds $G_1 \times_{s,\phi} H_0$ is defined as $\{(g, y) \mid g \in G_1, y \in H_0, s(g) = \phi(y)\}$.

- The square

$$\begin{tikzcd} H_1 \arrow[r, "\phi"] \arrow[d, "{(s,t)}"] & G_1 \arrow[d, "{(s,t)}"] \\ H_0 \times H_0 \arrow[r, "\phi \times \phi"] & G_0 \times G_0 \end{tikzcd}$$

  is a fibered product of manifolds.

The first condition implies that every object $x \in G_0$ can be connected by an arrow $g : \phi(y) \to x$ to an object in the image of ϕ, that is, ϕ is essentially surjective as a functor. The second condition implies that ϕ induces a diffeomorphism

$$H_1(y, z) \to G_1(\phi(y), \phi(z))$$

from the space of all arrows $y \to z$ in H_1 to the space of all arrows $\phi(y) \to \phi(z)$ in G_1. In particular ϕ is full and faithful as a functor.

A strict homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ induces continuous maps $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ and $B\phi : B\mathcal{H} \to B\mathcal{G}$. Moreover, if ϕ is an equivalence, $|\phi|$ is a homeomorphism and $B\phi$ is a homotopy equivalence. This follows from the fact that an equivalence induces an equivalence of categories.

Definition 2.2.2. A local equivalence $\mathcal{H} \longrightarrow \mathcal{G}$ is an equivalence with the additional property that each $g_0 \in G_0$ has a neighbourhood U admitting a lift to $\tilde{H}_0$ in the diagram

$$\begin{tikzcd}
 & \tilde{H}_0 \arrow[r] \arrow[d] & H_0 \arrow[d] \\
 & G_1 \arrow[r, "t"] \arrow[d, "s"] & G_0 \\
U \arrow[r] & G_0 &
\end{tikzcd}$$

in which the square is a pullback square.

Definition 2.2.3. Two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent if there exists a third groupoid $\mathcal{H}$ and two equivalences

$$\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'$$

Definition 2.2.4. Two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if there exists a third groupoid $\mathcal{H}$ and two local equivalences

$$\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'$$

2.3 Groupoid actions

Definition 2.3.1. Let $\mathcal{G}$ be a groupoid. A (right) $\mathcal{G}$-space is a manifold E equipped with an action by $\mathcal{G}$. Such an action is given by two maps $\pi : E \to G_0$ (called the anchor map) and $\mu : E \times_{G_0} G_1 \to E$. The latter map is defined on pairs (e, g) with $\pi(e) = t(g)$ and written $\mu(e, g) = e \cdot g$. They must satisfy $\pi(e \cdot g) = s(g)$, $e \cdot u(\pi(e)) = e$ and $(e \cdot g) \cdot h = e \cdot (gh)$.

Example 2.3.2. Let M be a $\mathcal{G}$-space. We can construct the action groupoid $\mathcal{H} = \mathcal{G} \rtimes M$ which has space of objects M and morphisms $M \times_{G_0} G_1$. This groupoid generalizes the earlier notion of action groupoid for a group action and the structure maps are formally the same as in that case.

Definition 2.3.3. Let $\mathcal{G}$ be a groupoid and let X, Y be $\mathcal{G}$-spaces. A map $f : X \longrightarrow Y$ is $\mathcal{G}$-equivariant if it commutes with the anchor maps and satisfies $f(x \cdot g) = f(x) \cdot g$ whenever one of the sides is defined.

- G_0 is a final object in the category of $\mathcal{G}$-spaces with the action given by $e \cdot g = s(g)$ and projection given by the identity.
- If X and Y are $\mathcal{G}$-spaces, the fibered product over G_0, $X \times_\pi Y = \{(x, y) \mid \pi(x) = \pi(y)\}$ becomes a $\mathcal{G}$-space with coordinate-wise action. In particular $X \times_\pi G_0 = X$.
- Similarly, if X is a $\mathcal{G}$-space and Y is any other space, $X \times Y$ is a $\mathcal{G}$-space with trivial action on the second factor. In fact, $X \times Y = X \times_\pi (Y \times G_0)$.

Let I denote the unit interval $[0, 1]$. Given a $\mathcal{G}$-space X with anchor map π_X, we give $X \times I$ the structure of a $\mathcal{G}$-space with anchor map $\pi_{X \times I}(x, \lambda) = \pi_X(x)$ and action $(x, \lambda) \cdot g = (x \cdot g, \lambda)$ when $\pi_{X \times I}(x, \lambda) = \pi_X(x) = t(g)$.

Definition 2.3.4. Let $\mathcal{G}$ be a groupoid and let $f, g : X \longrightarrow Y$ be two $\mathcal{G}$-equivariant maps between two $\mathcal{G}$-spaces X and Y. We say that f and g are $\mathcal{G}$-homotopic if there is a $\mathcal{G}$-map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We call H a $\mathcal{G}$-homotopy between f and g. This is an equivalence relation and we denote it by $f \simeq_{\mathcal{G}} g$.

Definition 2.3.5. We say two $\mathcal{G}$-spaces X and Y are $\mathcal{G}$-homotopy equivalent if there are $\mathcal{G}$- maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $fg \simeq_{\mathcal{G}} 1_Y$ and $gf \simeq_{\mathcal{G}} 1_X$.

2.4 Equivariant CW-complexes

Definition 2.4.1. An n-dimensional $\mathcal{G}$-cell is a space of the form $D^n \times U$ where U is a $\mathcal{G}$-space such that $\mathcal{G} \rtimes U$ is weakly equivalent to an action groupoid corresponding to a proper action of a compact Lie group G on a finite G-CW-complex.

Definition 2.4.2. A $\mathcal{G}$-CW-complex X is a $\mathcal{G}$-space together with an $\mathcal{G}$-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, that is, there exists a $\mathcal{G}$-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} U_i \times S^{n-1} \arrow[r, "\coprod_{i\in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} U_i \times D^n \arrow[r, "\coprod_{i\in I_n} Q_i^n"] & X_n
\end{tikzcd}$$

and X carries the colimit topology with respect to this filtration.

Definition 2.4.3. A $\mathcal{G}$-CW-complex X is finite if it is made up out of a finite number of cells. A $\mathcal{G}$-CW-complex X is finite-dimensional if there is a positive integer N such that $X = X_N$.

Definition 2.4.4. A $\mathcal{G}$-CW-pair (X, A) is a pair of $\mathcal{G}$-CW-complexes.

Definition 2.4.5. We say a groupoid $\mathcal{G}$ is finite if G_0 is a finite $\mathcal{G}$-CW-complex.

2.5 Equivariant cohomology theories

Definition 2.5.1. Let S be a Lie group and R a commutative ring. A proper S-cohomology theory H_S^* with values in R-modules is a collection of covariant functors H_S^n from the category of proper S-CW-pairs to the category of R-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\delta_S^n(X,A) : H_S^n(A) \longrightarrow H_S^{n+1}(X,A)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- S-homotopy equivariance. If f_0, $f_1 : (X,A) \to (Y,B)$ are S-homotopic maps of proper S-CW-pairs, then $H_S^n(f_0) = H_S^n(f_1)$ for $n \in \mathbb{Z}$.

- Long exact sequence of a pair. Given a pair (X,A) of proper S-CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{\delta_S^{n-1}} H_S^n(X,A) \xrightarrow{H_S^n(j)} H_S^n(X) \xrightarrow{H_S^n(i)} H_S^n(A) \xrightarrow{\delta_S^n} \dots$$

  where $i : A \to X$ and $j : X \to (X,A)$ are the inclusions.

- Excision. Let (X,A) be a proper S-CW-pair and let $f : A \to B$ be an S-map. Equip $(X \cup_f B, B)$ with the induced structure of a S-CW-pair. Then, the canonical map $(F,f) : (X,A) \to (X \cup_f B, B)$ induces an isomorphism

$$H_S^n(F,f) : H_S^n(X,A) \xrightarrow{\cong} H_S^n(X \cup_f B, B)$$

- Disjoint union axiom. Let $\{X_i \mid i \in I\}$ be a family of proper S-CW-complexes. Denote by $j_i : X_i \longrightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\prod_{i \in I} H_S^n(j_i) : H_S^n\Big(\coprod_{i \in I} X_i\Big) \xrightarrow{\cong} \prod_{i \in I} H_S^n(X_i)$$

  is an isomorphism.

Definition 2.5.2. Fix a groupoid $\mathcal{G}$ and a commutative ring R. A $\mathcal{G}$-cohomology theory $H_{\mathcal{G}}^*$ with values in R-modules is a collection of covariant functors $H^n_{\mathcal{G}}$ from the category of $\mathcal{G}$-CW-pairs to the category of R-modules indexed by integers $n \in \mathbb{Z}$ together with natural transformations $\delta^n_{\mathcal{G}}(X, A) : H^n_{\mathcal{G}}(A) \longrightarrow H^{n+1}_{\mathcal{G}}(X, A)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- $\mathcal{G}$-homotopy equivariance. If f_0, $f_1 : (X, A) \to (Y, B)$ are $\mathcal{G}$-homotopic maps of $\mathcal{G}$-CW-pairs, then $H^n_{\mathcal{G}}(f_0) = H^n_{\mathcal{G}}(f_1)$ for $n \in \mathbb{Z}$.

- Long exact sequence of a pair. Given a pair (X, A) of $\mathcal{G}$-CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{\delta^{n-1}_{\mathcal{G}}} H^n_{\mathcal{G}}(X, A) \xrightarrow{H^n_{\mathcal{G}}(j)} H^n_{\mathcal{G}}(X) \xrightarrow{H^n_{\mathcal{G}}(i)} H^n_{\mathcal{G}}(A) \xrightarrow{\delta^n_{\mathcal{G}}} \dots$$

  where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions.

- Excision. Let (X, A) be a $\mathcal{G}$-CW-pair and let $f : A \to B$ be a $\mathcal{G}$-map. Equip $(X \cup_f B, B)$ with the induced structure of a $\mathcal{G}$-CW-pair. Then, the canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$H^n_{\mathcal{G}}(F, f) : H^n_{\mathcal{G}}(X, A) \xrightarrow{\cong} H^n_{\mathcal{G}}(X \cup_f B, B)$$

- Disjoint union axiom. Let $\{X_i \mid i \in I\}$ be a family of $\mathcal{G}$-CW-complexes. Denote by $j_i : X_i \longrightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\prod_{i \in I} H^n_{\mathcal{G}}(j_i) : H^n_{\mathcal{G}}\Big(\coprod_{i \in I} X_i\Big) \xrightarrow{\cong} \prod_{i \in I} H^n_{\mathcal{G}}(X_i)$$

  is an isomorphism.

Let C_S be the category whose objects are pairs (X, P) where X is a proper S-CW-complex and P is an S-projective bundle on X. A morphism

$(X, P) \to (Y, Q)$ is a diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d] & Q \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where F is a map of S-projective bundles and f is an S-map.

Definition 2.5.3. Let S be a Lie group and R a commutative ring. A proper S-cohomology theory H^*_S on the category of proper S-spaces with S-projective bundles with values in R-modules is a collection of covariant functors H^n_S, $n \in \mathbb{Z}$, from the category C_S to the category of R-modules that take (X, P) to ${}^P H^n_S(X)$ together with boundary homomorphisms $d^n : {}^{P|_A} H^n_S(A) \longrightarrow {}^P H^{n+1}_S(X)$ for $n \in \mathbb{Z}$ for any pushout diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

where $P_A = (i_2)^*(P_2)$ and such that the following axioms are satisfied:

- S-homotopy equivariance. If f_0 and f_1 are S-homotopic maps $X \to Y$ of S-CW-complexes and P is a S-projective bundle on Y, then we have a commutative diagram for all $n \in \mathbb{Z}$.

$$\begin{tikzcd} {}^P H^n_S(Y) \arrow[r, "{}^P H^n_S(f_0)"] \arrow[dr, "{}^P H^n_S(f_1)"'] & {}^{f_0^*(P)} H^n_S(X) \arrow[d, "\cong"] \\ & {}^{f_1^*(P)} H^n_S(X) \end{tikzcd}$$

- Mayer-Vietoris sequence. For any pushout square of proper S-CW-complexes

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

  and any S-projective bundle P on X, let $P_k = j_k^*(P)$ for $k = 1, 2$ and $P_A = (i_2)^*(P_2)$. Then there is a natural exact sequence

$$\dots \xrightarrow{d^{-n-1}} {}^{P}H_S^{-n}(X) \xrightarrow{j_1^* \oplus j_2^*} {}^{P_1}H_S^{-n}(X_1) \oplus {}^{P_2}H_S^{-n}(X_2) \xrightarrow{i_1^* - i_2^*} {}^{P_A}H_S^{-n}(A) \xrightarrow{d^{-n}} \dots$$

  where $j_k^* = {}^{P}H_S^{-n}(j_k)$ and $i_k^* = {}^{P_k}H_S^{-n}(i_k)$ for $k = 1, 2$.

- Disjoint union axiom. Let $\{X_i \mid i \in I\}$ be a family of proper S-CW-complexes and P_i an S-projective bundle on X_i. Denote by $j_i : X_i \longrightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\prod_{i \in I} {}^{P_i}H_S^n(j_i) : {}^{\amalg_{i\in I} P_i}H_S^n\Big(\coprod_{i \in I} X_i\Big) \xrightarrow{\cong} \prod_{i \in I} {}^{P_i}H_S^n(X_i)$$

  is an isomorphism.

Let $C_{\mathcal{G}}$ be the category whose objects are pairs (X, P) where X is a $\mathcal{G}$-CW-complex and P is a $\mathcal{G}$-projective bundle on X. A morphism $(X, P) \to (Y, Q)$ is a diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d] & Q \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where F is a map of $\mathcal{G}$-projective bundles and f is a $\mathcal{G}$-map.

Definition 2.5.4. Fix a groupoid $\mathcal{G}$ and a commutative ring R. A $\mathcal{G}$-cohomology theory $H^*_{\mathcal{G}}$ on the category of $\mathcal{G}$-spaces with $\mathcal{G}$-projective bundles with values in R-modules is a collection of covariant functors $H^n_{\mathcal{G}}$, $n \in \mathbb{Z}$, from the category $C_{\mathcal{G}}$ to the category of R-modules that take (X, P) to ${}^P H^n_{\mathcal{G}}(X)$ indexed by $n \in \mathbb{Z}$ together with boundary homomorphisms $d^n : {}^{P|_A} H^n_{\mathcal{G}}(A) \longrightarrow {}^P H^{n+1}_{\mathcal{G}}(X)$ for $n \in \mathbb{Z}$ for any pushout diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

where $P_A = (i_2)^*(P_2)$ and such that the following axioms are satisfied:

- $\mathcal{G}$-homotopy equivariance. If f_0 and f_1 are $\mathcal{G}$-homotopic maps $X \to Y$ of $\mathcal{G}$-CW-complexes and P is a $\mathcal{G}$-projective bundle on Y, then we have a commutative diagram for all $n \in \mathbb{Z}$.

$$\begin{tikzcd} {}^P H^n_{\mathcal{G}}(Y) \arrow[r, "{}^P H^n_{\mathcal{G}}(f_0)"] \arrow[dr, "{}^P H^n_{\mathcal{G}}(f_1)"'] & {}^{f_0^*(P)} H^n_{\mathcal{G}}(X) \arrow[d, "\cong"] \\ & {}^{f_1^*(P)} H^n_{\mathcal{G}}(X) \end{tikzcd}$$

- Mayer-Vietoris sequence. For any pushout square of $\mathcal{G}$-CW-complexes

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

and any $\mathcal{G}$-projective bundle P on X, let $P_k = j_k^*(P)$ for $k = 1, 2$ and

$P_A = (i_2)^*(P_2)$. Then there is a natural exact sequence

$$\dots \xrightarrow{d^{-n-1}} {}^{P}H_{\mathcal{G}}^{-n}(X) \xrightarrow{j_1^*\oplus j_2^*} {}^{P_1}H_{\mathcal{G}}^{-n}(X_1)\oplus {}^{P_2}H_{\mathcal{G}}^{-n}(X_2) \xrightarrow{i_1^*-i_2^*} {}^{P_A}H_{\mathcal{G}}^{-n}(A) \xrightarrow{d^{-n}} \dots$$

where $j_k^* = {}^{P}H_{\mathcal{G}}^{-n}(j_k)$ and $i_k^* = {}^{P_k}H_{\mathcal{G}}^{-n}(i_k)$ for $k = 1, 2$.

- Disjoint union axiom. Let $\{X_i \mid i \in I\}$ be a family of $\mathcal{G}$-CW-complexes and P_i a $\mathcal{G}$-projective bundle on X_i. Denote by $j_i : X_i \longrightarrow \coprod_{i\in I} X_i$ the canonical inclusion. Then the map

$$\prod_{i\in I} {}^{P_i}H_{\mathcal{G}}^n(j_i) : {}^{\amalg_{i\in I} P_i}H_{\mathcal{G}}^n\Big(\coprod_{i\in I} X_i\Big) \xrightarrow{\cong} \prod_{i\in I} {}^{P_i}H_{\mathcal{G}}^n(X_i)$$

is an isomorphism.

2.6 Fiber bundles

Definition 2.6.1. Let X be a $\mathcal{G}$-space. A $\mathcal{G}$-fiber bundle on X is a fiber bundle $\pi : P \to X$ for which P is a $\mathcal{G}$-space and π is a $\mathcal{G}$-map.

Proposition 2.6.2. *Suppose that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a local equivalence. Then the pullback functor*

$$F^* : \{\mathcal{H} - \textit{Fiber bundles on } H_0\} \longrightarrow \{\mathcal{G} - \textit{Fiber bundles on } G_0\}$$

is an equivalence of categories

Proof. Suppose that P is a $\mathcal{G}$-fiber bundle over G_0. Since F is an equivalence of categories, the functor F^* has a left adjoint F_*, given by

$$F_*P(x) = \lim_{G_0 \to x} P$$

where $G_0 \to x$ are the elements of G_0 equipped with a morphism $Fy \to x$. Since F is an equivalence of groupoids, there is a unique map between any two objects of $G_0 \to x$, and so $F_*P(x)$ is isomorphic to P_y for any $y \in G_0 \to x$. For each $x \in H_0$, choose a neighbourhood $x \in U \subset H_0$, a map $t : U \to G_0$ and a family of morphisms $U \to H_1$ connecting $F \circ t$ to the inclusion $U \to H_0$. We topologize

$$\bigcup_{x \in H_0} F_* P_x$$

by requiring that the canonical map

$$t^* P \longrightarrow F_* P|_U$$

be a homeomorphism. This gives $F_* P$ the structure of a fiber bundle over H_0. Naturality provides $F_* P$ with the additional structure required to make it an $\mathcal{H}$-fiber bundle. One easily checks that the pair (F_*, F^*) is an adjoint equivalence of the category of $\mathcal{H}$-fiber bundles over H_0 with the category of $\mathcal{G}$-fiber bundles over G_0. □

Proposition 2.6.3. *Suppose that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a local equivalence. Then the pullback functor induces a homeomorphism from the space of $\mathcal{H}$-equivariant sections of an $\mathcal{H}$-fiber bundle on H_0 to the space of $\mathcal{G}$-equivariant sections of the pullback $\mathcal{G}$-fiber bundle on G_0.*

Proof. Assume we have a local equivalence $\mathcal{G} \to \mathcal{H}$. Given a section v of a fibre bundle $P \to H_0$, we can consider the section $F^*(v) : G_0 \to F^*(P)$ defined by $F^*(v)(x) = (x, v(F(x)))$. And on the other hand, given a section w of a fibre bundle $Q \to G_0$, we can consider the section $F_*(w) : H_0 \to F_*(Q)$ defined by $F_*(t)(x) = (x, w(y))$ where $y \in G_0$ is such that there is $h \in H_1$ that satisfies $F(y) = s(h)$ and $t(h) = x$. □

Definition 2.6.4. Let P be a $\mathcal{G}$-space and $T = P/\mathcal{G}$. Note that T is not a $\mathcal{G}$-space. We say that $P \longrightarrow T$ is a principal $\mathcal{G}$-bundle if it admits local sections and if the map $G_1 \times_{G_0} P \longrightarrow P \times_T P$ induced by the action and projection maps is a homeomorphism.

Definition 2.6.5. A principal $\mathcal{G}$-bundle $E \longrightarrow B$ is universal if every principal $\mathcal{G}$-bundle $P \longrightarrow X$ over a paracompact base admits a $\mathcal{G}$-bundle map

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

unique up to homotopy.

For any group G we can construct the universal G-space EG in the sense of Milnor [31] or Milgram [30]. We have analogous constructions for a groupoid.

The first construction is the analogue to the universal space of Milgram and it is also described in [20]. Given the groupoid $\mathcal{G}$, construct the translation groupoid $\bar{\mathcal{G}} = \mathcal{G} \rtimes G_1$, which has G_1 as its space of objects and only one arrow between two elements if they have the same source or none otherwise, i.e., the space of arrows is $G_1 \times_s G_1$. The nerve of this category is given by:

$$N\bar{\mathcal{G}}_k = G_1 \times_s G_1 \times_s \overset{k+1}{\cdots} \times_s G_1$$

There is a natural action of $\mathcal{G}$ on $N\bar{\mathcal{G}}_k$ with π given by the source map and $(f_1, ..., f_{k+1}) \cdot h = (f_1 h, ..., f_{k+1} h)$. With this action we have $N\bar{\mathcal{G}}_k/\mathcal{G} \cong G_k$, so $N\bar{\mathcal{G}}/\mathcal{G} \cong N\mathcal{G}$ and therefore $B\bar{\mathcal{G}}/\mathcal{G} \cong B\mathcal{G}$. It is clear that $B\bar{\mathcal{G}} \simeq G_0$ and $\mathcal{G}$ acts freely on $B\bar{\mathcal{G}}$.

The second construction imitates Milnor's universal G-space. We construct the $\mathcal{G}$-spaces:

$$E^n\mathcal{G} = G_1 *_s \overset{n}{\cdots} *_s G_1 = \{\sum_{i=1}^{n} \lambda_i g_i \mid s(g_1) = ... = s(g_n), \sum_{i=1}^{n} \lambda_i = 1\}$$

with the $\mathcal{G}$-action given by the anchor map $\pi(\Sigma_{i=1}^n \lambda_i g_i) = s(g_1)$, action map $(\Sigma_{i=1}^n \lambda_i g_i) \cdot g = \Sigma_{i=1}^n \lambda_i g_i g$ and the subspace topology from $G_1 * \overset{n}{\cdots} * G_1$.

Now define $E\mathcal{G}$ to be the direct limit of the sequence of $\mathcal{G}$-spaces $E^n\mathcal{G}$. Note $\mathcal{G}$ acts freely on $E^n\mathcal{G}$ for all n and thus on $E\mathcal{G}$. It can be checked that $E\mathcal{G}/\mathcal{G} = B\mathcal{G}$.

Both $E\mathcal{G}$ and $B\bar{\mathcal{G}}$ define universal $\mathcal{G}$-principal bundles. Therefore $E\mathcal{G}/\mathcal{G}$ and $B\bar{\mathcal{G}}/\mathcal{G} = B\mathcal{G}$ are homotopy equivalent. From this point on we will use $E\mathcal{G}$ as our universal $\mathcal{G}$-space and identify $E\mathcal{G}/\mathcal{G}$ with $B\mathcal{G}$.

Definition 2.6.6. Given a $\mathcal{G}$-space X, we define the Borel construction $X_{\mathcal{G}} = (X \times_\pi E\mathcal{G})/\mathcal{G}$.

In the case of group actions, this construction is also known in the literature as the homotopy orbit space.

Remark 2.6.7. Let $\mathcal{G} = G \rtimes M$, where G is a topological group and M is a G-space. Then, we have $E\mathcal{G} = M \times EH$ and $B\mathcal{G} = M_H$.

Remark 2.6.8. Let M be a $\mathcal{G}$-space. Consider the groupoid $\mathcal{H} = \mathcal{G} \rtimes M$. In this case, $E\mathcal{H} = M \times_\pi E\mathcal{G}$ and $B\mathcal{H} = M_{\mathcal{G}}$.

2.7 Vector bundles

Definition 2.7.1. A complex vector bundle over an orbifold groupoid $\mathcal{G}$ is a $\mathcal{G}$-space E for which $\pi : E \to G_0$ is a complex vector bundle, and the action of $\mathcal{G}$ on E is fibrewise linear. Namely, any arrow $g : x \to y$ induces a linear isomorphism $g^{-1} : E_y \to E_x$. In particular, E_x is a linear representation of the stabilizer $\mathcal{G}_x$.

We will only consider complex vector bundles, so will omit the word complex from now on.

Definition 2.7.2. Let $\mathcal{G}$ be a groupoid. A $\mathcal{G}$-vector bundle on a $\mathcal{G}$-space X is a vector bundle $p : E \longrightarrow X$ such that E is a $\mathcal{G}$-space with fibrewise linear action and p is a $\mathcal{G}$-equivariant map.

Definition 2.7.3. Let X be a $\mathcal{G}$-space and $V \longrightarrow X$ a $\mathcal{G}$-vector bundle. We say V is extendable if there is a $\mathcal{G}$-vector bundle $W \longrightarrow G_0$ such that V is a direct summand of π^*W.

- Direct sum of $\mathcal{G}$-extendable vector bundles induces an operation on the set of isomorphism classes of $\mathcal{G}$-extendable vector bundles on X, making this set a monoid. We can also tensor $\mathcal{G}$-extendable vector bundles.

- The pullback of a $\mathcal{G}$-extendable vector bundle by a $\mathcal{G}$-equivariant map is a $\mathcal{G}$-extendable vector bundle.

- All $\mathcal{G}$-vector bundles on G_0 are extendable. Note that $\mathcal{G}$-extendable vector bundles on G_0 are equivalent to orbifold vector bundles on $\mathcal{G}$.

Example 2.7.4. Let $\mathcal{G} = H \rtimes M$. Then, a $\mathcal{G}$-space is a H-space X with a H-equivariant map to M. In this case, $\mathcal{G}$-equivariant vector bundles on X correspond to H-vector bundles on X.

Example 2.7.5. Let M be a $\mathcal{G}$-space. Consider the groupoid $\mathcal{H} = \mathcal{G} \rtimes M$. An $\mathcal{H}$-space is a $\mathcal{G}$-space X with a $\mathcal{G}$-equivariant map to M. As in the previous example, $\mathcal{H}$-equivariant vector bundles on X are just $\mathcal{G}$-equivariant vector bundles on X.

Proposition 2.7.6. *All $\mathcal{G}$-vector bundles on a free $\mathcal{G}$-space X are extendable.*

Proof. It suffices to prove that vector bundles on $X/\mathcal{G}$ pull back to extendable $\mathcal{G}$-vector bundles. The anchor map $\pi_1 : X \longrightarrow G_0$ induces a map $\pi_2 : X/\mathcal{G} \longrightarrow |\mathcal{G}|$. These maps fit into a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[d, "p_1"'] & G_0 \arrow[d, "p_2"] \\ X/\mathcal{G} \arrow[r, "\pi_2"] & |\mathcal{G}| \end{tikzcd}$$

Given a vector bundle V on $X/\mathcal{G}$, there is a vector bundle W on $|\mathcal{G}|$ such that $\pi_2^* W = V \oplus A$ for some vector bundle A on $X/\mathcal{G}$. Consider $W' = p_2^* W$. We have

$$\pi_1^* W' = \pi_1^* p_2^* W = p_1^* \pi_2^* W = p_1^*(V \oplus A) = p_1^* V \oplus p_2^* A \qquad \square$$

2.8 Hilbert bundles

Definition 2.8.1. Let $\mathcal{G}$ be a Lie groupoid and X a $\mathcal{G}$-space. A $\mathcal{G}$-Hilbert bundle on X is a $\mathcal{G}$-space E with an equivariant map $p : E \longrightarrow X$ which is also a locally trivial Hilbert bundle with a continuous linear $\mathcal{G}$-action.

Definition 2.8.2. A universal $\mathcal{G}$-Hilbert bundle on X is a $\mathcal{G}$-Hilbert bundle E such that for each Hilbert bundle V on X there exists a $\mathcal{G}$-equivariant unitary embedding $V \subset E$.

Definition 2.8.3. A locally universal $\mathcal{G}$-Hilbert bundle on X is a $\mathcal{G}$-Hilbert bundle E such that there is a $\mathcal{G}$-equivariant countable open cover $\{U_i\}$ of X such that $E_{|U_i}$ is a universal $\mathcal{G}$-Hilbert bundle on U_i.

Definition 2.8.4. A local quotient groupoid is a groupoid $\mathcal{G}$ such that G_0 admits a $\mathcal{G}$-equivariant countable open cover $\{U_i\}$ with the property that $\mathcal{G} \rtimes U_i$ is weakly equivalent to an action groupoid corresponding to the proper action of a compact Lie group G on a finite G-CW-complex.

Corollary 2.8.5. *A finite Lie groupoid is a local quotient groupoid.*

Proposition 2.8.6. *If $\mathcal{G}$ is a local quotient groupoid, then there exists a locally universal $\mathcal{G}$-Hilbert bundle on G_0 that is unique up to unitary equivalence.*

Proof. See [17] $\square$

Corollary 2.8.7. *If $\mathcal{G}$ is a finite Lie groupoid, then there exists a locally universal $\mathcal{G}$-Hilbert bundle on G_0 that is unique up to unitary equivalence. We denote it by $U(\mathcal{G})$.*

Definition 2.8.8. Let $\mathcal{G}$ be a Lie groupoid with a locally universal $\mathcal{G}$-Hilbert bundle $U(\mathcal{G})$ on G_0 and X a $\mathcal{G}$-space. A $\mathcal{G}$-stable Hilbert bundle on X is a $\mathcal{G}$-Hilbert bundle $E \to X$ such that $E \oplus \pi_X^*(U(\mathcal{G})) \cong E$. In the case when $X = G_0$ we call E a stable representation of $\mathcal{G}$.

Proposition 2.8.9. *Suppose that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a local equivalence. Then the pullback functor induces an equivalence of categories, namely, from the category of locally universal $\mathcal{H}$-Hilbert bundles on H_0 to the category of locally universal $\mathcal{G}$-Hilbert bundles on G_0.*

Proof. See [17] □

Corollary 2.8.10. *Suppose that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a local equivalence. Then the pullback functor*

$$F^* : \{\textit{Stable representations of } \mathcal{H}\} \to \{\textit{Stable representations of } \mathcal{G}\}$$

is an equivalence of categories.

Definition 2.8.11. Let $\mathcal{G}$ be a Lie groupoid and X a $\mathcal{G}$-space. A $\mathcal{G}$-projective bundle on X is a $\mathcal{G}$-space P with a $\mathcal{G}$-equivariant map $p : P \longrightarrow X$ such that there exists an equivariant open covering $\{U_i\}$ of X for which $P_{|U_i} = U_i \times_\pi \mathbb{P}(E)$ for some $\mathcal{G}$-Hilbert bundle E on G_0. Moreover, we shall call P a $\mathcal{G}$-stable projective bundle if $P \cong P \otimes \pi^*\mathbb{P}(U(\mathcal{G}))$ for some locally universal $\mathcal{G}$-Hilbert bundle $U(\mathcal{G})$ on G_0.

Chapter 3

Equivariant K-theory

3.1 Extendable K-theory

Definition 3.1.1. Let X be a $\mathcal{G}$-space.

$$Vect_{\mathcal{G}}(X) = \{\text{isomorphism classes of extendable } \mathcal{G}\text{-vector bundles on } X\}$$

$$K_{\mathcal{G}}(X) = K(Vect_{\mathcal{G}}(X))$$

where $K(A)$ is the Grothendieck group of a monoid A. We call $K_{\mathcal{G}}(X)$ the extendable $\mathcal{G}$-equivariant K-theory of X.

Remark 3.1.2. If $\mathcal{G} = G \rtimes M$, then $K_{\mathcal{G}}(M) = K_G(M)$.

Remark 3.1.3. Let M be a $\mathcal{G}$-space and $\mathcal{H} = \mathcal{G} \rtimes M$, then $K_{\mathcal{H}}(M)$ does not necessarily coincide with $K_{\mathcal{G}}(M)$, as we will see later on.

We can now define the extendable K-groups as in [28]:

$$K_{\mathcal{G}}^{-n}(X) = Ker[K_{\mathcal{G}}(X \times S^n) \xrightarrow{i^*} K_{\mathcal{G}}(X)]$$

$$K_{\mathcal{G}}^{-n}(X, A) = Ker[K_{\mathcal{G}}^{-n}(X \cup_A X) \xrightarrow{j_2^*} K_{\mathcal{G}}^{-n}(X)]$$

where $i : X \to X \times S^n$ is the inclusion given by fixing a point in S^n and $j_2 : X \to X \cup_A X$ is one of the maps from X to the pushout. We equip $X \times S^n$ with a $\mathcal{G}$-action by taking as the anchor map the composition of the projection onto the first coordinate and the anchor map for X. Then let the groupoid act trivially on the sphere. The anchor map for $X \cup_A X$ is the unique map to G_0 making the pushout diagram commutative. The action is induced by the action of $\mathcal{G}$ on X.

The following lemma follows easily from the definitions:

Lemma 3.1.4. *Let (X, A) be a $\mathcal{G}$-pair. Suppose that $X = \coprod_{i \in I} X_i$, the disjoint union of open $\mathcal{G}$-invariant subspaces X_i and set $A_i = A \cap X_i$. Then there is a natural isomorphism*

$$K_{\mathcal{G}}^{-n}(X, A) \longrightarrow \prod_{i \in I} K_{\mathcal{G}}^{-n}(X_i, A_i)$$

From now on $\mathcal{G}$ will be a Lie groupoid.

Corollary 3.1.5. *If f_0, $f_1 : (X, A) \longrightarrow (Y, B)$ are $\mathcal{G}$-homotopic $\mathcal{G}$-maps between $\mathcal{G}$-pairs, then*

$$f_0^* = f_1^* : K_{\mathcal{G}}^{-n}(Y, B) \longrightarrow K_{\mathcal{G}}^{-n}(X, A)$$

for all $n \geq 0$.

Proof. It follows from the existence of a Haar system. □

3.2 The Mayer-Vietoris sequence

Lemma 3.2.1. *Let $\phi : (X_1, X_0) \longrightarrow (X, X_2)$ be a map of $\mathcal{G}$-spaces, set $\phi_0 = \phi|_{X_0}$, and assume that $X \cong X_2 \cup_{\phi_0} X_1$. Let $p_1 : E_1 \longrightarrow X_1$ and $p_2 : E_2 \longrightarrow X_2$ be $\mathcal{G}$-extendable vector bundles, let $\overline{\phi}_0 : E_1|_{X_0} \longrightarrow E_2$ be a strong map covering ϕ_0, and set $E = E_2 \cup_{\overline{\phi}_0} E_1$. Then $p = p_1 \cup p_2 : E \longrightarrow X$ is a $\mathcal{G}$-extendable vector bundle over X.*

Proof. We have to show that $p : E \to X$ is locally trivial. Since E_1 is locally trivial, so is $E|_{X - X_2} \cong E|_{X_1 - X_0}$. So it remains to find a neighbourhood of X_2 over which E is locally trivial. Choose a closed neighbourhood W_1 of X_0 in X_1 for which there is a strong deformation retraction $r : W_1 \to X_0$. By the homotopy invariance for nonequivariant vector bundles over paracompact spaces, r is covered by a strong map of vector bundles $\overline{r} : E_1|_{W_1} \to E_0$ which extends $\overline{i}_1$. Set $W = X_2 \cup_{\phi_0} W_1$. Then $\overline{r}$ extends, via the pushout, to a strong map of vector bundles $E|_W \to E_2$ which extends $\overline{i}_2$ and hence $E|_W$ is locally trivial. □

Lemma 3.2.2. *Let $\phi : X \longrightarrow Y$ be a $\mathcal{G}$-equivariant map and let $E' \longrightarrow X$ be a $\mathcal{G}$-extendable vector bundle. Then, there is a $\mathcal{G}$-extendable vector bundle $E \longrightarrow Y$ such that E' is a summand of $\phi^* E$.*

Proof. Consider $\pi : Y \longrightarrow G_0$. Now, $\pi\phi : X \longrightarrow G_0$. Since E' is extendable, there is a $\mathcal{G}$-vector bundle V on G_0 such that E' is a direct summand of $(\pi\phi)^* V$. Let $E = \pi^* V$. E is a $\mathcal{G}$-vector bundle on Y and it is the pullback of an extendable $\mathcal{G}$-vector bundle, hence it is extendable. And we have that E' is a direct summand of $(\pi\phi)^* V = \phi^* E$. $\square$

Lemma 3.2.3. *Let*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

be a pushout square of $\mathcal{G}$-spaces. Then there is a natural exact sequence, infinite to the left

$$\ldots \xrightarrow{d^{-n-1}} K_{\mathcal{G}}^{-n}(X) \xrightarrow{j_1^* \oplus j_2^*} K_{\mathcal{G}}^{-n}(X_1) \oplus K_{\mathcal{G}}^{-n}(X_2) \xrightarrow{i_1^* - i_2^*} K_{\mathcal{G}}^{-n}(A) \xrightarrow{d^{-n}} \ldots$$

$$\ldots \longrightarrow K_{\mathcal{G}}^{-1}(A) \xrightarrow{d^{-1}} K_{\mathcal{G}}^{0}(X) \xrightarrow{j_1^* \oplus j_2^*} K_{\mathcal{G}}^{0}(X_1) \oplus K_{\mathcal{G}}^{0}(X_2) \xrightarrow{i_1^* - i_2^*} K_{\mathcal{G}}^{0}(A) \qquad (3.1)$$

Proof. We first show that the sequence

$$K_{\mathcal{G}}(X) \xrightarrow{j_1^* \oplus j_2^*} K_{\mathcal{G}}(X_1) \oplus K_{\mathcal{G}}(X_2) \xrightarrow{i_1^* - i_2^*} K_{\mathcal{G}}(A) \qquad (3.2)$$

is exact; and hence the long sequence in the statement of the theorem is exact at $K_{\mathcal{G}}^{-n}(X_1) \oplus K_{\mathcal{G}}^{-n}(X_2)$ for all n. Clearly the composite is zero. So fix an element $(\alpha_1, \alpha_2) \in Ker(i_1^* - i_2^*)$. By the previous lemma, we can add an element of the form $([j_1^* E'], [j_1^* E'])$ for some $\mathcal{G}$-vector bundle $E' \to X$, and arrange that $\alpha_1 = [E_1]$ and $\alpha_2 = [E_2]$ for some pair of $\mathcal{G}$-vector bundles $E_k \to X_k$.

Then $i_1^*E_1$ and $i_2^*E_2$ are stably isomorphic, and after adding the restrictions of another bundle over X, we can arrange that $i_1^*E_1 \cong i_2^*E_2$. Lemma 3.2.1 now applies to show that there is a $\mathcal{G}$-vector bundle E over X such that $j_k^*E \cong E_k$ for $k = 1, 2$, and hence that $(\alpha_1, \alpha_2) = ([E_1], [E_2]) \in Im(j_1^* \oplus j_2^*)$.

Assume now that A is a retract of X_1. We claim that in this case,

$$Ker[K_{\mathcal{G}}(X) \xrightarrow{j_2^*} K_{\mathcal{G}}(X_2)] \xrightarrow{j_1^*} Ker[K_{\mathcal{G}}(X_1) \xrightarrow{i_1^*} K_{\mathcal{G}}(A)] \tag{3.3}$$

is an isomorphism. It is surjective by the exactness of (3.2). So fix an element $[E] - [E'] \in Ker(j_1^* \oplus j_2^*)$. To simplify the notation, we write $E|_{X_1} = j_1^*E$, $E|_A = i_2^*j_2^*E$, ... Let $p_1 : X_1 \to A$ be a retraction, and let $p : X \to X_2$ be its extension to X. By the previous lemma, we can arrange that $E|_{X_k} \cong E'|_{X_k}$ for $k = 1, 2$. Applying the same lemma to the retraction $p : X \to X_2$, we obtain a $\mathcal{G}$-vector bundle $F' \to X_2$ such that E' is a summand of p^*F'. Stabilizing again, we can assume that $E' \cong p^*F'$ and hence that $F' \cong E'|_{X_2}$ and $E'|_{X_1} \cong p_1^*(F'|_A) \cong p_1^*(E'|_A)$. Fix isomorphisms $\psi_k : E|_{X_k} \to E'|_{X_k}$ covering the identity on X. The automorphism $(\psi|_A) \circ (\psi_1|_A)^{-1}$ of $E'|_A$ pulls back, under p_1, to an automorphism ϕ of $E'|_{X_1}$. By replacing ψ_1 by $\phi \circ \psi_1$ we can arrange that $\psi_1|_A = \psi_2|_A$. Then $\psi_1 \cup \psi_2$ is an isomorphism from E to E', and this proves the exactness.

Now for each $n \geq 1$,

$$K_{\mathcal{G}}^{-n}(A) = Ker[K_{\mathcal{G}}(A \times S^n) \to K_{\mathcal{G}}(A)]$$

$$\cong Ker[K_{\mathcal{G}}(X \cup_{A \times pt} (A \times S^n)) \xrightarrow{incl^*} K_{\mathcal{G}}(X)]$$

$$\cong Ker[K_{\mathcal{G}}((X_1 \times D^n) \cup_{A \times S^{n-1}} (X_2 \times D^n)) \xrightarrow{(-,pt)^*} K_{\mathcal{G}}(X)]$$

the last step since $(X_1 \times pt \cup A \times D^n)$ is a strong deformation retract of $X_1 \times D^n$. Denote $Y = (X_1 \times D^n) \cup_{A \times S^{n-1}} (X_2 \times D^n)$ and define $d^{-n} : K_{\mathcal{G}}^{-n}(A) \to K_{\mathcal{G}}^{-n+1}(X)$ to be the homomorphism which makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & K_{\mathcal{G}}^{-n}(A) \arrow[r] \arrow[d, "d^{-n}"'] & K_{\mathcal{G}}(Y) \arrow[r, "(-,pt)^*"] \arrow[d, "incl^*"'] & K_{\mathcal{G}}(X) \arrow[r] \arrow[d, "Id"'] & 0 \\
0 \arrow[r] & K_{\mathcal{G}}^{-n+1}(X) \arrow[r] & K_{\mathcal{G}}(X \times S^{n-1}) \arrow[r, "(-,pt)^*"] & K_{\mathcal{G}}(X) \arrow[r] & 0
\end{tikzcd}$$

We have already shown that the long sequence (3.1) is exact at $K_{\mathcal{G}}^{-n}(X_1) \oplus K_{\mathcal{G}}^{-n}(X_2)$ for all n. Denote $Z = (X_1 \times D^n) \coprod (X_2 \times D^n)$ and $W = (X_1 \times D^n) \cup_{A \times pt} (X_2 \times D^n)$. To see exactness at $K_{\mathcal{G}}^{-n+1}(X)$ and $K_{\mathcal{G}}^{-n}(A)$ for any $n \geq 1$, apply the exactness of (3.3) to the following split inclusion of pushout squares:

$$\begin{tikzcd}
X_1 \coprod X_2 \arrow[r] \arrow[d] & X_1 \coprod X_2 \arrow[d] & (X_1 \coprod X_2) \times S^{n-1} \arrow[r] \arrow[d] & Z \arrow[d] \\
X \arrow[r] \arrow[d] & X \arrow[r, "incl"] \arrow[d] & X \times S^{n-1} \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & X & X \arrow[r] & W
\end{tikzcd}$$

The upper pair of squares induces a split surjection of exact sequence whose kernel yields the exactness of (3.1) at $K_{\mathcal{G}}^{-n+1}(X)$. And since

$$Ker[K_{\mathcal{G}}(W) \to K_{\mathcal{G}}(X)] \cong$$

$$\cong Ker\Big[K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(X_1 \coprod X_2)\Big] \cong K_{\mathcal{G}}^{-n}(X_1) \oplus K_{\mathcal{G}}^{-n}(X_2)$$

by (3.3), the lower pair of squares induces a split surjection of exact sequences whose kernel yields the exactness of (3.1) at $K_{\mathcal{G}}^{-n}(A)$. $\square$

Lemma 3.2.4. *Let $\phi : (X, A) \longrightarrow (Y, B)$ be a map of finite $\mathcal{G}$-CW-pairs* *such that* $Y \cong B \cup_{\phi|A} X$. *Then*

$$\phi^* : K_{\mathcal{G}}^{-n}(Y, B) \longrightarrow K_{\mathcal{G}}^{-n}(X, A)$$

is an isomorphism for all $n \geq 0$.

Proof. The square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \cup_A X \arrow[r] & Y \cup_B Y \end{tikzcd}$$

is a pushout, and X is a retract of $X \cup_A X$. So its Mayer-Vietoris sequence splits into short exact sequences

$$0 \to K_{\mathcal{G}}^{-n}(Y \cup_B Y) \to K_{\mathcal{G}}^{-n}(X \cup_A X) \oplus K_{\mathcal{G}}^{-n}(Y) \to K_{\mathcal{G}}^{-n}(X) \to 0$$

and so $K_{\mathcal{G}}^{-n}(Y, B) \cong K_{\mathcal{G}}^{-n}(X, A)$. □

Lemma 3.2.5. *Let* (X, A) *be a finite* $\mathcal{G}$*-CW-pair. Then the following sequence, extending infinitely far to the left, is natural and exact:*

$$\ldots \xrightarrow{\delta^{-n-1}} K_{\mathcal{G}}^{-n}(X, A) \xrightarrow{i^*} K_{\mathcal{G}}^{-n}(X) \xrightarrow{j^*} K_{\mathcal{G}}^{-n+1}(X, A) \xrightarrow{i^*} \ldots$$

$$\ldots \xrightarrow{\delta^{-1}} K_{\mathcal{G}}^{0}(X, A) \xrightarrow{i^*} K_{\mathcal{G}}^{0}(X) \xrightarrow{j^*} K_{\mathcal{G}}^{0}(A)$$

Proof. This follows immediately from the Mayer-Vietoris sequence for the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "i_2"'] \\ X \arrow[r, "i_1"] & X \cup_A X \end{tikzcd}$$

□

3.3 The Bott periodicity

We now consider products on $K^*_{\mathcal{G}}(X)$ and on $K^*_{\mathcal{G}}(X, A)$. We follow [28]. Tensor products of $\mathcal{G}$-extendable vector bundles makes $K_{\mathcal{G}}(X)$ into a commutative ring, and all induced maps $f^* : K_{\mathcal{G}}(Y) \longrightarrow K_{\mathcal{G}}(X)$ are ring homomorphisms. For each $n, m \geq 0$,

$$K_{\mathcal{G}}^{-n-m}(X) \cong Ker[K_{\mathcal{G}}^{-m}(X \times S^n) \longrightarrow K_{\mathcal{G}}^{-m}(X)] =$$

$$= Ker[K_{\mathcal{G}}(X \times S^n \times S^m) \longrightarrow K_{\mathcal{G}}(X \times S^n) \oplus K_{\mathcal{G}}(X \times S^m)]$$

where the first isomorphism follows from the usual Mayer-Vietoris sequences, hence

$$K_{\mathcal{G}}(X \times S^n) \otimes K_{\mathcal{G}}(X \times S^m) \overset{multo(p_1^* \otimes p_2^*)}{\longrightarrow} K_{\mathcal{G}}(X \times S^n \times S^m)$$

restricts to a homomorphism

$$K_{\mathcal{G}}^{-n}(X) \otimes K_{\mathcal{G}}^{-m}(X) \longrightarrow K_{\mathcal{G}}^{-n-m}(X)$$

By applying the above definition with $n = 0$ or $m = 0$, the multiplicative identity for $K_{\mathcal{G}}(X)$ is seen to be an identity for $K^*_{\mathcal{G}}(X)$. Associativity of the graded product is clear and graded commutativity follows upon showing that composition with a degree -1 map $S^n \to S^m$ induces multiplication by -1 on $K^{-n}(X)$. This product makes $K^*_{\mathcal{G}}(X)$ into a graded ring. Clearly, $f^* : K^*_{\mathcal{G}}(Y) \longrightarrow K^*_{\mathcal{G}}(X)$ is a ring homomorphism for any $\mathcal{G}$-map $f : X \longrightarrow Y$. This makes $K^*_{\mathcal{G}}(X)$ into a $K^*_{\mathcal{G}}(G_0)$-algebra, since G_0 is a final object in the category of $\mathcal{G}$-spaces.

We will now construct a Bott homomorphism. Recall that we have $\tilde{K}(S^2) = Ker[K(S^2) \longrightarrow K(pt)] \cong \mathbb{Z}$, and is generated by the Bott element $B \in \tilde{K}(S^2)$, the element $[S^2 \times \mathbb{C}] - [H] \in \tilde{K}(S^2)$, where H is the canonical complex line bundle over $S^2 = \mathbb{C}P^1$. For any $\mathcal{G}$-space X, there is an obvious pairing

$$K_{\mathcal{G}}^{-n}(X) \otimes \tilde{K}(S^2) \xrightarrow{\otimes} Ker[K_{\mathcal{G}}^{-n}(X \times S^2) \longrightarrow K_{\mathcal{G}}^{-n}(X)] \cong K_{\mathcal{G}}^{-n-2}(X)$$

induced by (external) tensor product of bundles. Evaluation at the Bott element now defines a homomorphism

$$b = b(X) : K_{\mathcal{G}}^{-n}(X) \longrightarrow K_{\mathcal{G}}^{-n-2}(X)$$

which by construction is natural in X. And this extends to a homomorphism

$$b = b(X, A) : K_{\mathcal{G}}^{-n}(X, A) \longrightarrow K_{\mathcal{G}}^{-n-2}(X, A)$$

defined for any $\mathcal{G}$-pair (X, A) and all $n \geq 0$.

Definition 3.3.1. A groupoid $\mathcal{G}$ is Bredon-compatible if given any $\mathcal{G}$-cell U, all $\mathcal{G}$-vector bundles on U are extendable.

Note that if U is a $\mathcal{G}$-cell and $\mathcal{G}$ is Bredon-compatible, then $K^*_{\mathcal{G}\rtimes U}(U) = K^*_{\mathcal{G}}(U)$.

Example 3.3.2. An example of a Bredon-compatible groupoid is $\mathcal{G} = G \rtimes M$, where G is a compact Lie group and M is a finite G-CW-complex. A $\mathcal{G}$-cell U is a finite G-space with an equivariant map to M and $\mathcal{G}$-vector bundles on U are just G-vector bundles. By [40], for any G-vector bundle A on U, there is another G-vector bundle B such that $A \oplus B$ is a trivial bundle, that is, the pullback of a G-vector bundle V over a point. Consider the unique map from M to a point. The pullback of V over this map is a G-vector bundle on M. If we pull it back to U we recover $A \oplus B$ and therefore $\mathcal{G}$ is Bredon-compatible.

Corollary 3.3.3. *If $\mathcal{G}$ is Bredon-compatible and U is a $\mathcal{G}$-cell, then $K^*_{\mathcal{G}}(U) \cong K^*_G(M)$ for some compact Lie group G and a finite G-CW-complex M.*

Theorem 3.3.4. *If $\mathcal{G}$ is Bredon-compatible, the Bott homomorphism*

$$b = b(X, A) : K_{\mathcal{G}}^{-n}(X, A) \longrightarrow K_{\mathcal{G}}^{-n-2}(X, A)$$

is a natural isomorphism for any finite $\mathcal{G}$-CW-pair (X, A) and all $n \geq 0$.

Proof. Assume first that $X = Y \cup_\phi (U \times D^m)$ where $U \times D^m$ is a $\mathcal{G}$-cell. Assume inductively that $b(Y)$ is an isomorphism. Since $K_{\mathcal{G}}^{-n}(U \times S^{m-1})$ is isomorphic to $K_G^{-n}(M \times S^{m-1})$ and $K_{\mathcal{G}}^{-n}(U \times D^m) \cong K_G^{-n}(M \times D^m)$, the Bott homomorphisms $b(U \times S^{m-1})$ and $b(U \times D^m)$ are isomorphisms by the equivariant Bott periodicity theorem for actions of compact Lie groups. The Bott map is natural and compatible with the boundary operators in the Mayer-Vietoris sequence for Y, X, $U \times S^{m-1}$ and $U \times D^m$ and so $b(X)$ is an isomorphism by the 5-lemma. The proof that $b(X, A)$ is an isomorphism follows immediately from the definitions of the relative groups. □

Based on the Bott isomorphism we just proved, we can now redefine for all $n \in \mathbb{Z}$

$$K_{\mathcal{G}}^n(X, A) = \begin{cases} K_{\mathcal{G}}^0(X, A) & \text{if } n \text{ is even} \\ K_{\mathcal{G}}^{-1}(X, A) & \text{if } n \text{ is odd} \end{cases}$$

For any finite $\mathcal{G}$-CW-pair (X, A), define the boundary operator $\delta^n : K_{\mathcal{G}}^n(A) \longrightarrow K_{\mathcal{G}}^{n+1}(X, A)$ to be $\delta : K_{\mathcal{G}}^{-1}(A) \longrightarrow K_{\mathcal{G}}^0(X, A)$ if n is odd, and to be the composite

$$K_{\mathcal{G}}^0(A) \xrightarrow{b} K_{\mathcal{G}}^{-2}(A) \xrightarrow{\delta^{-2}} K_{\mathcal{G}}^{-1}(X, A)$$

if n is even.

We can collect all the information we have so far about $\mathcal{G}$-equivariant K-theory in the following theorem:

Theorem 3.3.5. *If $\mathcal{G}$ is a Bredon-compatible Lie groupoid, the groups $K_{\mathcal{G}}^n(X, A)$ define a $\mathbb{Z}/2$-graded multiplicative cohomology theory on the category of finite $\mathcal{G}$-CW-pairs.*

Note that for a general Lie groupoid $\mathcal{G}$, $K_{\mathcal{G}}^*(-)$ is a multiplicative cohomology theory on the category of $\mathcal{G}$-spaces, but it is not clear whether we have Bott periodicity.

3.4 Equivariant representable K-theory

Let H be a stable representation of $\mathcal{G}$. Consider the associated bundle of Fredholm operators $Fred(H)$ on G_0, and the subbundle $Fred'(H)$ of operators A for which the action $g \to gAg^{-1}$ is continuous. See [16] for more details on the correct topology for this space.

Definition 3.4.1. We say that a $\mathcal{G}$-equivariant map $f : X \to Fred'(H)$ is extendable if there is another $\mathcal{G}$-equivariant map $g : X \to Fred'(H)$ such that $gf = v\pi_X$ for some section v of $Fred'(H) \to G_0$, where $\pi_X : X \to G_0$ is the anchor map.

Definition 3.4.2. We say that a homotopy $H : X \times I \to Fred'(H)$ of $\mathcal{G}$-equivariant maps is extendable if each H_t is an extendable $\mathcal{G}$-equivariant map $X \to Fred'(H)$.

Definition 3.4.3. Let X be a $\mathcal{G}$-space, H a stable representation of $\mathcal{G}$ and $n \geq 0$. Define the $\mathcal{G}$-equivariant representable K-theory groups of X to be

$$RK_{\mathcal{G}}^{-n}(X) = [X, \Omega^n Fred'(H)]_{\mathcal{G}}^{ext}$$

where this notation denotes the extendable homotopy classes of extendable $\mathcal{G}$-maps. For $\mathcal{G}$-pairs (X, A), define

$$RK_{\mathcal{G}}^{-n}(X, A) = Ker[RK_{\mathcal{G}}^{-n}(X \cup_A X) \xrightarrow{j_2^*} RK_{\mathcal{G}}^{-n}(X)]$$

where $j_2 : X \to X \cup_A X$ is one of the maps from X to the pushout.

We could have defined the representable K-groups as in [28]:

$$RK_{\mathcal{G}}^{-n}(X) = Ker[RK_{\mathcal{G}}(X \times S^n) \xrightarrow{i^*} RK_{\mathcal{G}}(X)]$$

where $RK_{\mathcal{G}}(X) = [X, Fred'(H)]_{\mathcal{G}}^{ext}$ and $i : X \to X \times S^n$ is the inclusion given by fixing a point in S^n. Both definitions are clearly equivalent.

We will now prove this defines a cohomology theory on the category of

$\mathcal{G}$-spaces. Since the definition is given by homotopy classes of maps, the next corollary follows from the definition.

Corollary 3.4.4. *If f_0, $f_1 : (X,A) \longrightarrow (Y,B)$ are $\mathcal{G}$-homotopic $\mathcal{G}$-maps between $\mathcal{G}$-pairs, then*

$$f_0^* = f_1^* : RK_{\mathcal{G}}^{-n}(Y,B) \longrightarrow RK_{\mathcal{G}}^{-n}(X,A)$$

for all $n \geq 0$.

The following lemma follows easily from the definitions:

Lemma 3.4.5. *Let (X,A) be a $\mathcal{G}$-pair. Suppose that $X = \coprod_{i\in I} X_i$, the disjoint union of open $\mathcal{G}$-invariant subspaces X_i and set $A_i = A \cap X_i$. Then there is a natural isomorphism*

$$RK_{\mathcal{G}}^{-n}(X,A) \longrightarrow \prod_{i\in I} RK_{\mathcal{G}}^{-n}(X_i, A_i)$$

Lemma 3.4.6. *Let $\phi : X \longrightarrow Y$ be a $\mathcal{G}$-equivariant map, H be a stable representation of $\mathcal{G}$ and let $s : X \longrightarrow Fred'(H)$ be a $\mathcal{G}$-extendable map. Then, there is a $\mathcal{G}$-extendable map $t : Y \longrightarrow Fred'(H)$ such that $s's = t\phi$ for some $\mathcal{G}$-extendable map $s' : X \longrightarrow Fred'(H)$.*

Proof. Since s is extendable, there is a $\mathcal{G}$-extendable map $s' : X \longrightarrow Fred'(H)$ such that $s's = v = \pi_X = v\pi_Y\phi$ for some section $v : G_0 \longrightarrow Fred'(H)$. Choose $t = v\pi_Y$. This is a $\mathcal{G}$-extendable map for it is the pullback of a $\mathcal{G}$-extendable map and we have $s's = t\phi$. $\square$

Lemma 3.4.7. *Let*

$$\begin{array}{ccc} A & \xrightarrow{i_1} & X_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a pushout square of $\mathcal{G}$-spaces, H a stable representation of $\mathcal{G}$ and i_1 a cofibration. Let $s_k : X_k \longrightarrow Fred'(H)$ be $\mathcal{G}$-extendable maps for $k = 1, 2$ such that $s_1 i_1$ and $s_2 i_2$ are $\mathcal{G}$-extendable homotopic maps from A to $Fred'(H)$. Then, there is a $\mathcal{G}$-extendable map $t : X \longrightarrow Fred'(H)$ such that tj_k is $\mathcal{G}$-extendable homotopic to s_k for $k = 1, 2$.

Proof. Let $F : A \times I \longrightarrow Fred'(H)$ be a $\mathcal{G}$-extendable homotopy with $F_0 = s_1 i_1$ and $F_1 = s_2 i_2$. There is $v_2 : G_0 \longrightarrow Fred'(H)$ such that $s_2' s_2 = v\pi_2$ for some $s_2' : X_2 \longrightarrow Fred'(H)$. By the previous lemma, there is a $\mathcal{G}$-extendable homotopy $\bar{F} : G_0 \times I \longrightarrow Fred'(H)$ such that $F'F = \bar{F} \circ (\pi_A \times id)$ for some $F' : A \times I \longrightarrow Fred'(H)$. We can also make it satisfy $F_1' = s_2' i_2$ by multiplying by a convenient constant homotopy for G_0.

Now, since $A \times I \xrightarrow{i_1 \times id} X_1 \times I$ is a $\mathcal{G}$-equivariant cofibration, there are $G, G' : X_1 \times I \longrightarrow Fred'(H)$ that extend F and F' respectively. Therefore $G'G$ must be an extension of $\bar{F} \circ (\pi_A \times id)$ to $X_1 \times I$. In fact, by the previous lemma, we can choose G and G' so that $G'G = \bar{F}(\pi_1 \times id)$. Therefore G is a $\mathcal{G}$-extendable homotopy.

Let $G_0 = s_1$ and $G_1 = \tilde{s}_1$. The extendable $\mathcal{G}$-homotopy classes of these two maps are equal, and $\tilde{s}_1 i_1 = G_1 i_1 = F_1 = s_2 i_2$. So we can easily extend this to a map $t : X \longrightarrow Fred'(H)$ such that $tj_1 = \tilde{s}_1$ and $tj_2 = s_2$. Therefore tj_k is $\mathcal{G}$-extendable homotopic to s_k for $k = 1, 2$. In fact, it is given by:

$$t(x) = \begin{cases} \tilde{s}_1(x_1) & \text{if } x = j_1(x_1) \\ s_2(x_2) & \text{if } x = j_2(x_2) \end{cases}$$

We only need to prove that t is extendable. Let $\tilde{s_1}' = G_1'$, $s_2' i_2 = F_1'$. We have $\tilde{s_1}' i_1 = G_1' i_1 = F_1' = s_2' i_2$. Consider

$$t'(x) = \begin{cases} \tilde{s_1}'(x_1) & \text{if } x = j_1(x_1) \\ s_2'(x_2) & \text{if } x = j_2(x_2) \end{cases}$$

Let $\bar{F}_1 = v_1$. Then we have $\tilde{s_1}'\tilde{s}_1 = v_1\pi_1$. Now consider the map:

$$v(x) = \begin{cases} v_1(\pi_1(x_1)) & \text{if } x = \pi_1(x_1) \\ v_2(\pi_2(x_2)) & \text{if } x = \pi_2(x_2) \end{cases}$$

This map is well defined. If $\pi_1(x_1) = \pi_2(x_2)$, then $x_1 = i_1a$ and $x_2 = i_2a$, and $v_1(\pi_1(x_1)) = v_1\pi_A a = \bar{F}_1\pi_A a = F_1\pi_A a = F_1\pi_2(x_2) = v_2(\pi_2(x_2))$. It is a routine check that $t't = v\pi_X$, thus t is extendable ☐

Lemma 3.4.8. *Let*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

be a pushout square of $\mathcal{G}$-spaces and i_1 a cofibration. Then there is a natural exact sequence, infinite to the left

$$\dots \xrightarrow{d^{-n-1}} RK_{\mathcal{G}}^{-n}(X) \xrightarrow{j_1^*\oplus j_2^*} RK_{\mathcal{G}}^{-n}(X_1) \oplus RK_{\mathcal{G}}^{-n}(X_2) \xrightarrow{i_1^*-i_2^*} RK_{\mathcal{G}}^{-n}(A) \xrightarrow{d^{-n}} \dots$$

$$\dots \longrightarrow RK_{\mathcal{G}}^{-1}(A) \xrightarrow{d^{-1}} RK_{\mathcal{G}}^{0}(X) \xrightarrow{j_1^*\oplus j_2^*} RK_{\mathcal{G}}^{0}(X_1) \oplus RK_{\mathcal{G}}^{0}(X_2) \xrightarrow{i_1^*-i_2^*} RK_{\mathcal{G}}^{0}(A)$$

Proof. It is a consequence of the two previous lemmas, the results in [8] and the proof of lemma 3.8 in [28]. ☐

For any stable representation H of $\mathcal{G}$ there is a $\mathcal{G}$-map $\Omega^n Fred'(H) \to \Omega^{n+2} Fred'(H)$, which therefore induces a Bott map $b(X) : RK_{\mathcal{G}}^{-n}(X) \to RK_{\mathcal{G}}^{-n-2}(X)$. By the definition of the relative groups, we also have Bott maps $b(X,A) : RK_{\mathcal{G}}^{-n}(X,A) \to RK_{\mathcal{G}}^{-n-2}(X,A)$. We will prove that these maps are isomorphisms for finite $\mathcal{G}$-CW-complexes.

Lemma 3.4.9. *Suppose that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a local equivalence. Then we have an isomorphism:*

$$F^* : RK_{\mathcal{H}}^*(H_0) \longrightarrow RK_{\mathcal{G}}^*(G_0)$$

Proof. $RK_{\mathcal{H}}^{-n}(H_0) = [H_0, \Omega^n Fred'(H)]_{\mathcal{H}}^{ext}$

$=$ Extendable $\mathcal{H}$-homotopy classes of extendable sections of $\Omega^n Fred'(H)$

$\cong$ Extendable $\mathcal{G}$-homotopy classes of extendable sections of $F^*(\Omega^n Fred'(H))$

$=$ Extendable $\mathcal{G}$-homotopy classes of extendable sections of $\Omega^n Fred'(F^*H)$

$= [G_0, \Omega^n Fred'(F^*H)]_{\mathcal{G}}^{ext}$

$= RK_{\mathcal{G}}^{-n}(G_0)$ □

Corollary 3.4.10. *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent, we have an isomorphism $RK_{\mathcal{H}}^*(H_0) \cong RK_{\mathcal{G}}^*(G_0)$*

Corollary 3.4.11. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid and U is a $\mathcal{G}$-cell, then $RK_{\mathcal{G}}^*(U) \cong K_G^*(M)$ for some compact Lie group G and a finite G-CW-complex M.*

Proof. Since U is a G-cell, we know that $\mathcal{G} \rtimes U$ is weakly equivalent to $G \rtimes M$ for some compact Lie group G and a finite G-CW-complex M. Therefore, by the previous corollary:

$$RK_{\mathcal{G}\rtimes U}^*(U) \cong RK_{G\rtimes M}^*(M)$$

Let H be a locally universal representation of $\mathcal{G}$. We want to see that $\pi_U^*(H) = U \times_\pi H$ is a locally universal $\mathcal{G} \rtimes U$-Hilbert bundle. Notice that if U is a $\mathcal{G}$-cell, so is any open $\mathcal{G}$-subspace of U. Therefore it is enough to prove the previous assertion with universal Hilbert bundles. So assume H is a universal $\mathcal{G}$-Hilbert bundle.

Now let V be a $\mathcal{G} \rtimes U$-vector bundle on U. This a $\mathcal{G}$-vector bundle on U, and since $\mathcal{G}$ is Bredon-compatible, there is a $\mathcal{G}$-vector bundle W on G_0 such that $\pi_U^*(W) = V \oplus V'$ for some other $\mathcal{G}$-vector bundle V' on U. Since H is universal, there is a unitary $\mathcal{G}$-embedding $W \hookrightarrow H$ and so

$\pi_U^*(W) \hookrightarrow \pi_U^*(H) = U \times_\pi H$. Since V is a direct summand of $\pi_U^*(W)$, we have a unitary $\mathcal{G} \rtimes U$-embedding $V \hookrightarrow U \times_\pi H$.

Thus, if E is a locally universal Hilbert representation of G, then $E \times M$ is a locally universal $G \rtimes M$-Hilbert bundle.

$RK^{-n}_{\mathcal{G}\rtimes U}(U) = [U, \Omega^n Fred'(U \times_\pi H)]^{ext}_{\mathcal{G}\rtimes U} =$

$= [U, U \times_\pi \Omega^n Fred'(H)]^{ext}_{\mathcal{G}\rtimes U} =$

$= (\mathcal{G} \rtimes U)$-extendable sections of $U \times_\pi \Omega^n Fred'(H)$ over $U =$

$= \mathcal{G}$-extendable sections of $U \times_\pi \Omega^n Fred'(H)$ over $U =$

$= [U, \Omega^n Fred'(H)]^{ext}_{\mathcal{G}} = RK^{-n}_{\mathcal{G}}(U)$

$RK^{-n}_{G\rtimes M}(M) = [M, \Omega^n Fred'(M \times E)]^{ext}_{G\rtimes M} =$

$= [M, M \times \Omega^n Fred'(E)]^{ext}_{G\rtimes M} =$

$= (G \rtimes M)$-extendable sections of $M \times \Omega^n Fred'(E)$ over $M =$

$= (G \rtimes M)$-sections of $M \times \Omega^n Fred'(E)$ over $M =$

$= [M, \Omega^n Fred'(E)]_G = K^{-n}_G(M)$

Therefore, $RK^{-n}_{\mathcal{G}}(U) \cong K^{-n}_G(M)$ □

Theorem 3.4.12. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, the Bott homomorphism*

$$b = b(X, A) : RK^{-n}_{\mathcal{G}}(X, A) \longrightarrow RK^{-n-2}_{\mathcal{G}}(X, A)$$

is a natural isomorphism for any finite $\mathcal{G}$-CW-pair (X, A) and all $n \geq 0$.

Proof. Assume first that $X = Y \cup_\phi (U \times D^m)$ where $U \times D^m$ is a $\mathcal{G}$-cell. Assume inductively that $b(Y)$ is an isomorphism. Since $RK_{\mathcal{G}}^{-n}(U \times S^{m-1})$ is isomorphic to $RK_G^{-n}(M \times S^{m-1})$ and $RK_{\mathcal{G}}^{-n}(U \times D^m) \cong RK_G^{-n}(M \times D^m)$, the Bott homomorphisms $b(U \times S^{m-1})$ and $b(U \times D^m)$ are isomorphisms by the equivariant Bott periodicity theorem for actions of compact Lie groups. The Bott map is natural and compatible with the boundary operators in the Mayer-Vietoris sequence for Y, X, $U \times S^{m-1}$ and $U \times D^m$ and so $b(X)$ is an isomorphism by the 5-lemma. The proof that $b(X, A)$ is an isomorphism follows immediately from the definitions of the relative groups. $\square$

Based on the Bott isomorphism we just proved, we can now redefine for all $n \in \mathbb{Z}$

$$RK_{\mathcal{G}}^n(X, A) = \begin{cases} RK_{\mathcal{G}}^0(X, A) & \text{if } n \text{ is even} \\ RK_{\mathcal{G}}^{-1}(X, A) & \text{if } n \text{ is odd} \end{cases}$$

For any finite $\mathcal{G}$-CW-pair (X, A), define the boundary operator $\delta^n : RK_{\mathcal{G}}^n(A) \longrightarrow RK_{\mathcal{G}}^{n+1}(X, A)$ to be $\delta : K_{\mathcal{G}}^{-1}(A) \longrightarrow K_{\mathcal{G}}^0(X, A)$ if n is odd, and to be the composite

$$RK_{\mathcal{G}}^0(A) \xrightarrow{b} RK_{\mathcal{G}}^{-2}(A) \xrightarrow{\delta^{-2}} RK_{\mathcal{G}}^{-1}(X, A)$$

if n is even.

We can collect all the information we have about $\mathcal{G}$-equivariant representable K-theory in the following theorem:

Theorem 3.4.13. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, the groups $RK_{\mathcal{G}}^n(X, A)$ define a $\mathbb{Z}/2$-graded multiplicative cohomology theory on the category of finite $\mathcal{G}$-CW-pairs.*

Note that for a general finite Lie groupoid $\mathcal{G}$, $RK_{\mathcal{G}}^*(-)$ is a multiplicative cohomology theory on the category of $\mathcal{G}$-spaces, but it is not clear whether we have Bott periodicity.

Corollary 3.4.14. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid and U a $\mathcal{G}$-cell. Then $K^*_{\mathcal{G}}(U) \cong RK^*_{\mathcal{G}}(U)$*

Proof. If U is a $\mathcal{G}$-cell, then $\mathcal{G} \rtimes U$ is weakly equivalent to $G \rtimes M$ for some compact Lie group G and a finite G-CW-complex M. By corollary 3.3.3, we also have $K^*_{\mathcal{G}}(U) \cong K^*_G(M)$. By corollary 3.4.11, we also have $RK^*_{\mathcal{G}}(U) \cong K^*_G(M)$. □

Theorem 3.4.15. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid and X a finite $\mathcal{G}$-CW-complex. Then $K^*_{\mathcal{G}}(X) \cong RK^*_{\mathcal{G}}(X)$*

Proof. Assume first that $X = Y \cup_\phi (U \times D^m)$ where $U \times D^m$ is a $\mathcal{G}$-cell. Assume inductively that we have an isomorphism $K^*_{\mathcal{G}}(Y) \xrightarrow{\cong} RK^*_{\mathcal{G}}(Y)$. We know that $K^*_{\mathcal{G}}(U \times S^{m-1})$ is isomorphic to $RK^*_{\mathcal{G}}(U \times S^{m-1})$ and $K^*_{\mathcal{G}}(U \times D^m) \cong RK^*_{\mathcal{G}}(U \times D^m)$ by the previous corollary. In fact, since these last two isomorphisms follow from choosing a weak equivalence from the same $\mathcal{G}$-cell to the action of a compact Lie group on a finite equivariant CW-complex, these isomorphisms are natural with respect to the Mayer-Vietoris sequences for $RK^*_{\mathcal{G}}$ and $K^*_{\mathcal{G}}$. Let us denote the corresponding groups by $RA^{-n} = RK^{-n}_{\mathcal{G}}(Y) \oplus RK^{-n}_{\mathcal{G}}(U \times D^m)$, $A^{-n} = K^{-n}_{\mathcal{G}}(Y) \oplus K^{-n}_{\mathcal{G}}(U \times D^m)$, $RB^{-n} = RK^{-n}_{\mathcal{G}}(U \times S^{m-1})$ and $B^{-n} = K^{-n}_{\mathcal{G}}(U \times S^{m-1})$, then:

$$\begin{tikzcd}
A^{-n-1} \arrow[r] \arrow[d, "\cong"'] & B^{-n-1} \arrow[r] \arrow[d, "\cong"'] & K^{-n}_{\mathcal{G}}(X) \arrow[r] \arrow[d] & A^{-n} \arrow[r] \arrow[d, "\cong"'] & B^{-n} \arrow[d, "\cong"'] \\
RA^{-n-1} \arrow[r] & RB^{-n-1} \arrow[r] & RK^{-n}_{\mathcal{G}}(X) \arrow[r] & RA^{-n} \arrow[r] & RB^{-n}
\end{tikzcd}$$

and so the result follows by the 5-lemma. □

In other words, we have just proved that the cohomology theory $K^*_{\mathcal{G}}(-)$ is representable by extendable maps.

Corollary 3.4.16. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid, X a* *finite $\mathcal{G}$-CW-complex and H a stable representation of $\mathcal{G}$, then:*

$$K^n_{\mathcal{G}}(X) = \begin{cases} [X, Fred'(H)]^{ext}_{\mathcal{G}} & \text{if } n \text{ is even} \\ [X, \Omega Fred'(H)]^{ext}_{\mathcal{G}} & \text{if } n \text{ is odd} \end{cases}$$

We would like to make two observations:

1. Note that all constructions and results in this sections remain true if we relax the condition of $\mathcal{G}$ being finite to $\mathcal{G}$ having a locally universal Hilbert representation $U(\mathcal{G})$.

2. This cohomology theory should not be confused with the $\mathcal{G}$-equivariant representable K-theory defined in [16]. In their paper, they define $\mathcal{G}$-equivariant representable K-theory of X as the KK-groups associated to $C_0(X)$ and show that this is actually representable (by all $\mathcal{G}$-equivariant continuous maps) by a corresponding Fredholm bundle. Note that in our case only a special class of maps are considered to have a correspondence with extendable vector bundles, but the Fredholm bundles in both cases are equivalent.

Chapter 4

Twisted equivariant K-theory

4.1 Twisted extendable K-theory

Let X be a $\mathcal{G}$-space and $P \to X$ a $\mathcal{G}$-projective bundle on X. We can then construct the bundle $End(P)$ on X whose fibre at x is the vector space $End(H_x)$ of endomorphisms of a Hilbert space H_x such that $P_x = \mathbb{P}(H_x)$. Similarly, we can replace $End(H_x)$ by $Fred(H_x)$, the space of Fredholm operators from H_x to H_x, and define in this way a bundle $Fred(P) \to X$. Now consider the subbundle $Fred'(P)$ of Fredholm operators A such that $g \to gAg^{-1}$ is continuous for all $g \in G_1$ for which the expression makes sense.

Definition 4.1.1. We say that a $\mathcal{G}$-equivariant section s of $Fred'(P) \to X$ is extendable if there is another $\mathcal{G}$-equivariant section t such that $ts = v\pi_X$ for some section v of $Fred'(P) \to G_0$.

Definition 4.1.2. We say that a homotopy $H : X \times I \to Fred'(P)$ of $\mathcal{G}$-equivariant sections is extendable if each H_t is an extendable $\mathcal{G}$-equivariant section of $Fred'(P) \to X$.

Definition 4.1.3. Let P be a $\mathcal{G}$-stable projective bundle and X a $\mathcal{G}$-space. We define the $\mathcal{G}$-equivariant twisted extendable K-theory of X with twisting P to be the group of extendable homotopy classes of extendable $\mathcal{G}$-equivariant sections of $Fred'(P)$ and we denote it by ${}^PK_{\mathcal{G}}(X)$

In order to define the rest of the twisted extendable K-groups, we need to introduce the fibrewise iterated loop-space $\Omega^n_X Fred'(P)$, which is a $\mathcal{G}$-bundle on X whose fibre at x is $\Omega^n Fred'(H_x)$.

Definition 4.1.4. The extendable homotopy classes of sections of this bundle will be denoted by ${}^{P}K_{\mathcal{G}}^{-n}(X)$.

The groups ${}^{P}K_{\mathcal{G}}^{-n}(X)$ are functorially associated to the pair (X, P) and so an isomorphism $P \to P'$ of $\mathcal{G}$-stable projective bundles on X induces an isomorphism ${}^{P}K_{\mathcal{G}}^{-n}(X) \to {}^{P'}K_{\mathcal{G}}^{-n}(X)$ for all $n \geq 0$

Corollary 4.1.5. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid and P is a trivial $\mathcal{G}$-stable projective bundle on a finite $\mathcal{G}$-CW-complex X, then ${}^{P}K_{\mathcal{G}}^{*}(X) \cong K_{\mathcal{G}}^{*}(X)$.*

Proof. It follows from the representability of $\mathcal{G}$-equivariant K-theory, that is, corollary 3.4.16. □

Corollary 4.1.6. *Let P be a $\mathcal{G}$-stable projective bundle on Y. If the maps $f_0, f_1 : X \longrightarrow Y$ are $\mathcal{G}$-homotopic $\mathcal{G}$-maps between $\mathcal{G}$-spaces, then $f_0^*(P)$ is isomorphic to $f_1^*(P)$ and we have a commutative diagram:*

$$\begin{tikzcd} {}^{P}K_{\mathcal{G}}^{-n}(Y) \arrow[r, "f_0^*", "\cong"'] \arrow[dr, "f_1^*", "\cong"'] & {}^{f_0^*(P)}K_{\mathcal{G}}^{-n}(X) \arrow[d, "\cong"'] \\ & {}^{f_1^*(P)}K_{\mathcal{G}}^{-n}(X) \end{tikzcd}$$

for all $n \geq 0$.

Lemma 4.1.7. *Let*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

be a pushout square of $\mathcal{G}$-spaces and P a $\mathcal{G}$-stable projective bundle on X. Let $P_k = j_k^*(P)$ for $k = 1, 2$ and $P_A = (i_2)^*(P_2)$. Then there is a natural *exact sequence, infinite to the left*

$$\dots \xrightarrow{d^{-n-1}} {}^{P}K_{\mathcal{G}}^{-n}(X) \xrightarrow{j_1^*\oplus j_2^*} {}^{P_1}K_{\mathcal{G}}^{-n}(X_1) \oplus {}^{P_2}K_{\mathcal{G}}^{-n}(X_2) \xrightarrow{i_1^*-i_2^*} {}^{P_A}K_{\mathcal{G}}^{-n}(A) \xrightarrow{d^{-n}} \dots$$

$$\dots \longrightarrow {}^{P_A}K_{\mathcal{G}}^{-1}(A) \xrightarrow{d^{-1}} {}^{P}K_{\mathcal{G}}^{0}(X) \xrightarrow{j_1^*\oplus j_2^*} {}^{P_1}K_{\mathcal{G}}^{0}(X_1) \oplus {}^{P_2}K_{\mathcal{G}}^{0}(X_2) \xrightarrow{i_1^*-i_2^*} {}^{P_A}K_{\mathcal{G}}^{0}(A)$$

Proof. The proof is essentially the same as that of lemma 3.4.8. □

If we use a mod 2 graded version $Fred^{(0)}(\hat{P})$ of the bundle of Fredholm operators associated to a projective bundle P, as in [6], we have a multiplication:

$${}^{P}K_{\mathcal{G}}^{0}(X) \otimes {}^{P'}K_{\mathcal{G}}^{0}(X) \longrightarrow {}^{P\otimes P'}K_{\mathcal{G}}^{0}(X)$$

coming from the map $(A, A') \to A \otimes 1 + 1 \otimes A'$ defined on the spaces of degree 1 self-adjoint Fredholm operators. This extends the multiplication in untwisted $\mathcal{G}$-equivariant K-theory and makes ${}^{P}K_{\mathcal{G}}^{*}(X)$ into a $K_{\mathcal{G}}^{*}(X)$-module.

Just like in the case of representable K-theory, for any $\mathcal{G}$-stable Hilbert bundle there is a $\mathcal{G}$-map $\Omega_X^n Fred'(H) \to \Omega_X^{n+2} Fred'(H)$. Therefore, for any $\mathcal{G}$-stable projective bundle P on X there is a Bott map:

$$b(X,P) : {}^{P}K_{\mathcal{G}}^{-n}(X) \longrightarrow {}^{P}K_{\mathcal{G}}^{-n-2}(X)$$

We do not know if this map is an isomorphism in general. Now we will prove that $b(X,P)$ is an isomorphism when X is a finite $\mathcal{G}$-CW-complex using a similar argument to the one used for untwisted $\mathcal{G}$-equivariant K-theory.

Proposition 4.1.8. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a local equivalence, and P a $\mathcal{H}$-stable projective bundle on H_0. Then F induces an isomorphism $F^* : {}^{P}K_{\mathcal{H}}^{*}(H_0) \longrightarrow {}^{F^*(P)}K_{\mathcal{G}}^{*}(G_0)$*

Proof. First of all, $F^*(P)$ is a $\mathcal{G}$-stable projective bundle by corollary 2.8.10 and proposition 2.6.2. Since these groups are defined using sections, the result follows from proposition 2.6.3. □

Corollary 4.1.9. *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent and P is a $\mathcal{H}$-stable projective bundle on H_0, then ${}^{P}K^*_{\mathcal{H}}(H_0) \cong {}^{F^*(P)}K^*_{\mathcal{G}}(G_0)$*

Corollary 4.1.10. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, U is a $\mathcal{G}$-cell and P is a $\mathcal{G}$-stable projective bundle on U, then ${}^{P}K^*_{\mathcal{G}}(U) \cong {}^{Q}K^*_{G}(M)$ for some compact Lie group G, some finite G-CW-complex M and some G-stable projective bundle Q on M.*

Proof. Since U is a G-cell, we know that $\mathcal{G} \rtimes U$ is weakly equivalent to $G \rtimes M$ for some compact Lie group G and a finite G-CW-complex M. Therefore, by the previous corollary:

$$ {}^{P}K^*_{\mathcal{G}\rtimes U}(U) \cong {}^{Q}K^*_{G\rtimes M}(M) $$

for some $G \rtimes M$-stable projective bundle Q on M. In the proof of corollary 3.4.11, we saw that if H is a locally universal representation of $\mathcal{G}$, then $U \times_{\pi} H$ is a locally universal $\mathcal{G} \rtimes U$-Hilbert bundle. And also that if E is a locally universal Hilbert representation of G, then $E \times M$ is a locally universal $G \rtimes M$-Hilbert bundle. It follows that if P is a $\mathcal{G} \rtimes U$-stable projective bundle on U, then P is a $\mathcal{G}$-stable projective bundle on U. Similarly, if Q is a $G \rtimes M$-stable projective bundle on M, then Q is a G-stable projective bundle on M.

${}^{P}K^*_{\mathcal{G}\rtimes U}(U) =$

$=$ $(\mathcal{G} \rtimes U)$-extendable sections of $\Omega^n Fred'(P)$ over $U =$

$=$ $\mathcal{G}$-extendable sections of $\Omega^n Fred'(P)$ over $U =$

$= {}^{P}K^*_{\mathcal{G}}(U)$

${}^{Q}K^*_{G\rtimes M}(M) =$

$=$ $(G \rtimes M)$-extendable sections of $\Omega^n Fred'(Q)$ over $M =$

$= G$-sections of $\Omega^n Fred'(Q)$ over $M =$

$= {}^QK^*_G(M)$

Therefore, ${}^PK^*_{\mathcal{G}}(U) \cong {}^QK^*_G(M)$

$\square$

Theorem 4.1.11. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, the Bott homomorphism*

$$b = b(X, P) : {}^PK_{\mathcal{G}}^{-n}(X) \longrightarrow {}^PK_{\mathcal{G}}^{-n-2}(X)$$

is an isomorphism for any finite $\mathcal{G}$-CW-complex X, all $\mathcal{G}$-stable projective bundles on X and all $n \geq 0$.

Proof. Assume that $X = Y \cup_\phi (U \times D^m)$ where $U \times D^m$ is a $\mathcal{G}$-cell. Let P be a $\mathcal{G}$-stable projective bundle. Assume inductively that $b(Y, P|Y)$ is an isomorphism. Since ${}^{P|(U\times D^m)}K_{\mathcal{G}}^{-n}(U \times D^m) \cong {}^QK_G^{-n}(M \times D^m)$ and ${}^{P|(U\times S^{m-1})}K_{\mathcal{G}}^{-n}(U \times S^{m-1}) \cong {}^{Q|(M\times S^{m-1})}K_G^{-n}(M \times S^{m-1})$, the Bott homomorphisms $b(U \times S^{m-1}, P_{|(U\times S^{m-1})})$ and $b(U \times D^m, P_{|(U\times D^m)})$ are isomorphisms by the Bott periodicity theorem in twisted equivariant K-theory for actions of compact Lie groups [6]. The Bott map is natural and compatible with the boundary operators in the Mayer-Vietoris sequence for Y, X, $U \times S^{m-1}$ and $U \times D^m$ and so $b(X, P)$ is an isomorphism by the 5-lemma. $\square$

Based on the Bott isomorphism we just proved, we can now redefine for all $n \in \mathbb{Z}$

$${}^PK^n_{\mathcal{G}}(X) = \begin{cases} {}^PK^0_{\mathcal{G}}(X) & \text{if } n \text{ is even} \\ {}^PK^{-1}_{\mathcal{G}}(X) & \text{if } n \text{ is odd} \end{cases}$$

We can collect all the information we have so far about twisted $\mathcal{G}$-equivariant K-theory in the following theorem:

Theorem 4.1.12. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, the groups ${}^PK^n_{\mathcal{G}}(X)$ define a $\mathbb{Z}/2$-graded cohomology theory on the category of finite*

$\mathcal{G}$-CW-complexes with $\mathcal{G}$-stable projective bundles, which is a module over untwisted $\mathcal{G}$-equivariant K-theory.

Note that for a general Lie groupoid $\mathcal{G}$, $K^*_{\mathcal{G}}(-)$ is a cohomology theory on the category of $\mathcal{G}$-spaces, but it is not clear whether we have Bott periodicity.

All constructions and results in this section are true if we relax the condition of $\mathcal{G}$ finite to $\mathcal{G}$ admitting a locally universal $\mathcal{G}$-representation.

4.2 The orbit category

Recall from section 2.4 that a $\mathcal{G}$-cell is a $\mathcal{G}$-space U for which the groupoid $\mathcal{G} \rtimes U$ is weakly equivalent to the action of a compact Lie group G on a finite G-CW-complex.

The orbit category $O_{\mathcal{G}}$ is a topological category with discrete object space formed by the $\mathcal{G}$-cells. The morphisms are the $\mathcal{G}$-maps, with a topology such that the evaluation maps $Hom_{\mathcal{G}}(U, V) \times U \to V$ are continuous for all $\mathcal{G}$-cells U, V. By an $O_{\mathcal{G}}$-space we shall mean a continuous contravariant functor from $O_{\mathcal{G}}$ to the category of topological spaces.

Definition 4.2.1. Let X be a $\mathcal{G}$-space. The fixed point set system of X, written ΦX, is an $O_{\mathcal{G}}$-space defined by $\Phi X(U) = Map_{\mathcal{G}}(U, X)$ and given $\Theta : U \to V$, $\Phi X(\Theta)(f) = f\Theta$. We also denote $X^U = Map_{\mathcal{G}}(U, X)$.

Definition 4.2.2. A CW-$O_{\mathcal{G}}$-space is an $O_{\mathcal{G}}$-space T such that each space $T(U)$ is a CW-complex and each structure map $T(U) \to T(V)$ is cellular.

Theorem 4.2.3. *There is a functor C : $O_{\mathcal{G}}$-spaces $\longrightarrow$ $\mathcal{G}$-spaces and a natural transformation $\eta : \Phi C \to Id$ such that for each $O_{\mathcal{G}}$-space T and each U, $\eta : (CT)^U \to T(U)$ is a homotopy equivalence, in fact a strong deformation retraction.*

Proof. We first construct the $\mathcal{G}$-space CT. Let O_T denote the topological category whose objects are triples (U, s, y) where U is a $\mathcal{G}$-cell,

$s \in J(U) \equiv U$ and $y \in T(U)$. Let us consider the nerve of this category as a topological simplicial space. This is the bar complex $B_*(T, O_{\mathcal{G}}, J)$, where $J : O_{\mathcal{G}} \to Top$ is the covariant functor which forgets the $\mathcal{G}$-action.

Then $B_n(T, O_{\mathcal{G}}, J)$ consist of $(n+2)$-tuples $(y, f_1, f_2, \dots, f_n, s)$ where the $f_i : U_i \to U_{i-1}$ are composable arrows in $O_{\mathcal{G}}$, $s \in J(U_n) \equiv U_n$ and $y \in T(U_0)$. The boundary maps are given by:

$$\partial_0(y, f_1, f_2, \dots, f_n, s) = (f_1^*(y), f_2, f_3, \dots, f_n, s)$$

$$\partial_n(y, f_1, f_2, \dots, f_n, s) = (y, f_1, f_2, \dots, f_{n-1}, (f_n)_*(s))$$

$$\partial_i(y, f_1, f_2, \dots, f_n, s) = (y, f_1, f_2, \dots, f_{i-1}, f_i f_{i+1}, f_{i+2}, \dots, f_n, s)$$

Degeneracies are the insertion of identity maps in the appropriate spots. The groupoid $\mathcal{G}$ acts simplicially on $B_*(T, O_{\mathcal{G}}, J)$ and consequently the geometric realization $B(T, O_{\mathcal{G}}, J)$ is a $\mathcal{G}$-space. We define $CT = B(T, O_{\mathcal{G}}, J)$.

We now require the homotopy equivalence $\eta : (CT)^U \to T(U)$ for each $\mathcal{G}$-cell U, natural in U. We have:

$$(CT)^U = B(T, O_{\mathcal{G}}, J)^U = B(T, O_{\mathcal{G}}, Hom_{\mathcal{G}}(U, -))$$

The second equality follows from the fact that $\mathcal{G}$ acts on the last coordinate only. Now it is a general property of the bar construction that for any topological category C, contravariant functor $F : C \to Top$ and object A of C, there is a natural map

$$\eta : B(F, C, Hom_C(A, -)) \longrightarrow F(A)$$

which is a strong deformation retraction. This map is induced by a simplicial map

$$\eta_* : B_*(F, C, Hom_C(A, -)) \longrightarrow F(A)_*$$

where $F(A)_*$ is the simplicial space all of whose components are $F(A)$ and all whose face and degeneracy maps are the identity. In our case, η_* is given by the formula:

$$\eta_n(y, f_1, f_2, \ldots, f_n, f) = (f_1 \circ \ldots \circ f_n \circ f)^*(y)$$

Now f is an element of $Hom_{O_{\mathcal{G}}}(U, U_n)$. The proof that η is a strong deformation retraction is a standard simplicial argument contained in chapter 12 of [29]. □

4.3 The classification of projective bundles

Now we follow [6] closely. Let us write $\mathrm{Pic}_{\mathcal{G}}(X)$ for the group of isomorphism classes of complex $\mathcal{G}$-line bundles on X (or equivalently, of principal S^1-bundles on X with $\mathcal{G}$-action), and $\mathrm{Proj}_{\mathcal{G}}(X)$ for the group of isomorphism classes of $\mathcal{G}$-stable projective bundles. Applying the Borel construction to line bundles and projective bundles gives us homomorphisms:

$$Pic_{\mathcal{G}}(X) \longrightarrow Pic(X_{\mathcal{G}}) \cong H^2_{\mathcal{G}}(X; \mathbb{Z})$$

$$Proj_{\mathcal{G}}(X) \longrightarrow Proj(X_{\mathcal{G}}) \cong H^3_{\mathcal{G}}(X; \mathbb{Z})$$

which we shall show are bijective.

Definition 4.3.1. A topological abelian $\mathcal{G}$-module is a $\mathcal{G}$-space such that each of the fibres of its anchor map is a topological abelian group and the action is linear.

Example 4.3.2. Given any topological abelian group B, $G_0 \times B$ is a $\mathcal{G}$-module with the anchor map given by projection on the first coordinate. When there is no danger of confusion we will denote it by B.

Let us introduce groups $H^*_{\mathcal{G}}(X; A)$ defined for any abelian $\mathcal{G}$-module A. These are the hypercohomology groups of a simplicial space χ whose realization is the space $X_{\mathcal{G}}$. Whenever a Lie groupoid $\mathcal{G}$ acts on a space X we can define the action groupoid whose space of objects is X and space of morphisms is $G_1 \times_\pi X$. Let χ be the nerve of this groupoid regarded as a simplicial space, that is, $\chi_p = G_p \times_\pi X$, where G_p is the space of composable p-tuples of arrows in $\mathcal{G}$.

For any simplicial space with an action of a Lie groupoid $\mathcal{G}$ and any topological abelian $\mathcal{G}$-module A we can define the hypercohomology $\mathbb{H}^*(\chi; sh(A))$ with coefficients in the sheaf of continuous equivariant A-valued functions. It is the cohomology of a double complex $C^{\cdot\cdot}$, where, for each $p \geq 0$, the cochain complex $C^{p\cdot}$ calculates $H^*(\chi_p; sh(A))$.

Definition 4.3.3. $H^*_{\mathcal{G}}(X; A) = \mathbb{H}^*(\chi; sh(A))$

These groups are the abutment of a spectral sequence with $E_1^{pq} = H^q(G_p \times_\pi X; sh(A))$.

Lemma 4.3.4. *If $\mathcal{G}$ is a Bredon-compatible Lie groupoid and X is a finite $\mathcal{G}$-CW-complex, $H^{p+1}_{\mathcal{G}}(X; \mathbb{Z}) \cong H^p_{\mathcal{G}}(X; S^1)$ for any $p > 0$.*

Proof. If we compare the spectral sequences for the $\mathcal{G}$-CW-structure of X with respect to the cohomology theories $H^{p+1}_{\mathcal{G}}(-; \mathbb{Z})$ and $H^p_{\mathcal{G}}(-; S^1)$, we notice that we have an isomorphism in each cell by the similar result in [6]. □

Proposition 4.3.5. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid and X a $\mathcal{G}$-space. Then we have:*

1. $H^2_{\mathcal{G}}(G_0; \mathbb{Z}) \cong Hom(\mathcal{G}, G_0 \times S^1)$
2. $H^3_{\mathcal{G}}(G_0; \mathbb{Z}) \cong Ext(\mathcal{G}, G_0 \times S^1)$, *the group of equivalence classes of central extensions* $1 \to G_0 \times S^1 \to \mathcal{H} \to \mathcal{G} \to 1$.
3. $H^2_{\mathcal{G}}(X; \mathbb{Z}) \cong Pic_{\mathcal{G}}(X)$
4. $H^3_{\mathcal{G}}(X; \mathbb{Z}) \cong Proj_{\mathcal{G}}(X)$

Proof. 1. We have $E_1^{0q} = H^q(G_0; sh(G_0 \times S^1)) \cong H^q(pt; sh(S^1)) = 0$ when $X = G_0$ and in the previous lemma we have seen that

$E_2^{p0} = H^p_{c.c.}(\mathcal{G}; S_1)$ is the cohomology of $\mathcal{G}$ defined by continuous Eilenberg-Maclane cochains. So

$$H^2_{\mathcal{G}}(G_0; \mathbb{Z}) = H^1_{\mathcal{G}}(G_0; S^1) \cong E_2^{10} \cong H^1_{c.c.}(\mathcal{G}; S_1) \cong Hom(\mathcal{G}, G_0 \times S^1)$$

2. In this case the spectral sequence gives us an exact sequence

$$0 \to E_2^{20} \to H^2_{\mathcal{G}}(G_0; S^1) \to E_2^{11} \to E_2^{30}$$

that is,

$$0 \to H^2_{c.c.}(\mathcal{G}; S_1) \to H^2_{\mathcal{G}}(G_0; S^1) \to Pic(\mathcal{G})_{prim} \to H^3_{c.c.}(\mathcal{G}; S_1)$$

for $E_1^{11} = H^1(\mathcal{G}; sh(G_0 \times S^1)) = Pic(\mathcal{G})$, and E_2^{11} is the subgroup of primitive elements, that is, of circle bundles $\mathcal{H}$ on $\mathcal{G}$ such that $m^*\mathcal{H} \cong pr_1^{\mathcal{H}} \otimes pr_2^*\mathcal{H}$, where $pr_1, pr_2, m : G_1 \times_{G_0} G_1 \longrightarrow G_1$ are the obvious maps. Equivalently, $Pic(\mathcal{G})_{prim}$ consists of circle bundles $\mathcal{H}$ on $\mathcal{G}$ equipped with bundle maps $\tilde{m} : H_1 \times_{G_0} H_1 \longrightarrow H_1$ covering the multiplication in $\mathcal{G}$. It is easy to see that the composite

$$Ext(\mathcal{G}, G_0 \times S^1) \to H^2_{\mathcal{G}}(G_0; G_0 \times S^1) \to Pic(\mathcal{G})$$

takes an extension to its class as a circle bundle. On the other hand $H^2_{c.c.}(\mathcal{G}; S^1)$ is just the group of equivalence classes of extensions $G_0 \times S^1 \to \mathcal{H} \to \mathcal{G}$ which as circle bundles admit a continuous section, so its image in $\mathrm{Ext}(\mathcal{G}, G_0 \times S^1)$ is precisely the kernel of this composite. It remains only to show that the image of $\mathrm{Ext}(\mathcal{G}, G_0 \times S^1)$ in $\mathrm{Pic}(\mathcal{G})_{prim}$ is the kernel of $Pic(\mathcal{G})_{prim} \longrightarrow H^3_{c.c.}(\mathcal{G}; S_1)$. This map associates to a bundle $\mathcal{H}$ with a bundle map $\tilde{m}$ as above precisely the obstruction to changing $\tilde{m}$ by a bundle map $G_1 \times_{G_0} G_1 \longrightarrow S^1$ to make it an associative product on $\mathcal{H}$.

3. The spectral sequence gives

$$0 \to E_2^{10} \to H^1_{\mathcal{G}}(X;S^1) \to E_2^{01} \to E_2^{20}$$

Now $E_1^{01} = Pic(X)$, and E_2^{01} is the subgroup of circle bundles $S \longrightarrow X$ which admit a bundle map $\tilde{m} : G_1 \times_{G_0} S \longrightarrow S$ covering the $\mathcal{G}$-action on X. As before, $\tilde{m}$ can be made into a $\mathcal{G}$-action on S if and only if an obstruction in $H^2_{c.c.}(\mathcal{G}; Map(X, G_0 \times S^1))$ vanishes. Finally, the kernel of $Pic_{\mathcal{G}}(X) \longrightarrow Pic(X)$ is the group of $\mathcal{G}$-actions on $X \times S^1$, and this is just $E_2^{10} = H^1_{c.c.}(\mathcal{G}; Map(X, G_0 \times S^1))$.

4. First we shall prove that the map $Proj_{\mathcal{G}}(X) \longrightarrow H^3_{\mathcal{G}}(X;\mathbb{Z})$ is injective. Consider the filtration

$$Proj_{\mathcal{G}}(X) \supseteq Proj^{(1)} \supseteq Proj^{(0)}$$

Here $Proj^{(1)}$ consists of the stable projective bundles which are trivial when the $\mathcal{G}$-action is forgotten, that is, those that can be described by cocycles $\alpha : G_1 \times_{G_0} X \longrightarrow PU(\mathcal{H})$ that satisfy the condition $\alpha(g_2, g_1 x)\alpha(g_1, x) = \alpha(g_2 g_1, x)$.

$Proj^{(0)}$ consists of those projective bundles for which α lifts to a map $\bar{\alpha} : G_1 \times_{G_0} X \longrightarrow U(\mathcal{H})$ satisfying the equality $\alpha(g_2, g_1 x)\alpha(g_1, x) = c(g_2, g_1, x)\alpha(g_2 g_1, x)$ for some $c : G_2 \times_\pi X \longrightarrow S^1$

We shall compare the filtration of $\mathrm{Proj}_{\mathcal{G}}(X)$ with the filtration

$$H^2_{\mathcal{G}}(X;S^1) \supset H^{(1)} \supset H^{(0)}$$

defined by the spectral sequence. By definition $H^{(1)}$ is the kernel of $H^2_{\mathcal{G}}(X;S_1) \longrightarrow E_1^{02} = H^2(X; sh(S_1)) = Proj(X)$, and the composite $Proj_{\mathcal{G}} \to H^2_{\mathcal{G}}(X;S_1) \to Proj(X)$ is clearly the map which forgets the $\mathcal{G}$-action. Thus $Proj_{\mathcal{G}}(X)/Proj^{(1)}$ maps injectively to $H^2_{\mathcal{G}}(X,S^1)/H^{(1)}$

Now let us consider the map $Proj^{(1)} \longrightarrow H^{(1)}$. The subgroup $H^{(0)}$ is the kernel of $H^{(1)} \longrightarrow E_2^{11}$, while $E_1^{11} = Pic(G_1 \times_{G_0} X)$. We readily check that an element of $Proj^{(1)}$ defined by the cocycle α maps to the element of $Pic(G_1 \times_{G_0} X)$ which is the pullback of the circle bundle $U(\mathcal{H}) \longrightarrow PU(\mathcal{H})$, and can conclude that α maps to zero in E_2^{11} if and only if it defines an element of $Proj^{(0)}$. Thus $Proj^{(1)}/Proj^{(0)}$ injects into $H^{(1)}/H^{(0)}$. Finally, assigning to an element α of $Proj^{(0)}$ the class in $E_2^{20} = H^2_{c.c.}(\mathcal{G}; Map(X, G_0 \times S^1))$ of the cocycle c, we see that if this class vanishes, then the projective bundle comes from a $\mathcal{G}$-Hilbert bundle, which is necessarily trivial, as we have already explained. So $Proj^{(0)}$ injects into $H^{(0)}$.

Now we will construct a universal $\mathcal{G}$-space $C(P)$ with a natural $\mathcal{G}$-stable projective bundle on it and show that the composite map

$$[X, C(P)]_{\mathcal{G}} \to Proj_{\mathcal{G}}(X) \to H^3_{\mathcal{G}}(X; \mathbb{Z})$$

is an isomorphism.

Let U be a $\mathcal{G}$-CW-cell and consider the spaces

$$P(U) = \coprod_{H \in Ext(U, S^1)} BPU(H)^U$$

where we represent an element of $Ext(U, S^1)$ by a Hilbert bundle H with a stable projective representation of $\mathcal{G} \rtimes U$ inducing the extension. Note that even though there may be different choices of H, they all determine the same class in $H^3_{\mathcal{G} \rtimes U}(U; \mathbb{Z})$, and so this space $P(U)$ is well-defined.

In fact, P is an $O_{\mathcal{G}}$-space. Now, we can use theorem 4.2.3 to construct the $\mathcal{G}$-space $C(P)$. This space satisfies $C(P)^U \simeq P(U)$ for every $\mathcal{G}$-cell U. Also, it carries a tautological $\mathcal{G}$-stable projective bundle, and so we have a $\mathcal{G}$-map $C(P) \to Map(E\mathcal{G}, BPU(H))$ into the space that represents the functor $X \to H^3_{\mathcal{G}}(X,\mathbb{Z})$. This map induces an isomorphism

$$[X, C(P)]_{\mathcal{G}} \longrightarrow H^3_{\mathcal{G}}(X,\mathbb{Z})$$

it is enough to check the cases $X = U \times S^i$, where U is a $\mathcal{G}$-cell. In fact, since finite G-CW-complexes are built out of spaces of the form $G/H \times S^i$, where H is a closed subgroup of G, it is enough to check this for the cases $X = U \times S^i$, where $\mathcal{G} \rtimes U$ is weakly equivalent to $G \rtimes G/H$. But this reduces to proving the isomorphism $\pi_i(P(U)) \cong H^{3-i}(B(\mathcal{G} \rtimes U),\mathbb{Z})$, which follows from the diagram:

$$\begin{tikzcd}
\pi_i(P(U)) \arrow[r] \arrow[d, "\cong"'] & H^{3-i}(B(\mathcal{G} \rtimes U),\mathbb{Z}) \arrow[d, "\cong"'] \\
\pi_i(P_H) \arrow[r, "\cong"] & H^{3-i}(BH,\mathbb{Z})
\end{tikzcd}$$

where the map in the bottom row is an isomorphism by the results in [6].

□

Chapter 5

The completion theorem

5.1 The completion map

Lemma 5.1.1. *Let $B^n\mathcal{G} = E^n\mathcal{G}/\mathcal{G}$. Any product of n elements in $K^*(B^n\mathcal{G}, G_0)$ is zero.*

Proof. Consider the subsets $U_i = \{\lambda_1 g_1 + ... + \lambda_n g_n \mid \lambda_i \neq 0\}$ in $E^n\mathcal{G}$ and let $\tilde{U}_i = U_i/\mathcal{G}$. These sets are open in $B^n\mathcal{G}$. They are also homotopy equivalent to G_0:

$$U_j \xrightarrow{f} G_1 \qquad\qquad G_1 \xrightarrow{h} U_j$$

$$[\Sigma\lambda_i g_i] \longrightarrow g_j \qquad\qquad g \longrightarrow g_j = g$$

$$U_j \times I \xrightarrow{H} U_j$$

$$([\Sigma\lambda_i g_i], t) \longrightarrow [\Sigma t\lambda_i g_i + (1-t)g_j]$$

These maps are $\mathcal{G}$-equivariant, so they define maps $\tilde{f} : \tilde{U}_j \longrightarrow G_0$, $\tilde{h} : G_0 \longrightarrow \tilde{U}_j$ and $\tilde{H} : \tilde{U}_j \times I \longrightarrow \tilde{U}_j$. We have $\tilde{f}\tilde{h} = 1_{G_0}$ and $\tilde{H}$ is a homotopy between $1_{\tilde{U}_j}$ and $\tilde{h}\tilde{f}$. We can see G_0 inside all the U_i's.

If $n = 2$, consider the following commutative diagram:

$$\begin{tikzcd}
K(B^2\mathcal{G}, U_1) \otimes K(B^2\mathcal{G}, U_2) \arrow[r] \arrow[d, "\cong"'] & K(B^2\mathcal{G}, U_1 \cup U_2) = 0 \arrow[d] \\
K(B^2\mathcal{G}, U_1) \otimes K(B^2\mathcal{G}, G_0) \arrow[r] \arrow[d, "\cong"'] & K(B^2\mathcal{G}, U_1) \arrow[d] \\
K(B^2\mathcal{G}, G_0) \otimes K(B^2\mathcal{G}, G_0) \arrow[r] & K(B^2\mathcal{G}, G_0)
\end{tikzcd}$$

Similarly, consider the diagram:

$$\begin{tikzcd}
K(B^n\mathcal{G}, U_1) \otimes K(B^n\mathcal{G}, U_2) \otimes \ldots \otimes K(B^n\mathcal{G}, U_n) \arrow[r] \arrow[d, "\cong"'] & K(B^n\mathcal{G}, U_1 \cup \ldots \cup U_n) = 0 \arrow[d] \\
\vdots \arrow[d, "\cong"'] & \vdots \arrow[d] \\
K(B^n\mathcal{G}, U_1) \otimes K(B^n\mathcal{G}, G_0) \otimes \ldots \otimes K(B^n\mathcal{G}, G_0) \arrow[r] \arrow[d, "\cong"'] & K(B^n\mathcal{G}, U_1) \arrow[d] \\
K(B^n\mathcal{G}, G_0) \otimes K(B^n\mathcal{G}, G_0) \otimes \ldots \otimes K(B^n\mathcal{G}, G_0) \arrow[r] & K(B^n\mathcal{G}, G_0)
\end{tikzcd}$$

□

Lemma 5.1.2. *If $\mathcal{G}$ acts freely on X and G_0 is a finite $\mathcal{G}$-CW-complex, then $K_{\mathcal{G}}(X) \cong K(X/\mathcal{G})$.*

Proof. Given $x \in X$, there exists a sufficiently small neighbourhood $U_{\pi(x)}$ of $\pi(x)$ such that $\mathcal{G}_{\pi(x)}$ acts on $U_{\pi(x)}$. Then $\mathcal{G}_{\pi(x)}$ acts on $U_x = \pi^{-1}(U_{\pi(x)})$. $\mathcal{G}_{\pi(x)}$ is a compact Lie group and so there is a local slice for that action at

x. □

We have an augmentation map $K^*_{\mathcal{G}}(G_0) \to K^*(G_0)$ given by forgetting the $\mathcal{G}$-action. Let $I_{\mathcal{G}}$ be the kernel of this map.

The composite homomorphism

$$K^*_{\mathcal{G}}(G_0) \to K^*_{\mathcal{G}}(E^n\mathcal{G}) \cong K^*(B^n\mathcal{G}) \to K^*(G_0)$$

is the augmentation map, whose kernel is $I_{\mathcal{G}}$. Therefore, the map $K^*_{\mathcal{G}}(G_0) \to K^*_{\mathcal{G}}(E^n\mathcal{G})$ factors through $K^*_{\mathcal{G}}(G_0)/I^n_{\mathcal{G}}$.

For any $\mathcal{G}$-space X, $K^*_{\mathcal{G}}(X)$ is a module over $K^*_{\mathcal{G}}(G_0)$ and by naturality the homomorphism $K^*_{\mathcal{G}}(X) \to K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})$ factorizes through

$$K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}K^*_{\mathcal{G}}(X) \to K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})$$

Conjecture 5.1.3. *Let $\mathcal{G}$ be a Lie groupoid and X a $\mathcal{G}$-space . Then we have an isomorphism of pro-rings*

$$\{K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}K^*_{\mathcal{G}}(X)\} \longrightarrow \{K^*(X \times_\pi E^n\mathcal{G}/\mathcal{G})\}$$

If a groupoid $\mathcal{G}$ satisfies this conjecture for $X = G_0$ we will say that $\mathcal{G}$ satisfies the completion theorem.

5.2 The completion theorem

Lemma 5.2.1. *Let $\mathcal{G} = G \rtimes X$, where G is a compact Lie group and X is a G-CW-complex such that $K^*_G(X)$ is finite over $R(G)$. Then $\mathcal{G}$ satisfies the completion theorem.*

Proof. Let I_X be the kernel of $K^*_G(X) \longrightarrow K^*(X)$. We would like to prove that there is an isomorphism $\{K^*_G(X)/I^n_X\} \cong \{K^*_G(X \times E^nG)\}$. By the Atiyah-Segal completion theorem, we have an isomorphism $\{K^*_G(X)/I^n_GK^*_G(X)\} \cong \{K^*_G(X \times E^nG)\}$. So it suffices to prove that the

I_G-adic topology and the I_X-adic topology are the same in $K^*_G(X)$. Since $K^*_G(X)$ is a module over $R(G)$, we have $I_G K^*_G(X) \subset I_X$.

Let K_n be the kernel of $\alpha_n : K^*_G(X) \longrightarrow K^*_G(X \times E^nG)$. As a corollary to the Atiyah-Segal completion theorem, we know that the sequence of ideals $\{K_n\}$ defines the I_G-adic topology on $K^*_G(X)$. In particular, there is $m \in \mathbb{N}$ such that $K_m \subset I_G K^*_G(X)$. Note that $K_1 = I_X$. Consider the composition $K^*_G(X) \longrightarrow K^*_G(X \times E^mG) \longrightarrow K^*_G(X \times E^1G) \cong K^*(X)$. Since $X \times E^mG$ is the union of m open sets which are G-homotopy equivalent to $X \times E^1G = X \times G$, we have $K^*_G(X \times E^mG, X \times E^1G)^m = 0$. Thus the first map factors through I^m_X, thus $I^m_X \subset K_m$. Hence $I^m_X \subset I_G K^*_G(X)$. □

Lemma 5.2.2. *If $\mathcal{G}$ and $\mathcal{H}$ are locally equivalent, then $\mathcal{G}$ satisfies the completion theorem if and only if $\mathcal{H}$ does.*

Proof. If $\mathcal{G}$ and $\mathcal{H}$ are locally equivalent by a local equivalence $\mathcal{H} \longrightarrow \mathcal{G}$, then we have an isomorphism $f : K^*_{\mathcal{G}}(G_0) \xrightarrow{\cong} K^*_{\mathcal{H}}(H_0)$. The following diagram is commutative:

$$\begin{array}{ccc} K^*_{\mathcal{G}}(G_0) & \xrightarrow{f} & K^*_{\mathcal{H}}(H_0) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ K^*(G_0) & \xrightarrow{\phi} & K^*(H_0) \end{array}$$

Therefore have $f(I_{\mathcal{G}}) \subset I_{\mathcal{H}}$. Let g be the inverse of f, $x \in I_{\mathcal{H}}$ and $y = g(x)$. Since $\beta(x) = 0$, we have $\phi\alpha(y) = 0$. But since $\alpha(y) = (n_y, a_y) \in \mathbb{Z} \oplus \tilde{K}^*(G_0) = K^*(G_0)$ and so $\phi\alpha(y) = (n_y, \tilde{\phi}(a_y))$. This implies in particular $n_y = 0$, that is, $\alpha g(I_{\mathcal{H}}) \subset \tilde{K}^*(G_0)$.

Let $m \in \mathbb{N}$ such that $\tilde{K}^*(G_0)^m = 0$. Then, $\alpha g(I^m_{\mathcal{H}}) \subset \tilde{K}^*(G_0)^m = 0$ and so $g(I^m_{\mathcal{H}}) \subset I_{\mathcal{G}}$. Thus $I^m_{\mathcal{H}} = fg(I^m_{\mathcal{H}}) \subset f(I_{\mathcal{G}})$ and the topologies induced by $I_{\mathcal{G}}$ and $I_{\mathcal{H}}$ on $K^*_{\mathcal{H}}(H_0)$ are the same. Therefore we have an isomorphism of pro-rings $\{K^*(G_0)/I^n_{\mathcal{G}}\} \cong \{K^*(H_0)/I^n_{\mathcal{H}}\}$.

The local equivalence also induces a homotopy equivalence between $B\mathcal{G}$ and $B\mathcal{H}$. If we consider the associated filtrations to each of these spaces, $\{B^n\mathcal{G}\}$ and $\{B^n\mathcal{H}\}$, the homotopy equivalence must take $B^n\mathcal{G}$ to some $B^{n+k}\mathcal{H}$ and $B^n\mathcal{H}$ to some $B^{n+k'}\mathcal{G}$. Hence we have an isomorphism of pro-rings $\{K^*(B^n\mathcal{G})\} \cong \{K^*(B^n\mathcal{H})\}$. The lemma follows then by looking at the diagram:

$$\begin{tikzcd} \{K^*(G_0)/I_{\mathcal{G}}^n\} \arrow[r, "\cong"] \arrow[d] & \{K^*(H_0)/I_{\mathcal{H}}^n\} \arrow[d] \\ \{K^*(B^n\mathcal{G})\} \arrow[r, "\cong"] & \{K^*(B^n\mathcal{H})\} \end{tikzcd} \quad \square$$

From the previous lemma, we obtain the following theorem:

Theorem 5.2.3. *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent, then $\mathcal{G}$ satisfies the completion theorem if and only if $\mathcal{H}$ does.*

In particular, we have this corollary:

Corollary 5.2.4. *If $\mathcal{G}$ is Bredon-compatible and U is a $\mathcal{G}$-cell, $K_{\mathcal{G}}^*(U)$ is a finitely generated abelian group and the groupoid $\mathcal{G} \rtimes U$ satisfies the completion theorem.*

This corollary tells us that the completion theorem is true for $\mathcal{G}$-cells. Now we move on to prove this for finite $\mathcal{G}$-CW-complexes.

Let X be a finite $\mathcal{G}$-CW-complex and consider the spectral sequences for the maps $f : X \to X/\mathcal{G}$ and $h : X \times_\pi E\mathcal{G} \to X/\mathcal{G}$ in $\mathcal{G}$-equivariant K-theory:

$$E_1^{pq} = \prod_{i \in I_p} K_{\mathcal{G}}^q(f^{-1}U_i) \Longrightarrow K_{\mathcal{G}}^{p+q}(X)$$

$$\bar{E}_1^{pq} = \prod_{i \in I_p} K_{\mathcal{G}}^q(h^{-1}U_i) \Longrightarrow K_{\mathcal{G}}^{p+q}(X_{\mathcal{G}})$$

Now, since $h^{-1}(U) = f^{-1}(U) \times_\pi E\mathcal{G}$ there is a map of spectral sequences

$E \to \bar{E}$ induced by the projections $f^{-1}(U) \times_\pi E\mathcal{G} \to f^{-1}(U)$.

The following lemma will be useful in what follows.

Lemma 5.2.5. *Fix any commutative Noetherian ring A, and any ideal $I \subset A$. Then for any exact sequence $M' \to M \to M''$ of finitely generated A-modules, the sequence:*

$$\{M'/I^n M'\} \longrightarrow \{M/I^n M\} \longrightarrow \{M''/I^n M''\}$$

of pro-groups is exact.

Proof. See [28], section 4. □

The spectral sequence E is a spectral sequence of $K^*_{\mathcal{G}}(G_0)$-modules. From this point we assume that $\mathcal{G}$ is finite. Note that this implies that $K^*_{\mathcal{G}}(G_0)$ is a Noetherian ring. All elements in these spectral sequences are finitely generated over $K^*_{\mathcal{G}}(G_0)$. By the previous lemma, the functor taking a $K^*_{\mathcal{G}}(G_0)$-module M to the pro-group $\{M/I^n_{\mathcal{G}} M\}$ is exact and so we can form the following spectral sequence of pro-rings:

$$F_1^{pq} = \{\prod_{i \in I_p} K^q_{\mathcal{G}}(f^{-1}U_i)/I^n_{\mathcal{G}} K^q_{\mathcal{G}}(f^{-1}U_i)\} \Longrightarrow \{K^{p+q}_{\mathcal{G}}(X)/I^n_{\mathcal{G}} K^{p+q}_{\mathcal{G}}(X)\}$$

Similarly consider the maps $h_n : X \times_\pi E^n\mathcal{G} \longrightarrow X/\mathcal{G}$. They give us another spectral sequence of pro-rings:

$$\bar{F}_1^{pq} = \{\prod_{i \in I_p} K^q_{\mathcal{G}}(h_n^{-1}U_i)\} \Longrightarrow \{K^{p+q}_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})\}$$

We have a map of spectral sequences $\phi : F \longrightarrow \bar{F}$.

If $\mathcal{G}$ is Bredon-compatible, the groupoids $\mathcal{G} \rtimes f^{-1}(U_i)$ satisfy the completion theorem for all i. Since we are taking quotient by the ideal $I_{\mathcal{G}}$ and not by $I_{\mathcal{G} \rtimes f^{-1}(U_i)}$, we need to check that both topologies coincide. We consider the long exact sequence in equivariant and non-equivariant K-theory for the pair (C_V, V) where V is any $f^{-1}(U_i)$ and C_V is the mapping cylinder of the map $\pi : V \longrightarrow G_0$. Note that C_V is $\mathcal{G}$-homotopy equivalent to G_0 and $V \subset C_V$.

$$\begin{tikzcd}
K_{\mathcal{G}}(C_U, U) \arrow[r] \arrow[d] & K_{\mathcal{G}}(G_0) \arrow[r] \arrow[d] & K_{\mathcal{G}}(U) \arrow[r] \arrow[d] & K^1_{\mathcal{G}}(C_U, U) \arrow[d] \\
K(C_U, U) \arrow[r] & K(G_0) \arrow[r] & K(U) \arrow[r] & K^1(C_U, U)
\end{tikzcd}$$

Let $I_V = I_{\mathcal{G} \rtimes f^{-1}(U_i)}$. It is clear that $I_{\mathcal{G}} K_{\mathcal{G}}(U) \subset I_V$. Now let $m \in \mathbb{N}$ such that $K(C_V, V)^m = 0$ and $n \in \mathbb{N}$ such that $K_{\mathcal{G}}(C_V, V)^n = 0$. Then $I_V^{nm} \subset I_{\mathcal{G}} K_{\mathcal{G}}(V)$ and so the topologies coincide.

This proves ϕ is an isomorphism when restricted to any particular element F^{ij} and therefore, it is an isomorphism of spectral sequences. In particular, we have $\{K^{p+q}_{\mathcal{G}}(X)/I^n_{\mathcal{G}} K^{p+q}_{\mathcal{G}}(X)\} \cong \{K^{p+q}_{\mathcal{G}}(X \times_\pi E^n \mathcal{G})\}$

Theorem 5.2.6. *Let $\mathcal{G}$ be a Bredon-compatible Lie finite groupoid and X a finite $\mathcal{G}$-CW-complex. Then we have an isomorphism of pro-rings*

$$\{K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}} K^*_{\mathcal{G}}(X)\} \longrightarrow \{K^*(X \times_\pi E^n \mathcal{G}/\mathcal{G})\}$$

Corollary 5.2.7. *Under the same circumstances, the homomorphism $K^*_{\mathcal{G}}(X) \to K^*(X_{\mathcal{G}})$ induces an isomorphism of the $I_{\mathcal{G}}$-adic completion of $K^*_{\mathcal{G}}(X)$ with $K^*(X_{\mathcal{G}})$.*

5.3 The twisted completion theorem

For any $\mathcal{G}$-stable projective bundle P on a $\mathcal{G}$-space X, consider the $\mathcal{G}$-stable projective bundle $P \times_\pi E^n \mathcal{G}$ on $X \times_\pi E^n \mathcal{G}$. The following diagram commutes:

$$\begin{tikzcd} P \times_\pi E^n\mathcal{G} \arrow[r] \arrow[d] & P \arrow[d] \\ X \times_\pi E^n\mathcal{G} \arrow[r] & X \end{tikzcd}$$

Therefore we have a map:

$${}^{P}K^*_{\mathcal{G}}(X) \longrightarrow {}^{P\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})$$

${}^{P}K^*_{\mathcal{G}}(X)$ is a module over $K^*_{\mathcal{G}}(G_0)$ and ${}^{P\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})$ is a module over $K^*_{\mathcal{G}}(E^n\mathcal{G})$. In fact we have a commutative diagram:

$$\begin{tikzcd} K^*_{\mathcal{G}}(G_0) \arrow[r] \arrow[d] & K^*_{\mathcal{G}}(E^n\mathcal{G}) \arrow[d] \\ {}^{P}K^*_{\mathcal{G}}(X) \arrow[r] & {}^{P\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G}) \end{tikzcd}$$

From the setup for untwisted K-theory in section 5.1, we know that the last map factors through $I^n_{\mathcal{G}}$ and therefore, by naturality we have a map:

$${}^{P}K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}{}^{P}K^*_{\mathcal{G}}(X) \longrightarrow {}^{P\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})$$

We can also look at these maps as a map of pro-$K^*_{\mathcal{G}}(G_0)$-modules:

$$\{{}^{P}K^*_{\mathcal{G}}(X)/I^n_{\mathcal{G}}{}^{P}K^*_{\mathcal{G}}(X)\} \longrightarrow \{{}^{P\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E^n\mathcal{G})\}$$

Taking limits we obtain a map of $K^*_{\mathcal{G}}(G_0)$-modules:

$${}^{P}K^*_{\mathcal{G}}(X)^{\wedge}_{I_G} \longrightarrow {}^{P\times_\pi E\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E\mathcal{G})$$

Conjecture 5.3.1. *Let $\mathcal{G}$ be a finite Lie groupoid, X a $\mathcal{G}$-space and P a $\mathcal{G}$-stable projective bundle. Then we have an isomorphism of $K^*_{\mathcal{G}}(G_0)$-modules:*

$${}^{P}K^*_{\mathcal{G}}(X)^{\wedge}_{I_G} \longrightarrow {}^{P\times_\pi E\mathcal{G}}K^*_{\mathcal{G}}(X \times_\pi E\mathcal{G})$$

If a groupoid $\mathcal{G}$ satisfies this conjecture for $X = G_0$ and all $\mathcal{G}$-stable projective bundles on G_0, we will say $\mathcal{G}$ satisfies the twisted completion theorem.

In what follows, we will use the following result from [26], stated here in a simpler form which suffices for our purpose:

Theorem 5.3.2. *Let X be a finite G-CW-complex, where G is a compact Lie group. Then ${}^{P}K_G^*(X)$ is finitely generated over $R(G)$ and the projection $\pi : EG \times X \to X$ induces an isomorphism:*

$$ {}^{P}K_G^*(X)^{\wedge}_{I_G} \xrightarrow{\cong} {}^{P\times EG}K_G^*(EG \times X) $$

for any G-stable projective bundle P on X.

Lemma 5.3.3. *Let $\mathcal{G} = G \rtimes X$, where G is a compact Lie group and X is a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Then $\mathcal{G}$ satisfies the twisted completion theorem.*

Proof. It follows from lemma 5.2.1 and the completion theorem for twisted equivariant K-theory for actions of compact Lie groups, that is, theorem 5.3.2. □

Lemma 5.3.4. *If $\mathcal{G}$ and $\mathcal{H}$ are locally equivalent, then $\mathcal{G}$ satisfies the twisted completion theorem if and only if $\mathcal{H}$ does.*

Proof. Let P be a $\mathcal{H}$-stable projective bundle on H_0. Then, $F^*(P)$ is $\mathcal{G}$-stable. If $\mathcal{G}$ and $\mathcal{H}$ are locally equivalent by a local equivalence $F : \mathcal{H} \longrightarrow \mathcal{G}$, then we have an isomorphism $f : {}^{F^*P}K_{\mathcal{G}}^*(G_0) \xrightarrow{\cong} {}^{P}K_{\mathcal{H}}^*(H_0)$. The following diagram is commutative:

$$\begin{tikzcd}
K_{\mathcal{G}}^*(G_0) \arrow[r, "\cong"] \arrow[d] & K_{\mathcal{H}}^*(H_0) \arrow[d] \\
{}^{F^*P}K_{\mathcal{G}}^*(G_0) \arrow[r, "\cong"] & {}^{P}K_{\mathcal{H}}^*(H_0)
\end{tikzcd}$$

By lemma 5.2.2, the topologies induced by $I_{\mathcal{G}}$ and $I_{\mathcal{H}}$ are the same and therefore we have an isomorphism of pro-rings:

$$\{{}^{F^*P}K^*_{\mathcal{G}}(G_0)/I^n_{\mathcal{G}}{}^{F^*P}K^*_{\mathcal{G}}(G_0)\} \cong \{{}^{P}K^*_{\mathcal{H}}(H_0)/I^n_{\mathcal{H}}{}^{P}K^*_{\mathcal{H}}(H_0)\}$$

The local equivalence also induces a local equivalence between the groupoids $\mathcal{G} \rtimes E\mathcal{G}$ and $\mathcal{H} \rtimes E\mathcal{H}$ and it takes $E^n\mathcal{H}$ to $E^n\mathcal{G}$. Therefore we have a homomorphism of pro-rings $\{{}^{F^*(P\times_\pi E^n\mathcal{H})}K^*_{\mathcal{G}}(E^n\mathcal{G})\} \cong \{{}^{P\times_\pi E^n\mathcal{H}}K^*_{\mathcal{H}}(E^n\mathcal{H})\}$, which is an isomorphism in the limit. It is also the case that $F^*(P \times_\pi E^n\mathcal{H}) = F^*(P) \times_\pi E^n\mathcal{G}$ and so we have a commutative diagram:

$$\begin{tikzcd}
\{{}^{F^*P}K^*_{\mathcal{G}}(G_0)/I^n_{\mathcal{G}}{}^{F^*P}K^*_{\mathcal{G}}(G_0)\} \arrow[r, "\cong"] \arrow[d] & \{{}^{P}K^*_{\mathcal{H}}(H_0)/I^n_{\mathcal{H}}{}^{P}K^*_{\mathcal{H}}(H_0)\} \arrow[d] \\
\{{}^{F^*(P)\times_\pi E^n\mathcal{G}}K^*_{\mathcal{G}}(E^n\mathcal{G})\} \arrow[r] & \{{}^{P\times_\pi E^n\mathcal{H}}K^*_{\mathcal{H}}(E^n\mathcal{H})\}
\end{tikzcd}$$

The lemma follows then by looking at the diagram:

$$\begin{tikzcd}
{}^{F^*P}K^*_{\mathcal{G}}(G_0)^{\wedge}_{I_{\mathcal{G}}} \arrow[r, "\cong"] \arrow[d] & {}^{P}K^*_{\mathcal{H}}(H_0)^{\wedge}_{I_{\mathcal{H}}} \arrow[d] \\
{}^{F^*(P)\times_\pi E\mathcal{G}}K^*_{\mathcal{G}}(E\mathcal{G}) \arrow[r, "\cong"] & {}^{P\times_\pi E\mathcal{H}}K^*_{\mathcal{H}}(E\mathcal{H})
\end{tikzcd} \quad \square$$

From the previous lemma, we obtain the following theorem:

Theorem 5.3.5. *If $\mathcal{G}$ and $\mathcal{H}$ are weakly equivalent, then $\mathcal{G}$ satisfies the twisted completion theorem if and only if $\mathcal{H}$ does.*

Now from this theorem, lemma 5.3.3 and theorem 5.3.2, we obtain this corollary:

Corollary 5.3.6. *If $\mathcal{G}$ is a Bredon-compatible finite Lie groupoid, U is a $\mathcal{G}$-cell and P is a stable $\mathcal{G}$-projective bundle on U, ${}^{P}K^*_{\mathcal{G}}(U)$ is a finitely generated abelian group and the groupoid $\mathcal{G} \rtimes U$ satisfies the completion theorem.*

This corollary tells us that the twisted completion theorem is true for $\mathcal{G}$-cells. Now we move on to prove this for finite $\mathcal{G}$-CW-complexes.

Let X be a finite $\mathcal{G}$-CW-complex and P a $\mathcal{G}$-stable projective bundle on X. Consider the spectral sequence for the maps $f : X \to X/\mathcal{G}$ in twisted $\mathcal{G}$-equivariant K-theory with twisting given by the restrictions of P:

$$E_1^{pq} = \prod_{i \in I_p} {}^{f^*(P_i)}K^q_{\mathcal{G}}(f^{-1}U_i) \Longrightarrow {}^{P}K^{p+q}_{\mathcal{G}}(X)$$

The spectral sequence E is a spectral sequence of $K^*_{\mathcal{G}}(G_0)$-modules. Assume $\mathcal{G}$ is a finite groupoid so that $K^*_{\mathcal{G}}(G_0)$ is a Noetherian ring. All elements in these spectral sequences are finitely generated over $K^*_{\mathcal{G}}(G_0)$. The functor taking a $K^*_{\mathcal{G}}(G_0)$-module M to the $K^*_{\mathcal{G}}(G_0)$-module $M^\wedge_{I_{\mathcal{G}}}$ is exact [28] and so we can form the following spectral sequence of $K^*_{\mathcal{G}}(G_0)$-modules.

$$F_1^{pq} = \prod_{i \in I_p} {}^{Q_i}K^q_{\mathcal{G}}(f^{-1}U_i)^\wedge_{I_{\mathcal{G}}} \Longrightarrow {}^{P}K^{p+q}_{\mathcal{G}}(X)^\wedge_{I_{\mathcal{G}}}$$

where $Q_i = f^*(P_i)$. Similarly consider the map $h : X \times_\pi E\mathcal{G} \longrightarrow X/\mathcal{G}$. It gives us another spectral sequence of $K^*_{\mathcal{G}}(G_0)$-modules:

$$\bar{F}_1^{pq} = \prod_{i \in I_p} {}^{Q_i \times_\pi E\mathcal{G}}K^q_{\mathcal{G}}(h^{-1}U_i) \Longrightarrow {}^{P \times_\pi E\mathcal{G}}K^{p+q}_{\mathcal{G}}(X \times_\pi E\mathcal{G})$$

since $h^*(P_i) = Q_i \times_\pi E\mathcal{G}$. We have $h^{-1}(U) = f^{-1}(U) \times_\pi E\mathcal{G}$ so there is a map of spectral sequences $F \to \bar{F}$ induced by the projections onto the first coordinate $f^{-1}(U) \times_\pi E\mathcal{G} \to f^{-1}(U)$.

If $\mathcal{G}$ is Bredon-compatible, the groupoids $\mathcal{G} \rtimes f^{-1}(U_i)$ satisfy the twisted completion theorem for all i. From the previous section, we know that the topologies determined by the groupoid $\mathcal{G}$ and $\mathcal{G} \rtimes U_i$ on $K^*_{\mathcal{G}}(f^{-1}U_i)$ are the same and therefore they are the same on ${}^{Q_i}K^*_{\mathcal{G}}(f^{-1}(U_i))$.

This proves ϕ is an isomorphism when restricted to any particular element F^{ij} and therefore, it is an isomorphism of spectral sequences. In particular, we have ${}^{P}K_{\mathcal{G}}^{p+q}(X)^{\wedge}_{I_{\mathcal{G}}} \cong {}^{P\times_{\pi}E\mathcal{G}}K_{\mathcal{G}}^{p+q}(X \times_{\pi} E\mathcal{G})$

Theorem 5.3.7. *Let $\mathcal{G}$ be a Bredon-compatible finite Lie groupoid, X a finite $\mathcal{G}$-CW-complex and P a $\mathcal{G}$-stable projective bundle on X. Then we have an isomorphism of $K_{\mathcal{G}}^{*}(G_0)$-modules:*

$${}^{P}K_{\mathcal{G}}^{n}(X)^{\wedge}_{I_{\mathcal{G}}} \longrightarrow {}^{P\times_{\pi}E\mathcal{G}}K_{\mathcal{G}}^{n}(X \times_{\pi} E\mathcal{G})$$

Chapter 6

Proper actions of Lie groups

Throughout this whole chapter S will be a Lie group, but not necessarily compact. To study proper actions of S, we can consider the groupoid $\mathcal{G} = S \rtimes \underline{E}S$, where $\underline{E}S$ is the universal space for proper actions of S as defined in [27]. This space is a proper S-CW-complex such that $\underline{E}S^G$ is contractible for all compact Lie subgroups G of S. The existence of $\underline{E}S$ is shown in [27]. It is also shown there that every proper S-CW-complex has an S-map to $\underline{E}S$ and this map is unique up to S-homotopy. Some immediate consequences follow:

- Proper S-CW-complexes are $\mathcal{G}$-CW-complexes.
- Extendable $\mathcal{G}$-bundles on a proper S-CW-complex X are extendable S-bundles for any S-map $X \longrightarrow \underline{E}S$, since all of them are S-homotopic.
- $E\mathcal{G} = \underline{E}S \times ES$ and this space is S-homotopy equivalent to ES, so $B\mathcal{G}$ is homotopy equivalent to BS.
- $\mathcal{G}$ is finite if and only if $\underline{E}S$ is a finite proper S-CW-complex.
- Extendable $\mathcal{G}$-sections on a proper S-CW-complex X are extendable S-sections for any S-map $X \longrightarrow \underline{E}S$, since all of them are S-homotopic.
- If H is a locally universal S-Hilbert representation, then $\underline{E}S \times H$ is a locally universal $\mathcal{G}$-Hilbert bundle.
- Stable $\mathcal{G}$-projective bundles on X are stable S-projective bundles on X.

By abuse of language, we say that proper actions of S are Bredon-compatible if the corresponding groupoid $\mathcal{G} = S \rtimes \underline{E}S$ is Bredon-compatible.

For such actions, the following results follow from the corresponding results for groupoids. We will denote $K^*_S(X) = K^*_{\mathcal{G}}(X)$ and ${}^PK^*_S(X) = {}^PK^*_{\mathcal{G}}(X)$.

Theorem 6.0.8. *If S is a Lie group with Bredon-compatible proper actions, the groups $K^n_S(X, A)$ define a $\mathbb{Z}/2$-graded multiplicative proper cohomology theory on the category of finite S-CW-pairs.*

Theorem 6.0.9. *If S is a Lie group with Bredon-compatible proper actions and a finite model for $\underline{E}S$, the groups $RK^n_S(X, A)$ define a $\mathbb{Z}/2$-graded multiplicative proper cohomology theory on the category of finite proper S-CW-pairs.*

Corollary 6.0.10. *Let S be a Lie group with Bredon-compatible proper actions and a finite model for $\underline{E}S$, X a finite S-CW-complex and H a stable representation of S, then:*

$$K^n_S(X) = \begin{cases} [X, Fred'(H)]^{ext}_S & \text{if } n \text{ is even} \\ [X, \Omega Fred'(H)]^{ext}_S & \text{if } n \text{ is odd} \end{cases}$$

Theorem 6.0.11. *If S is a Lie group with Bredon-compatible proper actions and a finite model for $\underline{E}S$, the groups ${}^PK^n_S(X)$ define a $\mathbb{Z}/2$-graded proper cohomology theory on the category of finite proper S-CW-complexes with S-stable projective bundles, which is a module over untwisted S-equivariant K-theory.*

Theorem 6.0.12. *Let S be a Lie group with Bredon-compatible proper actions and a finite model for $\underline{E}S$ and X a finite S-CW-complex. Then we have an isomorphism of pro-rings*

$$\{K^*_S(X)/I^n_S K^*_S(X)\} \longrightarrow \{K^*(X \times_\pi E^n S/S)\}$$

Theorem 6.0.13. *Let S be a Lie group with Bredon-compatible proper actions and a finite model for $\underline{E}S$, X a finite S-CW-complex and P a S-stable projective bundle on X. Then we have an isomorphism of $K^*_S(\underline{E}S)$-modules:*

$${}^PK^n_S(X)^\wedge_{I_S} \longrightarrow {}^{P\times_\pi ES}K^n_S(X \times_\pi ES)$$

All actions of finite groups and compact Lie groups are shown to be Bredon-compatible in [40]. Equivariant K-theory for these actions was introduced in [40]. It is a well-known fact that for these actions are Bredon-compatible [40]. The completion theorem in untwisted K-theory was proven in [5] and for twisted K-theory, it was recently proven in [26].

K-theory for proper actions of discrete groups was constructed in [28]. In this paper, actions of discrete groups are shown to be Bredon-compatible and a completion theorem is proven under some conditions. Twisted K-theory for proper actions of discrete groups for some particular twistings was defined in [14], but no completion theorem existed up to date in the literature.

In general, vector bundles may not be enough to construct an interesting equivariant cohomology theory for proper actions of second countable locally compact groups [35], but they suffice for two important families, almost compact groups and matrix groups [34].

Almost compact groups, that is, second countable locally compact groups whose group of connected components are compact, always have a maximal compact subgroup. Any space with a proper action of one of these groups is the induction of a space with an action of that compact subgroup and so the study of proper actions of almost compact groups are reduced to studying compact Lie group actions. This is carried on in [34], and so these action groupoids are Bredon-compatible and we have a completion theorem. In fact, this is also proved in [34], by showing that the completion maps are compatible with the reduction map to the maximal compact subgroup. With different techniques it is proven that proper actions of matrix groups, that is, closed subgroups of $GL(n,\mathbb{R})$, are Bredon-compatible and so a completion theorem follows. A particular instance of this case are proper actions of abelian Lie groups. Using the associated groupoids, we now can define twisted K-theory for actions of these groups and a completion theorem.

Proper actions of totally disconnected groups that are projective limits of discrete groups are shown to be Bredon-compatible in [38], where the corresponding K-groups are introduced. Since that theory coincides with the one constructed here, we now have a completion theorem for such actions. The previous constructions give a way of defining twisted K-theory and the corresponding results yield a completion theorem. One particular example is given by the group $SL_2(\hat{\mathbb{Z}}_p)$.

In general $\mathcal{G}$ need not be a Bredon-compatible groupoid. When $\mathcal{G} = S \rtimes \underline{E}S$ is a Bredon-compatible groupoid we must have $Vect_{\mathcal{G}}(S/G) = Vect_G(pt)$ for a compact subgroup G of S. Let S be a Kac-Moody group and T its maximal torus. Note that S is not a Lie group, but our constructions could be generalized to the context of topological groupoids.

There is an S-map $S/T \longrightarrow \underline{E}S$ which is unique, up to homotopy. Given an S-vector bundle V on $\underline{E}S$, the pullback to S/T is given by a finite-dimensional representation of T invariant under the Weyl group. This representation gives rise to a finite-dimensional representation of S. But this representation must be trivial. In particular, extendable S-vector bundles on S/T only come from trivial representations of T.

In order to deal with these groups, it is more convenient to use dominant K-theory, which was developed in [22]. Kac-Moody groups possess an important class of representations called dominant representations. A dominant representation of a Kac-Moody group in a Hilbert space is one that decomposes into a sum of highest weight representations. Equivariant K-theory for proper actions of Kac-Moody groups is defined as the representable equivariant cohomology theory modeled on the space of Fredholm operators on a Hilbert space which is a maximal dominant representation of the group. It is expected that twisted dominant K-theory can be defined in the same way using a corresponding Fredholm bundle over a projective bundle which is stable with respect to a suitable Hilbert space of dominant representations.

An example of a Kac-Moody group which is relevant to the work of Freed, Hopkins and Teleman [17, 18, 19] is the group $K(A) = T \rtimes \tilde{L}G$ ([22], section 8) associated to the loop group LG of a simply connected simple compact Lie group G, where T is a circle acting by rotation of the loops. The classifying spaces for proper actions of LG and $K(A)$ are the same. Let us call that space $X(A)$, following the notation in [22]. The $K(A)$-equivariant dominant K-theory of $X(A)$ can be identified with the Verlinde algebra of G, which is generated by projective representations of LG. The based loop group ΩG does not have any nontrivial compact subgroups and $X(A)/\Omega G$ is homeomorphic to G via the holonomy map [36]. This map carries the action of LG on $X(A)$ to the conjugation action of G on itself. Further calculations in section 5 of [22] imply that the $K(A)$-equivariant dominant K-theory of $X(A)$ is isomorphic to the twisted G-equivariant K-theory of G with the conjugation action.

The question arises whether it is possible to generalize this to generalized Kac-Moody algebras, introduced by R. Borcherds in [7]. These algebras have the potential to define topological groups (Borcherds groups) by amalgamation using the same tools as in [25], and also have highest weight representations, which would give rise to the dominant representations of the group. We could use these representations to construct an equivariant cohomology theory for proper actions of these groups. A completion map is also possible for dominant K-theory [23], although in this case the topology could not induced by an ideal of the base ring in general. This would be something worth studying in the case of Borcherds groups, to obtain some knowledge about the homotopy type of their classifying spaces.

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