A Quick Introduction to the Algebraic Theory of Anyons

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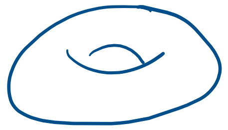
Introduction

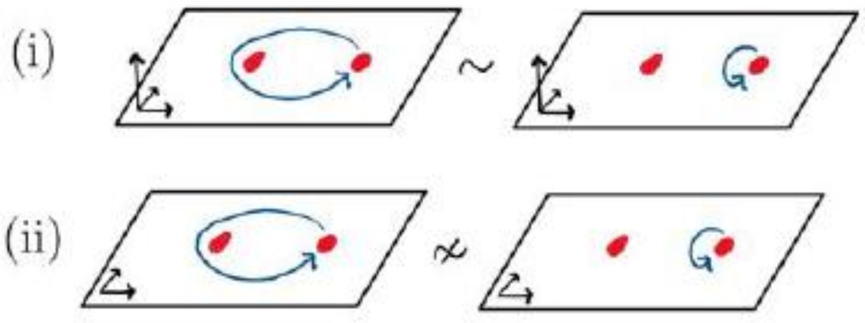
- Robbert Dijkgraaf : The liberation of quantum mechanics and the physics of what 'can be' rather than 'is'.
  - ↳ Topological phases of matter !
- Anyons : "Particles" in flatland ! (2 spatial dimensions)
  - ↳ Leinaas & Myrheim (Oslo, 1977) ↳ Nakamura et al. : abelian anyons (2020)
  - ↳ Wilczek (1982)
  - ↳ Störmer, Tsui & Gossard : fractional quantum hall effect (FQHE) (1982)
  - ↳ Moore & Read (1991)
  - ↳ Thouless, Haldane, Kosterlitz : topological phases (2016)

Introduction

- On the other side of the coin.
  - ↳ Lord Kelvin (19th century) : matter as knotted vortices in the aether? No.
  - ↳ Jones (1987) : Polynomial invariant of knots
  - ↳ Witten (1989) : Connecting Jones polynomial & Chern-Simons (2+1)-TQFTs
  - ↳ Quantum groups, quantum invariants, TQFTs, categories...
- Also,
  - ↳ Kitaev, Freedman, Larsen, Wang (late 90s, early 00s) : topological quantum computation
    - Fault-tolerant qubits
    - Topological quantum error-correction /memories



- Rough proof of concept for why 2D is special :



$$\boxed{(d \geq 3) \qquad \text{(i)} \quad \hat{P}^2 = 1 \Rightarrow \hat{P} = \pm 1}$$

$$(d = 2) \qquad \text{(ii)} \quad \hat{P} = e^{i\theta}$$

(i) All fundamental particles are either BOSONS or FERMIONS

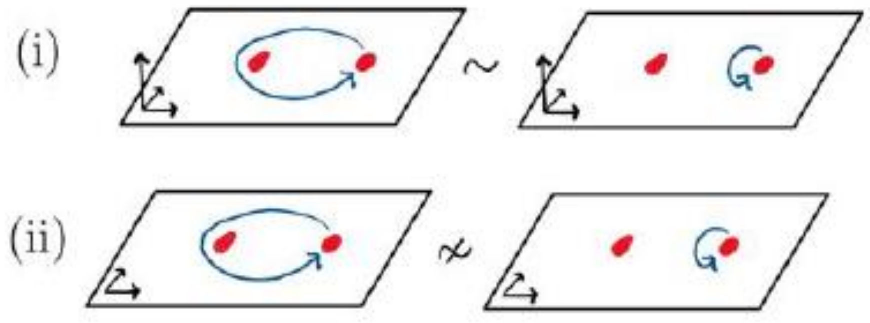
↳ Spin-statistics : Integer vs. half-integer spin s, $e^{i2\pi s} = \pm 1$

↳ Dirac belt trick

↳ Pauli exclusion principle

Introduction

- Rough proof of concept for why 2D is special :



$(d \geq 3)$ (i) $\hat{P}^2 = \mathbb{1} \Rightarrow \hat{P} = \pm \mathbb{1}$

$$\boxed{(d=2) \quad \text{(ii)} \ \hat{P} = e^{i\theta}}$$

(ii) Anyons : fractional statistics !

↳ Orientation of exchange matters

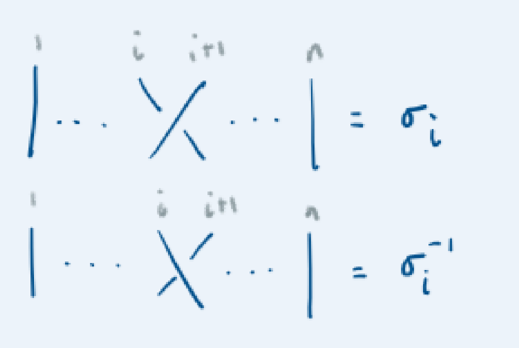
↳ Spin-statistics analogue (later...)

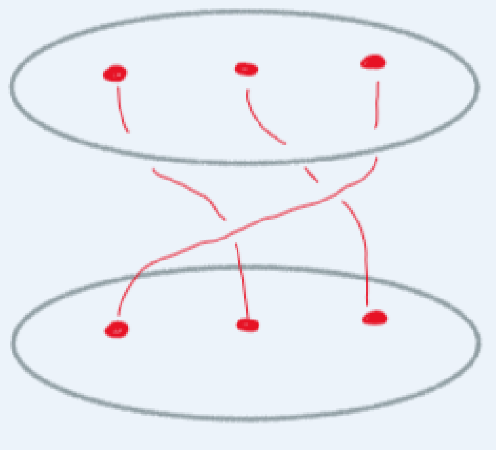
↳ Can have abelian vs. nonabelian statistics !

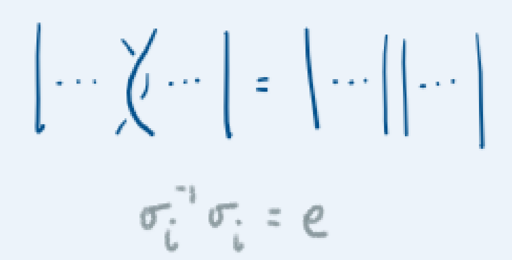
Introduction

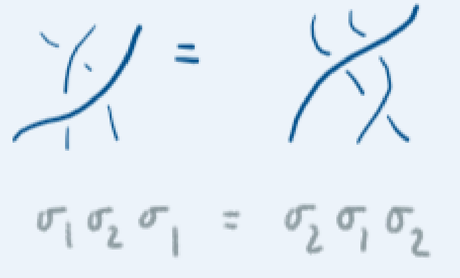
- Can represent the exchange trajectories of particles in spacetime using worldlines.
  - ↳ For $d=2$, these look like braids.

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \,, \quad |i-j| \geq 2 \end{array} \right\rangle$$









$$S_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{c} s_i^2 = e \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \,, \quad |i-j| \geq 2 \end{array} \right\rangle$$

  - ↳ For $d \geq 3$, we just have permutations.

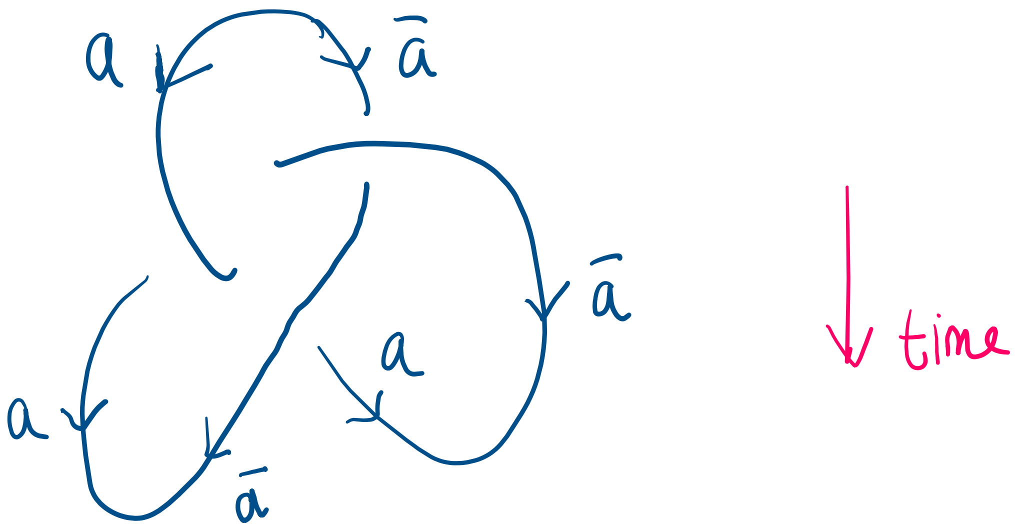
Insensitivity to orientation of exchange : $\eta : B_n \to S_n$, $\ker(\eta) = PB_n$

Introduction

- Anyons can be pair-created from or annihilate to the vacuum.

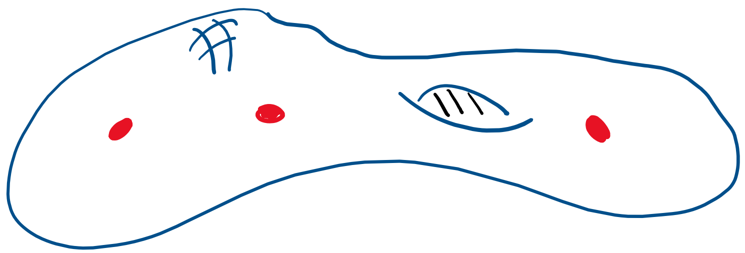
$\therefore$ Can get knots and links spanned by anyons in spacetime!

e.g.



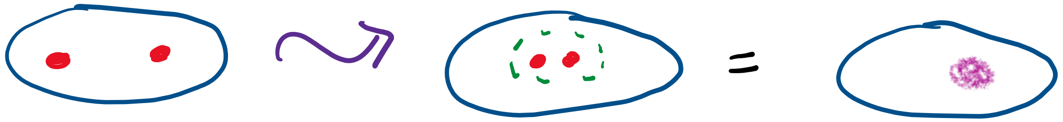
Anyon Diagrammatics

- Recall: Anyons can be thought of as mobile, localised 'blobs' sitting on a surface,



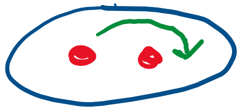
- There are 3 key operations that can be performed on anyons.

(1) FUSION.

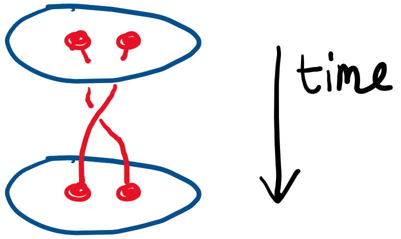


It's possible for there to be a superposition of fusion outcomes!

(2) BRAIDING.

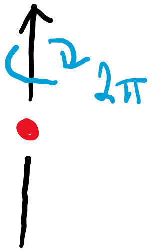


i.e.



Remember: exchange orientation matters in 2D!

(3) TWISTING.



i.e. 2π self-rotation of a specified orientation.

Anyon Diagrammatics

- We typically treat anyons as pointlike in our graphical calculus, and always represent processes / operations using worldlines (or 'string diagrams'): time flows downwards.

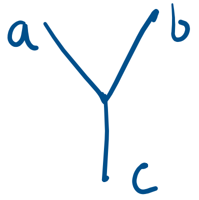
  ↳ Given any theory of anyons, we can have different "types" or "(topological) CHARGES" of anyons. There are finitely many, and we index them via a set of LABELS, $\mathcal{L}$.

  $$\mathcal{L} = \{\mathbb{1}, a, b, c, \dots\}$$

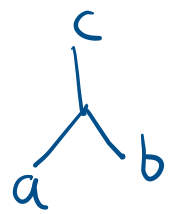
  $\mathbb{1}$ = vacuum (also written '0')

  $\therefore$ Must label all edges in our diagrams!

- Fusion is represented by a TRIVALENT VERTEX

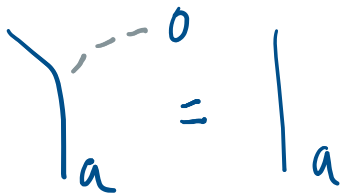


Similarly, the time-reversal of fusion is called SPLITTING and is represented as e.g.



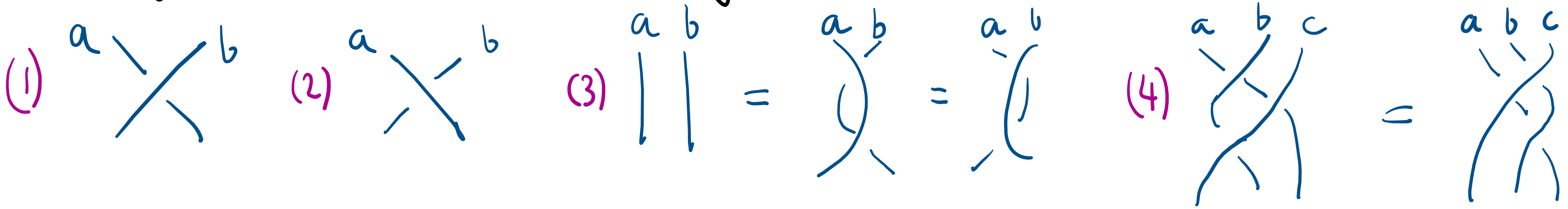
Anyon Diagrammatics

- Fusion with $\mathbb{1}$ is equivalent to doing nothing!



$\therefore$ freely add/remove dashed lines!

- Braiding n anyons is represented by a "coloured" (i.e. labelled) n-braid.



(1) Clockwise exchange operator : R^{ab}

(2) Anticlockwise " " : $(R^{-1})^{ab}$

(3) $id_{ab} = R^{ba} \circ (R^{-1})^{ab} = (R^{-1})^{ba} \circ R^{ab}$

i.e. $(R^{-1})^{ab} = (R^{ba})^{-1} = (R^{ba})^{\dagger}$ } acting on a Hilbert space...

(4) $(R^{bc} \otimes id_a) \circ (id_b \otimes R^{ac}) \circ (R^{ab} \otimes id_c) = (id_c \otimes R^{ab}) \circ (R^{ac} \otimes id_b) \circ (id_a \otimes R^{bc})$

i.e. YANG-BAXTER EQ^{N}.

- Braiding w\ the vacuum = doing nothing i.e. $R^{a0} = R^{0a} = id_a$

Anyon Diagrammatics

- The input data for a theory of anyons is given by

  (i) $\mathcal{L} = \{\mathbb{1}, a, b, c, \ldots\}$ finite.

  (ii) FUSION COEFFICIENTS $\{N_c^{ab}\}_{a,b,c\in\mathcal{L}}$, $N_c^{ab} \in \mathbb{Z}_{\geq 0}$ s.t. $N_c^{ab} < \infty$

↳ The fusion coefficients satisfy conditions (F1) – (F5).

↳ They describe the FUSION RULES of the theory : $\boxed{a * b = \sum_c N_c^{ab}\, c}$ (❀)

↳ LHS of (❀) : fuse a with b (a on left, b on right)

↳ RHS of (❀) : Concatenation of all possible fusion outcomes

↳ $N_c^{ab} = K$: a & b can be fused in K physically distinguishable ways

Anyon Diagrammatics

(F1) $a * 0 = 0 * a = a$ i.e. $N^{a0}_{b} = N^{0a}_{b} = \delta_{ab}$ (Existence of identity)

(F2) Given $a \in \mathcal{L}$, there exists a unique $\bar{a} \in \mathcal{L}$ (i.e. a unique dual charge) s.t.

$$a * \bar{a} = \bar{a} * a = \mathbb{1} + \ldots \quad \text{i.e.} \quad N^{ab}_{0} = N^{ba}_{0} = \delta_{b\bar{a}}$$

(F3) For $a, b \in \mathcal{L}$, $\sum_c N^{ab}_{c} \geq 1$ (Existence of fusion channel)

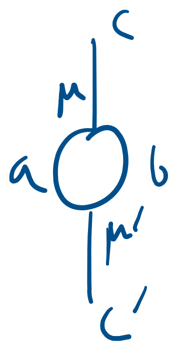
(F4) $(a * b) * c = a * (b * c)$ (Associativity of fusion)

(F5) $a * b = b * a$ i.e. $N^{ab}_{c} = N^{ba}_{c}$ (Commutativity of fusion)

Anyon Diagrammatics

- A pair of charges $a, b \in \mathcal{L}$ has an associated FUSION (HILBERT) SPACE V^{ab}, where

$$V^{ab} = \bigoplus_{c \in \mathcal{L}} V^{ab}_c \quad , \quad \dim\left(V^{ab}_c\right) = N^{ab}_c$$

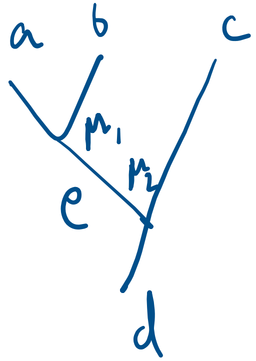
- Can fix an orthonormal basis ("ONB") $\{ |ab \to c; \mu\rangle \}_{\mu=1}^{N^{ab}_c}$ on V^{ab}_c.
  - ↳ These vectors are called FUSION STATES.
  - ↳ Graphically, $|ab \to c; \mu\rangle =$  $\in V^{ab}_c$; $\langle ab \to c; \mu | =$  $\in V^c_{ab} := \left(V^{ab}_c\right)^*$
  - ↳ Completeness relation :  $= \sum_{c,\mu}$  , Orthogonality :  $= \delta_{\mu\mu'} \delta_{cc'}$ 

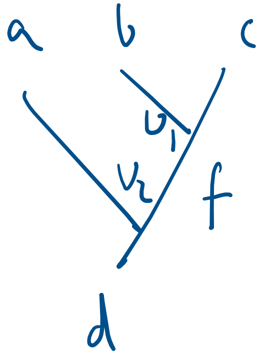
Anyon Diagrammatics

- For multiple charges, have $V^{q_1 q_2 \cdots q_n} = \bigoplus_{Q \in \mathcal{Z}} V_Q^{q_1 q_2 \cdots q_n}$ (by associativity)

  ↳ Can decompose into triangular spaces e.g.

$$\boxed{V_d^{abc} \cong \underbrace{\bigoplus_e V_e^{ab} \otimes V_d^{ec}} \cong \underbrace{\bigoplus_f V_d^{af} \otimes V_f^{bc}}}$$

$\mathrm{span}_{\mathbb{C}} \Big\{$ 

$\Big\}_{e, \mu_1, \mu_2}$; $\mathrm{span}_{\mathbb{C}} \Big\{$ 

$\Big\}_{f, \nu_1, \nu_2}$

Specifying decomposition

||

Specifying fusion order of particles

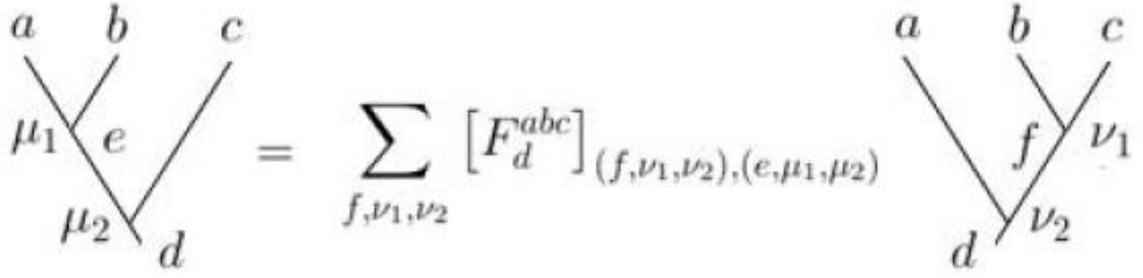
(i.e. measurement basis)

"fusion basis"

||

Choose full rooted n-leaf binary tree

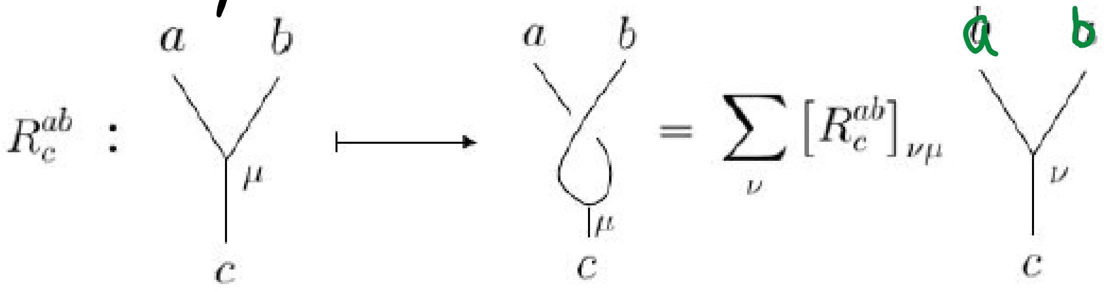
- We can move between any 2 decompositions of an n-particle space using a sequence of F-MOVES. These are described by isomorphisms called F-MATRICES:



F_d^{abc} is a unitary operator on V_d^{abc}

(The entries of an F-matrix are called F-SYMBOLS or 6j-SYMBOLS)

- The exchange operator R^{ab} is called an R-MOVE. In our ONB,

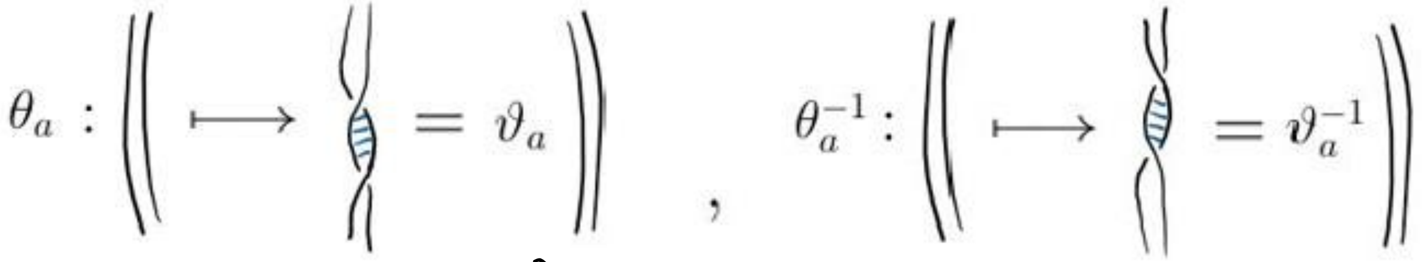


$R^{ab} = \bigoplus_c R_c^{ab}$ is an isomorphism,

$R^{ab} : V^{ab} \xrightarrow{\sim} V^{ba}$

(Entries of R-MATRIX called R-SYMBOLS)

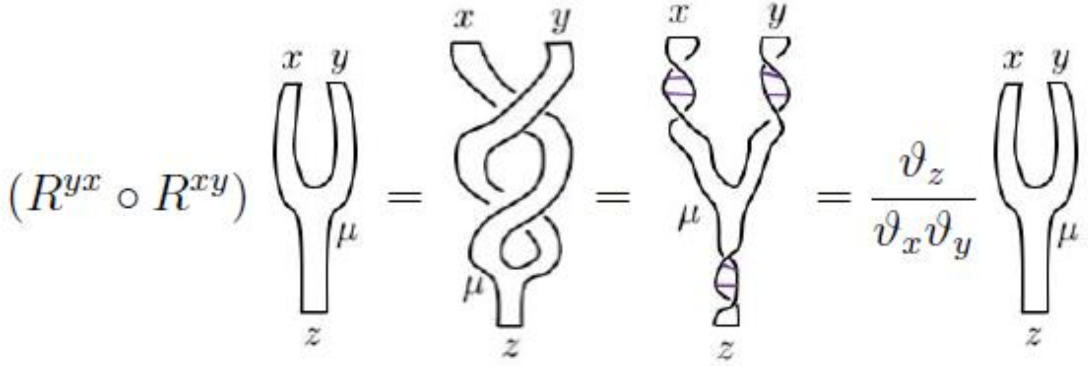
- TWISTING is described by maps



(Note: We've promoted worldlines to worldribbons!)

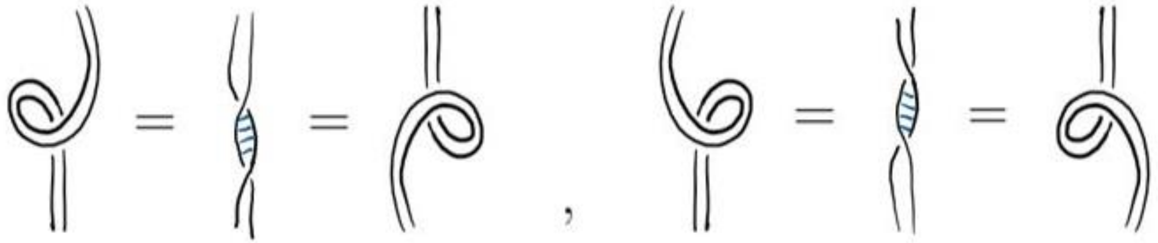
↳ $\vartheta_a \in U(1)$ is called the TOPOLOGICAL SPIN of $a \in \mathcal{L}$

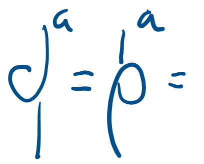
↳ We have,



Spin-statistics relation for anyons!

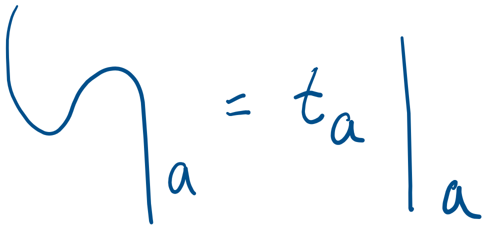
↳ Note that

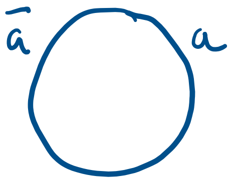
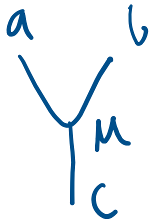


∴ draw  $= \vartheta_a \big|_a$

etc. for worldlines

Anyon Diagrammatics

- For any $a \in \mathcal{L}$, a 'wiggle' in spacetime satisfies  , $t_a \in U(1)$.
  - ↳ t_a is called the PIVOTAL COEFFICIENT of a.
  - ↳ If $\bar{a} = a$, then $t_a =: \varkappa_a \in \{\pm 1\}$ is a property of charge a called its FROBENIUS-SCHUR INDICATOR.

- The value of an (unnormalised) loop  is called the QUANTUM DIMENSION of $a \in \mathcal{L}$.
  - ↳ $d_a \in \mathbb{R}_{>0}$ is the dominant eigenvalue of $[N^a]_{bc} =: N^{ab}_c$.
  - ↳ $\mathcal{D} = \sum_{q \in \mathcal{L}} \sqrt{d_q^2}$ is called the TOTAL QUANTUM DIMENSION of the theory.
  - ↳ Trivalent vertices  are normalised by $\sqrt[4]{\frac{d_c}{d_a d_b}}$.

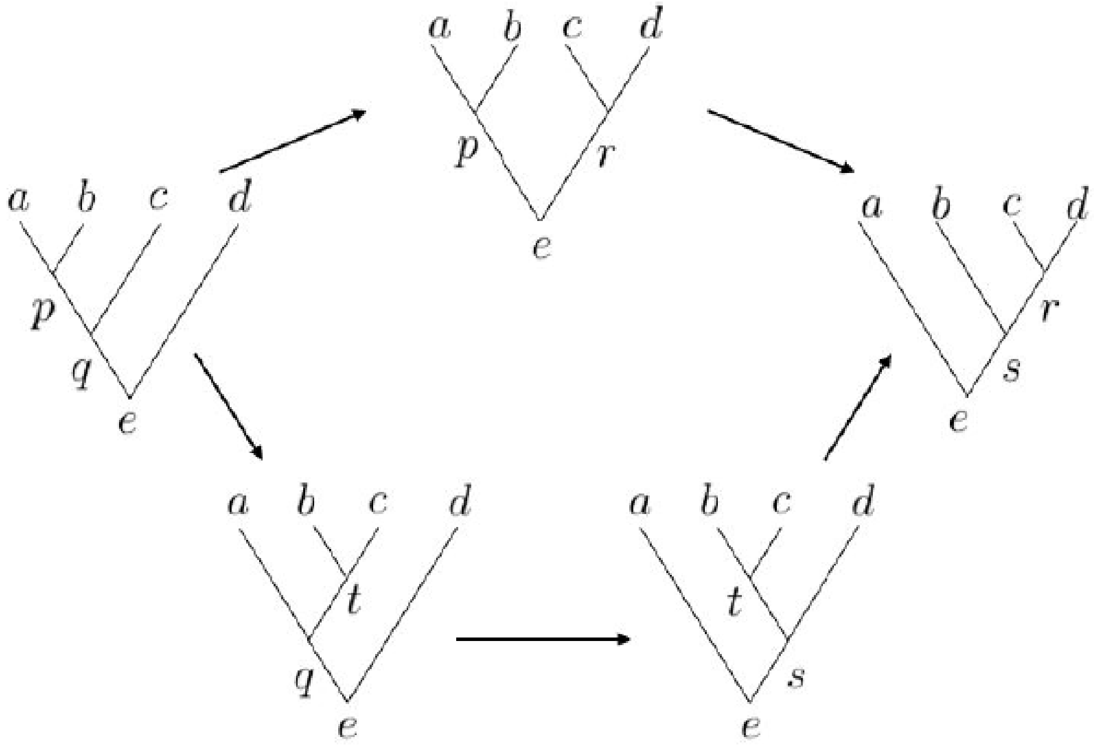
- F-moves and R-moves must satisfy some coherence conditions for consistency!

(1) PENTAGON EQUATION:

$$\begin{tikzcd}[column sep=small]
 & & \bigoplus_{p,r} V^{ab}_p \otimes V^{pr}_e \otimes V^{cd}_r \arrow[drr, "\sum_r F^{abr}_e \otimes \mathrm{id}_{V^{cd}_r}"] & & \\
\bigoplus_{p,q} V^{ab}_p \otimes V^{pc}_q \otimes V^{qd}_e \arrow[urr, "\sum_p \mathrm{id}_{V^{ab}_p} \otimes F^{pcd}_e"] \arrow[dr, "\sum_q F^{abc}_q \otimes \mathrm{id}_{V^{qd}_e}"'] & & & & \bigoplus_{r,s} V^{as}_e \otimes V^{br}_s \otimes V^{cd}_r \\
 & \bigoplus_{q,t} V^{at}_q \otimes V^{bc}_t \otimes V^{qd}_e \arrow[rr, "\sum_t F^{atd}_e \otimes \mathrm{id}_{V^{bc}_t}"'] & & \bigoplus_{s,t} V^{as}_e \otimes V^{bc}_t \otimes V^{td}_s \arrow[ur, "\sum_s \mathrm{id}_{V^{as}_e} \otimes F^{bcd}_s"'] &
\end{tikzcd}$$

commutes for all $a, b, c, d, e \in \mathfrak{L}$.

Anyon Diagrammatics



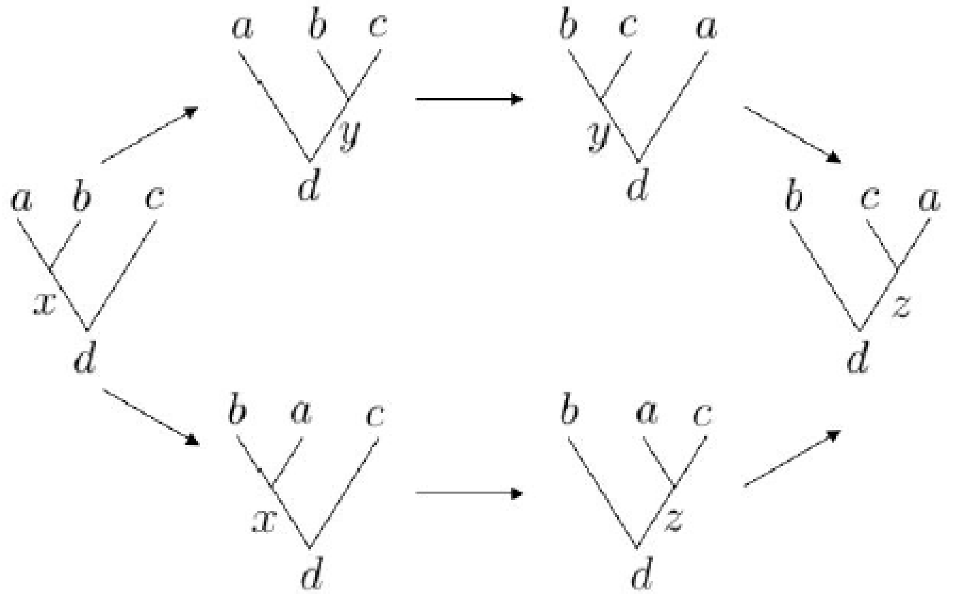
$$\left.\begin{aligned}&\sum_{\sigma}\left[F^{abr}_{e}\right]_{(s,\delta,\rho)(p,\alpha,\sigma)}\left[F^{pcd}_{e}\right]_{(r,\gamma,\sigma)(q,\beta,\lambda)}\\ =&\sum_{t,\mu,\nu,\eta}\left[F^{bcd}_{s}\right]_{(r,\gamma,\delta)(t,\mu,\eta)}\left[F^{atd}_{e}\right]_{(s,\eta,\rho)(q,\nu,\lambda)}\left[F^{abc}_{q}\right]_{(t,\mu,\nu)(p,\alpha,\beta)}\end{aligned}\right\}\text{(matrix form)}$$

(2) HEXAGON EQUATIONS:

$$
\begin{tikzcd}
 & \bigoplus_y V_d^{ay} \otimes V_y^{bc} \arrow[rr, "\bigoplus_y \left(\mathrm{id}_{V_y^{bc}} \otimes R_d^{ay}\right) \cdot \mathrm{swap}"] & & \bigoplus_y V_y^{bc} \otimes V_d^{ya} \arrow[dr, "F_d^{bca}"] & \\
\bigoplus_x V_x^{ab} \otimes V_d^{xc} \arrow[ur, "F_d^{abc}"] \arrow[dr, "\bigoplus_x R_x^{ab} \otimes \mathrm{id}_{V_d^{xc}}"'] & & & & \bigoplus_z V_d^{bz} \otimes V_z^{ca} \\
 & \bigoplus_x V_x^{ba} \otimes V_d^{xc} \arrow[rr, "F_d^{bac}"'] & & \bigoplus_z V_d^{bz} \otimes V_z^{ac} \arrow[ur, "\bigoplus_z \mathrm{id}_{V_d^{bz}} \otimes R_z^{ac}"'] &
\end{tikzcd}
$$

Commutes for all $a, b, c, d \in \mathcal{L}$ (and similarly for R^{-1}).

Anyon Diagrammatics



$$\sum_{y,\beta,\mu,\sigma} \left[F^{bca}_d\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[R^{ay}_d\right]_{\sigma\mu} \left[F^{abc}_d\right]_{(y,\beta,\mu)(x,\alpha,\lambda)}$$

$$= \sum_{\delta,\epsilon} \left[R^{ac}_z\right]_{\gamma\epsilon} \left[F^{bac}_d\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[R^{ab}_x\right]_{\delta\alpha}$$

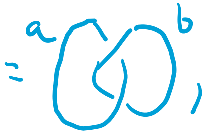
$$\sum_{y,\beta,\mu,\sigma} \left[F^{bca}_d\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[\left(R^{-1}\right)^{ay}_d\right]_{\sigma\mu} \left[F^{abc}_d\right]_{(y,\beta,\mu)(x,\alpha,\lambda)}$$

$$= \sum_{\delta,\epsilon} \left[\left(R^{-1}\right)^{ac}_z\right]_{\gamma\epsilon} \left[F^{bac}_d\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[\left(R^{-1}\right)^{ab}_x\right]_{\delta\alpha}$$

matrix form
for each equation

Anyon Diagrammatics

- Pent./hex. equations are a machine: given a set of fusion rules, each unitary solution corresponds to a theory of anyons.

$S_{ab} :=$ 

, $[S]_{ab} = \frac{1}{D} S_{ab}$

↳ When $|\mathcal{L}|$ grows, solving these equations is notoriously hard!

↳ There are finitely many theories of anyons for any set of fusion rules. ("Ocneanu Rigidity")

↳ When $|\mathcal{L}|$ is fixed, there are finitely many sets of fusion rules that yield unitary solutions. ("Rank-Finiteness") [Bruillard et al., Jones et al.]

↳ ∴ Finitely many theories of anyons of a fixed rank! Motivates classification programme.

↳ Modular theories have a prime factorisation structure! ⇝ Periodic table of bosonic theories? (Classified up to rank 5/6 [Rowell et al., Creamer]).

- Example : FIBONACCI THEORIES

$$\mathcal{L} = \{\mathbb{1}, \tau\} \quad , \quad \tau * \tau = \mathbb{1} + \tau$$

$$F^{\tau\tau\tau}_{\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \quad , \quad R^{\tau\tau} = \begin{pmatrix} e^{-i\frac{4\pi}{5}} & 0 \\ 0 & e^{i\frac{3\pi}{5}} \end{pmatrix} \quad , \quad \phi = \frac{1+\sqrt{5}}{2}$$

↳ ϕ is the golden ratio, $d_\tau = \phi$ & τ is called a Fibonacci anyon

↳ There is also a mirror theory. Both are modular.

↳ Fibonacci anyons are universal for quantum computation.

↳ $(G_2, \pm 1)$ Chern-Simons QFTs ; realisation expected in low-energy excitations of FQHE at $\nu = 12/5$.

Some poetry

Anyon, anyon, where do you roam?
Braid for a while before you go home.

Though you're condemned just to slide on a table,
A life in 2D also means that you're able
To be of a type neither Fermi nor Bose
And to know left from right --- that's a kick, I suppose.

You and your buddy were made in a pair
Then wandered around, braiding here, braiding there.
You'll fuse back together when braiding is through
Well bid you adieu as you vanish from view.

No one can say, not at this early juncture
If someday we'll store quantum data in punctures
With quantum states hidden where no one can see,
Protected from damage through topology.

Anyon, anyon, where do you roam?
Braid for a while before you go home.

John Preskill
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