Theory X and Geometric Representation Theory I

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There is a mysterious quantum field theory that physicists call the 6d $(2,0)$ superconformal field theory. It hasn't been constructed. It doesn't have a Lagrangian. We just have some lore about it. We'll call it Theory X. We'll learn some things by compactifying it to get many other lower-dimensional field theories.

Theory X takes as input an ADE Lie algebra $\mathfrak{g}$; to each such Lie algebra it assigns a 6d field theory. After dimensional reduction on S^1, we get 5d super Yang-Mills associated to a Lie group G with Lie algebra $\mathfrak{g}$. (This is one of the confusing parts of the theory.) Which G?

X is not a plain QFT; instead, it is a relative QFT in the sense of Freed and Teleman, and in particular to describe it we need to give some extra data. Geometrically the data looks like a Lagrangian subspace of $H^3(X_G, \mathbb{Z})$. In particular the extra data supplies a choice of G. Also, we can get Lie algebras of non-ADE type by twisting by automorphisms around the circle; this corresponds to taking the fixed points of some corresponding diagram automorphisms.

Following Gaitto-Moore-Neitzke, we'll restrict our attention to 6-manifolds of the form a 2-manifold cross a 4-manifold. Theory X will be conformal in the 2-manifold but topological in the 4-manifold; equivalently, given a Riemann surface Σ Theory X supplies a 4d TFT. These are the Class S theories, and some of them have specific names:

1. When $\Sigma = T^2$ we get (the topological twist of) $N = 4$ super Yang-Mills. This is the geometric Langlands TFT.
2. When $\Sigma = S^2$ (with some extra data) we get (the topological twist of) $N = 2$ super Yang-Mills. This is the TFT related to Donaldson theory.
3. When $\Sigma = D^2$ (with some boundary conditions) we get the Khovanov-Witten TFT. This is the TFT that explains Khovanov homology. A further reduction on S^1 should get us the Chern-Simons TFT.

We will think about 4d TFT in the framework of the cobordism hypothesis. For us, a (fully extended) n-dimensional TFT is a symmetric monoidal functor from a (higher) cobordism category of manifolds (possibly with some extra structure, e.g. framings) to some target symmetric monoidal (higher) category. The source has

1. Objects 0-dimensional manifolds with n-dimensional tangent spaces (e.g. stabilized by $\mathbb{R}^n$)
2. Morphisms 1-dimensional cobordisms between objects with n-dimensional tangent spaces (e.g. stabilized by $\mathbb{R}^{n-1}$).
3. 2-morphisms 2-dimensional cobordisms, etc.

One way to think about the extra data is that for k-morphisms we want germs of k-manifolds in $\mathbb{R}^n$, e.g. so that we can glue cobordisms (possibly with some extra structure) appropriately. It's also important that we remember the homotopy type of the space of cobordisms between two manifolds M, N, which is the disjoint union of the classifying spaces of the diffeomorphism groups of all diffeomorphism classes of cobordisms between M and N.

This higher category Bord_n is equipped with a symmetric monoidal structure given by disjoint union, and a TFT Z must in particular send disjoint unions to the symmetric monoidal structure of the target. It must also respect the homotopy types of the spaces of cobordisms.

We will not be very precise about the target category $(C, \otimes)$. Heuristically, to an n-dimensional manifold, Z should assign the path integral

$$Z(M) = \int_{F(M)} e^{iS(\varphi)}\, d\varphi \tag{1}$$

where $F(M)$ is a mythical space of fields and $S(\varphi)$ is a mythical action. In particular, $Z(M)$ is a number, so n-morphisms in C should be numbers.

To an $(n-1)$-dimensional manifold N (which we will always implicitly cross with $\mathbb{R}$), Z should assign a Hilbert space of functionals on fields $F(N)$ equipped with a differential coming from a supercharge and with a grading coming from a $U(1)$ charge or fermion number. In particular we'll get dg vector spaces, so $(n-1)$-morphisms in C should be dg vector spaces.

If $N = \partial M$ for some n-manifold M, then $Z(M)$ should be a vector in $Z(N)$. Heuristically it should be given by the path integral

$$Z(M) : \varphi_N \mapsto \int_{\varphi|_N = \varphi_N} e^{iS(\varphi)}\, d\varphi. \tag{2}$$

More generally, if M is a cobordism between manifolds N_1, N_2, $Z(M)$ is a linear map between functionals on fields given by an integral transform

$$Z(M)(f_{N_1}) : \varphi_{N_2} \mapsto \int_{\varphi|_{N_2} = \varphi_{N_2}} f_{N_1}(\varphi\mid_{N_1}) e^{iS(\varphi)}\, d\varphi \tag{3}$$

or, less confusingly, as a push-pull

$$Z(M) = (\pi_2)_* \left(\pi_1^*(-) e^{iS}\right). \tag{4}$$

Functoriality, together with keeping homotopy types, requires that this construction is locally constant over $B\mathrm{Diff}(M)$, and hence as we vary M we get a map

$$C_\bullet(B\mathrm{Diff}(M)) \otimes Z(N_1) \to Z(N_2). \tag{5}$$

To an $(n-2)$-manifold we should assign objects in a 2-category (perhaps really an $(\infty, 2)$-category) which deloops dg vector spaces in the sense that endomorphisms of the identity are dg vector spaces (with composition given by tensor product). Two popular choices are

1. The 2-category of dg categories, dg functors, and dg natural transformations.

2. The 2-category of dg algebras, dg bimodules, and dg morphisms of bimodules.

The second 2-category embeds into the first by associating to a dg algebra A its dg modules and associating to a dg bimodule the functor given by tensor product. In other words we should think of dg algebras as presenting particularly nice dg categories (equipped with a distinguished object which generates the dg category in some sense).

So if P is an $(n-2)$-manifolds, $Z(P)$ is some category (of sheaves, or vector space valued functions, on fields $F(P)$) or some dg algebra (of functions on fields $F(P)$).

If N is an $(n-1)$-dimensional cobordism between two $(n-2)$-manifolds P_1, P_2, then $Z(N)$ is again an integral / push-pull transform, but now on the level of sheaves

$$Z(N) = (\pi_2)_* \left(\pi_1^*(-) \otimes e^{iS} \right) \tag{6}$$

where e^{iS} is now some distinguished bimodule.

One way to state the cobordism hypothesis is that if Z is a framed TFT, then $Z(\text{pt})$ (secretly $Z(\mathbb{R}^n)$) uniquely determines the rest of the field theory; we will call the procedure giving us $Z(M)$ from $Z(\mathbb{R}^n)$ integration. Moreover, given an object in the target category C, there is a list of finiteness conditions (k-dualizability) such that if the k^{th} finiteness condition is satisfied then the integral $Z(M^k)$ converges.

Compactification or dimensional reduction is the following procedure: if M is a k-dimensional manifold, then $Z(M \times (-))$ is an $(n-k)$-dimensional field theory. By the cobordism hypothesis, the data of this field theory is equivalent to the data of $Z(M)$.

The field theories we care about are not framed but depend on less data like an orientation or a spin, and this requires supplying extra data to $Z(\text{pt})$ (homotopy fixed point data).