~~The Tripleableness Theorem~~ by Jon Beck

"UNTITLED MANUSCRIPT".

1. Let $\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}$ with $F \dashv U$. Write $\eta : 1_{\underline{A}} \to FU$, $\epsilon : UF \to 1_{\underline{B}}$ for the unit and counit of the adjointness. Then $\mathbb{T} = (T, \eta, \mu)$ is a triple in $\underline{A}$, where $T = FU$, $\eta : 1_{\underline{A}} \to T$, $\mu = F\epsilon U : T^2 \to T$. We have the category of $\mathbb{T}$-algebras $\underline{A}^{\mathbb{T}}$ as defined by Eilenberg-Moore, $F^{\mathbb{T}} : \underline{A} \to \underline{A}^{\mathbb{T}}$ by $X \rightsquigarrow (XT, X\mu)$, $U^{\mathbb{T}} : \underline{A}^{\mathbb{T}} \to \underline{A}$ by $(X, \xi) \rightsquigarrow X$, and $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$.

$$\begin{tikzcd}
\underline{A}^{\mathbb{T}} \arrow[dr, "U^{\mathbb{T}}"'] & & \underline{B} \arrow[ll, "\Phi"'] \arrow[dl, "U"] \\
& \underline{A} &
\end{tikzcd}$$

is defined by $Y\Phi = (YU, Y\epsilon U)$. The adjoint pair $F \dashv U$ is <u>tripleable</u> if $\Phi^{\vee} \dashv \Phi$ exists such that the unit and counit are isomorphisms: $1_{\underline{A}^{\mathbb{T}}} \xrightarrow{\sim} \Phi^{\vee}\Phi$, $\Phi\Phi^{\vee} \xrightarrow{\sim} 1_{\underline{B}}$. Given U, this property is independent of which left adjoint F is used, so we also say: <u>U is tripleable</u> in this situation. It seems to be too much to ask for $\Phi^{\vee}\Phi = \underline{A}^{\mathbb{T}}$, $\Phi\Phi^{\vee} = \underline{B}$. On the other hand, in category theory, the usual "equivalences" of categories should be replaced by adjoint equivalences.

2. <u>Crude tripleableness theorem.</u> If $\underline{B}$ has coequalizers and U preserves and reflects coequalizers, then U is tripleable. (It is assumed $F \dashv U$ exists.)

<u>Proof</u>. $\Phi^{\vee}$ is the coequalizer: $XFUF \underset{XF\epsilon}{\overset{\xi F}{\rightrightarrows}} XF \xrightarrow{k} (X, \xi)\Phi^{\vee}$.

One way of proving this is by verifying the sequence of set isomorphisms

maps $(X, \xi) \xrightarrow{f} Y\Phi$

$\longrightarrow$ maps $X \xrightarrow{f} YU$ such that $\xi f = fFU . Y\epsilon U$

$\longrightarrow$ maps $XF \xrightarrow{g} Y$ such that $\xi F = XF\epsilon . g$

$\longrightarrow$ maps $(X, \xi)\Phi^{\vee} \xrightarrow{g} Y$.

If $(X,\xi) \xrightarrow{\varphi} (X,\xi)\phi^\vee\phi$ denotes the unit of $\phi^\vee \dashv \phi$, then $\varphi U^\Pi = X\eta . kU$

$$\begin{tikzcd}
 & & X \arrow[d, "X\eta"] & \\
XFUFU \arrow[rr, shift left, "\xi FU"] \arrow[rr, shift right, "XF\epsilon U"'] & & XFU \arrow[r, "\xi"] \arrow[dr, "kU"'] & X \arrow[d, "\varphi U^\Pi"] \\
 & & & (X,\xi)\check{\phi}U
\end{tikzcd}$$

Now, $\xi = \mathrm{coeq}(\xi FU, XF\epsilon U)$ for if some $XFU \xrightarrow{z} Z$ coequalizes ξFU and $XF\epsilon U$, then $X \xrightarrow{X\eta . z} Z$ is the unique map such ... But: $kU = \mathrm{co\text{-}eq}(\xi FU, XF\epsilon U)$ since U preserves coequalizers. Moreover, $\dot{\xi}(\varphi U^\Pi) = \xi . X\eta . kU = XFU\eta . \xi FU . kU = XFU\eta . XF\epsilon U . kU = kU$. Therefore φU^Π is an isomorphism, and since U^Π reflects isomorphism, so is φ. The counit $Y\phi\phi^\vee \xrightarrow{\psi} Y$ is defined by its appearance in the diagram below. We proved above that the Π-structure of an algebra is a coequalizer, so if U is applied to $(Y\epsilon UF, YUF\epsilon, Y\epsilon)$ we get a coequalizer diagram in $\underline{A}$ ($Y\epsilon U$ is the Π-structure of the algebra $Y\phi$). But U reflects coequalizers, so, $Y\epsilon = \mathrm{coeq}(Y\epsilon UF, YUF\epsilon)$. Therefore ψ is an isomorph.

$$\begin{tikzcd}
YUFUF \arrow[r, shift left, "Y\epsilon UF"] \arrow[r, shift right, "YUF\epsilon"'] & YUF \arrow[r, "k"] \arrow[d, "Y\epsilon"'] & Y\phi\phi^\vee \arrow[dl, "\psi"] \\
 & Y &
\end{tikzcd}$$

3. <u>Contractible coequalizers.</u> A diagram $Y_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y_0 \xrightarrow{d} Y$ with $d_0 d = d_1 d$ looks like the 1-skeleton of an augmented simplicial object. (Here degeneracies will be ignored.) A <u>contraction</u> of a simplicial objects is a sequence of maps $h_n : Y_n \longrightarrow Y_{n+1}$ such that $h_n d_i = d_i h_{n-1}$ for $0 \leq i \leq n$ and $h_n d_{n+1} = Y_n$. (You can also use $h_n d_0 = Y_n$, $h_n d_i = d_{i-1} h_{n}$.) We are led to look at diagrams such that $d_0 d = d_1 d$, $h_{-1} d = Y$, $h_0 d_0 = d h_{-1}$, $h_0 d_1 = Y_0$. In this case $d = \mathrm{coeq}(d_0, d_1)$, for if $d_0 z = d_1 z$ for $Y_0 \xrightarrow{z} Z$ then $h_{-1} z : Y \longrightarrow Z$ is the unique map such ... Thus we call such a diagram a <u>contractible coequalizer diagram.</u>

$$\begin{tikzcd}
Y_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y_0 \arrow[l, bend right=60, "h_0"'] \arrow[r, "d"'] & Y \arrow[l, bend right=60, "h_{-1}"']
\end{tikzcd}$$

If $\underline{A} \xleftarrow{U} \underline{B}$, we call coequalizer data $Y_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y_0$ <u>U-contractible</u> if there are Z, d, h_{-1}, h_0 in $\underline{A}$ such that

$$\begin{tikzcd} Y_1U \arrow[r, shift left, "d_0U"] \arrow[r, shift right, "d_1U"'] & Y_0U \arrow[l, bend right, "h_0"'] \arrow[r, "d"'] & Z \arrow[l, bend right, "h_{-1}"'] \end{tikzcd}$$

is a contractible coequalizer diagram.

We say: $\underline{B}$ <u>has U-contractible coequalizers</u> if all U-contractible coequalizer data in $\underline{B}$ have coequalizers in $\underline{B}$; U <u>preserves U-contractible coequalizers</u> if whenever $Y_1 \rightrightarrows Y_0$ is U-contractible and has a coequalizer $Y_0 \to Y$ in $\underline{B}$, then the canonical map $Z \to YU$ is an isomorphism; U <u>reflects</u> <u>U-contractible coequalizers</u> if $Y_1 \rightrightarrows Y_0 \to Y$ being mapped into a contractible coequalizer diagram by U implies that $Y_1 \rightrightarrows Y_0 \to Y$ is a coequalizer diagram in $\underline{B}$.

$$\left[\begin{tikzcd} Y_1U \arrow[r, shift left, "d_0U"] \arrow[r, shift right, "d_1U"'] & Y_0U \arrow[l, bend right, "h_0"'] \arrow[r, "dU"'] & YU \arrow[l, bend right, "h_0"'] \end{tikzcd}\right]$$

($Y_1 \rightrightarrows Y_0 \to Y$ will not necessarily be contractible in $\underline{B}$.)

4. <u>Precise tripleableness theorem.</u> U is tripleable $\Longleftrightarrow$ $\underline{B}$ has, and U preserves and reflects, U-contractible coequalizers.

<u>Proof.</u> $\Leftarrow$ is clear. One only has to notice that all coequalizers arising in the proof of the crude theorem were U-contractible.

$\Rightarrow$: We can assume $\underline{B} = \underline{A}^{\mathbb{T}}$ and prove that $\underline{A}^{\mathbb{T}}$ has $U^{\mathbb{T}}$-contractible coequalizers. (The (dual) example of comodules over a non-flat coalgebra shows that $\underline{A}^{\mathbb{T}}$ need not have all coequalizers. But it follows from a result of Linton's alluded to below that $\underline{A}^{\mathbb{T}}$ has all coequalizers if $\underline{A}$ = sets.) Let $(X_1, \xi_1) \overset{d_0}{\underset{d_1}{\rightrightarrows}} (X_0, \xi_0)$ be $U^{\mathbb{T}}$-contractible, i.e. we have the accompanying diagram in $\underline{A}$. Let $XT \xrightarrow{\xi} X$ be $h_{-1}T . \xi_0 d$. Then $dT . \xi = \xi_0 d$. For

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[l, bend right, "h_0"'] \arrow[r, "d"'] & X \arrow[l, bend right, "h_{-1}"'] \end{tikzcd}$$

$$\begin{tikzcd} X_0T \arrow[r, "dT"] \arrow[d, "\xi_0"'] & XT \arrow[d, "\xi"] \\ X_0 \arrow[r, "d"'] & X \end{tikzcd}$$

$dT . \xi = dT . h_{-1}T . \xi_0 d = (dh_{-1})T . \xi_0 d =$

$= (h_0 d_0)T \cdot \xi_0 d = h_0 T \cdot d_0 T \cdot \xi_0 d = h_0 T \cdot \xi_1 d_0 d = h_0 T \cdot \xi_1 d_1 d$
$= h_0 T \cdot d_1 T \cdot \xi_0 d = (h_0 d_1)T \cdot \xi_0 d = \xi_0 d.$

This shows that $d : X_0 \longrightarrow X$ is compatible with $\mathbb{T}$-structures. Since $h_{-1} d = X$, it follows that (X, ξ) is a $\mathbb{T}$-algebra. Also if a different contraction h_0', h_{-1}' were used, and ξ' defined as $h_{-1}'T \cdot \xi_0 d$, then $\xi' = \xi$, since $\xi = (h_{-1} d)T \cdot \xi = h_{-1}T \cdot dT \cdot \xi = h_{-1}T \cdot \xi_0 d$, and $\xi' = (h_{-1} d)T \cdot$ $= h_{-1}T \cdot dT \cdot \xi' = h_{-1}T \cdot \xi_0 d$ also. Thus the $\mathbb{T}$-structure ξ is well-defined. Finally, $d = \mathrm{coeq}(d_0, d_1)$ for if $(X_0, \xi_0) \xrightarrow{y} (Y, \theta)$ coequalizes d_0, d_1, then $(X, \xi) \xrightarrow{h_{-1} y} (Y, \Theta)$ is the unique .. *) The above construction shows that $U^{\mathbb{T}}$ preserves and reflects $U^{\mathbb{T}}$-contractible coequalizers.

*) Note that h_{-1} is not an algebra map, but $h_{-1} y$ is.

5. Remarks. It should be possible to improve the above theorem (apart from streamlining the exposition). Conditions implying tripleableness should be found which are easier to verify in practice. For instance, the following is true:

U is tripleable $\Longleftrightarrow$ $\underline{B}$ has and U preserves U-contractible coequalizers, and U reflects isomorphisms.

It seems to follow without much difficulty, from this, that algebraic or varietal categories are tripleable $/\mathcal{S}$ (and Linton can prove tripleable categories are varietal).