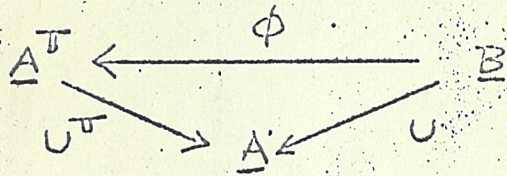


"UNTITLED MANUSCRIPT"

1. Let $\underline{A} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{U} \end{matrix} \underline{B}$ with $F \dashv U$. Write $\eta: 1_{\underline{A}} \rightarrow FU$, $\epsilon: UF \rightarrow 1_{\underline{B}}$ for the unit and counit of the adjointness. Then $\mathbb{T} = (T, \eta, \mu)$ is a triple in \underline{A} , where $T = FU$, $\eta: 1_{\underline{A}} \rightarrow T$, $\mu = F\epsilon U: T^2 \rightarrow T$. We have the category of \mathbb{T} -algebras $\underline{A}^{\mathbb{T}}$ as defined by Eilenberg-Moore, $F^{\mathbb{T}}: \underline{A} \rightarrow \underline{A}^{\mathbb{T}}$ by $X \mapsto (XF, X\eta)$, $U^{\mathbb{T}}: \underline{A}^{\mathbb{T}} \rightarrow \underline{A}$ by $(X, \xi) \mapsto X$, and $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$.



is defined by $\gamma\phi = (\gamma U, \gamma\epsilon U)$. The adjoint pair $F \dashv U$ is tripleable if $\phi^{\vee} \dashv \phi$ exists such that the unit and counit are isomorphisms $1_{\underline{A}^{\mathbb{T}}} \xrightarrow{\sim} \phi^{\vee}\phi$, $\phi\phi^{\vee} \xrightarrow{\sim} 1_{\underline{B}}$. Given U , this property is independent of which left adjoint F is used, so we also say, U is tripleable in this situation. It seems to be too much to ask for $\phi^{\vee}\phi = 1_{\underline{A}^{\mathbb{T}}}$, $\phi\phi^{\vee} = 1_{\underline{B}}$. On the other hand, in category theory, the usual "equivalences" of categories should be replaced by adjoint equivalences.

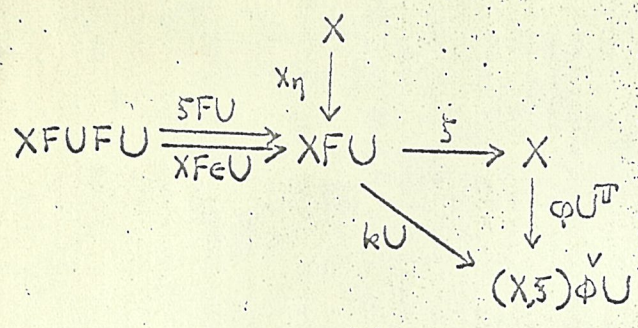
2. Crude tripleability theorem. If \underline{B} has coequalizers and U preserves and reflects coequalizers, then U is tripleable. (It is assumed $F \dashv U$ exists.)

Proof. ϕ^{\vee} is the coequalizer: $XFU \xrightarrow[XF\epsilon]{\xi F} XF \xrightarrow{k} (X, \xi)\phi^{\vee}$.

One way of proving this is by verifying the sequence of set isomorphisms

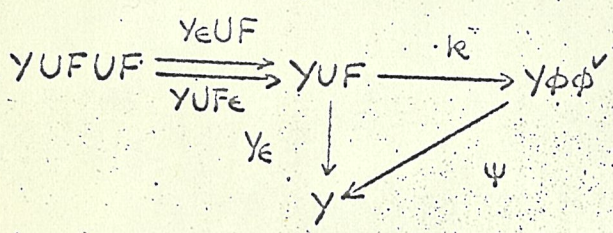
- maps $(X, \xi) \xrightarrow{f} \gamma\phi$
- \longrightarrow maps $X \xrightarrow{f} \gamma U$ such that $\xi f = f F U \cdot \gamma\epsilon U$
- \longrightarrow maps $X F \xrightarrow{g} \gamma$ such that $\xi F = X F \epsilon \cdot g$
- \longrightarrow maps $(X, \xi)\phi^{\vee} \xrightarrow{g} \gamma$.

If $(X, \xi) \xrightarrow{\varphi} (X, \xi) \overset{\vee}{\phi} \phi$ denotes the unit of $\phi^{-1}\phi$, then $\varphi U^\pi = X\eta \cdot kU$



Now, $\xi = \text{coeq}(\xi FU, XFEU)$ for if some $XFU \xrightarrow{z} Z$ coequalizes ξFU and $XFEU$, then $X \xrightarrow{x_{\eta, z}} Z$ is the unique map such ... But: $kU = \text{coeq}(\xi FU, XFEU)$ since U preserves coequalizers. Moreover,

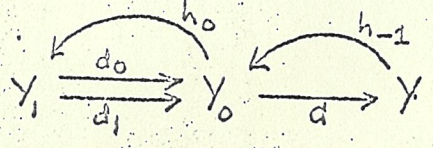
$\xi(\varphi U^\pi) = \xi \cdot X\eta \cdot kU = XFU\eta \cdot \xi FU \cdot kU = XFU\eta \cdot XFEU \cdot kU = kU$. Therefore φU^π is an isomorphism, and since U^π reflects isomorphism, so is φ . The counit $Y\phi\phi^{-1} \xrightarrow{\psi} Y$ is defined by its appearance in the diagram below. We proved above that the π -structure of an algebra is a coequalizer, so if U is applied to $(Y\epsilon UF, YUF\epsilon, Y\epsilon)$ we get a coequalizer diagram in \underline{A} ($Y\epsilon U$ is the π -structure of the algebra $Y\phi$). But U reflects coequalizers, so $Y\epsilon = \text{coeq}(Y\epsilon UF, YUF\epsilon)$. Therefore ψ is an isomorphism.



we get a coequalizer diagram in \underline{A} ($Y\epsilon U$ is the π -structure of the algebra $Y\phi$). But U reflects coequalizers, so $Y\epsilon = \text{coeq}(Y\epsilon UF, YUF\epsilon)$. Therefore ψ is an isomorphism.

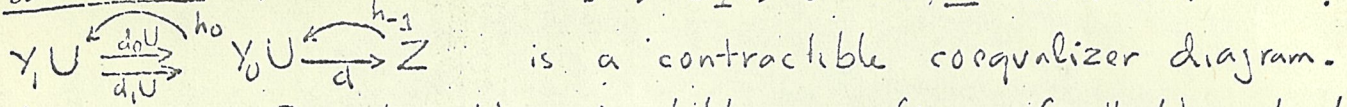
3. Contractible coequalizers.

A diagram $Y_1 \xrightleftharpoons[d_1]{d_0} Y_0 \xrightarrow{d} Y$ with $d_0 d = d_1 d$ looks like the 1-skeleton of an augmented simplicial object. (Here degeneracies will be ignored.) A contraction of a simplicial object is a sequence of maps $h_n: Y_n \rightarrow Y_{n+1}$ such that $h_n d_i = d_i h_{n-1}$ for $0 \leq i \leq n$ and $h_n d_{n+1} = Y_n$. (You can also use $h_n d_0 = Y_n, h_n d_i = d_{i-1} h_n$.) We are led to look at diagrams

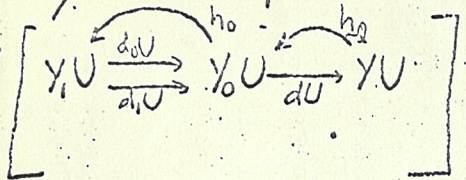


such that $d_0 d = d_1 d, h_{-1} d = Y, h_0 d_0 = d h_{-1}, h_0 d_1 = Y_0$. In this case $d = \text{coeq}(d_0, d_1)$, for if $d_0 z = d_1 z$ for $Y_0 \xrightarrow{z} Z$ then $h_{-1} z: Y \rightarrow Z$ is the unique map such ... Thus we call such a diagram a contractible coequalizer diagram.

If $A \xleftarrow{U} B$, we call coequalizer data $Y_1 \rightrightarrows_{d_i} Y_0 \xrightarrow{U} \dots$ contractible if there are Z, d, h_1, h_0 in A such that



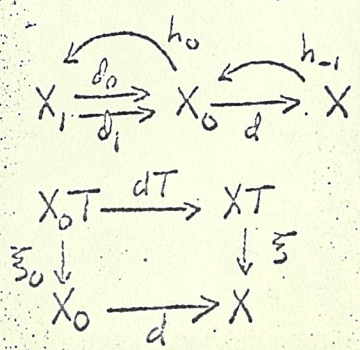
We say: B has U-contractible coequalizers if all U-contractible coequalizer data in B have coequalizers in B ; U preserves U-contractible coequalizers if whenever $Y_1 \rightrightarrows Y_0$ is U-contractible and has a coequalizer $Y_0 \rightarrow Y$ in B , then the canonical map $Z \rightarrow YU$ is an isomorphism; U reflects U-contractible coequalizers if $Y_1 \rightrightarrows Y_0 \rightarrow Y$ being mapped into a contractible coequalizer diagram by U implies that $Y_1 \rightrightarrows Y_0 \rightarrow Y$ is a coequalizer diagram in B .



($Y_1 \rightrightarrows Y_0 \rightarrow Y$ will not necessarily be contractible in B .)

4. Precise tripleability theorem. U is tripleable $\iff B$ has, and U preserves and reflects, U-contractible coequalizers.

Proof. \Leftarrow is clear. One only has to notice that all coequalizers arising in the proof of the crude theorem were U-contractible. \Rightarrow : We can assume $B = A^U$ and prove that A^U has U^U -contractible coequalizers. (The (dual) example of comodules over a non-flat coalgebra shows that A^U need not have all coequalizers. But it follows from a result of Linton's alluded to below that A^U has all coequalizers if $A = \text{sets}$.) Let $(X_1, \xi_1) \xrightarrow{d_0} (X_0, \xi_0)$ be U^U -contractible, i.e. we have



the accompanying diagram in A . Let $X T \xrightarrow{\xi} X$ be $h_1 T \cdot \xi_0 d$. Then $d T \cdot \xi = \xi_0 d$. For

$$d T \cdot \xi = d T \cdot h_1 T \cdot \xi_0 d = (d h_1) T \cdot \xi_0 d =$$

$$\begin{aligned}
 &= (h_0 d_0) T. \xi_0 d = h_0 T. d_0 T. \xi_0 d = h_0 T. \xi_1 d_0 d = h_0 T. \xi_1 d_1 d \\
 &= h_0 T. d_1 T. \xi_0 d = (h_0 d_1) T. \xi_0 d = \xi_0 d.
 \end{aligned}$$

This shows that $d: X_0 \rightarrow X$ is compatible with \mathbb{T} -structures. Since $h_1 d = X$, it follows that (X, ξ) is a \mathbb{T} -algebra. Also if a different contraction h'_0, h'_1 were used, and ξ' defined as $h'_1 T. \xi_0 d$, then $\xi' = \xi$, since $\xi = (h_1 d) T. \xi = h_1 T. d T. \xi = h_1 T. \xi_0 d$, and $\xi' = (h_1 d) T. \xi = h_1 T. d T. \xi' = h_1 T. \xi_0 d$ also. Thus the \mathbb{T} -structure ξ is well-defined.

Finally, $d = \text{coeq}(d_0, d_1)$ for if $(X_0, \xi_0) \xrightarrow{y} (Y, \theta)$ coequalizes d_0, d_1 , then $(X, \xi) \xrightarrow{h_1 y} (Y, \theta)$ is the unique...*) The above construction shows that $U^{\mathbb{T}}$ preserves and reflects $U^{\mathbb{T}}$ -contractible coequalizers.

*) Note that h_1 is not an algebra map, but $h_1 y$ is.

5. Remarks. It should be possible to improve the above theorem (apart from streamlining the exposition). Conditions implying triple-ability should be found which are easier to verify in practice. For instance, the following is true:

U is tripleable $\iff \underline{B}$ has and U preserves U -contractible coequalizers, and U reflects isomorphisms.

It seems to follow without much difficulty, from this, that algebraic or varietal categories are tripleable / \mathcal{S} (and Linton can prove tripleable categories are varietal).