Lecture 6
QUANTUM CHANNELS

Stéphane ATTAL

Abstract This lecture is devoted to some of the most important tools in the theory of Quantum Open Systems, that is, quantum channels, completely positive maps, and their Krauss representations. We discuss dilations and physical examples of quantum channels, but also non-uniqueness of the Krauss representation. We show the close relation between completely positive maps and dual maps of quantum channels. We end this course by exhibiting a close parallel with the situation of dynamical systems and classical Markov chains.

We assume the reader is familiar with the basic axioms of Quantum Mechanics, with their extension to bipartite quantum systems, with density matrices and the way they are obtained as partial traces of pure states on larger systems (see Lecture 5 if necessary). The usual mathematical and physical literature, either deal with only the finite dimensional case or the general setup of C^*-algebras and von Neumann algebras. We have chosen to stick to a setup in between, that is, operators on Hilbert spaces, but eventually infinite dimensional Hilbert spaces.

Stéphane ATTAL
Institut Camille Jordan, Université Lyon 1, France
e-mail: attal@math.univ-lyon1.fr

6.1 Quantum Channels

A very important actor in the theory of open quantum systems, as well as in Quantum Information Theory, is the notion of *quantum channel*. They are the most general transform of a quantum state that are physically reasonable. They are the transforms of a quantum state resulting from any kind of interaction with a quantum environment. In Quantum Information Theory they represent the possible transforms of a quantum system after it has been transported via a non-perfect communication channel (whence the origin of the name "quantum channel").

6.1.1 Introduction

Let us recall the setup of quantum open systems. We are given two quantum systems interacting together, with state space $\mathcal{H}$ and $\mathcal{K}$ respectively. Our approach is in discrete time only in this section, that is, we shall look at the evolution of the two systems together for a fixed time duration τ. This is to say that if the Hamiltonian of the coupled system $\mathcal{H}\otimes\mathcal{K}$ is a self-adjoint operator $\mathsf{H}_{\rm tot}$ on $\mathcal{H}\otimes\mathcal{K}$, then the unitary evolution operator is

$$\mathsf{U}=\mathrm{e}^{-\mathrm{i}\tau\,\mathsf{H}_{\rm tot}}\,.$$

Such a unitary operator can be any kind of unitary operator U on $\mathcal{H}\otimes\mathcal{K}$, hence our ingredient for one step of the time evolution of the system $\mathcal{H}\otimes\mathcal{K}$ is just any given unitary operator U on $\mathcal{H}\otimes\mathcal{K}$.

In this context, the most general possible transform for the state of a quantum system $\mathcal{H}$ is obtained as follows:
– the quantum system $\mathcal{H}$, in the initial state ρ, is coupled to another quantum system $\mathcal{K}$ in the state ω,
– we let them evolve together for a fixed time duration under the unitary evolution operator U,
– we look at the state of $\mathcal{H}$ after this interaction.

This is clearly the most general transform one could think of for a quantum system: starting from an initial state, couple it to any kind of environment, let them evolve along any unitary evolution, finally look at the resulting state for $\mathcal{H}$.

Translating this in a more mathematical langage, we are given a state ω on $\mathcal{K}$, a unitary operator U on $\mathcal{H}\otimes\mathcal{K}$. Given any initial state ρ on $\mathcal{H}$, we couple the system $\mathcal{H}$ to $\mathcal{K}$ so that to obtain the state $\rho\otimes\omega$. After the evolution driven by U the state of the whole system is $\mathsf{U}(\rho\otimes\omega)\mathsf{U}^*$. Finally, ignoring the environment and focusing on the system $\mathcal{H}$, the resulting state is finally

$$\mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\rho\otimes\omega)\mathsf{U}^*\right)\,.$$

It is thus the aim of this section to study those mappings

$$\begin{array}{rcl} \mathcal{L} : \mathcal{L}_1(\mathcal{H}) & \longrightarrow & \mathcal{L}_1(\mathcal{H}) \\ \rho & \longmapsto & \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\rho \otimes \omega)\mathsf{U}^*\right) , \end{array} \tag{6.1}$$

to characterize them, to find useful representations of them. In particular we would like to find a representation of $\mathcal{L}$ which makes use only of ingredients coming from $\mathcal{H}$,in the same way as for the density matrices whose strong point is that they are a given ingredient of $\mathcal{H}$ from which one can compute probabilities of measurements and time evolutions on $\mathcal{H}$, independently of the fact we know or not the full environment or the full pure state on $\mathcal{H} \otimes \mathcal{K}$. Here it would be nice to have a way of computing $\mathcal{L}(\rho)$ without having to know $\mathcal{K}$, ω or U.

Definition 6.1. We call *quantum channel* any (linear) mapping on $\mathcal{L}_1(\mathcal{H})$ which is of the form

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\mathsf{T} \otimes \omega)\mathsf{U}^*\right)$$

for some auxiliary Hilbert space $\mathcal{K}$, some density matrix ω on $\mathcal{K}$ and some unitary operator U on $\mathcal{H} \otimes \mathcal{K}$.

Note that such a mapping automatically preserves the positivity of operators and preserves the trace. As a consequence, if T is a density matrix then $\mathcal{L}(\mathsf{T})$ is a density matrix too. Even though quantum channels are naturally defined on $\mathcal{L}_1(\mathcal{H})$, one usually consider them only as acting on density matrices, expressing this way the original idea underlying the definition: those mapping are the most general transforms of quantum states into quantum states.

6.1.2 Technicalities

Before characterizing completely the quantum channels, we shall need some technical results.

First, to the usual decomposition of bounded self-adjoint operators into positive and negative parts we add here a more precise result.

Lemma 6.2. *If T is a self-adjoint trace-class operator, then its positive and negative parts, T^+ and T^-, are also trace-class operators.*

Proof. If T is trace-class then $\mathrm{Tr}\,(|\mathsf{T}|) < \infty$. But

$$\mathrm{Tr}\,(|\mathsf{T}|) = \mathrm{Tr}\,(\mathsf{T}^+ + \mathsf{T}^-) = \mathrm{Tr}\,(\mathsf{T}^+) + \mathrm{Tr}\,(\mathsf{T}^-)$$

for T^+ and T^- are positive operators. Hence $\mathrm{Tr}\,(\mathsf{T}^+)$ and $\mathrm{Tr}\,(\mathsf{T}^-)$ are both finite quantities. The operators T^+ and T^- are thus trace-class. □

We now develop technical results which will be used to ensure the good convergence of the series of operators associated to quantum channels.

Proposition 6.3. *Let (M_n) be a sequence of bounded operators on a separable Hilbert space $\mathcal{H}$ such that the series $\sum_n \mathsf{M}_n^*\mathsf{M}_n$ converges weakly to a bounded operator X. If T is any trace-class operator on $\mathcal{H}$ then the series*

$$\sum_{n\in\mathbb{N}} \mathsf{M}_n\,\mathsf{T}\,\mathsf{M}_n^*$$

is trace-norm convergent and we have

$$\mathrm{Tr}\left(\sum_{n\in\mathbb{N}} \mathsf{M}_n\,\mathsf{T}\,\mathsf{M}_n^*\right) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})\,.$$

Proof. As a first step we assume that T is positive. Each of the operators $\mathsf{M}_n\,\mathsf{T}\,\mathsf{M}_n^*$ is then positive and trace-class. Put $\mathsf{Y}_n = \sum_{i\leq n} \mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^*$, for all $n\in\mathbb{N}$. For all $n<m$ the operator $\mathsf{Y}_m - \mathsf{Y}_n$ is positive and trace-class. Put $\mathsf{X}_n = \sum_{i\leq n} \mathsf{B}_i^*\,\mathsf{B}_i$. Then, for all $n<m$ we have

$$\begin{aligned}
\|\mathsf{Y}_m - \mathsf{Y}_n\|_1 &= \mathrm{Tr}\,(\mathsf{Y}_m - \mathsf{Y}_n)\\
&= \sum_{n<i\leq m} \mathrm{Tr}\,(\mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^*)\\
&= \mathrm{Tr}\,\big(\mathsf{T}\,(\mathsf{X}_m - \mathsf{X}_n)\big)\,.
\end{aligned}$$

As T can be decomposed into $\sum_{i\in\mathbb{N}} \lambda_i |e_i\rangle\langle e_i|$, with $\sum_{i\in\mathbb{N}} \lambda_i < \infty$, the last term above is equal to

$$\sum_{i\in\mathbb{N}} \lambda_i\,\langle e_i\,,\,(\mathsf{X}_m - \mathsf{X}_n)\,e_i\rangle\ .$$

As the X_n's converge weakly to X, each of the terms $\langle e_i\,,\,(\mathsf{X}_m - \mathsf{X}_n)\,e_i\rangle$ converges to 0 as n and m go to $+\infty$. Each of the sequences $(\mathsf{X}_n\,e_i)_{n\in\mathbb{N}}$ is bounded, for every weakly convergent sequence is bounded. Hence the terms $\langle e_i\,,\,(\mathsf{X}_m - \mathsf{X}_n)\,e_i\rangle$ are all bounded independently of n and m. By Lebesgue's Theorem $\|\mathsf{Y}_m - \mathsf{Y}_n\|_1$ tends to 0 when n and m tend to $+\infty$. In other words, the sequence (Y_n) is a Cauchy sequence in $\mathcal{L}_1(\mathcal{H})$, hence it converges to some trace-class operator Y.

The identity $\mathrm{Tr}\,(\mathsf{Y}) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})$ is now easy and left to the reader. We have proved the theorem for positive trace-class operators T.

As a second step, if T is a self-adjoint trace-class operator, then T can be decomposed as $\mathsf{T} = \mathsf{T}^+ - \mathsf{T}^-$ where T^+ and T^- are positive trace-class operators (Lemma 6.2) It is then easy to extend the above result to that case.

Finally, if T is any trace-class operator, one can then decompose it as $\mathsf{T} = \mathsf{A} + \mathrm{i}\,\mathsf{B}$ where $\mathsf{A} = (\mathsf{T} + \mathsf{T}^*)/2$ and $\mathsf{B} = -\mathrm{i}\,(\mathsf{T} - \mathsf{T}^*)/2$ are both self-adjoint and trace-class. It is easy to conclude now. □

We continue with some important results and structures exploring the properties of the sandwich-sum above.

Proposition 6.4. *Let $\mathcal{K}$ be an infinite dimensional Hilbert space with a given orthonormal basis (e_n).*
1) Let (M_n) be a sequence of bounded operators on $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \mathsf{M}_n^*\mathsf{M}_n$ converges weakly to a bounded operator X on $\mathcal{H}$. Then the series

$$\mathsf{M} = \sum_{n\in\mathbb{N}} (\mathsf{M}_n \otimes \mathrm{I})\, |e_n\rangle_{\mathcal{K}} \tag{6.2}$$

is strongly convergent and defines a bounded operator

$$\begin{aligned} \mathsf{M} : \mathcal{H} &\longrightarrow \quad \mathcal{H}\otimes\mathcal{K} \\ h &\longmapsto \sum_{n\in\mathbb{N}} \mathsf{M}_n\, h \otimes e_n \,. \end{aligned}$$

In particular we have

$$\mathsf{M}_n = {}_{\mathcal{K}}\langle e_n|\, \mathsf{M}\,. \tag{6.3}$$

This operator M satisfies satisfies

$$\mathsf{M}^*\mathsf{M} = \mathsf{X} \tag{6.4}$$

and

$$\mathsf{M}^* = \sum_{n\in\mathbb{N}} {}_{\mathcal{K}}\langle e_n|\, (\mathsf{M}_n^* \otimes \mathrm{I})\,, \tag{6.5}$$

where the series is weakly convergent.
2) Conversely, let M be any bounded operator from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{K}$. For all $n \in \mathbb{N}$ put

$$\mathsf{M}_n = {}_{\mathcal{K}}\langle e_n|\, \mathsf{M}\,.$$

Then the M_n's are bounded operators on $\mathcal{H}$, the sum $\sum_n \mathsf{M}_n^*\mathsf{M}_n$ converges strongly to $\mathsf{M}^*\mathsf{M}$ and the operator M is given by

$$\mathsf{M} = \sum_{n\in\mathbb{N}} (\mathsf{M}_n \otimes \mathrm{I})\, |e_n\rangle_{\mathcal{K}}\,,$$

where the sum is strongly convergent.
3) Finally, in any of two cases above, the mapping

$$\mathsf{T} \mapsto \mathcal{L}(\mathsf{T}) = \sum_{n\in\mathbb{N}} \mathsf{M}_n\, \mathsf{T}\, \mathsf{M}_n^*\,,$$

on $\mathcal{L}_1(\mathcal{H})$, is given by

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*\right)\,.$$

In particular, the mapping $\mathcal{L}$ is a bounded operator on $\mathcal{L}_1(\mathcal{H})$ with

$$\|\mathcal{L}\| \leq \|\mathsf{M}^*\mathsf{M}\| \ .$$

Proof. Let us prove 1). As we have

$$\left\|\sum_{i=n}^{m} \mathsf{M}_i\, h \otimes e_i\right\|^2 = \sum_{i=n}^{m} \|\mathsf{M}_i\, h\|^2 = \left\langle h\,,\, \left(\sum_{i=n}^{m} \mathsf{M}_i^*\mathsf{M}_i\right) h\right\rangle\,,$$

then the series $\sum_{n\in\mathbb{N}} \mathsf{M}_n\, h \otimes e_n$ is convergent in $\mathcal{H}\otimes\mathcal{K}$. This shows that the series

$$\mathsf{M} = \sum_{n\in\mathbb{N}} (\mathsf{M}_n \otimes \mathrm{I})\, |e_n\rangle_{\mathcal{K}}\,,$$

is strongly convergent. By the way we get

$$\begin{aligned}
\|\mathsf{M}\, h\|^2 &= \left\|\sum_{n\in\mathbb{N}} \mathsf{M}_n\, h \otimes e_n\right\|^2 \\
&= \left\langle h\,,\, \sum_{n\in\mathbb{N}} \mathsf{M}_n^*\mathsf{M}_n\, h\right\rangle \\
&= \langle h\,,\, \mathsf{X}\, h\rangle \\
&\leq \|\mathsf{X}\|\ \|h\|^2\,.
\end{aligned}$$

This shows that M is a bounded operator. Hence the adjoint of M is given by

$$\mathsf{M}^* = \sum_{n\in\mathbb{N}} {}_{\mathcal{K}}\langle e_n|(\mathsf{M}_n^* \otimes \mathrm{I})\,,$$

where the series is weakly convergent. Now, for all $g, h \in \mathcal{H}$ we have

$$\begin{aligned}
\langle g\,,\, \mathsf{M}^*\mathsf{M}\, h\rangle &= \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{m} \left\langle g\,,\, {}_{\mathcal{K}}\langle e_j|(\mathsf{M}_j^*\otimes\mathrm{I})(\mathsf{M}_i\otimes\mathrm{I})|e_i\rangle_{\mathcal{K}}\, h\right\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{m} \left\langle g\,,\, {}_{\mathcal{K}}\langle e_j|(\mathsf{M}_j^*\mathsf{M}_i\, h\otimes e_i)\right\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=0}^{n\wedge m} \langle g\,,\, \mathsf{M}_i^*\mathsf{M}_i\, h\rangle \\
&= \langle g\,,\, \mathsf{X}\, h\rangle \ .
\end{aligned}$$

This proves that $\mathsf{M}^*\mathsf{M} = \mathsf{X}$ and we have proved the part 1).

Let us prove 2) now. Let M be a bounded operator from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{K}$ and put $\mathsf{M}_n = {}_{\mathcal{K}}\langle e_n|\,\mathsf{M}$ for all $n\in\mathbb{N}$. Each M_n is a bounded operator on $\mathcal{H}$ and we have

$$\begin{aligned}\sum_{n=0}^{N} \mathsf{M}_n^*\mathsf{M}_n &= \sum_{n=0}^{N} \mathsf{M}^*|e_n\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle e_n|\,\mathsf{M}\\ &= \mathsf{M}^*(\mathrm{I}\otimes \mathsf{P}_N)\mathsf{M}\,,\end{aligned}$$

where P_N is the orthogonal projector of $\mathcal{K}$ onto the space generated by $e_0, \ldots, e_N$. This proves that $\sum_{n=0}^N \mathsf{M}_n^*\mathsf{M}_n$ converges strongly to $\mathsf{M}^*\mathsf{M}$ when N tends to $+\infty$.

Let us prove that M and the M_n's are related by (6.2). We have for all $h \in \mathcal{H}$

$$\begin{aligned}\sum_{n=0}^{N} (\mathsf{M}_n\otimes \mathrm{I})|e_n\rangle_{\mathcal{K}}\, h &= \sum_{n=0}^{N} \big({}_{\mathcal{K}}\langle e_n|\mathsf{M}\, h\big)\otimes e_n\\ &= \sum_{n=0}^{N} |e_n\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle e_n|\,\mathsf{M}\, h\\ &= (\mathrm{I}\otimes \mathsf{P}_N)\mathsf{M}\, h\,.\end{aligned}$$

This proves the strong convergence of $\sum_{n=0}^N (\mathsf{M}_n\otimes \mathrm{I})|e_n\rangle_{\mathcal{K}}$ to M. We have proved 2).

Let us prove 3) finally. The quantity $\mathrm{Tr}_{\mathcal{K}}(\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*)$ is the $\|\cdot\|_1$-convergent sum

$$\sum_{n\in\mathbb{N}} {}_{\mathcal{K}}\langle e_n|\,\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*\,|e_n\rangle_{\mathcal{K}}\,,$$

that is,

$$\mathrm{Tr}_{\mathcal{K}}(\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*) = \sum_{n\in\mathbb{N}} \mathsf{M}_n\,\mathsf{T}\,\mathsf{M}_n^* = \mathcal{L}(\mathsf{T})\,.$$

This proves the announced identity. We prove the norm estimate, we have

$$\begin{aligned}\|\mathcal{L}(\mathsf{T})\|_1 &= \|\mathrm{Tr}_{\mathcal{K}}(\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*)\|_1 \leq \|\mathsf{M}\,\mathsf{T}\,\mathsf{M}^*\|_1\\ &\leq \|\mathsf{M}\|^2\ \|\mathsf{T}\|_1 = \|\mathsf{M}^*\mathsf{M}\|\ \|\mathsf{T}\|_1\,.\end{aligned}$$

All the assertions of the proposition have been proved. □

6.1.3 Krauss Representation

We are now technically ready to obtain the most important result concerning quantum channels on $\mathcal{L}_1(\mathcal{H})$: they admit a particular representation, the *Krauss representation*, involving only operators on $\mathcal{H}$. This is a crucial point for using quantum channels in everyday life (see the discussion at the end of Subsection 6.1.1).

Theorem 6.5 (Krauss representation of quantum channels). *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Let U be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$ and let ω be a density matrix on $\mathcal{K}$. Define the quantum channel*

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}_{\mathcal{K}}(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*)$$

on $\mathcal{L}_1(\mathcal{H})$. Then there exists an at most countable family $(\mathsf{M}_i)_{i\in I}$ of bounded operators on $\mathcal{H}$ such that

$$\boxed{\sum_{i\in I} \mathsf{M}_i^*\,\mathsf{M}_i = \mathrm{I}\,,} \tag{6.6}$$

in the strong convergence sense, and satisfying

$$\boxed{\mathcal{L}(\mathsf{T}) = \sum_{i\in I} \mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^*\,,} \tag{6.7}$$

for all $\mathsf{T}\in\mathcal{L}_1(\mathcal{H})$ and where the series above is $\|\cdot\|_1$-convergent.

Proof. Assume first that ω is a pure state $|\psi\rangle\langle\psi|$. Let $(f_i)_{i\in I}$ be an orthonormal basis of $\mathcal{K}$. The partial trace $\mathrm{Tr}_{\mathcal{K}}(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*)$ is given by

$$\sum_{i\in I} {}_{\mathcal{K}}\langle f_i|\,\mathsf{U}\,(\mathsf{T}\otimes|\psi\rangle\langle\psi|)\,\mathsf{U}^*\,|f_i\rangle_{\mathcal{K}}\,,$$

where this series is $\|\cdot\|_1$-convergent. But note that we have

$$\mathsf{T}\otimes|\psi\rangle\langle\psi| = |\psi\rangle_{\mathcal{K}}\;\mathsf{T}\;{}_{\mathcal{K}}\langle\psi|\,,$$

as can be checked easily. Hence we have

$$\mathrm{Tr}_{\mathcal{K}}(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*) = \sum_{i\in I} {}_{\mathcal{K}}\langle f_i|\,\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\;\mathsf{T}\;{}_{\mathcal{K}}\langle\psi|\,\mathsf{U}^*\,|f_i\rangle_{\mathcal{K}}\,.$$

Let us put

$$\mathsf{M}_i = {}_{\mathcal{K}}\langle f_i|\,\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\,,$$

for all $i\in I$. This clearly defines bounded operators on $\mathcal{H}$ and we have

$$\mathsf{M}_i^* = {}_{\mathcal{K}}\langle\psi|\,\mathsf{U}^*\,|f_i\rangle_{\mathcal{K}}\,.$$

We have proved that $\mathrm{Tr}_{\mathcal{K}}(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*)$ can be written as $\sum_i \mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^*$ where the series is $\|\cdot\|_1$-convergent.

Now, if we compute

$$\sum_{i\in I} \mathsf{M}_i^*\,\mathsf{M}_i = \sum_{i\in I} {}_{\mathcal{K}}\langle\psi|\,\mathsf{U}^*\,|f_i\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle f_i|\,\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\,,$$

we first notice that this series is strongly convergent, for $\sum_{i\in I} |f_i\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle f_i|$ is strongly convergent. But also, as $\sum_{i\in I} |f_i\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle f_i|$ converges to the identity operator I, we get that $\sum_{i\in I} \mathsf{M}_i^*\mathsf{M}_i$ converges strongly to

$$ {}_{\mathcal{K}}\langle\psi|\, \mathsf{U}^*\, \mathsf{U}\, |\psi\rangle_{\mathcal{K}} = {}_{\mathcal{K}}\langle\psi|\, \mathrm{I}\, |\psi\rangle_{\mathcal{K}} = \mathrm{I}_{\mathcal{H}} \,. $$

We have proved the theorem when ω is a pure state. Now consider a general density matrix ω on $\mathcal{K}$, it can be decomposed into

$$ \omega = \sum_{k\in I} \mu_k |\psi_k\rangle\langle\psi_k| $$

for some orthonormal basis $(\psi_k)_{k\in I}$ of $\mathcal{K}$. For all $i, k \in I$ we put

$$ \mathsf{M}_i^k = \sqrt{\mu_k}\, {}_{\mathcal{K}}\langle f_i|\, \mathsf{U}\, |\psi_k\rangle_{\mathcal{K}} \,. $$

We then have

$$ \sum_{i,k\in I} \mathsf{M}_i^{k\,*}\mathsf{M}_i^k = \sum_{i,k\in I} \mu_k\, {}_{\mathcal{K}}\langle\psi_k|\, \mathsf{U}^*\, |f_i\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle f_i|\, \mathsf{U}\, |\psi_k\rangle_{\mathcal{K}} = \mathrm{I}_{\mathcal{H}} \,, $$

where the series above is obviously strongly convergent. By Proposition 6.3, for every $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$ the series

$$ \sum_{i,k\in I} \mathsf{M}_i^k\, \mathsf{T}\, \mathsf{M}_i^{k\,*} $$

is $\|\cdot\|_1$-convergent. Its sum is equal to

$$ \begin{aligned} \sum_{i,k\in I} \mu_k\, {}_{\mathcal{K}}\langle f_i|\, \mathsf{U}\, |\psi_k\rangle_{\mathcal{K}}\, \mathsf{T}\, {}_{\mathcal{K}}\langle\psi_k|\, \mathsf{U}^*\, |f_i\rangle_{\mathcal{K}} &= \\ &= \sum_{i\in I} {}_{\mathcal{K}}\langle f_i|\, \mathsf{U} \left(\mathsf{T} \otimes \sum_{k\in I} \mu_k\, |\psi_k\rangle_{\mathcal{K}}\,{}_{\mathcal{K}}\langle\psi_k| \right) \mathsf{U}^*\, |f_i\rangle_{\mathcal{K}} \\ &= \sum_{i\in I} {}_{\mathcal{K}}\langle f_i|\, \mathsf{U}\, (\mathsf{T}\otimes\omega)\, \mathsf{U}^*\, |f_i\rangle_{\mathcal{K}} \\ &= \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}\, (\mathsf{T}\otimes\omega)\, \mathsf{U}^* \right) \,, \end{aligned} $$

where we leave details to the reader. This proves the theorem. □

Definition 6.6. The decomposition (6.7) of a quantum channel $\mathcal{L}$ is called a *Krauss decomposition* of the quantum channel.

6.1.4 Unitary Dilations

We have defined quantum channels as associated to partial traces of unitary conjugations. We have then proved that they admit a Krauss representation. We shall now prove a reciprocal: every mapping on the trace-class operators given by a Krauss decomposition as above comes from the partial trace of some unitary conjugation on some larger space, that is, it defines a quantum channel.

Theorem 6.7. *Let $\mathcal{H}$ be some separable Hilbert space. Let $\mathcal{L}$ be a linear mapping on $\mathcal{L}_1(\mathcal{H})$ of the form*

$$\mathcal{L}(\mathsf{T}) = \sum_{i\in I} \mathsf{M}_i\, \mathsf{T}\, \mathsf{M}_i^*$$

for some bounded operators M_i on $\mathcal{H}$ satisfying

$$\sum_{i\in I} \mathsf{M}_i^*\, \mathsf{M}_i = \mathrm{I}\,,$$

in the strong convergence sense. Then there exists a separable Hilbert space $\mathcal{K}$, a density matrix ω on $\mathcal{K}$ and a unitary operator U on $\mathcal{H}\otimes\mathcal{K}$ such that

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}\,(\mathsf{T}\otimes\omega)\,\mathsf{U}^*\right)\,,$$

for all $\mathsf{T}\in\mathcal{L}_1(\mathcal{H})$.

Furthermore, it is always possible to choose $\mathcal{K}, \mathsf{U}$ and ω is such a way that ω is a pure state.

Proof. Consider a mapping $\mathcal{L}$ on $\mathcal{L}_1(\mathcal{H})$ of the form

$$\mathcal{L}(\mathsf{T}) = \sum_{i\in I} \mathsf{M}_i\, \mathsf{T}\, \mathsf{M}_i^*\,,$$

with the M_i's being bounded operators on $\mathcal{H}$ and

$$\sum_{i\in I} \mathsf{M}_i^*\, \mathsf{M}_i = \mathrm{I}\,,$$

in the strong convergence sense. We consider the Hilbert space $\mathcal{K}$ with some orthonormal basis $(f_i)_{i\in I}$ indexed by the same set I (which may be finite or countable infinite). Define the linear operator

$$\begin{aligned} \mathsf{V} : \mathcal{H}\otimes\mathbb{C}\,|f_0\rangle &\longrightarrow \quad \mathcal{H}\otimes\mathcal{K} \\ x\otimes f_0 &\longmapsto \sum_{j\in I} \mathsf{M}_j\, x\otimes f_j\,, \end{aligned}$$

Then V is isometric for

$$\|\mathsf{V}(x\otimes f_0)\|^2=\sum_{j\in I}\|\mathsf{M}_j\,x\|^2=\left\langle x\,,\,\sum_{j\in I}\mathsf{M}_j^*\,\mathsf{M}_j\,x\right\rangle=\|x\|^2=\|x\otimes f_0\|^2\,.$$

We now wish to extend the operator V into a unitary operator from $\mathcal{H}\otimes\mathcal{K}$ to $\mathcal{H}\otimes\mathcal{K}$. We shall consider several cases.

Assume first that $\mathcal{H}$ is finite dimensional, then the range of V is a finite dimensional subspace of $\mathcal{H}\otimes\mathcal{K}$, with same dimension as $\mathcal{H}$. If $\mathcal{K}$ is finite dimensional then $\mathcal{H}\otimes(\mathcal{K}\ominus\mathbb{C}|f_0\rangle)$ is of same dimension as $(\mathrm{Ran}\,\mathsf{V})^\perp$; if $\mathcal{K}$ is infinite dimensional then $\mathcal{H}\otimes(\mathcal{K}\ominus\mathbb{C}|f_0\rangle)$ and $(\mathrm{Ran}\,\mathsf{V})^\perp$ are both infinite dimensional separable Hilbert spaces. In both cases the space $(\mathrm{Ran}\,\mathsf{V})^\perp$ can be mapped unitarily to $\mathcal{H}\otimes(\mathcal{K}\ominus\mathbb{C}|f_0\rangle)$ through a unitary operator W. The operator $\widehat{\mathsf{V}}$ from $\mathcal{H}\otimes\mathcal{K}$ to itself, which acts as V on $\mathcal{H}\otimes\mathbb{C}|f_0\rangle$ and as W^* on $\mathcal{H}\otimes(\mathcal{K}\ominus\mathbb{C}|f_0\rangle)$ is then unitary and extends V.

If $\mathcal{H}$ is infinite dimensional it may happen that $\mathrm{Ran}\,\mathsf{V}$ is the whole of $\mathcal{H}\otimes\mathcal{K}$, so we cannot directly extend V into a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. We embed $\mathcal{K}$ into a larger Hilbert space $\mathcal{K}'$ by adding one new vector, f_{-1} say, orthogonal to all $\mathcal{K}$. The space $(\mathrm{Ran}\,\mathsf{V})^\perp$ in $\mathcal{H}\otimes\mathcal{K}'$ is then infinite dimensional (separable), as is $\mathcal{H}\otimes(\mathcal{K}'\ominus\mathbb{C}|f_0\rangle)$. Once again one can unitarily map $\mathcal{H}\otimes(\mathcal{K}'\ominus\mathbb{C}|f_0\rangle)$ onto $(\mathrm{Ran}\,\mathsf{V})^\perp$ and this way obtain an extension of V into a unitary operator $\widehat{\mathsf{V}}$ from $\mathcal{H}\otimes\mathcal{K}'$ onto itself.

In any case, we have obtained a Hilbert space which we denote by $\mathcal{K}$, with an orthonormal basis $(f_i)_{i\in J}$ where the set J contains the original set I; we have obtained a unitary extension $\widehat{\mathsf{V}}$ of V from $\mathcal{H}\otimes\mathcal{K}$ onto itself.

We want now to prove that they provide the announced quantum channel. Let T be a trace class operator, which we first assume to be a pure state $|\psi\rangle\langle\psi|$. If we compute

$$\mathrm{Tr}_{\mathcal{K}}\left(\widehat{\mathsf{V}}\,(\mathsf{T}\otimes|f_0\rangle\langle f_0|)\,\widehat{\mathsf{V}}^*\right)$$

we obtain, by considering the orthonormal basis $(f_i)_{i\in J}$ of $\mathcal{K}$ as described above

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{K}}\left(\widehat{\mathsf{V}}(\mathsf{T}\otimes|f_0\rangle\langle f_0|)\widehat{\mathsf{V}}^*\right)&=\sum_{i\in J}{}_{\mathcal{K}}\langle f_i|\,\widehat{\mathsf{V}}\,(\mathsf{T}\otimes|f_0\rangle\langle f_0|)\,\widehat{\mathsf{V}}^*\,|f_i\rangle_{\mathcal{K}}\\
&=\sum_{i\in J}{}_{\mathcal{K}}\langle f_i|\,\widehat{\mathsf{V}}\,(|\psi\otimes f_0\rangle\langle\psi\otimes f_0|)\,\widehat{\mathsf{V}}^*\,|f_i\rangle_{\mathcal{K}}\\
&=\sum_{\substack{i\in J\\ k,l\in I}}{}_{\mathcal{K}}\langle f_i|\,(|\mathsf{M}_k\psi\otimes f_k\rangle\langle\mathsf{M}_l\,\psi\otimes f_l|)\,|f_i\rangle_{\mathcal{K}}\\
&=\sum_{i\in I}|\mathsf{M}_i\,\psi\rangle\langle\mathsf{M}_i\,\psi|\\
&=\sum_{i\in I}\mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^*\,.
\end{aligned}$$

This proves the announced result for this kind of operator T. Extending this result to general trace class operator is now easy and left to the reader.

The last remark at the end of the theorem is now obvious for ω is a pure state in our construction above. □

6.1.5 Properties

The different characterizations we have obtained for quantum channels allow to derive some basic properties.

Proposition 6.8. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}_1, \mathcal{L}_2$ be two quantum channels on $\mathcal{L}_1(\mathcal{H})$ with Krauss decompositions*

$$\mathcal{L}_1(\mathsf{T}) = \sum_{i\in\mathbb{N}} \mathsf{A}_i \, \mathsf{T} \, \mathsf{A}_i^* \qquad \textit{and} \qquad \mathcal{L}_2(\rho) = \sum_{i\in\mathbb{N}} \mathsf{B}_i \, \mathsf{T} \, \mathsf{B}_i^* \, ,$$

respectively.

1) The composition $\mathcal{L}_2 \circ \mathcal{L}_1$ is a quantum channel on $\mathcal{L}_1(\mathcal{H})$. It admits a Krauss decomposition given by

$$\mathcal{L}_2 \circ \mathcal{L}_1(\mathsf{T}) = \sum_{i,j\in\mathbb{N}} \mathsf{B}_j \mathsf{A}_i \, \mathsf{T} \, \mathsf{A}_i^* \mathsf{B}_j^* \, .$$

2) Any convex combination $\lambda\mathcal{L}_1 + (1-\lambda)\mathcal{L}_2$ (with $0 \leq \lambda \leq 1$) is a quantum channel on $\mathcal{L}_1(\mathcal{H})$ with Krauss decomposition

$$\begin{aligned}(\lambda\mathcal{L}_1 + (1-\lambda)\mathcal{L}_2)\,(\mathsf{T}) &= \\ &= \sum_{i\in\mathbb{N}} \left(\sqrt{\lambda}\,\mathsf{A}_i\right) \mathsf{T} \left(\sqrt{\lambda}\,\mathsf{A}_i\right)^* + \sum_{i\in\mathbb{N}} \left(\sqrt{1-\lambda}\,\mathsf{B}_i\right) \, \mathsf{T} \, \left(\sqrt{1-\lambda}\,\mathsf{B}_i\right)^* \, .\end{aligned}$$

Proof.

1) By definition, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are quantum channels on $\mathcal{L}_1(\mathcal{H})$, this means that for $i = 1, 2$ there exists Hilbert spaces $\mathcal{K}_i$, states ω_i on $\mathcal{K}_i$ and unitary operators U_i on $\mathcal{K}_i$ such that

$$\mathcal{L}_i(\mathsf{T}) = \mathrm{Tr}_{\mathcal{K}_i}(\mathsf{U}_i(\rho \otimes \omega_i)\mathsf{U}_i^*) \, ,$$

for all $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$.

We shall now consider the Hilbert space $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$, the quantum state $\omega = \omega_1 \otimes \omega_2$. We consider the natural ampliations $\widehat{\mathsf{U}}_i$ of U_i to $\mathcal{H} \otimes \mathcal{K}$ by tensorizing U_i with the identity operator on the space $\mathcal{K}_j$ where it is not initially defined. Finally, put $\mathsf{U} = \widehat{\mathsf{U}}_2 \, \widehat{\mathsf{U}}_1$, it is obviously a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Now we have, using basic properties of the partial traces

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*\right) &= \mathrm{Tr}_{\mathcal{K}_1\otimes\mathcal{K}_2}\left(\widehat{\mathsf{U}}_2\,\widehat{\mathsf{U}}_1(\mathsf{T}\otimes\omega_1\otimes\omega_2)\widehat{\mathsf{U}}_1^*\widehat{\mathsf{U}}_2^*\right)\\
&= \mathrm{Tr}_{\mathcal{K}_2}\left(\mathrm{Tr}_{\mathcal{K}_1}\left(\widehat{\mathsf{U}}_2\left(\mathsf{U}_1(\mathsf{T}\otimes\omega_1)\mathsf{U}_1^*\otimes\omega_2\right)\widehat{\mathsf{U}}_2^*\right)\right)\\
&= \mathrm{Tr}_{\mathcal{K}_2}\left(\mathsf{U}_2\left(\,\mathrm{Tr}_{\mathcal{K}_1}(\mathsf{U}_1(\mathsf{T}\otimes\omega_1)\mathsf{U}_1^*)\otimes\omega_2\right)\mathsf{U}_2^*\right)\\
&= \mathrm{Tr}_{\mathcal{K}_2}\left(\mathsf{U}_2\left(\mathcal{L}_1(\mathsf{T})\otimes\omega_2\right)\mathsf{U}_2^*\right)\\
&= \mathcal{L}_2\left(\mathcal{L}_1(\mathsf{T})\right)\,.
\end{aligned}$$

We have obtained $\mathcal{L}_2\circ\mathcal{L}_1$ as the partial trace $\mathsf{T}\mapsto\mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\mathsf{T}\otimes\omega)\mathsf{U}^*\right)$. By definition, it is a quantum channel.

The unitary dilations of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be chosen in such a way that ω_1 and ω_2 are pure states $|\psi_i\rangle\langle\psi_i|$ (Theorem 6.7). We are given orthonormal bases $(e_i)_{i\in\mathbb{N}}$ and $(f_j)_{j\in\mathbb{N}}$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. In the proof of Theorem 6.5 it is shown that the coefficients of a Krauss representation of $\mathcal{L}_1$ can be obtained as

$$\mathsf{A}_i = {}_{\mathcal{K}_1}\langle e_i|\,\mathsf{U}_1\,|\psi_1\rangle_{\mathcal{K}_1}$$

and, in the same way, those of $\mathcal{L}_2$ are obtained as

$$\mathsf{B}_i = {}_{\mathcal{K}_2}\langle f_j|\,\mathsf{U}_2\,|\psi_2\rangle_{\mathcal{K}_2}\,.$$

Now if we compute those of $\mathcal{L}_2\circ\mathcal{L}_1$ we get

$$\begin{aligned}
\mathsf{M}_{ij} &= {}_{\mathcal{K}_1\otimes\mathcal{K}_2}\langle e_i\otimes f_j|\,\mathsf{U}\,|\psi_1\otimes\psi_2\rangle_{\mathcal{K}_1\otimes\mathcal{K}_2}\\
&= {}_{\mathcal{K}_2}\langle f_j|\,{}_{\mathcal{K}_1}\langle e_i|\,\widehat{\mathsf{U}}_2\,\widehat{\mathsf{U}}_1\,|\psi_1\rangle_{\mathcal{K}_1}\,|\psi_2\rangle_{\mathcal{K}_2}\\
&= {}_{\mathcal{K}_2}\langle f_j|\,\mathsf{U}_2\,{}_{\mathcal{K}_1}\langle e_i|\,\widehat{\mathsf{U}}_1\,|\psi_1\rangle_{\mathcal{K}_1}\,|\psi_2\rangle_{\mathcal{K}_2}\\
&= {}_{\mathcal{K}_2}\langle f_j|\,\mathsf{U}_2\,|\psi_2\rangle_{\mathcal{K}_2}\,{}_{\mathcal{K}_1}\langle e_i|\,\mathsf{U}_1\,|\psi_1\rangle_{\mathcal{K}_1}\\
&= \mathsf{B}_j\,\mathsf{A}_i\,.
\end{aligned}$$

This gives the announced Krauss representation, together with a proof of the strong convergence of $\sum_{i,j\in\mathbb{N}}\mathsf{A}_i^*\mathsf{B}_j^*\mathsf{B}_j\mathsf{A}_i$. We have proved 1).

The property 2) is very easy to prove, using directly their Krauss decompositions. □

6.1.6 Examples of Quantum Channels

In this subsection we describe some concrete physical examples of quantum channels. These examples come from Quantum Physics or from Quantum Information Theory. The setup here is the one of the simplest (non-trivial) Hilbert space, that is, $\mathbb{C}^2$. This describes the simplest quantum system, usually called *qubit*, that is, the quantum state space of a two-level system.

We make use here of the usual notations for two-level quantum systems, in particular of the Pauli matrices and the Bloch sphere representation. In this subsection we adopt notations which are typical of Quantum Physics or Quantum Information Theory for qubits, that is, our chosen orthonormal basis of $\mathbb{C}^2$ is denoted by $(|0\rangle, |1\rangle)$.

6.1.6.1 The Depolarizing Channel

The quantum channel that we describe here is part of the so-called *noisy channels* in Quantum Information Theory. The noisy channels describe what occurs to a qubit which is transmitted to someone else and which is affected by the fact that the transmission is not perfect: the communication channel has to undergo some perturbations (some noise) coming from the environment. Hence, the noisy channel tries to describe the typical defects that the quantum bit may undergo during its transmission.

The noisy channel that we shall describe is the *depolarizing channel*. It describes the fact that the qubit may be left unchanged with probability $q = 1 - p \in [0, 1]$, or may undergo, with probability $p/3$, one the three following transformations:

– bit flip: $\begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_x|\psi\rangle\,,$

– phase flip: $\begin{cases} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto |-1\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_z|\psi\rangle\,,$

– both: $\begin{cases} |0\rangle \mapsto i|1\rangle \\ |1\rangle \mapsto -i|0\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_y|\psi\rangle\,.$

This channel can be represented through a unitary evolution U staking a four dimensional environment $\mathcal{H}_E$ with orthonormal basis denoted by $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. More precisely, the small system is $\mathcal{H}_A = \mathbb{C}^2$, which we identify to the subspace $\mathcal{H}_A \otimes |0\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_E$. The operator U acts on $\mathcal{H}_A \otimes \mathcal{H}_E$ by

$$\begin{aligned}\mathsf{U}\,(|\psi\rangle \otimes |0\rangle) = \sqrt{1-p}\;|\psi\rangle \otimes |0\rangle + \sqrt{\frac{p}{3}}\Big[\sigma_x|\psi\rangle \otimes |1\rangle + \sigma_y|\psi\rangle \otimes |2\rangle + \\ + \sigma_z|\psi\rangle \otimes |3\rangle\Big]\end{aligned}$$

and U is completed in any way as a unitary operator on $\mathcal{H}_A \otimes \mathcal{H}_E$.

The effect of the transform U when seen only from the small system $\mathcal{H}_A$ is then

$$\mathcal{L}(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\rho \otimes |0\rangle\langle 0|)\mathsf{U}^*\right)\,.$$

An easy computation gives the Krauss representation

$$\mathcal{L}(\rho) = \sum_{i=0}^{3} \mathsf{M}_i \, \rho \, \mathsf{M}_i^*$$

with

$$\mathsf{M}_0 = \sqrt{1-p}\ \mathrm{I}\,,\ \ \mathsf{M}_1 = \sqrt{\frac{p}{3}}\ \sigma_x\,,\ \ \mathsf{M}_2 = \sqrt{\frac{p}{3}}\ \sigma_y\,,\ \ \mathsf{M}_3 = \sqrt{\frac{p}{3}}\ \sigma_z\,.$$

Another interesting way to see this map acting on density matrices is to see it acting on the Bloch sphere. Recall that any qubit state can be represented as a point (x, y, z) in the 3 dimensional ball:

$$\rho = \frac{1}{2}\left(\mathrm{I} + x\sigma_x + y\sigma_y + z\sigma_z\right),$$

with $x^2 + y^2 + z^2 \leq 1$. An easy computation shows that

$$\mathcal{L}(\rho) = \frac{1}{2}\left(\mathrm{I} + \left(1 - \frac{4p}{3}\right)(x\sigma_x + y\sigma_y + z\sigma_z)\right).$$

As a mapping of the ball, the depolarizing channel acts simply as an homothetic transformation with rate $1 - 4p/3$.

This mapping illustrates very easily an interesting fact about quantum channels. This fact will be discussed more deeply in Subsection 6.4.3 but we can already make an easy remark here. The point is that quantum channels, when they are non-trivial, are non-invertible; there happens a loss of the invertibility of the unitary conjugation with the loss of information that occurred when taking the partial trace. A quantum channel can be invertible as a linear map, but not as a quantum channel. The inverse in general does not map quantum states to quantum states, it fails in preserving the positivity.

The depolarizing channel illustrates very clearly this fact: it coincides with the homothetic transformation of the unit ball, with a factor $1 - 4p/3$. Its inverse exists for $p \neq 3/4$, it is the inflation of the unit ball with factor $(1 - 4p/3)^{-1}$. This defines a linear map, but it does not map states to states, for it may map a point of the unit ball outside of the unit ball. From the point of view of quantum states this means that it maps some density matrices to some trace-class operators with strictly negative eigenvalues, that is, non-physical quantum states.

6.1.6.2 The Phase-Damping Channel

The second type of channel we shall describe is the *phase-damping channel*. It is advantageously defined through a unitary evolution involving the small system $\mathcal{H}_A = \mathbb{C}^2$ and the environment $\mathcal{H}_E$ which is now 3 dimensional. This unitary operator is given by

$$\mathsf{U}\,(|0\rangle \otimes |0\rangle) = \sqrt{1-p}\,\,|0\rangle \otimes |0\rangle + \sqrt{p}\,\,|0\rangle \otimes |1\rangle$$

and

$$\mathsf{U}\,(|1\rangle \otimes |0\rangle) = \sqrt{1-p}\,\,|1\rangle \otimes |0\rangle + \sqrt{p}\,\,|0\rangle \otimes |2\rangle\,.$$

The action of U on the other types of vectors of $\mathcal{H}_A \otimes \mathcal{H}_E$ is not necessary to describe.

In this interaction with the environment, the small system is not changed, only the environment, in contact with $\mathcal{H}_A$ may scatter (with probability p) from the ground state $|0\rangle$ to an excited state $|1\rangle$ or $|2\rangle$, depending on the state of $\mathcal{H}_A$. In some way, the environment "reads" the state of $\mathcal{H}_A$ and may be influenced by it.

The Krauss decomposition resulting from this unitary transform is then given by

$$\mathcal{L}(\rho) = \sum_{i=0}^{2} \mathsf{M}_i\,\rho\,\mathsf{M}_i^*$$

with

$$\mathsf{M}_0 = \sqrt{1-p}\,\mathrm{I}\,, \qquad \mathsf{M}_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}\,, \qquad \mathsf{M}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}\,.$$

An easy computation shows that the associated action on the Bloch sphere is the transform:

$$(x, y, z) \longmapsto ((1-p)x, (1-p)y, z)\ .$$

In particular, repeated applications of this quantum channel make any initial quantum state

$$\rho_0 = \begin{pmatrix} a & z \\ \overline{z} & b \end{pmatrix}$$

converge exponentially fast to the state

$$\rho_\infty = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\ .$$

This can understood as follows. As we analyzed above, the environment acts as a watcher of the state of $\mathcal{H}_A$, at least with a certain probability p. In the end, after numerous applications of the same action, the environment ends up in completely measuring the state of $\mathcal{H}_A$ along the basis $\{|0\rangle\,, |1\rangle\}$. The quantum state ρ_0 loses its off-diagonal coefficients and ends up completely diagonal, that is, as a classical mixture of the states $|0\rangle$ and $|1\rangle$.

6.1.6.3 Spontaneous Emission

The third channel to be presented here is the *amplitude-damping channel* or *spontaneous emission*. Here the environment is 2 dimensional and the unitary evolution is given by

$$\begin{aligned} \mathsf{U}\,(|0\rangle \otimes |0\rangle\,) &= |0\rangle \otimes |0\rangle \\ \mathsf{U}\,(|1\rangle \otimes |0\rangle) &= \sqrt{1-p}\;|1\rangle \otimes |0\rangle + \sqrt{p}\;|0\rangle \otimes |1\rangle\,. \end{aligned}$$

In other words, if the small system is in the ground state $|0\rangle$ then nothing happens, if it is in the excited state $|1\rangle$ then it may emit this energy into the environment with probability p. This is the simplest model of spontaneous emission of an excited particle: the excited particle goes down to the ground state, emiting a photon into the environment.

In this model there are only two Krauss operators for the associated completely positive map:

$$\mathsf{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad \mathsf{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Successive applications of the associated completely positive map $\mathcal{L}$ make any initial state ρ_0 converge exponentially fast to the ground state

$$\rho_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|\,.$$

That is, as we explained above, the system $\mathcal{H}_A$ ends up emitting all its energy into the environment and hence converges to the ground state.

6.2 Heisenberg Picture

The presentation we made of quantum channels, as being the resulting transformation of a quantum state of $\mathcal{H}$ after a contact and an evolution with some environment, is typical of the Schrödinger picture of Quantum Mechanics. As usual, for all quantum evolutions there is a dual picture, an Heisenberg picture, where the evolution is seen from the point of view of observables instead of states. It so happens that in the case of quantum channels this dual picture opens the door to a vast and interesting field: the notion of *completely positive maps* that we shall explore in next section. But before hands we simply detail the situation for the dual picture of quantum channels.

6.2.1 Technicalities Again

Again we shall start with some technicalities adapted to the context we want to develop. This context is that of "sandwich-sums" as in previous section but in the converse direction:

$$\sum_{n\in\mathbb{N}} \mathsf{M}_n^* \, \mathsf{X} \, \mathsf{M}_n \,.$$

Before hands, here is a very useful result.

Lemma 6.9. *Let* M *and* X *be any bounded operators on* $\mathcal{H}$*, with* X *self-adjoint. Then we have*

$$\mathsf{M}^* \, \mathsf{X} \, \mathsf{M} \leq \|\mathsf{X}\| \; \mathsf{M}^* \, \mathsf{M} \,.$$

Proof. As X is self-adjoint then $\langle g \,,\, \mathsf{X}\, g\rangle$ is real. By Cauchy-Schwarz Inequality we have

$$\langle g \,,\, \mathsf{X}\, g\rangle \leq \|\mathsf{X}\| \; \|g\|^2$$

for all $g \in \mathcal{H}$. This gives the result by taking $g = \mathsf{M}\, f$. □

We can now be clear with the convergence of the sandwich-sums.

Proposition 6.10. *Let* (M_n) *be a sequence of bounded operators on a separable Hilbert space* $\mathcal{H}$ *such that the series* $\sum_n \mathsf{M}_n^* \, \mathsf{M}_n$ *converges weakly to a bounded operator* Y*. If* X *is a bounded operator on* $\mathcal{H}$ *then the series*

$$\sum_{n\in\mathbb{N}} \mathsf{M}_n^* \, \mathsf{X} \, \mathsf{M}_n$$

is strongly convergent.

Proof. We first assume that X is a positive bounded operator on $\mathcal{H}$. Put $\mathsf{Y}_n = \sum_{i\leq n} \mathsf{M}_i^* \, \mathsf{M}_i$ and $\mathsf{S}_n(\mathsf{X}) = \sum_{i\leq n} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i$. Note that $\mathsf{S}_n(\mathsf{X})$ is a positive operator and, even more, all the operators $\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})$ are positive operators, for all $n < m$. Thus, for all $f \in \mathcal{H}$, all $n < m$ we have, using the Functional Calculus and Lemma 6.9 several times

$$\begin{aligned}
&\|(\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X}))\, f\|^2 = \\
&\quad = \left\langle \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \,,\, (\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X}))\sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \right\rangle \\
&\quad \leq \|\mathsf{X}\| \left\langle \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \,,\, (\mathsf{Y}_m - \mathsf{Y}_n)\sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \right\rangle \\
&\quad \leq \|\mathsf{X}\| \left\langle \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \,,\, \mathsf{Y}\, \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \right\rangle \\
&\quad = \|\mathsf{X}\| \left\langle f \,,\, \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, \mathsf{Y}\, \sqrt{\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X})}\, f \right\rangle \\
&\quad \leq \|\mathsf{X}\| \; \|\mathsf{Y}\| \; \langle f \,,\, (\mathsf{S}_m(\mathsf{X}) - \mathsf{S}_n(\mathsf{X}))\, f\rangle \\
&\quad \leq \|\mathsf{X}\|^2 \; \|\mathsf{Y}\| \; \langle f \,,\, (\mathsf{Y}_m - \mathsf{Y}_n)\, f\rangle \,.
\end{aligned}$$

By hypothesis, this term converges to 0 as n and m tend to $+\infty$, hence we have proved the strong convergence of $(\mathsf{S}_n(\mathsf{X}))$.

In the same way as for Proposition 6.3, we then easily extend this property to bounded self-adjoint operators and then to general bounded operators. □

We can now define the associated map $\mathcal{N}$ on $\mathcal{B}(\mathcal{H})$. For the following result, the reader needs to recall the results and notations of Proposition 6.4.

Proposition 6.11. *Let (M_n) be a sequence of bounded operators on a separable Hilbert space $\mathcal{H}$ such that the series $\sum_n \mathsf{M}_n^* \mathsf{M}_n$ converges weakly to a bounded operator Y. Consider the linear map $\mathcal{N}$ on $\mathcal{B}(\mathcal{H})$ defined by*

$$\mathsf{X} \mapsto \mathcal{N}(\mathsf{X}) = \sum_{n\in\mathbb{N}} \mathsf{M}_n^* \, \mathsf{X} \, \mathsf{M}_n \, .$$

Then $\mathcal{N}$ is a bounded operator on $\mathcal{B}(\mathcal{H})$, with

$$\|\mathcal{N}\| \leq \|\mathsf{Y}\| \, .$$

The operator $\mathcal{N}$ is the dual of the operator $\mathcal{L}$ defined in Proposition 6.4. In particular the operator $\mathcal{N}$ is $*$-weakly continuous on $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}$ is the predual map of $\mathcal{N}$.

Finally, using the same operator $\mathsf{M} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ as in Proposition 6.4, the operator $\mathcal{N}$ is given by

$$\mathcal{N}(\mathsf{X}) = \mathsf{M}^*(\mathsf{X} \otimes \mathrm{I})\mathsf{M} \, , \tag{6.8}$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$.

Proof. Let T be a self-adjoint trace-class operator on $\mathcal{H}$, but with a finite canonical decomposition:

$$\mathsf{T} = \sum_{i=1}^n \lambda_i \, |\phi_i\rangle\langle\phi_i| \, .$$

Then, as the sandwich-sum in $\mathcal{N}(\mathsf{X})$ is strongly convergent, we have

$$\begin{aligned}
\mathrm{Tr}\,(\mathsf{T}\,\mathcal{N}(\mathsf{X})) &= \lim_{m\to\infty} \sum_{i=1}^n \sum_{j=0}^m \lambda_i \, \mathrm{Tr}\, \big(|\phi_i\rangle\langle\phi_i| \, \mathsf{M}_j^* \, \mathsf{X} \, \mathsf{M}_j\big) \\
&= \lim_{m\to\infty} \sum_{i=1}^n \sum_{j=0}^m \lambda_i \, \mathrm{Tr}\, \big(\mathsf{M}_j \, |\phi_i\rangle\langle\phi_i| \, \mathsf{M}_j^* \, \mathsf{X}\big) \\
&= \lim_{m\to\infty} \sum_{j=0}^m \mathrm{Tr}\, \big(\mathsf{M}_j \, \mathsf{T} \, \mathsf{M}_j^* \, \mathsf{X}\big) \, .
\end{aligned}$$

We know that the series $\sum_j \mathsf{M}_j \, \mathsf{T} \, \mathsf{M}_j^*$ is $\|\cdot\|_1$-convergent, hence the quantity above converges to

$$\mathrm{Tr}\left(\left(\sum_{j=0}^{\infty} \mathsf{M}_j\,\mathsf{T}\,\mathsf{M}_j^*\right)\mathsf{X}\right) = \mathrm{Tr}\left(\mathcal{L}(\mathsf{T})\,\mathsf{X}\right)\,.$$

Now if T is a general self-adjoint trace-class operator, it is the $\|\cdot\|_1$-limit of a sequence $(\mathsf{T}_n)_{n\in\mathbb{N}}$ of self-adjoint trace-class operators, each of which with finite canonical representation as above. We have proved in Proposition 6.4 that $\mathcal{L}$ is $\|\cdot\|_1$-continuous, hence the identity $\mathrm{Tr}\left(\mathsf{T}\,\mathcal{N}(\mathsf{X})\right) = \mathrm{Tr}\left(\mathcal{L}(\mathsf{T})\,\mathsf{X}\right)$ passes to the limit.

Finally it is easy to extend this relation by linearity to the case of any trace-class operator T. This proves that $\mathcal{N}$ is the dual of $\mathcal{L}$.

As a consequence of this duality relation, the norm estimate on $\mathcal{N}$ is immediate from the corresponding estimate on $\mathcal{L}$ (Proposition 6.4). The σ-weak continuity is also obvious, for $\mathcal{N}$ is a dual operator, and $\mathcal{L}$ is obviously the predual of $\mathcal{N}$.

Let us finally prove the last identity (6.8). For $\mathsf{X}\in\mathcal{B}(\mathcal{H})$, let us compute $\mathsf{M}^*(\mathsf{X}\otimes\mathrm{I})\mathsf{M}$, which is then a bounded operator on $\mathcal{H}$. We have for all $f, g\in\mathcal{H}$

$$\begin{aligned}
&\langle g\,,\,\mathsf{M}^*(\mathsf{X}\otimes\mathrm{I})\mathsf{M}\,f\rangle = \\
&\quad= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\left\langle g\,,\,{}_{\mathcal{K}}\langle e_j|(\mathsf{M}_j^*\otimes\mathrm{I})(\mathsf{X}\otimes\mathrm{I})(\mathsf{M}_i\otimes\mathrm{I})|e_i\rangle_{\mathcal{K}}\,f\right\rangle \\
&\quad= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=0}^{n\wedge m}\langle g\,,\,\mathsf{M}_i^*\,\mathsf{X}\,\mathsf{M}_i f\rangle \\
&\quad= \left\langle g\,,\,\left(\sum_{i\in\mathbb{N}}\mathsf{M}_i^*\,\mathsf{X}\,\mathsf{M}_i\right)f\right\rangle\,.
\end{aligned}$$

We have proved that $\mathsf{M}^*(\mathsf{X}\otimes\mathrm{I})\mathsf{M} = \mathcal{N}(\mathsf{X})$. □

For the series $\sum_{n\in\mathbb{N}}\mathsf{M}_n^*\,\mathsf{X}\,\mathsf{M}_n$ we have obtained the strong convergence above. One may be tempted to ask for the operator-norm convergence instead. This can be obtained if we have stronger conditions, as is proved below.

Proposition 6.12. *Let* (M_n) *be a sequence of bounded operators on a separable Hilbert space* $\mathcal{H}$ *such that the series* $\sum_n \mathsf{M}_n^*\,\mathsf{M}_n$ *converges in operator-norm to a bounded operator* Y*. If* X *is a bounded operator on* $\mathcal{H}$ *then the series*

$$\sum_{n\in\mathbb{N}}\mathsf{M}_n^*\,\mathsf{X}\,\mathsf{M}_n$$

is operator-norm convergent.

Proof. Consider the polar decomposition $\mathsf{X} = \mathsf{U}\,|\mathsf{X}|$ of the bounded operator X. We then have

$$\left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i \right\| = \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \; |\mathsf{X}| \; \mathsf{M}_i \right\| . \tag{6.9}$$

But we have, for all $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$

$$\begin{aligned}
&\left| \left\langle y \, , \, \left(\sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \; |\mathsf{X}| \; \mathsf{M}_i \right) x \right\rangle \right| \\
&\quad \leq \sum_{i=n}^{m} |\langle \mathsf{U}^* \, \mathsf{M}_i \, y \, , \, |\mathsf{X}| \; \mathsf{M}_i \, x \rangle| \\
&\quad \leq \sum_{i=n}^{m} \|\mathsf{U}^* \, \mathsf{M}_i \, y\| \; \||\mathsf{X}| \; \mathsf{M}_i \, x\| \\
&\quad \leq \left(\sum_{i=n}^{m} \|\mathsf{U}^* \, \mathsf{M}_i \, y\|^2 \right)^{1/2} \left(\sum_{i=n}^{m} \||\mathsf{X}| \; \mathsf{M}_i \, x\|^2 \right)^{1/2} \\
&\quad \leq \left(\sum_{i=n}^{m} \langle y \, , \, \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \, y \rangle \right)^{1/2} \left(\sum_{i=n}^{m} \left\langle x \, , \, \mathsf{M}_i^* \; |\mathsf{X}|^2 \; \mathsf{M}_i \, x \right\rangle \right)^{1/2} \\
&\quad \leq \left(\left\langle y \, , \, \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \, y \right\rangle \right)^{1/2} \left(\left\langle x \, , \, \sum_{i=n}^{m} \mathsf{M}_i^* \; |\mathsf{X}|^2 \; \mathsf{M}_i \, x \right\rangle \right)^{1/2} \\
&\quad \leq \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \right\|^{1/2} \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \; |\mathsf{X}|^2 \; \mathsf{M}_i \right\|^{1/2} .
\end{aligned}$$

This shows the inequality

$$\left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \; |\mathsf{X}| \; \mathsf{M}_i \right\| \leq \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \right\|^{1/2} \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \; |\mathsf{X}|^2 \; \mathsf{M}_i \right\|^{1/2} ,$$

which is a particular case of the so-called *operator-norm Cauchy-Schwarz inequality* (cf [Cas05], for example). When applied to Identity (6.9), this gives

$$\left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i \right\| \leq \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \right\|^{1/2} \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \; |\mathsf{X}|^2 \; \mathsf{M}_i \right\|^{1/2} .$$

But, using Lemma 6.9 again, we get

$$
\begin{aligned}
\left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \right\| &= \sup_{\|x\|=1} \left\langle x \, , \, \left(\sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \right) x \right\rangle \\
&= \sup_{\|x\|=1} \sum_{i=n}^{m} \langle x \, , \, \mathsf{M}_i^* \, \mathsf{U} \, \mathsf{U}^* \, \mathsf{M}_i \, x \rangle \\
&\leq \|\mathsf{U} \, \mathsf{U}^*\| \, \sup_{\|x\|=1} \sum_{i=n}^{m} \langle x \, , \, \mathsf{M}_i^* \, \mathsf{M}_i \, x \rangle \\
&= \|\mathsf{U} \, \mathsf{U}^*\| \, \sup_{\|x\|=1} \left\langle x \, , \, \left(\sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{M}_i \right) x \right\rangle \\
&= \|\mathsf{U}\|^2 \, \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{M}_i \right\| \, .
\end{aligned}
$$

This gives finally

$$
\left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i \right\| \leq \|\mathsf{U}\| \, \|\mathsf{X}\| \, \left\| \sum_{i=n}^{m} \mathsf{M}_i^* \, \mathsf{M}_i \right\| \, .
$$

By hypothesis the last term tends to 0 when n and m tend to $+\infty$. This means that the sequence $(\sum_{i=0}^{n} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i)_{n \in \mathbb{N}}$ is a Cauchy sequence in operator-norm, hence it converges in operator-norm. $\square$

6.2.2 Dual of Quantum Channels

Coming back now to quantum channels and their dual picture, we are ready to introduce the corresponding theorem for the Heisenberg picture.

Theorem 6.13. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$ and let ω be a density matrix on $\mathcal{K}$. Define*

$$
\boxed{\mathcal{N}(\mathsf{X}) = \mathrm{Tr}_\omega(\mathsf{U}^*(\mathsf{X} \otimes \mathrm{I})\mathsf{U})}
$$

for all bounded operator X on $\mathcal{H}$. Then there exists an at most countable family $(\mathsf{M}_i)_{i \in I}$ of bounded operators on $\mathcal{H}$, such that

$$
\boxed{\sum_{i \in I} \mathsf{M}_i^* \mathsf{M}_i = \mathrm{I} \, ,} \tag{6.10}
$$

in the sense of strong convergence, satisfying

$$
\boxed{\mathcal{N}(\mathsf{X}) = \sum_{i \in I} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i \, ,} \tag{6.11}
$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$ *and where the series above is strongly convergent.*

The mapping $\mathcal{N}$ *defined this way is related to the mapping* $\mathcal{L}$ *of Theorem 6.5 by the relation*

$$\boxed{\operatorname{Tr}\left(\mathcal{L}(\mathsf{T})\,\mathsf{X}\right) = \operatorname{Tr}\left(\mathsf{T}\,\mathcal{N}(\mathsf{X})\right),} \tag{6.12}$$

for all trace-class operator T *and all bounded operator* X *on* $\mathcal{H}$*. In other words the mapping* $\mathcal{N}$ *acting on* $\mathcal{B}(\mathcal{H})$ *is the dual of* $\mathcal{L}$ *acting on* $\mathcal{L}_1(\mathcal{H})$*.*

Proof. We first assume that ω is a pure state $\omega = |\psi\rangle\langle\psi|$. We have, taking $(f_i)_{i\in I}$ to be some orthonormal basis of $\mathcal{K}$

$$\begin{aligned}
\operatorname{Tr}_\omega\left(\mathsf{U}^*\left(\mathsf{X}\otimes \mathrm{I}\right)\mathsf{U}\right) &= {}_{\mathcal{K}}\!\langle\psi|\,\mathsf{U}^*\left(\mathsf{X}\otimes \mathrm{I}\right)\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\\
&= \sum_{i\in I} {}_{\mathcal{K}}\!\langle\psi|\,\mathsf{U}^*\left(\mathsf{X}\otimes |f_i\rangle\langle f_i|\right)\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\\
&= \sum_{i\in I} {}_{\mathcal{K}}\!\langle\psi|\,\mathsf{U}^*\,|f_i\rangle_{\mathcal{K}}\,\mathsf{X}\,{}_{\mathcal{K}}\!\langle f_i|\,\mathsf{U}\,|\psi\rangle_{\mathcal{K}}\,.
\end{aligned}$$

This gives the required form, putting $\mathsf{M}_i = {}_{\mathcal{K}}\!\langle f_i|\,\mathsf{U}\,|\psi\rangle_{\mathcal{K}}$, exactly in the same way as in Theorem 6.5. In particular the relation (6.10) is already proved.

The case where ω is a general density matrix is treated in the same way as in Theorem 6.5, decomposing it as a convex combination of pure states.

We now prove the relation (6.12). By definition of the mappings $\mathcal{L}$ and $\mathcal{N}$, by definitions of the two types of partial traces, we have

$$\begin{aligned}
\operatorname{Tr}\left(\mathsf{T}\,\mathcal{N}(X)\right) &= \operatorname{Tr}\,\left(\mathsf{T}\ \operatorname{Tr}_\omega\left(\mathsf{U}^*\left(\mathsf{X}\otimes \mathrm{I}\right)\mathsf{U}\right)\right)\\
&= \operatorname{Tr}\,\left(\left(\mathsf{T}\otimes\omega\right)\mathsf{U}^*\left(\mathsf{X}\otimes \mathrm{I}\right)\mathsf{U}\right)\\
&= \operatorname{Tr}\,\left(\mathsf{U}\left(\mathsf{T}\otimes\omega\right)\mathsf{U}^*\left(\mathsf{X}\otimes \mathrm{I}\right)\right)\\
&= \operatorname{Tr}\,\left(\operatorname{Tr}_{\mathcal{K}}\left(\mathsf{U}\left(\mathsf{T}\otimes\omega\right)\mathsf{U}^*\right)\,\mathsf{X}\right)\\
&= \operatorname{Tr}\,\left(\mathcal{L}(\mathsf{T})\,\mathsf{X}\right)\,.
\end{aligned}$$

This proves the announced relation. □

Remark 6.14. One can understand clearly the relation

$$\mathcal{N}(\mathsf{X}) = \operatorname{Tr}_\omega(\mathsf{U}^*(\mathsf{X}\otimes \mathrm{I})\mathsf{U})$$

as the Heisenberg picture of the corresponding formula for quantum channels. Indeed, here the scheme is described as follows. Given an observable X on $\mathcal{H}$, we consider the corresponding observable $\mathsf{X}\otimes \mathrm{I}$ on $\mathcal{H}\otimes\mathcal{K}$, that is, as part of a larger world. After the action of the unitary evolution, the corresponding observable in the Heisenberg picture is $\mathsf{U}^*(\mathsf{X}\otimes \mathrm{I})\mathsf{U}$. Now tracing out over the environment $\mathcal{K}$ via the reference state ω gives the resulting observable $\mathcal{N}(\mathsf{X})$ on $\mathcal{H}$.

In the same way as for quantum channels we have a dilation theorem for linear maps on $\mathcal{B}(\mathcal{H})$ of the form $\mathcal{N}(\mathsf{X}) = \sum_{i\in I} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i$.

Theorem 6.15. *Let $\mathcal{H}$ be some separable Hilbert space. Let $\mathcal{N}$ be a linear mapping on $\mathcal{B}(\mathcal{H})$ of the form*

$$\mathcal{N}(\mathsf{X}) = \sum_{i\in I} \mathsf{M}_i^* \, \mathsf{X} \, \mathsf{M}_i$$

for some bounded operators M_i on $\mathcal{H}$ satisfying

$$\sum_{i\in I} \mathsf{M}_i^* \, \mathsf{M}_i = \mathrm{I} \,,$$

for the strong convergence. Then there exists a separable Hilbert space $\mathcal{K}$, a density matrix ω on $\mathcal{K}$ and a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ such that

$$\mathcal{N}(\mathsf{X}) = \mathrm{Tr}_\omega \left(\mathsf{U}^* \left(\mathsf{X} \otimes \mathrm{I} \right) \mathsf{U} \right) \,,$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$.

Proof. By Theorem 6.7 we know that there exists a Hilbert space $\mathcal{K}$, a state ω on $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ such that the quantum channel

$$\mathsf{T} \mapsto \mathcal{L}(\mathsf{T}) = \sum_{i\in I} \mathsf{M}_i \, \mathsf{T} \, \mathsf{M}_i^*$$

is given by

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}_\mathcal{K} \left(\mathsf{U}(\mathsf{T} \otimes \omega)\mathsf{U}^* \right) \,,$$

for all $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$.

By Proposition 6.11 we know that the dual of $\mathsf{T} \mapsto \sum_{i\in I} \mathsf{M}_i \, \mathsf{T} \, \mathsf{M}_i^*$ is the mapping $\mathcal{N}$. By Theorem 6.13 we know that the dual of $\mathsf{T} \mapsto \mathrm{Tr}_\mathcal{K} \left(\mathsf{U}(\mathsf{T} \otimes \omega)\mathsf{U}^* \right)$ is the mapping $\mathsf{X} \mapsto \mathrm{Tr}_\omega \left(\mathsf{U}^* \left(\mathsf{X} \otimes \mathrm{I} \right) \mathsf{U} \right)$. This gives the result. □

6.3 Complete Positivity

We have introduced those dual maps of quantum channels as maps on $\mathcal{B}(\mathcal{H})$ arising as partial tracing a unitary conjugation on one component of a bipartite quantum system. This is very natural in the context and the language of open quantum systems. Actually there is another way to introduce this type of maps on $\mathcal{B}(\mathcal{H})$, a way which is also very intuitive physically but which is non-obviously equivalent to the first definition. They are the so-called *completely positive maps* on $\mathcal{B}(\mathcal{H})$. It is our goal now to introduce this new notion and to prove the equivalence with the definitions of previous sections.

6.3.1 Completely Positive Maps

For a moment, let us forget the discussions we have had in the previous sections and let us wonder what should be physically (and mathematically) the most general type of transforms $\mathcal{L}$ which would map physicals states of $\mathcal{H}$ to physical states of $\mathcal{H}$. Such a family of maps $\mathcal{L}$ should be able to represent the most general transformations that a quantum system can undergo. The image $\mathcal{L}(\rho)$ of a state ρ represent the state of the system after a (maybe complicated) transform, a "kick" given to it. Note that our discussion is not time-dependent, we have fixed an interval of time (1, say) and we discuss how the state can change between time 0 and time 1.

A map $\mathcal{L}$ on $\mathcal{L}_1(\mathcal{H})$ such as described above must have the following properties:

i) it should be linear,

ii) it should preserve the positivity,

iii) it should preserve the trace.

Even though the discussion above is more intuitive when dealing with states, it is actually more confortable to work with the dual map $\mathcal{L}^*$ acting on $\mathcal{B}(\mathcal{H})$, as we shall see later on with Stinespring's Theorem. The three conditions above for $\mathcal{L}$ are equivalent to the following three for $\mathcal{L}^*$ acting on $\mathcal{B}(\mathcal{H})$, as can be checked easily (left to the reader). The map $\mathcal{N} = \mathcal{L}^*$ on $\mathcal{B}(\mathcal{H})$ should satisfy the following:

i') it should be linear,

ii') it should preserve the positivity,

iii') it should preserve the identity operator I.

One can wonder if the conditions i')-iii') here are sufficient to characterize the physically interesting transforms. Actually, the answer is negative. The condition ii') is slightly too weak for describing a physically reasonable transform. Let us explain this point.

Consider a linear map $\mathcal{N} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(\mathsf{A})$ is positive for every positive $\mathsf{A} \in \mathcal{B}(\mathcal{H})$. Now imagine that outside of the system represented by $\mathcal{H}$ is another quantum system $\mathcal{K}$, which is independent of $\mathcal{H}$, in the sense that the two quantum systems have no interaction whatsoever. Consider the whole system $\mathcal{H}\otimes\mathcal{K}$ and the transformation $\widehat{\mathcal{N}}$ which consists in applying the transformation $\mathcal{N}$ to $\mathcal{H}$ and ignoring $\mathcal{K}$. That is, applying the transformation $\mathcal{N}$ to the $\mathcal{H}$-part and the identity to $\mathcal{K}$-part. In other words, this means considering the mapping

$$\begin{aligned}\widehat{\mathcal{N}} = \mathcal{N} \otimes \mathrm{I} \ : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) &\longrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \\ \mathsf{A} \otimes \mathsf{B} \quad &\longmapsto \mathcal{N}(\mathsf{A}) \otimes \mathsf{B}\,.\end{aligned}$$

Then, surprisingly enough, the mapping $\mathcal{N} \otimes \mathrm{I}$ needs not preserve positivity anymore! This is quite surprising, but it may not map positive elements of

$\mathcal{H} \otimes \mathcal{K}$ to positive elements; its predual map does not map quantum states to quantum states.

Let us show a counter-example. Take $\mathcal{N}$ to be the transpose mapping, that is,

$$\mathcal{N}(\mathsf{A}) = {}^{t}\mathsf{A} = \overline{\mathsf{A}^*}\,,$$

on $\mathcal{B}(\mathcal{H})$. This mapping $\mathcal{N}$ is clearly a positivity-preserving map. Now, consider the space $\mathcal{K} = \mathbb{C}^2$ and let us view the algebra $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ as the algebra of 2 by 2 matrices with coefficients in $\mathcal{B}(\mathcal{H})$. Then the mapping $\widehat{\mathcal{N}}$ acts as follows:

$$\widehat{\mathcal{N}}\ :\ \begin{pmatrix} \mathsf{A}_0^0 & \mathsf{A}_0^1 \\ \mathsf{A}_0^1 & \mathsf{A}_1^1 \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{N}(\mathsf{A}_0^0) & \mathcal{N}(\mathsf{A}_0^1) \\ \mathcal{N}(\mathsf{A}_0^1) & \mathcal{N}(\mathsf{A}_1^1) \end{pmatrix}.$$

In particular, if $\mathcal{H} = \mathbb{C}^2$ then $\widehat{\mathcal{N}}$ maps

$$\begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&1 \end{pmatrix} \qquad \text{to} \qquad \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}.$$

The first matrix is positive, whereas the second one is not. The mapping $\widehat{\mathcal{N}}$ is not positivity preserving.

This is clearly physically unsatisfactory to consider state transforms which are not state transforms anymore when considered as part of a larger world, even though one does not interact with the environment! These remarks justify the following definition.

Definition 6.16. A linear map $\mathcal{N}$ on $\mathcal{B}(\mathcal{H})$ is *completely positive* if, for every $n \in \mathbb{N}$, the natural ampliation $\mathcal{N}_n$ of $\mathcal{N}$ to $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ given by

$$\mathcal{N}_n(\mathsf{A} \otimes \mathsf{B}) = \mathcal{N}(\mathsf{A}) \otimes \mathsf{B}$$

is positivity-preserving.

A completely positive map $\mathcal{N}$ on $\mathcal{B}(\mathcal{H})$ is called *normal* if it is σ-weakly continuous.

6.3.2 Stinespring Theorem

The aim of this subsection is to prove the celebrated characterization of completely positive maps due to Stinespring. In the next subsection this would lead to its very useful consequence obtained by Krauss; we shall recover the Krauss decompositions that we introduced in the previous sections.

We first start with the notion of *representation* for operator algebras.

Definition 6.17. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A *representation* π of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ is a linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ satisfying

$$\pi(\mathrm{I}) = \mathrm{I}$$

and

$$\pi(\mathsf{A}^*\mathsf{B}) = \pi(\mathsf{A})^*\pi(\mathsf{B})$$

for all $\mathsf{A}, \mathsf{B} \in \mathcal{B}(\mathcal{H})$. In other words π is a unital *-algebra homomorphism.

If furthermore π is σ-weakly continuous, then π is called a *normal representation.*

Note that in the case where $\mathcal{K} = \mathcal{H}$ there exists a trivial representation of $\mathcal{B}(\mathcal{H})$ on $\mathcal{B}(\mathcal{H})$ given by $\pi(\mathsf{X}) = \mathsf{X}$. We shall call it the *standard representation* of $\mathcal{B}(\mathcal{H})$.

Proposition 6.18. *A representation π of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ automatically preserves positivity and is always continuous.*

Proof. If $\mathsf{A} = \mathsf{B}^*\mathsf{B}$ is any positive element of $\mathcal{B}(\mathcal{H})$ then $\pi(\mathsf{A}) = \pi(\mathsf{B})^*\pi(\mathsf{B})$ is a positive element of $\mathcal{B}(\mathcal{K})$. This proves the first part.

If A is a self-adjoint element of $\mathcal{B}(\mathcal{H})$ then $\pi(\mathsf{A})$ is a self-adjoint element of $\mathcal{B}(\mathcal{K})$. In particular

$$\|\pi(\mathsf{A})\| = \sup\{|\lambda|\ ;\ \lambda \in \sigma(\pi(\mathsf{A}))\}\ .$$

But note that if r belongs to the resolvent set $\rho(\mathsf{A})$ of A then $\mathsf{A} - r\mathrm{I}$ is invertible in $\mathcal{B}(\mathcal{H})$, hence $\pi(\mathsf{A} - r\mathrm{I}) = \pi(\mathsf{A}) - r\mathrm{I}$ is invertible in $\mathcal{B}(\mathcal{K})$. This is to say that r belongs to the resolvent set $\rho(\pi(\mathsf{A}))$. This proves that $\sigma(\pi(\mathsf{A}))$ is included in $\sigma(\mathsf{A})$ and this gives the estimate

$$\|\pi(\mathsf{A})\| \leq \sup\{|\lambda|\ ;\ \lambda \in \sigma(\mathsf{A})\} = \|\mathsf{A}\|\ .$$

Now if A is any element of $\mathcal{B}(\mathcal{H})$ then

$$\|\pi(\mathsf{A})\|^2 = \|\pi(\mathsf{A})^*\pi(\mathsf{A})\| = \|\pi(\mathsf{A}^*\mathsf{A})\| \leq \|\mathsf{A}^*\mathsf{A}\| = \|\mathsf{A}\|^2\ .$$

This proves the continuity. □

Theorem 6.19 (Stinespring's Theorem). *Let $\mathcal{M} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Then $\mathcal{M}$ is completely positive if and only if $\mathcal{M}$ has the form*

$$\mathcal{M}(\mathsf{A}) = \mathsf{V}^*\pi(\mathsf{A})\,\mathsf{V} \tag{6.13}$$

for some representation π of $\mathcal{B}(\mathcal{H})$ on some separable Hilbert space $\mathcal{K}$ and for some bounded linear map $\mathsf{V} : \mathcal{H} \to \mathcal{K}$.

If $\mathcal{M}$ is normal then π can be chosen to be normal.

Proof. First, the easy direction: assume that $\mathcal{M}$ is of the form (6.13). Let $\mathsf{A} = \mathsf{B}^*\mathsf{B}$ be any positive element in $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$. Let us be given an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. The operator B can be represented as a block-matrix with coefficients in $\mathcal{B}(\mathcal{H})$:

$$\mathsf{B} = \sum_{i,j=1}^{n} \mathsf{B}_j^i \otimes |e_j\rangle\langle e_i| \, .$$

Let $v = \sum_{i=1}^n v_i \otimes e_i$ be any element of $\mathcal{H} \otimes \mathbb{C}^n$. We have

$$\begin{aligned}
\langle v \, , \, \mathcal{M}(\mathsf{A})v\rangle &= \sum_{i',j',i,j,k=1}^{n} \Big\langle v_{i'} \otimes e_{i'} \, , \, \Big(\mathcal{M}(\mathsf{B}_k^{i\,*}\,\mathsf{B}_k^j) \otimes |e_i\rangle\langle e_j|\Big) \, (v_{j'} \otimes e_{j'})\Big\rangle \\
&= \sum_{i,j,k=1}^{n} \Big\langle v_i \, , \, \mathcal{M}(\mathsf{B}_k^{i\,*}\,\mathsf{B}_k^j) \, v_j\Big\rangle \\
&= \sum_{i,j,k=1}^{n} \Big\langle \mathsf{V}v_i \, , \, \pi(\mathsf{B}_k^{i\,*}\,\mathsf{B}_k^j)\mathsf{V}v_j\Big\rangle \\
&= \sum_{i,j,k=1}^{n} \Big\langle \pi(\mathsf{B}_k^i) \, \mathsf{V}v_i \, , \, \pi(\mathsf{B}_k^j) \, \mathsf{V}v_j\Big\rangle \\
&= \sum_{k=1}^{n} \left\| \sum_{i=1}^{n} \pi(\mathsf{B}_k^i) \, \mathsf{V}v_i \right\|^2
\end{aligned}$$

which is positive. This proves that $\mathcal{M}$ is completely positive.

Conversely, assume that $\mathcal{M}$ is completely positive. Let $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{H}$ denote the algebraic tensor product of $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$. Let

$$\phi = \sum_{i=1}^{n} \mathsf{A}_i \otimes x_i \qquad \text{and} \qquad \psi = \sum_{i=1}^{n} \mathsf{B}_i \otimes y_i$$

be two elements of $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{H}$. Consider the elements

$$x = \sum_{i=1}^{n} x_i \otimes e_i \qquad \text{and} \qquad y = \sum_{i=1}^{n} y_i \otimes e_i$$

of $\mathcal{H} \otimes \mathbb{C}^n$. Consider the elements

$$\mathsf{A}' = \sum_{i=1}^{n} \mathsf{A}_i \otimes |e_i\rangle\langle e_1| \qquad \text{and} \qquad \mathsf{B}' = \sum_{i=1}^{n} \mathsf{B}_i \otimes |e_i\rangle\langle e_1|$$

of $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$. We define a sesquilinear form $\langle \cdot \, , \, \cdot \rangle$ on $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{H}$ by putting

$$\langle \psi \, , \, \phi\rangle = \langle y \, , \, \mathcal{M}_n(\mathsf{B}'^* \mathsf{A}')x\rangle \ .$$

In particular, the quantity

$$\langle \phi\,,\,\phi\rangle = \langle x\,,\,\mathcal{M}_n(\mathsf{A}'^*\mathsf{A}')x\rangle$$

is positive by the complete positivity of $\mathcal{M}$. This means that our sesquilinear form is positive. It misses being definite positive only by the fact that the kernel space

$$\mathcal{R} = \{\phi \in \mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{H}\,;\ \langle \phi\,,\,\phi\rangle = 0\}$$

may not be reduced to $\{0\}$. We put $\mathcal{K}$ to be the Hilbert space obtained by completion of the quotient $(\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{H})/\mathcal{R}$ for the induced scalar product.

We wish to construct a representation π of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ by putting

$$\pi(\mathsf{X})\sum_{i=1}^n \mathsf{A}_i\otimes x_i = \sum_{i=1}^n (\mathsf{X}\,\mathsf{A}_i)\otimes x_i\,.$$

For every $\mathsf{X}\in\mathcal{B}(\mathcal{H})$, we put $\mathsf{X}'' = \mathsf{X}\otimes\mathrm{I}\in\mathcal{B}(\mathcal{H}\otimes\mathbb{C}^n)$. With these definitions we get

$$\|\pi(\mathsf{X})\|^2 = \langle x\,,\,\mathcal{M}_n(\mathsf{A}'^*\,\mathsf{X}''^*\,\mathsf{X}''\mathsf{A}')\,x\rangle\,.$$

Since

$$0\le \mathsf{A}'^*\,\mathsf{X}''^*\,\mathsf{X}''\mathsf{A}' \le \|\mathsf{X}''\|^2\mathsf{A}'^*\,\mathsf{A}'$$

we get, by the positivity-preserving property of $\mathcal{M}_n$

$$\|\pi(\mathsf{X})\|^2 \le \|\mathsf{X}''\|^2\,\langle x\,,\,\mathcal{M}_n(\mathsf{A}'^*\,\mathsf{A}')\,x\rangle = \|\mathsf{X}''\|^2\,\|\phi\|^2\,.$$

As a consequence, the map $\pi(\mathsf{X})$ extends to a bounded operator on $\mathcal{K}$. The mapping $\pi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$, defined this way, is clearly a unital *-algebra homomorphism, that is, it is a representation of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$.

Finally, let $\mathsf{V}:\mathcal{H}\to\mathcal{K}$ be defined by $\mathsf{V}x = \mathrm{I}\otimes x$. Then

$$\|\mathsf{V}x\|^2 = \langle x\,,\,\mathcal{L}(\mathrm{I})\,x\rangle \le \|\mathcal{L}\|\,\|x\|^2$$

and V is a bounded linear map. With this map V we get finally, for every $x\in\mathcal{H}$ and every $\mathsf{A}\in\mathcal{B}(\mathcal{H})$

$$\langle \mathsf{V}x\,,\,\pi(\mathsf{A})\mathsf{V}x\rangle = \langle \mathrm{I}\otimes x\,,\,\mathsf{A}\otimes x\rangle = \langle x\,,\,\mathcal{L}(\mathsf{A})\,x\rangle\,.$$

This proves that $\mathcal{L}(\mathsf{A}) = \mathsf{V}^*\pi(\mathsf{A})\mathsf{V}$. We have obtained the required representation for $\mathcal{L}$.

As $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ are both generated by countable dense sets, then $\mathcal{K}$ is generated by a countable set and it is a separable Hilbert space.

Finally, we prove the σ-weak continuity. Let (X_n) be a sequence in $\mathcal{B}(\mathcal{H})$ converging weakly to $\mathsf{X}\in\mathcal{B}(\mathcal{H})$. Note that for all $\mathsf{A},\mathsf{B}\in\mathcal{B}(\mathcal{H})$, by the σ-weak continuity of $\mathcal{L}$, we have

$$w-\lim_{n\to\infty}\mathcal{L}(\mathsf{A}^*\mathsf{X}_n\,\mathsf{B}) = \mathcal{L}(\mathsf{A}^*\mathsf{X}\,\mathsf{B})\,.$$

If $\phi = \sum_{i=1}^{k} \mathsf{A}_i \otimes x_i$ we get

$$\begin{aligned}
\lim_{n\to\infty} \langle \phi \,,\, \pi(\mathsf{X}_n)\phi \rangle &= \lim_{n\to\infty} \sum_{i,j=1}^{k} \langle \mathsf{A}_j \otimes x_j \,,\, (\mathsf{X}_n\, \mathsf{A}_i) \otimes x_i \rangle \\
&= \lim_{n\to\infty} \sum_{i,j=1}^{k} \langle x_j \,,\, \mathcal{L}(\mathsf{A}_i^* \mathsf{X}_n\, \mathsf{A}_i)\, x_i \rangle \\
&= \sum_{i,j=1}^{k} \langle x_j \,,\, \mathcal{L}(\mathsf{A}_i^* \mathsf{X}\, \mathsf{A}_i)\, x_i \rangle \\
&= \langle \phi \,,\, \pi(\mathsf{X})\phi \rangle \ .
\end{aligned}$$

This proves that π is σ-weakly continuous. □

The representation of $\mathcal{L}$ as in (6.13) is not unique in general. It becomes unique, up to unitary transform, if it satisfies a supplementary minimality condition.

Definition 6.20. Let $\mathcal{L}$ be a completely positive map on $\mathcal{B}(\mathcal{H})$. A triple $(\mathcal{K}, \pi, \mathsf{V})$ made of a Hilbert space $\mathcal{K}$, a representation π of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ and a bounded operator $\mathsf{V} : \mathcal{H} \to \mathcal{K}$, satisfying

$$\mathcal{L}(\mathsf{A}) = \mathsf{V}^* \pi(\mathsf{A})\, \mathsf{V}$$

for all $\mathsf{A} \in \mathcal{B}(\mathcal{H})$, is called a *Stinespring representation* of $\mathcal{L}$.

If furthermore this representation satisfies the condition that the set

$$\{\pi(\mathsf{A})\mathsf{V}\, u \,;\ \mathsf{A} \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}\}$$

is total in $\mathcal{K}$, then the Stinespring representation is called *minimal*.

Proposition 6.21. *If $\mathcal{L}$ be a completely positive map on $\mathcal{B}(\mathcal{H})$. If $(\mathcal{K}_1, \pi_1, \mathsf{V}_1)$ and $(\mathcal{K}_2, \pi_2, \mathsf{V}_2)$ are two minimal Stinespring representations of $\mathcal{L}$, then there exists a unitary map $\mathsf{U} : \mathcal{K}_1 \to \mathcal{K}_2$ such that*

$$\mathsf{U}\, \mathsf{V}_1 = \mathsf{V}_2 \qquad \textit{and}\ \mathsf{U}\, \pi_1(\mathsf{A}) = \pi_2(\mathsf{A})\, \mathsf{U}\,,$$

for all $\mathsf{A} \in \mathcal{B}(\mathcal{H})$.

Proof. Let $\mathsf{U} : \mathcal{K}_1 \to \mathcal{K}_2$ be the densely defined operator:

$$\mathsf{U}\left(\sum_{j=1}^{n} \pi_1(\mathsf{A}_j)\mathsf{V}_1\, u_j \right) = \sum_{j=1}^{n} \pi_2(\mathsf{A}_j)\mathsf{V}_2\, u_j \,.$$

This operator satisfies

$$\begin{aligned}\langle \mathsf{U}\,\pi_1(\mathsf{B})\,\mathsf{V}_1\,v\,,\,\mathsf{U}\,\pi_1(\mathsf{A})\,\mathsf{V}_1\,u\rangle &= \langle \pi_2(\mathsf{B})\,\mathsf{V}_2\,v\,,\,\pi_2(\mathsf{A})\,\mathsf{V}_2\,u\rangle\\ &= \langle v\,,\,V_2^*\pi_2(\mathsf{B}^*\mathsf{A})\,\mathsf{V}_2\,u\rangle\\ &= \langle v\,,\,\mathcal{L}(\mathsf{B}^*\,\mathsf{A})\,u\rangle\\ &= \langle \mathsf{V}_1\,v\,,\,\pi_1(\mathsf{B}^*\mathsf{A})\,\mathsf{V}_1\,u\rangle\\ &= \langle \pi_1(\mathsf{B})\,\mathsf{V}_1\,v\,,\,\pi_1(\mathsf{A})\,\mathsf{V}_1\,u\rangle\ .\end{aligned}$$

Therefore U is an isometry on a dense set of $\mathcal{K}_1$, it extends to an isometry on $\mathcal{K}_1$. Its range is dense in $\mathcal{K}_2$, thus its extension is a unitary operator.

Our definition of U specializes into

$$\mathsf{U}\,\mathsf{V}_1\,u = \mathsf{U}\,\pi_1(\mathrm{I})\,\mathsf{V}_1\,u = \pi_2(\mathrm{I})\,\mathsf{V}_2\,u = \mathsf{V}_2\,u\,.$$

Finally, if we put $x = \sum_{j=1}^n \pi_1(\mathsf{A}_j)\mathsf{V}_1\,u_j$ then

$$\begin{aligned}\mathsf{U}\,\pi_1(\mathsf{A})\,x &= \mathsf{U}\left(\sum_{j=1}^n \pi_1(\mathsf{A}\mathsf{A}_j)\mathsf{V}_1\,u_j\right)\\ &= \sum_{j=1}^n \pi_2(\mathsf{A}\mathsf{A}_j)\mathsf{V}_2\,u_j\\ &= \pi_2(\mathsf{A})\left(\sum_{j=1}^n \pi_2(\mathsf{A}\mathsf{A}_j)\mathsf{V}_2\,u_j\right)\\ &= \pi_2(\mathsf{A})\mathsf{U}\,x\,.\end{aligned}$$

This gives easily the announced relations. □

6.3.3 Krauss Theorem

We now turn to Krauss version of Stinespring's Theorem. Before hand we need the following preliminary result.

Lemma 6.22. *Let π be a normal representation of $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{K}$. Then there exists a direct sum decomposition*

$$\mathcal{K} = \bigoplus_{n\in\mathbb{N}} \mathcal{K}_n$$

where the subspaces $\mathcal{K}_n$ are invariant under π and the restriction of π to each $\mathcal{K}_n$ is unitarily equivalent to the standard representation of $\mathcal{B}(\mathcal{H})$.

Proof. If e is a unit vector in $\mathcal{H}$ then the projection $\mathsf{P} = \pi(|e\rangle\langle e|)$ is non zero, for if U_n are unitary operators in $\mathcal{B}(\mathcal{H})$ such that $e_n = \mathsf{U}_n\,e$ form a maximal orthonormal set in $\mathcal{H}$ and if $\mathsf{P} = 0$ then

$$\pi(|e_n\rangle\langle e_n|) = \pi(\mathsf{U}_n)\mathsf{P}\,\pi(\mathsf{U}_n) = 0$$

and, by normality of π

$$\pi(\mathrm{I}) = \sum_n \pi(|e_n\rangle\langle e_n|) = 0$$

instead of being equal to I as it should be.

If ψ is a unit vector in $\mathcal{K}$ such that $\mathsf{P}\,\psi = \psi$ then

$$\begin{aligned}\langle \psi\,,\,\pi(\mathsf{X})\psi\rangle &= \langle \mathsf{P}\,\psi\,,\,\pi(\mathsf{X})\mathsf{P}\,\psi\rangle\\ &= \langle \psi\,,\,\pi\left(|e\rangle\langle e|\,\mathsf{X}\,|e\rangle\langle e|\right)\psi\rangle\\ &= \langle e\,,\,\mathsf{X}\,e\rangle\ \langle \psi\,,\,\pi\left(|e\rangle\langle e|\right)\psi\rangle\\ &= \langle e\,,\,\mathsf{X}\,e\rangle\ .\end{aligned}$$

Consider the sub-Hilbert space of $\mathcal{K}$

$$\mathcal{K}_1 = \overline{\{\pi(\mathsf{X})\,\psi\,;\ \mathsf{X}\in\mathcal{B}(\mathcal{H})\}}.$$

It is clearly stable under all the operators $\pi(\mathsf{A})$. Hence so is the space $\mathcal{K}_1^\perp$.

Consider the map $\mathsf{U} : \pi(\mathsf{X})\,\psi \mapsto \mathsf{X}\,e$, it is easy to see that it is isometric and hence extends into a unitary operator from $\mathcal{K}_1$ to $\mathcal{H}$. Computing $\mathsf{U}\,\pi(\mathsf{Y})\,\mathsf{U}^*$ on elements of the form $\mathsf{X}\,e$, shows that

$$\mathsf{U}\,\pi(\mathsf{Y})\,\mathsf{U}^* = \mathsf{Y}\,.$$

We have decomposed $\mathcal{K}$ into $\mathcal{K}_1 \oplus \mathcal{K}_1^\perp$, where $\mathcal{K}_1$ is stable under π and on which π is unitarily equivalent to the standard representation of $\mathcal{H}$. We are left with a normal representation on $\mathcal{K}_1^\perp$. We can apply the same procedure again and again.

By Zorn's Lemma there exists a maximal class of subspaces $\mathcal{K}_n$ of $\mathcal{K}$ with unitary maps $\mathsf{U}_n : \mathcal{K}_n \to \mathcal{H}$ such that $\mathcal{K}_n$ is invariant by π and

$$\mathsf{X} = \mathsf{U}_n\,\pi(\mathsf{X})\,\mathsf{U}_n^*$$

for all $\mathsf{X}\in\mathcal{B}(\mathcal{H})$. This maximal family must satisfy

$$\mathcal{K} = \bigoplus_{n\in\mathbb{N}} \mathcal{K}_n$$

for otherwise we can repeat the same construction as for $\mathcal{K}_1$ inside the space

$$\left(\bigoplus_{n\in\mathbb{N}} \mathcal{K}_n\right)^\perp$$

and contradict the maximality. $\square$

We are now able to prove our final characterization.

Theorem 6.23 (Krauss Theorem). *Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{M}$ a σ-weakly continuous linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. Then $\mathcal{M}$ is completely positive if and only if it is of the form*

$$\mathcal{M}(\mathsf{A}) = \sum_n \mathsf{M}_n^* \, \mathsf{A} \, \mathsf{M}_n \tag{6.14}$$

for a sequence (M_n) of bounded linear operators on $\mathcal{H}$ such that the series $\sum_n \mathsf{M}_n^* \, \mathsf{M}_n$ is strongly convergent.

Proof. Assume that $\mathcal{M}$ is completely positive. By Stinespring's Theorem (Theorem 6.19), there exists a Stinespring representation $(\mathcal{K}, \pi, \mathsf{V})$ of $\mathcal{M}$ with π being normal. By Lemma 6.22 there exists a decomposition $\mathcal{K} = \oplus_{n\in\mathbb{N}}\mathcal{K}_n$ such that each $\mathcal{K}_n$ is stable under π and such that π restricted to $\mathcal{K}_n$ is unitarily equivalent to the standard representation. Let P_n be the orthogonal projectors from $\mathcal{K}$ onto $\mathcal{K}_n$, let $\mathsf{U}_n : \mathcal{K}_n \to \mathcal{H}$ denote the unitary operator ensuring the equivalence. The operators P_n commute with the representation π, hence

$$\begin{aligned}
\mathcal{M}(\mathsf{X}) &= \mathsf{V}^* \, \pi(\mathsf{X}) \, \mathsf{V} \\
&= \sum_n \mathsf{V}^* \, \mathsf{P}_n \, \pi(\mathsf{X}) \, \mathsf{P}_n \, \mathsf{V} \\
&= \sum_n \mathsf{V}^* \mathsf{P}_n \, \mathsf{U}_n \, \mathsf{X} \, \mathsf{U}_n^* \, \mathsf{P}_n \, \mathsf{V} \\
&= \sum_n \mathsf{M}_n^* \, \mathsf{X} \, \mathsf{M}_n
\end{aligned}$$

where we have put $\mathsf{M}_n = \mathsf{U}_n^* \, \mathsf{P}_n \, \mathsf{V}$. This gives the announced form.

The converse is easy to check and left to the reader. □

Here we are! We have completed the loop. Starting from the notion of quantum channels as being the most general transforms of a quantum state, obtained by a contact and an evolution with any kind of quantum environment, we had obtained the Krauss representation of quantum channels. By duality we have obtained the Krauss decomposition of their adjoint acting on the observables.

But another approach, more abstract, has led us to define what should be a reasonable state transform: the completely positive maps. The last theorem above shows that all these objects are one and the same thing. This has been quite a long and non-trivial path, but in the end this result is very reassuring!

The Krauss decomposition also authorizes a nice form for the Stinespring representation : this is just applying Proposition 6.11 actually! Indeed, the representation (6.8) now immediately gives the following.

Theorem 6.24. *Let $\mathcal{M}$ be a normal completely positive map on $\mathcal{B}(\mathcal{H})$ then there exists a Hilbert space $\mathcal{K}$ and a bounded operator M from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{K}$ such that*

$$\mathcal{M}(\mathsf{X}) = \mathsf{M}^* \, (\mathsf{X} \otimes \mathrm{I}) \, \mathsf{M} \tag{6.15}$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$. One can choose $\mathcal{K}$ in such a way that the set

$$\{(\mathsf{X} \otimes \mathrm{I}) \, \mathsf{M} \, u \, ; \; u \in \mathcal{H} \, , \; \mathsf{X} \in \mathcal{B}(\mathcal{H})\}$$

is total in $\mathcal{K}$.

6.3.4 Minimality and Ambiguity

Here we discuss particular properties of the coefficients of the Krauss representation obtained above. The first property is a minimality property, as was discussed for the Stinespring representation. On the coefficients of the Krauss representation the minimality is very nicely characterized by a kind of linear independence property.

Definition 6.25. Let $(\mathsf{M}_i)_{i \in I}$ be a family of elements of $\mathcal{B}(\mathcal{H})$ such that

$$\sum_{i \in I} \mathsf{M}_i^* \mathsf{M}_i$$

converges strongly. This family satisfies a *tempered linear independence* if for every family $(c_i)_{i \in I}$ of complex numbers satisfying

$$\sum_{i \in I} |c_i|^2 < \infty$$

and

$$\sum_{i \in I} c_i \, \mathsf{M}_i = 0 \, ,$$

then necessarily all the c_i vanish.

Note that with the above conditions the series $\sum_{i \in I} c_i \, \mathsf{M}_i$ is automatically strongly convergent.

Proposition 6.26. *Let $(\mathsf{M}_i)_{i \in I}$ be a family of elements of $\mathcal{B}(\mathcal{H})$ such that*

$$\sum_{i \in I} \mathsf{M}_i^* \, \mathsf{M}_i$$

converges strongly. Let M be the associated operator $\mathsf{M} = \sum_{i \in I} (\mathsf{M}_i \otimes \mathrm{I}) \, |f_i\rangle_{\mathcal{K}}$, as in Proposition 6.4. Then the space generated by

$$\{(\mathsf{X} \otimes \mathrm{I}) \mathsf{M} u \, ; \; \mathsf{X} \in \mathcal{B}(\mathcal{H}) \, , \; u \in \mathcal{H}\}$$

is dense in $\mathcal{H}\otimes\mathcal{K}$ if and only if the family $(\mathsf{M}_i)_{i\in I}$ satisfies a tempered linear independence.

Proof. Assume first that $\{(\mathsf{X}\otimes \mathrm{I})\mathsf{M}u\,;\ \mathsf{X}\in\mathcal{B}(\mathcal{H})\,,\ u\in\mathcal{H}\}$ is total in $\mathcal{H}\otimes\mathcal{K}$. Let v be any element of $\mathcal{H}$. If $(c_i)_{i\in I}$ satisfies $\sum_{i\in I} c_i\,\mathsf{M}_i=0$, then

$$\left\langle v\,,\,\mathsf{X}\sum_{i\in I} c_i\,\mathsf{M}_i\,u\right\rangle$$

vanishes for every $\mathsf{X}\in\mathcal{B}(\mathcal{H})$ and every $u\in\mathcal{H}$. But with the computation

$$\begin{aligned}\left\langle \sum_{i\in I}\overline{c_i}\,v\otimes f_i\,,\,\sum_{i\in I}\mathsf{X}\,\mathsf{M}_i\,u\otimes f_i\right\rangle &= \sum_{i\in I}\langle \overline{c_i}\,v\,,\,\mathsf{X}\,\mathsf{M}_i\,u\rangle\\ &=\left\langle v\,,\,\mathsf{X}\sum_{i\in I}c_i\,\mathsf{M}_i\,u\right\rangle\,,\end{aligned}$$

we get that

$$\left\langle \sum_{i\in I}\overline{c_i}\,v\otimes f_i\,,\,\sum_{i\in I}\mathsf{X}\,\mathsf{M}_i\,u\otimes f_i\right\rangle$$

vanishes for all $\mathsf{X}\in\mathcal{B}(\mathcal{H})$ and all $u\in\mathcal{H}$. The density hypothesis implies that necessarily $\sum_{i\in I}\overline{c_i}\,v\otimes f_i$ vanishes. This shows that all the c_i's are null. We have proved the result in one direction.

Conversely, let $\sum_{i\in I} v_i\otimes f_i$ be an element of $\mathcal{H}\otimes\mathcal{K}$ such that

$$\left\langle \sum_{i\in I} v_i\otimes f_i\,,\,\sum_{i\in I}\mathsf{X}\,\mathsf{M}_i\,u\otimes f_i\right\rangle=0$$

for all $\mathsf{X}\in\mathcal{B}(\mathcal{H})$, all $u\in\mathcal{B}(\mathcal{H})$. Let us fix i_0 in I and take any $v\in\mathcal{H}$. Consider the operator $X=|v_{i_0}\rangle\langle v|\in\mathcal{B}(\mathcal{H})$. We get

$$0=\sum_{i\in I}\langle v_i\,,\,|v_{i_0}\rangle\langle v|\,\mathsf{M}_i\,u\rangle=\sum_{i\in I}\langle v\,,\,\langle v_i\,,\,v_{i_0}\rangle\ \mathsf{M}_i\,u\rangle\ ,$$

that is,

$$\left\langle v\,,\,\sum_{i\in I}\langle v_i\,,\,v_{i_0}\rangle\ \mathsf{M}_i\,u\right\rangle=0\,.$$

As this holds for all $u,v\in\mathcal{H}$, this implies that

$$\sum_{i\in I}\langle v_i\,,\,v_{i_0}\rangle\ \mathsf{M}_i=0$$

(note that $\sum_{i\in I} |\langle v_i\,,\,v_{i_0}\rangle|^2$ is obviously finite). By hypothesis this implies that the scalars $\langle v_i\,,\,v_{i_0}\rangle$ vanish for all $i\in I$. In particular $\|v_{i_0}\|^2=0$ and $v_{i_0}=0$. As this holds for all i_0 we have proved that the element $\sum_{i\in I} v_i\otimes f_i$ must vanish. We have proved that the set $\{(\mathsf{X}\otimes \mathrm{I})\mathsf{M}u\,;\ \mathsf{X}\in\mathcal{B}(\mathcal{H})\,,\ u\in\mathcal{H}\}$ generates a dense set. □

Definition 6.27. The Krauss representation

$$\mathcal{N}(\mathsf{X})=\sum_{i\in I}\mathsf{M}_i^*\,\mathsf{X}\,\mathsf{M}_i$$

of a normal completely positive map $\mathcal{N}$ on $\mathcal{B}(\mathcal{H})$ is called *minimal* if the family $(\mathsf{M}_i)_{i\in I}$ satisfies one of the two equivalent conditions of Proposition 6.26.

Theorem 6.28. *Let $\mathcal{M}$ be a normal completely positive map on $\mathcal{B}(\mathcal{H})$ then there always exist a Krauss representation of $\mathcal{M}$ which is minimal.*

Consider any two minimal representations of $\mathcal{M}$, with coefficients $(\mathsf{L}_i)_{i\in I}$ and $(\mathsf{M}_i)_{i\in I}$, respectively (the two sets of indices have been made equal by eventually adding 0's to the families). Then there exist a complex unitary matrix $(u_{ij})_{i,j\in I}$, eventually of infinite size, such that

$$\boxed{\mathsf{M}_i=\sum_{j\in I}u_{ij}\,\mathsf{L}_j\,,}\tag{6.16}$$

for all $i\in I$ (the sums are all strongly convergent).

Proof. The existence of a minimal Krauss representation comes from Theorem 6.24.

Let us now prove the relation between two minimal representations. Denote by L and M the operators associated to the families (L_i) and (M_i), as in Proposition 6.4. If $\mathsf{X}\mapsto\mathsf{L}^*(\mathsf{X}\otimes\mathrm{I})\mathsf{L}$ and $\mathsf{X}\mapsto\mathsf{M}^*(\mathsf{X}\otimes\mathrm{I})\mathsf{M}$ are two minimal Krauss representations, with associated Hilbert spaces $\mathcal{H}\otimes\mathcal{K}$ and $\mathcal{H}\otimes\mathcal{K}'$ respectively, then the mapping U which maps $(\mathsf{X}\otimes\mathrm{I})\mathsf{L}\,u$ to $(\mathsf{X}\otimes\mathrm{I})\mathsf{M}\,u$ is isometric for

$$\begin{aligned}\langle(\mathsf{X}'\otimes\mathrm{I})\mathsf{M}\,u'\,,\,(\mathsf{X}\otimes\mathrm{I})\mathsf{M}\,u\rangle&=\langle u'\,,\,\mathsf{M}^*(\mathsf{X}'\mathsf{X}\otimes\mathrm{I})\mathsf{M}\,u\rangle\\&=\langle u'\,,\,\mathcal{M}(\mathsf{X}'\mathsf{X})\,u\rangle\\&=\langle u'\,,\,\mathsf{L}^*(\mathsf{X}'\mathsf{X}\otimes\mathrm{I})\mathsf{L}\,u\rangle\\&=\langle(\mathsf{X}'\otimes\mathrm{I})\mathsf{L}\,u'\,,\,(\mathsf{X}\otimes\mathrm{I})\mathsf{L}\,u\rangle\ .\end{aligned}$$

Hence it extends to a unitary operator U from $\mathcal{H}\otimes\mathcal{K}$ to $\mathcal{H}\otimes\mathcal{K}'$.

Obviously, this unitary operator U commutes with all the operators of the form $\mathsf{X}\otimes\mathrm{I}$, $\mathsf{X}\in\mathcal{B}(\mathcal{H})$. It is then easy to see that U has to be of the form $\mathrm{I}\otimes\mathsf{W}$ for some unitary operator W from $\mathcal{K}$ to $\mathcal{K}'$. We leave to the reader to check that this relation between L and M exactly corresponds to (6.16). □

Before proving the general theorem about the ambiguity of the Krauss representation, we prove a useful equivalent form of the density criterion of Proposition 6.26.

Lemma 6.29. *With the same notations as above, the space*

$$\mathcal{V} = \mathit{Vect}\{(\mathsf{X}\otimes \mathrm{I})\,\mathsf{M}\,u\,;\ \mathsf{X}\in\mathcal{B}(\mathcal{H})\,,\ u\in\mathcal{H}\}$$

is dense in $\mathcal{H}\otimes\mathcal{K}$ if and only if the space

$$\mathcal{W} = \mathit{Vect}\left\{{}_{\mathcal{H}}\langle\phi|\,\mathsf{M}u\,;\ u,\phi\in\mathcal{H}\right\}$$

is dense in $\mathcal{K}$.

Proof. Let $\phi\in\mathcal{H}$ and $\psi\in\mathcal{K}$ be fixed. For any $\mathsf{X}\in\mathcal{B}(\mathcal{H})$, any $u\in\mathcal{H}$, we have

$$\begin{aligned}\langle\phi\otimes\psi\,,\,(\mathsf{X}\otimes\mathrm{I})\,\mathsf{M}\,u\rangle &= \left\langle|\phi\rangle_{\mathcal{H}}\psi\,,\,(\mathsf{X}\otimes\mathrm{I})\,\mathsf{M}\,u\right\rangle\\ &= \left\langle\psi\,,\,{}_{\mathcal{H}}\langle\phi|(\mathsf{X}\otimes\mathrm{I})\,\mathsf{M}\,u\right\rangle\\ &= \left\langle\psi\,,\,{}_{\mathcal{H}}\langle\mathsf{X}^*\phi|\,\mathsf{M}\,u\right\rangle\\ &= \left\langle\psi\,,\,{}_{\mathcal{H}}\langle\phi'|\,\mathsf{M}\,u\right\rangle\,,\end{aligned}$$

where one notices that $\phi'=\mathsf{X}^*\phi$ can be any element of $\mathcal{H}$.

Assume that the space $\mathcal{V}$, as defined above, is dense in $\mathcal{H}\otimes\mathcal{K}$. Let ψ be an element of $\mathcal{K}$ which is orthogonal to all the ${}_{\mathcal{H}}\langle\phi|\,\mathsf{M}\,u$. Then by the computation above, any element $\phi\otimes\psi$ is orthogonal to all $\mathcal{V}$. By the density hypothesis this means that $\phi\otimes\psi=0$ for all $\phi\in\mathcal{H}$, that is, $\psi=0$. We have proved that the orthogonal of $\mathcal{W}$ is reduced to $\{0\}$, hence $\mathcal{W}$ is dense in $\mathcal{K}$.

Conversely, assume that $\mathcal{W}$ is dense in $\mathcal{K}$, then consider an element x of $\mathcal{H}\otimes\mathcal{K}$ which is orthogonal to $\mathcal{V}$. Taking $\mathsf{X}=|\phi_1\rangle\langle\phi_2|\in\mathcal{B}(\mathcal{H})$, we get

$$\begin{aligned}0 &= \langle x\,,\,(|\phi_1\rangle\langle\phi_2|\otimes\mathrm{I})\ \mathsf{M}\,u\rangle\\ &= \left\langle{}_{\mathcal{H}}\langle\phi_1|\,x\,,\,{}_{\mathcal{H}}\langle\phi_2|\,\mathsf{M}\,u\right\rangle\,.\end{aligned}$$

This would imply that ${}_{\mathcal{H}}\langle\phi_1|\,x$ is orthogonal to all the ${}_{\mathcal{H}}\langle\phi_2|\,\mathsf{M}\,u$, that is, to the whole $\mathcal{W}$. Hence ${}_{\mathcal{H}}\langle\phi_1|\,x=0$. But as this holds for all ϕ_1 this means that $x=0$. We have proved that the orthogonal of $\mathcal{V}$ is $\{0\}$. The lemma is proved. □

We can now state the main result concerning the ambiguity of the Krauss representation.

Theorem 6.30 (GHJW Theorem for Krauss representations). *Let $(\mathsf{A}_n)_{n\in\mathcal{N}}$ and $(\mathsf{B}_m)_{m\in\mathcal{M}}$ be families of bounded operators on $\mathcal{H}$ such that*

$$\sum_{n\in\mathcal{N}}\mathsf{A}_n^*\mathsf{A}_n = \sum_{m\in\mathcal{M}}\mathsf{B}_m^*\mathsf{B}_m = \mathrm{I}\,,$$

where the sums are strongly convergent if infinite. Assume that they are indexed by the same set $\mathcal{N}$ by adding eventually 0's to the smallest list. Consider the quantum channels

$$\rho \mapsto \mathcal{L}_1(\rho) = \sum_{n\in\mathcal{N}} \mathsf{A}_n\, \rho\, \mathsf{A}_n^*$$

and

$$\rho \mapsto \mathcal{L}_2(\rho) = \sum_{m\in\mathcal{N}} \mathsf{B}_m\, \rho\, \mathsf{B}_m^* \, .$$

1) If there exists a complex unitary matrix (eventually of infinite size) $(u_{nm})_{n,m\in\mathcal{N}}$ such that

$$\boxed{\mathsf{A}_n = \sum_{m\in\mathcal{N}} u_{nm}\, \mathsf{B}_m \, ,} \tag{6.17}$$

for all n (the sums are automatically strongly convergent), then $\mathcal{L}_1$ and $\mathcal{L}_2$ define the same quantum channel.
2) Assume that $\mathcal{N}$ is a finite set. If $\mathcal{L}_1$ and $\mathcal{L}_2$ define the same quantum channel then there exists a complex unitary matrix $(u_{nm})_{n,m\in\mathcal{N}}$ such that

$$\boxed{\mathsf{A}_n = \sum_{m\in\mathcal{N}} u_{nm}\, \mathsf{B}_m \, ,} \tag{6.18}$$

for all n.
3) Assume that $\mathcal{N}$ is infinite and that the lists $(\mathsf{A}_n)_{n\in\mathcal{N}}$ and $(\mathsf{B}_m)_{m\in\mathcal{N}}$ contain an infinite number of 0's (this can always be achieved by adding eventually 0's to the lists). If the mappings $\mathcal{L}_1$ and $\mathcal{L}_2$ define the same quantum channel then there exists a complex unitary matrix (of infinite size) $(u_{nm})_{n,m\in\mathcal{N}}$ such that

$$\boxed{\mathsf{A}_n = \sum_{m\in\mathcal{N}} u_{nm}\, \mathsf{B}_m \, ,} \tag{6.19}$$

for all n (the sums are automatically strongly convergent).

Proof. We first prove 1). The difficulty really come for the infinite dimensional case; so this is the case we shall treat here.

Let (B_n) be a sequence of bounded operators on $\mathcal{H}$ such that

$$\sum_{n\in\mathbb{N}} \mathsf{B}_n^* \mathsf{B}_n = \mathrm{I}$$

for the strong convergence. Let

$$\mathcal{L}_2(\rho) = \sum_{n\in\mathbb{N}} \mathsf{B}_n\, \rho\, \mathsf{B}_n^*$$

be the associated quantum channel. If $(u_{nm})_{n,m\in\mathbb{N}}$ is a complex unitary matrix then the series

$$\sum_{m\in\mathbb{N}} u_{nm}\,\mathsf{B}_m\,,$$

is strongly convergent, for

$$\begin{aligned}\sum_{m\in\mathbb{N}} \|u_{nm}\,\mathsf{B}_m\,h\| &\leq \left(\sum_{m\in\mathbb{N}} |u_{nm}|^2\right)^{1/2} \left(\sum_{m\in\mathbb{N}} \|\mathsf{B}_m h\|^2\right)^{1/2}\\ &= \left(\sum_{m\in\mathbb{N}} \langle h\,,\,\mathsf{B}_m^*\,\mathsf{B}_m h\rangle\right)^{1/2}\\ &= \|h\|^2\,.\end{aligned}$$

Hence this series defines a bounded operator A_n. Let B be the operator on $\mathcal{H}\otimes\mathcal{K}$ associated to the family (B_n) as in Proposition 6.4, that is,

$$\mathsf{B} = \sum_{n\in\mathbb{N}} (\mathsf{B}_n\otimes \mathrm{I})|e_n\rangle_{\mathcal{K}}\,.$$

By Proposition 6.4 the operator B is an isometry. Consider the operator U on $\mathcal{K}$ given by

$$\mathsf{U} = \sum_{m,n\in\mathbb{N}} u_{mn}\,|e_m\rangle\langle e_n|\,.$$

It is clearly a unitary operator on $\mathcal{K}$. Define the operator $\mathsf{C} = (\mathrm{I}\otimes\mathsf{U})\mathsf{B}$ from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{K}$, it is then an isometry. The associated operators C_n, as defined in Proposition 6.4, are given by

$$\mathsf{C}_n = {}_{\mathcal{K}}\langle e_n|\,(\mathrm{I}\otimes\mathsf{U})\mathsf{B}\,.$$

This gives for all $h\in\mathcal{H}$

$$\begin{aligned}\mathsf{C}_n h &= \sum_{m\in\mathbb{N}}\sum_{i,j\in\mathbb{N}} {}_{\mathcal{K}}\langle e_n|\,(\mathrm{I}\otimes u_{ji}|e_j\rangle\langle e_i|)\,(\mathsf{B}_m h\otimes e_m)\\ &= \sum_{m\in\mathbb{N}}\sum_{j\in\mathbb{N}} {}_{\mathcal{K}}\langle e_n|\,u_{jm}\,(\mathsf{B}_m h\otimes e_j)\\ &= \sum_{m\in\mathbb{N}} u_{nm}\,\mathsf{B}_m h\\ &= \mathsf{A}_n h\,.\end{aligned}$$

Hence, by Proposition 6.4 again, the series $\sum_{n\in\mathbb{N}} \mathsf{A}_n^*\mathsf{A}_n$ converges strongly to the operator I and the family (A_n) defines a quantum channel

$$\mathcal{L}_1(\rho) = \sum_{n\in\mathbb{N}} \mathsf{A}_n\, \rho\, \mathsf{A}_n^* \,.$$

By Proposition 6.4 again, we know that

$$\mathcal{L}_1(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{C}\,\rho\,\mathsf{C}^*\right)\,,$$

hence we get

$$\begin{aligned}
\mathcal{L}_1(\rho) &= \mathrm{Tr}_{\mathcal{K}}\left((\mathsf{I}\otimes\mathsf{U})\,\mathsf{B}\,\rho\,\mathsf{B}^*(\mathsf{I}\otimes\mathsf{U}^*)\right)\\
&= \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{B}\,\rho\,\mathsf{B}^*(\mathsf{I}\otimes\mathsf{U}^*)(\mathsf{I}\otimes\mathsf{U})\right)\\
&= \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{B}\,\rho\,\mathsf{B}^*\right)\\
&= \mathcal{L}_2(\rho)\,.
\end{aligned}$$

The quantum channels $\mathcal{L}_1$ and $\mathcal{L}_2$ define the same quantum channels. We have proved 1).

We now prove 2) and 3) at the same time. Let $\mathcal{H}\otimes\mathcal{K}$ and A be associated to $\mathcal{L}_1$ as in 1). Let $\mathcal{K}_1$ denote the closed subspace of $\mathcal{K}$ generated by the ${}_{\mathcal{H}}\langle\phi|\,\mathsf{A}\,u$, for all $\phi, u\in\mathcal{H}$. In particular $\mathrm{Ran}\,\mathsf{A}$ is included in $\mathcal{H}\otimes\mathcal{K}_1$. One can consider the operator A as acting from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{K}_1$. Constructing $\mathcal{K}_2$ in the same way, associated to B and by the same trick as in Theorem 6.28 we construct a unitary operator $\mathsf{U}:\mathcal{K}_1\to\mathcal{K}_2$ such that $\mathsf{A}=(\mathsf{I}\otimes\mathsf{U})\mathsf{B}$.

The point now is to extend U to $\mathcal{K}$. One wishes to extend U by completing it with a unitary operator from $\mathcal{K}_1^\perp$ to $\mathcal{K}_2^\perp$, which would do the job and put an end to the proof.

– If $\mathcal{N}$ is finite, then $\mathcal{K}$ is finite dimensional. Thus so are $\mathcal{K}_1$ and $\mathcal{K}_2$. As they are unitarily isomorphic they are of same dimension, thus so are $\mathcal{K}_1^\perp$ and $\mathcal{K}_2^\perp$. It is therefore easy to extend U into a unitary operator from $\mathcal{K}$ to $\mathcal{K}$.

– If $\mathcal{N}$ is infinite, this is where the condition, that the A_n's and B_n's contain an infinite number of 0's, enters into the game. Indeed this hypothesis implies that $\mathcal{K}_1^\perp$ and $\mathcal{K}_2^\perp$ are infinite dimensional (and separable, of course), hence unitarily isomorphic. One completes the proof in the same way as above. □

6.3.5 Terminology

This small subsection is here in order to make a few points clear regarding the terminology around these different maps. We have felt it could be necessary, for the terminology is sometimes misused the literature, some confusion is coming from the finite dimensional case which is the one most often studied.

6.3.5.1 The infinite Dimensional Case

Definition 6.31. Recall that if (M_i) is a family of bounded operators on $\mathcal{H}$ such that $\sum_i \mathsf{M}_i^*\, \mathsf{M}_i$ is strongly convergent then the mapping

$$\mathcal{M}(\mathsf{X}) = \sum_i \mathsf{M}_i^*\, \mathsf{X}\, \mathsf{M}_i \tag{6.20}$$

is well-defined on $\mathcal{B}(\mathcal{H})$ (strongly convergent by Proposition 6.10). It is then a *completely positive map* on $\mathcal{B}(\mathcal{H})$ and all the normal completely positive maps on $\mathcal{B}(\mathcal{H})$ are of this form, as we have seen Theorem 6.23.

If $\mathcal{M}$ furthermore satisfies the relation

$$\mathcal{M}(\mathrm{I}) = \mathrm{I}\,,$$

that is, if

$$\sum_i \mathsf{M}_i^*\, \mathsf{M}_i = \mathrm{I}\,,$$

then $\mathcal{M}$ is called a *unital completely positive map*. We have seen, in Subsection 6.2.2, that they are exactly the adjoints of the quantum channels.

Quantum channels, as we have seen, also admit a Krauss decomposition

$$\mathcal{L}(\mathsf{T}) = \sum_i \mathsf{M}_i\, \mathsf{T}\, \mathsf{M}_i^*$$

and one may be tempted to say that it is a special case of the Krauss decomposition of completely positive maps, with M_i^* playing the role of M_i in (6.20). But it is important to notice that under the only assumption that $\sum_i \mathsf{M}_i^*\, \mathsf{M}_i$ is strongly convergent then the series $\sum_i \mathsf{M}_i\, \mathsf{X}\, \mathsf{M}_i^*$ may not be convergent for some bounded operator X. We have proved in this lecture that it is $\|\cdot\|_1$-convergent when specializing to those X that are trace-class, but it may not be convergent for more general bounded operators X. Let us see that on a counter-example.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Define the bounded operators

$$\mathsf{M}_i = |e_0\rangle\langle e_i|\,.$$

We then have

$$\sum_{i\in\mathbb{N}} \mathsf{M}_i^*\, \mathsf{M}_i = \sum_{i\in\mathbb{N}} |e_i\rangle\langle e_0|\; |e_0\rangle\langle e_i| = \sum_{i\in\mathbb{N}} |e_i\rangle\langle e_i| = \mathrm{I}\,.$$

But in the other direction

$$\sum_{i\in\mathbb{N}} \mathsf{M}_i\, \mathsf{M}_i^* = \sum_{i\in\mathbb{N}} |e_0\rangle\langle e_i|\; |e_i\rangle\langle e_0| = \sum_{i\in\mathbb{N}} |e_0\rangle\langle e_0|\,,$$

which converges in no way. Hence, in this example, $\sum_{i\in\mathbb{N}} \mathsf{M}_i\,\mathsf{X}\,\mathsf{M}_i^*$ does not converge when $\mathsf{X} = \mathrm{I}$.

Let us continue to explore this example. The map

$$\mathsf{X} \mapsto \mathcal{L}^*(\mathsf{X}) = \sum_{i\in\mathbb{N}} \mathsf{M}_i^*\,\mathsf{X}\,\mathsf{M}_i = \sum_{i\in\mathbb{N}} |e_i\rangle\langle e_0|\,\mathsf{X}\,|e_0\rangle\langle e_i| = \langle e_0|\,\mathsf{X}\,|e_0\rangle\,\mathrm{I}$$

is a well-defined completely positive map on $\mathcal{B}(\mathcal{H})$. The map

$$\mathsf{T} \mapsto \mathcal{L}(\mathsf{T}) = \sum_{i\in\mathbb{N}} \mathsf{M}_i\,\mathsf{T}\,\mathsf{M}_i^* = \sum_{i\in\mathbb{N}} |e_0\rangle\langle e_i|\,\mathsf{T}\,|e_i\rangle\langle e_0| = \mathrm{Tr}\,(\mathsf{T})\,|e_0\rangle\langle e_0|$$

is well-defined only on $\mathcal{L}_1(\mathcal{H})$ and it is a quantum channel.

The notion of complete positive map could in fact be defined not only on $\mathcal{B}(\mathcal{H})$ but on any $*$-algebra. In particular it could be defined on $\mathcal{L}_1(\mathcal{H})$ in the same way as for $\mathcal{B}(\mathcal{H})$ by asking all the natural dilations $\mathcal{L}_n$ of $\mathcal{L}$ to $\mathcal{L}_1(\mathcal{H}\otimes\mathbb{C}^n) = \mathcal{L}_1(\mathcal{H})\otimes M_n(\mathbb{C})$, given by

$$\mathcal{L}_n(\mathsf{A}\otimes\mathsf{B}) = \mathcal{L}(\mathsf{A})\otimes\mathsf{B}\,,$$

to be positivity-preserving. Quantum channels would then be completely positive in this sense. But they do not admit in general a Stinespring representation. Let us see that with the counterexample above. The first obstacle is the definition of *Stinespring representation* in this case, one should at least adjust to the fact that I is not in general an element of $\mathcal{L}_1(\mathcal{H})$ so the condition $\pi(\mathrm{I}) = \mathrm{I}$ should be be forgotten.

But even so, this does not work. With our counterexample, if one had

$$\mathcal{L}(\mathsf{T}) = \mathrm{Tr}\,(\mathsf{T})|e_0\rangle\langle e_0| = \mathsf{V}^*\,\pi(\mathsf{T})\,\mathsf{V}$$

for some ($*$-algebra) representation π of $\mathcal{L}_1(\mathcal{H})$ then, putting $\Psi = \mathsf{V}e_0$, we would have

$$\mathrm{Tr}\,(\mathsf{T}) = \langle\Psi\,,\,\pi(\mathsf{T})\,\Psi\rangle\ ,$$

for all $\mathsf{T}\in\mathcal{L}_1(\mathcal{H})$. If T is any rank one orthogonal projector $|x\rangle\langle x|$ then $\mathsf{P} = \pi(|x\rangle\langle x|)$ is an orthogonal projector too, by the $*$-algebra morphism property of π. We have

$$1 = \langle\Psi\,,\,\mathsf{P}\,\Psi\rangle\ .$$

This implies $\mathsf{P}\Psi = \Psi$, for we are in the case of equality in the Cauchy-Schwarz inequality.

Repeating the argument with some y orthogonal to x we would get $\mathsf{Q}\Psi = \Psi$, where $\mathsf{Q} = \pi(|y\rangle\langle y|)$. This is impossible, for $\mathsf{PQ} = 0$ by the morphism property of π.

Hence there is no Stinespring representation whatsoever for $\mathcal{L}$ even if we try to make effort to adapt it to $\mathcal{L}_1(\mathcal{H})$.

The denomination “completely positive” should be in general used only on $\mathcal{B}(\mathcal{H})$ (actually the natural context is the one of unital C^*-algebras). The quantum channels are preduals of completely positive maps, they also admit a Krauss representation, but one should be careful with the direction of the sandwich-sum in this case. In the infinite dimensional case, quantum channels should not be called ”completely positive maps” on $\mathcal{L}_1(\mathcal{H})$, even though we know that these maps satisfies similar positivity-preserving properties.

6.3.5.2 The Finite Dimensional Case

The finite dimensional case simplifies everything but also introduces the confusion. When $\mathcal{H}$ is finite dimensional then

1) $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}_1(\mathcal{H})$ coincide

2) the sums

$$\sum_i \mathsf{M}_i^* \mathsf{M}_i$$

and

$$\sum_i \mathsf{M}_i \mathsf{M}_i^*$$

are finite and thus obviously convergent in any topology.

Hence in that case the mapping

$$\mathsf{T} \mapsto \mathcal{L}(\mathsf{T}) = \sum_i \mathsf{M}_i\, \mathsf{T}\, \mathsf{M}_i^*$$

is also a completely positive map (with M_i playing the role of M_i^*) on $\mathcal{B}(\mathcal{H})$.

Note that this completely positive map is not unital anymore, but the fact that

$$\mathrm{Tr}\,(\mathcal{L}(\mathsf{T})) = \mathrm{Tr}\,(\mathsf{T})$$

for all T makes it called a *trace-preserving completely positive map*. Hence, in finite dimension, quantum channels are exactly the trace-preserving completely positive maps!

This is the source of the confusion, in finite dimension quantum channels are just a special case of completely positive maps, whereas in infinite dimension they are not.

Here we have made the choice tofo distinguishing between quantum channels which are defined for elements of $\mathcal{L}_1(\mathcal{H})$ and completely positive maps which are better defined on $\mathcal{B}(\mathcal{H})$.

6.4 Properties of CP Maps

In this section we prove several interesting properties shared by completely positive maps.

6.4.1 Basic Properties

First of all the result which is exactly parallel to the first one of Proposition 6.8.

Proposition 6.32. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two completely positive maps on $\mathcal{B}(\mathcal{H})$. Then the composition $\mathcal{M}_2 \circ \mathcal{M}_1$ is a completely positive map on $\mathcal{B}(\mathcal{H})$.*

If furthermore $\mathcal{M}_1$ and $\mathcal{M}_2$ are normal and admit Krauss decompositions

$$\mathcal{M}_1(\mathsf{X}) = \sum_{i\in\mathbb{N}} \mathsf{A}_i^* \,\rho\, \mathsf{A}_i \qquad \textit{and} \qquad \mathcal{M}_2(\mathsf{X}) = \sum_{i\in\mathbb{N}} \mathsf{B}_i^* \,\rho\, \mathsf{B}_i \,,$$

respectively, then $\mathcal{M}_2 \circ \mathcal{M}_1$ admits a Krauss decomposition given by

$$\mathcal{M}_2 \circ \mathcal{M}_1(\mathsf{X}) = \sum_{i,j\in\mathbb{N}} \mathsf{B}_j^* \mathsf{A}_i^* \,\mathsf{X}\, \mathsf{A}_i \mathsf{B}_j \,.$$

Proof. The fact that $\mathcal{M}_2 \circ \mathcal{M}_1$ is completely positive comes immediately from the initial definition. Indeed we have $(\mathcal{M}_2 \circ \mathcal{M}_1) \otimes \mathrm{I}_n = (\mathcal{M}_2 \otimes \mathrm{I}_n) \circ (\mathcal{M}_1 \otimes \mathrm{I}_n)$ and the n-positivity preservation property holds obviously.

Assume now that both $\mathcal{M}_1$ and $\mathcal{M}_2$ admit a Krauss representation as described in the Proposition statements. The associated operators and spaces such as described in Proposition 6.4 are denoted by A, B and $\mathcal{K}_1, \mathcal{K}_2$, respectively. Consider the operator $\mathsf{C} = (\mathsf{A} \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{B}$ from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{K}_1 \otimes \mathcal{K}_2$. It is a bounded operator and it satisfies

$$\begin{aligned}
\mathsf{C}^*(\mathsf{X} \otimes \mathrm{I}_{\mathcal{K}_1} \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{C} &= \mathsf{B}^*(\mathsf{A}^* \otimes \mathrm{I}_{\mathcal{K}_2})(\mathsf{X} \otimes \mathrm{I}_{\mathcal{K}_1} \otimes \mathrm{I}_{\mathcal{K}_2})(\mathsf{A} \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{B} \\
&= \mathsf{B}^*(\mathsf{A}^*(\mathsf{X} \otimes \mathrm{I}_{\mathcal{K}_1})\mathsf{A} \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{B} \\
&= \mathsf{B}^*(\mathcal{M}_1(\mathsf{X}) \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{B} \\
&= \mathcal{M}_2(\mathcal{M}_1(\mathsf{X}))\,.
\end{aligned}$$

This means that we have obtained a Krauss representation of $\mathcal{M}_2 \circ \mathcal{M}_1$. As in Proposition 6.4, let (f_n) and (g_n) be the orthonormal basis of $\mathcal{K}_1$ and $\mathcal{K}_2$ from which the coefficients A_n and B_n are deduced from A and B. Then the Krauss coefficients of $\mathcal{M}_2 \circ \mathcal{M}_1$ are given by

$$\begin{aligned}
\mathsf{C}_{nm} &= {}_{\mathcal{K}_1\otimes\mathcal{K}_2}\langle f_n \otimes g_m | \mathsf{C} \\
&= {}_{\mathcal{K}_2}\langle g_m |\, {}_{\mathcal{K}_1}\langle f_n | (\mathsf{A} \otimes \mathrm{I}_{\mathcal{K}_2})\mathsf{B} \\
&= {}_{\mathcal{K}_2}\langle g_m | \left({}_{\mathcal{K}_1}\langle f_n | \mathsf{A} \otimes \mathrm{I}_{\mathcal{K}_2} \right) \mathsf{B} \\
&= {}_{\mathcal{K}_2}\langle g_m | \,(\mathsf{A}_n \otimes \mathrm{I}_{\mathcal{K}_2})\, \mathsf{B} \\
&= \mathsf{A}_n \, {}_{\mathcal{K}_2}\langle g_m | \mathsf{B} \\
&= \mathsf{A}_n \, \mathsf{B}_m \,.
\end{aligned}$$

This gives the right coefficients for the Krauss representation and by the way this proves the strong convergence of $\sum_{n,m} \mathsf{B}_m^* \mathsf{A}_n^* \mathsf{A}_n \mathsf{B}_m$. □

Our second easy property is a Cauchy-Schwarz inequality for completely positive maps.

Proposition 6.33 (Cauchy-Schwarz Inequality for Unital Completely Positive Maps). *Let $\mathcal{M}$ be a unital completely positive map on $\mathcal{B}(\mathcal{H})$. Then*

$$\mathcal{M}(\mathsf{X}^*\mathsf{X}) \geq \mathcal{M}(\mathsf{X})^*\mathcal{M}(\mathsf{X})$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$.

Proof. By Stinespring's Theorem this unital completely positive map is of the form $\mathcal{M}(\mathsf{X}) = \mathsf{V}^*\pi(\mathsf{X})\mathsf{V}$ for some isometry V and some representation π of $\mathcal{B}(\mathcal{H})$.

In particular we have

$$\begin{aligned}
\mathcal{M}(\mathsf{X}^*\,\mathsf{X}) - \mathcal{M}(\mathsf{X})^*\mathcal{M}(\mathsf{X}) &= \mathsf{V}^*\pi(\mathsf{X}^*\,\mathsf{X})\mathsf{V} - \mathsf{V}^*\pi(\mathsf{X})^*\,\mathsf{V}\,\mathsf{V}^*\pi(\mathsf{X})\mathsf{V} \\
&= \mathsf{V}^*\pi(\mathsf{X})^*(\mathrm{I} - \mathsf{V}\,\mathsf{V}^*)\pi(\mathsf{X})\mathsf{V}\,.
\end{aligned}$$

As $\mathrm{I} - \mathsf{V}\,\mathsf{V}^*$ is a positive operator, the expression above is a positive operator too. □

6.4.2 Defect of Morphism Property

The points we discuss here and in next subsection have in general no much application, they are just interesting points to note about completely positive maps. We show two interesting results which have in common to quantify what has been lost when passing from a dynamics of the form $\mathsf{U}\cdot\mathsf{U}^*$ to a non-trivial quantum channel. The first result is a very nice one, to my opinion, which shows that the difference between closed and open quantum systems is exactly characterized by the loss of $*$-morphism property.

The second result, presented in the next subsection, which somehow refers to the first one, has to do with the information lost from the environment when tracing out: the dynamics is not reversible anymore.

We first start our discussion with the finite dimensional case, that is, the Hilbert space $\mathcal{H}$ is finite dimensional for a while.

Consider the Krauss representation of a quantum channels $\mathcal{L}$ on $\mathcal{H}$:

$$\mathcal{L}(\rho) = \sum_{i=1}^{n} \mathsf{M}_i \, \rho \, \mathsf{M}_i^* \, .$$

In general there is more than one term in this sum, but note that when there is only one term

$$\mathcal{L}(\rho) = \mathsf{M}_1 \, \rho \, \mathsf{M}_1^* \, ,$$

then the condition $\mathsf{M}_1^* \mathsf{M}_1 = \mathrm{I}$ implies that M_1 is unitary and thus $\mathcal{L}$ is the usual unitary conjugation dynamics of a closed quantum system. This corresponds to the case where the environment is either trivial, or does not exist, or does not interact with the system $\mathcal{H}$. In this case the dynamics $\mathcal{L}$ or better its adjoint $\mathcal{N} = \mathcal{L}^*$ acting on $\mathcal{B}(\mathcal{H})$ has the property

$$\mathcal{N}(\mathsf{X}^* \, \mathsf{Y}) = \mathcal{N}(\mathsf{X})^* \, \mathcal{N}(\mathsf{Y})$$

and

$$\mathcal{N}(\mathrm{I}) = \mathrm{I} \, .$$

In other words $\mathcal{N}$ is a morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$. The point with our first result is that the distinction between a truly unitary evolution and a non-trivial action of the environment exactly correspond to the loss of this morphism property.

Theorem 6.34. *Let $\mathcal{L}$ be a quantum channel acting on $\mathcal{L}_1(\mathcal{H})$ for a finite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{N} = \mathcal{L}^*$ be its dual map acting on $\mathcal{B}(\mathcal{H})$, that is, a unital completely positive map on $\mathcal{B}(\mathcal{H})$. Then the mapping $\mathcal{N}$ is a morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{N}$ is of the form*

$$\mathcal{N}(\mathsf{X}) = \mathsf{U}^* \, \mathsf{X} \, \mathsf{U}$$

for some unitary operator U on $\mathcal{H}$.

Proof. Obviously, if $\mathcal{N}$ is of the form $\mathcal{N}(\mathsf{X}) = \mathsf{U}^* \, \mathsf{X} \, \mathsf{U}$ for some unitary operator U, then is satisfies

$$\mathcal{N}(\mathsf{X}^* \, \mathsf{Y}) = \mathcal{N}(\mathsf{X})^* \, \mathcal{N}(\mathsf{Y}) \, ,$$

for all $\mathsf{X}, \mathsf{Y} \in \mathcal{B}(\mathcal{H})$. We have proved one direction.

Conversely, assume that $\mathcal{N}$ is a unital completely positive map and a morphism of $\mathcal{B}(\mathcal{H})$. If P is an orthogonal projector on $\mathcal{H}$ then

$$\mathcal{N}(\mathsf{P})^2 = \mathcal{N}(\mathsf{P}^2) = \mathcal{N}(\mathsf{P})$$

and

$$\mathcal{N}(\mathsf{P})^* = \mathcal{N}(\mathsf{P})\,.$$

Hence $\mathcal{N}(\mathsf{P})$ is an orthogonal projector too. Furthermore, if P and Q are orthogonal projectors with orthogonal ranges then so do $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{Q})$, for

$$\mathcal{N}(\mathsf{P})\,\mathcal{N}(\mathsf{Q}) = \mathcal{N}(\mathsf{PQ}) = 0\,.$$

Take an orthonormal basis $\{e_i\,;\ i = 1,\dots,d\}$ of $\mathcal{H}$. The operators

$$\mathsf{P}_i = \mathcal{N}\left(|e_i\rangle\langle e_i|\right)\,,$$

$i = 1,\dots,d$, are orthogonal projectors, with mutually orthogonal ranges. Furthermore the fact that $\mathcal{N}(\mathrm{I}) = \mathrm{I}$ shows that

$$\sum_{i=1}^{d} \mathsf{P}_i = \mathsf{I}\,.$$

One cannot yet conclude that the P_i's are all of rank 1, for some of them might be null.

Consider the operators $\mathsf{M}_{ij} = \mathcal{N}\left(|e_i\rangle\langle e_j|\right)$. By the morphism property they satisfy

$$\mathsf{M}_{ij}^*\mathsf{M}_{ij} = \mathsf{P}_j$$

and

$$\mathsf{M}_{ij}\mathsf{M}_{ij}^* = \mathsf{P}_i\,.$$

In other words M_{ij} is a partial isometry with initial space $\mathrm{Ran}\,\mathsf{P}_j$ and final space $\mathrm{Ran}\,\mathsf{P}_i$. In particular this means that the two spaces $\mathrm{Ran}\,\mathsf{P}_j$ and $\mathrm{Ran}\,\mathsf{P}_i$ are unitarily equivalent; they have same dimension.

Now there is no other possibility than having all the P_i's to be rank one:

$$\mathsf{P}_i = |\phi_i\rangle\langle\phi_i|\,,$$

say. The ϕ_i's form an orthonormal basis of $\mathcal{H}$, hence the mapping U which maps ϕ_i to e_i for all i is a unitary operator.

We have proved that

$$\mathcal{N}(|e_i\rangle\langle e_i|) = |\phi_i\rangle\langle\phi_i| = \mathsf{U}^*\,|e_i\rangle\langle e_i|\,\mathsf{U}\,,$$

for all $i = 1,\dots,d$. Even more, the fact that M_{ij} is a partial isometry with initial space $\mathbb{C}\phi_j$ and final space $\mathbb{C}\phi_i$ shows that

$$\mathsf{M}_{ij} = |\phi_i\rangle\langle\phi_j|\,.$$

In other words

$$\mathcal{N}(|e_i\rangle\langle e_j|) = \mathsf{U}^*\,|e_i\rangle\langle e_j|\,\mathsf{U}\,.$$

This is clearly enough for proving that

$$\mathcal{N}(\mathsf{X}) = \mathsf{U}^* \, \mathsf{X} \, \mathsf{U}$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$. $\square$

Remark 6.35. This is such a nice and net result, which characterizes in a completely algebraical way what is the difference between the unitary dynamics of an isolated quantum system and the quantum channel (or its adjoint) associated to an open quantum system. One can even think of using this theorem in order to quantify the defect of morphism, which also would quantify the defect of unitarity of the dynamics, or else the defect of being a closed system. For example, for a unital completely positive map $\mathcal{N}$ put

$$\mathsf{D}(\mathcal{N}) = \sup\{\|\mathcal{N}(\mathsf{X}^* \, \mathsf{X}) - \mathcal{N}(\mathsf{X})^* \, \mathcal{N}(\mathsf{X})\| \; ; \; \mathsf{X} \in \mathcal{B}(\mathcal{H}) \, , \; \|\mathsf{X}\| = 1\} \, .$$

This quantity vanishes if and only if $\mathcal{N}$ is of the form $\mathsf{U}^* \cdot \mathsf{U}$ and it should be representative of how the map $\mathcal{N}$ is far or not from a unitary conjugation.

The result of Theorem 6.34 does not extend to the infinite dimensional case as it is. Let us detail here a counter-example. We consider $\mathcal{H}$ an infinite dimensional Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Consider also the two following operators on $\mathcal{H}$:

$$\mathsf{M}_1 = \sum_{k=0}^{\infty} |e_k\rangle\langle e_{2k}|$$
$$\mathsf{M}_2 = \sum_{k=0}^{\infty} |e_k\rangle\langle e_{2k+1}| \, .$$

Proposition 6.36. *The linear map $\mathcal{L}$ defined on $\mathcal{B}(\mathcal{H})$ by*

$$\mathcal{L}(\mathsf{X}) = \mathsf{M}_1^* \, \mathsf{X} \, \mathsf{M}_1 + \mathsf{M}_2^* \, \mathsf{X} \, \mathsf{M}_2$$

is a unital completely positive map on $\mathcal{B}(\mathcal{H})$. It is also a morphism of $\mathcal{B}(\mathcal{H})$, but there exist no unitary operator V on $\mathcal{H}$ such that

$$\mathcal{L}(\mathsf{X}) = \mathsf{V}^* \, \mathsf{X} \, \mathsf{V} \, ,$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$.

Proof. The operators M_1 and M_2 satisfy the relation

$$\mathsf{M}_1^*\mathsf{M}_1 + \mathsf{M}_2^*\mathsf{M}_2 = \sum_{k=0}^{\infty} |e_{2k}\rangle\langle e_{2k}| + \sum_{k=0}^{\infty} |e_{2k+1}\rangle\langle e_{2k+1}| = \mathrm{I} \, .$$

This implies that $\mathcal{L}$ is a unital completely positive map. They also satisfy the relations

$$\mathsf{M}_1\mathsf{M}_1^* = \mathsf{M}_2\mathsf{M}_2^* = \mathrm{I} \, , \qquad \mathsf{M}_1\mathsf{M}_2^* = \mathsf{M}_2\mathsf{M}_1^* = 0 \, ,$$

as can be checked easily. This implies

$$\begin{aligned}\mathcal{L}(\mathsf{X})^*\mathcal{L}(\mathsf{Y}) &= \mathsf{M}_1^*\,\mathsf{X}^*\,\mathsf{M}_1\mathsf{M}_1^*\,\mathsf{Y}\,\mathsf{M}_1 + \mathsf{M}_2^*\,\mathsf{X}^*\,\mathsf{M}_2\mathsf{M}_1^*\,\mathsf{Y}\,\mathsf{M}_1 + \mathsf{M}_1^*\,\mathsf{X}^*\,\mathsf{M}_1\mathsf{M}_2^*\,\mathsf{Y}\,\mathsf{M}_2 + \\ &\quad + \mathsf{M}_2^*\,\mathsf{X}^*\,\mathsf{M}_2\mathsf{M}_2^*\,\mathsf{Y}\,\mathsf{M}_2 \\ &= \mathsf{M}_1^*\,\mathsf{X}^*\,\mathsf{Y}\,\mathsf{M}_1 + \mathsf{M}_2^*\,\mathsf{X}^*\,\mathsf{Y}\,\mathsf{M}_2 \\ &= \mathcal{L}(\mathsf{X}^*\,\mathsf{Y})\,.\end{aligned}$$

In other words $\mathcal{L}$ is a morphism of $\mathcal{B}(\mathcal{H})$.

Now, if there exists a unitary operator V on $\mathcal{H}$ such that

$$\mathcal{L}(\mathsf{X}) = \mathsf{V}^*\,\mathsf{X}\,\mathsf{V}\,,$$

for all $\mathsf{X} \in \mathcal{B}(\mathcal{H})$, we get, putting $f_k = \mathsf{V}^* e_k$ for all k,

$$|f_k\rangle\langle f_l| = \mathcal{L}\left(|e_k\rangle\langle e_l|\right) = |e_{2k}\rangle\langle e_{2l}| + |e_{2k+1}\rangle\langle e_{2l+1}|\,.$$

This is clearly impossible (for example, this would imply

$$\langle f_l\,,\, e_{2l}\rangle\ f_k = e_{2k} \qquad \text{and} \qquad \langle f_l\,,\, e_{2l+1}\rangle\ f_k = e_{2k+1}$$

at the same time). This proves that the $*$-algebra morphism $\mathcal{L}$ is not an obvious unitary conjugation. □

In order to obtain a result similar to Theorem 6.34 in the infinite dimensional case one needs to add one condition. This condition is automatically satisfied in the finite dimensional case, it has to be explicitly added in the infinite dimensional case.

Theorem 6.37. *Let $\mathcal{L}$ be a quantum channel acting on $\mathcal{L}_1(\mathcal{H})$ for an infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{N} = \mathcal{L}^*$ be its dual map acting on $\mathcal{B}(\mathcal{H})$, that is, a unital completely positive map on $\mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent.*

i) The map $\mathcal{N}$ is of the form

$$\mathcal{N}(\mathsf{X}) = \mathsf{U}^*\,\mathsf{X}\,\mathsf{U}$$

for some unitary operator U on $\mathcal{H}$.

ii) The mapping $\mathcal{N}$ is an invertible morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$.

iii) The mapping $\mathcal{N}$ is a morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$, it preserves $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ and it preserves the trace on $\mathcal{L}_1(\mathcal{H})$.

iv) The mapping $\mathcal{N}$ is a morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$ and the mapping$\mathcal{L}$ is a morphism of the $*$-algebra $\mathcal{L}_1(\mathcal{H})$.

Proof. Obviously i) implies ii), iii) and iv).

Let us prove that ii) implies i). As in the proof of Theorem 6.34. The operators $\mathsf{P}_i = \mathcal{N}(|e_i\rangle\langle e_i|)$ are orthogonal projectors, pairwise orthogonal to each other, their sum is the identity operator.

The rank one orthogonal projectors, such as $|e_i\rangle\langle e_i|$, are *minimal* in $\mathcal{B}(\mathcal{H})$, that is, they are the only orthogonal projectors P such that if Q is any orthogonal projector satisfying

$$\mathsf{QP} = \mathsf{PQ} = \mathsf{P}\,,$$

then

$$\mathsf{Q} = \mathsf{P}\,.$$

This statement is very easy to check and left to the reader.

We claim that if $\mathcal{N}$ is invertible then $\mathsf{P}_i = \mathcal{N}(|e_i\rangle\langle e_i|)$ has to be minimal too. Indeed, take Q to be any orthogonal projector satisfying $\mathsf{QP}_i = \mathsf{P}_i\mathsf{Q} = \mathsf{Q}$. Put $\widehat{\mathsf{Q}} = \mathcal{N}^{-1}(\mathsf{Q})$, then $\widehat{\mathsf{Q}}$ is an orthogonal projector too, for $\mathcal{N}^{-1}$ is automatically a morphism of $\mathcal{B}(\mathcal{H})$ too (as can be checked easily). We then have

$$\mathcal{N}(\widehat{\mathsf{Q}})\,\mathcal{N}(|e_i\rangle\langle e_i|) = \mathcal{N}(|e_i\rangle\langle e_i|)\,\mathcal{N}(\widehat{\mathsf{Q}}) = \mathcal{N}(\widehat{\mathsf{Q}})\,,$$

that is

$$\widehat{\mathsf{Q}}\,|e_i\rangle\langle e_i| = |e_i\rangle\langle e_i|\,\widehat{\mathsf{Q}} = \widehat{\mathsf{Q}}\,.$$

The minimality of $|e_i\rangle\langle e_i|$ implies that $\widehat{\mathsf{Q}} = |e_i\rangle\langle e_i|$ and thus $\mathsf{Q} = \mathsf{P}_i$. We have proved that the P_i are minimal in $\mathcal{B}(\mathcal{H})$, hence they are rank one projectors.

We now conclude exactly in the same way as the end of the proof of Theorem 6.34.

Let us prove that iii) implies i), now. As $\mathcal{N}$ is a morphism of the algebra $\mathcal{B}(\mathcal{H})$ we can start the proof in exactly the same way as the one of Theorem 6.34. The operators $\mathsf{P}_i = \mathcal{N}(|e_i\rangle\langle e_i|)$ are orthogonal projectors, pairwise orthogonal to each other, their sum is the identity operator. The operators $\mathsf{M}_{ij} = \mathcal{N}(|e_i\rangle\langle e_j|)$ are partial isometries with initial space $\mathrm{Ran}\,\mathsf{P}_j$ and final space $\mathrm{Ran}\,\mathsf{P}_i$. Hence the spaces $\mathrm{Ran}\,\mathsf{P}_i$ are all unitarily isomorphic. But the counterexample above shows that this is not sufficient to conclude that the P_i's are rank one operators.

It is where the assumption that $\mathcal{N}$ is trace-preserving on $\mathcal{L}_1(\mathcal{H})$ comes into the game and makes everything trivial: as $|e_i\rangle\langle e_i|$ belongs to $\mathcal{L}_1(\mathcal{H})$ we have

$$\mathrm{Tr}\,(\mathsf{P}_i) = \mathrm{Tr}\,\left(\mathcal{N}(|e_i\rangle\langle e_i|)\right) = \mathrm{Tr}\,(|e_i\rangle\langle e_i|) = 1\,,$$

then P_i is rank one and one concludes exactly in the same way as the end of the proof of Theorem 6.34.

We finally prove iv) implies i). Considering the map $\mathcal{L}$ acting on $\mathcal{L}_1(\mathcal{H})$ and being a morphism of the $*$-algebra $\mathcal{L}_1(\mathcal{H})$. We can use the same trick as for $\mathcal{N}$ above: consider the operators $\mathsf{Q}_i = \mathcal{L}(|e_i\rangle\langle e_i|)$, they are pairwise orthogonal projectors and they are rank one, for $\mathcal{L}$ preserves the trace. This means that

$$\mathcal{L}(|e_i\rangle\langle e_i|) = |\phi_i\rangle\langle\phi_i|$$

for some orthonormal family $(\phi_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$. One cannot yet say that this orthonormal family is an orthonormal basis, for one does not know that $\mathcal{L}$ preserves the identity operator (which is not even an element of $\mathcal{L}_1(\mathcal{H})$!). With the same reasoning as previously, the operators $\mathcal{L}(|e_i\rangle\langle e_j|)$ are partial isometries mapping $\operatorname{Ran}\mathsf{Q}_j$ onto $\operatorname{Ran}\mathsf{Q}_i$, hence

$$\mathcal{L}(|e_i\rangle\langle e_j|) = |\phi_i\rangle\langle\phi_j|\,.$$

The partial isometries $\mathsf{M}_{kl} = \mathcal{N}(|e_k\rangle\langle e_l|)$ satisfy

$$\begin{aligned}
\mathrm{Tr}\,\big(|e_i\rangle\langle e_j|\,\mathsf{M}_{kl}\big) &= \mathrm{Tr}\,\big(\mathcal{L}(|e_i\rangle\langle e_j|)\,|e_l\rangle\langle e_k|\big)\\
&= \mathrm{Tr}\,\big(|\phi_i\rangle\langle\phi_j|\,|e_l\rangle\langle e_k|\big)\\
&= \langle\phi_j\,,\,e_l\rangle\,\langle e_k\,,\,\phi_i\rangle\ .
\end{aligned}$$

In other words, the operator M_{kl} is given by

$$\begin{aligned}
\mathsf{M}_{kl} &= \sum_{i,j\in\mathbb{N}} \mathrm{Tr}\,\big(|e_i\rangle\langle e_j|\,\mathsf{M}_{kl}\big)\,|e_j\rangle\langle e_i|\\
&= \sum_{i,j\in\mathbb{N}} \langle\phi_j\,,\,e_l\rangle\,\langle e_k\,,\,\phi_i\rangle\ |e_j\rangle\langle e_i|\\
&= |\psi_l\rangle\langle\psi_k|\,,
\end{aligned}$$

where we have put

$$\psi_l = \sum_{j\in\mathbb{N}} \langle\phi_j\,,\,e_l\rangle\,e_j\,.$$

This means that the orthogonal projectors $\mathsf{P}_k = \mathsf{M}_{kk}$ are rank one projectors and one concludes in the same way as for ii). □

6.4.3 Loss of Invertibility

We can now pass to the second announced result. This one puts the emphasis on the fact that if the quantum channel is not a unitary conjugation, there has been a true action of the environment and then the tracing out implies a loss of information which makes the dynamics not reversible anymore.

Theorem 6.38. *Let $\mathcal{L}$ be a unital completely positive map on $\mathcal{B}(\mathcal{H})$. If there exists a completely positive map $\mathcal{S}$ on $\mathcal{B}(\mathcal{H})$ such that*

$$\mathcal{S}\circ\mathcal{L} = \mathrm{I}$$

then $\mathcal{L}$ is a morphism of the unital $*$-algebra $\mathcal{B}(\mathcal{H})$. In particular if $\mathcal{H}$ is finite dimensional then $\mathcal{L}$ is a unitary conjugation.

Proof. Let $\mathsf{A} \in \mathcal{B}(\mathcal{H})$, by the Cauchy-Schwarz inequality for completely positive maps we have

$$\mathsf{A}^*\mathsf{A} = \mathcal{S} \circ \mathcal{L}(\mathsf{A}^*\mathsf{A}) \geq \mathcal{S}(\mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{A})) \geq \mathcal{S} \circ \mathcal{L}(\mathsf{A})^*\, \mathcal{S} \circ \mathcal{L}(\mathsf{A}) = \mathsf{A}^*\mathsf{A}\,.$$

Hence all the inequalities above are equalities.

Let ρ be an invertible density matrix on $\mathcal{H}$ and let $\lambda : \mathsf{A} \mapsto \mathrm{Tr}\,(\rho\,\mathcal{S}(\mathsf{A}))$. Note that if X is a positive operator such that $\lambda(\mathsf{X}) = 0$ then $\mathsf{X} = 0$. We have

$$\lambda(\mathcal{L}(\mathsf{A}^*\mathsf{A}) - \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{A})) = \mathrm{Tr}\,\left(\rho\,\mathcal{S}\big(\mathcal{L}(\mathsf{A}^*\mathsf{A}) - \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{A})\big)\right) = 0\,.$$

With the remark above, this means that

$$\mathcal{L}(\mathsf{A}^*\mathsf{A}) = \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{A})$$

for all $\mathsf{A} \in \mathcal{B}(\mathcal{H})$.

Now we have to prove that this implies $\mathcal{L}(\mathsf{A}^*\mathsf{B}) = \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{B})$ for all $\mathsf{A}, \mathsf{B} \in \mathcal{B}(\mathcal{H})$. Developing $\mathcal{L}((\mathsf{A}+\mathsf{B})^*(\mathsf{A}+\mathsf{B}))$ in two different ways this gives

$$\mathcal{L}(\mathsf{A}^*\mathsf{B}) + \mathcal{L}(\mathsf{B}^*\mathsf{A}) = \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{B}) + \mathcal{L}(\mathsf{B})^*\mathcal{L}(\mathsf{A}),.$$

Replacing B by iB shows that

$$\mathcal{L}(\mathsf{A}^*\mathsf{B}) = \mathcal{L}(\mathsf{A})^*\mathcal{L}(\mathsf{B})\,.$$

We have shown that $\mathcal{L}$ is a morphism of $\mathcal{B}(\mathcal{H})$.

The conclusion on the unitary conjugation, in the finite dimensional case, is just an application of Theorem 6.34. □

In the infinite dimensional case, the property $\mathcal{S} \circ \mathcal{L}$ proves the $*$-morphism property but is not enough for forcing $\mathcal{L}$ to be a unitary conjugation. Indeed, our counter-example of previous subsection satisfies

$$\frac{1}{2}\mathcal{L}_* \circ \mathcal{L} = \mathrm{I}\,,$$

as can be checked easily, but we know that $\mathcal{L}$ is not a unitary conjugation. Once again in the infinite dimensional case one needs more hypothesis.

Theorem 6.39. *Let $\mathcal{L}$ be a unital completely positive map. If there exists a completely positive map $\mathcal{S}$ on $\mathcal{B}(\mathcal{H})$ such that*

$$\mathcal{S} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{S} = \mathrm{I} \tag{6.21}$$

then $\mathcal{L}$ is unitary conjugation.

Proof. This is now very easy. By Theorem 6.38 the completely positive map $\mathcal{L}$ is a morphism of $\mathcal{B}(\mathcal{H})$. But the relations (6.21) show that it is invertible. Hence by Theorem 6.37 it is a unitary conjugation. □

6.5 Markov Chains and Dynamical Systems

This section aims at establishing a close parallel between the situation we have seen in the previous sections (unitary evolution of bipartite quantum systems, partial traces, completely positive maps) and the situation of classical dynamical systems. We shall establish many similar results in the context of classical dynamical systems and Markov chains. We show that Markov chains appear from any dynamical system on a product space, when averaging out one of the two components. This way, Markov chains are interpreted as what remains on one system when it interacts with some environment but we do not have access to that environment. The randomness appears directly from the determinism, solely by the fact that we have lost some information. We show that any Markov chain can be obtained this way. We also show two results which characterize what properties are lost when going from a deterministic dynamical system to a Markov chain: typically the loss of algebra morphism property and the loss of reversibility, exactly in the same way as for completely positive maps!

6.5.1 Basic Definitions

Let us recall some basic definitions concerning dynamical systems and Markov chains.

Definition 6.40. Let $(E, \mathcal{E})$ be a measurable space. Let $\widetilde{T}$ be a measurable function from E to E. We then say that $\widetilde{T}$ is a *dynamical system* on E. Such a mapping $\widetilde{T}$ induces a natural mapping T on $\mathcal{L}^\infty(E)$ defined by

$$\mathsf{T}f(x) = f\big(\widetilde{T}(x)\big)\,.$$

Note that this mapping clearly satisfies the following properties (proof left to the reader).

Proposition 6.41.
i) T *is a morphism of the* $*$*-algebra* $\mathcal{L}^\infty(E)$*,*
ii) $\mathsf{T}(\mathbb{1}_E) = \mathbb{1}_E$*,*
iii) $\|\mathsf{T}\| = 1$*.*

Definition 6.42. What is called *dynamical system* is actually the associated discrete-time semigroup $(\widetilde{T}^n)_{n\in\mathbb{N}}$, when acting on points, or $(\mathsf{T}^n)_{n\in\mathbb{N}}$, when acting on functions.

When the mapping $\widetilde{T}$ is invertible, then so is the associated operator T. In this case, the two semigroups $(\widetilde{T}^n)_{n\in\mathbb{N}}$ and $(\mathsf{T}^n)_{n\in\mathbb{N}}$ can be easily extended into one-parameter groups $(\widetilde{T}^n)_{n\in\mathbb{Z}}$ and $(\mathsf{T}^n)_{n\in\mathbb{Z}}$, respectively.

Definition 6.43. Let us now recall basic definitions concerning Markov chains. Let $(E, \mathcal{E})$ be a measurable space. A mapping ν from $E \times \mathcal{E}$ to $[0,1]$ is a *Markov kernel* if

i) $x \mapsto \nu(x, A)$ is a measurable function, for all $A \in \mathcal{E}$,

ii) $A \mapsto \nu(x, A)$ is a probability measure, for all $x \in E$.

When E is a finite set, then ν is determined by the quantities

$$P(i,j) = \nu(i, \{j\})$$

which form a *stochastic matrix*, i.e. a square matrix with positive entries and sum of each row being equal to 1.

Definition 6.44. A Markov kernel ν acts on $\mathcal{L}^\infty(E)$ as follows:

$$\nu \circ f(x) = \int_E f(y)\, \nu(x, \mathrm{d}y)\,.$$

A linear operator T on $\mathcal{L}^\infty(E, \mathcal{E})$ which is of the form

$$\mathsf{T} f(x) = \int_E f(y)\, \nu(x, \mathrm{d}y)\,,$$

for some Markov kernel ν, is called a *Markov operator*.

In a dual way, a Markov kernel ν acts on probability measures on $(E, \mathcal{E})$. Indeed, if $\mathbb{P}$ is a probability measure on $(E, \mathcal{E})$ then so is the measure $\mathbb{P} \circ \nu$ defined by

$$\mathbb{P} \circ \nu(A) = \int_E \nu(x, A)\, \mathbb{P}(\mathrm{d}x)\,.$$

Definition 6.45. Markov kernels can be composed. If ν_1 and ν_2 are two Markov kernels on $(E, \mathcal{E})$ then so is

$$\nu_1 \circ \nu_2(x, A) = \int_E \nu_2(y, A)\, \nu_1(x, \mathrm{d}y)\,.$$

This kernel represents the Markov kernel resulting from making a first step following ν_1 and then another step following ν_2.

Definition 6.46. A *Markov chain* with *state space* $(E, \mathcal{E})$ is a discrete-time stochastic process $(X_n)_{n \in \mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that each $X_n : \Omega \to E$ is measurable and

$$\mathbb{E}\left[f(X_{n+1}) \,|\, X_0, X_1, \dots, X_n \right] = \mathbb{E}\left[f(X_{n+1}) \,|\, X_n \right]$$

for all bounded function $f : E \to \mathbb{R}$ and all $n \in \mathbb{N}$. In particular, if $\mathcal{F}_n$ denotes the σ-algebra generated by $X_0, X_1, \dots, X_n$, then the above implies

$$\mathbb{E}\left[f(X_{n+1}) \,|\, \mathcal{F}_n \right] = \mathsf{L}_n f(X_n)$$

for some function $\mathsf{L}_n f$. The Markov chain is *homogeneous* if furthermore L_n does not depend on n. We shall be interested only in this case and we denote by L this unique value of L_n:

$$\mathbb{E}\left[f(X_{n+1})\,|\,\mathcal{F}_n\right] = \mathsf{L}f(X_n)\,. \tag{6.22}$$

Applying successive conditional expectations, one gets

$$\mathbb{E}\left[f(X_n)\,|\,\mathcal{F}_0\right] = \mathsf{L}^n f(X_0)\,.$$

If $\nu(x, dy)$ denotes the conditional law of X_{n+1} knowing $X_n = x$, which coincides with the conditional law of X_1 knowing $X_0 = x$, then ν is a Markov kernel and one can easily see that

$$\mathsf{L}f(x) = \int_E f(y)\,\nu(x, \mathrm{d}y) = \nu \circ f(x)\,.$$

Hence L is the Markov operator associated to ν.

With our probabilistic interpretation we get easily that $\nu \circ f(x)$ is the expectation of $f(X_1)$ when $X_0 = x$ almost surely. The measure $\mathbb{P} \circ \nu$ is the distribution of X_1 if the distribution of X_0 is $\mathbb{P}$.

Definition 6.47. We end up this section with the following last definition. A Markov kernel ν is said to be *deterministic* if for all $x \in E$ the measure $\nu(x, \cdot\,)$ is a Dirac mass. This is to say that there exists a measurable mapping $\widetilde{T} : E \to E$ such that

$$\nu(x, \mathrm{d}y) = \delta_{\widetilde{T}(x)}(\mathrm{d}y)\,.$$

In other words, the Markov chain associated to ν is not random at all, it maps with probability 1, each point x to $\widetilde{T}(x)$. In other words it is a dynamical system.

6.5.2 Reduction of Dynamical Systems

Now consider two measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$, together with a dynamical system $\widetilde{T}$ on $E \times F$, equipped with the product σ-field. As above, consider the lifted mapping T acting on $\mathcal{L}^\infty(E \times F)$.

For any bounded measurable function f on E, we consider the bounded (measurable) function $f \otimes 1\!\!1$ on $E \times F$ defined by

$$(f \otimes 1\!\!1)(x, y) = f(x)\,,$$

for all $x \in E$, $y \in F$.

Assume that $(F, \mathcal{F})$ is equipped with a probability measure μ. We shall be interested in the mapping L of $\mathcal{L}^\infty(E)$ defined by

$$\boxed{\mathsf{L}f(x) = \int_F \mathsf{T}(f \otimes \mathbb{1})(x,y)\,\mathrm{d}\mu(y) = \int_F (f \otimes \mathbb{1})\big(\widetilde{T}(x,y)\big)\,\mathrm{d}\mu(y)\,.} \tag{6.23}$$

In other words, we have a deterministic dynamical system on a product space, but we place ourselves from one component point of view only: we have access to E only. Starting from a point $x \in E$ and a function f on E we want to see how they evolve according to T, but seen from the E point of view. The function f on E is naturally lifted into a function $f \otimes \mathbb{1}$ on $E \times F$, that is, it still acts on E only, but it is now part of a "larger world". We make $f \otimes \mathbb{1}$ evolve according to the deterministic dynamical system T. Finally, in order to come back to E we project the result onto E, by taking the average on F according to a fixed measure μ on F. This is to say that, from the set E, what we see of the action of the "environment" F is just an average with respect to some measure μ.

Theorem 6.48. *The mapping L is a Markov operator on E.*

Proof. As $\widetilde{T}$ is a mapping from $E \times F$ to $E \times F$, there exist two measurable mappings:

$$X \ : \ E \times F \longrightarrow E \qquad \text{and} \qquad Y \ : \ E \times F \longrightarrow F\,,$$

such that

$$\widetilde{T}(x,y) = (X(x,y), Y(x,y))$$

for all $(x,y) \in E \times F$.

Let us compute the quantity $\mathsf{L}f(x)$, with these notations. We have

$$\begin{aligned}\mathsf{L}f(x) &= \int_F \mathsf{T}(f \otimes \mathbb{1})(x,y)\,\mathrm{d}\mu(y)\\ &= \int_F (f \otimes \mathbb{1})\big(X(x,y), Y(x,y)\big)\,\mathrm{d}\mu(y)\\ &= \int_F f\big(X(x,y)\big)\,\mathrm{d}\mu(y)\,.\end{aligned}$$

Denote by $\nu(x, dz)$ the probability measure on E, which is the image of μ by the function $X(x, \cdot\,)$ (which goes from F to E, for each fixed x). By a standard result from Measure Theory, the Transfer Theorem, we get

$$\mathsf{L}f(x) = \int_E f(z)\,\nu(x, \mathrm{d}z)\,.$$

Hence L acts on $\mathcal{L}^\infty(E)$ as the Markov operator associated to the transition kernel $\nu(x, \mathrm{d}z)$. □

Remark 6.49. Note the following important fact: *the mapping Y played no role at all in the proof above.*

Remark 6.50. Note that the Markov kernel ν associated to $\widetilde{T}$ restricted to E is given by

$$\boxed{\nu(x,A) = \mu\left(\{y \in F;\ X(x,y) \in A\}\right)\,.} \tag{6.24}$$

In particular, when E is finite (or even countable), the transition kernel ν is associated to a Markovian matrix P whose coefficients are given by

$$P(i,j) = \nu(i,\{j\}) = \mu(\{k; X(i,k) = j\})\,.$$

What we have obtained here is important and deserves more explanation. Mathematically, we have obtained a commuting diagram:

$$\begin{array}{ccc} \mathcal{L}^\infty(E\times F) & \xrightarrow{\mathsf{T}} & \mathcal{L}^\infty(E\times F) \\ \otimes\mathbb{1}\Big\uparrow & & \Big\downarrow\mu \\ \mathcal{L}^\infty(E) & \xrightarrow[\mathsf{L}]{} & \mathcal{L}^\infty(E)\,. \end{array}$$

In more physical language, what we have obtained here can be interpreted in two different ways. If we think of the dynamical system $\widetilde{T}$ first, we have emphasized the fact that losing the information of a deterministic dynamics on one of the components creates a random behavior on the other component. The randomness here appears only as a lack of knowledge of deterministic behavior on a larger world. A part of the universe interacting with our system E is inaccessible to us (or at least we see a very small part of it: an average) which results in random behavior on E.

In the converse direction, that is, seen from the Markov kernel point of view, what we have obtained is a *dilation* of a Markov transition kernel into a dynamical system. Consider the kernel L on the state space E. It does not represent the dynamics of a closed system, it is not a dynamical system. In order to see L as coming from a true dynamical system, we have enlarged the state space E with an additional state space F, which represents the environment. The dynamical system $\widetilde{T}$ represents the true dynamics of the closed system "E+environment". Equation (6.23) says exactly that the effective pseudo-dynamics L that we have observed on E is simply due to the fact that we are looking only at a subpart of a true dynamical system and an average of the F part of the dynamics.

These observations would be even more interesting if one could prove the converse: *every Markov transition kernel can be obtained this way.* This is what we prove now, with only a very small restriction on E.

Definition 6.51. A *Lusin space* is a measurable space which is homeomorphic (as a measurable space) to a Borel subset of a compact metrisable space. This condition is satisfied for example by all the spaces $\mathbb{R}^n$.

Theorem 6.52. *Let $(E, \mathcal{E})$ be a Lusin space and ν a Markov kernel on E. Then there exists a measurable space $(F, \mathcal{F})$, a probability measure μ on $(F, \mathcal{F})$ and a dynamical system $\widetilde{T}$ on $E \times F$ such that the Markov operator L associated to the restriction of $\widetilde{T}$ to E is the one associated to ν.*

Proof. Let $\nu(x, dz)$ be a Markov kernel on $(E, \mathcal{E})$. Let F be the set of functions from E to E. For every finite subset $\sigma = \{x_1, \ldots, x_n\} \subset E$ and every $A_1, \ldots, A_n \in \mathcal{E}$ consider the set

$$F(x_1, \ldots, x_n\,;\, A_1, \ldots, A_n) = \{y \in F\,;\; y(x_1) \in A_1, \ldots, y(x_n) \in A_n\}\;.$$

By the Kolmogorov Consistency Theorem (which applies for E is a Lusin space!) there exists a unique probability measure μ on F such that

$$\mu\left(F(x_1, \ldots, x_n; A_1, \ldots, A_n)\right) = \prod_{i=1}^{n} \nu(x_i, A_i)\,.$$

Indeed, it is easy to check that the above formula defines a consistent family of probability measures on the finitely-based cylinders of F, then apply Kolmogorov's Theorem.

Now define the dynamical system

$$\begin{aligned} \widetilde{T}\;:\; E \times F &\longrightarrow \quad E \times F \\ (x, y) &\longmapsto (y(x), y)\,. \end{aligned}$$

With the same notations as in the proof of Theorem 6.48, we have $X(x, y) = y(x)$ in this particular case and hence

$$\mu(\{y \in F\,;\; X(x, y) \in A\}) = \mu(\{y \in F\,;\; y(x) \in A\}) = \nu(x, A)\,.$$

This proves our claim by (6.24). □

Note that in this dilation of L, the dynamical system T has no reason to be invertible in general. It is worth noticing that one can always construct a dilation where T is invertible.

Proposition 6.53. *Every Markov kernel ν on a Lusin space admits a dilation $\widetilde{T}$ which is an invertible dynamical system.*

Proof. Consider the construction and notations of Theorem 6.52. Consider the space $F' = E \times F$. Let x_0 be a fixed element of E and define the mapping $\widetilde{T}'$ on $E \times F'$ by

$$\begin{cases} \widetilde{T}'(x, (x_0, y)) = (y(x), (x, y)), & \\ \widetilde{T}'(x, (y(x), y)) = (x_0, (x, y)), & \\ \widetilde{T}'(x, (z, y)) = (z, (x, y)), & \text{if } z \neq x_0 \text{ and } z \neq y(x)\,. \end{cases}$$

It is easy to check that $\widetilde{T}'$ is a bijection of $E\times F'$. Now extend the measure μ on F to the measure $\delta_{x_0}\otimes\mu$ on F'. Then the dynamical system $\widetilde{T}'$ is invertible and dilates the same Markov kernel as $\widetilde{T}$. $\square$

6.5.3 Iterating the Dynamical System

We have shown that every dynamical system on a product set gives rise to a Markov kernel when restricted to one of the sets. We have seen that every Markov kernel can be obtained this way. But one has to notice that our construction allows the dynamical system T to dilate the Markov kernel L as a single mapping only. This is to say that iterations of the dynamical system T^n do not in general dilate the semigroup L^n associated to the Markov process. Let us check this with a simple counter-example.

Put $E = F = \{1,2\}$. On F define the probability measure $\mu(1) = 1/4$ and $\mu(2) = 3/4$. Define the dynamical system $\widetilde{T}$ on $E\times F$ which is the "anticlockwise rotation":

$$\widetilde{T}(1,1)=(2,1),\quad \widetilde{T}(2,1)=(2,2),\quad \widetilde{T}(2,2)=(1,2),\quad \widetilde{T}(1,2)=(1,1)\,.$$

With the same notations as above, the associated map X is thus given by

$$X(1,1)=2,\quad X(2,1)=2,\quad X(2,2)=1,\quad X(1,2)=1\,.$$

Hence, we get

$$\mu(X(1,\cdot\,)=1)=\frac{3}{4}\,,\qquad \mu(X(1,\cdot\,)=2)=\frac{1}{4}\,,$$
$$\mu(X(2,\cdot\,)=1)=\frac{3}{4}\,,\qquad \mu(X(2,\cdot\,)=2)=\frac{1}{4}\,.$$

The Markovian matrix associated to the restriction of $\widetilde{T}$ to E is

$$L=\begin{pmatrix}3/4 & 1/4\\ 3/4 & 1/4\end{pmatrix}\,.$$

In particular

$$L^2=L\,.$$

Let us compute $\widetilde{T}^2$. We get

$$\widetilde{T}^2(1,1)=(2,2),\quad \widetilde{T}^2(2,1)=(1,2),\quad \widetilde{T}^2(2,2)=(1,1),\quad \widetilde{T}^2(1,2)=(2,1)\,.$$

Hence the associated X-mapping, which we shall denote by X_2, is given by

$$X_2(1,1)=2,\quad X_2(2,1)=1,\quad X_2(2,2)=1,\quad X_2(1,2)=2\,.$$

This gives the Markovian matrix

$$L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

which is clearly not equal to L^2.

It would be very interesting if one could find a dilation of the Markov kernel L by a dynamical system T such that any power T^n would also dilate L^n. We would have realized the whole Markov chain as the restriction of iterations of a single dynamical system on a larger space.

This can be performed in the following way (note that this is not the only way, nor the more economical). Let L be a Markov operator on a Lusin space E with kernel ν and let T be a dynamical system on $E \times F$ which dilates L. Consider the set $\widehat{F} = F^{\mathbb{N}^*}$ equipped with the usual cylinder σ-field $\mathcal{F}^{\otimes \mathbb{N}^*}$ and the product measure $\widehat{\mu} = \mu^{\otimes \mathbb{N}^*}$. The elements of $\widehat{F}$ are thus sequences $(y_n)_{n \in \mathbb{N}^*}$ in F. Put

$$\begin{aligned} \widetilde{S} : E \times \widehat{F} &\longrightarrow \quad E \times \widehat{F} \\ (x, y) &\longmapsto (X(x, y_1), \Theta(y)) \end{aligned}$$

where X is as in the the proof of Theorem 6.48 and Θ is the usual *shift* on $\widehat{F}$: $\Theta(y) = (y_{n+1})_{n \in \mathbb{N}^*}$.

Then $\widetilde{S}$ can be lifted into a morphism S of $\mathcal{L}^\infty(E \times \widehat{F})$, as previously. Furthermore, any function f in $\mathcal{L}^\infty(E)$ can be lifted into $f \otimes \mathbb{1}$ on $\mathcal{L}^\infty(E \times \widehat{F})$, with $(f \otimes \mathbb{1})(x, y) = f(x)$.

Theorem 6.54. *For all $n \in \mathbb{N}^*$, all $x \in E$ and all $f \in \mathcal{L}^\infty(E)$ we have*

$$\int_{\widehat{F}} \mathsf{S}^n (f \otimes \mathbb{1})(x, y) \, \mathrm{d}\widehat{\mu}(y) = (\mathsf{L}^n f)(x) \, .$$

Proof. Recall that we noticed in the proof of Theorem 6.48, that the mapping Y associated to $\widetilde{T}$ played no role in the proof of this theorem, only the mapping X was of importance. In particular this implies that Theorem 6.54 is true for $n = 1$, for the dynamical systems $\widetilde{T}$ and $\widetilde{S}$ share the same X-mapping.

By induction, let us assume that the relation

$$\int_{\widehat{F}} \mathsf{S}^k (f \otimes \mathbb{1})(x, y) \, \mathrm{d}\widehat{\mu}(y) = (\mathsf{L}^k f)(x)$$

holds true for all $f \in \mathcal{L}^\infty(E)$, all $x \in E$ and all $k \leq n$. Set $\widehat{F}_{[2}$ to be the set of sequences $(y_n)_{n \geq 2}$ with values in $\widehat{F}$ and $\widehat{\mu}_{[2}$ the restriction of $\widehat{\mu}$ to $\widehat{F}_{[2}$. We have

$$\begin{aligned}
&\int_{\widehat{F}} \mathsf{S}^{n+1}(f\otimes \mathbb{1})(x,y)\,\mathrm{d}\widehat{\mu}(y) = \\
&\qquad = \int_{\widehat{F}} \mathsf{S}^n(f\otimes \mathbb{1})\left(X(x,y_1),\Theta(y)\right)\,\mathrm{d}\widehat{\mu}(y)\\
&\qquad = \int_F\int_{\widehat{F}_{[2}} S^n(f\otimes \mathbb{1})\left(X(x,y_1),y\right)\,\mathrm{d}\widehat{\mu}_{[2}(y)\,\mathrm{d}\mu(y_1)\,.
\end{aligned}$$

Put $\widetilde{x} = X(x,y_1)$, the above is equal to

$$\begin{aligned}
&\int_F\int_{\widehat{F}_{[2}} \mathsf{S}^n(f\otimes \mathbb{1})\left(\widetilde{x},y\right)\,\mathrm{d}\widehat{\mu}_{[2}(y)\,\mathrm{d}\mu(y_1)\\
&= \int_F \mathsf{L}^n(f)(\widetilde{x})\,\mathrm{d}\mu(y_1) \qquad \text{(by induction hypothesis)}\\
&= \int_F \mathsf{L}^n(f)(X(x,y_1))\,\mathrm{d}\mu(y_1)\\
&= \mathsf{L}^{n+1}(f)(x)\,.
\end{aligned}$$

This proves the announced relation by induction. □

With this theorem and with Theorem 6.52, we see that every Markov chain on E can be realized as the restriction on E of the iterations of a deterministic dynamical system $\widetilde{T}$ acting on a larger set.

The physical interpretation of the construction above is very interesting. It represents a scheme of "repeated interactions". That is, we know that the result of the deterministic dynamics associated to $\widetilde{T}$ on $E\times F$ gives rises to the Markov operator L on E. The idea of the construction above is that the environment is now made of a chain of copies of F, each of which is going to interact, one after the other, with E. After, the first interaction between E and the first copy of F has happened, following the dynamical system $\widetilde{T}$, the first copy of F stops interacting with E and is replaced by the second copy of F. This copy now interacts with E following $\widetilde{T}$. And so on, we repeat these interactions. The space E keeps the memory of the different interactions, while each copy of F arrives independently in front of E and induces one more step of evolution following $\widetilde{T}$.

As a result of this procedure, successive evolutions restricted to E correspond to iterations of the Markov operator L. This gives rise to behavior as claimed: an entire path of the homogeneous Markov chain with generator L.

6.5.4 Defect of Determinism and Loss of Invertibility

We end up this section with some algebraic characterizations of determinism for Markov chains. The point is to characterize what exactly is lost when going from the deterministic dynamics T on $E\times F$ to the Markov operator L on

E. Of course, the reader will notice the exact parallel with the corresponding theorems for completely positive maps.

Theorem 6.55. *Let $(E, \mathcal{E})$ be a Lusin space. Let (X_n) be a Markov chain with state space $(E, \mathcal{E})$ and with transition kernel ν. Let L be the Markov operator on $\mathcal{L}^\infty(E)$ associated to ν:*

$$\mathsf{L}f(x) = \int_E f(y)\, \nu(x, \mathrm{d}y)\,.$$

Then the Markov chain (X_n) is deterministic if and only if L is a morphism of the unital $*$-algebra $\mathcal{L}^\infty(E)$.

Proof. If the Markov chain is deterministic, then L is associated to a dynamical system and hence it is a morphism of $\mathcal{L}_\infty(\mathcal{H})$ (Proposition 6.41).

Conversely, suppose that L is a morphism of $\mathcal{L}^\infty(\mathcal{H})$. We shall first consider the case where $(E, \mathcal{E})$ is a Borel subset of a compact metric space.

Take any $A \in \mathcal{E}$, any $x \in E$ and recall that we always have

$$\nu(x, A) = \mathsf{L}(\mathbb{1}_A)(x)\,.$$

The morphism property gives

$$\mathsf{L}(\mathbb{1}_A)(x) = \mathsf{L}(\mathbb{1}_A^2)(x) = \mathsf{L}(\mathbb{1}_A)^2(x) = \nu(x, A)^2\,.$$

Hence $\nu(x, A)$ satisfies $\nu(x, A)^2 = \nu(x, A)$. This means that $\nu(x, A)$ is equal to 0 or 1, for all $x \in E$ and all $A \in \mathcal{E}$.

Consider a covering of E with a countable family of balls $(B_i)_{i\in\mathbb{N}}$, each of which with diameter smaller than 2^{-n} (this is always possible as E is separable). From this covering one can easily extract a partition $(S_i)_{i\in\mathbb{N}}$ of E by measurable sets, each of which with diameter smaller than 2^{-n}. We shall denote by $\mathcal{S}^n$ this partition.

Let $x \in E$ be fixed. As we have $\sum_{E\in\mathcal{S}^n} \nu(x, E) = 1$ we must have $\nu(x, E) = 1$ for one and only one $E \in \mathcal{S}^n$. Let us denote by $E^{(n)}(x)$ this unique set. Clearly, the sequence $\left(E^{(n)}(x)\right)_{n\in\mathbb{N}}$ is decreasing (for otherwise there will be more than one set $E \in \mathcal{S}^n$ such that $\nu(x, E) = 1$). Let $A = \cap_n E^{(n)}(x)$. The set A satisfies $\nu(x, A) = 1$, hence A is non-empty. But also, the diameter of A has to be 0, for it is smaller than 2^{-n} for all n. As a consequence A has to be a singleton $\{y(x)\}$, for some $y(x) \in E$. Hence we have proved that for each $x \in E$ there exists a $y(x) \in E$ such that $n(x, \{y(x)\}) = 1$. This proves the deterministic character of our chain.

The case where E is only homeomorphic to a Borel subset E' of a compact metric space is obtained by using the homeomorphism to transfer suitable partitions $\mathcal{S}^n$ of E' to E. □

Another strong result on determinism of Markov chains is the way it is related to non-invertibility.

Theorem 6.56. *Let $(E, \mathcal{E})$ be a Lusin space. Let L be a Markov operator on $\mathcal{L}^\infty(E)$ associated to a Markov chain (X_n). If L is invertible in the category of Markov operators then (X_n) is deterministic.*

Proof. Recall that a Markov operator L maps positive functions to positive functions. Hence, in the same way as one proves Cauchy-Schwarz inequality, we always have

$$\overline{\mathsf{L}(f)} = \mathsf{L}(\overline{f})$$

and

$$\mathsf{L}(|f|^2) \geq \mathsf{L}(\bar{f})\mathsf{L}(f)$$

(hint: write the positivity of $\mathsf{L}(\overline{(f+\lambda g)}(f+\lambda g))$ for all $\lambda \in \mathbb{C}$).

Let M be a Markov operator such that $\mathsf{M}\,\mathsf{L} = \mathsf{L}\,\mathsf{M} = \mathrm{I}$. We have

$$|f|^2 = \bar{f}f = \mathsf{M} \circ \mathsf{L}(\bar{f}f) \geq \mathsf{M}(\mathsf{L}(\bar{f})\mathsf{L}(f)) \geq \mathsf{M} \circ \mathsf{L}(\bar{f})\, \mathsf{M} \circ \mathsf{L}(f) = \bar{f}f = |f|^2\ .$$

Hence we have equalities everywhere above. In particular

$$\mathsf{M} \circ \mathsf{L}(\bar{f}f) = \mathsf{M}(\mathsf{L}(\bar{f})\mathsf{L}(f))\,.$$

Applying L to this equality, gives

$$\mathsf{L}(\bar{f}f) = \mathsf{L}(\bar{f})\mathsf{L}(f)\,,$$

for all $f \in \mathcal{L}^\infty(E)$.

By polarization it is easy to prove now that L is a morphism. By Theorem 6.55 it is the Markov operator associated to a deterministic chain. □

The result above is more intuitive than the one of Theorem 6.55, from the point of view of open systems. If the dynamical system $\widetilde{T}$ on the large space $E \times F$ is invertible, this invertibility is always lost when projecting on E. The fact we do not have access to one component of the coupled system makes that we lose all chance of invertibility.

Notes

Writing this course has been quite a difficult task for me. The literature is very important in the finite dimensional case, mainly in connection with Quantum Information Theory. The infinite dimensional case is not treated in general outside a general C^*-algebra setup. My choice to present everything in the context of $\mathcal{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$ has made it necessary for me to find original proofs and paths for this lecture. In particular I have obtained many important informations and pieces of proofs from discussions with colleagues. I think of and I thank in particular: Ivan Bardet, Gilles

Cassier, Franco Fagnola, Hans Maassen, Petru Mironescu and Mikael de la Salle.

Nevertheless, I have also been inspired by many references. For the finite dimensional context, with applications to Quantum Information Theory, the nice lecture notes by J. Preskil [Pre04]. From his course I took the construction of the Krauss representation of a quantum channel and the three examples. For the C^*-algebraic framework, the course by R. Rebolledo [Reb06], from which I took (and adapted) the proof of Stinespring Theorem and Krauss representation for completely positive maps. For the context $\mathcal{B}(\mathcal{H})$ with general $\mathcal{H}$, one of the only reference I have found, and which inspired me for the proofs of minimality and ambiguity of the Krauss representation, is K.R. Parthasarathy's book [Par92]. The last section on Markov chains and dynamical is directly taken from an article by S. Attal [Att10].

References

[Att10] Stéphane Attal. Markov chains and dynamical systems: the open system point of view. *Commun. Stoch. Anal.*, 4(4):523–540, 2010.

[Cas05] Gilles Cassier. Generalized Toeplitz operators, restrictions to invariant subspaces and similarity problems. *J. Operator Theory*, 53(1):49–89, 2005.

[Par92] K. R. Parthasarathy. *An introduction to quantum stochastic calculus*, volume 85 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[Pre04] John Preskil. *A Course on Quantum Computation.* Course on internet: http://www.theory.caltech.edu/people/preskill/ph229/#lecture. Caltech Institute, 2004.

[Reb06] Rolando Rebolledo. Complete positivity and the Markov structure of open quantum systems. In *Open quantum systems. II*, volume 1881 of *Lecture Notes in Math.*, pages 149–182. Springer, Berlin, 2006.