# THE VARIATIONAL BICOMPLEX

BY

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†preliminary draft, not complete.

‡In preparation

# PREFACE

The variational bicomplex is a double complex of differential forms defined on the infinite jet bundle of any fibered manifold $\pi\colon E \to M$. This double complex of forms is called the variational bicomplex because one of its differentials (or, more precisely, one of the induced differentials in the first term of the first spectral sequence) coincides with the classical Euler-Lagrange operator, or variational derivative, for arbitrary order, multiple integral problems in the calculus of variations. Thus, the most immediate application of the variational bicomplex is that of providing a simple, natural, and yet general, differential geometric development of the variational calculus. Indeed, the subject originated within the last fifteen years in the independent efforts of W. M. Tulczyjew and A. M. Vinogradov to resolve the Euler-Lagrange operator and thereby characterize the kernel and the image of the the variational derivative. But the utility of this bicomplex extents well beyond the domain of the calculus of variations. Indeed, it may well be that the more important aspects of our subject are those aspects which pertain either to the general theory of conservation laws for differential equations, as introduced by Vinogradov, or to the theory of characteristic (and secondary characteristic) classes and Gelfand-Fuks cohomology, as suggested by T. Tsujishita. All of these topics are part of what I. M. Gelfand, in his 1970 address to the International Congress in Nice, called formal differential geometry. The variational bicomplex plays the same ubiquitous role in formal differential geometry, that is, in the geometry of the infinite jet bundle for the triple $(E, M, \pi)$ that the de Rham complex plays in the geometry of a single manifold $M$.

The purpose of this book is to develop the basic general theory of the variational bicomplex and to present a variety of applications of this theory in the areas of differential geometry and topology, differential equations, and mathematical physics.

This book is divided, although not explicitly, into four parts. In part one, which consists of Chapters One, Two, and Three, the differential calculus of the variational bicomplex is presented. Besides the usual operations involving vector fields and forms on manifolds, there are two additional operations upon which much of the general theory rests. The first of these is the operation of prolongation which lifts generalized vector fields on the total space $E$ to vector fields on the infinite jet bundle $J^\infty(E)$. The second operation is essentially an invariant "integration by parts" operation which formalizes and extends the familiar process of forming

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the formal adjoint of a linear differential operator. Chapters Four and Five form the second part of the book. Here, the local and global cohomological properties of the variational bicomplex are studied in some detail. While the issues, methods and applications found in these two chapters differ considerable from one another, they both pertain to what may best be called the theory of the "free" variational bicomplex — no restrictions are imposed on the domain of definition of the differential forms in the variational bicomplex nor do we restrict our attention to sub-bicomplexes of invariant forms. In short, part two of this book does for the variational bicomplex what, by analogy, the Poincaré Lemma does for the de Rham complex. In Chapter Six, the third part of the book, we let $G$ be a group of fiber preserving transformations on $E$. The action of $G$ on $E$ lifts to a group action $\mathcal{G}$ on $J^\infty(E)$. Because $\mathcal{G}$ respects the structure of the variational bicomplex, we can address the problem of computing its $\mathcal{G}$ equivariant cohomology. The Gelfand-Fuks cohomology of formal vector fields is computed anew from this viewpoint. We also show how characteristic and secondary characteristic classes arise as equivariant cohomology classes on the variational bicomplex for the bundle of Riemannian structures. In the final part, Chapter Seven, we look at systems of differential equations $\mathcal{R}$ on $E$ as subbundles $\mathcal{R}^\infty$ of $J^\infty(E)$ and investigate the cohomology of the variational bicomplex restricted to $\mathcal{R}^\infty$. Cohomology classes now represent various deformation invariants of the given system of equations — first integrals and conservation laws, integral invariants, and variational principles. Vinogradov's Two Line Theorem is extended to give a sharp lower bound for the dimension of the first nonzero cohomology groups in the variational bicomplex for $\mathcal{R}$. A new perspective is given to J. Douglas' solution to the inverse problem to the calculus of variations for ordinary differential equations.

A major emphasis throughout the entire book is placed on specific examples, problems and applications. These are test cases against which the usefulness of this machinery can, at least for now, be judged. Through different choices of the bundle $E$, the group $\mathcal{G}$ and the differential relations $\mathcal{R}$, these examples also illustrate how the variational bicomplex can be adapted to model diverse phenomena in differential geometry and topology, differential equations, and mathematical physics. They suggest possible avenues for future research.

The general prequisites for this book include the usual topics from the calculus on manifolds and a modest familiarity with the classical variational calculus and its role in mechanics, classical field theory, and differential geometry. In the early chapters, some acquaintance with symmetry group methods in differential equations would surely be helpful. Indeed, we shall find ourselves referring often to *Applica-*

*tions of Lie Groups to Differential Equations*, P. Olver's fine text on this subject. In Chapter Five, the global properties of the variational bicomplex are developed using the generalized Mayer-Vietoris argument, as explained in the wonderful book, *Differential Forms in Algebraic Topology*, by R. Bott and L. Tu. The classical invariant theory of the general linear and orthogonal groups play a central role in our calculations of equivariant cohomology in Chapter Six.

Some of the material presented here represents new, previously unpublished research by the author. This includes the results in §3B on cochain maps between bicomplexes, the entire theory of minimal weight forms developed in §4B, the analysis of locally variational operators in §4C, the existence of global homotopy operators in §5D, and the calculation of the equivariant cohomology of the variational bicomplex on Riemannian structures in Chapter Six. New proofs of some of the basic properties of the variational bicomplex are given in §4A, §5A, and §5B and §7B. Also, many of the specific examples and applications of the variational bicomplex are presented here for the first time.

ACKNOWLEDGMENTS

It has taken a long time to research and to write this book. It is now my pleasure to thank those who helped to initiate this project and to bring this work to completion. Its genesis occurred, back in the fall of 1983, with a seemingly innocuous suggestion by Peter Olver concerning the use of the higher Euler operators to simplify the derivation of the horizontal homotopy operators for the variational bicomplex. Over time this idea developed into the material for sections §2A, §2B and §5D. The general outline of the book slowly emerged from series of lectures given at the University of Utah in the fall of 1984, at the University of Waterloo in January, 1987 and during the Special Year in Differential Geometry at the University of North Carolina, Chapel Hill, 1987–88. My thanks to Hugo Rossi, Niky Kamran, and Robbie Gardner for providing me with these opportunities.

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## INTRODUCTION

Although the formal introduction of our subject can be attributed to to the work of Tulczyjew [*70*], Vinogradov [*74*], [*75*], and Tsujishita [*68*], it may nevertheless be argued that the origins of the variational bicomplex are to be found in an article, written just over a hundred years ago, by the mathematical physicist H. Helmholtz [*32*]. Helmholtz sought new applications of the powerful Hamiltonian-Jacobi method for the integration of the equations of mechanics and so, to this end, he formulated the problem of characterizing those systems of differential equations which are derivable from a variational principle. This is the inverse problem to the calculus of variations. While Helmholtz restricted his attention to the inverse problem for second order ordinary differential equations, others soon thereafter ([*33*], [*11*], [*40*]) treated the inverse problem for higher order systems of partial differential equations. By the turn of the century the following facts were known.

(i) The Lagrangian $L$ for the given system of equations is not unique. In fact, since the Euler-Lagrange operator $E$ annihilates divergences of vector fields (*e.g.*, total time derivatives in the case of mechanics) another Lagrangian, with identical Euler-Lagrange expressions, can be obtained by adding a divergence to the original Lagrangian $L$.

(ii) There are certain necessary local integrability conditions, henceforth called the *Helmholtz conditions* $H$, which a system of equations must satisfy in order to be derivable from a variational principle. For example, in the case of linear differential equations, the Helmholtz conditions coincide with those conditions for formal self-adjointness. For special classes of equations, it was shown that the Helmholtz conditions are sufficient for the local existence of a Lagrangian.

These two facts can be summarized symbolically by the following naive sequence of spaces and maps:

$$\{\text{Vector Fields}\} \xrightarrow{\text{Div}} \{\text{Lagrangians}\} \xrightarrow[\text{Lagrange}]{\text{Euler}} \{\text{Diff. Equations}\} \xrightarrow[\text{holtz}]{\text{Helm-}} \{\text{Diff. Operators}\}. \tag{0.1}$$

In the parlance of homological algebra, this sequence is a cochain complex in that the composition of successive maps yields zero, *i.e.*,

$$E(\text{Div} X) = 0 \quad \text{and} \quad H(E(L)) = 0.$$

The variational bicomplex is the full extension and proper differential geometric realization of the complex (0.1).

The first step towards a complete definition of the variational bicomplex is a description of the mathematical data from which it is constructed. This data varies in accordance with the specific application at hand; however, for most situations the following is prescribed:

(i) a fibered manifold $\pi\colon E \to M$;

(ii) a transformation group $G$ on $E$; and

(iii) A set $\mathcal{R}$ of differential relations on the local sections of $E$.

The purpose of this introductory survey is to illustrate how this data arises in some familiar contexts and to present some specific examples of cohomology classes in the variational bicomplex. But, in order to do justice to these examples, it is necessary to be to briefly introduce some notation and definitions and to be more precise with regards to the sequence (0.1). A detailed description of our basic notation and definitions is given in Chapter One.

Given the fibered manifold $\pi\colon E \to M$, we construct the infinite jet bundle

$$\pi^\infty_M\colon J^\infty(E) \to M$$

of jets of local sections of $M$. If $x \in M$, then the fiber $(\pi^\infty_M)^{-1}(x)$ in $J^\infty(E)$ consists of equivalence classes, denoted by $j^\infty(s)(x)$, of local sections $s$ on $E$. If $V_1$ and $V_2$ are two neighborhoods of $x$ in $M$ and if

$$s_1\colon V_1 \to E \quad \text{and} \quad s_2\colon V_2 \to E$$

are local sections, then $s_1$ and $s_2$ are equivalent local sections if their partial derivatives to all orders agree at $x$. If the dimension of $M$ is $n$ and that of $E$ is $m+n$, then on $E$ we can use local coordinates

$$\pi\colon (x^i, u^\alpha) \to (x^i),$$

where $i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$. The induced coordinates on $J^\infty(E)$ are

$$(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots),$$

where

$$u^\alpha_i\bigl(j^\infty(s)(x)\bigr) = \frac{\partial s^\alpha}{\partial x^i}(x), \qquad u^\alpha_{ij}\bigl(j^\infty(s)(x)\bigr) = \frac{\partial^2 s^\alpha}{\partial x^i \partial x^j}(x),$$

and so on.

The variational bicomplex is a double complex of differential forms on $J^\infty(E)$ and while it is not difficult to define (see Chapter 1) it suffices for the purposes of this introduction to focus on that edge of the variational bicomplex consisting of *horizontal forms*. A horizontal $p$ form $\omega$ is a differential $p$ form on $J^\infty(E)$ which, in any system of local coordinates, is of the form

$$\omega = A_{j_1 j_2 \cdots j_p}\, dx^{j_1} \wedge dx^{j_2} \wedge \dots dx^{j_p},$$

where the coefficients

$$A_{\cdots} = A_{\cdots}(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)$$

are smooth functions on $J^\infty(U)$. In particular, a horizontal $n$ form

$$\lambda = L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\, \nu,$$

where $\nu = dx^1 \wedge dx^2 \cdots \wedge dx^n$, is a Lagrangian for a variational problem on $E$. The associated fundamental integral is defined on all sections $s\colon M \to E$ (assuming for the moment that such sections exist and, in addition, that $M$ is compact) by

$$I\,[s] = \int_M \bigl(j^\infty(s)(x)\bigr)^* \lambda.$$

We denote the space of all horizontal $p$ forms on $J^\infty(E)$ by $\mathcal{E}^p(J^\infty(E))$. For $p < n$, there is a differential

$$d_H\colon \mathcal{E}^p(J^\infty(E)) \to \mathcal{E}^{p+1}(J^\infty(E))$$

called the *horizontal exterior derivative* or total exterior derivative. On functions $d_H$ coincides with the familiar process of total differentiation; if

$$f = f(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)$$

then

$$d_H f = [\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial u^\alpha} u^\alpha_j + \frac{\partial f}{\partial u^\alpha_i} u^\alpha_{ij} + \cdots]\, dx^j.$$

We also introduce the space $\mathcal{E}^{n+1}(J^\infty(E))$ of *source forms* on $J^\infty(E)$. A source form $\Delta$ is an $n+1$ form on $J^\infty(E)$ which, in any local system of coordinates, is of the form

$$\Delta = P_\beta(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\, du^\beta \wedge \nu. \tag{0.2}$$

The Euler-Lagrange operator $E$ can now be defined as an $\mathbf{R}$ linear map

$$E\colon \mathcal{E}^n(J^\infty(E)) \to \mathcal{E}^{n+1}(J^\infty(E)).$$

If $\lambda = L(x^i, u^\alpha, u^\alpha_i, \dots)\,\nu$ is a Lagrangian on $J^\infty(E)$, then

$$E(\lambda) = E_\beta(L)\,du^\beta \wedge \nu,$$

where

$$E_\beta(L) = \frac{\delta L}{\delta u^\beta} = \frac{\partial L}{\partial u^\beta} - \frac{d}{dx^i}\Big(\frac{\partial L}{\partial u^\beta_i}\Big) + \frac{d^2}{dx^i dx^j}\Big(\frac{\partial L}{\partial u^\beta_{ij}}\Big) - \cdots .$$

The spaces $\mathcal{E}^p(J^\infty(E))$, for $p \le n+1$, and the maps $d_H$ and $E$ form a cochain complex

$$0 \longrightarrow \mathcal{E}^0 \xrightarrow{d_H} \mathcal{E}^1 \cdots \xrightarrow{d_H} \mathcal{E}^{n-1} \xrightarrow{d_H} \mathcal{E}^n \xrightarrow{E} \mathcal{E}^{n+1} \longrightarrow \cdots .$$

That $E \circ d_H = 0$ is simply a restatement of the aforementioned fact the the Euler-Lagrange operator annihilates divergences. This complex is called the *Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$ on the infinite jet bundle of the fibered manifold $\pi\colon E \to M$.* The Euler-Lagrange complex continues indefinitely. The next differential in the Euler-Lagrange complex,

$$H\colon \mathcal{E}^{n+1}(J^\infty(E)) \to \mathcal{E}^{n+2}(J^\infty(E)),$$

is the Helmholtz operator from the inverse problem to the calculus of variations. Thus the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$ is the sought after formalization of the informal sequence (0.1). The space $\mathcal{E}^{n+2}(J^\infty(E))$ is a subspace of the space of all $n+2$ forms on $J^\infty(E)$ but it is *not*, as one might anticipate in analogy with (0.2), the subspace of forms of the type

$$\eta = P_{\alpha\beta}\,du^\alpha \wedge du^\alpha \wedge \nu.$$

To properly define $\mathcal{E}^{n+p}(J^\infty(E))$, for $p \ge 2$, and to define the map $H$ and the subsequent differentials in the Euler-Lagrange complex, we first need the full definition of the variational bicomplex. These definitions are given in Chapters One and Two.

Lagrangians $\lambda \in \mathcal{E}^n(J^\infty(E))$ which lie in the kernel of the Euler-Lagrange operator, that is, which have identically vanishing Euler-Lagrange form

$$E(\lambda) = 0$$

are called *null* or *variationally trivial* Lagrangians. Every total divergence or $d_H$ exact Lagrangian

$$\lambda = d_H \eta$$

is variationally trivial but the converse is not always true. The $n^{\text{th}}$ cohomology group

$$\begin{aligned} H^n(\mathcal{E}^*(J^\infty(E))) &= \frac{\ker\{E\colon \mathcal{E}^n \to \mathcal{E}^{n+1}\}}{\operatorname{im}\{d_H\colon \mathcal{E}^{n-1} \to \mathcal{E}^n\}} \\ &= \frac{\{\text{variationally trivial Lagrangians } \lambda\ \}}{\{\text{ exact Lagrangians } \lambda = d_H\eta\}} \end{aligned}$$

of the Euler-Lagrange complex will, in general, be non-zero.

We call source forms $\Delta \in \mathcal{E}^{n+1}$ which satisfy the Helmholtz conditions $H(\Delta) = 0$ *locally variational source forms*. All Euler-Lagrange forms are locally variational but again the cohomology group

$$\begin{aligned} H^{n+1}(\mathcal{E}^*(J^\infty(E))) &= \frac{\ker\{H\colon \mathcal{E}^{n+1} \to \mathcal{E}^{n+2}\}}{\operatorname{im}\{E\colon \mathcal{E}^n \to \mathcal{E}^{n+1}\}} \\ &= \frac{\{\text{locally variational source forms}\}}{\{\text{Euler-Lagrange forms}\}} \end{aligned}$$

will in general be non-trivial. In Chapter Five, we shall prove that the cohomology of the Euler-Lagrange complex $\mathcal{E}^*$ is isomorphic to the de Rham cohomology of $E$. In particular, a locally variational source form $\Delta$ is always the Euler-Lagrange form of a Lagrangian $\lambda$, $\Delta = E(\lambda)$, whenever $H^{n+1}(E) = 0$.

In summary, from the first item on the above list of data, namely the fibered manifold $\pi\colon E \to M$, we can construct in a canonical fashion the infinite jet bundle $J^\infty(E)$, the variational bicomplex of differential forms on $J^\infty(E)$ and its edge complex, the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$. At this point, the cohomology groups of the variational bicomplex and the Euler-Lagrange complex are well understood. The remaining data, namely the group $G$ and the differential relations $\mathcal{R}$ are used to modify this basic step-up.

The role of the group $G$ is easily described. It is a symmetry group for the problem at hand. Let $\mathcal{G}$ be the prolongation of $G$ to $J^\infty(E)$. If we denote by $\mathcal{E}^p_G(J^\infty(E))$ the $G$ invariant (or more precisely $\mathcal{G}$ invariant) horizontal $p$ forms on $J^\infty(E)$ and by $\mathcal{E}^{n+1}_G(J^\infty(E))$ the $G$ invariant source forms, then we can form the $G$ invariant Euler-Lagrange complex

$$0 \longrightarrow \mathcal{E}^0_G \xrightarrow{d_H} \mathcal{E}^1_G \cdots \xrightarrow{d_H} \mathcal{E}^{n-1}_G \xrightarrow{d_H} \mathcal{E}^n_G \xrightarrow{E} \mathcal{E}^{n+1}_G \longrightarrow \cdots .$$

For example, if $\Delta$ is a source form which is invariant under the group $G$ and if $\Delta$ is the Euler-Lagrange form for some Lagrangian $\lambda$, that is, if $\Delta = E(\lambda)$, then it natural to ask whether $\Delta$ is the Euler-Lagrange form of a $G$ invariant Lagrangian. The answer to this question is tantamount to the calculation of the $G$ equivariant cohomology of the Euler-Lagrange complex:

$$H^{n+1}(\mathcal{E}^*_G) = \frac{\{\text{locally variational, } G \text{ invariant source forms}\}}{\{\text{source forms of } G \text{ invariant Lagrangians}\}}.$$

Although the equivariant cohomology of the variational bicomplex has been computed in some special cases, it is fair to say that there are few, if any, general results. This can be a difficult problem.

The differential relations $\mathcal{R}$ may represent open conditions on the jets of local sections of $E$ or they may represent systems of differential equations. These equations may be classical deterministic (well-posed) systems or they may be the kind of underdetermined or overdetermined systems that are often encountered in differential geometry. We prolong $\mathcal{R}$ to a set of differential equations $\mathcal{R}^\infty$ on $J^\infty(E)$ and then restrict (or pullback) the variational bicomplex on $J^\infty(E)$ to $\mathcal{R}^\infty$. The cohomology group $H^{n-1}(\mathcal{E}^*(\mathcal{R}))$ can now be identified with the vector space of conservation laws for $\mathcal{R}$ ( first integrals when $\mathcal{R}$ is a system of ordinary differential equations) while elements of $H^n(\mathcal{E}^*(\mathcal{R}))$ characterize the possible variational principles for $\mathcal{R}$. In Chapter Seven we obtain a lower bound on the dimension $p$ of the non-zero cohomology groups $H^p(\mathcal{E}^*(\mathcal{R}))$ (for classical, well-posed problems Vinogradov showed that this bound is $p = n - 1$) and present some explicit general techniques for the calculation of $H^p(\mathcal{E}^*(\mathcal{R}))$. But here too, as anyone who has studied Jesse Douglas' paper on variational problems for ordinary differential equations will attest, the general problem of computing the cohomology of the entire bicomplex for a given system of equations $\mathcal{R}$ can be a difficult one.

Because of the ability to make these modifications to the free variational bicomplex on $J^\infty(E)$, a surprising diversity of phenomena from geometry and topology, differential equations, and mathematical physics, including many topics *not* directly related to the calculus of variations, can be studied in terms of the cohomological properties of the variational bicomplex. Our goal is to develop the machinery of the variational bicomplex to the point that this cohomological viewpoint becomes a useful one.

Some specific examples of cohomology classes in the variational bicomplex can now be presented.

EXAMPLE 1. For our first example, we consider a system of autonomous, second order ordinary differential equations defined by the functions

$$P_\alpha = \ddot{u}^\alpha - F_{\alpha\beta}(u^\gamma)\dot{u}^\beta - V_\alpha(u^\gamma),$$

for $\alpha = 1, 2, \ldots, m$. The differential equations

$$P_\alpha = 0 \tag{0.3}$$

are (affine) linear in the first derivatives of the dependent variables $u^\beta$. The coefficient functions $F_{\alpha\beta}$ and $V_\alpha$ are smooth functions of the dependent variable and are defined on some open domain $F \subset \mathbf{R}^m$.

The variational bicomplex which we use to study these equations is defined over the trivial bundle $E$ given by

$$\pi\colon \mathbf{R} \times F \to \mathbf{R}, \tag{0.4}$$

with coordinates $(x, u^\alpha) \to x$. Thus, for ordinary differential equations the configuration space is, from our viewpoint, the fiber space $F$ and the dynamics are prescribed by the source form

$$\Delta = P_\alpha \, du^\alpha \wedge dx \tag{0.5}$$

on the infinite jet bundle of $E$. This is notably different from the more standard geometric treatment of second order ordinary differential equations wherein the configuration space is viewed as the base manifold and the dynamics are specified by a vector field on the tangent space of the configuration space. For the problems we wish to treat in this example, our formulation is the better one — it extends in a natural and obvious way to non-autonomous systems, to higher order equations and to partial differential equations. For ordinary differential equations which are periodic in the independent variable $x$, we can replace the base space $\mathbf{R}$ in (0.4) by the circle $S^1$.

For the source form (0.5), the Helmholtz equations $H(\Delta) = 0$ are equivalent to the algebraic condition

$$F_{\alpha\beta} = -F_{\beta\alpha},$$

and the differential conditions

$$\frac{\partial F_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial u^\beta} = 0,$$

and

$$\frac{\partial V_\alpha}{\partial u^\beta} - \frac{\partial V_\beta}{\partial u^\alpha} = 0.$$

(See §3A) In short, the 2 form

$$F = \frac{1}{2} F_{\alpha\beta}\, du^\alpha \wedge du^\beta$$

and the 1 form

$$V = V_\alpha\, du^\alpha$$

must be closed forms on $E$. If $F$ and $V$ are closed, then the local exactness of the Euler-Lagrange complex (see Chapter Four) implies that there is a coordinate neighborhood $U$ of each point in $E$ and a first order Lagrangian $\lambda$ defined on the jet space over $U$ such that

$$\Delta_{\mid J^\infty(U)} = E(\lambda).$$

There may be obstructions to the existence of a global Lagrangian — because the base space is one dimensional, these obstructions will lie in $H^2(E)$ which, in this example, is isomorphic to $H^2(F)$.

**Case 1.** For simplicity, suppose that $V_\alpha = 0$ so that the system of source equations (0.3) becomes

$$\ddot{u}^\alpha = F_{\alpha\beta}\, \dot{u}^\beta. \tag{0.6}$$

If $F$ is a closed 2 form, then $\Delta$ is a locally variational source form. As a representative of a cohomology class in $H^2(\mathcal{E}^*(J^\infty(E)))$, $\Delta$ is mapped by the isomorphism between $H^2(\mathcal{E}^*(J^\infty(E)))$ and $H^2(E)$ to the class represented by the two form $F$. Thus $\Delta$ admits a global Lagrangian if and only if $F$ is exact.

For example, with $F = \mathbf{R}^3 - \{0\}$, the equations (in vector notation)

$$\ddot{\mathbf{u}} = -\frac{1}{||\mathbf{u}||}\, \mathbf{u} \times \dot{\mathbf{u}} \tag{0.7}$$

are locally variational but not globally variational because the associated 2 form

$$F = \frac{u^1\, du^2 \wedge du^3 - u^2\, du^1 \wedge du^3 + u^3\, du^1 \wedge du^2}{[(u^1)^2 + (u^2)^2 + (u^3)^2]^{\frac{3}{2}}}$$

is not exact on $E$.

Despite the fact that there may be no global Lagrangian for the locally variational source form $\Delta$, it may still be possible, via an appropriate formulation of Noether's theorem, to obtain global conservation laws for $\Delta$ from global symmetries. This is because the obstructions for the existence of global conservation laws lie in a different cohomology group, namely $H^n(E)$, where $n = \dim M$. For instance, because the equations (0.7) are

(i) locally variational,

(ii) invariant with respect to rotations about the $u^3$ axis, and because

(iii) $H^1(E) = H^1(\mathbf{R}^3 - \{0\}) = 0$,

the global version of Noether's Theorem enables us to construct the first integral

$$I = -u^1\dot{u}^2 + u^2\dot{u}^1 + \frac{u^3}{||\mathbf{u}||},$$

which is defined on all of $J^\infty(E)$.

**Case 2.** Let us now suppose that $F = 0$ and that the 1 form $V$ is closed. The de Rham cohomology class on $E$ now determined by $\Delta$ is represented by the 2 form $V \wedge dx$. This form is always exact whether or not $V$ itself is exact. For example, with $m = 2$ and $F = \mathbf{R}^2 - \{0\}$ the equations

$$\ddot{u} = -\frac{v}{u^2+v^2} \quad \text{and} \quad \ddot{v} = \frac{u}{u^2+v^2} \tag{0.8}$$

must admit a a global variational principle even though the 1 form

$$V = \frac{v\,du - u\,du}{u^2+v^2}$$

is not exact on $E$. Indeed, one readily checks that

$$\lambda = [-\frac{1}{2}(\dot{u}^2 + \dot{v}^2) + x\frac{v\dot{u} - u\dot{v}}{u^2+v^2}]\,dx \tag{0.9}$$

is a Lagrangian for the system (0.8). But, because $H^1(E) = \mathbf{R}$, there is now a potential obstruction to the construction of global conservation laws via Noether's Theorem and, in fact, the rotational symmetry of (0.8) does not lead to a global first integral.

Observe that while (0.8) is translational invariant in $x$, the Lagrangian (0.9) contains an explicit $x$ dependence and is therefore not translational invariant. If we let $G$ be the group of translations in $x$, then any source $\Delta$ defining an autonomous system of equations belongs to $\mathcal{E}^2_G(J^\infty(E))$. The problem of determining if a locally variational, $G$ invariant source form admits a global $G$ invariant Lagrangian is, by definition, the same as that of calculating the equivariant cohomology group $H^2(\mathcal{E}^*_G(J^\infty(E)))$. A theorem of Tulczyjew [*71*] states that

$$H^2(\mathcal{E}^*_G(J^\infty(E))) = H^1(E) \oplus H^2(E).$$

In particular, we shall see that the source form (0.5) admits a global, translational invariant Lagrangian if and only if both forms $F$ and $V$ are exact. Thus the system (0.8) does not admit a translational invariant Lagrangian. This is perhaps the simplest illustration of how the introduction of a symmetry group $G$ can change the cohomology of the Euler-Lagrange complex.

EXAMPLE 2. Let $\pi\colon E \to M$ be a fibered manifold over a $n$ dimensional base manifold $M$. As we have already mentioned, there is an isomorphism between $H^*(E)$ and $H^*(\mathcal{E}^*(J^\infty(E)))$. In particular, to every representative $\omega \in \Omega^n(E)$ of a cohomology class in $H^n(E)$, we can associate a Lagrangian on $J^\infty(E)$ which is variationally trivial but which is not globally $d_H$ exact.

Suppose that $M$ and $F$ are compact oriented $n$ manifolds and that

$$h = h_{\alpha\beta}\, du^\alpha \otimes du^\beta$$

is a Riemannian metric on $F$. Let $E : M \times F \to M$. Then the volume form

$$\nu = \sqrt{\det h}\; du^1 \wedge du^2 \cdots \wedge du^n$$

on $F$ pulls back to a closed form on $E$ which represents a nontrivial cohomology class in $H^n(E)$. The associated Lagrangian $\lambda$ on $J^\infty(E)$ is found to be

$$\lambda = \sqrt{\det h}\; \det\Big[\frac{\partial u^\alpha}{\partial x^j}\Big]\, dx^1 \wedge dx^2 \cdots \wedge dx^n. \tag{0.10}$$

It is an amusing exercise to verify directly that for any metric $h$, $E(\lambda) = 0$. The Lagrangian $\lambda$ is not a global divergence and represents a non-trivial cohomology class in the Euler-Lagrange complex. The corresponding fundamental integral, defined on sections of $E$, or equivalently on maps $s\colon M \to F$, is

$$I\,[s] = \int_M ((j^\infty(s))^*(\lambda) = \int_M s^*(\nu),$$

and coincides, apart from a numerical factor, with the topological degree of the map $s$. This example illustrates how cohomology classes in the variational bicomplex on $J^\infty(E)$ may lead to topological invariants for the sections $s$ of $E$.

Consider now the special case where $M$ is the two sphere $S^2$ and $F$ is the two torus $S^1 \times S^1$. Let $\alpha = du$ and $\beta = dv$, where $(u, v)$ are the standard angular coordinates on $F$ and let $\nu = \alpha \wedge \beta$. The Lagrangian (0.10) becomes

$$\lambda = (u_x v_y - u_y v_x)\, dx \wedge dy \tag{0.11}$$

and, on sections $s$ on $E$,

$$(j^\infty(s))^*(\lambda) = s^*(\alpha \wedge \beta) = s^*(\alpha) \wedge s^*(\beta).$$

But $s^*(\alpha)$ is a closed one form on $S^2$ and is hence exact on $f$. If we write

$$s^*(\alpha) = d\,f, \tag{0.12}$$

where $f$ is a smooth real-valued function on $S^2$, then

$$(j^\infty(s))^*(\lambda) = d\,(f s^*(\beta)). \tag{0.13}$$

This proves that $\lambda$ is exact on all sections $s$ of $E$.

This example underscores an important point — that the cohomology of the variational bicomplex is "local" cohomology. The Lagrangian (0.11) defines a nontrivial cohomology class in the Euler-Lagrange complex because it cannot be expressed as the derivative of a one form whose values on sections $s$ can be computed pointwise from the jets of $s$. Indeed, because the function $f$ in (0.13) is the solution to the partial differential equation (0.12) on the two sphere $S^2$, the value of $f$ at any point $p \in S^2$ cannot be computed from the knowledge of $j^\infty(s)$ at the point $p$ alone.

Example 3. In this example we consider the variational bicomplex for regular plane curves. Let

$$\gamma\colon [0,1] \to \mathbf{R}^2$$

be a smooth, closed curve parametrized by

$$\gamma(x) = (u(x), v(x)).$$

We say that $\gamma$ is regularly parametrized or, equivalently, that $\gamma$ is an immersed plane curve, if the velocity vector

$$\dot\gamma(x) = (\dot u(x), \dot v(x)) \neq 0$$

for all $x$. The *rotation index* of $\gamma$ is defined by the functional

$$R\,[\gamma] = \frac{1}{2\pi}\int_0^1 \frac{\dot u \ddot v - \ddot u \dot v}{\dot u^2 + \dot v^2}\,dx.$$

The rotation index is integer valued. It is also an isotopy invariant of the curve $\gamma$. That is, if $\tilde\gamma\colon [0,1] \to \mathbf{R}^2$ is another smooth, regularly parametrized closed curve and if

$$H\colon [0,1] \times [0,1] \to \mathbf{R}^2$$

is a smooth homotopy of $\gamma$ to $\tilde{\gamma}$ through regular curves, then

$$R\,[\gamma] = R\,[\tilde{\gamma}].$$

This latter fact is easily proved directly. Let

$$\gamma_\epsilon(x) = H(x,\epsilon) = (u_\epsilon(x), v_\epsilon(x))$$

and let

$$u' = \frac{\partial u_\epsilon}{\partial \epsilon} \quad \text{and} \quad v' = \frac{\partial v_\epsilon}{\partial \epsilon}.$$

We then deduce, by direct calculation, that

$$\begin{aligned} \frac{d}{d\epsilon} R\,[\gamma_\epsilon] &= \frac{1}{2\pi} \int_0^1 \frac{d}{d\epsilon} \Big[\frac{\dot{u}_\epsilon \ddot{v}_\epsilon - \ddot{u}_\epsilon \dot{v}_\epsilon}{\dot{u}_\epsilon^2 + \dot{v}_\epsilon^2}\Big]\, dx \\ &= \frac{1}{2\pi} \int_0^1 \frac{d}{dx} \Big[\frac{\dot{u}_\epsilon \dot{v}'_\epsilon - \dot{u}'_\epsilon \dot{v}_\epsilon}{\dot{u}_\epsilon^2 + \dot{v}_\epsilon^2}\Big]\, dx = 0, \end{aligned} \tag{0.14}$$

and therefore $R\,[\gamma_0] = R\,[\gamma_1]$, as required.

The calculation in (0.14) can be interpreted from the viewpoint of the calculus of variations. Let

$$\lambda = L(x, u, v, \dot{u}, \dot{v}, \ddot{u}, \ddot{v})\, dx \tag{0.15}$$

be a second order Lagrange function for a variational problem for plane curves and let $\gamma_\epsilon(x) = (u_\epsilon(x), v_\epsilon(x))$ be a 1 parameter family of such plane curves. Then the *first variational formula* for the Lagrangian (0.15) is the identity

$$\frac{d}{d\epsilon} L(x, u, v, \dot{u}_\epsilon, \dot{v}_\epsilon, \ddot{u}_\epsilon, \ddot{v}_\epsilon) = u'_\epsilon E_u(L) + y'_\epsilon E_v(L) + \frac{d\,f}{dx} \tag{0.16}$$

where (with $u^1 = u$ and $u^2 = v$)

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - \frac{d}{dx}\Big(\frac{\partial L}{\partial \dot{u}^\alpha}\Big) + \frac{d^2}{dx^2}\Big(\frac{\partial L}{\partial \ddot{u}^\alpha}\Big)$$

are the components of the Euler-Lagrange expressions of $L$ and $f$ is the function

$$f = \Big[\frac{\partial L}{\partial \dot{u}} - \frac{d}{dx}\Big(\frac{\partial L}{\partial \ddot{u}}\Big)\Big] u' + \Big(\frac{\partial L}{\partial \ddot{u}}\Big)\dot{u}' + \Big[\frac{\partial L}{\partial \dot{v}} - \frac{d}{dx}\Big(\frac{\partial L}{\partial \ddot{v}}\Big)\Big] v' + \Big(\frac{\partial L}{\partial \ddot{v}}\Big)\dot{v}'.$$

To apply this formula to the rotation index, we consider the specific Lagrangian

$$L_{\mathrm{R}} = \frac{\dot{u}\ddot{v} - \ddot{u}\dot{v}}{\dot{u}^2 + \dot{v}^2}$$

so that the corresponding fundamental integral is the rotation index

$$R\,[\gamma] = \frac{1}{2\pi}\int_0^1 L_{\mathrm{R}}(x, u, v, \dot{u}, \dot{v}, \ddot{u}, \ddot{v})\,dx.$$

The calculation (0.14) is now, in essence, the calculation of the first variation of $R\,[\gamma]$. Indeed, one can readily verify that

$$E_u(L_{\mathrm{R}}) = 0 \quad \text{and} \quad E_v(L_{\mathrm{R}}) = 0.$$

The general first variational formula (0.16) now reduces to

$$\frac{d}{d\epsilon} L_{\mathrm{R}}(x, u, v, \dot{u}_\epsilon, \dot{v}_\epsilon, \ddot{u}_\epsilon, \ddot{v}_\epsilon) = \frac{d\,f}{dx}$$

where

$$f = \frac{\dot{u}_\epsilon \dot{v}'_\epsilon - \dot{u}'_\epsilon \dot{v}_\epsilon}{\dot{u}^2_\epsilon + \dot{v}^2_\epsilon}.$$

Upon integration with respect to $x$, this becomes (0.14). Thus, from the viewpoint of the variational calculus, one proves that the rotation index is an isotopy invariant by (i) showing that the Lagrangian $\lambda_{\mathrm{R}}$ defining the rotation index is variationally trivial, and then (ii) applying the first variational formula.

Because $E(\lambda_{\mathrm{R}}) = 0$, it can be expressed as the total derivatives

$$\lambda_{\mathrm{R}} = d_H\big[\arctan(\frac{\dot{v}}{\dot{u}})\big] = -d_H\big[\arctan(\frac{\dot{u}}{\dot{v}})\big]. \tag{0.17}$$

But, because the domain of $\lambda_{\mathrm{R}}$ is the open set

$$\mathcal{U} = \{\,(x, u, , v, \dot{u}, \dot{v}, \ddot{u}, \ddot{v}) \mid \dot{u}^2 + \dot{v}^2 \neq 0\,\}$$

and because neither one of the arctan functions in (0.17) is defined on all of $\mathcal{U}$, equation (0.17) is only a local formula. In fact, $\lambda_{\mathrm{R}}$ cannot be the horizontal exterior derivative of any function on all of $\mathcal{U}$ since this would imply that the rotation index vanishes for all closed curves. Hence the Lagrangian $\lambda_{\mathrm{R}}$ defines a nontrivial cohomology class in the Euler-Lagrange complex for regular plane curves.

To make this last statement more precise, we take for the fibered manifold $E$ the trivial bundle $\mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ with coordinates $(x, u, v) \to x$. (We could also take the base space to be the circle $S^1$ — then all sections of $E$ would automatically correspond to closed curves in the plane.) The jet bundle $J^\infty(E)$ is the bundle of infinite jets of all plane curves $\gamma$ and has coordinates

$$j^\infty(\gamma)(x) = (x, u, v, \dot u, \dot v, \ddot u, \ddot v, \dots).$$

Because we are interested in regularly parametrized closed curves we restrict the variational bicomplex to the open submanifold defined by

$$\mathcal{R} = \{\, j^\infty(\gamma)(x) \mid \dot u^2 + \dot v^2 \neq 0 \,\}.$$

Now $\mathcal{R}$ has the same homotopy type as the circle $S^1$ which has 1 dimensional de Rham cohomology in dimension $p = 1$. Our general theory implies that $H^p(\mathcal{E}^*(\mathcal{R}))$ is therefore zero for $p \neq 1$ while $H^1(\mathcal{E}^*(\mathcal{R}))$ is the one dimensional vector space generated by $L_{\mathrm{R}}$. This means that if

$$\lambda = L(x, u, v, \dot u, \dot v, \ddot u, \ddot v, \dots)\, dx$$

is any variational trivial Lagrangian defined on $\mathcal{R}$, then there is a constant $a$ and a function

$$f = f(x, u, v, \dot u, \dot v, \ddot u, \ddot v, \dots)$$

such that

$$\lambda = \lambda_{\mathrm{R}} + d_H f.$$

Apart from the multiplicative factor $a$, the fundamental integral

$$I\,[\gamma] = \int_0^1 L(j^\infty(\gamma)(x))\, dx,$$

which we know *a priori* to be an isotopy invariant of $\gamma$, must coincide with the rotation index of $\gamma$.

In our discussion thus far, we have not considered possible group actions on the bundle $E : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$. The group $G$ that naturally arises in geometric problems for regular plane curves consists of

(i) the group of Euclidean motions in the fiber $\mathbf{R}^2$; and

(ii) the group of local, orientation preserving, diffeomorphism $\bar x = f(x)$ of the base space $\mathbf{R}$.

Since the Lagrangian $\lambda_0$ can be expressed as

$$\lambda_{\mathrm{R}} = \kappa\, ds,$$

where $\kappa$ is the curvature function and $ds$ is the arclength differential, it follows that $\lambda_{\mathrm{R}}$ is $G$ invariant, *i.e.*,

$$\lambda_{\mathrm{R}} \in \mathcal{E}^1_G(\mathcal{R}).$$

Moreover, $\lambda_{\mathrm{R}}$ is patently a cohomology class in $H^*(\mathcal{E}^1_G(\mathcal{R}))$ since it is not the total derivative of any function on $\mathcal{R}$, let alone a $G$ invariant one. As we shall see, a theorem of Cheung [*17*] asserts that $\lambda_0$ generates the only class in $H^1(\mathcal{E}^*_G(\mathcal{R}))$. We shall also see that the $G$ invariant source form

$$\Delta = (-\dot v\, du + \dot u\, dv)\, dx,$$

which is the Euler-Lagrange form for the Lagrangian

$$\lambda = \frac{1}{2}(u\dot v - v\dot u)\, dx$$

is not the Euler-Lagrange form of any $G$ invariant Lagrangian and, in fact, generates $H^2(\mathcal{E}^*(\mathcal{R}))$. We hasten to remark that while the calculation of $H^p(\mathcal{E}^*(\mathcal{R}))$ is a simple consequence of the general theory, the calculation of $H^1(\mathcal{E}^*_G(\mathcal{R}))$ and $H^2(\mathcal{E}^*_G(\mathcal{R}))$ is based upon more *ad hoc* arguments.

Example 4. In this example we examine the Gauss-Bonnet theorem from the viewpoint of the variational bicomplex. This theorem states that if $S$ is a compact oriented surface with Gaussian curvature $K$ and Euler characteristic $\chi(S)$, then

$$\chi(S) = \frac{1}{2\pi}\int_S K\, dA.$$

We use the Gauss-Bonnet theorem to illustrate many of the issues with which we shall be concerned. Specifically, we use the Gauss-Bonnet theorem to show

(i) how the data $\{E, G, \mathcal{R}\}$ can fitted together, sometimes in quite different ways, to define a variational bicomplex to model a given situation;

(ii) that there is a rich local theory of the variational bicomplex which impinges upon recent developments in the theory of determinantal ideals;

(iii) the need for a global first variational formula;

(iv) the role of the first variational formula in Chern's proof of the generalized Gauss-Bonnet theorem;

(v) how characteristic and secondary characteristic classes arise as cohomology classes in the variational bicomplex; and

(vi) that there is a novel connection between the cohomology of the variational bicomplex for surfaces and the Gelfand-Fuks cohomology of formal vector fields.

We begin with the local theory of surfaces and so, to this end, we let $E$ be the trivial bundle $E\colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^2$ with coordinates

$$(x, y, R) \to (x, y),$$

where $R$ is the position vector $R = (u, v, w)$ in $\mathbf{R}^3$. A section of $E$ defined by a map $\phi\colon \mathbf{R}^2 \to \mathbf{R}^3$ determines a local, regularly parametrized surface if $\phi_x \times \phi_y \neq 0$ and, accordingly we restrict our considerations to the open set $\mathcal{R}$ of $J^\infty(E)$ defined by

$$\mathcal{R} = \{\, (x, y, R, R_x, R_y, R_{xx}, R_{xy}, R_{yy}, \dots) \mid R_x \times R_y \neq 0 \,\}.$$

The unit normal vector $N$, the first and second fundamental forms and the Gaussian curvature are defined by

$$N = \frac{R_x \times R_y}{||R_x \times R_y||}, \qquad \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle R_x, R_x\rangle & \langle R_x, R_y\rangle \\ \langle R_y, R_x\rangle & \langle R_y, R_y\rangle \end{bmatrix},$$

$$\begin{bmatrix} l & m \\ m & n \end{bmatrix} = \begin{bmatrix} \langle N, R_{xx}\rangle & \langle N, R_{xy}\rangle \\ \langle N, R_{xy}\rangle & \langle N, R_{yy}\rangle \end{bmatrix}, \qquad \text{and} \qquad K = \frac{ln - m^2}{EG - F^2}.$$

We emphasize that these expressions are all to be thought of as functions on $\mathcal{R}$ — it is on jets of sections of $E$ that these equations take on their usual meanings from the local theory of surfaces.

The integrand in the Gauss-Bonnet formula defines a second order Lagrangian on $\mathcal{R}$, namely

$$\lambda_{\text{GB}} = L_{\text{GB}}(x, y, R, R_x, R_y, R_{xx}, R_{xy}, R_{yy})\, dx \wedge dy$$

where

$$L_{\text{GB}} = K\sqrt{EG - F^2}.$$

One can show that $\lambda_{\rm GB}$ is variational trivial, *i.e.*,

$$E(\lambda_{\rm GB}) = 0.$$

Now the Euler-Lagrange form of any second order Lagrangian $\lambda$ is generally a fourth order source form which is linear in the derivatives of order 4 and quadratic in the derivatives of order 3. If $\lambda$ is variational trivial, then the vanishing of these higher derivative terms in $E(\lambda)$ severely restrict the functional dependencies of $\lambda$ on its second order derivatives. These dependencies can be completely characterized — the highest derivative dependencies of a variationally trivial Lagrangian, of any order, must occur via Jacobian-like determinants. In the case of the Gauss-Bonnet Lagrangian $\lambda_{\rm GB} = K dA$, these dependencies manifest themselves in the following formula ( Struik [*64*] p.112):

$$L_{\rm GB} = \frac{1}{D^3}\cdot$$
$$\left\{ \begin{vmatrix} \langle R_{xx}, R_{yy}\rangle & \langle R_{xx}, R_x\rangle & \langle R_{xx}, R_y\rangle \\ \langle R_x, R_{yy}\rangle & E & F \\ \langle R_y, R_{yy}\rangle & F & G \end{vmatrix} - \begin{vmatrix} \langle R_{xy}, R_{xy}\rangle & \langle R_{xy}, R_x\rangle & \langle R_{xy}, R_y\rangle \\ \langle R_{xy}, R_x\rangle & E & F \\ \langle R_{xy}, R_y\rangle & F & G \end{vmatrix} \right\}.$$

By expanding these two determinants, one finds that $\lambda_{\rm GB}$ belongs to the determinantal ideal generated ( over the ring of functions on the first order jet bundle $J^1(E)$) by the nine $2\times 2$ Jacobians

$$\frac{\partial(R_x, R_y)}{\partial(x,y)} = \begin{bmatrix} \dfrac{\partial(u_x,u_y)}{\partial(x,y)} & \dfrac{\partial(u_x,v_y)}{\partial(x,y)} & \dfrac{\partial(u_x,w_y)}{\partial(x,y)} \\ \dfrac{\partial(v_x,u_y)}{\partial(x,y)} & \dfrac{\partial(v_x,v_y)}{\partial(x,y)} & \dfrac{\partial(v_x,w_y)}{\partial(x,y)} \\ \dfrac{\partial(w_x,u_y)}{\partial(x,y)} & \dfrac{\partial(w_x,v_y)}{\partial(x,y)} & \dfrac{\partial(w_x,w_y)}{\partial(x,y)} \end{bmatrix}.$$

This observation applies to characteristic forms in general; because these forms are closed forms in the appropriate Euler-Lagrange complex, their highest derivative dependencies can always be expressed in terms of Jacobian determinants.

Source forms which are locally variational must likewise exhibit similar functional dependencies in their highest order derivatives. The Monge-Ampere equation with source form

$$\Delta = (u_{xx}u_{yy} - u_{xy}^2)\, du\wedge dx\wedge dy.$$

typifies these dependencies.

Because the Gauss-Bonnet Lagrangian is variational trivial, it is possible to express $\lambda_{\text{GB}}$, at least locally, as the total exterior derivative of a horizontal 1 form $\eta$. The general techniques provided by our local theory (see §4B and §4C) can be applied, in a straightforward and elementary fashion, to find that

$$\lambda_{\text{GB}} = d_H \eta$$

where

$$\eta = \frac{F\langle R_x, d_H R_x\rangle - E\langle R_y, d_H R_x\rangle}{E\sqrt{EG - F^2}}. \tag{0.18}$$

This coincides with another formula found in Struik for the Gaussian curvature and attributed to Liouville. We now observe that the 1 form $\eta$ is actually defined on all of $\mathcal{R}$ and so the Gauss-Bonnet Lagrangian $\lambda_{\text{GB}}$ is a trivial cohomology class in the Euler-Lagrange complex $\mathcal{E}^*(\mathcal{R})$. In fact, because $\mathcal{R}$ has the same homotopy type as $\mathbf{SO}(3)$, $H^2(\mathcal{E}^*(\mathcal{R})) = 0$ and hence all variational trivial Lagrangians on $\mathcal{R}$ are $d_H$ exact.

This does not contradict the Gauss-Bonnet theorem because the data given to this point, namely the bundle $E\colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}$ and the regularity condition defining the open domain $\mathcal{R} \subset J^\infty(E)$ only models the local theory of surfaces. To capture the global aspects of the Gauss-Bonnet theorem, we now observe that the Lagrangian $\lambda_{\text{GB}}$ is invariant under the group $G$ consisting of

(i) the Euclidean group of motions in the fiber $\mathbf{R}^3$; and

(ii) the group (or properly speaking, pseudo-group) of local orientation preserving diffeomorphism of the base space $\mathbf{R}^2$, that is, $\lambda_{\text{GB}}$ is invariant under coordinate transformations of the surface.

We call $\mathcal{E}^*_G(J^\infty(E))$ the $G$ equivariant Euler-Lagrange complex for surfaces in $\mathbf{R}^3$.

Let $S$ be any compact, oriented surface in $\mathbf{R}^3$. By restricting the Euler-Lagrange complex on $\mathcal{R}$ to forms which are $G$ invariant we effectively restrict our considerations to forms on $\mathcal{E}^*$ which, when pulled back by coordinate charts to $S$, will automatically patch on overlapping coordinate charts to define global forms. The Lagrangian $\lambda_{\text{GB}} \in \mathcal{E}^2_G(\mathcal{R})$ but the form $\eta$, as defined by (0.18), does not transform invariantly under change of coordinates and hence $\eta \notin \mathcal{E}^1_G(\mathcal{R})$. Indeed, $\lambda_{\text{GB}}$ now represents a nontrivial cohomology class in $H^2(\mathcal{E}^*_G(\mathcal{R}))$ and in fact generates $H^2(\mathcal{E}^*_G(\mathcal{R}))$.

There are three other ways in which a variational bicomplex can be constructed so as to study some property of the Gaussian curvature. To describe the first of these alternative ways, we fix an abstract 2 dimensional manifold $M$, construct the bundle $E_1\colon M \times \mathbf{R}^3 \to M$ and consider the open set $\mathcal{R} \subset J^\infty(E_1)$ of jets of immersions from $M$ to $\mathbf{R}^3$. For sections $\phi$ of $E_1$ with $j^\infty(\phi) \in \mathcal{R}$, we can define the functional

$$I\,[\phi] = \frac{1}{2\pi}\int_M (j^\infty(\phi))^*\lambda.$$

The Gauss-Bonnet theorem implies that $I\,[\phi]$ is independent of $\phi$ and depends only on the choice of the base manifold $M$. To prove this result directly, that is, to prove that $I\,[\phi]$ is a deformation invariant of the immersion $\phi$, we turn, just as in the previous example, to the first variational formula. Let

$$\lambda = L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij})\, dx \wedge dy$$

be any second order Lagrangian on $J^\infty(E_1)$. The independent variables $(x^i) = (x, y)$ are now local coordinates on $M$. If

$$\phi_\epsilon\colon M \;\to \mathbf{R}^3$$

is a smooth 1 parameter family of immersions, then the classical first variation formula states that

$$\frac{d}{d\epsilon}L(j^2(\phi_\epsilon)) = \frac{\partial u^\alpha}{\partial \epsilon}E_\alpha(L)(j^\infty(\phi_\epsilon)) + [\frac{dV^i}{dx^i}](j^\infty(\phi_\epsilon), \tag{0.19}$$

where

$$V^i = [\frac{\partial L}{\partial u^\alpha_i} - \frac{\partial}{\partial x^j}(\frac{\partial L}{\partial u^\alpha_{ij}})]\frac{du^\alpha}{d\epsilon} + [\frac{\partial L}{\partial u^\alpha_{ij}}]\frac{du^\alpha_j}{d\epsilon}.$$

Since $\lambda_{\text{GB}}$ has vanishing Euler-Lagrange form, this implies that

$$\frac{d}{d\epsilon}[(j^\infty(\phi_\epsilon))^*\lambda_{\text{GB}}] = (j^\infty(\phi_\epsilon))^*(d_H\eta) = d\,[(j^\infty(\phi_\epsilon))^*\eta],$$

where

$$\eta = V^1\, dy - V^2\, dx.$$

Before we can integrate this equation over all of $M$ and thereby conclude that $I[\phi_1] = I[\phi_2]$ we must check that the local first variational formula (0.19) holds globally. In other words, it is necessary to verify that the 1 form $\eta$ transforms properly under coordinate transforms on $\mathcal{R}$ to insure that $(j^\infty(\phi_\epsilon))^*\eta$ patches together to define a 1 form on all of $S$. For second order Lagrangians, this is indeed the case and thus $I\,[\phi]$ is deformation invariant of the immersion $\phi$. However, for higher order variational problems, the standard extension of (0.19) does *not* provide us with a global first variational formula. This problem has been identified and resolved by a number of authors — for us the existence of a global first variational formula is an easy consequence of the general global theory of the variational bicomplex developed in Chapter Five.

Gauss' Theorema Egregium states that the Gaussian curvature is an intrinsic quantity, computable pointwise from the two jets of the first fundamental form $g_{ij}$ by the formula

$$K = \frac{1}{\det(g_{ij})} R_{1212},$$

where $R_i{}^j{}_{hk}$ is the curvature tensor of $g_{ij}$. This theorem motivates the second alternative to our original bicomplex. This time we let $Q$ be the manifold of positive definite quadratic forms on $\mathbf{R}^2$ and let $E_2 \colon \mathbf{R}^2 \times Q \to \mathbf{R}^2$. A section $g$ of $E_2$ can be identified with a metric $g(x,y) = (g_{ij}(x,y))$ on $\mathbf{R}^2$. Lagrangians on $J^\infty(E_2)$ take the form

$$\lambda = L(x^h, g_{ij}, g_{ij,h}, g_{ij,hk}, \dots)\, dx \wedge dy.$$

Let $M$ be any compact oriented Riemannian 2 manifold with metric $g$. Then, on any coordinate chart $(U,(x,y))$ of $M$, we can use $g$ to pull $\lambda$ back to a two form $\tilde{\lambda}$ on $U$:

$$\tilde{\lambda} = (j^\infty(g))^* \lambda.$$

A condition sufficient to insure that the $\tilde{\lambda}$ will patch together to define a 2 form on all of $M$ is that $\lambda$ be $G$ invariant, where $G$ is now just the group of local orientation preserving diffeomorphisms of $\mathbf{R}^2$. The Gauss-Bonnet Lagrangian

$$\lambda_{\mathrm{GB}} = \sqrt{g} R_{1212}\, dx \wedge dy$$

belongs to $\mathcal{E}^2_G(J^\infty(E))$ and, once again, generates all the 2 dimensional cohomology of the Euler-Lagrange complex.

This approach immediately generalizes to higher dimensions by letting $E_2 : \mathbf{R}^n \times Q \to \mathbf{R}^n$. The bundle $E_2$ is called the bundle of Riemannian structures on $\mathbf{R}^n$. The forms on $J^\infty(E_2)$ which are invariant under the group of orientation-preserving local diffeomorphisms on $\mathbf{R}^n$ are called natural differential forms on the bundle of Riemannian structures. A well-known theorem of Gilkey [*28*] asserts that the cohomology of the Euler-Lagrange complex $H^p(\mathcal{E}^*_G(J^\infty(E_2)))$ is generated, for $p \leq n$ by the Pontryagin forms and the Euler form. It is in this context that characteristic forms first appear in general as cohomology classes in the variational bicomplex.

Secondary characteristic classes can be identified with the next cohomology group $H^{n+1}(\mathcal{E}^*_G(J^\infty(E)))$. At this time, this is best illustrated by an example. Let $n = 3$. Then, with respect to the coordinate frame $\{\dfrac{\partial}{\partial x^i}\}$, the first Chern-Simons form is

$$\lambda = (\frac{1}{2}\Gamma^j_{ih}\Gamma^i_{jk,l} + \Gamma^j_{ih}\Gamma^m_{jk}\Gamma^i_{ml})\, dx^h \wedge dx^k \wedge dx^l.$$

Here $\Gamma^j_{ih}$ are the components of the Christoffel symbols for the metric $g$. This 3 form is a second order Lagrangian on $E$ but it is not a $G$ invariant one. Nevertheless, its Euler-Lagrange form

$$\Delta = (\nabla_k R^i_h)\, dg_{ij}\, dx^i \wedge dx^h \wedge dx^k,$$

where $\nabla_k$ denotes partial covariant differentiation and $R^i_h$ is the Ricci tensor, is manifestly $G$ invariant, *i.e.*, $\Delta \in \mathcal{E}^4_G(J^\infty(E))$. Because $\Delta$ is the Euler-Lagrange form for some Lagrangian, it satisfies the Helmholtz conditions and therfore $\Delta$ defines a cohomology class in $H^4(\mathcal{E}^*_G(J^\infty(E)))$. The form $\Delta$ actually determines a nontrivial class ( meaning that it is not the Euler- Lagrange form of any natural Lagrangian) and in fact generates $H^4(\mathcal{E}^*_G(J^\infty(E)))$. We shall rederive and generalize all these results in Chapter Six by computing the $G$ equivariant cohomology of the entire variational bicomplex on Riemannian structures.

Because the curvature tensor is defined solely in terms of the connection coefficients and their first derivatives, we can also think of the Lagrangian $\lambda_{\text{GB}} = K\, dA$ as a first order Lagrangian on yet another bundle; the bundle $E_3$ of metrics $g$ and affine connections $\Gamma$ on $\mathbf{R}^2$, *i.e.*,

$$\lambda_{\text{GB}} = L_{\text{GB}}(g_{ij}, \Gamma^i_{jh}, \Gamma^i_{jh,k})\, dx \wedge dy$$

From this viewpoint $\lambda_{\text{GB}}$ is no longer variationally trivial with respect to the independent variations of the metric and connection. Nevertheless, we shall see (Chapter Four, Example 4.14) that

$$E(\lambda_{\text{GB}})(j^\infty(g), j^\infty(\Gamma)) = 0$$

whenever $\Gamma$ is a Riemannian connection for $g$. Consequently, if $(g_1, \Gamma_1)$ and $(g_0, \Gamma_0)$ are two pairs of metrics and Riemannian connections, the first variational formula can still be used to deduce that

$$\lambda_{\text{GB}}((j^\infty(g_1), j^\infty(\Gamma_1)) - \lambda_{\text{GB}}((j^\infty(g_0), j^\infty(\Gamma_0)) = d_H \eta, \tag{0.20}$$

where $\eta$ is a manifestly invariant form depending smoothly on the one jets of $g_0, g_1, \Gamma_0, \Gamma_1$. Again this proves that the integral

$$I\,[g, \Gamma] = \int_M \lambda_{\text{GB}}(j^\infty(g), j^\infty(\Gamma))$$

is independent of the choice of metric $g$ and Riemannian connection $\Gamma$ on the two dimensional manifold $M$. This result is not restricted to two dimensional manifolds but applies equally well to the Euler form

$$\lambda = \sqrt{g} K_n \, dx^1 \wedge \, dx^2 \cdots \wedge dx^n,$$

where $K_n$ is the total curvature function of a metric $g$ on an $n$ dimensional manifold $M$. Moreover, given a vector field $X$ on $M$, one can define a connection $\Gamma_0$ away from the zeros of $X$ in such a way that $\lambda(j^\infty(g), j^\infty(\Gamma_0)) = 0$. Thus, with $g_0 = g_1 = g$ and $\Gamma_1$ the Christoffel symbols of $g$, (0.20) reduces to

$$\sqrt{g} K_n \, dx^1 \wedge \, dx^2 \cdots \wedge dx^n = d_H \eta.$$

This first variational formula reproduces exactly the formula needed by Chern in his celebrated proof of the generalized Gauss-Bonnet theorem.

Finally, we briefly mention the connection with Gelfand-Fuks cohomology. Let $E$ be the trivial bundle $E\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$, let $\mathcal{R} \subset J^\infty(E)$ be the set of jets of immersions (*i.e.*, maps whose Jacobian is of maximal rank) and let $G$ be the pseudo-group of orientation preserving local diffeomorphism of the base space $\mathbf{R}^n$. When $n = 1$ and $m = 2$, we saw that the rotation index arises as a cohomology class in $\mathcal{E}^*(\mathcal{R})$. When $n = 2$ and $m = 3$, the cohomology of the Euler-Lagrange complex detects the Gaussian curvature. When $n = m$, *i.e.*, when $\mathcal{R}$ is the set of jets of local diffeomorphisms of $\mathbf{R}^n$ to $\mathbf{R}^n$, we shall find (Chapter Six) that the Euler-Lagrange complex computes the Gelfand-Fuks cohomology of formal vector fields on $\mathbf{R}^n$.

Example 5. *Outline, to be completed after writing Chapter Seven.*

We now turn to some examples of cohomology classes for variational bicomplexes for differential equations.

- Cauchy's Integral Theorem as a conservation law for the Cauchy-Riemman equations.
- The Godbillon-Vey form as a conservation law for the Frobenius equations

$$d\,\omega \wedge \omega = 0.$$

- Variational principles as the $E_2$ in the spectral sequence for the variational bicomplex and Douglas' solution to the inverse problem for second order ODE.

This completes our introductory survey of the variational bicomplex. Additional introductory remarks, of a more specific nature, can be found at the beginning of each chapter.

Chapter One

# VECTOR FIELDS AND FORMS ON INFINITE JET BUNDLES

In this chapter we introduce the variational bicomplex for a fibered manifold and we develop the requisite calculus of vector fields and differential forms on the infinite jet bundle of such spaces. Our objectives in this regard are simply to gather together those definitions, basic results and formulas which will be used throughout this book and to fix our notational conventions. We pay particular attention to the notion of generalized vector fields and the interplay between the prolongations of these fields and the contact ideal on the infinite jet bundle.

Detailed accounts of the geometry of finite jet bundles have been provided by numerous authors including, for example, Goldschmidt [*29*], Pommaret [*59*] and Saunders [*62*], and accordingly we do not dwell on this subject here. However, since infinite jet bundles are, strictly speaking, not manifolds some care is given to the development of the calculus of vector fields and forms on these spaces. We follow, with some modifications, the presentations of Saunders [*62*] and Takens [*65*]. The material on generalized vector fields has been adopted from the recent text by Olver [*55*].

**A. Infinite Jet Bundles.** Let $\pi\colon E \to M$ be a smooth fibered manifold with total space $E$ of dimension $n+m$ and base space $M$ of dimension $n$. The projection map $\pi$ is a smooth surjective submersion. The fiber $\pi^{-1}(x)$ over a point $x \in M$ may change topologically as $x$ varies over $M$; for example, let $E$ be $\mathbf{R}^2 - \{(1,0)\}$ and let $\pi$ be the projection onto the $x$ axis. In many situations $E$ will actually be a fiber bundle over $M$ but this additional structure is not needed to define the variational bicomplex. We assume that $M$ is connected.

We refer to the fibered manifold $E$ locally by coordinate charts $(\varphi, U)$ where, for $p \in U \subset E$,

$$\varphi(p) = (x(p), u(p))$$

and

$$x(p) = (x^i(p)) = (x^1, x^2, \dots, x^n), \quad u(p) = (u^\alpha(p)) = (u^1, u^2, \dots, u^m).$$

These coordinates are always taken to be adapted to the fibration $\pi$ in the sense that $(\varphi_0, U_0)$, where $\varphi_0 = \varphi \circ \pi$ and $U_0 = \pi(U)$, is a chart on the base manifold $M$ and that the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbf{R}^n \times \mathbf{R}^m \arrow[d, "\mathrm{proj}"] \\ U_0 \arrow[r, "\varphi_0"] & \mathbf{R}^n, \end{tikzcd}$$

where $\mathrm{proj}((x,u)) = (x)$, commutes. Throughout this book latin indices range from 1 to $n$ and greek indices from 1 to $m$ unless otherwise indicated. The summation convention, by which repeated indices are assumed to be summed, is in effect. If $(\psi, V)$ is an overlapping coordinate system and $\psi(p) = (y(p), v(p))$, then on the overlap $U \cap V$ we have the change of coordinates formula

$$y^j = y^j(x^i) \qquad \text{and} \qquad v^\beta = v^\beta(x^i, u^\alpha). \tag{1.1}$$

If $\rho\colon F \to N$ is another fibered manifold, then a map $\phi\colon E \to F$ is said to be fiber-preserving if it is covers a map $\phi_0\colon M \to N$, *i.e.*, the diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & F \arrow[d, "\rho"] \\ M \arrow[r, "\phi_0"] & N \end{tikzcd}$$

commutes. Thus, the fiber over $x \in M$ in $E$ is mapped by $\phi$ into the fiber over $y = \phi_0(x) \in N$ in $F$. We shall, on occasion, consider arbitrary maps between fibered bundles although the general theory of the variational bicomplex is to be developed within the category of fibered manifolds and fiber-preserving maps.

Denote by $\pi^k\colon J^k(E) \to M$ the fiber bundle of $k$-jets of local sections of $E$. The fiber $(\pi^k)^{-1}(x)$ of $x \in M$ in $J^k(E)$ consists of equivalence classes, denoted by $j^k(s)(x)$, of local sections $s$ of $E$ at $x$; two local sections $s_1$ and $s_2$ about $x$ are equivalent if with respect to some adapted coordinate chart (and hence any adapted chart) all the partial derivatives of $s_1$ and $s_2$ agree up to order $k$ at $x$. Each projection $\pi^l_k\colon J^l(E) \to J^k(E)$, defined for $l \geq k$ by

$$\pi^l_k[j^l(s)(x)] = j^k(s)(x),$$

is a smooth surjection and, in fact, for $l = k+1$ defines $J^l(E)$ as an affine bundle over $J^k(E)$. This implies that for all $l \geq k$, $J^l(E)$ is smoothly contractible to $J^k(E)$. We shall often write, simply for the sake of notational clarity,

$$\pi^k_E = \pi^k_0 \quad \text{and} \quad \pi^k_M = \pi^k$$

for the projections from $J^k(E)$ to $E$ and $M$.

An adapted coordinate chart $(\varphi, U)$ on $E$ lifts to a coordinate chart $(\widetilde{\varphi}, \widetilde{U})$ on $J^k(E)$. Here $\widetilde{U} = (\pi^k_E)^{-1}(U)$ and, if $s\colon U_0 \to U$ is the section $s(x) = (x^i, s^\alpha(x^i))$, then the coordinates of the the point $j^k(s)(x)$ are

$$\widetilde{\varphi}\,[j^k(s)(x)] = (x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \cdots i_k}), \tag{1.2}$$

where, for $l = 0, 1, \ldots,$ k,

$$u^\alpha_{i_1 i_2 \cdots i_l} = \frac{\partial^l s^\alpha}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_l}}(x),$$

and where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq n$. This notation becomes cumbersome when discussing specific examples for which the dimension of $E$ is small. For these situations we reserve the symbols $x, y, z$ for base coordinates and $u, v, w$ for fiber coordinates. We shall write $u_x, u_y, u_{xx}, u_{xy}, u_{yy} \cdots$ for the jet coordinates.

The inverse sequence of topological spaces $\{J^k(E), \pi^l_k\}$ determine an inverse limit space $J^\infty(E)$ together with projection maps

$$\pi^\infty_k\colon J^\infty(E) \to J^k(E) \quad \text{and} \quad \pi^\infty_E\colon J^\infty(E) \to E,$$

and

$$\pi^\infty_M\colon J^\infty(E) \to M.$$

The topological space $J^\infty(E)$ is called the infinite jet bundle of the fibered manifold $E$. A point in $J^\infty(E)$ can be identified with an equivalence class of local sections around a point $x \in M$ — local sections $s$ around $x$ define the same point $j^\infty(s)(x)$ in $J^\infty(E)$ if they have the same Taylor coefficients to all orders at $x$. A basis for the inverse limit topology on $J^\infty(E)$ consists of all sets $\widetilde{W} = (\pi^\infty_k)^{-1}(W)$ , where $W$ is any open set in $J^k(E)$ and $k = 0, 1, 2, \ldots$.

If $\sigma$ is a point in $J^\infty(E)$, it will be convenient to write

$$\sigma^k = \pi^\infty_k(\sigma)$$

for its projection into $J^k(E)$.

The notion of a smooth function on the infinite jet bundle must be defined. Let $P$ be any manifold and let $C^\infty(J^k(E), P)$ be the set of smooth maps from $J^k(E)$ to $P$. For $l \geq k$, there are the obvious connecting maps

$$\tilde{\pi}^l_k\colon C^\infty(J^k(E), P) \to C^\infty(J^l(E), P)$$

which define the direct sequence $\{C^\infty(J^k(E), P),\, \tilde{\pi}^l_k\}$. The set of smooth functions from $J^\infty(E)$ to $P$ is then defined to be the direct limit of this sequence and is denoted by $C^\infty(J^\infty(E), P)$. If $f \in C^\infty(J^\infty(E), P)$ then, by definition of the direct limit, $f$ must factor through a smooth map $\hat{f}$ from $J^k(E)$ to $P$ for some $k$, *i.e.*,

$$f = \hat{f} \circ \pi^\infty_k. \tag{1.3}$$

We call $k$ the order of $f$. If $f$ is of order $k$, then it also of any order greater than $k$. In particular, the projection maps $\pi^\infty_k$ are themselves smooth functions of order $k$. If $U$ is a coordinate chart on $E$ and the restriction of $f$ to $J^\infty(U)$ factors through $J^l(U)$ then we say that $f$ has order $l$ on $U$. We remark that if $W$ is an open set in $P$, then $f^{-1}(W) = (\pi^\infty_k)^{-1}(\hat{f}^{-1}(W))$ is an open set in $J^\infty(E)$. Therefore all smooth functions on $J^\infty(E)$ are continuous. This conclusion would be false had we enlarged the class of smooth functions to include those of locally finite order.

We let $C^\infty(J^\infty(E))$ denote the set of smooth, real-valued functions on $J^\infty(E)$. If $f$ is a smooth, real-valued function on $J^\infty(E)$ which is represented by a smooth function $\hat{f}$ on $J^k(E)$, then on each coordinate neighborhood $(\pi^\infty_E)^{-1}(U)$ and for each point $\sigma = j^\infty(s)(x) \in (\pi^\infty_E)^{-1}(U)$ with $k$-jet coordinates given by (1.2),

$$f(\sigma) = \hat{f}(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \cdots i_k}). \tag{1.4}$$

As a matter of notational convenience, we shall often use square brackets, for example, $f = f[x, u]$, to indicate that the function $f$ is a function on the infinite jet bundle over $U$. Here the order of $f$ is finite but unspecified. To indicate that a function $f$ is a function of order $k$, *i.e.*, a function on $J^k(U)$, we shall write $f = f[x, u^{(k)}]$.

Unless it is necessary to do so, we shall not distinguish between a function on $J^\infty(E)$ and its representatives on finite dimensional jet bundles.

A map $f\colon P \to J^\infty(E)$ is said to be smooth if for any manifold $Q$ and any smooth map $g\colon J^\infty(E) \to Q$, the composition $g \circ f$ from $P$ to $Q$ is a smooth map. Likewise, if $\rho\colon F \to N$ is another fibered manifold we declare that a map $\Phi\colon J^\infty(E) \to J^\infty(F)$ is smooth if for every smooth map $g\colon J^\infty(F) \to Q$ the composition $g \circ \Phi$ from $J^\infty(E)$ to $Q$ is smooth. The following proposition furnishes us with representations of these maps by smooth maps on finite jet bundles.

Proposition 1.1. *(i) If $f\colon P \to J^\infty(E)$ is a smooth map, then for each $k = 0, 1, 2, \ldots$ maps $f_k\colon P \to J^k(E)$ defined by*

$$f_k = \pi^\infty_k \circ f \tag{1.5}$$

*are smooth. Conversely, given a sequence of smooth maps $f_k\colon P \to J^k(E)$ such that $f_k = \pi^l_k \circ f_l$ for all $l \geq k$, there exists a unique smooth map $f$ from $P$ to $J^\infty(E)$ satisfying (1.5).*

*(ii) If $\Phi\colon J^\infty(E) \to J^\infty(F)$ is a smooth map, then for each $k = 0, 1, 2, \ldots$ there exists an integer $m_k$ with $m_l \geq m_k$ whenever $l \geq k$ and smooth maps*

$$\Phi^{m_k}_k\colon J^{m_k}(E) \to J^k(F) \tag{1.6}$$

*satisfying*

$$\rho^\infty_k \circ \Phi = \Phi^{m_k}_k \circ \pi^\infty_{m_k}. \tag{1.7}$$

*Conversely, given a sequence of smooth maps (1.6) such that*

$$\rho^l_k \circ \Phi^{m_l}_l = \Phi^{m_k}_k \circ \pi^{m_l}_{m_k} \tag{1.8}$$

*for all $l \geq k$, there exists a unique smooth map $\Phi$ from $J^\infty(E)$ to $J^\infty(F)$ satisfying (1.7).*

PROOF: These statements are direct consequences of the definitions. To prove (ii), observe that for each $k = 0, 1, 2, \ldots$ the map $\rho^\infty_k \circ \Phi$ from $J^\infty(E)$ to $J^k(F)$ is required to be smooth and therefore must factor through $J^{m_k}(E)$ for some $m_k$. With no loss in generality it can be assumed that the order $m_k$ of $\rho^\infty_k \circ \Phi$ is an increasing function of $k$. ∎

A smooth map $\Phi$ from $J^\infty(E)$ to $J^\infty(F)$ described by a sequence of maps (1.6) is said to be of type $(m_0, m_1, m_2, \ldots)$. We emphasize that a smooth map $\Phi$ from $J^\infty(E)$ to $J^\infty(F)$ need not factor through $J^k(E)$ for any $k$. Indeed, such a restriction would preclude the identity map on $J^\infty(E)$ from being smooth.

A map $\Phi : J^\infty(E) \to J^\infty(F)$ called projectable if it is covers maps from $J^k(E)$ to $J^k(F)$ for each $k$, *i.e.*,

$$\begin{array}{ccc} J^\infty(E) & \xrightarrow{\Phi} & J^\infty(F) \\ {\scriptstyle \pi^\infty_k}\big\downarrow & & \big\downarrow{\scriptstyle \rho^\infty_k} \\ J^k(E) & \xrightarrow{\Phi_k} & J^k(F). \end{array}$$

Such a map is of type $(0, 1, 2, \ldots)$.

Although the fibered manifold $\pi\colon E \to M$ may not admit any global sections, the bundle $\pi^\infty_E\colon J^\infty(E) \to E$ always admits global sections. These can be readily constructed using partitions of unity.

An important class of smooth maps from $J^\infty(E)$ to $J^\infty(F)$ are those which arise as the prolongation of maps from $E$ to $F$.

Definition 1.2. *Let $\phi$ be a map from $E$ to $F$ which covers a local diffeomorphism $\phi_0$. Then the infinite prolongation of $\phi$ is the map*

$$\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(F)$$

*defined by*

$$\operatorname{pr}\phi\bigl(j^\infty(s)(x)\bigr) = [j^\infty(\phi \circ s \circ \phi_0^{-1})]\bigl(\phi_0(x)\bigr), \tag{1.9}$$

*where $s$ is a local section of $E$ defined on a neighborhood of $x$ on which $\phi_0$ is a diffeomorphism.*

The prolongation of $\phi$ is a smooth, projectable map. Moreover, if $\phi$ is a diffeomorphism, then so is $\operatorname{pr}\phi$.

**B. Vector Fields and Generalized Vector Fields.** The tangent bundle to the infinite jet bundle $J^\infty(E)$ can be defined in various (equivalent) ways. One possibility is to consider the inverse system of tangent bundles $T(J^k(E))$ with the projections $(\pi^l_k)_*$ from $T(J^l(E))$ to $T(J^k(E))$ for all $l \geq k$ as connecting maps and to designate $T(J^\infty(E))$ as the inverse limit of these vector bundles. In this way $T(J^\infty(E))$ inherits the structure of a topological vector bundle over $J^\infty(E)$. Alternatively, the tangent space $T_\sigma(J^\infty(E))$ at a point $\sigma \in J^\infty(E)$ may be defined directly as the vector space of real-valued $\mathbf{R}$ linear derivations on $J^\infty(E)$. The tangent bundle $T(J^\infty(E))$ can then be constructed from the union of all individual tangent spaces $T_\sigma(J^\infty(E))$ in the usual fashion. These two approaches are equivalent. Indeed, a derivation $X_\sigma$ on $J^\infty(E)$ at the point $\sigma$ determines a sequence of derivations $X_{k,\sigma^k}$ to $T(J^k(E))$ at $\sigma^k = \pi^\infty_k(\sigma)$ — if $f$ is a smooth function on $J^k(E)$, then

$$X_{k,\sigma^k}(f) = X_\sigma(f \circ \pi^\infty_k). \tag{1.10}$$

These derivations satisfy

$$(\pi^l_k)_* X_{l,\sigma^l} = X_{k,\sigma^k} \tag{1.11}$$

for all $l \geq k$ and therefore define a tangent vector in the inverse limit space $T(J^\infty(E))$ at $\sigma$. Conversely, every sequence of vectors $X_{k,\sigma^k} \in T_{\sigma^k}(J^\infty(E))$ satisfying (1.11) defines a derivation $X_\sigma$ on $J^\infty(E)$ at $\sigma$ — if $f$ is a function on $J^\infty(E)$ which is represented by a function $\hat f$ on $J^k(E)$, then

$$X_\sigma(f) = X_{k,\sigma^k}(\hat f).$$

The projection property (1.11) ensures that this is a well-defined derivation, independent of the choice of representative $\hat f$ of $f$.

If $\Phi\colon J^\infty(E) \to J^\infty(F)$ is a smooth map, then the Jacobian

$$\Phi_*\colon T(J^\infty(E)) \to T(J^\infty(F))$$

is defined in the customary manner, *viz.*, if $X_\sigma$ is a tangent vector to $J^\infty(E)$ at the point $\sigma$, then for any smooth function $f$ on $J^\infty(F)$

$$(\Phi_* X_\sigma)(f) = X_\sigma(f \circ \Phi).$$

If $X_\sigma$ is represented by the sequence of vectors $X_k$ at $\sigma^k$ for $k = 0, 1, 2, \ldots$ and $\Phi$ is represented, in accordance with Proposition 1.1, by functions $\Phi_k^{m_k}$ then $\Phi_*(X_\sigma)$ is represented by the sequence of vectors $(\Phi_k^{m_k})_*(X_{m_k})$.

A vector field $X$ on $J^\infty(E)$ is defined to be a $C^\infty(J^\infty(E))$ valued, $\mathbf{R}$-linear derivation on $C^\infty(J^\infty(E))$. Thus, for any real-valued function $f$ on $J^\infty(E)$, $X(f)$ is a smooth function on $J^\infty(E)$ and must therefore be of some finite order. Although the order of the function $X(f)$ may exceed that of $f$, the order of $X(f)$ is nevertheless bounded for all functions $f$ of a given order.

PROPOSITION 1.3. *Let $X$ be a vector field on $J^\infty(E)$. Then for each $k = 0, 1, 2, \ldots$, there exists an integer $m_k$ such that for all functions $f$ of order $k$, the order of $X(f)$ does not exceed $m_k$.*

PROOF: The case $k = 0$ and $E$ compact is easily treated. For $k = 0$ and $E$ non-compact or for $k > 0$, we argue by contradiction. First, pick a sequence of points $p_i$, $i = 1, 2, 3, \ldots$ in $J^k(E)$ with no accumulation points. Let $U_i$ be a collection of disjoint open sets in $J^k(E)$ containing $p_i$. Let $\phi_i$ be smooth functions on $J^k(E)$ which are 1 on a neighborhood of $p_i$ and have support inside of $U_i$.

Now suppose, contrary to the conclusion of the proposition, that there are functions $f_i$ on $J^k(E)$ for $i = 1, 2, 3, \ldots$ such that the order of $X(f_i)$ exceeds $i$. We can assume that the order of $X(f_i)$ exceeds $i$ in a neighborhood of a point $\tilde{p}_i$, where $\tilde{p}_i \in (\pi_k^\infty)^{-1}(p_i)$. If this is not the case, if the maximum order of $X(f_i)$ is realized about a point $\tilde{q}_i \notin (\pi_k^\infty)^{-1}(p_i)$, then we can simply redefine $f_i$ to be the composition of $f_i$ with any diffeomorphism of $J^k(E)$ which carries $p_i$ to the point $q_i = \pi_k^\infty(\tilde{q}_i)$.

Define $f = \sum_i \phi_i f_i$. Then $f$ is a smooth function on $J^k(E)$ but $X(f)$ is not a smooth function on $J^\infty(E)$ since it is not of global finite order. This contradiction proves the lemma. ∎

We say that a vector field $X$ on $J^\infty(E)$ is of type $(m_0, m_1, m_2, \ldots)$ if for all functions $f$ of order $k$ the order of $X(f)$ is $m_k$. With no loss in generality, we shall suppose the sequence $m_k$ increases with $k$. A vector field on $J^\infty(E)$ is projectable if it projects under $\pi_k^\infty$ to a vector field on $J^k(E)$ for each $k$. Projectable vector fields are of type $(0, 1, 2, \dots)$.

With respect to our induced local coordinates on $J^\infty(U)$, a vector field $X$ takes the form

$$X = a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{p=1}^{\infty} \Big[ \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq n} b^\alpha_{i_1 i_2 \cdots i_p} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_p}} \Big]. \tag{1.12}$$

The components $a^i, b^\alpha$ and $b^\alpha_{i_1 i_2 \cdots i_p}$ are all smooth functions on $J^\infty(U)$. If $f$ is a smooth function on $J^\infty(U)$, then $f$ is of finite order and so $X(f)$ involves only finitely many terms from (1.12). The vector field $X$ is projectable if the $a^i$ and $b^\alpha$ are smooth functions on $U$ and the $b^\alpha_{i_1 i_2 \cdots i_k}$ are smooth functions on $J^k(U)$, $k = 1, 2, \dots$.

The sets of sections of $T(J^k(E))$ for $k = 0, 1, 2, \dots$ do not constitute an inverse system (since it is not possible to project an arbitrary vector field on $J^l(E)$ to one on $J^k(E)$ for $k < l$) and, for this reason, it is not possible to represent a given vector field on the infinite jet bundle by a sequence of vector fields on finite dimensional jet bundles. To circumvent this problem we introduce the notion of generalized vector fields. Generalized vector fields first appeared as generalized or higher order symmetries of the KdV equation. They play a central role in both the theory and applications of the variational bicomplex. First recall that if $P$ and $Q$ are finite dimensional manifolds and $\phi\colon P \to Q$ is a smooth map, then a vector field along $\phi$ is a smooth map $Z\colon P \to T(Q)$ such that for all $p \in P$, $Z(p)$ is a tangent vector to $Q$ at the point $\phi(p)$.

Definition 1.4. *A generalized vector field $Z$ on $J^k(E)$ is a vector field along the projection $\pi_k^\infty$, i.e., $Z$ is a smooth map*

$$Z\colon J^\infty(E) \to T(J^k(E))$$

*such that for all $\sigma \in J^\infty(E)$, $Z(\sigma) \in T_{\sigma^k}(J^k(E))$.*

*Similarly, a generalized vector field $Z$ on $M$ is a vector field along the projection $\pi_M^\infty$, i.e., $Z$ is a smooth map*

$$Z\colon J^\infty(E) \to T(M)$$

*such that for all $\sigma = j^\infty(s)(x)$, $Z(\sigma) \in T_x(M)$.*

Since a generalized vector field $Z$ on $J^k(E)$ is a smooth map from the infinite jet bundle to a finite dimensional manifold, it must factor through $J^m(E)$ for some

$m \geq k$. Thus there is a vector field $\hat{Z}$ along $\pi^m_k$, *i.e.*, a map

$$\hat{Z}\colon J^m(E) \to T(J^k(E))$$

such that

$$Z = \hat{Z} \circ \pi^\infty_m.$$

We call $m$ the order of the generalized vector field $Z$. If $f$ is a function on $J^k(E)$, then $Z(f)$ is the smooth function on $J^\infty(E)$ defined by

$$Z(f)(\sigma) = \hat{Z}(\sigma^m)(f).$$

The order of the function $Z(f)$ is $m$. Note that a generalized vector field on $J^k(E)$ of order $k$ is simply a vector field on $J^k(E)$.

Generalized vector fields are projectable. If $Z$ is a generalized vector field on $J^l(E)$, then for $k \leq l$ the map $\overline{Z}\colon J^\infty(E) \to T(J^k(E))$ defined by

$$\overline{Z}(\sigma) = (\pi^l_k)_*(\sigma^l)[Z(\sigma)]$$

is a generalized vector field on $J^k(E)$. We write $(\pi^l_k)_*(Z)$ for $\overline{Z}$.

Proposition 1.5. *Let $X$ be a vector field on $J^\infty(E)$ of type $(m_0, m_1, m_2, \dots)$. Then there exist generalized vector fields $X_k$ on $J^k(E)$ of order $m_k$ such that*

$$(\pi^l_k)_*(X_l) = X_k \tag{1.13}$$

*and, for all functions $f$ of order $k$,*

$$X(f)(\sigma) = X_k(\sigma)(f). \tag{1.14}$$

*Conversely, given a sequence of generalized vector fields $X_k$ on $J^k(E)$ satisfying (1.13), there exists a unique vector field $X$ on $J^\infty(E)$ satisfying (1.14).*

Proof: Given $X$, simply define the generalized vector fields $X_k$ by $X_k(\sigma) = (\pi^\infty_k)_*(\sigma)(X_\sigma)$. ∎

We remark that if the vector field $X$ on $J^\infty(E)$ is given locally by (1.12), then the associated generalized vector fields $X_k$ on $J^k(E)$ are given by truncating the infinite sum on $p$ in (1.12) at $p = k$.

The Lie bracket of two vector fields $X$ and $Y$ on $J^\infty(E)$ is the vector field $[X, Y]$ given by

$$[X, Y](f) = X\bigl(Y(f)\bigr) - Y\bigl(X(f)\bigr).$$

If $X$ is of type $(m_0, m_1, m_2, \dots)$ and $Y$ is of type $(n_0, n_1, n_2, \dots)$, then $[X, Y]$ is of type $(t_0, t_1, t_2, \dots)$, where $t_k = \max\{m_{n_k}, n_{m_k}\}$. In fact, the projection of $[X, Y]$ to $J^k(E)$ is the generalized vector field $[X, Y]_k$ on $J^k(E)$ given by

$$[X, Y]_k(\sigma)(f) = X_{n_k}(\sigma)\bigl(Y_k(f)\bigr) - Y_{m_k}(\sigma)\bigl(X_k(f)\bigr),$$

where $f$ is any function on $J^k(E)$. If $X$ and $Y$ are projectable, then $[X, Y]$ is projectable and $[X, Y]_k = [X_k, Y_k]$.

This is a convenient point at which to fix our multi-index notation. For arbitrary, unordered values of the $k$-tuple $I = i_1 i_2 \cdots i_k$, let $\{i_1 i_2 \cdots i_k\}$ denote the rearrangement of these indices in non-decreasing order. Set

$$u^\alpha_I = u^\alpha_{i_1 i_2 \cdots i_k} = u^\alpha_{\{i_1 i_2 \cdots k\}}.$$

With this convention $u^\alpha_I$ is symmetric in the individual indices $i_1 i_2 \cdots i_k$ which make up the multi-index $I$. The length $k$ of the multi-index $I$ is denoted by $|I|$. Next, let $l_j$ be the number of occurrences of the integer $j$ amongst the $i_1 i_2 \cdots i_k$ and define the symmetric partial derivative operator $\partial^I_\alpha$ by

$$\partial^I_\alpha = \partial^{i_1 i_2 \cdots i_k}_\alpha = \frac{l_1!\, l_2! \cdots l_n!}{k!} \frac{\partial}{\partial u^\alpha_{\{i_1 i_2 \cdots i_k\}}}. \tag{1.15}$$

For $J = j_1 j_2 \cdots j_k$ it is easily seen that

$$\partial^I_\alpha(u^\beta_J) = \delta^\beta_\alpha \delta^I_J,$$

where

$$\delta^I_J = \delta^{(i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_k)}_{j_k}$$

and $(i_1 i_2 \ldots i_k)$ denotes symmetrization on the enclosed indices, *e.g.*,

$$\delta^{(i}_j \delta^{h)}_k = \frac{1}{2}[\delta^i_j \delta^h_k + \delta^h_j \delta^i_k].$$

Thus we have that

$$\frac{\partial u_{xx}}{\partial u_{xx}} = 1 \qquad \text{and} \qquad \frac{\partial u_{xy}}{\partial u_{xy}} = 1$$

whereas

$$\partial^{xx}_u u_{xx} = 1 \qquad \text{and} \qquad \partial^{xy}_u u_{xy} = \frac{1}{2}.$$

As an illustration of this notation, suppose that $f$ is a smooth function on $J^\infty(U)$ which is homogeneous in the fiber variables of degree p, *i.e.*, for all $\lambda \geq 0$

$$f(x^i, \lambda u^\alpha, \lambda u^\alpha_i, \lambda u^\alpha_{ij}, \ldots) = \lambda^p f[x, u].$$

Then Euler's equation for homogenous functions can be expressed in terms of these symmetrized partial derivatives as

$$\sum_{|I|=0}^{\infty} (\partial_\alpha^I f)\, u_I^\alpha = pf[x,u],$$

where, as usual, the summation convention applies to each individual index $i_1 i_2 \dots i_k$ in the repeated multi-index $I$. The numerical factors introduced in (1.15) compensate precisely for all the repeated terms that occur in these sums.

This multi-index notation differs from that which is commonly encountered in the literature but it is well suited to our purposes. It ensures that the operator $\partial_\alpha^I$ is completely symmetric in its upper indices $i_1 i_2 \cdots i_k$ and this, in turn, simplifies many of the formulas in our variational calculus. It eliminates the need to order the sums which occur in formulas such as (1.12). Indeed, we can now rewrite this equation as

$$X = a^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} b_I^\alpha \partial_\alpha^I,$$

where the coefficients $b_I^\alpha$ are taken to be symmetric in the indices of $I$. Finally, this notation eliminates the presence of many unwieldy multi-nomial coefficients which would otherwise explicitly occur. Note that, with these conventions in effect,

$$X(u_J^\beta) = b_J^\beta.$$

In general, our multi-index notation is restricted to quantities which are totally symmetric in the individual indices represented by the multi-indices. For example, if $I = rst$ and $J = hk$, then

$$a^{IJ} = a^{rst\,hk}$$

may be assumed to be symmetric in the indices $rst$ and $hk$. The one exception to this rule arises when we write

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} = dx^I.$$

In this case the multi-index $I$ is skew-symmetric in the indices $i_1 i_2 \dots i_k$.

**C. Differential Forms and the Variational Bicomplex.** The $p^{\text{th}}$ exterior product bundles $\Lambda^p(J^k(E))$ together with the pullback maps

$$(\pi_k^l)^* \colon \Lambda^p(J^k(E)) \to \Lambda^p((J^l(E)),$$

defined for all $l \geq k \geq 0$, form a direct system of vector bundles whose direct limit is designated as the $p^{\text{th}}$ exterior product bundle $\Lambda^p(J^\infty(E))$ of $J^\infty(E)$. Let $\sigma \in J^\infty(E)$. Then each $\omega \in \Lambda^p_\sigma(J^\infty(E))$ admits a representative $\hat{\omega} \in \Lambda^p_{\sigma^k}(J^k(E))$ for some $k = 0,\, 1,\, 2,\, \ldots$ and $\omega = (\pi^\infty_k)^*\hat{\omega}$. We call $k$ the order of $\omega$. If $X^1, X^2, \ldots, X^p$ are tangent vectors to $J^\infty(E)$ at $\sigma$ then, by definition,

$$\omega(X^1, X^2, \ldots, X^p) = \hat{\omega}\big((\pi^\infty_k)_*X^1, (\pi^\infty_k)_*X^2, \ldots, (\pi^\infty_k)_*X^p\big).$$

Observe that this is well-defined, that is independent of the choice of representative $\hat{\omega}$ of $\omega$. Evidently, if $\omega$ is of order $k$ and one of the vector fields $X^1, X^2, \ldots, X^p$ is $\pi^\infty_k$ vertical, then $\omega(X^1, X^2, \ldots, X^p) = 0$.

A section of $\Lambda^p(J^k(E))$ is a differential $p$-form on $J^k(E)$. We denote the vector space of all differential forms on $J^k(E)$ by $\Omega^p(J^k(E))$. These spaces of differential $p$ forms also constitute a direct limit system whose direct limit is the vector space of all differential $p$ forms on $J^\infty(E)$ and is denoted by $\Omega^p(J^\infty(E))$. Again, every smooth differential $p$ form $\omega$ on $J^\infty(E)$ is represented by a $p$ form $\hat{\omega}$ on $J^k(E)$ for some $k$. In local coordinates $(x, u, U)$ a $p$-form $\omega$ on $J^\infty(U)$ is therefore a *finite* sum of terms of the type

$$A[x,u]\, du^{\alpha_1}_{I_1} \wedge du^{\alpha_2}_{I_2} \wedge \cdots \wedge du^{\alpha_a}_{I_a} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_b}, \tag{1.16}$$

where $a + b = p$ and where the coefficient $A$ is a smooth function on $J^\infty(U)$. The order of the term (1.16) is the maximum of the orders of the coefficient function $A[x, u]$ the differentials $du^\alpha_{I_l}$. For example, the form $u_{xx} du_x \wedge dx$ is of order 2 and $u_x du_{xxx} \wedge dx$ is of order 3.

If $\hat{\omega}$ is a $p$-form on $J^k(E)$ and $X^1, X^2, \ldots, X^p$ are generalized vector fields on $J^k(E)$ of type $m_1, m_2, \ldots, m_p$ respectively, then the function $\hat{\omega}(X^1, X^2, \ldots, X^p)$ is a smooth function on $J^\infty(E)$ the order of which is equal to the maximum of $m_1, m_2, \ldots, m_p$. If $\omega$ is a differential form on $J^\infty(E)$ which is represented by a form $\hat{\omega}$ on $J^k(E)$ and $X^1, X^2, \ldots, X^p$ are vector fields on $J^\infty(E)$ represented by sequences of generalized vector fields $\{X^1_l\}, \{X^2_l\}, \ldots, \{X^p_l\}$ for $l = 0,\, 1,\, 2,\, \ldots,$ then

$$\omega(X^1, X^2, \ldots, X^p) = \hat{\omega}(X^1_k, X^2_k, \ldots, X^p_k).$$

With these definitions in hand, much of the standard calculus of differential forms on finite dimensional manifolds readily extends to the infinite jet bundle. Let $\omega$ be a differential $p$ form on $J^\infty(E)$ which is represented by the form $\hat{\omega}$ on $J^k(E)$. If $X$ is a vector field of type $(m_0, m_1, m_2, \ldots)$ on $J^\infty(E)$ which is represented by the sequence of generalized vector fields $X_k$ on $J^k(E)$, then $X \mathbin{\lrcorner} \omega$ is the $p - 1$ form on $J^\infty(E)$

represented by the form $X_k \mathbin{\lrcorner} (\pi_k^{m_k})^*(\omega)$ on $J^{m_k}(E)$. Hence, $X \mathbin{\lrcorner} \omega$ is a differential form of order $m_k$. If $\Phi\colon J^\infty(E) \to J^\infty(F)$ is a smooth map represented by the sequence of maps $\Phi_k^{m_k}\colon J^{m_k}(E) \to J^k(F)$ and $\omega$ is a form on $J^\infty(F)$ represented by a form $\hat\omega$ on $J^k(F)$, then the pullback form $\Phi^*(\omega)$ is represented by the form $(\Phi_k^{m_k})^*(\hat\omega)$ of order $m_k$ Exterior differentiation

$$d\colon \Omega^p(J^\infty(E)) \to \Omega^{p+1}(J^\infty(E))$$

is similarly defined via representatives — if $\omega$ is a $p$ form on $J^\infty(E)$ represented by $\hat\omega$ on $J^k(E)$, then $d\,\omega$ is the $p+1$ form on $J^\infty(E)$ represented by $d\,\hat\omega$. In local coordinates, the differential $df$ of a function of order $k$ is given by

$$\begin{aligned} d\,f &= \frac{\partial f}{\partial x^i}\,dx^i + (\partial_\alpha f)\,du^\alpha + (\partial^{\,i}_\alpha f)\,du^\alpha_i + \cdots + (\partial^{\,i_1 i_2\ldots i_k}_\alpha f)\,du^\alpha_{i_1 i_2\ldots i_k} \\ &= \frac{\partial f}{\partial x^i}\,dx^i + \sum_{|I|=0}^{k} (\partial^{\,I}_\alpha f)\,du^\alpha_I. \end{aligned} \tag{1.17}$$

When the order of $f$ is unspecified, we simply extend the summation in (1.17) from $|I| = k$ to $|I| = \infty$ and bear in mind that sum is indeed a finite one.

Let $X$ and $Y$ be vector fields on $J^\infty(E)$ and suppose that $\omega$ is a one form. It follows from the above definitions and the invariant definition of the exterior derivative $d$ on finite dimensional manifolds, that

$$\begin{gathered} (d\,f)(X) = X(f), \\ (d\,\omega)(X,Y) = X\bigl(\omega(Y)\bigr) - Y\bigl(\omega(X)\bigr) - \omega\bigl([X,Y]\bigr), \end{gathered}$$

and so on.

Lie differentiation of differential forms on the infinite jet bundle is exceptional in this regard. This is due to the fact that for an arbitrary vector field $X$ on $J^\infty(E)$, there is no general existence theorem for the integral curves of $X$ and hence even the short-time flow of $X$ may not be defined. However, when $X$ is a projectable vector field on $J^\infty(E)$, then the flow of each projection $X_k$ is a well-defined local diffeomorphism $\phi_k(t)$ on $J^k(E)$ for each $k$. If $\omega$ is represented by the form $\hat\omega$ on $J^k(E)$, define

$$[\mathcal{L}_X(\omega)](\sigma) = [\mathcal{L}_{X_k}\hat\omega](\sigma^k) = \Bigl[\frac{d}{dt}[(\phi_k(t))^*(\hat\omega)](\sigma)\ \Bigr]\Big|_{t=0}.$$

From this definition, it can be proved that for vector fields $X_1$, $X_2$, ..., $X_p$,

$$\mathcal{L}_X\,\omega(X_1,X_2,\dots,X_p) \tag{1.18}$$
$$= X\big(\omega(X_1,X_2,\dots,X_p)\big) + \sum_{i=1}^{p}(-1)^{i+1}\omega([X,X_i],X_1,\dots,\widehat{X}_i,\dots,X_p).$$

For a non-projectable vector field $X$ the right-hand side of this equation is still a well-defined derivation on $\Omega^p(J^\infty(E))$ and so, for such vector fields, we simply adopt (1.18) as the definition of Lie differentiation.

From (1.18) and the previous formula for the exterior derivative $d$, it follows in the customary manner that

$$\mathcal{L}_X\omega = d\,(X \lrcorner\, \omega) + X \lrcorner\, d\,\omega. \tag{1.19}$$

Henceforth we shall not, as a general rule, distinguish between a differential form on $J^\infty(E)$ and its representatives on finite dimensional jet bundles.

Now let $\Omega^*(J^\infty(E))$ be the full exterior algebra of differential forms on $J^\infty(E)$. The contact ideal $\mathcal{C}(J^\infty(E))$ is the ideal in $\Omega^*(J^\infty(E))$ of forms $\omega$ such that for all $\sigma\in J^\infty(E)$ and local sections $s$ of $E$ around $\sigma^0=\pi^\infty_E(\sigma)$,

$$[j^\infty(s)]^*(x)\omega(\sigma)=0.$$

If $\omega\in\mathcal{C}$, then $d\,\omega\in\mathcal{C}$ so that $\mathcal{C}$ is actually a differential ideal.

A local basis for $\mathcal{C}$ is provided by the contact one forms

$$\theta^\alpha_I = du^\alpha_I - u^\alpha_{Ij}\,dx^j,$$

where $|I|=0,\ 1,\ 2,\ \dots$ . We call $|I|$ the order of the contact form $\theta^\alpha_I$ even though this form is defined on the $(|I|+1)$-st jet bundle over $U$. For example, with respect to the coordinates $(x,y,u)$ on $\mathbf{R}^3\to\mathbf{R}^2$ the contact one forms of order zero and one are $\theta = du - u_x dx - u_y dy$ and

$$\theta_x = du_x - u_{xx}dx - u_{xy}dy \quad\text{and}\quad \theta_y = du_y - u_{xy}dx - u_{yy}dy.$$

If $\pi\colon U\to U_0$ is a local coordinate neighborhood for $E$ and $\Xi\colon U_0\to J^\infty(U)$ satisfies

$$\Xi^*(\omega)=0$$

for all $\omega\in\mathcal{C}$, then there exists a local section $s: U_0\to U$ such that

$$\Xi(x)=j^\infty(s)(x)$$

for all $x\in U_0$.

PROPOSITION 1.6. *Let $\pi\colon E \to M$ and $\rho\colon F \to N$ be two fibered manifolds and let $\phi\colon E \to F$ be a smooth map which covers a local diffeomorphism $\phi_0\colon M \to N$.*

*(i) The prolongation of $\phi$, $\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(F)$ preserves the ideal of contact forms, i.e.,*

$$[\operatorname{pr}\phi]^*\mathcal{C}(J^\infty(F)) \subset \mathcal{C}(J^\infty(E)).$$

*(ii) Let $\Phi\colon J^\infty(E) \to J^\infty(F)$ be a smooth map which covers $\phi$. If $\Phi$ preserves the contact ideal, then $\Phi = \operatorname{pr}\phi$.*

PROOF: To prove (i), let $\omega \in \mathcal{C}(J^\infty(F))$ and let $\eta = [\operatorname{pr}\phi]^*(\omega)$. We show that $\eta \in \mathcal{C}(J^\infty(E))$. Let $\sigma = j^\infty(s)(x)$ be a point in $J^\infty(E)$, where $s$ is a local section of $E$ around $x$ and let $\tilde{s} = \phi \circ s \circ \phi_0^{-1}$ be the induced local section of $F$ around the point $y = \phi_0(x)$. Let $\tilde{\sigma} = j^\infty(\tilde{s})(y)$. The definition (1.9) of $\operatorname{pr}\phi$ implies that

$$\operatorname{pr}\phi \circ j^\infty(s) = j^\infty(\tilde{s}) \circ \phi_0.$$

The chain rule now gives

$$\begin{aligned}
[j^\infty(s)]^*(x)\,\eta(\sigma) &= [j^\infty(s)]^*(x)\,\big([\operatorname{pr}\phi]^*(\sigma)\,\omega(\tilde{\sigma})\big)\\
&= [\operatorname{pr}\phi \circ j^\infty(s)]^*(x)\,[\omega(\tilde{\sigma})]\\
&= [j^\infty(\tilde{s}) \circ \phi_0]^*(x)\,[\omega(\tilde{\sigma})]\\
&= \phi_0^*(x)\,\big([j^\infty(\tilde{s})]^*(y)\,\omega(\tilde{\sigma})\big).
\end{aligned}$$

This last expression vanishes since $\omega$ lies in the contact ideal of $J^\infty(F)$. Therefore $\eta$ belongs to the contact ideal of $J^\infty(E)$.

To prove (ii), let $\pi\colon U \to U_0$ and $\rho\colon V \to V_0$ be coordinate neighborhoods on $E$ and $F$ such that $\phi_0\colon U_0 \to V_0$ is a diffeomorphism. Let $s : U_0 \to U$ be any local section and let $\Xi\colon V_0 \to J^\infty(V)$ be defined by

$$\Xi(y) = (\Phi \circ j^\infty(s) \circ \phi_0^{-1})(y).$$

Because $\Phi$ is assumed to preserve the contact ideal, $\Xi^*(\omega) = 0$ for any $\omega \in \mathcal{C}(J^\infty(F))$. This implies that there is a section $\bar{s}\colon V_0 \to V$ such that $\Xi(y) = j^\infty(\bar{s})(y)$ for all $y \in V_0$, *i.e.*,

$$\Phi \circ j^\infty(s) \circ \phi_0^{-1} = j^\infty(\bar{s}).$$

Since $\Phi$ covers $\phi$, it follows immediately that $\bar{s} = \phi \circ s \circ \phi_0^{-1}$ and hence $\Phi = \operatorname{pr}\phi$, as required. ∎

One forms in $\mathcal{C}$ are said to be vertical one forms on $J^\infty(E)$. More generally, let $\mathcal{C}^s$ be the $s^{\text{th}}$ wedge product of $\mathcal{C}$ in $\Omega^*$, *i.e.*, $\omega \in \mathcal{C}^s$ if and only if it is a sum of terms of the form

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_s \wedge \eta,$$

where each $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathcal{C}$ and $\eta \in \Omega^*$. Set, for $s = 0, 1, \ldots, p+1$,

$$\Omega^{s,p}_V = \mathcal{C}^s \cap \Omega^p.$$

Then every form in $\Omega^{s,p}_V$ consists of terms containing at least $s$ contact one forms. There are clearly inclusions

$$\Omega^p = \Omega^{0,p}_V \supset \Omega^{1,p}_V \supset \cdots \supset \Omega^{p,p}_V \supset \Omega^{p+1,p}_V = 0$$

and, because $\mathcal{C}$ is a differential ideal,

$$d\,\Omega^{s,p}_V \subset \Omega^{s,p+1}_V. \tag{1.20}$$

Let $\omega$ be a $p$ form on $J^\infty(E)$. Then $\omega$ is said to be horizontal if at each point $\sigma \in J^\infty(E)$ and for each $\pi^\infty_M$ vertical tangent vector $Y \in T_\sigma(J^\infty(E))$, *i.e.*, $(\pi^\infty_M)_*(Y) = 0$,

$$Y \lrcorner\, \omega(\sigma) = 0.$$

More generally, fix $0 \leq r \leq p+1$ and let let $s = p - r$. Define $\Omega^{r,p}_H$ to be the subspace of $\Omega^p$ of forms $\omega$ such that for all points $\sigma \in J^\infty(E)$

$$\omega(X_1, X_2, \ldots, X_p) = 0,$$

whenever at least $s+1$ of the tangent vectors to $J^\infty(E)$ at $\sigma$ are $\pi^\infty_M$ vertical. A local section of $\Omega^{r,p}_H$ is a sum of terms of the form (1.16) provided $b \geq r$ — *i.e.*, there are at least r horizontal differentials $dx$ in each term of $\omega$. There are the obvious inclusions

$$\Omega^p = \Omega^{0,p}_H \supset \Omega^{1,p}_H \supset \cdots \supset \Omega^{p,p}_H \supset \Omega^{p+1,p}_H = 0$$

and

$$d\,\Omega^{r,p}_H \subset \Omega^{r,p+1}_H. \tag{1.21}$$

Definition 1.7. *The space $\Omega^{r,s}(J^\infty(E))$ of type $(r,s)$ differential forms on $J^\infty(E)$ is defined to be the intersection*

$$\Omega^{r,s}(J^\infty(E)) = \Omega^{r,p}_H(J^\infty(E)) \cap \Omega^{s,p}_V(J^\infty(E)),$$

*where $r+s=p$. The horizontal degree of a form in $\Omega^{r,s}$ is $r$ and the vertical degree is $s$.*

A $p$ form belongs to $\Omega^{r,s}(J^\infty(E))$ if and only if it is locally a sum of terms of the form

$$A[x,u]\,\theta^{\alpha_1}_{I_1}\wedge\theta^{\alpha_2}_{I_2}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}.$$

For example, if $\alpha$, $\beta$ and $\gamma$ are forms of order $k$ and type $(r,0)$, $(r,1)$ and $(r,2)$ respectively then with respect to our local coordinates on $J^\infty(U)$,

$$\alpha = A_{j_1j_2\cdots j_r}[x,u^{(k)}]\,dx^{j_1}\wedge d^{j_2}\cdots\wedge dx^{j_r},$$

$$\beta = \sum_{|I|=0}^{k} A^{I}_{\alpha\, j_1j_2\cdots j_r}[x,u^{(k)}]\,\theta^\alpha_I\wedge dx^{j_1}\wedge dx^{j_2}\cdots\wedge dx^{j_r},$$

$$\gamma = \sum_{|I|,|J|=0}^{k} A^{IJ}_{\alpha\beta\, j_1j_2\cdots j_r}[x,u^{(k)}]\,\theta^\alpha_I\wedge\theta^\beta_J\wedge dx^{j_1}\wedge dx^{j_2}\cdots\wedge d^{j_r}.$$

In particular, a form $\lambda\in\Omega^{n,0}(J^\infty(U))$ assumes the form

$$\lambda = L[x,u]\,\nu,$$

where $\nu = dx^1\wedge dx^2\wedge\cdots dx^n$. Hence each element of $\Omega^{n,0}(J^\infty(E))$ defines a Lagrangian for a variational problem on $E$. The fundamental integral or action for this variational problem is the functional $I[s]$, defined on compactly supported sections $s$ of $E$ by

$$I[s] = \int_M [j^\infty(s)]^*(\lambda).$$

Observe that

$$\Omega^{r,s} = 0 \qquad \text{if} \qquad r\geq n.$$

Let $\Omega^r_M$ denote the space of $r$ forms on $M$. The projection $\pi^\infty_M\colon J^\infty(E)\to M$ induces inclusions

$$(\pi^\infty_M)^*\colon \Omega^r_M\to\Omega^{r,0}.$$

It follows easily from the definitions that

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p}\Omega^{r,s}(J^\infty(E)).$$

We denote the projection map to each summand by

$$\pi^{r,s}\colon \Omega^p(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E)). \tag{1.22}$$

If $\omega \in \Omega^p$ is given by a sum of terms of the form (1.16), then the projection $\pi^{r,s}(\omega)$ can be computed by first substituting

$$du_I^\alpha = \theta_I^\alpha + u_{Ij}^\alpha dx^j \tag{1.23}$$

and then collecting together those terms of horizontal degree $r$ and vertical degree $s$.

Owing to (1.20) and (1.21), it follows that the exterior derivative $d$ on $J^\infty(E)$, when restricted to $\Omega^{r,s}$, maps into $\Omega^{r+1,s} \oplus \Omega^{r,s+1}$. Thus $d$ splits into horizontal and vertical components

$$d = d_H + d_V,$$

where

$$d_H\colon \Omega^{r,s}(J^\infty(E)) \longrightarrow \Omega^{r+1,s}(J^\infty(E))$$

and

$$dV\colon \Omega^{r,s}(J^\infty(E)) \longrightarrow \Omega^{r,s+1}(J^\infty(E)).$$

Specifically, for functions $f$ on $J^\infty(U)$ of order $k$, the substitution of (1.23) into (1.17) leads to

$$\begin{aligned} d\,f &= \frac{\partial f}{\partial x^i}\,dx^i + (\partial_\alpha f)\,du^\alpha + (\partial^i_\alpha f)\,du^\alpha_i + \cdots \\ &= [\frac{\partial f}{\partial x^i} + (\partial_\alpha f)u^\alpha_i + (\partial^j_\alpha f)u^\alpha_{ij} + \cdots]\,dx^i + \\ &+ [(\partial_\alpha f)\,\theta^\alpha(\partial^i_\alpha f)\,\theta^\alpha_i + (\partial^{ij}_\alpha f)\,\theta^\alpha_{ij} + \cdots]. \end{aligned}$$

The terms in brackets are of type $(1,0)$ and $(0,1)$ respectively and therefore define $d_H f$ and $d_V f$, *i.e.*,

$$d_H f = [\frac{\partial f}{\partial x^i} + (\partial_\alpha f)u^\alpha_i + (\partial^j_\alpha f)u^\alpha_{ij} + \cdots]\,dx^i$$

and

$$\begin{aligned} d_V f &= (\partial_\alpha f)\,\theta^\alpha + (\partial^i_\alpha f)\,\theta^\alpha_i + (\partial^{ij}_\alpha f)\;\theta^\alpha_{ij} + \cdots \\ &= \sum_{|I|=0}^{k} (\partial^I_\alpha f)\,\theta^\alpha_I. \end{aligned}$$

The $dx^j$ component of the horizontal one form $d_H f$ is the total derivative of $f$ with respect to $x^j$ and is denoted by $D_j f$:

$$\begin{aligned} D_j f &= \frac{\partial f}{\partial x^j} + (\partial_\alpha f)u^\alpha_j + (\partial^h_\alpha f)u^\alpha_{hj} + (\partial^{hk}_\alpha f)u^\alpha_{hkj} + \cdots \\ &= \frac{\partial f}{\partial x^j} + \sum_{|I|=0}^{k} (\partial^I_\alpha f)u^\alpha_{Ij}. \end{aligned} \tag{1.24}$$

For example, on $\mathbf{R}^3 \to \mathbf{R}^2$ with coordinates $(x, y, u)$

$$D_x u^2 = 2uu_x \qquad \text{and} \qquad D_{xy}u^2 = 2u_xu_y + 2uu_{xy}.$$

Note that total differentiation agrees with ordinary partial differentiation on the jets of local sections $s$ of $E$, *i.e.*,

$$[(D_j f)](j^\infty(s)) = \frac{\partial}{\partial x^j}[f(j^\infty(s))].$$

Note also that if $f$ is a function of order $k$, then $D_j f$ is of order $k+1$.

Since $d\,(dx^i) = 0$, we have that

$$d_H(dx^i) = 0 \qquad \text{and} \qquad d_V(dx^i) = 0.$$

Likewise, because

$$d\,\theta^\alpha_I = -du^\alpha_{Ij} \wedge dx^j = -\theta^\alpha_{Ij} \wedge dx^j,$$

we can conclude that

$$d_H\theta^\alpha_I = -\theta^\alpha_{Ij} \wedge dx^j \qquad \text{and} \qquad d_V\theta^\alpha_I = 0.$$

Of course, $d^2 = 0$ implies that

$$d_H^2 = d_V^2 = 0 \qquad \text{and} \qquad d_Hd_V = -d_Vd_H.$$

These formulas, together with the formula for $d_H f$ show that if $\omega$ is of order $k$, then the order of $d_H\omega$ in general increases to order $k+1$.

DEFINITION 1.8. *The variational bicomplex for the fibered manifold $\pi : E \to M$ is the double complex $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty(E)$ of $E$:*

$$
\begin{array}{ccccccccccccc}
 & & & & \uparrow d_V & & & & & & & & \uparrow d_V \\
 & & 0 & \longrightarrow & \Omega^{0,3} & & & & & \cdots & & & \Omega^{n,3} \\
 & & & & \uparrow d_V & & \uparrow & & \uparrow & & \uparrow & & \uparrow d_V \\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \Omega^{2,2} & \xrightarrow{d_H} \cdots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V \\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \Omega^{2,1} & \xrightarrow{d_H} \cdots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \Omega^{2,0} & \xrightarrow{d_H} \cdots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0}
\end{array}
\tag{1.25}
$$

To (1.25) we append the de Rham complex of $M$, *viz.*,

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \Omega^{2,0} & \xrightarrow{d_H} \cdots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} \\
 & & & & \uparrow (\pi^\infty_M)^* & & \uparrow (\pi^\infty_M)^* & & \uparrow (\pi^\infty_M)^* & & \uparrow (\pi^\infty_M)^* & & \uparrow (\pi^\infty_M)^* \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^0_M & \xrightarrow{d} & \Omega^1_M & \xrightarrow{d} & \Omega^2_M & \xrightarrow{d} \cdots & \Omega^{n-1}_M & \xrightarrow{d} & \Omega^n_M
\end{array}
$$

However, in general, this will not be done explicitly.

We record the following elementary fact.

PROPOSITION 1.9. *Let $\omega \in \Omega^{r,0}(J^\infty(E))$. Then $d_V\omega = 0$ if and only if $\omega$ is the pullback, by $\pi^\infty_M$, of an $r$ form on $M$.*

PROOF: It suffices to prove this proposition locally since the global result then follows by an elementary partition of unity argument. Let $U$ be a coordinate chart on $E$ and let

$$\omega = A_J[x, u^{(k)}]\, dx^J$$

on $J^\infty(U)$. Then

$$d_V\omega = \sum_{|I|=0}^{k} (\partial_\alpha^I A_J)\, \theta_I^\alpha \wedge dx^J$$

and this vanishes if and only if all the coefficients $A_J$ of $\omega$ are functions of the base variables $x^i$ alone ∎

All forms in $\Omega^{n,s}$ are obviously $d_H$ closed but they are not, in general, $d_H$ exact (even locally). One can then introduce the cohomology vector spaces

$$E_1^{n,s}(J^\infty(E)) = \Omega^{n,s}(J^\infty(E)) / \, d_H\{\Omega^{n-1,s}(J^\infty(E))\}.$$

These spaces are part of the so-called $E_1$ term of the spectral sequence for the variational bicomplex (1.25) and they play a central role in development of the subject. A slightly different approach will be adopted here. In the next chapter we shall introduce the *interior Euler operator*

$$I\colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E)) \qquad \text{for } s \geq 1$$

which is defined in local coordinates by

$$I(\omega) = \frac{1}{s}\, \theta^\alpha \wedge [\sum_{|I|=0}^{\infty} (-D)_I [\partial_\alpha^I \lrcorner\, \omega].$$

This operator satisfies $I \circ d_H = 0$ and is a projection operator in the sense that $I^2 = I$. We set

$$\mathcal{F}^s(J^\infty(E)) = I(\Omega^{n,s}(J^\infty(E))) = \{\, \omega \in \Omega^{n,s} \mid I(\omega) = \omega \,\}.$$

For instance, it is not difficult to see that $\mathcal{F}^1(J^\infty(E))$ consists of those type $(n,1)$ which are locally of the form

$$\omega = P_\alpha[x,u]\, \theta^\alpha \wedge \nu.$$

For reasons to presented in Chapter 3, we call $\mathcal{F}^s(J^\infty(E))$ the space of type $s$ *functional forms* on $J^\infty(E)$. The induced vertical differential

$$\delta_V\colon \mathcal{F}^s(J^\infty(E)) \to \mathcal{F}^{s+1}(J^\infty(E))$$

is defined by $\delta_V = I \circ d_V$ Thus, with the interior Euler operator $I$ in hand, we can construct the *augmented variational bicomplex* for the fibered manifold $E$:

$$
\begin{tikzcd}
 & & {} & & & & {} & {} & \\
 & 0 \arrow[r] & \Omega^{0,3} \arrow[u, "d_V"'] & & \cdots & & \Omega^{n,3} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^3 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,2} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,2} \arrow[r, "d_H"] & \cdots & \Omega^{n-1,2} \arrow[r, "d_H"] & \Omega^{n,2} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^2 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{n,1} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^1 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{R} \arrow[r] & \Omega^{0,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{n,0} \arrow[u, "d_V"'] & &
\end{tikzcd}
\tag{1.26}
$$

It turns out (see Chapter 5) that

$$\mathcal{F}^s(J^\infty(E)) \cong E_1^{n,s}(J^\infty(E))$$

so that this distinction is really just one of terminology. Nevertheless, the subspaces $\mathcal{F}^s \subset \Omega^{n,s}$ and the projection operators $I$ are very useful in both theoretical and practical considerations. It is the utility of these operators which we wish to emphasize.

As we shall see in the next chapter, the map $E = I \circ d_V \colon \Omega^{n,0} \to \mathcal{F}^1$ is precisely the Euler-Lagrange operator from the calculus of variations.

Definition 1.10. *The Euler-Lagrange complex $\mathcal{E}(J^\infty(E))$ associated to a fibered manifold $\pi \colon E \to M$ is the the edge complex of the augmented variational bicomplex on $J^\infty(E)$:*

$$
\begin{aligned}
&0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \\
&\qquad\qquad \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \xrightarrow{\delta_V} \cdots
\end{aligned}
\tag{1.27}
$$

For many problems, the Euler-Lagrange complex is the object of ultimate interest — the variational bicomplex provides us with the means by which the Euler-Lagrange complex can be studied.

**D. Prolongations of Generalized Vector Fields.** To each generalized vector field $X$ on $E$ there is an associated vector field on $J^\infty(E)$ called the prolongation of $X$ and denoted by $\operatorname{pr} X$. To motivate this general construction, consider first the classical case of a projectable vector field $X$ on $E$ with projection $\overline{X} = \pi_* X$ on $M$. Then the flow $X_t$ of $X$ on $E$ covers the flow $\overline{X}_t$ of $\overline{X}$ on $M$ and so, by prolongation to $J^\infty(E)$ we obtain a local, one parameter group of transformations $F_t$ on $J^\infty(E)$, *i.e.*,

$$F_t = \operatorname{pr} X_t.$$

The prolongation of the vector field $X$ is the vector field on $J^\infty(E)$ associated to this flow,*i.e.*,

$$\operatorname{pr} X = \frac{d}{dt}\big[F_t\,\big]\big|_{t=0}.$$

The salient property of $\operatorname{pr} X$ is that it preserves the contact ideal $\mathcal{C}$.

Lemma 1.11. *Let $X$ be a projectable vector field on $E$ and let $\operatorname{pr} X$ be its prolongation to $J^\infty(E)$. Then*

(i) $\operatorname{pr} X$ *projects to $X$,i.e., $(\pi^\infty_E)_*(\operatorname{pr} X) = X$, and*

(ii) $\mathcal{L}_{\operatorname{pr} X}\,\mathcal{C} \subset \mathcal{C}$.

Proof: Property (i) is obvious — in fact $\operatorname{pr} X$ is a projectable vector field whose projection to $J^k(E)$ is the flow of the prolongation of $X_t$ to $J^k(E)$, *i.e.*,

$$F^k_t\big(j^k(s)(x)\big) = j^k\big(X_t \circ s \circ \overline{X}_{-t}\big)(y).$$

By virtue of 1.6, the flow $F_t$ preserves the contact ideal and this suffices to prove (ii). ∎

Proposition 1.12. *Let $X$ be a generalized vector field on $E$. Then there exists a unique vector field $Z$ on $J^\infty(E)$ such that*

(i) *$Z$ projects to $X$, i.e., $(\pi^\infty_E)_*(Z) = X$, and*

(ii) *$Z$ preserves the contact ideal, i.e., $\mathcal{L}_Z\,\mathcal{C} \subset \mathcal{C}$.*

*The vector field $Z$ is called the prolongation of $X$ to $J^\infty(E)$ and is denoted by* $\operatorname{pr} X$.

*Conversely, if $Z$ is any vector field on $J^\infty(E)$ which satisfies (ii), then $Z$ is the prolongation of its projection onto $E$, i.e., $Z = \operatorname{pr} Z_0$ where $Z_0 = (\pi^\infty_E)(Z)$.*

Proof: It suffices to prove local uniqueness and existence. Let $X$ and $Z$ be given locally by

$$X = a^i\frac{\partial}{\partial x^i} + b^\alpha\frac{\partial}{\partial u^\alpha} \tag{1.28}$$

and

$$Z = Z^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} Z_I^\alpha \partial_\alpha^I, \tag{1.29}$$

where the coefficients $a^i$, $b^\alpha$, $Z^i$ and $Z_I^\alpha$ are all smooth functions on $J^\infty(U)$. Property (i) requires that

$$Z^i = a^i \quad \text{and} \quad Z^\alpha = b^\alpha.$$

Since $\mathcal{L}_Z$ is a derivation, property (ii) holds for all contact forms on $J^\infty(U)$ if and only if it holds for the local basis of contact forms $\theta_I^\alpha$, *i.e.*,

$$\mathcal{L}_Z\, \theta_I^\alpha \in \mathcal{C}. \tag{1.30}$$

Direct calculation yields

$$\begin{aligned} \mathcal{L}_Z \theta_I^\alpha &= d\, Z_I^\alpha - Z_{Ij}^\alpha\, dx^j - u_{Ij}^\alpha d\, Z^j \\ &= d_H Z_I^\alpha + d_V Z_I^\alpha - Z_{Ij}^\alpha\, dx^j - u_{Ij}^\alpha d_H Z^j - u_{Ij}^\alpha d_V Z^j. \end{aligned}$$

The second and last terms in this last equation are in $\mathcal{C}$. The remaining terms are all horizontal and must therefore vanish. Consequently (1.30) holds if and only if

$$Z_{Ij}^\alpha = D_j Z_I^\alpha - u_{Ih}^\alpha D_j Z^h. \tag{1.31}$$

This equation furnishes us with a recursive formula for the coefficients of $Z$. It is clear that this formula uniquely determines the coefficients $Z_I^\alpha$ in terms of the coefficients $a^i$ and $b^\alpha$. ∎

In fact, (1.31) is easily solved to give rise to the explicit prolongation formula

$$Z_I^\alpha = D_I(b^\alpha - u_j^\alpha a^j) + u_{jI}^\alpha a^j, \tag{1.32}$$

where $D_I$ indicates repeated total differentiation,

$$D_I = D_{i_1} D_{i_2} \cdots D_{i_k}$$

for $I = i_1 i_2 \ldots i_k$.

We remark that if $X$ is of order $m$, then $Z$ is of type $(m, m+1, m+2, \ldots)$. For example, if $X$ is of order 0, the first prolongation coefficient is given by

$$\begin{aligned} Z_i^\alpha &= D_i b^\alpha - u_j^\alpha D_i a^j \\ &= \frac{\partial b^\alpha}{\partial x^i} + \frac{\partial b^\alpha}{\partial u^\beta} u_i^\beta - u_j^\alpha \big[ \frac{\partial a^j}{\partial x^i} + \frac{\partial a^j}{\partial u^\beta} u_i^\beta \big]. \end{aligned}$$

For projectable vector fields on $E$, Lemma 1.11 and Proposition 1.12 yield a proof different than that presented in Olver [55] of the prolongation formula (1.32).

We now show that every tangent vector to $J^\infty(E)$ can be realized pointwise as the prolongation of a vector field on $E$.

PROPOSITION 1.13. *Let $\sigma$ be a point in $J^\infty(E)$ and let $Z_\sigma$ be a tangent vector to $J^\infty(E)$ at $\sigma$. Then there exists a vector field $X$ on $E$ such that*

$$(\operatorname{pr} X)(\sigma) = X_\sigma.$$

PROOF: It suffices to construct $X$ locally on a coordinate neighborhood $U$ of $E$. Let

$$Z_\sigma = c^i \frac{\partial}{\partial x^i} + \sum_{k=0}^{\infty} c_I^\alpha \partial_\alpha^I$$

and

$$X = a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha}.$$

Take $a^i$ to be the constants $c^i$ and take $b^\alpha$ to be functions of the base coordinates $x^i$ alone and such that

$$\frac{\partial b^\alpha}{\partial x^I}(x_0) = c_I^\alpha,$$

where $x_0 = \pi_M^\infty(\sigma)$. Such functions $b^\alpha$ exist by virtue of a theorem of Borel (see Kahn [*38*], pp 31–33 ). The prolongation formula (1.32) shows that the coefficient $X_I^\alpha$ of $\partial_\alpha^I$ in $\operatorname{pr} X$ at $\sigma$ equals $c_I^\alpha$, as required. ∎

COROLLARY 1.14. *If $\omega \in \Omega^p(J^\infty(E))$ and*

$$\operatorname{pr} X \lrcorner\, \omega = 0$$

*for all vector fields $X$ on $E$, then $\omega = 0$.*

DEFINITION 1.15. *A generalized vector field on $\pi : E \to M$ which is $\pi$ vertical is called an evolutionary vector field.*

In local coordinates an evolutionary vector field $Y$ takes the form

$$Y = Y^\alpha \frac{\partial}{\partial u^\alpha},$$

where the coefficients are functions on $J^\infty(U)$. This terminology reflects the fact that, at least within a fixed coordinate chart, the components $Y^\alpha$ define the system of evolution equations

$$\frac{\partial s^\beta}{\partial \epsilon} = Y^\beta(x^i, s^\alpha, \frac{\partial s^\alpha}{\partial x^{i_1}}, \ldots, \frac{\partial s^\alpha}{\partial x^{i_1} x^{i_2} \cdots x^{i_k}})$$

for a one parameter family $s^\alpha(x,\epsilon)$ of local sections of $E$. For evolutionary vector fields the prolongation formula (1.32) simplifies to

$$\operatorname{pr} Y = \sum_{|I|=0}^{\infty} [D_I Y^\alpha]\, \partial_\alpha^I. \tag{1.33}$$

PROPOSITION 1.16. *Suppose that $Y$ is an evolutionary vector field on $J^\infty(E)$ and $\omega \in \Omega^{r,s}(J^\infty(E))$. Then $\mathcal{L}_{\operatorname{pr} Y}\, \omega \in \Omega^{r,s}$,*

$$\mathcal{L}_{\operatorname{pr} Y}\, \omega = d_V(\operatorname{pr} Y \mathbin{\lrcorner} \omega) + \operatorname{pr} Y \mathbin{\lrcorner} (d_V \omega) \tag{1.34}$$

*and*

$$\operatorname{pr} Y \mathbin{\lrcorner} (d_H \omega) = -d_H(\operatorname{pr} Y \mathbin{\lrcorner} \omega). \tag{1.35}$$

PROOF: The statement that $\mathcal{L}_{\operatorname{pr} Y}\omega \in \Omega^{r,s}$ is a local one which can be verified on generators. First, by Proposition 1.12, the Lie derivative with respect to $\operatorname{pr} Y$ of the contact one form $\theta_I^\alpha$ is again a contact one form. Second, since $Y$ is $\pi$ vertical, the Lie derivative $\mathcal{L}_{\operatorname{pr} Y} dx^i = 0$. These two observations show that $\mathcal{L}_{\operatorname{pr} Y}$ preserves horizontal and vertical type.

To prove (1.34) and (1.35), we simply expand the Lie derivative formula (1.19) in terms of $d_H$ and $d_V$ to arrive at

$$\mathcal{L}_{\operatorname{pr} Y}\omega = \{d_V(\operatorname{pr} Y \mathbin{\lrcorner} \omega) + \operatorname{pr} Y \mathbin{\lrcorner} (d_V \omega)\} + \{d_H(\operatorname{pr} Y \mathbin{\lrcorner} \omega) + \operatorname{pr} Y \mathbin{\lrcorner} (d_H \omega)\}.$$

Because $Y$ is vertical, $\operatorname{pr} Y \mathbin{\lrcorner} \omega$ is of degree $(r, s-1)$. The first group of terms is therefore of degree $(r,s)$ while the second group is of degree $(r+1, s-1)$. Consequently the second group must vanish. This proves (1.35) which, in turns, proves (1.34). ∎

Owing to (1.34), it is easily verified that $\mathcal{L}_{\operatorname{pr} Y}$ commutes with $d_V$ and hence with $d_H$.

COROLLARY 1.17. *If $Y$ is an evolutionary vector field and $\omega \in \Omega^{r,s}(J^\infty(E))$, then*

$$\mathcal{L}_{\operatorname{pr} Y}(d_H\omega) = d_H(\mathcal{L}_{\operatorname{pr} Y}\omega)$$

*and*

$$\mathcal{L}_{\operatorname{pr} Y}(d_V\omega) = d_V(\mathcal{L}_{\operatorname{pr} Y}\omega).$$

Generalized vector fields on $M$ can also be lifted to vector fields on $J^\infty(E)$.

PROPOSITION 1.18. *Let $X$ be a generalized vector field on $M$. There exists a unique vector field $Z$ on $J^\infty(E)$ such that*

(i) *$Z$ projects to $X$, i.e., $(\pi^\infty_M)_*(Z) = X$, and*

(ii) *$Z$ annihilates all contact one forms, i.e., if $\omega \in \Omega^{0,s}$ then $Z \lrcorner\, \omega = 0$.*

*The vector field $Z$ is called the total vector field for $X$ and is denoted by $Z = \operatorname{tot} X$.*

*Conversely, if $Z$ is any vector field on $J^\infty(E)$ satisfying (ii), then it is the total vector field for its projection onto $M$*

PROOF: As in the proof of Proposition 1.12, it suffices to work locally. Let

$$X = a^i[x,u]\frac{\partial}{\partial x^i}$$

and

$$Z = Z^i\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} Z^\alpha_I \partial^I_\alpha.$$

Then (i) implies that $Z^i = a^i$. Property (ii) holds if and only if

$$Z \lrcorner\, \theta^\alpha_I = 0$$

and this implies that $Z^\alpha_I = u^\alpha_{Ij} a^j$. Thus $Z$ is uniquely given in terms of $X$ by

$$Z = a^i D_i, \tag{1.36}$$

where $D_i$ is total differentiation with respect to $x^i$. ∎

We remark that if $X$ is of order $m$, then $\operatorname{tot} X$ is of type $(m_0, m_1, m_2, \dots)$, where $m_i = m$ for $i \le m$ and $m_i = i$ for $i \ge m$. Note also that $\operatorname{tot} X$ is the prolongation of its projection onto $E$, *i.e.*, with $X_E = (\pi^\infty_E)_*(\operatorname{tot} X)$, we have

$$\operatorname{tot} X = \operatorname{pr} X_E. \tag{1.37}$$

This result is immediate from our local coordinate formula (1.32) This implies that $\operatorname{tot} X$ preserves the contact ideal, *i.e.*,

$$\mathcal{L}_{\operatorname{tot} X}\, \mathcal{C} \subset \mathcal{C}.$$

We also remark that the total differentiation operator $D_j$ is the total vector field for partial differentiation $\dfrac{\partial}{\partial x^j}$, *i.e.*,

$$\begin{aligned} D_j &= \operatorname{tot}\frac{\partial}{\partial x^j} \\ &= \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty} u^\alpha_{Ij}\, \partial^\alpha_I. \end{aligned} \tag{1.38}$$

By Proposition 1.18, $D_j$ annihilates all contact one forms:

$$D_j \lrcorner\, \theta_I^\alpha = 0. \tag{1.39}$$

Definition 1.19. *Let $X$ be a generalized vector field on $E$. The associated total vector field is the total vector field for the projection of $X$ onto $M$:*

$$\operatorname{tot} X = \operatorname{tot}[(\pi)_* X].$$

*The associated evolutionary vector field is the vertical, generalized vector field*

$$X_{\mathrm{ev}} = X - (\pi_E^\infty)_*(\operatorname{tot} X). \tag{1.40}$$

If, in local coordinates,

$$X = a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha}$$

then

$$X_{ev} = (b^\alpha - u_i^\alpha a^i)\frac{\partial}{\partial u^\alpha}.$$

We shall use the next proposition repeatedly in subsequent chapters.

Proposition 1.20. *Let $X$ be a generalized vector field on $E$. Then the prolongation of $X$ splits into the sum*

$$\operatorname{pr} X = \operatorname{pr} X_{\mathrm{ev}} + \operatorname{tot} X. \tag{1.41}$$

Proof: Take the prolongation of (1.40) and apply (1.37). ∎

We conclude this chapter with formulas for the Lie brackets of prolonged vector fields and total vector fields.

Proposition 1.21. *(i) Let $X$ and $Y$ be generalized vector fields on $E$. Then*

$$[\operatorname{pr} X, \operatorname{pr} Y] = \operatorname{pr} Z_1, \tag{1.42a}$$

*where $Z_1$ is the generalized vector field on $E$ defined, for any function $f$ on $E$ by*

$$Z_1(f) = \operatorname{pr} X\big(Y(f)\big) - \operatorname{pr} Y\big(X(f)\big). \tag{1.42b}$$

*(ii) Let $X$ and $Y$ be generalized vector fields on $M$. Then*

$$[\operatorname{tot} X, \operatorname{tot} Y] = \operatorname{tot} Z_2, \tag{1.43a}$$

*where $Z_2$ is the generalized vector field on $M$ defined, for all functions $f$ on $M$ by*

$$Z_2(f) = \operatorname{tot} X\big(Y(f)\big) - \operatorname{tot} Y\big(X(f)\big). \tag{1.43b}$$

*(iii) Let $X$ be a generalized vector field on $M$ and let $Y$ be an evolutionary vector field on $E$. Then*

$$[\operatorname{tot} X, \operatorname{pr} Y] = \operatorname{tot} Z_3, \tag{1.44a}$$

*where $Z_3$ is the generalized vector field on $M$ defined, for all functions $f$ on $M$, by*

$$Z_3(f) = \operatorname{pr} Y\big(X(f)\big). \tag{1.44b}$$

*In particular, if $X$ is a vector field on $M$, then*

$$[\operatorname{tot} X, \operatorname{pr} Y] = 0. \tag{1.45}$$

PROOF: To prove (i) it suffices to observe that since $\operatorname{pr} X$ and $\operatorname{pr} Y$ preserve the contact ideal, $[\operatorname{pr} X, \operatorname{pr} Y]$ must preserve the contact ideal and so, in accordance with Proposition 1.12, this Lie bracket is the prolongation of its projection onto $E$. To prove (ii) it suffices to check, by virtue of Proposition 1.18, that $[\operatorname{tot} X, \operatorname{tot} Y]$ annihilates all the contact forms $\theta^\alpha_I$. This follows from the identity

$$\begin{aligned}
&(d\,\theta^\alpha_I)(\operatorname{tot} X, \operatorname{tot} Y) = \\
&\qquad (\operatorname{tot} X)\big((\operatorname{tot} Y) \lrcorner\, \theta^\alpha_I\big) - (\operatorname{tot} Y)\big((\operatorname{tot} X) \lrcorner\, \theta^\alpha_I\big) - [\operatorname{tot} X, \operatorname{tot} Y] \lrcorner\, \theta^\alpha_I.
\end{aligned}$$

A similar argument proves (iii). Note that $Z_3$ is indeed a well-defined generalized vector field on $M$ because, for all functions $f$ on $M$, $\operatorname{pr} Y(f) = 0$. ∎

## Chapter Two

# EULER OPERATORS

In the variational calculus, various local differential operators, similar in construction to the Euler-Lagrange operator, occur repeatedly and play a distinguished role. These so-called higher Euler operators first arose in the classification of the conservation laws for the KdV equation [*45*] and the BBM equation [*51*] and they occur naturally in the solution to the inverse problem to the calculus of variations. The general properties of these operators have been well documented by various authors including Aldersley [*1*], Wantanabe [*79*] and Tu [*69*]. In this chapter a general framework is introduced in which the higher Euler operators naturally emerge as a special case. Many properties of the higher Euler operators can be effortlessly derived from this viewpoint. These properties will be used in the next chapter to prove the local exactness of the rows of the variational bicomplex.

This general framework also leads immediately to the construction of the projection operators

$$I\colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E)),$$

introduced in §2C, and used to construct the augmented variational bicomplex. The map $I$ first appeared explicitly in the papers of Kuperschmidt [*46*], Decker and Tulczyjew [*21*], and Bauderon [*8*] (but denoted by $\tau^+$, $\tau$, and $D^*$ respectively). It plays a central role in both the theoretical developments and practical applications of the variational bicomplex. Salient properties of the operator $I$ are established. In particular, we show that the Euler-Lagrange operator $E$ factors as the composition of the operator $I$ and the vertical differential $d_V$, *i.e.*, if $\lambda \in \Omega^{n,0}(J^\infty(E))$ is a Lagrangian for a variational problem on $E$, then

$$E(\lambda) = I(d_V\lambda).$$

We illustrate the utility of this invariant decomposition of the Euler-Lagrange operator by calculating the Euler-Lagrange equations for some of the interesting geometrical variational problems as described in Griffiths [*31*]. These calculations also highlight the role of moving frames in applications of the variational bicomplex.

**A. Total Differential Operators.** We begin with a general construction. Let $\mathcal{E}v(J^\infty(E))$ be the vector space of evolutionary vector fields on $J^\infty(E)$. Consider a differential operator

$$P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$$

which is locally of the form

$$P(Y) = \sum_{|I|=0}^{k} (D_I Y^\alpha) P^I_\alpha[x,u], \tag{2.1}$$

where $Y$ is an evolutionary vector field given locally by

$$Y = Y^\alpha[x,u]\frac{\partial}{\partial u^\alpha}.$$

Each coefficient $P^I_\alpha$ is a smooth form of type $(r,s)$. We call such an operator a *total differential operator* since it involves only total derivatives of the evolutionary vector field $Y$. To define this class of operators intrinsically, let $T_V \to J^\infty(E)$ be the bundle of $\pi^\infty_M$ vertical vectors on $J^\infty(E)$. Let $\Lambda^{r,s} \to J^\infty(E)$ be the bundle of type $(r,s)$ forms on $J^\infty(E)$. Let $L$ be a linear bundle map from the bundle $T_V$ to the bundle $\Lambda^{r,s}$ which covers the identity map on $J^\infty(E)$, *i.e.*,

$$\begin{array}{ccc} T_V & \xrightarrow{L} & \Lambda^{r,s} \\ \downarrow & & \downarrow \\ J^\infty(E) & \xrightarrow{\text{id}} & J^\infty(E). \end{array}$$

Then a differential operator $P$ on $\mathcal{E}v(J^\infty(E))$ is a total differential operator if there exists a linear map $L$ such that

$$P(Y) = L(\operatorname{pr} Y).$$

Two examples of such operators which we shall study in detail are

$$P_\omega(Y) = \operatorname{pr} Y \mathbin{\lrcorner} \omega,$$

where $\omega$ is a fixed, type $(r, s+1)$ form on $J^\infty(E)$, and

$$P_\eta(Y) = \mathcal{L}_{\operatorname{pr} Y}\, \eta = \operatorname{pr} Y \mathbin{\lrcorner} d_V \eta,$$

where $\eta$ is a fixed form of type $(r, 0)$. If $\eta$ is of type $(r, s)$ for $s \geq 1$, then $P_\eta$ is not a total differential operator.

If, at a point $\sigma \in J^\infty(E)$, one of the type $(r, s)$ forms $P^I_\alpha(\sigma)$ is nonzero for $|I| = k$, then the operator $P$ is said to be of order $k$ at $\sigma$. The collection of forms $P^I_\alpha$, $|I| = k$, determine the *symbol* of $P$ at $\sigma$. The order of the operators $P_\omega$ and $P_\eta$ coincide with the orders of $\omega$ and $\eta$ as differential forms on $J^\infty(E)$. The symbol of $P_\eta$ is

$$(P_\eta)^I_\alpha = \partial^I_\alpha(\eta).$$

The standard approach the linear differential operators on finite dimensional manifolds (see, *e.g.*, Kahn [*38*], Chapter 6) can be adopted without change to give invariant definitions to the order and symbol of our total differential operators. In particular, a zeroth order total differential operator is one which is linear over smooth functions on $J^\infty(E)$ and is therefore given locally by

$$P(Y) = Y^\alpha Q_\alpha.$$

Proposition 2.1. *Let $P \colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$ be a total differential operator which is given locally by (2.1). Then $P$ can be rewritten locally as*

$$\begin{aligned} P(Y) &= Y^\alpha Q_\alpha + D_i(Y^\alpha Q^i_\alpha) + D_{ij}(Y^\alpha Q^{ij}_\alpha) + \cdots \\ &= \sum_{|I|=0}^{k} D_I(Y^\alpha Q^I_\alpha). \end{aligned} \tag{2.2}$$

*The coefficients $Q^I_\alpha$ are smooth, type $(r, s)$ forms on $J^\infty(U)$. They are uniquely defined in terms of the original coefficients $P^I_\alpha$ of $P$ by*

$$Q^I_\alpha = \sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|I|} (-D)_J P^{IJ}_\alpha. \tag{2.3}$$

Proof: This proposition is a simple exercise in repeated "integration by parts". To establish this proposition, we expand the right-hand side of (2.2) by the product rule to find that

$$\begin{aligned} \sum_{l=0}^{k} D_{I_l}\bigl(Y^\alpha Q^{I_l}_\alpha\bigr) &= \sum_{l=0}^{k} \sum_{r=0}^{l} \tbinom{l}{r} \bigl(D_{I_r} Y^\alpha\bigr) D_{J_{l-r}} Q^{I_r J_{l-r}}_\alpha \\ &= \sum_{r=0}^{k} \bigl(D_{I_r} Y^\alpha\bigr) \bigl[\sum_{l=r}^{k} \tbinom{l}{r} D_{J_{l-r}} Q^{I_r J_{l-r}}_\alpha\bigr]. \end{aligned}$$

Here the subscripts on the multi-indices denote their length, *e.g.*, $|I_l| = l$. Consequently, (2.2) holds if and only if

$$P_\alpha^I = \sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|J|} D_J Q_\alpha^{IJ}, \tag{2.4}$$

for all $|I| = 0, 1, 2, \ldots, k$. For example, with $k = 2$, this set of equations is

$$\begin{aligned} P_\alpha^{ij} &= Q_\alpha^{ij}, \\ P_\alpha^{i} &= Q_\alpha^{i} + 2D_j Q_\alpha^{ij}, \quad \text{and} \\ P_\alpha &= Q_\alpha + D_i Q_\alpha^i + D_{ij} Q_\alpha^{ij}. \end{aligned}$$

The uniqueness of this representation is apparent from (2.4) — if all the $P_\alpha^I$ vanish, then by examining these equations in the order $|I| = k$, $|I| = k-1$, $\ldots$ $|I| = 0$, it follows immediately that all the $Q_\alpha^I = 0$. Consequently, to verify (2.3), it suffices to check that the $Q_\alpha^I$ given by (2.3) satisfy (2.4). Substitution of (2.3) into the right-hand side of (2.4) with $|I| = r$ leads to

$$\begin{aligned} \sum_{l=0}^{k-r} \tbinom{r+l}{r} D_{J_l} Q_\alpha^{I_r J_l} &= \sum_{l=0}^{k-r} \tbinom{r+l}{r} D_{J_l} \Big[ \sum_{s=0}^{k-r-l} \tbinom{r+l+s}{r+l} (-D)_{K_s} P^{I_r J_{l-s} K_s} \Big] \\ &= \sum_{l=0}^{k-r} \Big[ \sum_{s=0}^{k-r-l} \tbinom{r+l}{r} \tbinom{r+l+s}{r+l} (-1)^s D_{J_{l+s}} P_\alpha^{I_r J_{l+s}} \Big]. \end{aligned}$$

To simplify this last expression, we replace the summation on $s$ by one on $s' = l + s$ and interchange the order of summations. This leads to the expression

$$\sum_{s'=0}^{k-r} \Big[ \sum_{l=0}^{s'} (-1)^{l+s} \tbinom{r+l}{r} \tbinom{r+s'}{r+l} \Big] D_{J_{s'}} P_\alpha^{I_r J_{s'}}.$$

Since the summation in square brackets vanishes if $s' = 0$ and equals 1 if $s' = 0$, this expression reduces to $P_\alpha^{I_r}$, as required. ∎

Proposition 2.1 implies that if $Q_\alpha^I$ and $\widetilde{Q}_\alpha^I$ are two collections of forms of type $(r, s)$ and

$$\sum_{|I|=0}^{k} D_I (Y^\alpha Q_\alpha^I) = \sum_{|I|=0}^{k} D_I (Y^\alpha \widetilde{Q}_\alpha^I)$$

for all evolutionary vector fields $Y$, then $Q^I_\alpha = \widetilde{Q}^I_\alpha$. We shall use this simple fact repeatedly in what follows.

Our next task is to examine the transformation properties of the coefficients $Q^I_\alpha$. First note that for $|I| = k$,

$$Q^I_\alpha = P^I_\alpha$$

and that these forms determine the symbol of $P$. The lower order coefficients $Q^I_\alpha$, $|I| < k$ do not, in general have an intrinsic meaning, *i.e.*, their vanishing in one coordinate system does not imply their vanishing in an overlapping one. For example, let

$$\omega = A_i \, dx^i$$

be a type $(1,0)$ form on $J^1(E)$ and consider the differential operator

$$\begin{aligned} P(Y) = \mathcal{L}_{\mathrm{pr}\,Y}\, \omega &= [Y^\alpha \frac{\partial A_i}{\partial u^\alpha} + D_j Y^\alpha \frac{\partial A_i}{\partial u^\alpha_j}]\, dx^i \\ &= Y^\alpha Q_\alpha + D_j\, [Y^\alpha \frac{\partial A_i}{\partial u^\alpha_j}]\, dx^i, \end{aligned}$$

where

$$Q_\alpha = [\frac{\partial A_i}{\partial u^\alpha} - D_k \frac{\partial A_i}{u^\alpha_k}]\, dx^i.$$

Under the change of coordinates

$$y^j = y^j(x^i) \quad \text{and} \quad v^\alpha = u^\alpha$$

one has that

$$v^\alpha_j = \frac{\partial x^i}{\partial y^j} u^\alpha_i$$

and

$$\overline{A}_h(y^j, v^\alpha, v^\alpha_j) \frac{\partial y^h}{\partial x^i} = A_i(x^j, u^\alpha, u^\alpha_j),$$

where $\overline{A}_h$ are the components of $\omega$ in the $(y, v)$ coordinate system. A simple calculation then shows that

$$\begin{aligned} \overline{Q}_\alpha &= [\frac{\partial \overline{A}_i}{\partial v^\alpha} - \overline{D}_k \frac{\partial \overline{A}_i}{\partial v^\alpha_k}]\, dy^i \\ &= Q_\alpha + \frac{\partial \overline{A}_h}{\partial v^\alpha_j} \frac{\partial^2 x^k}{\partial y^j \partial y^l} [\frac{\partial y^h}{\partial x^k} \frac{\partial y^l}{\partial x^i} - \frac{\partial y^l}{\partial x^k} \frac{\partial y^h}{\partial x^i}]\, dx^i. \end{aligned}$$

Therefore the zeroth order total differential operator $Q$ defined locally by

$$Q(Y) = Y^\alpha Q_\alpha$$

is not invariantly defined. Notice however, that the second term in the above equation vanishes if $\dim M = 1$, *i.e.*, if $P(Y)$ is a top dimensional form. In this instance one can explicitly verify that $Q$ is a invariantly defined operator in the sense that under arbitrary changes of coordinates $y = y(x)$ and $v^\beta = v^\beta(x, u^\alpha)$ the $Q_\alpha$ transform as

$$\overline{Q}_\beta = \frac{\partial u^\alpha}{\partial v^\beta} Q_\alpha.$$

The next proposition shows this to be true in general.

PROPOSITION 2.2. *Let $P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$, where $\dim M = n$, be a total differential operator. Then there exists a unique, globally defined, zeroth order operator*

$$Q\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$$

*with the following property. On each coordinate chart $J^\infty(U)$ there is a locally defined total differential operator $R\colon \mathcal{E}v(J^\infty(U)) \to \Omega^{n-1,s}(J^\infty(U))$ such that*

$$P(Y) = Q(Y) + d_H R(Y). \tag{2.5}$$

PROOF: We first prove the uniqueness of $Q(Y)$. Suppose then, that in addition to the decomposition (2.5), we have that

$$P(Y) = \widetilde{Q}(Y) + d_H \widetilde{R}(Y)$$

and consequently

$$Q(Y) - \widetilde{Q}(Y) = d_H[\widetilde{R}(Y) - R(Y)]. \tag{2.6}$$

Since $Q$ and $\widetilde{Q}$ are both zeroth order operators, there are type $(n, s)$ forms $\Sigma_\alpha$ on $J^\infty(U)$ such that

$$Q(Y) - \widetilde{Q}(Y) = Y^\alpha \Sigma_\alpha.$$

We prove that $\Sigma_\alpha = 0$ thereby establishing the uniqueness of the operator $Q(Y)$.

Let $Y_1$, $Y_2$, ..., $Y_s$ be evolutionary vector fields on $J^\infty(U)$. Inner evaluation of (2.6) with $\operatorname{pr} Y_1$, $\operatorname{pr} Y_2$, ..., $\operatorname{pr} Y_s$ and repeated application of Proposition 1.16 leads to

$$Y^\alpha \eta_\alpha = (-1)^s d_H \bigl\{\operatorname{pr} Y_1 \lrcorner \operatorname{pr} Y_2 \lrcorner \cdots \operatorname{pr} Y_s \lrcorner [\widetilde{R}(Y) - R(Y)]\bigr\}, \tag{2.7}$$

where the $\eta_\alpha$ are the type $(n,0)$ forms given by

$$\eta_\alpha = \operatorname{pr} Y_1 \lrcorner \operatorname{pr} Y_2 \lrcorner \cdots \operatorname{pr} Y_s \lrcorner \Sigma_\alpha.$$

From (2.7) it is a simple matter to repeat standard, elementary arguments from the calculus of variations to conclude that $\Sigma_\alpha = 0$. Indeed, pick an open set $W \subset \pi(U)$ with $\overline{W}$ compact, pick $Y^\alpha = Y^\alpha(x^i)$ with support $V \subset W$, evaluate (2.7) on a local section $s\colon \pi(U) \to U$ and integrate the resulting $n$ form on $\pi(U)$ over $\overline{W}$. Since the right-hand side of (2.7) is linear in $Y$, it vanishes outside of the support of $Y$. It then follows from Stokes Theorem that

$$\int_{\overline{W}} Y^\alpha \bigl\{[j^\infty(s)]^* \eta_\alpha\bigr\} = 0.$$

Since the functions $Y^\alpha$ and the local section $s$ are arbitrary, $\eta_\alpha$ must vanish at each point at each point in $J^\infty(U)$. Since the evolutionary vector fields $Y_1$, $Y_2$, $\ldots$, $Y_s$ are arbitrary, $\Sigma_\alpha$ must vanish. Alternatively, one can argue formally by recalling that the Euler-Lagrange operator annihilates Lagrangians which are locally $d_H$ exact, *i.e.*, which are local divergences. Therefore the application to (2.7) of the Euler-Lagrange operator with respect to the variables $Y^\alpha$ leads directly to $\Sigma_\alpha = 0$. This proves the uniqueness of the operator $Q(Y)$.

Next we establish the local existence of the operator $Q(Y)$. Write $P(Y)$ in the form (2.2). Since the coefficients $Q^I_\alpha$ are of top horizontal degree it is a simple exercise to check that

$$Q^I_\alpha = dx^{(i} \wedge R^{I')}_\alpha,$$

where $I = iI'$ and

$$R^{I'} = D_j \lrcorner Q^{jI'}_\alpha.$$

(Recall that $D_j = \operatorname{tot} \dfrac{\partial}{\partial x^j}$ is the total vector field defined by (1.38).) Accordingly, (2.2) becomes

$$P(Y) = Y^\alpha Q_\alpha + D_i \Bigl[ \sum_{|I'|=0}^{k-1} D_{I'}(dx^i \wedge Y^\alpha R^{I'}_\alpha) \Bigr]$$

and consequently (2.5) holds with

$$Q(Y) = Y^\alpha Q_\alpha, \qquad \text{where} \quad Q_\alpha = \sum_{|I|=0}^{k} (-D)_I P^I_\alpha \tag{2.8}$$

and

$$R(Y) = \sum_{|I|=0}^{k-1} D_I(Y^\alpha R^I_\alpha). \tag{2.9}$$

The local uniqueness and existence of $Q(Y)$ suffice to imply that this operator is globally well-defined. ∎

Definition 2.3. *Let $P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$ be a total differential operator described locally by*

$$P(Y) = \sum_{|I|=0}^{k} (D_I Y^\alpha) P^I_\alpha.$$

*Then the associated zeroth order operator $E(P)$ defined by*

$$E(P)(Y) = Y^\alpha E_\alpha(P), \qquad \text{where} \quad E_\alpha(P) = \sum_{|I|=0}^{k} (-D)_I P^I_\alpha \tag{2.10}$$

*is called the Euler operator associated to $P$.*

Before turning to the applications of Propositions 2.1 and 2.2, several remarks are in order. First, we note that the decomposition (2.5) is *not* valid (even locally) for operators $P\colon \mathcal{E}v \to \Omega^{r,s}$ for $r < n$. For example, on $E : \mathbf{R}^2 \times \mathbf{R} \xrightarrow{\pi} \mathbf{R}^2$ with coordinates $(x, y, u) \xrightarrow{\pi} (x, y)$, the operator

$$P(Y) = Y(u_x\, dy - u_y\, dx) - (D_y Y)u\, dx + (D_x Y)u\, dy$$

mapping $\mathcal{E}v$ to $\Omega^{1,0}$ does not admit such a decomposition. Together with a previous example, this suggests that for $r < n$ it is not possible to canonically construct, by some universal formula linear in the coefficients of $P$, a globally well-defined zeroth order linear operator $Q\colon \mathcal{E}v \to \Omega^{r,s}$ while, for $r = n$, the only such operator is given by (2.10).

Secondly, although we have established that $E(P)$ is a globally well-defined operator, we can assert presently only that the decomposition

$$P(Y) = E(P)(Y) + d_H R(Y) \tag{2.11}$$

holds locally. The global existence of the operator $R$ has not yet been established. It is obvious that the operator $R$ is not uniquely determined by this decomposition (unless $n = 1$). Hence, without additional restrictions on $R$, the elementary uniqueness and local existence arguments of Proposition 2.2 will not prove the global existence of the operator $R$. To this end, let

$$R(Y) = \sum_{|I|=0}^{l} (D_I Y^\alpha) R^I_\alpha \tag{2.12}$$

be a total differential operator from $\mathcal{E}v$ to $\Omega^{r,s}$. Let $p$ be a point in $J^\infty(E)$ and suppose that the order of $R$ is $l$ at $p$. We say that $R$ is *trace-free* at $p$ is either $l = 0$ or for $l \geq 1$

$$D_i \mathbin{\lrcorner} R^{iI'}_\alpha(p) = 0, \qquad |I'| = l - 1. \tag{2.13}$$

This is an invariantly defined condition on the symbol of $R$.

Proposition 2.4. *For $r < n$, let $R\colon \mathcal{E}v \to \Omega^{r,s}$ be a trace-free total differential operator. If*

$$d_H[R(Y)] = 0$$

*for all evolutionary vector fields $Y$, then $R = 0$.*

Proof: Suppose, in order to derive a contradiction, that there is some coordinate neighborhood $J^\infty(U)$ on which $R$ is non-zero and that the order of $R$ equals $l \geq 0$ on this neighborhood. Thus, with $R$ is given by (2.12) on $J^\infty(U)$, (2.13) holds if $l \geq 1$ and $R^I_\alpha \neq 0$ for $|I| = l$. An simple calculation shows that

$$\begin{aligned} d_H[R(Y)] = Y^\alpha \wedge d_H R_\alpha + \sum_{|I|=1}^{k} (D_I Y^\alpha)\bigl[dx^i \wedge R^{I'}_\alpha + d_H R^I_\alpha\bigr] \\ + (D_{i_1 i_2 \cdots i_l i} Y^\alpha)\, dx^i \wedge R^{i_1 i_2 \cdots i_l}_\alpha. \end{aligned}$$

Because of the hypothesis, this vanishes for all $Y^\alpha$ and hence it follows that

$$dx^{(i} \wedge R^{i_1 i_2 \cdots i_l)}_\alpha = 0.$$

Written out in full this equation becomes

$$dx^i \wedge R^{i_1 i_2 \cdots i_l}_\alpha + dx^{i_1} \wedge R^{i i_2 \cdots i_l}_\alpha + dx^{i_2} \wedge R^{i_1 i \cdots i_l}_\alpha + \cdots = 0.$$

On account of the fact that $R^I_\alpha$ is trace-free, interior evaluation of this equation with $D_i$ gives rise to

$$(n + l - r) R^{i_1 i_2 \cdots i_l}_\alpha = 0.$$

Since $r < n$ this contradicts the assumption that the order of $R$ is $l$ on $J^\infty(U)$ and proves that $R$ must vanish identically. ∎

If $P\colon \mathcal{E}v \to \Omega^{n,s}$ is a *second* order total differential operator, then $R\colon \mathcal{E}v \to \Omega^{n-1,s}$, as given by (2.9), is a first order, trace-free operator. Since the difference of two first order, trace-free operators is again trace-free, Proposition 2.4 can be used to prove that $P$ decomposes uniquely into the form (2.11), where $R$ is first order and trace-free. Thus this decomposition holds globally and both operators $E(P)$ and $R$ are constructed canonically from $P$. If $P$ is of order 3 or higher, then $R$ is of order at least 2 and trace-free. However, the difference of two second order trace-free operators is no longer necessarily trace-free (if the symbols of the two operators coincide then the difference is a first order operator which need not be trace-free) and Proposition 2.4 cannot be used to prove the uniqueness of $R$. It is possible to prove that for operators $P$ of order 3 or higher that there does not exist a canonical, universal decomposition of the type (2.11). Nevertheless, we shall prove in Chapter 5 (by a partition of unity argument) that the decomposition (2.11) does indeed hold globally. Thus the operator $R$ always exists globally — it just cannot in general be canonically fashioned from the coefficients of $P$.

If we allow ourselves to interpret (2.11) as a geometric version of the integration by parts formula, then we can summarize this state of affairs by saying that *there exists a global integration by parts formula for total differential operators* $P\colon \mathcal{E}v \to \Omega^{r,s}$ *only for* $r = n$ *which is canonical only for operators* $P$ *of order two.*

**B. Euler Operators.** For our first application of Propositions 2.1 and 2.2, let $\lambda = L\,\nu$ be a type $(n,0)$ form on $J^\infty(E)$, *i.e.*, a Lagrangian for a variational problem on $E$. We consider the operator $P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,0}(J^\infty(E))$ given by

$$P_\lambda(Y) = \mathcal{L}_{\operatorname{pr} Y}\lambda = \sum_{|I|=0}^{k} (D_I Y^\alpha)(\partial_\alpha^I L)\,\nu.$$

In this instance we shall denote the coefficients $Q_\alpha^I$ introduced in Proposition 2.1 by $E_\alpha^I(L)$ so that

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} Y}\lambda &= \bigl[Y^\alpha E_\alpha(L) + D_i(Y^\alpha E_\alpha^i(L)) + D_{ij}(Y^\alpha E_\alpha^{ij}(L)) + \cdots\bigr]\,\nu \\ &= \bigl[\sum_{|I|=0}^{k} D_I(Y^\alpha E_\alpha^I(L))\bigr]\,\nu,\end{aligned} \tag{2.14}$$

where

$$E_\alpha^I(L) = \sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|I|}(-D)_J(\partial_\alpha^{IJ}L). \tag{2.15}$$

When $|I| = 0$

$$\begin{aligned}E_\alpha(L) &= \partial_\alpha L - D_i\bigl(\partial_\alpha^i L\bigr) + D_{ij}\bigl(\partial_\alpha^{ij}L\bigr) - \cdots \\ &= \sum_{|I|=0}^{k} (-D)_I(\partial_\alpha^I L)\end{aligned} \tag{2.16}$$

is the classical Euler-Lagrange operator for higher order, multiple integral problems in the calculus of variations. Aldersley [*1*] refers to the operators $E_\alpha^I(L)$, $|I| > 0$, as higher Euler operators — we shall refer to them as *Lie-Euler* operators in order to emphasize that they arise naturally as the coefficients of the Lie derivative operator $P_\lambda(Y) = \mathcal{L}_{\operatorname{pr} Y}\lambda$ in the representation (2.14) and to distinguish them from another set of similar operators to be introduced later.

Definition 2.5. *The Euler-Lagrange operator*

$$E\colon \Omega^{n,0}(J^\infty(E)) \to \Omega^{n,1}(J^\infty(E))$$

*is the linear differential operator*

$$E(\lambda) = E_\alpha(L)\,\theta^\alpha \wedge \nu.$$

*The type $(n,1)$ form $E(\lambda)$ is called the Euler-Lagrange form for the Lagrangian $\lambda$.*

Equation (2.5) can be rewritten in terms of the Euler-Lagrange form as

$$\mathcal{L}_{\operatorname{pr} Y}\lambda = (\operatorname{pr} Y) \,\lrcorner\, E(\lambda) + d_H[(\operatorname{pr} Y) \,\lrcorner\, \sigma], \tag{2.17a}$$

where $\sigma$ is the type $(n-1,1)$ form given locally by

$$\sigma = \sum_{|I|=0}^{k-1} D_I[E_\alpha^{jI}(L)\,\theta^\alpha \wedge \nu_j], \tag{2.17b}$$

where

$$\nu_j = D_j \,\lrcorner\, \nu.$$

Thus, in the context of the present example, Proposition 2.1 leads to the formula for the first variation in the calculus of variations while Proposition 2.2 asserts that the Euler-Lagrange operator is globally well-defined. Our earlier remarks concerning the global validity of (2.5) now imply that there is a global first variational formula (again, see Chapter 5 for details) in the sense that there exists a type $(n-1,1)$ form $\beta$, defined on all of $J^\infty(E)$, such that

$$\mathcal{L}_{\operatorname{pr} Y}\lambda = (\operatorname{pr} Y) \,\lrcorner\, E(\lambda) + d_H[(\operatorname{pr} Y) \,\lrcorner\, \beta].$$

However, the form $\beta$ need not, in general, agree with $\sigma$ on any given coordinate chart.

From (2.4), we find that

$$\partial_\alpha^I = \sum_{|J|=0}^{\infty} \tbinom{|I|+|J|}{|J|} (D)_J E_\alpha^{IJ} \tag{2.18a}$$

and therefore any computation involving the partial differentiation operators $\partial_\alpha^I$ can also be carried out using the Lie-Euler operators. In fact, calculations in the variational calculus are often simplified by the use of these latter operators.

We now use Propositions 2.1 and 2.2 to give elementary proofs of two important properties of the Lie-Euler operators. Recall that round parenthesis indicate symmetrization on the enclosed indices so that, for example,

$$\delta^{(i}_j E^{hk)}_\alpha = \frac{1}{3}(\delta^i_j E^{hk}_\alpha + \delta^h_j E^{ik}_\alpha + \delta^k_j E^{hi}_\alpha).$$

PROPOSITION 2.6. *Let $f$ be an function on $J^\infty(U)$. The Lie-Euler operators $E^I_\alpha$ and the total differentiation operators $D_j$ satisfy the commutation relations*

$$E^I_\alpha(D_j f) = \delta^{(i}_j E^{I')}_\alpha(f), \qquad I = iI' \tag{2.19a}$$

*for $|I| > 0$ and*

$$E_\alpha(D_j f) = 0. \tag{2.19b}$$

PROOF: On account of the fact that $\mathcal{L}_{\mathrm{pr}\,Y}$ commutes with $d_H$ (see Corollary 1.17) we can conclude that

$$\mathcal{L}_{\mathrm{pr}\,Y}(D_j f) = D_j(\mathcal{L}_{\mathrm{pr}\,Y} f).$$

Due to (2.14) we can rewrite this equation in the form (assuming that $f$ is of order $k$)

$$\begin{aligned}\sum_{|I|=0}^{k+1} D_I[Y^\alpha E^I_\alpha(D_j f)] &= D_j\{\sum_{|I|=0}^{k} D_I[Y^\alpha E^I_\alpha(f)]\}\\ &= \sum_{|I|=1}^{k+1} D_I[Y^\alpha \delta^{(i}_j E^{I')}_\alpha(f)].\end{aligned}$$

By virtue of Proposition 2.1, we can match the coefficients in this equation and thereby arrive at (2.19). ∎

COROLLARY 2.7. *Suppose f is a local $l^{\text{th}}$ order divergence in the sense that there are functions $A^J, |J| = l$ on $J^\infty(U)$ for which $f = D_J A^J$. Then $E^I_\alpha(f) = 0$ for all $|I| \leq l - 1$.*

PROOF: This follows immediately from the repeated application of (2.19). Incidentally, this result first appeared in [*45*]. ∎

PROPOSITION 2.8. *Let $f$ and $g$ be two functions on $J^k(U)$. Then the Euler-Lagrange operator satisfies the local product rule*

$$E_\alpha(fg) = \sum_{|I|=0}^{k} [(-D)_I f] E^I_\alpha(g) + \sum_{|I|=0}^{k} E^I_\alpha(f)[(-D)_I(g)]. \tag{2.20}$$

PROOF: From the product rule for the Lie derivative

$$\mathcal{L}_{\operatorname{pr} Y}(fg) = (\mathcal{L}_{\operatorname{pr} Y} f)g + f(\mathcal{L}_{\operatorname{pr} Y} g)$$

and (2.14), we deduce that

$$\sum_{|I|=0}^{k} D_I[Y^\alpha E^I_\alpha(fg)]$$
$$= \sum_{|I|=0}^{k} D_I[Y^\alpha E^I_\alpha(f)]\, g + f \sum_{|I|=0}^{k} D_I[Y^\alpha E^I_\alpha(g)]. \tag{2.21}$$

Moreover, Proposition 2.1 can be used to infer that

$$D_I[Y^\alpha E^I_\alpha(f)]g = Y^\alpha E^I_\alpha(f)[(-D)_I g] + D_i[R^i_1(Y)]$$

and

$$f D_I[Y^\alpha E^I_\alpha(g)] = Y^\alpha[(-D)_I f] E^I_\alpha(g) + D_i[R^i_2(Y)].$$

for some choice of linear total differential operators $R^i_1$ and $R^i_2$. These two equations are substituted into (2.21). Comparison of the coefficients of $Y^\alpha$ yields (2.20). ∎

Had we explicitly exhibited the operators $R^i_1$ and $R^i_2$, then we could have compared the coefficients of the higher order derivatives of $Y^\alpha$ in (2.21) to recover Aldersley's product rule for the Lie-Euler operators $E^I_\alpha$, $|I| > 0$.

COROLLARY 2.9. *If $f$ is a function on $J^\infty(U)$ and*

$$E_\alpha(fg) = 0 \tag{2.22}$$

*for all functions $g = g(x^i)$ on $\pi(U)$, then $f = f(x^i)$. If (2.22) holds for all functions $g$ on $U$, then $f = 0$.*

PROOF: First take $g = g(x^i)$. Then the product rule (2.20) reduces to

$$E_\alpha(fg) = \sum_{|I|=0}^{k} [(-D)_I(g)] E^I_\alpha(f) = 0$$

Since $g$ is arbitrary, this implies that $E^I_\alpha(f) = 0$ for all $|I|$. In view of (2.18), this shows that $f$ is a function of the base variables $x^i$ alone. The second statement now follows by letting $g = u^\beta$ for some $\beta$. ∎

For our next application of Propositions 2.1 and 2.2, let $\omega$ be a type $(r,s)$ form on $J^\infty(E)$ and let

$$P_\omega(Y) = \operatorname{pr} Y \mathbin{\lrcorner} \omega = \sum_{|I|=0}^{k} (D_I Y^\alpha)(\partial_\alpha^I \mathbin{\lrcorner} \omega).$$

By Proposition 2.1, we can rewrite this operator in the form

$$\operatorname{pr} Y \mathbin{\lrcorner} \omega = \sum_{|I|=0}^{k} D_I[Y^\alpha F_\alpha^I(\omega)] \tag{2.23}$$

where the operators $F_\alpha^I$ are defined by

$$F_\alpha^I(\omega) = \sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|J|} (-D)_J (\partial_\alpha^{IJ} \mathbin{\lrcorner} \omega). \tag{2.24}$$

We call these operators *interior Euler operators* since they arise from the representation (2.23) of the interior product operator. Note that $F_\alpha^I(\omega)$ is a form of type $(r, s-1)$. These operators were defined, at least recursively, by Tulczyjew [*70*]. We shall see in Chapter 4 that they play a key role in the proof of the local exactness of the interior rows of the variational bicomplex.

PROPOSITION 2.10. *Let $\omega$ be a type $(r,s)$ form on $J^\infty(U)$. Then the interior Euler operators $F_\alpha^I$ and the horizontal differential $d_H$ satisfy the commutation relations*

$$F_\alpha^I(d_H\omega) = F_\alpha^{(I'}(dx^{i)} \wedge \omega) \tag{2.25a}$$

*for $|I| > 0$ and $I = I'i$, and*

$$F_\alpha(d_H\omega) = 0. \tag{2.25b}$$

*Moreover, if $Y$ is any evolutionary vector field, then*

$$\operatorname{pr} Y \mathbin{\lrcorner} F_\alpha^I(\omega) = -F_\alpha^I(\operatorname{pr} Y \mathbin{\lrcorner} \omega). \tag{2.26}$$

PROOF: To prove (2.25) we use the commutation relation

$$\operatorname{pr} Y \mathbin{\lrcorner} (d_H\omega) = -d_H(\operatorname{pr} Y \mathbin{\lrcorner} \omega)$$

and repeat the calculations used in the proof of Proposition 2.6.

To prove (2.26), let $Z$ be another evolutionary vector field. Since interior evaluation by $\operatorname{pr} Y$ commutes with $D_i$, it follows immediately from the defining relations (2.23) for the interior Euler operators that

$$\begin{aligned} \operatorname{pr} Y \lrcorner\, (\operatorname{pr} Z \lrcorner\, \omega) &= \operatorname{pr} Y \lrcorner \sum_{|I|=0}^{k} D_I[Z^\alpha F^I_\alpha(\omega)] \\ &= \sum_{|I|=0}^{k} D_I[Z^\alpha \big(\operatorname{pr} Y \lrcorner\, F^I_\alpha(\omega)\big)] \end{aligned} \tag{2.27}$$

and

$$- \operatorname{pr} Z \lrcorner\, (\operatorname{pr} Y \lrcorner\, \omega) = - \sum_{|I|=0}^{k} [Z^\alpha F^I_\alpha(\operatorname{pr} Y \lrcorner\, \omega)]. \tag{2.28}$$

The left-hand sides of (2.27) and (2.28) are equal and therefore the coefficients on the right-hand sides must coincide. ∎

With $r = n$, Proposition 2.2 can be applied to the operator $P_\omega(Y)$ to deduce that

$$\operatorname{pr} Y \lrcorner\, \omega = E(P_\omega)(Y) + d_H[R(Y)], \tag{2.29}$$

where $E(P_\omega)(Y) = Y^\alpha F_\alpha(\omega)$ is an invariantly defined operator and $R(Y)$ is the locally defined, total differential operator

$$R(Y) = \sum_{|I|=0}^{k-1} D_I[Y^\alpha \big(D_j \lrcorner\, F^{Ij}_\alpha(\omega)\big)]. \tag{2.30}$$

Definition 2.11. *For $s \geq 1$, the linear differential operator*

$$I \colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$$

*defined by*

$$I(\omega) = \frac{1}{s}\, \theta^\alpha \wedge F_\alpha(\omega) = \frac{1}{s}\, \theta^\alpha \wedge \Big[ \sum_{|I|=0}^{\infty} (-D)_I(\partial^I_\alpha \lrcorner\, \omega) \Big] \tag{2.31}$$

*is called the interior Euler operator on $\Omega^{n,s}(J^\infty(E))$.*

Since we have defined $I$ in local coordinates, we need to verify that it is actually globally well-defined. This can be done in one of two ways. The first way is obtain the change of variables formula for the operator $F_\alpha(\omega)$. Let $(x, u, U)$ and $(y, v, V)$ be two overlapping coordinate charts on $E$. Let $\omega_{|U}$ and $\omega_{|V}$ be the restrictions of a type $(n, s)$ form $\omega$ on $J^\infty(E)$ to $J^\infty(U)$ and $J^\infty(V)$. Let

$$F_\alpha(\omega_{|U}) = \sum_{|I|=0}^{k} (-D)_I(\partial_\alpha^I \lrcorner\, \omega_{|U})$$

and

$$\overline{F}_\alpha(\omega_{|V}) = \sum_{|I|=0}^{k} (-\overline{D})_I(\overline{\partial}_\alpha^I \lrcorner\, \omega_{|V}),$$

where $\overline{D}_i$ denotes total differentiation with respect to $y^i$ and $\overline{\partial}_\alpha^I$ is the symmetrized partial differentiation with respect to $v_I^\alpha$. Then, because $E(P_\omega)$ is an invariant differential operator, we have, on $U \cap V$,

$$\overline{Y}^\beta \overline{F}_\beta(\omega_{|V}) = Y^\alpha F_\alpha(\omega_{|U})$$

and hence

$$\overline{F}_\beta(\omega_{|V}) = \frac{\partial u^\alpha}{\partial v^\beta}\, F_\alpha(\omega_{|U}). \tag{2.32}$$

It is now evident that the form $\theta^\alpha \wedge F_\alpha(\omega)$ is a globally well-defined, type $(n, s)$ form on $J^\infty(E)$.

The other way to verify that the differential form $I(\omega)$ is well-defined globally is to give a coordinate-free expression for its value on the prolongations of evolutionary vector fields. For instance, when $s = 2$, we find, on account of (2.26) and (2.31), that

$$\begin{aligned} I(\omega)(\operatorname{pr} Y_1, \operatorname{pr} Y_2) &= \frac{1}{2} \operatorname{pr} Y_2 \lrcorner\, \operatorname{pr} Y_1 \lrcorner\, [\theta^\alpha \wedge F_\alpha(\omega)] \\ &= \frac{1}{2} \operatorname{pr} Y_2 \lrcorner\, [Y_1^\alpha F_\alpha(\omega) + \theta^\alpha F_\alpha(\operatorname{pr} Y_1 \lrcorner\, \omega)] \\ &= \frac{1}{2}[-Y_1^\alpha F_\alpha(\operatorname{pr} Y_2 \lrcorner\, \omega) + Y_2^\alpha F_\alpha(\operatorname{pr} Y_1 \lrcorner\, \omega)] \\ &= \frac{1}{2}[-E(P_\alpha)(Y_1) + E(P_\beta)(Y_2)], \end{aligned}$$

where $\alpha = \operatorname{pr} Y_2 \lrcorner\, \omega$ and $\beta = \operatorname{pr} Y_1 \lrcorner\, \omega$. In general, if $\omega$ is of type $(r, s)$, if $Y_1$, $Y_2$, $\dots$, $Y_s$ are evolutionary vector fields, and if we set

$$\omega_i = \operatorname{pr} Y_s \lrcorner \cdots \operatorname{pr} Y_{i-1} \lrcorner\, \operatorname{pr} Y_{i+1} \cdots \lrcorner\, \operatorname{pr} Y_1 \lrcorner\, \omega,$$

then

$$I(\omega)(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_s) = \frac{1}{s}\sum_{i=1}^{s}(-1)^{s+i} E(P_{\omega_i})(Y_i). \tag{2.33}$$

Alternatively, we have that

$$\operatorname{pr} Y \lrcorner\, I(\omega) = E(P_\omega)(Y) + I(\operatorname{pr} Y \lrcorner\, \omega) \tag{2.34}$$

which furnishes us with an inductive definition of $I$ by vertical degree.

THEOREM 2.12. *The interior Euler operator* $I\colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$ *enjoys the following properties.*

*(i) The kernel of $I$ contains all locally $d_H$ exact forms in $\Omega^{n,s}(J^\infty(E))$, i.e.,*

$$I \circ d_H = 0.$$

*(ii) The interior Euler operator $I$ is a projection operator, i.e.,*

$$I^2 = I.$$

*(iii) The difference $I(\omega) - \omega$ is locally $d_H$ exact. Thus, on each coordinate chart $J^\infty(U)$, there is a type $(n-1, s)$ form $\eta$ such that*

$$\omega = I(\omega) + d_H(\eta). \tag{2.35}$$

*This decomposition is unique in the sense that if*

$$\omega = \tilde{\omega} + d_H(\tilde{\eta})$$

*and $\tilde{\omega}$ is in the image of $I$, then $\tilde{\omega} = I(\omega)$.*

*(iv) The interior Euler operator is a natural differential operator. If $\phi\colon E \to E$ is a fiber-preserving map which prolongs to $\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(E)$, then*

$$I[(\operatorname{pr}\phi)^*(\omega)] = (\operatorname{pr}\phi)^*[I(\omega)].$$

*(v) If $X$ is a projectable vector field on $E$, then*

$$I[\mathcal{L}_{\operatorname{pr} X}\omega] = \mathcal{L}_{\operatorname{pr} X}[I(\omega)]. \tag{2.36}$$

*(vi) The induced differential* $\delta\colon \mathcal{F}^s \to \mathcal{F}^{s+1}$ *defined by* $\delta = I \circ d_V$ *satisfies* $\delta^2 = 0$.

PROOF: Property (i) follows immediately from (2.25b). To prove (ii), first observe that if $\omega$ is of vertical degree $s$, then by virtue of (2.23),

$$\omega = \frac{1}{s}[\sum_{|I|=0}^{\infty} \theta_I^\alpha \wedge (\partial_\alpha^I \lrcorner\, \omega)] = \frac{1}{s}\sum_{|I|=0}^{\infty} D_I[\theta^\alpha \wedge F_\alpha^I(\omega)].$$

Because $\omega$ is of top horizontal degree, this equation can be rewritten as (2.35), with

$$\eta = \sum_{|I|=0}^{\infty} D_I[\theta^\alpha \wedge F_\alpha^{Ij}(D_j \lrcorner\, \omega)]. \tag{2.37}$$

To (2.35), apply the interior Euler operator $I$ and invoke property (i) to conclude that $I(\omega) = I^2(\omega)$. This proves (ii) and then (iii) follows immediately.

Property (iv) follows from the naturality of the operator $E(P_\omega)$. Property (v) is simply the infinitesimal version of (iv). We can also give a direct proof of (v) which, in addition, serves as a good illustration of the variational calculus which we have developed thus far.

To begin, let the vector field $X$ on $E$ be given locally by

$$X = a^i(x)\frac{\partial}{\partial x^i} + b^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Since

$$\mathcal{L}_{\mathrm{pr}\,X}\,\theta^\alpha = d_V b^\alpha = \frac{\partial b^\alpha}{\partial u^\beta}\,\theta^\beta,$$

it is a straightforward matter to check that (2.36) is equivalent to the commutation rule

$$F_\alpha(\mathcal{L}_{\mathrm{pr}\,X}\,\omega) = \mathcal{L}_{\mathrm{pr}\,X}\big(F_\alpha(\omega)\big) + \frac{\partial b^\beta}{\partial u^\alpha}F_\beta(\omega). \tag{2.38}$$

To derive (2.38), let $Y$ be an evolutionary vector field with local components

$$Y = Y^\alpha[x,u]\frac{\partial}{\partial u^\alpha}.$$

Then, on the one hand, since $d_H$ commutes with $\mathcal{L}_{\mathrm{pr}\,X}$, the application of $\mathcal{L}_{\mathrm{pr}\,X}$ to (2.29) gives rise to

$$\mathcal{L}_{\mathrm{pr}\,X}(\mathrm{pr}\,Y \lrcorner\, \omega) = [\mathrm{pr}\,X(Y^\alpha)]F_\alpha(\omega) + Y^\alpha[\mathcal{L}_{\mathrm{pr}\,X}F_\alpha(\omega)] + d_H\eta_1.$$

On the other hand, the product rule implies that

$$\mathcal{L}_{\mathrm{pr}\,X}(\mathrm{pr}\,Y \mathbin{\lrcorner} \omega) = [\mathcal{L}_{\mathrm{pr}\,X}(\mathrm{pr}\,Y)] \mathbin{\lrcorner} \omega + \mathrm{pr}\,Y \mathbin{\lrcorner} [\mathcal{L}_{\mathrm{pr}\,X}\,\omega]. \tag{2.39}$$

By virtue of Proposition 1.20,

$$\mathcal{L}_{\mathrm{pr}\,X}(\mathrm{pr}\,Y) = [\mathrm{pr}\,X, \mathrm{pr}\,Y] = \mathrm{pr}\,Z,$$

where

$$Z = \big[\mathrm{pr}\,X(Y^\alpha) - Y^\beta \frac{\partial b^\alpha}{\partial u^\beta}\big] \frac{\partial}{\partial u^\alpha}.$$

Note that the generalized vector field $Z$ on $E$ is $\pi$ vertical but that this would not have been true had $X$ not been projectable.

To each term on the right-hand side of (2.39) we apply (2.23) to conclude that

$$[\mathcal{L}_{\mathrm{pr}\,X}(\mathrm{pr}\,Y)] \mathbin{\lrcorner} \omega = Z^\alpha F_\alpha(\omega) + d_H \eta_2 \tag{2.40}$$

and

$$\mathrm{pr}\,Y \mathbin{\lrcorner} \mathcal{L}_{\mathrm{pr}\,X}\omega = Y^\alpha F_\alpha(\mathcal{L}_{\mathrm{pr}\,X}\,\omega) + d_H \eta_3. \tag{2.41}$$

The combination of (2.39)–(2.41) leads immediately to (2.38), as required.

Finally, to prove (vi), let $\omega \in \mathcal{F}^s(J^\infty(E))$. Then $d_V\omega \in \Omega^{n,s+1}$ and therefore, at least locally,

$$d_V\omega = I(d_V\omega) + d_H\eta,$$

for some form $\eta$ of type $(n-1, s+1)$. The application of $d_V$ to this equation leads to

$$d_V \circ I \circ d_V(\omega) = d_H d_V(\omega)$$

from which it now follows that

$$\delta_V^2 = I \circ d_V \circ I \circ d_V = 0.$$

∎

Corollary 2.13. *The Euler-Lagrange operator* $E\colon \Omega^{n,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$ *enjoys the following properties.*

*(i) For any Lagrangian* $\lambda \in \Omega^{n,0}(J^\infty(E))$,

$$E(\lambda) = I \circ d_V(\lambda). \tag{2.42}$$

*(ii) For any type* $(n-1,0)$ *form* $\eta$,

$$E(d_H\eta) = 0. \tag{2.43}$$

*(iii) If* $\phi \colon E \to E$ *is a fiber-preserving map which prolongs to* $\operatorname{pr}\phi \colon J^\infty(E) \to J^\infty(E)$, *then*

$$E((\operatorname{pr}\phi)^*\lambda) = (\operatorname{pr}\phi)^* E(\lambda).$$

*(iv) If* $X$ *is a projectable vector field on* $E$, *then*

$$E(\mathcal{L}_{\operatorname{pr}X}\lambda) = \mathcal{L}_{\operatorname{pr}X} E(\lambda). \tag{2.44}$$

PROOF: Properties (ii)–(iv) follow directly from (i) and the corresponding properties of the interior Euler operator $I$. To prove (i), let $Y$ be an arbitrary evolutionary vector field on $J^\infty(E)$. Then (2.24) implies that

$$\mathcal{L}_{\operatorname{pr}Y}\lambda = \operatorname{pr}Y \lrcorner\, d_V\lambda.$$

The representation (2.14) of the Lie derivative operator shows that the left-hand side of this equation equals $\operatorname{pr}Y \lrcorner\, E(\lambda) + d_H\eta_1$ while the representation (2.23) of the interior product operator shows that the right-hand side equals $\operatorname{pr}Y \lrcorner\, I(d_V\lambda) + d_H\eta_2$. Equation (2.42) now follows from Proposition 2.2. Alternatively, we find in local coordinates that

$$\begin{aligned} I \circ (d_V\lambda) &= I\big[\sum_{|I|=0}^{k} (\partial_\alpha^I L)\, \theta_I^\alpha \wedge \nu\big] \\ &= \theta^\alpha \wedge \big[\sum_{|I|=0}^{k} (-D)_I(\partial_\alpha^I)L\big] \wedge \nu \\ &= E(\lambda), \end{aligned}$$

as required. ∎

**C. Some Geometric Variational Problems.** In this section we use the variational bicomplex to compute the Euler-Lagrange forms for some geometric variational problems for curves and surfaces. The idea is a simple one. In each instance we use a moving frame adopted to the problem to construct an invariant basis for the contact ideal and then we compute the components of the Euler-Lagrange form with respect to this basis. The problems considered here are taken from Griffiths' book [*31*] on exterior differential systems and the calculus of variations.

EXAMPLE 2.14. *Space curves in* $\mathbf{R}^3$.

Let $E$ be the trivial bundle $E\colon \mathbf{R}\times\mathbf{R}^3 \to \mathbf{R}$ with Cartesian coordinates $(x, R) \to x$, where $R = (u^1, u^2, u^3)$ is the position vector in $\mathbf{R}^3$. The two inequalities

$$|R'| \neq 0 \quad \text{and} \quad |R' \times R''| \neq 0.$$

define a open subset $\mathcal{R} \subset J^\infty(E)$. A section $\gamma$ of $E$ is a space curve in $\mathbf{R}^3$. The space curve $\gamma$ is called regular if $j^\infty(\gamma)(x) \in \mathcal{R}$ for all $x$. We restrict our considerations to the variational bicomplex over $\mathcal{R}$.

A Lagrangian for a variational problem on $\mathcal{R}$ is a type $(1, 0)$ form

$$\lambda = L(x, R, R', R'', \dots, R^{(k)})\, dx.$$

For geometric variational problems, we consider only Lagrangians $\lambda$ which are *natural* in the sense that they are invariant under the pseudo-group of local, orientation-preserving diffeomorphisms of the base $\mathbf{R}$ (*i.e.*, under arbitrary reparameterizations of the curve) and invariant under the group of Euclidean motions of the fiber $\mathbf{R}^3$. Call the group of all such transformations $G$.

DEFINITION 2.15. *The variational bicomplex* $\{\Omega^{*,*}_G(\mathcal{R}), d_H, d_V\}$ *of* $G$ *invariant forms on* $\mathcal{R}$ *is called the natural variational bicomplex for regular space curves.*

We shall describe the forms in $\Omega^{r,s}_G(\mathcal{R})$ explicitly. Let $\{T, N, B\}$ be the Frenet frame for a regular space curve. From the jet bundle viewpoint, we treat the frame $\{T, N, B\}$ as a smooth map from the infinite jet space $\mathcal{R}$ to $SO(3)$. The function $T$ factors through the first jet bundle while both $N$ and $B$ factor through the second jet bundle. The curvature $\kappa$ and torsion $\tau$ are $G$ invariant functions on $\mathcal{R}$ defined by the Frenet formula

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.45}$$

Now define contact one forms $\Theta^1$, $\Theta^2$, and $\Theta^3$ by

$$\theta = \Theta^1 T + \Theta^2 N + \Theta^3 B, \tag{2.46}$$

where $\theta = dR - R'dx$ are the usual, vector-valued contact one forms on $\mathcal{R}$. The forms $\Theta^i$, together with all of their derivatives $\dot\Theta^i, \ddot\Theta^i, \ldots$ with respect to arclength $s$ form a $G$ invariant basis for the contact ideal — if $\phi\colon E \to E$ belongs to $G$ and $\operatorname{pr}\phi\colon \mathcal{R} \to \mathcal{R}$ is the prolongation of $\phi$, then

$$(\operatorname{pr}\phi)^*\,\Theta^i = \Theta^i \quad \text{and} \quad (\operatorname{pr}\phi)^*\,\dot\Theta^i = \dot\Theta^i$$

and so on. We also define the $G$ invariant horizontal form

$$\sigma = |R'|\,dx, \quad \text{where} \quad |R'| = \sqrt{\langle R', R'\rangle},$$

and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbf{R}^3$. Consequently, every form in $\Omega^{*,*}_G(\mathcal{R})$ can be expressed as wedge products of $\sigma$, the forms $\Theta^i$ and their derivatives, with coefficients which are smooth $G$ invariant functions on $\mathcal{R}$. Such coefficients are necessarily functions of $\kappa$, $\tau$ and their derivatives. In particular, a Lagrangian $\lambda \in \Omega^{1,0}_G(\mathcal{R})$ assumes the form

$$\lambda = L(\kappa, \tau, \dot\kappa, \dot\tau, \ddot\kappa, \ddot\tau, \ldots, \kappa^{(p)}, \tau^{(p)})\,\sigma. \tag{2.47}$$

The next step in our analysis of the natural variational bicomplex for space curves is to compute the vertical differentials of $\sigma$, $\Theta^1$, $\Theta^2$, $\Theta^3$, $\kappa$ and $\tau$. To begin, we totally differentiate (2.46) with respect to $s$ (where $d/ds = |R'|d/dt$) and apply the Frenet formula to arrive at

$$\dot\theta = \alpha\,T + \beta\,N + \gamma\,B, \tag{2.48}$$

where

$$\alpha = \dot\Theta^1 - \kappa\Theta^2, \tag{2.49a}$$

$$\beta = \kappa\Theta^1 + \dot\Theta^2 - \tau\Theta^3, \quad \text{and} \tag{2.49b}$$

$$\gamma = \tau\Theta^2 + \dot\Theta^3. \tag{2.49c}$$

Next we compute the vertical differential of $\dfrac{ds}{dt}$:

$$\begin{aligned} d_V\Bigl(\frac{ds}{dt}\Bigr) &= d_V|R'| = \frac{1}{|R'|}\langle\theta', R'\rangle \\ &= \langle\dot\theta, T\rangle = \alpha. \end{aligned} \tag{2.50}$$

This formula leads to

$$d_V \sigma = \alpha \wedge \sigma \tag{2.51}$$

and the commutation rule

$$d_V \frac{d}{ds} = -\alpha \frac{d}{ds} + \frac{d}{ds} d_V. \tag{2.52}$$

We shall make repeated use of this result.

Indeed, this commutation rule, together with (2.48), immediately yields

$$\begin{aligned} d_V T = d_V \frac{dR}{ds} &= -\alpha T + \dot{\theta} \\ &= \beta N + \gamma B. \end{aligned} \tag{2.53}$$

Next we apply $d_V$ to the first Frenet formula to obtain

$$d_V \frac{dT}{ds} = (d_V \kappa) N + \kappa \, d_V N. \tag{2.54}$$

To evaluate the left-hand side of this equation, we use the commutation rule (2.52), (2.53), and the Frenet formula to find that

$$\begin{aligned} d_V \frac{dT}{ds} &= -\alpha \frac{dT}{ds} + \frac{d}{ds}(d_V T) \\ &= -\kappa\beta T + (-\kappa\alpha + \dot{\beta} - \tau\gamma) N + (\tau\beta + \dot{\gamma}) B. \end{aligned} \tag{2.55}$$

A comparison of (2.54) and (2.55) implies that

$$d_V \kappa = -\kappa\alpha + \dot{\beta} - \tau\gamma \tag{2.56}$$

and

$$d_V N = -\beta T + \frac{1}{\kappa}(\tau\beta + \dot{\gamma}) B.$$

Next, the vertical differential $d_V B$ can be computed by differentiating the orthonormality relations $\langle T, B \rangle = \langle N, B \rangle = 0$ and $\langle B, B \rangle = 1$. We conclude that the vertical differential of the Frenet frame is

$$d_V \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \frac{1}{\kappa}(\tau\beta + \dot{\gamma}) \\ -\gamma & -\frac{1}{\kappa}(\tau\beta + \dot{\gamma}) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.57}$$

The formula for $d_V\tau$ can now be derived by differentiating the last Frenet formula $\dot{B} = -\tau N$. This gives

$$d_V\tau = -\tau\alpha + \kappa\gamma + \frac{d}{ds}[\frac{1}{\kappa}(\tau\beta + \dot{\gamma})]. \tag{2.58}$$

Finally, by applying $d_V$ to (2.48) and by substituting from (2.57), we conclude that

$$d_V \begin{bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{bmatrix} = \begin{bmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \frac{1}{\kappa}(\tau\beta + \dot{\gamma}) \\ -\gamma & -\frac{1}{\kappa}(\tau\beta + \dot{\gamma}) & 0 \end{bmatrix} \wedge \begin{bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{bmatrix}. \tag{2.59}$$

Equations (2.51), (2.52), (2.56), (2.58) and (2.59) constitute the basic computational formulas for the variational bicomplex $\Omega_G^{*,*}(\mathcal{R})$. For the natural variational bicomplex for regular plane curves, formulas (2.51) and (2.52) remain valid while the others simplify to

$$d_V\kappa = -\kappa\alpha + \dot{\beta} \tag{2.60}$$

and

$$d_V \begin{bmatrix} \Theta^1 \\ \Theta^2 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \wedge \begin{bmatrix} \Theta^1 \\ \Theta^2 \end{bmatrix}. \tag{2.61}$$

We are now ready to compute the Euler-Lagrange form for the natural Lagrangian

$$\lambda = L(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ddot{\kappa}, \ddot{\tau}, \ldots, \kappa^{(p)}, \tau^{(p)})\, \sigma.$$

First, let

$$P_\kappa^{(p)} = \frac{\partial L}{\partial \kappa^{(p)}}, \qquad P_\tau^{(p)} = \frac{\partial L}{\partial \tau^{(p)}}$$

and define recursively, for $j = p-1, \ldots, 1, 0$,

$$P_\kappa^{(j)} = \frac{\partial L}{\partial \kappa^{(j)}} - \frac{d}{ds}\big(P_\kappa^{(j+1)}\big), \qquad P_\tau^{(j)} = \frac{\partial L}{\partial \tau^{(j)}} - \frac{d}{ds}\big(P_\tau^{(j+1)}\big).$$

Define the Euler-Lagrange operators $E_\kappa(L)$ and $E_\tau(L)$ and the Hamiltonian operator H(L) by

$$E_\kappa(L) = P_\kappa^{(0)} = \frac{\partial L}{\partial \kappa} - \frac{d}{ds}\big(\frac{\partial L}{\partial \dot{\kappa}}\big) + \frac{d^2}{ds^2}\big(\frac{\partial L}{\partial \ddot{\kappa}}\big) - \cdots\ ,$$

$$E_\tau(L) = P_\tau^{(0)} = \frac{\partial L}{\partial \tau} - \frac{d}{ds}\big(\frac{\partial L}{\partial \dot{\tau}}\big) + \frac{d^2}{ds^2}\big(\frac{\partial L}{\partial \ddot{\tau}}\big) \cdots\ ,$$

and

$$H(L) = -L + P^{(1)}_\kappa \dot\kappa + P^{(2)}_\kappa \ddot\kappa + \cdots + P^{(p)}_\kappa \kappa^{(p)}$$
$$+ P^{(1)}_\tau \dot\tau + P^{(2)}_\tau \ddot\tau + \cdots + P^{(p)}_\tau \tau^{(p)}.$$

Bear in mind that $E_\kappa(L)$ and $E_\tau(L)$ are the Euler-Lagrange expression obtained from $L$ thorough the variation of $\kappa$ and $\tau$ thought of as new dependent variables whereas our objective is to compute the Euler-Lagrange expression for $L$ derived from the variation of the underlying curves on $\mathbf{R}^3$. A direct calculation leads to

$$\frac{dH}{ds} = -\dot\kappa E_\kappa - \dot\tau E_\tau. \tag{2.62}$$

This identity may also be derived from Noether's theorem (see Theorem 3.24) for autonomous Lagrangians.

We now compute $d_V\lambda$:

$$\begin{aligned} d_V\lambda =& \Big[\frac{\partial L}{\partial\kappa} d_V\kappa + \frac{\partial L}{\partial\dot\kappa} d_V\dot\kappa + \cdots + \frac{\partial L}{\partial\kappa^{(p)}} d_V\kappa^{(p)}\Big] \wedge \sigma \\ &+ \Big[\frac{\partial L}{\partial\tau} d_V\tau + \frac{\partial L}{\partial\dot\tau} d_V\dot\tau + \cdots + \frac{\partial L}{\partial\tau^{(p)}} d_V\tau^{(p)}\Big] \wedge \sigma \\ &+ L\,\alpha \wedge \sigma. \end{aligned} \tag{2.63}$$

We use the commutation rule (2.52) to "integrate by parts", that is, to cast off expressions which are total derivatives and which therefore lie in the kernel of $I$. For example,

$$\begin{aligned} \frac{\partial L}{\partial\kappa^{(p)}} d_V\kappa^{(p)} \wedge \sigma &= P^{(p)}_\kappa d_V\big(\frac{d}{ds}\kappa^{(p-1)}\big) \wedge \sigma \\ &= P^{(p)}_\kappa \big[-\kappa^{(p)}\alpha + \frac{d}{ds}(d_V\kappa^{(p-1)})\big] \wedge \sigma \\ &= \big[-P^{(p)}_\kappa \kappa^{(p)}\big]\alpha \wedge \sigma - \big[\big(\frac{d}{ds}P^{(p)}_\kappa\big)\, d_V\kappa^{(p-1)}\big] \wedge \sigma + d_H[\cdots]. \end{aligned}$$

The first term on the right-hand side of this last equation can be combined with the last term $L\,\alpha \wedge \sigma$ in (2.63) to give two of the terms in the formula for $H(L)$; the next term is this equation can be combined with the term $\dfrac{\partial L}{\partial\kappa^{(p-1)}} d_V\kappa^{(p-1)}$ in (2.63) to become $P^{(p-1)}_\kappa\, d_V\kappa^{(p-1)}$. By continuing in this fashion we deduce that

$$d_V\lambda = \big[E_\kappa(L) d_V\kappa + E_\tau(L) d_V\tau - H(L)\alpha\big] \wedge \sigma + d_H[\cdots].$$

Already, this formula insures that the Euler-Lagrange form $E(\lambda)$ will be expressed in terms of the Euler-Lagrange expressions $E_\kappa(L)$ and $E_\tau(L)$ and the Hamiltonian $H(L)$. To complete the calculation of $E(\lambda)$, we use (2.53) and (2.48) to evaluate $E_\kappa d_V\kappa\wedge\sigma$:

$$\begin{aligned}
E_\kappa d_V\kappa\wedge\sigma &= E_\kappa[-\kappa\alpha+\dot\beta-\tau\gamma]\wedge\sigma = -[\kappa E_\kappa\alpha+\dot E_\kappa\beta+\tau E_\kappa\gamma]\wedge\sigma+d_H[E_\kappa\beta] \\
&= [-\kappa\dot E_\kappa\Theta^1-\kappa E_\kappa\dot\Theta^1]\wedge\sigma+[(\kappa^2-\tau^2)E_\kappa-\dot E_\kappa\dot\Theta^2]\wedge\sigma \\
&\qquad+[\tau\dot E_\kappa\Theta^3-\tau E_\kappa\dot\Theta^3]\wedge\sigma+d_H[\cdots] \\
&= [\dot\kappa E_\kappa]\Theta^1\wedge\sigma+[(\kappa^2-\tau^2)E_\kappa+\ddot E_\kappa]\Theta^2\wedge\sigma+[2\tau\dot E_\kappa+\dot\tau E_\kappa]\Theta^3\wedge\sigma \\
&\qquad+d_H[\cdots].
\end{aligned}$$

A similar but slightly longer calculation is required to evaluate $E_\tau d_V\tau\wedge\sigma$ and thereby complete the proof of the following proposition.

PROPOSITION 2.16. *Let* $\lambda\in\Omega^{1,0}_G(\mathcal R)$,

$$\lambda = L(\kappa,\tau,\dot\kappa,\dot\tau,\ddot\kappa,\ddot\tau,\ldots,\kappa^{(p)},\tau^{(p)})\,\sigma,$$

*be a natural Lagrangian on the variational bicomplex for regular space curves. Let $E_\kappa$, $E_\tau$ and $H$ be the Euler-Lagrange expressions and Hamiltonian function for $L$ viewed as a function of $\kappa$, $\tau$ and their derivatives. Then*

$$E(\lambda) = [E_1\Theta^1+E_2\Theta^2+E_3\Theta^3]\wedge\sigma$$

*where*

$$E_1 = \dot H+\dot\kappa E_\kappa+\dot\tau E_\tau, \tag{2.64a}$$

$$\begin{aligned}E_2 &= \kappa H+(\kappa^2-\tau^2)E_\kappa+\ddot E_\kappa+2\kappa\tau E_\tau \\ &\qquad+\Big[\frac{\kappa\dot\tau-2\tau\dot\kappa}{\kappa^2}\Big]\dot E_\tau+2\frac{\tau}{\kappa}\ddot E_\tau,\end{aligned} \tag{2.64b}$$

*and*

$$\begin{aligned}E_3 &= \dot\tau E_\kappa+2\tau\dot E_\kappa-\dot\kappa E_\tau+\Big[\frac{\tau^2\kappa^2-\kappa^4-2\dot\kappa^2+\kappa\ddot\kappa}{\kappa^3}\Big]\dot E_\tau \\ &\qquad+2\frac{\dot\kappa}{\kappa^2}\ddot E_\tau-\frac{1}{\kappa}\dddot E_\tau.\end{aligned} \tag{2.64c}$$

*On account of (2.62), the tangential component $E_1$ vanishes identically.*

A couple of special cases are noteworthy. First if $\lambda$ is independent the torsion $\tau$ and its derivatives, then these formula for the Euler-Lagrange form simplify to

$$E_2 = \kappa H + (\kappa^2 - \tau^2)E_\kappa + \ddot{E}_\kappa,$$

and

$$E_3 = \dot{\tau}E_\kappa + 2\tau\dot{E}_\kappa.$$

Since

$$E_\kappa\, E_3 = \frac{d}{ds}(\tau E_\kappa^2)$$

the Euler-Lagrange equations $E_2 = E_3 = 0$ always admit the first integral

$$\tau E_\kappa^2 = c_1,$$

where $c_1$ is a constant. In particular, for the Lagrangian $\lambda = \frac{1}{2}\kappa^2\sigma$, we have that

$$E_2 = \ddot{\kappa} + \frac{1}{2}\kappa^3 - \tau^2\kappa,$$

and

$$E_3 = \dot{\tau}\kappa + 2\tau\dot{\kappa}.$$

First integrals for this system are

$$\dot{\kappa}^2 + \tau^2\kappa^2 + \frac{1}{4}\kappa^4 = c_2 \quad \text{and} \quad \tau\kappa^2 = c_1.$$

For variational problems in the plane, there is no binormal component ( *i.e.*, $\Theta^3$) for the Euler-Lagrange form and we find that

$$E(\lambda) = \big[\kappa H + \kappa^2 E_\kappa + \ddot{E}_\kappa\big]\Theta^2 \wedge \sigma. \tag{2.65}$$

For example, the Euler-Lagrange equation for the Lagrangian $\lambda = \frac{1}{2}\dot{\kappa}^2\sigma$ is

$$\ddddot{\kappa} + \kappa^2\ddot{\kappa} - \frac{1}{2}\kappa\dot{\kappa}^2 = 0.$$

The Euler-Lagrange form for the Lagrangian $\lambda = \kappa\sigma$ vanishes identically as, of course, it must — the integral of $\lambda$ around any closed curve is the rotation number of that curve and this is a deformation invariant of $\gamma$.

Example 2.17. *Surfaces in* $\mathbf{R}^3$.

This example follows the same general lines as the previous one. We now take as our bundle $E\colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^2$ with Cartesian coordinates $(x^i, R) \to (x^i)$, where $i = 1, 2$ and $R = (u^1, u^2, u^3)$ is again the position vector in $\mathbf{R}^3$. We restrict our considerations to the open set $\mathcal{R} \subset J^\infty(E)$ where the one-jets $R_i$ satisfy

$$|R_1 \times R_2| \neq 0.$$

A section $\Sigma\colon \mathbf{R}^2 \to \mathbf{R}^3$ whose infinite jet lies in $\mathcal{R}$ defines a regularly parametrized surface in $\mathbf{R}^3$. Furthermore, we shall consider only those forms on $\mathcal{R}$ which are invariant under (i) all local oriented diffeomorphisms of the base $\mathbf{R}^2$, and (ii) under the action of the Euclidean group on the fiber $\mathbf{R}^3$.

Defintion 2.18. *The bicomplex* $(\Omega^{*,*}_G(\mathcal{R}), d_H, d_V)$ *is called the natural variational bicomplex for regular surfaces in* $\mathbf{R}^3$.

Observe that if $\lambda \in \Omega^{2,0}_G(\mathcal{R})$, then the integral

$$I = \int_M \lambda$$

is well defined for any compact surface $M$ in $\mathbf{R}^3$ — the diffeomorphism invariance of $\lambda$ insures that $\lambda$ pulls back via the coordinate charts of $M$ in a consistent, unambiguous fashion.

We now view the local differential geometry of surfaces as being defined over $\mathcal{R}$. Indeed, the normal vector $N$, the first and second fundamental forms $g_{ij}$ and $h_{ij}$ and the Christoffel symbols $\Gamma^i_{jl}$, as defined by

$$N = R_1 \times R_2 / |R_1 \times R_2|, \tag{2.66a}$$

$$g_{ij} = \langle R_i, R_j \rangle, \quad \text{and} \tag{2.66b}$$

$$R_{ij} = \Gamma^l_{ij} R_l + h_{ij} N. \tag{2.66c}$$

are all functions on the second jet bundle of $E$. Let

$$h^i_j = g^{il} h_{jl} \quad \text{and} \quad h^{ij} = g^{il} g^{jk} h_{lk},$$

where $g^{il}$ are the components of the inverse of the metric $g_{ij}$. Let $\nabla_j$ denote covariant differentiation with respect to the Christoffel symbols in the direction $D_j$. Equation (2.66c) can be rewritten as

$$\nabla_j R_i = h_{ij} N.$$

The Weingarten equations and the Codazzi equations

$$D_j N = -h^i_j R_i \tag{2.67}$$

and

$$\nabla_k h_{ij} = \nabla_j h_{ik} \tag{2.68}$$

are identities on $\mathcal{R}$. Finally, let

$$H_1 = \tfrac{1}{2}\operatorname{tr} h = \frac{1}{2} h^i_i,$$

and

$$H_2 = \tfrac{1}{2}\operatorname{tr} h^2 = 2H^2 - K,$$

where $K = \det(h_{ij})$. Both $H$ and $K$ are $G$ invariant functions on the second jet bundle of $E$. On sections $\Sigma$ of $E$ they are, of course, the mean and Gauss curvatures of the surface $\Sigma$.

We shall compute the Euler-Lagrange form for G invariant Lagrangians $\lambda \in \Omega^{2,0}_G(\mathcal{R})$. The invariance of $\lambda$ under the Euclidean transformations of the fiber $\mathbf{R}^3$ implies that $\lambda$ can be expressed in the form

$$\lambda = L(j^k(g_{ij}, h_{ij}))\, \nu = L(g_{ij},\, h_{ij},\, g_{ij,l},\, h_{ij,l}, \dots)\, \nu \tag{2.69}$$

for some smooth real-valued function $L$ defined on the $k$-jet bundle of maps from $\mathbf{R}^2$ to $\operatorname{Sym}^2_+(T^*(\mathbf{R}^2)) \times \operatorname{Sym}^2(T^*(\mathbf{R}^2))$. To insure that $\lambda$ is invariant with respect to all local diffeomorphism of the base $\mathbf{R}^2$, we shall suppose that $L$ is a *natural*[1] Lagrangian in the metric $g_{ij}$ and in the symmetric $(0,2)$ tensor field $h_{ij}$. Examples of $G$ invariant Lagrangians include the Gauss-Bonnet integrand

$$\lambda = \sqrt{g} K\, \nu,$$

the Willmore Lagrangian

$$\lambda = \frac{1}{2}\sqrt{g}(H^2 - K)\, \nu$$

as well as higher order Lagrangians such as

$$\lambda = \frac{1}{2}\sqrt{g}(\Delta H)^2 \nu,$$

[1]Natural variational principles are discussed in greater detail in Chapter 3C and Chapter 6

where $\Delta$ is the Laplacian computed with respect to the metric $g$. As a cautionary note we observe that, because of the Gauss-Codazzi equations, different functions $L$ can define the same $G$ invariant Lagrangian $\lambda$. For example, we can define the Gauss-Bonnet integrand by either a function on the 2-jets of the metric alone, *viz.*,

$$\sqrt{g}K = \frac{1}{\sqrt{g}} R_{1212},$$

where $R_{ijhk}$ is the curvature tensor of $g_{ij}$, or alternatively, by a zeroth order Lagrangian in $g$ and $h$, *viz.*,

$$\sqrt{g}K = \sqrt{g}(2H_1^2 - H_2).$$

Of course, as functions on the jets of the position vector $R$ both formulas coincide.

We introduce contact one forms $\Theta^i$ and $\Xi$ by

$$\theta = \Theta^i R_i + \Xi N, \tag{2.70}$$

where $\theta = dR - R_i dx^i$. The form $\Xi$ is $G$ invariant whereas $\Theta^i$ is invariant under the group of Euclidean transformations in the fiber but transforms as a vector-valued form under local diffeomorphisms of the base $\mathbf{R}^2$. Differentiation of (2.70) with respect to $D_j$ leads, by virtue of (2.66b) and (2.67), to

$$\theta_j = \alpha^i_j R_i + \beta_j N, \tag{2.71}$$

where

$$\alpha^i_j = \nabla_j \Theta^i - h^i_j \Xi,$$

and

$$\beta_j = h_{ij}\Theta^i + D_j \Xi.$$

Here the covariant derivative of $\Theta^i$ is the type $(1,1)$ tensor-valued form defined by

$$\nabla_j \Theta^i = D_j \Theta^i + \Gamma^i_{lj}\Theta^l.$$

Next, we compute $d_V g_{ij}$, $d_V N$ and $d_V h^i_j$. From (2.67) and (2.71), it immediately follows that

$$\begin{aligned} d_V g_{ij} &= \langle \theta_i, R_j \rangle + \langle \theta_j, R_i \rangle \\ &= g_{il}\,\alpha^l_j + g_{jl}\,\alpha^l_i. \end{aligned} \tag{2.72}$$

From the orthonormality relations $\langle N, N\rangle = 1$ and $\langle N, R_i\rangle = 0$, we deduce that

$$d_V N = -g^{ij}\beta_i R_j. \tag{2.73}$$

To compute $d_V h^i_j$, we apply $d_V$ to the Weingarten equation (2.67) to obtain

$$D_j\big(d_V N\big) = -\big(d_V h^i_j\big) R_j - h^i_j\,\theta_j.$$

We substitute into this equation from (2.71) and (2.73) to conclude, after some calculation, that

$$d_V h^i_j = g^{il}(\nabla_j \beta_l) - h^l_j\,\alpha^i_l.$$

This, in turn, gives rise to

$$d_V h_{ij} = h_{lj}\,\alpha^l_i + \nabla_j\beta_i. \tag{2.74}$$

We are now ready to compute the Euler-Lagrange form for the Lagrangian $\lambda$. Define

$$A^{ij} = E^{ij}_{(g)} = \frac{\partial L}{\partial g_{ij}} - D_l\big(\frac{\partial L}{\partial g_{ij,l}}\big) + \cdots \tag{2.75a}$$

and

$$B^{ij} = E^{ij}_{(h)} = \frac{\partial L}{\partial h_{ij}} - D_l\big(\frac{\partial L}{\partial h_{ij,l}}\big) + \cdots \tag{2.75b}$$

and set

$$A^j_i = g_{il}A^{jl} \quad \text{and} \quad B^j_i = h_{il}B^{jl}.$$

Because $L$ is a natural, diffeomorphism invariant Lagrangian Noether's theorem implies that the two Euler-Lagrange expressions $A^{ij}$ and $B^{ij}$ are related by the identity

$$2\nabla_j A^j_i + 2\nabla_j B^j_i - (\nabla_i h_{jk})B^{jk} = 0. \tag{2.76}$$

We shall give a direct proof of this result in the next chapter.

The same "integration by parts " argument that we used in the previous example is repeated here to yield

$$\begin{aligned} d_V\lambda &= \big[A^{ij} d_V g_{ij} + B^{ij} d_V h_{ij}\big] \wedge \nu + d_H[\cdots] \\ &= \big[2A^j_i \alpha^i_j + B^j_i\alpha^i_j + B^{ij}\nabla_j\beta_i\big] \wedge \nu + d_H[\cdots] \\ &= -\big[2\nabla_j A^j_i + \nabla_j B^j_i + h_{li}(\nabla_j B^{lj})\big]\Theta^i \wedge \nu \\ &\qquad + \big[\nabla_{ij}B^{ij} - B^j_i h^i_j - 2A^j_i h^i_j\big]\Xi \wedge \nu + d_H[\cdots]. \end{aligned}$$

But, by virtue of the Codazzi equations (2.68),

$$\big[h_{li}(\nabla_j B^{lj})\big]\Theta^i \wedge \nu = \big[\nabla_j(B^j_i) - (\nabla_i h_{lj})B^{lj}\big]\Theta^i \wedge \nu + d_H[\cdots]$$

and hence, on account of the (2.76), the components of $\Theta^i$ in the foregoing expression for $d_V\lambda$ vanishes identically.

PROPOSITION 2.19. *The Euler-Lagrange form for the $G$-invariant Lagrangian $\lambda \in \Omega^{2,0}_G(\mathcal{R})$ given by (2.69) is*

$$E(\lambda) = \left[\nabla_{ij} B^{ij} - B^j_i h^i_j - 2A^j_i h^i_j\right] \Xi \wedge \nu$$

*where $A^{ij}$ and $B^{ij}$ are the Euler-Lagrange expressions (2.75)*

For example, with $L = \sqrt{g} f(H_1, H_2)$ we have that

$$B^{ij} = \sqrt{g}\left[\tfrac{1}{2} f_1 g^{ij} + f_2 h^{ij}\right]$$

and

$$A^j_i = \sqrt{g}\left[\tfrac{1}{2} f \delta^j_i - \frac{1}{2} f_1 h^j_i - f_2 (h^l_i h^j_l)\right],$$

where $f_1$ and $f_2$ are the partial derivatives of $f$ with respect to $H_1$ and $H_2$ respectively. The Euler-Lagrange form for the Lagrangian

$$\lambda = \sqrt{g} f(H_1, H_2)\, \nu$$

is therefore

$$E(\lambda) = \sqrt{g}\left[\tfrac{1}{2}\Delta f_1 + \nabla_{ij}(h^{ij} f_2) - 2H_1 f + H_2 f_1 + 2H_1(3H_2 - 2H_1^2) f_2\right] \Xi \wedge \nu.$$

With $f = \frac{1}{2}(H_2 - H_1^2)$, this gives

$$E(\lambda) = \sqrt{g}\left[\tfrac{1}{2}\Delta H - KH^2 + H^3\right] \Xi \wedge \nu$$

as the Euler-Lagrange form for the Willmore Lagrangian. For the Gauss-Bonnet integrand $f = 2H_1^2 - H_2$, this gives $E(\lambda) = 0$.

EXAMPLE 2.20. *Curves on Surfaces.*

Let $M$ be a two dimensional Riemannian manifold with metric $g$ and constant scalar curvature $R$. In this final example, we wish to compute the Euler-Lagrange equations for the natural variational problems for curves on such surfaces. The Lagrangians for such variational problems take the form

$$\lambda = L(\kappa_g, \dot{\kappa}_g, \ddot{\kappa}_g, \dots)\, \sigma, \tag{2.77}$$

where $\kappa_g$ is the geodesic curvature of the curve, computed with respect to the metric $g$, and

$$\sigma = |u'|\, dx, \quad \text{where} \quad |u'| = \sqrt{g_{ij} \frac{du^i}{dx} \frac{du^j}{dx}}.$$

To facilitate the calculations for this example, it is very helpful to introduce covariant horizontal and vertical differential $D_H$ and $D_V$. If $D$ denotes the usual covariant differential defined on tensor-valued forms on $J^\infty(E)$, then

$$D = D_H + D_V.$$

For example, if $A$ is a type $(1,1)$ tensor-valued type $(r,s)$ form on $J^\infty(E)$, then $DA$ is the type $(1,1)$ tensor-valued $(r+s+1)$ form with components

$$DA^i_j = dA^i_j + du^k \wedge (A^l_j\,\Gamma^i_{lk} - A^i_l\,\Gamma^l_{jk}).$$

Here $\Gamma^l_{ij}$ are the components of the Christoffel symbols for the metric $g_{ij}$. Since $d = d_H + d_V$ and $du^k = \theta^k + \dot{u}^k dx$, this equation decomposes by type to give

$$D_H A^i_j = d_H A^i_j + dx \wedge \dot{u}^k (A^l_j\,\Gamma^i_{lk} - A^i_l\,\Gamma^l_{jk})$$

and

$$D_V A^i_j = d_V A^i_j + \theta^k \wedge \big(A^l_j\,\Gamma^i_{lk} - A^i_l\,\Gamma^l_{jk}\big).$$

Of course, on scalar valued forms $d_H$ and $D_H$, and $d_V$ and $D_V$, coincide.

The curvature two form $\Omega^i_j$ decomposes according to

$$\begin{aligned}
\Omega^i_j &= -\tfrac{1}{2} R_j{}^i{}_{hk} du^h \wedge du^k \\
&= \big[-R_j{}^i{}_{hk} \dot{u}^k \theta^h \wedge dx\big] + \big[-\tfrac{1}{2} R_j{}^i{}_{hk} \theta^h \wedge \theta^k\big] \\
&= \overset{(1,1)}{\Omega}{}^i_j + \overset{(0,2)}{\Omega}{}^i_j.
\end{aligned}$$

Consequently, the Ricci identity

$$D^2 A^i_j = \Omega^i_l \wedge A^l_j - \Omega^l_j \wedge A^i_l$$

decomposes by type to yield

$$D^2_H A^i_j = 0, \tag{2.78a}$$

$$D_V D_H A^i_j + D_H D_V A^i_j = \overset{(1,1)}{\Omega}{}^i_l \wedge A^l_j - \overset{(1,1)}{\Omega}{}^l_j \wedge A^i_l \tag{2.78b}$$

and

$$D_V^2 A^i_j = \overset{(0,2)}{\Omega}{}^i{}_l \wedge A^l_j - \overset{(0,2)}{\Omega}{}^l{}_j \wedge A^i_l. \tag{2.78c}$$

With this formalism in hand, the calculation of the Euler-Lagrange equation for (2.77) proceeds along the same lines as that of our first example. Let $\{T, N\}$ be the Frenet frame and define contact one forms $\Theta^1$ and $\Theta^2$ by

$$\theta = \Theta^1 T + \Theta^2 N.$$

The Frenet formula

$$\frac{DT}{ds} = \kappa_g N \tag{2.79}$$

leads to

$$\frac{D\theta}{ds} = \alpha T + \beta N,$$

where

$$\alpha = \dot{\Theta}^1 - \kappa_g \Theta^2 \quad \text{and} \quad \beta = \kappa_g \Theta^1 + \dot{\Theta}^2.$$

These equations imply that

$$d_V |u'| = \alpha \quad \text{and} \quad d_V \sigma = \alpha \wedge \sigma \tag{2.80}$$

and

$$D_V \begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}.$$

To compute $d_V \kappa_g$, we apply $D_V$ to the Frenet formula (2.79) to obtain

$$D_V \big(\frac{DT}{ds}\big) = (d_V \kappa) N - \kappa \beta T.$$

To evaluate the left-hand side of this equation, we use (2.78b) and (2.80) to deduce that

$$\begin{aligned} D_V \big(\frac{DT^i}{ds}\big) &= D_V \big(\frac{DT^i}{Dt} \frac{1}{|u'|}\big) \\ &= -\alpha \frac{DT^i}{ds} + \frac{D}{ds}\big(D_V T^i\big) + \overset{(1,1)}{\Omega}{}^i{}_l T^l. \end{aligned} \tag{2.81}$$

On a two-dimensional manifold the curvature tensor satisfies

$$R_l{}^i{}_{hk} = R(g_{lh} \delta^i_k - g_{lk} \delta^i_h).$$

and so the curvature term in equation (2.81) simplifies to

$$\overset{(1,1)}{\Omega}{}^{i}_{\,l}\, T^l = R\, N^i \Theta^2.$$

From the normal component of (2.81), we therefore find that

$$d_V \kappa_g = -\kappa\alpha + \dot{\beta} + R\Theta^2.$$

For the Lagrangian (2.77) we define the Euler-Lagrange operator $E_{\kappa_g}(L)$ and the Hamiltonian $H(L)$ as in our first example. The calculations there can now be repeated without modification.

PROPOSITION 2.21. *The Euler-Lagrange form for the natural Lagrangian (2.77) for curves on a surface of constant curvature $R$ is*

$$E(\lambda) = \big[\ddot{E}_{\kappa_g}(L) + E_{\kappa_g}(L)(\kappa_g^2 + R) + \kappa_g H(L)\big]\Theta^2 \wedge \sigma.$$

Chapter Three

# FUNCTIONAL FORMS AND COCHAIN MAPS

We begin this chapter by studying the subspaces of functional forms $\mathcal{F}^s(J^\infty(E)) \in \Omega^{n,s}(J^\infty(E))$. These spaces were briefly introduced in Chapter One as the image of $\Omega^{n,s}(J^\infty(E))$ under the interior Euler operator $I$:

$$\mathcal{F}^s(J^\infty(E)) = I\big(\Omega^{n,s}(J^\infty(E))\big) = \{\, \omega \in \Omega^{n,s}(J^\infty(E)) \,|\, I(\omega) = \omega \,\}.$$

We give local normal forms for functional forms of degree 1 and 2; for $s > 2$, the determination of local normal forms seems to be a difficult problem. As an elementary application of the theory of functional forms, we show that for

$$\Delta = P_\alpha[x,u]\, \theta^\alpha \wedge \nu \in \mathcal{F}^1(J^\infty(E))$$

the equation $\delta_V(\Delta) = I(d_V\Delta) = 0$ coincides with the classical Helmholtz conditions for the inverse problem to the calculus of variations. A simplification of these conditions is presented in the case of one dependent variable. The problem of classifying Hamiltonian operators for scalar evolution equations is formulated in terms of functional 3 forms. Our discussion indicates that the complexity of this problem is due, in part, to the problem of finding normal forms for functional 3 forms.

In section B, we classify explicitly those maps on the infinite jet bundle $J^\infty(E)$ which induce cochain maps for either the vertical or the horizontal subcomplexes in the variational bicomplex. Attention is also paid to maps which commute with the interior Euler operator $I$ although only partial results are obtained. We are able, however, to completely solve the infinitesimal version of these problems, that is, the characterization of those vector fields on $J^\infty(E)$ whose Lie derivatives commute with either $d_H$, $d_V$, or $I$. As an immediate application of these considerations a general change of variable formula for the Euler-Lagrange operator is derived.

In section C, we derive a Cartan-like formula for the Lie derivative of functional forms with respect to a generalized vector field in terms of interior products and the induced vertical differential $\delta_V$. Suitably interpreted, this formula yields one version of Noether's theorem. To illustrate these results, the problem of finding conservation laws for the geodesic equation is studied. We also show how Noether's second theorem can be derived, at least in the special case of natural variational principles on Riemannian structures, from the same Lie derivative formula.

**A. Functional Forms.** Our first task in this section is to explain our nomenclature, that is, to explain why we call type $(n,s)$ forms in $\mathcal{F}^s(J^\infty(E))$ functional forms. Let $\omega$ be a $p$-form on $J^\infty(E)$, where $p > n$. Then, given a coordinate neighborhood $\pi : U_0 \to U$ of $E$, a compact set $V \subset U_0$ and generalized vector fields $X_1, X_2, \ldots, X_q$ on $U$, where $q = p - n$, we can define a functional on local sections $s : U_0 \to U$ by

$$\mathcal{W}(X_1, X_2, \ldots, X_q)[s] = \int_V [j^\infty(s)]^* \omega(\operatorname{pr} X_1, \operatorname{pr} X_2, \ldots, \operatorname{pr} X_q), \tag{3.1}$$

where

$$\omega(\operatorname{pr} X_1, \operatorname{pr} X_2, \ldots, \operatorname{pr} X_q) = \operatorname{pr} X_q \mathbin{\lrcorner} \ldots \operatorname{pr} X_2 \mathbin{\lrcorner} \operatorname{pr} X_1 \mathbin{\lrcorner} \omega$$

Observe that $\mathcal{W}$ is $\mathbf{R}$ linear and alternating but that it is is not linear over functions on $J^\infty(E)$. We call $q$ the degree of the functional $\mathcal{W}$.

In considering functionals of the type (3.1) there are two simplifications that can be made. First, decompose $\omega$ by horizontal and vertical degrees to obtain

$$\omega = \omega^{(n,q)} + \omega^{(n-1,q+1)} + \cdots + \omega^{(0,n+q)},$$

where each form $\omega^{(r,s)}$ is of type $(r,s)$. Because contact forms are annihilated by the pullback of the map $j^\infty(s)$, only the first term $\omega^{(n,q)}$ will survive in (3.1) *i.e.*,

$$[j^\infty(s)]^* \omega(\operatorname{pr} X_1, \operatorname{pr} X_2, \ldots, \operatorname{pr} X_q) = [j^\infty(s)]^* \omega^{(n,q)}(\operatorname{pr} X_1, \operatorname{pr} X_2, \ldots, \operatorname{pr} X_q).$$

Second, decompose the prolongation of each generalized vector field $X_i$ into its evolutionary and total components (see Proposition 1.20) so that

$$\operatorname{pr} X_i \mathbin{\lrcorner} \omega^{(n,q)} = \operatorname{pr}(X_i)_{\mathrm{ev}} \mathbin{\lrcorner} \omega^{(n,q)} + \operatorname{tot} X_i \mathbin{\lrcorner} \omega^{(n,q)}.$$

By Proposition 1.18, the second term on the right-hand side of this equation is of type $(n-1,q)$ and therefore it too will not contribute to the integrand in (3.1). We therefore conclude that the functional $q$ form (3.1) is completely determined by the type $(n,q)$ form $\omega^{(n,q)}$ and by its values on arbitrary $q$-tuples of evolutionary vector fields.

Thus, with no loss in generality, we can view functionals of the type (3.1) as multi-linear, alternating maps from the space of evolutionary vector fields on $E$ to the space of functionals on $E$. If $\omega \in \Omega^{n,q}(J^\infty(E))$, then the corresponding functional $\mathcal{W}$ is given by

$$\mathcal{W}(Y_1, Y_2, \ldots, Y_q)[s] = \int_V [j^\infty(s)]^* \omega(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_q). \tag{3.2}$$

where $Y_1, Y_2, \ldots, Y_q$ are evolutionary vector fields on $E$.

Now, just as in the calculus of variations where the fundamental integral does not uniquely determine the Lagrangian, so it is with functional $q$-forms. Indeed, because $d_H$ and $\operatorname{pr} Y_i$ anti-commute, both $\omega$ and $\omega + d_H\eta$, where $\eta \in \Omega^{n-1,q}$, determine the same functional $\mathcal{W}$. The next proposition shows that this non-uniqueness can be eliminated if we restrict our attention to type $(n, q)$ differential forms in $\mathcal{F}^q$.

Proposition 3.1. *Let $\omega$ and $\tilde{\omega}$ be two type $(n, q)$ forms in $\mathcal{F}^q(J^\infty(E))$. Then the corresponding functionals $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are equal if and only $\omega = \tilde{\omega}$.*

It is because of the one-to-one correspondence between functionals $\mathcal{W}$ of the type (3.1) and forms $\omega$ in $\mathcal{F}^s(J^\infty(E))$ that we call the latter functional forms.

The proof of Proposition 3.1 rests on the following lemma.

Lemma 3.2. *Let $\omega$ be a type $(n, q)$ form on $J^\infty(E)$. Then $I(\omega) = 0$ if and only if for all evolutionary vector fields $Y_1, Y_2, \ldots, Y_q$ the type $(n, 0)$ form*

$$\lambda = \omega(Y_1, Y_2, \ldots, Y_q)$$

*is variational trivial, i.e., $E(\lambda) = 0$.*

Proof: It suffices to work locally. Recall that for $\omega \in \Omega^{n,q}(J^\infty(U))$,

$$I(\omega) = \frac{1}{q}\,\theta^\alpha \wedge F_\alpha(\omega),$$

where $F_\alpha(\omega)$ is defined by (2.23). Also recall that if $Y_1 = Y_1^\alpha \dfrac{\partial}{\partial u^\alpha}$ is any evolutionary vector field, then

$$\operatorname{pr} Y_1 \lrcorner\ \omega = Y_1^\alpha F_\alpha(\omega) + d_H\eta,$$

where $\eta$ is a form of type $(n-1, q)$. Consequently, we find that

$$\lambda = Y_1^\alpha \sigma_\alpha + d_H\tilde{\eta},$$

where

$$\sigma_\alpha = [F_\alpha(\omega)](\operatorname{pr} Y_2, \operatorname{pr} Y_3, \ldots, \operatorname{pr} Y_q),$$

and where $\tilde{\eta}$ is the form of type $(n-1, 0)$ obtained from $\eta$ by interior evaluation by $\operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_q$. This implies that

$$E(\lambda) = E(Y_1^\alpha \sigma_\alpha).$$

If $E(\lambda) = 0$ for all $Y_1$, then we can invoke Corollary 2.9 to conclude that $\sigma_\alpha = 0$ and hence that $I(\omega) = 0$.

Conversely, if $I(\omega) = 0$ then $\omega = d_H\eta$ for some type $(n-1, q)$ form $\eta$. This implies that $\lambda$ is also locally $d_H$ exact and so $E(\lambda) = 0$. ∎

Proof of Proposition 3.2: By linearity, it suffices to prove that if $\omega \in \mathcal{F}^q$ and if the corresponding functional $\mathcal{W}$ vanishes identically, then $\omega = 0$. If $\mathcal{W} \equiv 0$, then standard arguments from the calculus of variations imply that the type $(n, 0)$ form

$$\lambda = \omega(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_q)$$

is variationally trivial for all evolutionary vector fields $Y_1, Y_2, \ldots, Y_q$. From the foregoing lemma we can infer that $I(\omega) = 0$. But, by hypothesis, $\omega$ is in the image of the projection operator $I$ and so $\omega = 0$. ∎

We remark that Takens [*65*] introduced an equivalence relation $\sim$ on $\Omega^{n,q}(J^\infty(E))$ whereby $\omega_1 \sim \omega_2$ if for all evolutionary vector fields $Y_1$, $Y_2$, $\ldots$,$Y_q$ the two Lagrangians

$$\lambda_1 = \omega_1(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_q)$$

and

$$\lambda_2 = \omega_2(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_q)$$

have identical Euler-Lagrange expressions. On account of Lemma 3.2 we have that the quotient spaces $\Omega^{n,q}/\sim$ are isomorphic, via the map $[\omega] \to I(\omega)$, to the subspaces $\mathcal{F}^q$.

A differential $\delta$ can be defined for functionals $\mathcal{W}$ of the type (3.2). First, let $\mathcal{V}$ be a functional on degree zero given by

$$\mathcal{V}[s] = \int_V [j^\infty(s)]^* \lambda,$$

where $\lambda$ is a type $(n, 0)$ form. If $Y$ is an evolutionary vector field, then we define the functional $Y(\mathcal{V})$ by

$$Y(\mathcal{V})[s] = \int_V [j^\infty(s)]^* \bigl(\mathcal{L}_{\operatorname{pr} Y} \lambda\bigr). \tag{3.3}$$

If $\mathcal{W}$ is a functional of degree 1, we define the functional $\delta\mathcal{W}$ of degree 2 by

$$\bigl(\delta\mathcal{W}\bigr)(Y_1, Y_2) = Y_1\bigl(\mathcal{W}(Y_2)\bigr) - Y_2\bigl(\mathcal{W}(Y_1)\bigr) - \mathcal{W}([Y_1, Y_2]). \tag{3.4}$$

The differential of functionals of degree greater than 1 is similarly defined.

PROPOSITION 3.3. *Let $\omega$ be a type $(n,q)$ form in $\mathcal{F}^q(J^\infty(E))$ and let $\mathcal{W}$ be the associated functional $q$ form. The $\delta\mathcal{W}$ is the functional $q+1$ form associated to the type $(n,q+1)$ form $\delta_V\omega$ in $\mathcal{F}^{q+1}(J^\infty(E))$.*

PROOF: The proof is based upon the decomposition

$$d_V\omega = I(d_V\omega) + d_H\eta = \delta_V\omega + d_H\eta, \tag{3.5}$$

which we have already established in Theorem 2.12. For simplicity, we consider only the case $q=1$. In view of the definitions (3.1) and (3.4), we find that

$$\big(\delta\mathcal{W}\big)(Y_1,Y_2)[s] = \int_V [j^\infty(s)]^*\big\{\mathcal{L}_{\mathrm{pr}\,Y_1}\big(\omega(Y_2)\big) - \mathcal{L}_{\mathrm{pr}\,Y_2}\big(\omega(Y_1)\big) - \omega\big([Y_1,Y_2]\big)\big\}. \tag{3.6}$$

Since, by Propositions 1.16 and 1.21,

$$\begin{aligned} \mathcal{L}_{\mathrm{pr}\,Y_1}\big(\omega(Y_2)\big) &= \big(\mathcal{L}_{\mathrm{pr}\,Y_1}(\mathrm{pr}\,Y_2)\big) \lrcorner\, \omega \,+\, \mathrm{pr}\,Y_2 \lrcorner\, \big(\mathcal{L}_{\mathrm{pr}\,Y_1}\omega\big) \\ &= \big(\mathrm{pr}\,[Y_1,Y_2]\big) \lrcorner\, \omega \,+\, \mathrm{pr}\,Y_2 \lrcorner\, \mathrm{pr}\,Y_1 \lrcorner\, d_V\omega \\ &\qquad + \mathrm{pr}\,Y_2 \lrcorner\, d_V\big(\mathrm{pr}\,Y_1 \lrcorner\, \omega\big) \end{aligned}$$

and

$$\mathcal{L}_{\mathrm{pr}\,Y_2}\big(\omega(Y_1)\big) = \mathrm{pr}\,Y_2 \lrcorner\, d_V\big(\mathrm{pr}\,Y_1 \lrcorner\, \omega\big)$$

equation (3.6) simplifies to

$$\big(\delta\mathcal{W}\big)(Y_1,Y_2)[s] = \int_V [j^\infty(s)]^* d_V\omega(Y_1,Y_2).$$

Finally, by virtue of (3.5), we can replace the form $d_V\omega$ in this integral by the form $\delta_V\omega$. ∎

We now turn to the problem of explicitly describing local basis for the spaces $\mathcal{F}^s$. The case $s=1$ is somewhat special.

PROPOSITION 3.4. *Let $\omega$ be a type $(n,1)$ form in $\mathcal{F}^1(J^\infty(E))$. Then, on any coordinate chart $J^\infty(U)$, there exist unique functions $P_\alpha[x,u]$ such that*

$$\omega = P_\alpha[x,u]\,\theta^\alpha \wedge \nu. \tag{3.7}$$

*Conversely, every type $(n,1)$ form which is locally of form (3.7) belongs to $\mathcal{F}^1$.*

PROOF: If $\omega \in \mathcal{F}^1(J^\infty(E))$ then $\omega = I(\sigma)$ for some form $\sigma \in \Omega^{n,1}(J^\infty(E))$. If, on $J^\infty(U)$

$$\sigma = \sum_{|I|=0}^{k} B^I_\alpha\,\theta^\alpha_I \wedge \nu,$$

then by direct calculation $\omega$ is given by (3.7), where

$$P_\alpha = \sum_{|I|=0}^{k} (-D)_I B^I_\alpha.$$

Conversely, if in any coordinate chart $\omega$ assumes the form (3.7), then $I(\omega) = \omega$ and hence $\omega \in \mathcal{F}^1(J^\infty(E))$. ∎

This characterization of $\mathcal{F}^1(J^\infty(E))$ shows that this space is a module over $C^\infty$ functions on $J^\infty(E)$. As is easily checked by example, this is not true for the spaces $\mathcal{F}^s(J^\infty(E))$ when $s > 1$. Every form $\omega \in \mathcal{F}^1(J^\infty(E))$ defines a system of $m$ partial differential equations on the space of sections on $E$ — if $\omega$ is given locally by (3.7) and is of order $k$, then in these coordinates the equations are

$$P_\alpha(j^k(s)) = 0.$$

To distinguish systems of equations which arise in this manner we follow Takens [*65*] to make the following definition.

Definition 3.5. *Forms $\omega$ in $\mathcal{F}^1$ are called source forms on $J^\infty(E)$ and the partial differential equations on $E$ defined by $\omega$ are called source equations.*

In particular, for a Lagrangian $\lambda = L\nu \in \Omega^{n,0}(J^\infty(E))$, the Euler-Lagrange form

$$E(\lambda) = I(d_V\lambda) = E_\alpha(L)\,\theta^\alpha \wedge \nu$$

is always a source form.

Forms in $\mathcal{F}^2$ admit the following local characterization.

Proposition 3.6. *Fix $|I| = l$ and let $A_{\alpha\beta}{}^I$ be a collection of smooth functions on $J^\infty(U)$ which satisfy*

$$A_{\alpha\beta}{}^I = (-1)^{l+1} A_{\beta\alpha}{}^I.$$

*Then the type $(n,2)$ form $\omega^{(l)}$ defined by*

$$w^{(l)} = \theta^\alpha \wedge [A_{\alpha\beta}{}^I\,\theta^\beta_I + D_I(A_{\beta\alpha}{}^I\,\theta^\beta)] \wedge \nu \tag{3.8}$$

*belongs to $\mathcal{F}^2(J^\infty(U))$. Furthermore, every form $\omega \in \mathcal{F}^2(J^\infty(U))$ can be written uniquely as a sum of such forms, i.e.,*

$$\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \cdots \omega^{(k)}. \tag{3.9}$$

PROOF: We show that $I(\omega^{(l)}) = \omega^{(l)}$. Write

$$\omega^{(l)} = \phi + \psi,$$

where

$$\phi = \theta^\alpha \wedge [A_{\alpha\beta}{}^{I}\, \theta^\beta_I] \wedge \nu \quad \text{and} \quad \psi = \theta^\alpha \wedge [D_I(A_{\beta\alpha}{}^{I}\, \theta^\beta)] \wedge \nu.$$

Direct calculation using the coordinate definition of $I$ leads to

$$\begin{aligned} I(\phi) &= \frac{1}{2}\theta^\alpha \wedge [A_{\alpha\beta}{}^{I}\, \theta^\beta_I - (-1)^l D_I(A_{\beta\alpha}{}^{I}\, \theta^\beta)] \wedge \nu \\ &= \frac{1}{2}\phi + \frac{1}{2}\psi. \end{aligned}$$

The easiest way to evaluate $I(\psi)$ is to first "integrate by parts" and rewrite $\psi$ in the form

$$\psi = (-1)^l \theta^\alpha_I \wedge \theta^\beta A_{\alpha\beta}{}^{I} + d_H\eta = \phi + d_H\eta.$$

Then the $d_H\eta$ term does not contribute to $I(\psi)$ and so

$$I(\omega^{(l)}) = 2I(\phi) = \omega^{(l)},$$

as required.

It remains to verify (3.9). Any type $(n, 2)$ form $\omega$ in $\mathcal{F}^2$ locally assumes the form

$$\omega = \theta^\alpha \wedge \big[\sum_{|I|=0}^{l} P_{\alpha\beta}{}^{I}\, \theta^\beta_I\big] \wedge \nu.$$

From the coefficient of $\theta^\alpha \wedge \theta^\beta_I$, $|I| = l$, in the identity $I(\omega) = \omega$ we conclude that

$$P_{\alpha\beta}{}^{I} = (-1)^{l+1} P_{\beta\alpha}{}^{I}.$$

Hence, with $P_{\alpha\beta}{}^{I} = \frac{1}{2} A_{\alpha\beta}{}^{I}$ and $\omega^{(l)}$ defined by (3.8) we can write

$$\omega = \widetilde{\omega} + \omega^{(l)},$$

where $\widetilde{\omega}$ is of order $l - 1$ in the contact forms. Since both $\omega$ and $\omega^{(l)}$ belong to $\mathcal{F}^2$, the same is true of $\widetilde{\omega}$. The validity of (3.9) can now be established by induction on the order $l$. ∎

For $s > 2$, little general progress has been made towards the explicit local characterization of the spaces $\mathcal{F}^s$. We can, however, offer the following alternative description of these spaces. First, it is evident from the definition of $I$ that every form $\omega \in \mathcal{F}^s(J^\infty(E))$ must locally assume the form

$$\omega = \theta^\alpha \wedge P_\alpha,$$

where each $P_\alpha$ is a form of type $(n, s-1)$. For $s > 1$ these forms are not arbitrary and additional conditions must be imposed upon the $P_\alpha$ to insure that the $\omega$ belong to $\mathcal{F}^s$.

Let $P$ be a $C^\infty(J^\infty(E))$ linear map (or equivalently, a zeroth order total differential operator)

$$P \colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,s-1}(J^\infty(E))$$

on the space of evolutionary vector fields $\mathcal{E}v(J^\infty(E))$. We say that $P$ is *formally skew-adjoint* if for every pair of evolutionary vector fields $Y$ and $Z$ and every coordinate chart $U$ on $E$ there is a type $(n, s-2)$ form $\rho$ on $J^\infty(U)$ such that

$$\operatorname{pr} Z \lrcorner P(Y) + \operatorname{pr} Y \lrcorner P(Z) = d_H \rho. \tag{3.10}$$

Proposition 3.7. *A type $(n, s)$ form $\omega$ on $J^\infty(E)$ belongs to $\mathcal{F}^s(J^\infty(E))$ if and only if there exists a formally skew-adjoint operator $P(Y) = Y^\alpha P_\alpha$ such that in any coordinate system*

$$\omega = \theta^\alpha \wedge P_\alpha. \tag{3.11}$$

*This representation of $\omega$ is unique; if $\widetilde{P}$ is another linear, formally skew-adjoint operator and $\omega = \theta^\alpha \wedge \widetilde{P}_\alpha$, then $P = \widetilde{P}$.*

Proof: Suppose that $\omega \in \mathcal{F}^s$. Then $I(\omega) = \omega$ and so

$$\omega = \theta^\alpha \wedge P_\alpha,$$

where $P_\alpha = \dfrac{1}{s} F_\alpha(\omega)$. Equation (2.29) can therefore be rewritten as

$$\operatorname{pr} Y \lrcorner \omega = sP(Y) + d_H \eta. \tag{3.12}$$

Interior evaluate this equation with $\operatorname{pr} Z$. Since the left-hand side of the resulting equation changes sign under the interchange of $Y$ and $Z$ it follows immediately that $P$ is formally skew-adjoint.

Conversely, suppose that (3.11) holds where $P$ is formally skew-adjoint. We show that $P_\alpha = \dfrac{1}{s} F_\alpha(\omega)$ which proves that $\omega \in \mathcal{F}^s$ and that the representation (3.11) is unique. Let $Z_1, Z_2, \ldots, Z_{s-1}$ be evolutionary vector fields on $J^\infty(E)$. Because $P$ is skew-adjoint, we find that

$$P(Y)(\operatorname{pr} Z_1, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Z_{s-1}) + P(Z_1)(\operatorname{pr} Y, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Z_{s-1}) = d_H \eta_1.$$

Substitution from (3.11) and the repeated use of this equation yields

$$\begin{aligned}
&(\operatorname{pr} Z_1 \lrcorner \operatorname{pr} Z_2 \lrcorner \cdots \lrcorner \operatorname{pr} Z_{s-1}) \lrcorner \big[\operatorname{pr} Y \lrcorner \omega\big] \\
&\qquad = P(Y)(\operatorname{pr} Z_1, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Z_{s-1}) - P(Z_1)(\operatorname{pr} Y, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Z_{s-1}) \\
&\qquad\qquad - \cdots - P(Z_{s-1})(\operatorname{pr} Z_1, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Y) \\
&\qquad = sP(Y)(\operatorname{pr} Z_1, \operatorname{pr} Z_2, \ldots, \operatorname{pr} Z_{s-1}) + d_H \eta_2 \\
&\qquad = (\operatorname{pr} Z_1 \lrcorner \operatorname{pr} Z_2 \lrcorner \cdots \lrcorner \operatorname{pr} Z_{s-1}) \lrcorner \big[sP(Y)\big] + d_H \eta_2.
\end{aligned}$$

By virtue of Lemma 3.2 (with $\lambda = \operatorname{pr} Y \lrcorner \omega - sP(Y)$ ) this implies that

$$\operatorname{pr} Y \lrcorner \omega = sP(Y) + d_H \eta_3. \tag{3.13}$$

But according to Proposition 2.2 and equation (2.29), the Euler operator

$$E(P_\omega)(Y) = Y^\alpha F_\alpha(\omega)$$

is uniquely defined by this very condition and so $sP_\alpha = F_\alpha$, as required. ∎

Corollary 3.8. *The type $(n, s)$ form $\omega = \theta^\alpha \wedge P_\alpha$ belongs to $\mathcal{F}^s$ if and only if the coefficients of $P_\alpha(Z) = \operatorname{pr} Z \lrcorner P_\alpha$, viz.,*

$$P_\alpha(Z) = \sum_{|I|=0}^{k} P_{\alpha\beta}{}^{I} (D_I Z^\beta)$$

*satisfy the the differential conditions*

$$(-1)^{|I|+1} P_{\alpha\beta}{}^{I} = \sum_{|J|=0}^{k-|I|} \binom{|I|+|J|}{|J|} (-D)_J P_{\beta\alpha}{}^{IJ} \tag{3.14}$$

*for all* $|I| = 0, 1, \ldots, k$.

PROOF: Proposition 2.1 implies that

$$[Y^\alpha P_\alpha(Z) + Z^\alpha P_\alpha(Y)] = Y^\alpha [P_\alpha(Z) + \sum_{|I|=0}^{k} (-D)_I (Z^\beta P_{\beta\alpha}{}^I)] + d_H \rho.$$

Proposition 2.2 then implies that (3.10) holds if and only if the expression in brackets on the right-hand side of this last equation vanishes for all $Z$. We set the coefficient of $D_I Z^\alpha$ in the resulting equation to zero to arrive at (3.14). ∎

EXAMPLE 3.9. We offer the following cautionary example. Let

$$\omega = A[x,u]\, \theta \wedge \theta_x \wedge \theta_{xx} \wedge dx.$$

Then, a direct calculation yields

$$\begin{aligned} I(\omega) &= \frac{1}{3}\theta \wedge [A\, \theta_x \wedge \theta_{xx} + D_x(A\, \theta \wedge \theta_{xx}) + D_{xx}(A\, \theta \wedge \theta_x] \wedge dx \\ &= \omega \end{aligned}$$

so that this type $(3,1)$ form belongs to $\mathcal{F}^3$. According to Proposition 3.7, we can express this form uniquely in the form

$$\omega = \theta \wedge P$$

where $P$ satisfies (3.10). The apparent choice for $P$, namely $P = A\, \theta_x \wedge \theta_{xx} \wedge dx$, is not formally skew-adjoint and is therefore incorrect. Indeed, P(Z) is a second order operator in $Z$ therefore cannot satisfy (3.14) with $|I| = 2$. The correct choice for $P$ is

$$P = \frac{1}{3} F(\omega) = \frac{1}{3}[2A\, \theta \wedge \theta_{xxx} + 3A\, \theta_x \wedge \theta_{xx} + 3A_x\, \theta \wedge \theta_{xx} + A_{xx} \theta \wedge \theta_x].$$

∎

This is an appropriate point at which to make a few, relatively elementary remarks concerning the inverse problem to the calculus of variations. In its simplest form, the inverse problem is to determine when a given source form $\Delta \in \mathcal{F}^1(J^\infty(E))$ is the Euler-Lagrange form for some Lagrangian $\lambda \in \Omega^{n,0}(J^\infty(E))$. We have already observed that a necessary condition for $\Delta = E(\lambda)$ is given by $\delta_V \Delta = 0$. In the chapters that follow we shall establish the local sufficiency of this condition and identify the topological obstructions to the construction of global Lagrangians. For now, we direct our attention to the nature of these necessary conditions themselves.

DEFINITION 3.10. *A source form $\Delta \in \mathcal{F}^1(J^\infty(E))$ is said to be locally variational if*

$$\delta_V \Delta = 0.$$

This definition reflects the fact, which we shall prove in Chapter Four, that if $\delta_V \Delta = 0$ then, at least locally there is a Lagrangian $\lambda$ such that $\Delta = E(\lambda)$.

To examine the conditions $\delta_V \Delta = 0$ in coordinates, let $\Delta = P_\alpha[x, u]\,\theta^\alpha \wedge \nu$. Then

$$d_V \Delta = \sum_{|J|=0}^{k} \big(\partial^J_\beta P_\alpha\big)\, \theta^\beta_J \wedge \theta^\alpha \wedge \nu$$

and hence, using the definition of $I$, we have that

$$\begin{aligned} \delta_V \Delta &= \frac{1}{2}\theta^\gamma \wedge \big[\sum_{|J|=0}^{k} (-D)_I[\partial^I_\gamma \lrcorner\, (d_V \Delta)]\big] \\ &= \frac{1}{2}\theta^\gamma \wedge \big[- \sum_{|I|=0}^{k} (\partial^I_\beta P_\gamma)\theta^\beta_I + \sum_{|I|=0}^{k} (-D)_I(\partial^I_\gamma P_\alpha \theta^\alpha)\big]\,\nu. \end{aligned} \tag{3.15}$$

But, according to the defining property (2.14) of the Lie-Euler operators $E^I_\beta$, we can write

$$\sum_{|I|=0}^{k} \partial^I_\beta P_\gamma\, \theta^\beta_I = \sum_{|I|=0}^{k} D_I[E^I_\beta(P_\gamma)\,\theta^\beta]$$

and therefore (3.15) simplifies to

$$\delta_V \Delta = \frac{1}{2}\theta^\gamma \wedge \big[\sum_{|I|=0}^{k} D_I(H^I_{\gamma\beta}\,\theta^\beta)\big],$$

where

$$H^I_{\gamma\beta} = -E^I_\beta(P_\gamma) + (-1)^{|I|}\partial^I_\gamma P_\beta.$$

Thus the conditions $\delta_V \Delta = 0$ are given explicitly by

$$(-1)^{|I|}\partial^I_\gamma(P_\beta) = E^I_\beta(P_\gamma), \tag{3.16}$$

for $|I| = k$, $k-1$, ..., $0$. For example, when $\Delta$ is of order 2, this system of equations becomes

$$\partial^{ij}_{\gamma} P_{\beta} = \partial^{ij}_{\beta} P_{\gamma}, \tag{3.17a}$$

$$-\partial^{i}_{\gamma} P_{\beta} = \partial^{i}_{\beta} P_{\gamma} - 2D_j\bigl(\partial^{ij}_{\beta} P_{\gamma}\bigr), \tag{3.17b}$$

and

$$\partial_{\gamma} P_{\beta} = \partial_{\beta} P_{\gamma} - D_i\bigl(\partial^{i}_{\beta} P_{\gamma}\bigr) - D_{ij}\bigl(\partial^{ij}_{\beta} P_{\gamma}\bigr). \tag{3.17c}$$

Historically, the equations (3.17) were first derived in 1887 by Helmholtz for the special case of one independent variable and soon thereafter generalized to (3.16) by [*40*]. Since then, the equations (3.16) have been rederived by many authors. In any event, we shall call the differential

$$\delta_V : \mathcal{F}^1(J^\infty(E)) \to \mathcal{F}^2(J^\infty(E))$$

the *Helmholtz operator* and refer to the full system of necessary conditions (3.16) as the Helmholtz equations.

Example 3.11. *The Helmholtz conditions for scalar equations.*

Because $\delta_V \Delta \in \mathcal{F}^2$, the components $H^I_{\gamma\beta}$ of $\delta_V \Delta$ are not independent but are related by the conditions (3.14). These conditions represent certain interdependencies amongst the Helmholtz conditions themselves. These become particularly manifest when the number of dependent variables is one. For example, if

$$\Delta = P(x^i, u, u_i, u_{ij})\, \theta \wedge \nu$$

is a second order source form, then (3.17a) is an identity, (3.17b) reduces to

$$\partial^i P = D_j(\partial^{ij} P) \tag{3.18}$$

and (3.17c) becomes the divergence of (3.18). For higher order scalar equations a similar reduction occurs but the explicit form of the reduced system does not seem to have appeared in the literature. The coefficients of the reduced system are given in terms of the coefficients of the Euler polynomial. The $n$-th Euler polynomial $E_n(x)$ is a polynomial of degree $n$,

$$E_n(x) = \sum_{k=0}^{n} \tbinom{n}{k} a_k x^{n-r},$$

with generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{p=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

For $k > 1$, $a_k = 0$ if $k$ is even while

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_3 = \frac{1}{4}, \quad a_5 = \frac{1}{2}, \quad a_7 = \frac{17}{8}, \quad \ldots .$$

The coefficients $a_k$ are the Taylor series coefficients of the function

$$\frac{2}{e^t+1} = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}.$$

PROPOSITION 3.12. *Let $\pi\colon E \to M$ be a fibered manifold with one dimensional fiber and let*

$$\Delta = P[x, u^{(2k)}]\, \theta \wedge \nu$$

*be a source form of order $2k$. Then the $2k+1$ Helmholtz conditions (3.16) are equivalent to the $k$ conditions*

$$\partial^I P + \sum_{|J|=0}^{2k-i} \tbinom{|I|+|J|}{|J|} a_{|J|}\, D_J\big(\partial^{IJ} P\big) = 0 \tag{3.19}$$

*for $|I| = 2k-1,\ 2k-3,\ \ldots,\ 1$.*

PROOF: According to Corollary 3.8, the components

$$H^I = -E^I(P) + (-1)^{|I|} \partial^{|I|} P$$

of the functional form $\delta_V \Delta \in \mathcal{F}^2$ are related by the identities (3.14), *viz.*,

$$(-1)^{|I|+1} H^I = \sum_{|J|=0}^{2k-|I|} \tbinom{|I|+|J|}{|J|} (-D)_J H^{IJ}, \tag{3.20}$$

for all $|I| = 2k,\ 2k-1,\ \ldots,\ 0$. For the argument that follows, it is possible to suppress our multi-index notation and simply write $P^{(l)}$ for $\partial^I P$, $D_l$ for $D_I$, and $H^{(l)}$ for $H^I$, where $|I| = l$. In terms of this abbreviated notation, the system of equations (3.20) can be expressed as

$$\begin{aligned}
-H^{(2k)} &= H^{(2k)}, \\
H^{(2k-1)} &= H^{(2k-1)} - 2k D_1 H^{(2k)}, \\
-H^{(2k-2)} &= H^{(2k-2)} - (2k-1) D_1 H^{(2k-1)} + \tbinom{2k}{2} D_2 H^{(2k)},
\end{aligned}$$

and so on. These equations evidently imply that the Helmholtz condition $H^{(2l)}=0$ is a consequence of the conditions $H^{(2l+1)}=0$, $H^{(2l+3)}=0, \ldots$, $H^{(2k-1)}=0$. Hence $\Delta$ is locally variational if and only if

$$H^{(2k-1)}=0, \quad H^{(2k-3)}=0, \quad \ldots, \quad H^{(1)}=0. \tag{3.21}$$

Written out in full, these equations become

$$2P^{(2k-1)} = \tbinom{2k}{1} D_1 P^{(2k)}, \tag{3.22a}$$

$$2P^{(2k-3)} = \tbinom{2k-2}{1} D_1 P^{(2k-2)} - \tbinom{2k-1}{2} D_2 P^{(2k-1)} + \tbinom{2k}{3} D_3 P^{(2k)}, \tag{3.22b}$$

$$\begin{aligned} 2P^{(2k-5)} = {} & \tbinom{2k-4}{1} D_1 P^{(2k-2)} - \tbinom{2k-3}{2} D_2 P^{(2k-3)} + \tbinom{2k-2}{3} D_3 P^{(2k-2)} \\ & + \tbinom{2k-1}{4} D_4 P^{(2k-1)} + \tbinom{2k}{5} D_5 P^{(2k)}, \end{aligned} \tag{3.22c}$$

and so on. We substitute (3.22a) into (3.22b) to eliminate the odd derivative term $P^{(2k-1)}$ from (3.22b). Then we substitute (3.22a) and (3.22b) into (3.22c) in order to eliminate the odd derivative terms $P^{(2k-3)}$ and $P^{(2k-1)}$. In short, it is clear that the remaining Helmholtz conditions (3.21) are equivalent to a system of equations of the form

$$P^{(i)} + \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k-i} c_{ij} D_j P^{(i+j)} = 0, \tag{3.23}$$

where $i$ is odd. In anticipation of the result which we wish to prove, let us write the constants $c_{ij}$ in the form

$$c_{ij} = b_{i+j,j} \tbinom{i+j}{j}.$$

The index $j$ is also odd and ranges from 1 to $2k-i$. To prove (3.19), we must show that

$$b_{i+j,j} = a_j. \tag{3.24}$$

To this end, we use the identity

$$E_n(x+1) + E_n(x) = 2x^n$$

to deduce that the constants $a_j$ satisfy

$$2a_j + \sum_{l=1}^{j} \tbinom{j}{l} a_{j-l} = 0.$$

This, in fact, provides us with a recursion formula for these constants; we define $a_0 = 1$, $a_j = 0$ if $j > 0$ and even, and compute $a_j$ for $j$ odd from the formula

$$2a_j + 1 + \sum_{\substack{l=2 \\ l \text{ even}}}^{j-1} \tbinom{j}{l} a_{j-l} = 0. \tag{3.25}$$

We prove (3.24) by showing that the coefficients $b_{k,j}$ also satisfy (3.25) for each fixed $k$.

We substitute (3.23) back into the original Helmholtz condition

$$2P^{(m)} + \sum_{l=1}^{2k-m} \tbinom{l+m}{l} (-D)_l P^{(m+l)} = 0,$$

where $m$ is odd, in order to eliminate *all* odd derivatives $P^{(m)}$, $P^{(m+2)}$ $\ldots$ . The result is

$$2\sum_{\substack{j=1 \\ j \text{ odd}}}^{2k-m} b_{m+j,j} \tbinom{m+j}{j} D_j P^{(m+j)} + \sum_{\substack{l=1 \\ l \text{ odd}}}^{2k-m} \tbinom{m+l}{l} D_l P^{(m+l)}$$
$$+ \sum_{\substack{l=2 \\ l \text{ even}}}^{2k-m-1} \Big[ \sum_{\substack{j=1 \\ j\text{odd}}}^{2k-l-m} b_{m+l+j,j} \tbinom{m+l}{l} \tbinom{m+l+l}{j} D_{l+j} P^{(m+l+j)} \Big] = 0.$$

We change the sum on $l$ in the second summation to one on $j$, we change the sum on $j$ in the fourth summation to one on $j' = j + l$ and we interchange the order of the double summation to find that

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{2k-m} \tbinom{m+j}{j} \Big[ 2b_{m+j,j} + 1 + \sum_{\substack{l=2 \\ l\text{even}}}^{2k-m} \tbinom{j}{l} b_{m+j,j-l} \Big] D_j P^{(m+j)} = 0. \tag{3.26}$$

We now argue that, because this equation is an identity for all Euler-Lagrange expressions $P = E(L)$, the expressions in brackets in (3.26) must vanish for each $m$ odd, $m = 1, 3, \ldots, 2k-1$ and each $j = 1, 3, \ldots, 2k - m$. But this gives precisely the same recursion formula as (3.25) for the coefficients $a_j$, as required.

To complete the proof, consider the Lagrangian

$$L = \frac{1}{2} f(x) u^2_{(q)}.$$

The Euler-Lagrange expression for this Lagrangian is

$$E(L) = f(x)u_{(2q)} + \{\,\text{lower order terms}\,\}.$$

By successively modifying $L$ we can eliminate all of the lower order, even order terms to obtain a Lagrangian $\tilde{L}$ whose Euler-Lagrange expression is of the form

$$E(\tilde{L}) = f(x)u_{(2q)} + \{\,\text{odd order terms}\,\}.$$

With $P = E(\tilde{L})$, the only term that survives in (3.26) is that for which $j = 2q - m$. ∎

This form of the Helmholtz conditions provides us with an explicit form for all linear, locally variational source forms in one dependent variable. For $p$ even, let $B_p = (B^{i_1 i_2 \cdots i_p})$ be a collection of smooth functions on $\mathbf{R}^n$ and define a linear $p$-th order source form $\Delta_{B_p}$ by

$$\Delta_{B_p} = \bigl[B^I u_I - \sum_{\substack{|J|=1 \\ |J| \text{ odd}}}^{|I|-1} \tbinom{|I|}{|J|} a_{|J|}\,(D_J B^{JK}) u_K\bigr]\,\theta \wedge \nu, \tag{3.27}$$

where $|K| = p - |J|$.

Corollary 3.13. *For $p$ even, the linear source forms $\Delta_{B_p}$ are all locally variational. Moreover, every linear, locally variational source form $\Delta$ of order $2k$ is a unique sum of source forms of this type, i.e., there exist functions $B_{2k}$, $B_{2k-2}$, ..., $B_2$, $B_0$ such that*

$$\Delta = \Delta_{B_{2k}} + \Delta_{B_{2k-2}} + \cdots + \Delta_{B_2} + \Delta_{B_0}.$$

In the special case $n = 1$, this corollary simplifies the formula due to Krall [*43*] for the most general, linear, formally self-adjoint, scalar ordinary differential operator.

Example 3.14. *Hamiltonian operators for scalar evolution equations.*

We close this section by briefly describing the role played by functional forms in the theory of infinite dimensional Hamiltonian systems. For simplicity, we consider only the case of scalar evolution equations in one spatial variable. Let $E : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ with coordinates $(x, u) \to x$. Then a Hamiltonian operator can be identified, at least formally, with a linear differential operator

$$\mathcal{D} \colon C^\infty(J^\infty(E)) \to C^\infty(J^\infty(E))$$

such that the bracket

$$\{\cdot\,,\,\cdot\}\colon C^\infty(J^\infty(E)) \times C^\infty(J^\infty(E)) \to C^\infty(J^\infty(E))$$

defined by

$$\{P, Q\} = \int E(P)\mathcal{D}(E(Q))\,dx,$$

is a Poisson bracket. Here $E(P)$ and $E(Q)$ are the Euler-Lagrange expressions of $P$ and $Q$. At this point, the integral in this definition need not be taken literally; rather it simply serves to indicate that we are to calculate modulo exact 1 forms. Thus $\mathcal{D}$ is Hamiltonian if for all functions $P$, $Q$, and $R$

$$E(P)\mathcal{D}\left(E(Q)\right) + E(Q)\mathcal{D}\left(E(P)\right) = d_H f_1 \tag{3.28}$$

and

$$\begin{aligned} E(P)\mathcal{D}\big(E(Q)\mathcal{D}\left(E(R)\right)\big) + E(Q)\mathcal{D}\big(E(R)\mathcal{D}\left(E(P)\right)\big) \\ + E(R)\mathcal{D}\big(E(P)\mathcal{D}\left(E(Q)\right)\big) = d_H f_2 \end{aligned} \tag{3.29}$$

for some functions $f_1$ and $f_2$ on $J^\infty(E)$. See Vinogradov [*73*], Kuperschmidt [*46*] and Kosmann-Schwarzbach [*42*] for more rigorous and general definitions of the concept of Hamiltonian operator.

For a given operator $\mathcal{D}$, the skew-symmetry condition (3.28) is easily verified but the Jacobi identity (3.29) can be quite difficult to check directly, even when the operator $\mathcal{D}$ is a simple one. Olver [*55*] (pp. 424–436, in particular Theorem 7.8) devised a rather simple test to check the Jacobi identity. To each operator $\mathcal{D}$, he explicitly constructs a certain functional 3 form $\omega_{\mathcal{D}} \in \mathcal{F}^3$ and shows, in effect, that $\mathcal{D}$ satisfies the Jacobi identity if and only if $I(\omega_{\mathcal{D}}) = 0$.

For example, for the KdV equation with the Hamiltonian operator

$$\mathcal{D} = D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$$

the corresponding functional 3 form is

$$\omega_{\mathcal{D}} = \theta \wedge \theta_x \wedge \theta_{xxx}$$

which is easily seen to satisfy $I(\omega_{\mathcal{D}}) = 0$.

Conversely, in order to characterize all Hamiltonian operators and to classify their normal forms — in other words, to obtain the analogue of the Darboux theorem, one is confronted with the the analysis of the equation $I(\omega) = 0$ for type $(1, 3)$ forms of a certain prescribed form. For a Hamiltonian operator on $E$ of *odd* order $l$ it so happens that the relevant $(1,3)$ form to consider is

$$\begin{aligned}\omega = \theta \wedge [A_1\theta_x \wedge \theta_{p-1} + A_3\theta_{xxx} \wedge \theta_{p-3} + A_5\theta_{xxxxx} \wedge \theta_{p-5} + \cdots \\ + A_l\theta_l \wedge \theta_{p-l}] \wedge dx + \{\text{ lower order terms}\},\end{aligned} \tag{3.30}$$

where the coefficients $A_i$ are functions on $J^\infty(E)$ and $p \geq l+1$. The value of the integer $p$ is not specified *a priori* but is to be determined as a consequence of the equation $I(\omega) = 0$. I conjecture that $p \leq 3l+1$ and that this bound is sharp.

To appreciate the combinatorial complexity of this apparently innocuous problem, consider the case $l = 5$. We suppose, therefore, that $A_5 \neq 0$ and we must prove that this is possible only if $p \leq 16$. From the coefficients of $\theta \wedge \theta_k \wedge \theta_{p-k}$ in the equation $I(\omega) = 0$, for $k = 1, 2, \ldots, 6$ we obtain the following system of linear homogeneous equations for $A_1$, $A_3$, and $A_5$:

$$\begin{bmatrix} \binom{p-1}{1} - 1 + 2\epsilon & \binom{p-3}{1} + \epsilon\binom{3}{1} & \binom{p-5}{1} + \epsilon\binom{5}{1} \\ \binom{p-1}{2} - \binom{p-1}{1} & \binom{p-3}{2} + \epsilon\binom{3}{2} & \binom{p-5}{2} + \epsilon\binom{5}{2} \\ \binom{p-1}{3} - \binom{p-1}{2} & \binom{p-3}{3} - 1 + 2\epsilon & \binom{p-5}{3} + \epsilon\binom{5}{3} \\ \binom{p-1}{4} - \binom{p-1}{3} & \binom{p-3}{4} - \binom{p-3}{1} & \binom{p-5}{4} + \epsilon\binom{5}{4} \\ \binom{p-1}{5} - \binom{p-1}{4} & \binom{p-3}{5} - \binom{p-3}{2} & \binom{p-5}{5} - 1 + 2\epsilon \\ \binom{p-1}{6} - \binom{p-1}{5} & \binom{p-3}{6} - \binom{p-3}{3} & \binom{p-5}{6} - \binom{p-5}{1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \end{bmatrix} = 0.$$

Here $\epsilon = (-1)^{p+1}$. To complete this analysis, it therefore suffices to check that the coefficient matrix of this system has maximum rank only when $p > 16$! For $p$ odd the determinant formed from columns 1, 2 and 4 has value

$$-\frac{1}{6}p(p-6)(p-11)(p-13)$$

which is non-zero for $p > 16$. For $p$ even the determinant formed from columns 1, 3 and 5 has values

$$-\frac{1}{360}(p-3)(p-5)(p-10)(p-12)(p-14)(p-16)$$

which does again not vanish for $p > 16$.

The form

$$\omega = \theta \wedge [\frac{37}{36}\theta_{(1)} \wedge \theta_{(15)} - \frac{30}{13}\theta_{(3)} \wedge \theta_{(13)} + \theta_{(5)} \wedge \theta_{(11)}],$$

where $\theta_{(i)} = D_i\theta$, is of the form (3.30) and so, at least when $l = 5$, the conjectured bound is sharp.

Once the total order $p$ of $\omega$ is determined, the remaining conditions that arise from the equation $I(\omega_{\mathcal{D}}) = 0$ are complicated nonlinear differential conditions which must solved in order to characterize that particular class of Hamiltonian operators. Since our purpose here is merely to draw attention to the role that the variational bicomplex can play in this problem and to highlight some of the complexities of the action of the interior Euler operator $I$ on functional three forms, it is inappropriate for us to continue this analysis here. For a complete classification of Hamiltonian operators of low order on low dimensional spaces see Astashov and Vinogradov [*7*], Olver [*56*] and Cooke [*19*].

**B. Cochain Maps on the Variational Bicomplex.** In this section we classify those maps between infinite jet bundles whose differentials commute with either $d_V$ or $d_H$ or $\delta_V$ and thereby define cochain maps on either the vertical subcomplexes or the horizontal subcomplexes of the variational bicomplex or on the complex of functional forms.

Let $\pi\colon E \to M$ and $\rho\colon F \to N$ be two fibered manifolds and let

$$\Phi\colon J^\infty(E) \to J^\infty(F)$$

be a smooth map.

The map $\Phi$ need not be the prolongation of a map from $E$ to $F$ nor do we suppose that $\Phi$ covers a map from $E$ to $F$ or from $M$ to $N$. Recall that by Proposition 1.1, there are, for each $k = 0, 1, 2, \dots$ and some $m_k \geq k$, smooth maps

$$\Phi_k^{m_k}\colon J^{m_k}(E) \to J^k(F)$$

such that

$$\rho_k^\infty \circ \Phi = \Phi_k^{m_k} \circ \pi_{m_k}^\infty.$$

If $\omega \in \Omega^p(J^\infty(F))$ is a differential $p$ form on $J^\infty(F)$ which is represented by a form of order $k$, then the pullback $\Phi^*(\omega) \in \Omega^p(J^\infty(E))$ is represented by the form $(\Phi_k^{m_k})^*(\omega)$ of order $m_k$.

In general, the pullback map

$$\Phi^*\colon \Omega^p(J^\infty(F)) \to \Omega^p(J^\infty(E))$$

will not preserve the horizontal and vertical bigrading of forms and therefore will not induce a map from the variational bicomplex on $J^\infty(F)$ to the variational bicomplex on $J^\infty(E)$. To circumvent this obvious difficulty, let

$$\Phi^\sharp \colon \Omega^{r,s}(J^\infty(F)) \to \Omega^{r,s}(J^\infty(E))$$

be the map defined, for $\omega \in \Omega^{r,s}(J^\infty(F))$, by

$$\Phi^\sharp(\omega) = \pi^{r,s}[\Phi^*(\omega)],$$

where $\pi^{r,s}$ is the projection map from $\Omega^p(J^\infty(E))$ to $\Omega^{r,s}(J^\infty(E))$. We shall give necessary and sufficient conditions under which the *projected pullback* $\Phi^\sharp$ commutes with either $d_V$ or $d_H$.

At first glance, the introduction of the maps $\Phi^\sharp$ may seem somewhat artificial. However, in the calculus of variations it is precisely the map $\Phi^\sharp$, and not $\Phi^*$, which is used to pullback Lagrangians. Indeed, suppose that $\dim M = \dim N$ and that $\lambda$ is a Lagrangian on $J^\infty(F)$. Then $\Phi^*(\lambda)$ is an $n$ form on $J^\infty(E)$ but only the type $(n,0)$ component of $\Phi(\lambda)$ will contribute to the fundamental integral $\displaystyle\int_M [j^\infty(s)]^*\Phi^*(\lambda)$. Hence, the transformed Lagrangian is defined to be

$$\Phi^\sharp(\lambda) = \pi^{n,0}[\Phi^*(\lambda)].$$

Locally, if $(y^j, v^\mu)$ are adapted coordinates on $F$ and

$$\Phi_0^{m_0}[x,u] = \bigl(f^i[x,u], g^\mu[x,u]\bigr),$$

then (with n=2 for simplicity)

$$\begin{aligned}
\Phi^*\bigl(L[y,v]\,dy^1\wedge dy^2\bigr) &= (L\circ\Phi)[df^1\wedge df^2] \\
&= (L\circ\Phi)[d_H f^1\wedge d_H f^2 + (d_H f^1\wedge d_V f^2 + d_V f^1\wedge d_H f^2) \\
&\qquad + d_V f^1\wedge d_V f^2]
\end{aligned}$$

and thus

$$\Phi^\sharp(L\,dy^1\wedge dy^2) = (L\circ\Phi)\det(D_j f^i)\,dx^1\wedge dx^2.$$

We emphasize that $\det(D_j f^i)$ is the Jacobian of the functions $f^i[x,u]$ with respect to the total derivatives $D_j$.

THEOREM 3.15. *Let $\Phi$ be a smooth map from $J^\infty(E)$ to $J^\infty(F)$.*

*(i) The projected pullback $\Phi^\sharp$ commutes with $d_V$ if and only if $\Phi$ covers a smooth map $\phi_0$ from $M$ to $N$, i.e.,*

$$\begin{array}{ccc} J^\infty(E) & \xrightarrow{\Phi} & J^\infty(F) \\ {\scriptstyle \pi^\infty_M}\downarrow & & {\scriptstyle \rho^\infty_N}\downarrow \\ M & \xrightarrow{\phi_0} & N. \end{array}$$

*(ii) The projected pullback $\Phi^\sharp$ commutes with $d_H$ if and only if $\Phi^*$ is a contact transformation, i.e., for every $\omega \in \mathcal{C}(J^\infty(F))$*

$$\Phi^*(\omega) \in \mathcal{C}(J^\infty(E)).$$

*(iii) The projected pullback $\Phi^\sharp$ commutes with both $d_V$ and $d_H$ if and only if it coincides with the pullback $\Phi^*$, i.e.,*

$$\Phi^\sharp = \Phi^*.$$

PROOF: (i) First suppose that $\Phi$ covers a map $\phi_0$ from $M$ to $N$. Then the Jacobian $\Phi_*\colon T(J^\infty(E)) \to T(J^\infty(F))$ satisfies

$$(\rho^\infty_N)_* \circ \Phi_* = (\phi_0)_* \circ (\pi^\infty_M)_*.$$

Consequently, if $Y$ is a $\pi^\infty_M$ vertical vector at a point $\sigma \in J^\infty(E)$, then $\Phi_*(Y)$ is a $\rho^\infty_N$ vertical vector at the point $\tilde\sigma = \Phi(\sigma)$. Now consider a $p$ form $\omega \in \Omega^{r,s}(J^\infty(F))$. Then $\omega \in \Omega^{r,p}_H(J^\infty(F))$ (see §2C). Hence, if $X_1, X_2, \ldots, X_p$ are tangent vectors at $\sigma$, at least $s+1$ of which are $\pi^\infty_M$ vertical, we can conclude that

$$[\Phi^*(\omega)(\sigma)]\bigl(X_1, X_2, \ldots, X_p\bigr) = \omega(\tilde\sigma)\bigl(\Phi_*(X_1), \Phi_*(X_2), \ldots, \Phi_*(X_p)\bigr) = 0.$$

This shows that $\Phi^*(\omega) \in \Omega^{r,p}_H(J^\infty(E))$ and therefore

$$\begin{aligned} &\Phi^*\Omega^{r,s}(J^\infty(F)) \\ &\qquad \subset \Omega^{r,s}(J^\infty(E)) \oplus \Omega^{r+1,s-1}(J^\infty(E)) \oplus \Omega^{r+2,s-2}(J^\infty(E)) \oplus \cdots . \end{aligned} \tag{3.31}$$

From this inclusion we can easily deduce, for a type $(r,s)$ form $\omega$ on $J^\infty(F)$, that

$$\pi^{r,s+1}[d\,\Phi^*(\omega)] = d_V[\Phi^\sharp(\omega)]$$

and

$$\pi^{r,s+1}[\Phi^*(d_H\omega)] = 0.$$

These two equations, together with the fact that $\Phi^*$ commutes with $d$, lead to

$$\begin{aligned}\Phi^\sharp[d_V\omega] &= \pi^{r,s+1}[\Phi^*(d_V\omega)] \\ &= \pi^{r,s+1}[\Phi^*(d\,\omega - d_H\omega)] \\ &= \pi^{r,s+1}[d\,(\Phi^*\omega)] = d_V[\Phi^\sharp(\omega)],\end{aligned}$$

as required.

Conversely, suppose that $\Phi^\sharp$ commutes with $d_V$. Let $(y^a, v^\mu)$ be local adapted coordinates on $F$ and suppose that the map $\phi_0 = \rho^\infty_N \circ \Phi$ is given in these coordinates by functions $f^a = y^a \circ \phi_0$, *i.e.*,

$$\phi_0[x,u] = (f^a[x,u]).$$

To complete the proof of (i), we must show that functions $f^a$ are independent of all the fiber variables $u^\alpha_I$, $|I| \geq 0$.

If $g\colon J^\infty(F) \to \mathbf{R}$ is any smooth function then, by hypothesis,

$$\Phi^\sharp(d_V g) = d_V(g \circ \Phi).$$

In particular, if $g_0\colon N \to \mathbf{R}$ and $g = g_0 \circ \rho^\infty_N$, then $d_V g = 0$ and therefore

$$d_V[g_0 \circ \rho^\infty_N \circ \Phi] = 0.$$

By choosing $g_0$ to be a coordinate function $y^a$, this equation becomes $d_V f^a = 0$ which implies that $f^a = f^a(x^i)$. This completes the proof of (i).

The proof of (ii) is similar. If $\Phi^*$ preserves the contact ideal then, because $\Omega^{r,s}(J^\infty(F)) \subset \mathcal{C}^s(J^\infty(F))$, we have that

$$\Phi^*\Omega^{r,s}(J^\infty(E)) \subset \mathcal{C}^s(J^\infty(E))$$

and therefore

$$\begin{aligned}&\Phi^*\Omega^{r,s}(J^\infty(F)) \\ &\qquad \subset \Omega^{r,s}(J^\infty(E)) \oplus \Omega^{r-1,s+1}(J^\infty(E)) \oplus \Omega^{r-2,s+2}(J^\infty(E)) \oplus \cdots .\end{aligned} \tag{3.32}$$

From this inclusion we can deduce, for $\omega \in \Omega^{r,s}(J^\infty(F))$, that

$$\pi^{r+1,s}[d\,\Phi^*(\omega)] = d_H[\Phi^\sharp(\omega)]$$

and

$$\pi^{r+1,s}[\Phi^*(d_V\omega)] = 0.$$

It is now a simply matter to verify that

$$\Phi^\sharp\big(d_H\omega\big) = d_H\big(\Phi^\sharp\omega\big).$$

Conversely, suppose that $\Phi^\sharp$ commutes with $d_H$. Then for every real-valued function $g\colon J^\infty(F) \to \mathbf{R}$ we have

$$(\pi^{1,0}\circ\Phi^*)[d_H g] = d_H[g\circ\Phi].$$

From this equation it is easily seen that

$$(\pi^{1,0}\circ\Phi^*)[d_V g] = 0$$

and hence

$$\Phi^*[d_V g] \in \Omega^{0,1}(J^\infty(E)) \subset \mathcal{C}[J^\infty(E)].$$

Since the contact ideal $\mathcal{C}(J^\infty(F))$ is locally generated by the vertical differentials of the coordinate functions, *viz.*, $d_V v^\mu_J = \bar\theta^\mu_J$, this proves that

$$\Phi^*\mathcal{C}(J^\infty(F)) \subset \mathcal{C}(J^\infty(E)).$$

Finally, the third statement in the proposition is a direct consequence of the two inclusions (3.31) and (3.32). ∎

Corollary 3.16. *Let $E$, $F$ and $G$ be fibered manifolds and suppose that*

$$\Phi\colon J^\infty(E) \to J^\infty(F) \qquad \text{and} \qquad \Psi\colon J^\infty(F) \to J^\infty(G)$$

*are smooth maps.*

*(i) If $\Phi$ and $\Psi$ both cover maps between the base spaces, then*

$$(\Psi\circ\Phi)^\sharp = \Phi^\sharp\circ\Psi^\sharp.$$

*(ii) If $\Phi$ and $\Psi$ are contact transformations, then*

$$(\Psi\circ\Phi)^\sharp = \Phi^\sharp\circ\Psi^\sharp.$$

Proof: Suppose that $\Phi$ and $\Psi$ cover maps between the base spaces. Let $\omega$ be a type $(r,s)$ form on $J^\infty(G)$. Then, on account of (3.31),

$$\Psi^*(\omega) = \alpha_0 + \alpha_1 + \alpha_2 + \dots,$$

where $\alpha_i$ is a form on $J^\infty(F)$ of type $(r+i, s-i)$. For the same reason we obtain

$$\Phi^*(\alpha_i) = \beta_{i0} + \beta_{i1} + \beta_{i2} + \dots,$$

where $\beta_{ij}$ is a form on $J^\infty(E)$ of type $(r+i+j, s-i-j)$. Thus, on the one hand, it follows that

$$(\Phi^\sharp \circ \Psi^\sharp)(\omega) = \Phi^\sharp(\alpha_0) = \beta_{00}$$

while, on the other hand,

$$\begin{aligned}(\Psi \circ \Phi)^\sharp(\omega) &= \pi^{r,s} \circ (\Psi \circ \Phi)^*(\omega) \\ &= \pi^{r,s}\big[\Phi^*\big(\Psi^*(\omega)\big)\big] = \pi^{r,s}\big[\sum_{i,j} \beta_{ij}\big] \\ &= \beta_{00}.\end{aligned}$$

This proves (i). The proof of (ii) is similar. ∎

Proposition 3.15 has an infinitesimal analogue in terms of Lie derivatives. Let $X$ be an arbitrary vector field on $J^\infty(E)$. In general, the Lie derivative $\mathcal{L}_X$ does not preserve the bidegree of a form on $J^\infty(E)$ and so we introduce the *projected Lie derivative* by defining, for $\omega \in \Omega^{r,s}(J^\infty(E))$,

$$\mathcal{L}_X^\sharp(\omega) = \pi^{r,s}[\mathcal{L}_X(\omega)]. \tag{3.33}$$

PROPOSITION 3.17. *Let $X$ be a vector field on $J^\infty(E)$.*

*(i) The projected Lie derivative $\mathcal{L}_X^\sharp$ commutes with $d_V$ if and only if $X$ is $\pi^\infty_M$ related to a vector field $X_0$ on $M$.*

*(ii) The projected Lie derivative $\mathcal{L}_X^\sharp$ commutes with $d_H$ if and only if $X$ is the prolongation of a generalized vector field on $E$.*

PROOF: We cannot appeal directly to the previous theorem since the vector field $X$ may not define a flow on $J^\infty(E)$. Nevertheless, the proof is similar and so we shall omit the details. Let us remark, however, that if $(\pi^\infty_M)_* X = X_0$ and if $\omega \in \Omega^{r,s}$, then $\mathcal{L}_X \omega \in \Omega^{r,p}_H$. To prove this, we use (1.18) and note that if $Y$ is a $\pi^\infty_M$ vertical vector field on $J^\infty(E)$, then the Lie bracket $[X, Y]$ is also $\pi^\infty_M$ vertical. ∎

Let us now investigate under what conditions the maps $\Phi^\sharp$ will induce a cochain map on the complex of functional forms. Since functional forms are of top horizontal degree, this problem becomes meaningful only when the dimensions of the base spaces $M$ and $N$ coincide. This we shall assume for the remainder of this section. It is obvious that if $\Phi^\sharp$ commutes with both $d_V$ and $I$, then $\Phi^\sharp$ will restrict to a map from $\mathcal{F}^s(J^\infty(F))$ to $\mathcal{F}^s(J^\infty(E))$ which will commute with the differential $\delta_V$. But, as the next proposition shows, the requirement that $\Phi^\sharp$ commute with both $d_V$ and $I$ is very restrictive.

Proposition 3.18. *Let $\Phi\colon J^\infty(E) \to J^\infty(F)$ be a smooth map for which $\Phi^\sharp$ commutes with both $d_V$ and $I$ and for which the map*

$$\Phi^\sharp\colon \mathcal{F}^1(J^\infty(F)) \to \mathcal{F}^1(J^\infty(E))$$

*is injective. Then $\Phi$ is the prolongation of a fiber-preserving local diffeomorphism $\phi\colon E \to F$, i.e., $\Phi = \operatorname{pr}\phi$.*

Proof: We work locally. Let $(x^i, u_I^\alpha)$ and $(y^j, v_J^\mu)$ be local coordinates around a point $\sigma \in J^\infty(E)$ and around the point $\tilde{\sigma} = \Phi(\sigma) \in J^\infty(F)$. By Theorem 3.15 the map $\Phi$ covers a map $\phi_0\colon M \to N$ and hence $\Phi$ is described in these coordinates by functions

$$y^j = y^j(x^i) \qquad \text{and} \qquad v_J^\mu = v_J^\mu(x^i, u_I^\alpha).$$

To prove the proposition it suffices to show, in view of Proposition 1.6,

(A) that $v^\mu = v^\mu(x^i, u^\alpha)$ so that $\Phi$ covers a map $\phi\colon E \to F$; and

(B) that $\Phi$ is a contact transformation. To prove this it suffices to show, in view of Theorem 3.15, (or more precisely the proof of Theorem 3.15) that $\Phi$ commutes with $d_H$ acting on functions, *i.e.*,

$$\Phi^*(d_H f) = d_H(f \circ \Phi). \tag{3.34}$$

To prove (A), let $\omega = \bar{\theta}^\mu \wedge \bar{\nu}$, where

$$\bar{\theta}^\mu = dv^\mu - v_j^\mu dy^j \qquad \text{and} \qquad \bar{\nu} = dy^1 \wedge dy^2 \wedge \ldots dy^n.$$

Then $I(\omega) = \omega$ and therefore, because $I$ commutes with $\Phi^\sharp$, we must have

$$\begin{aligned} I\bigl(\Phi^\sharp(\omega)\bigr) &= \Phi^\sharp(\omega) = \Phi^\sharp(dv^\mu \wedge \bar{\nu}) \\ &= J\bigl[\sum_{|I|=0}^{k} (\partial_\beta^I v^\mu)\, \theta_I^\beta \wedge \nu\bigr]. \end{aligned} \tag{3.35}$$

where $J = \det\bigl(\dfrac{\partial y^j}{\partial x^i}\bigr)$. Since the left-hand side of this equation belongs to $\mathcal{F}^1$, the coefficients of $\theta_I^\alpha \wedge \nu$ on the right-hand side must vanish for $|I| \geq 1$, *i.e.*,

$$J \frac{\partial v^\mu}{\partial u_I^\beta} = 0, \tag{3.36}$$

in which case equation (3.35) reduces to

$$I\big(\Phi^\sharp(\omega)\big) = J\frac{\partial v^\mu}{\partial u^\alpha}\,\theta^\alpha \wedge \nu.$$

The assumption that $\Phi^\sharp$ is injective on $\mathcal{F}^1$ now establishes that the two Jacobians $J$ and $\det\big(\dfrac{\partial v^\mu}{\partial u^\alpha}\big)$ do not vanish. Consequently, equation (3.36) shows that the functions $v^\mu$ are independent of the jet variables $u^\alpha_I$ and (A) is proved.

To prove (B), we consider the form

$$\omega = f(y^k)\bar\theta^\mu_j \wedge \bar\nu,$$

where $\bar\theta^\mu_j = dv^\mu_j - v^\mu_{jk}dy^k$ and the coefficient $f$ is a function of the base coordinates $y^k$ alone. In this case we find, on the one hand, that

$$I(\omega) = -\frac{\partial f}{\partial y^j}\,\bar\theta^\mu \wedge \bar\nu$$

and

$$\Phi^\sharp\big(I(\omega)\big) = -\frac{\partial f}{\partial y^j}[y(x)]J\frac{\partial v^\mu}{\partial u^\beta}\,\theta^\beta \wedge \nu. \tag{3.37}$$

On the other hand,

$$\Phi^\sharp(\omega) = f\big(y(x)\big)J d_V v^\mu_j \wedge \nu$$

and consequently

$$I\big(\Phi^\sharp(\omega)\big) = \sum_{|I|=0}^{k}(-D)_I\big[f\big(y(x)\big)J\partial^I_\beta v^\mu_j\big]\,\theta^\beta \wedge \nu. \tag{3.38}$$

Equate the right-hand sides of (3.37) and (3.38). Since the function $f$ is arbitrary, we can equate the coefficients of the various derivatives of $f$ to obtain

$$\partial^I_\beta v^\mu_j = 0 \qquad\qquad \text{for } |I| \geq 2, \tag{3.39a}$$

$$\frac{\partial v^\mu_j}{\partial u^\beta_i} = \frac{\partial x^i}{\partial y^j}\frac{\partial v^\mu}{\partial u^\beta}, \quad\text{ and }\quad J\frac{\partial v^\mu_j}{\partial u^\alpha} = D_i\big[J\frac{\partial v^\mu_j}{\partial u^\alpha_i}\big]. \tag{3.39b}$$

Let

$$C^i_j = \big[\text{cofactor of } \frac{\partial y^j}{\partial x^i}\big] = \frac{\partial x^i}{\partial y^j}J.$$

It is easily seen that this matrix is divergence-free, *i.e.*,

$$\frac{\partial C^i_j}{\partial x^i} = 0.$$

On combining this fact with (3.39a) and (3.39b), we arrive at

$$\Phi^*(\bar\theta^\mu_j \wedge \bar\nu) = D_i\big[C^i_j \frac{\partial v^\mu}{\partial u^\alpha}\,\theta^\alpha\big] \wedge \nu. \tag{3.40}$$

We shall use this result momentarily.

Finally, consider the form

$$\omega = d_H[f\,\bar\theta^\mu \wedge \bar\nu_j] = \frac{d\,f}{dy^j}\,\bar\theta^\mu \wedge \bar\nu + f\,\bar\theta^\mu_j \wedge \bar\nu,$$

where $f = f(y^j, v^\mu_J)$ is now an arbitrary function. By using (3.40), we calculate that

$$\begin{aligned}\Phi^\sharp(\omega) &= \big[\big(\frac{d\,f}{dy^j}\circ\Phi\big)\frac{\partial v^\mu}{\partial u^\alpha}\big]J\,\theta^\alpha \wedge \nu + (f\circ\Phi)D_i\big[C^i_j\frac{\partial v^\mu}{\partial u^\alpha}\,\theta^\alpha \wedge \nu\big] \\ &= \big[\frac{\partial y^k}{\partial x^i}\big(\frac{d\,f}{dy^k}\circ\Phi\big) - \frac{d}{dx^i}\big(f\circ\Phi\big)\big]C^i_j\frac{\partial v^\mu}{\partial u^\alpha}\,\theta^\alpha \wedge \nu \\ &\qquad + d_H\big[(f\circ\Phi)C^i_j\frac{\partial v^\mu}{\partial u^\alpha}\,\theta^\alpha \wedge \nu_i\big].\end{aligned} \tag{3.41}$$

Since $\omega$ is $d_H$ exact, $I(\omega) = 0$ and consequently $I(\Phi^\sharp(\omega)) = 0$. In view of (3.41), this implies that

$$\frac{\partial y^k}{\partial x^i}\big(\frac{d\,f}{dy^k}\circ\Phi\big) = \frac{d}{dx^i}\big(f\circ\Phi\big).$$

Because $f$ is arbitrary, this proves that $\Phi$ commutes with $d_H$ acting on functions. The map $\Phi$ must therefore be a contact transformation. This proves (B) and completes the proof of Proposition 3.18. ∎

Because of the ubiquitous role that the operator $I$ plays in the development of the variational bicomplex, it is of some interest to characterize those maps $\Phi$ for which $\Phi^\sharp$ commutes with $I$. This condition involves the pullback $\Phi^*$ acting on forms of total degree $n+1$ or higher and is therefore expressed by complicated non-linear equations in the derivatives of $\Phi$. My attempts to analyze these equations have been unsuccessful. However, the corresponding infinitesimal problem for the projected Lie derivative $\mathcal{L}^\sharp_X$, where $X$ is a vector field on $J^\infty(E)$, results in linear conditions on $X$ which are easily solved. It is surprising that the characterization of vector fields $X$ on the *infinite* jet bundle $J^\infty(E)$ for which $\mathcal{L}^\sharp_X \circ I = I \circ \mathcal{L}^\sharp_X$ coincides precisely with the characterization of vector fields on the *finite* dimensional jet bundle $J^k(E)$ which preserve the contact ideal (see R. Anderson and N. Ibragimov [5], pp. 37–46 and, in particular, Theorem 10.1).

Proposition 3.19. *Let $X$ be an arbitrary vector field on $J^\infty(E)$.*

*(i) If $m > 1$, (where $m$ is the fiber dimension of $E$), then $\mathcal{L}_X^\sharp$ commutes with $I$ if and only if $X$ is the prolongation of a vector field on $E$.*

*(ii) If $m = 1$, then $\mathcal{L}_X^\sharp$ commutes with $I$ if and only if $X$ is locally the prolongation of a generalized vector field $\widetilde{X}$ on $E$ of the form*

$$\widetilde{X} = -\frac{\partial S}{\partial u_i}\frac{\partial}{\partial x^i} + \left[S - u_i\frac{\partial S}{\partial u_i}\right]\frac{\partial}{\partial u}, \tag{3.42}$$

*where $S = S(x^i, u, u_i)$ is an arbitrary function on $J^1(U)$.*

Proof: We begin with the proof of sufficiency. This involves a slight generalization of the calculations presented in the proof of Theorem 2.12, part (v). We assume that the vector field $X$ is the prolongation of

$$\widetilde{X} = a^i\frac{\partial}{\partial x^i} + b^\alpha\frac{\partial}{\partial u^\alpha}.$$

In part (i) of the theorem the coefficients $a^i$ and $b^\alpha$ are functions of the coordinates $x^i$ and $u^\alpha$ only while, in part (ii), $a^i = -\dfrac{\partial S}{\partial u_i}$ and $b = S - u_i\dfrac{\partial S}{\partial u_i}$. Observe that

$$\pi^{0,1}(\mathcal{L}_X\theta^\alpha) = d_V b^\alpha - u_i^\alpha d_V a^i$$

and that, in either case (i) or (ii), this simplifies to

$$\pi^{0,1}(\mathcal{L}_X\theta^\alpha) = Q_\beta^\alpha\theta^\beta, \tag{3.43}$$

where

$$Q_\beta^\alpha = \frac{\partial b^\alpha}{\partial u^\beta} - u_i^\alpha\frac{\partial a^i}{\partial u^\beta}.$$

From the definition 2.11 of $I$ in terms of the interior product operator $F_\alpha$, we therefore conclude that $\mathcal{L}_X^\sharp$ will commute with $I$ in either case (i) or (ii) if, for all type $(n, s)$ forms $\omega$,

$$F_\alpha(\mathcal{L}_X^\sharp\omega) = Q_\alpha^\beta F_\beta(\omega) + \pi^{n,s-1}[\mathcal{L}_X\big(F_\alpha(\omega)\big)]. \tag{3.44}$$

To derive this formula, recall that the defining property of $F_\alpha(\omega)$ is the relation

$$\mathrm{pr}\, Y \mathbin{\lrcorner} \omega = Y^\alpha F_\alpha(\omega) + d_H(\eta).$$

where $Y$ is an arbitrary evolutionary vector field

$$Y = Y^\alpha[x,u]\frac{\partial}{\partial u^\alpha}.$$

To this last equation we apply $\mathcal{L}^\sharp_X$. Because $X$ is the prolongation of a generalized vector field on $E$, the projected Lie derivative commutes with $d_H$. This gives rise to

$$\pi^{n,s-1}[\mathcal{L}_X(\operatorname{pr} Y \lrcorner\, \omega)] = X(Y^\alpha)F_\alpha(\omega) + Y^\alpha\big[\pi^{n,s-1}\mathcal{L}_X F_\alpha(\omega)\big] + d_H\eta_1. \tag{3.45}$$

Expansion of the left-hand side of this equation by the product rule for Lie differentiation yields

$$\pi^{n,s-1}\big[\mathcal{L}_X(\operatorname{pr} Y \lrcorner\, \omega)\big] = \pi^{n,s-1}\big[\mathcal{L}_X(\operatorname{pr} Y) \lrcorner\, \omega\big] + \operatorname{pr} Y \lrcorner\, \big(\mathcal{L}^\sharp_X \omega\big). \tag{3.46}$$

The next step is to analyze each term on the right-hand side on equation (3.46). By virtue of the definition of $F_\alpha$, the second term becomes

$$\operatorname{pr} Y \lrcorner\, \big(\mathcal{L}^\sharp_X \omega\big) = (Y^\alpha)F_\alpha(\mathcal{L}^\sharp_X\omega) + d_H\eta_2. \tag{3.47}$$

To analyze the first term we use Propositions 1.20 and 1.21 to write

$$\begin{aligned}\mathcal{L}_X(\operatorname{pr} Y) &= [\operatorname{pr}\widetilde{X}, \operatorname{pr} Y] = \operatorname{pr}\,[\widetilde{X}, Y] \\ &= \operatorname{pr}\{[\widetilde{X}, Y]_{\mathrm{ev}}\} + \operatorname{tot}\,[\widetilde{X}, Y].\end{aligned}$$

Since $\omega$ is of type $(n,s)$, $\operatorname{tot}[\widetilde{X}, Y] \lrcorner\, \omega$ is of type $(n-1,s)$ and therefore

$$\begin{aligned}\pi^{n,s-1}\big[\mathcal{L}_X(\operatorname{pr} Y \lrcorner\, \omega)\big] &= \{\operatorname{pr}\,[\widetilde{X}, Y]_{\mathrm{ev}}\} \lrcorner\, \omega \\ &= [\widetilde{X}, Y]^\alpha_{\mathrm{ev}} F_\alpha(\omega) + d_H\eta_3 &(3.48)\\ &= \big[X(Y^\alpha) - Q^\alpha_\beta Y^\beta\big]F_\alpha(\omega) + d_H\eta_3. &(3.49)\end{aligned}$$

In deriving (3.48) we have once again used the defining property of $F_\alpha(\omega)$. In deriving (3.49), we have used the definition (1.42) for the bracket of two generalized vector fields and the special form of $\widetilde{X}$, as postulated in cases (i) and (ii).

The combination of (3.45), (3.46), (3.47), and (3.49) leads to (3.44). This proves sufficiency.

To prove that the vector fields described in cases (i) and (ii) are the only vector fields on $J^\infty(E)$ whose projected Lie derivatives commute with $I$, we need the following result, valid for any $m$.

Lemma 3.20. *Let $Y$ be a $\pi^\infty_E$ vertical vector field on $J^\infty(E)$. If $\mathcal{L}^\sharp_Y$ commutes with $I$, then $Y = 0$.*

Proof: Let

$$Y = \sum_{|I|=1}^{\infty} Y^\alpha_I[x,u]\partial^I_\alpha$$

and let $\omega = f(x,u)\,\theta^\alpha_I \wedge \nu$, where $f$ is a function on $U$. We find that

$$\mathcal{L}^\sharp_Y[I(\omega)] = \mathcal{L}^\sharp_Y\bigl[(-1)^{|I|}(D_I f)\,\theta^\alpha \wedge \nu\bigr] = (-1)^{|I|}\mathcal{L}_Y[D_I(f)]\,\theta^\alpha \wedge \nu,$$

while

$$\mathcal{L}_Y\omega = f(d_V Y^\alpha_I) \wedge \nu = f \sum_{|J|=0}^{k} (\partial^J_\beta Y^\alpha_I)\,\theta^\beta_J \wedge \nu,$$

and thus

$$I\bigl(\mathcal{L}^\sharp_Y(\omega)\bigr) = E_\beta(fY^\alpha_I)\,\theta^\beta \wedge \nu,$$

where $E_\beta$ is the Euler-Lagrange operator. Hence the hypothesis that $L^\sharp_Y$ commutes with $I$ leads to the conclusion that

$$E_\beta(fY^\alpha_I) = (-1)^{|I|}\mathcal{L}_Y\bigl[D_I(f)\bigr]\delta^\alpha_\beta. \tag{3.50}$$

When $f$ is a function of the base variables $x^i$ alone, the right-hand side of (3.50) vanishes and so, by virtue of Corollary 2.9, $Y^\alpha_I = Y^\alpha_I(x^j)$. With $f = u^\gamma$, (3.50) becomes

$$\delta^\gamma_\beta Y^\alpha_I = (-1)^{|I|} Y^\gamma_I \delta^\alpha_\beta$$

from which it readily follows, for $m > 1$, that $Y^\alpha_I = 0$. When $m = 1$, the result follows from (3.50) with $f(x^i,u) = g(x^i)u$. ∎

To complete the proof of Proposition 3.19, let $X$ be an arbitrary vector field on $J^\infty(E)$ and let

$$\widetilde{X} = (\pi^\infty_E)_* X = a^i[x,u]\frac{\partial}{\partial x^i} + b^\alpha[x,u]\frac{\partial}{\partial u^\alpha}.$$

The vector field $\widetilde{X}$ is a generalized vector field on $J^\infty(E)$ so that the coefficients $a^i$ and $b^\alpha$ are smooth functions on $J^\infty(U)$. We shall prove that if $\mathcal{L}^\sharp_X$ commutes with $I$ then, for $m > 1$, $\widetilde{X}$ is actually a vector field on $E$ while, for $m = 1$, $\widetilde{X}$ is given by (3.42) for some choice of function $S$. In either case, $\mathcal{L}_{\operatorname{pr}\widetilde{X}}$ commutes with $I$. The vector field $Y = X - \operatorname{pr}\widetilde{X}$ is therefore a $\pi^\infty_E$ vertical vector field whose Lie derivative commutes with $I$. The previous lemma implies that $Y = 0$ and so $X = \operatorname{pr}\widetilde{X}$, as required.

Let $\omega = \theta^\alpha \wedge \nu$. Then $I(\omega) = \omega$ so that the vector field $X$ must satisfy

$$I(\mathcal{L}^\sharp_X \omega) = \mathcal{L}^\sharp_X \omega. \tag{3.51}$$

Since

$$\mathcal{L}^\sharp_X \omega = d_V b^\alpha \wedge \nu - u^\alpha_j d_V a^j \wedge \nu + (D_j a^j)\theta^\alpha \wedge \nu,$$

it immediately follows that (3.51) holds if and only if

$$\partial^I_\beta b^\alpha = u^\alpha_j \partial^I_\beta a^j \tag{3.52}$$

for all $|I| \geq 1$. With $|I| = 1$, this equation is

$$\frac{\partial b^\alpha}{\partial u^\beta_i} = u^\alpha_j \frac{\partial a^j}{\partial u^\beta_i}. \tag{3.53}$$

The analysis of this equation depends on the value of $m$.

If $m > 1$, we obtain from (3.53) the integrability condition

$$\delta^\alpha_\gamma \frac{\partial a^k}{\partial u^\beta_i} = \delta^\alpha_\beta \frac{\partial a^i}{\partial u^\gamma_k}. \tag{3.54}$$

This implies that $\dfrac{\partial a^j}{\partial u^\beta_j} = 0$ in which case (3.53) reduces to $\dfrac{\partial b^\alpha}{\partial u^\beta_i} = 0$. Equation (3.52) now forces both $a^i$ and $b^\alpha$ to be independent of the derivatives $u^\alpha_I$ for all $|I| \geq 1$. This proves that $\widetilde{X}$ is a vector field on $E$.

For the case $m = 1$, we return to (3.52) which, with $|I| > 1$, implies that $b - u_j a^j$ is independent of the derivatives $u_I$ for all $|I| > 1$, *i.e.*,

$$b - u_j a^j = S(x^i, u, u_i).$$

We differentiate this equation with respect to $u_i$ and substitute from (3.53) to deduce that

$$a^i = -\frac{\partial S}{\partial u_i}.$$

This proves (3.54) and completes the proof of Proposition 3.19. ∎

We began this section with the observation that, for an arbitrary map $\Phi$ between infinite jet bundles, the pullback map $\Phi^*$ does not preserve the bidegree of forms. To circumvent this problem, we introduced the projected pullback maps $\Phi^\sharp$. A similar problem has now risen in the case of functional forms since, in general, $\Phi^\sharp$ does not map $\mathcal{F}^s(J^\infty(F))$ to $\mathcal{F}^s(J^\infty(E))$. Accordingly, let us now define

$$\Phi^\natural \colon \Omega^{n,s}(J^\infty(F)) \to \Omega^{n,s}(J^\infty(E))$$

by

$$\Phi^\natural(\omega) = I[\Phi^\sharp(\omega)] = (I \circ \pi^{n,s})[\Phi^*(\omega)].$$

Evidently, $\Phi^\natural$ restricts to a map

$$\Phi^\natural \colon \mathcal{F}^s(J^\infty(F)) \to \mathcal{F}^s(J^\infty(E)).$$

Likewise, if $X$ is any vector field on $J^\infty(E)$, then we define, for $\omega \in \Omega^{n,s}(J^\infty(E))$,

$$\mathcal{L}^\natural_X \omega = (I \circ \pi^{n,s})[\mathcal{L}_X \omega].$$

We now establish the following important result.

THEOREM 3.21. *Let $\Phi \colon J^\infty(E) \to J^\infty(F)$.*

*(i) If $\Phi$ is a contact transformation, then $\Phi^\natural$ commutes with both $I$ and $\delta_V$.*

*(ii) Let $X$ be a generalized vector field on $E$. Then $\mathcal{L}^\natural_{\operatorname{pr} X}$ commutes with both $I$ and $\delta_V$.*

PROOF: We shall prove (i). The proof of (ii) is similar and is therefore omitted. Let $\omega \in \Omega^{n,s}(J^\infty(F))$. Since $\omega = I(\omega) + d_H\eta$ (at least locally) and since, by Theorem 3.15, $\Phi^\sharp$ commutes with $d_H$, we have that

$$\Phi^\sharp(\omega) = \Phi^\sharp(I(\omega)) + d_H(\Phi^\sharp\eta).$$

To this equation we apply $I$ to conclude, because $I \circ \Phi^\natural = \Phi^\natural$, that $\Phi^\natural$ commutes with $I$.

To prove that $\Phi^\natural$ commutes with $\delta_V$, let $\omega \in \mathcal{F}^s(J^\infty(F))$. Decompose $\Phi^*\omega$ by type, *i.e.*, let

$$\Phi^*\omega = \bar\omega^0 + \bar\omega^1 + \bar\omega^2 + \cdots ,$$

where $\bar\omega^i$ is a type $(n-i, s+i)$ form on $J^\infty(E)$. Note that $\bar\omega^0 = \Phi^\sharp\omega$. The following sequence of elementary equalities completes the proof:

$$\begin{aligned}
\Phi^\natural(\delta_V\omega) = \Phi^\natural(d_V\omega) = \Phi^\natural(d\,\omega) &= (I \circ \pi^{n,s+1}\big(d\,(\Phi^*\omega)\big) \\
&= (I \circ \pi^{n,s+1})\big(d\,(\bar\omega^0 + \bar\omega^1 + \bar\omega^2 + \dots)\big) \\
&= I(d_V\bar\omega^0) = I\big(d_V(I\bar\omega^0)\big) = \delta_V(\Phi^\natural\omega).
\end{aligned}$$

∎

COROLLARY 3.22. *(i) If $\Phi\colon J^\infty(E) \to J^\infty(F)$ is a contact transformation, then for any Lagrangian $\bar\lambda \in \Omega^{n,0}(J^\infty(F))$,*

$$E(\Phi^\sharp \bar\lambda) = \Phi^\natural[E(\bar\lambda)]. \tag{3.55}$$

*(ii) If $X$ is a generalized vector field on $E$ and $\lambda \in \Omega^{n,0}(J^\infty(E))$, then*

$$E(\mathcal{L}^\sharp_{\operatorname{pr} X}\lambda) = \mathcal{L}^\natural_{\operatorname{pr} X}[E(\lambda)]. \tag{3.56}$$

*(iii) If $\overline{\Delta} \in \mathcal{F}^1(J^\infty(F))$ is a locally variational source form, then so is*

$$\Delta = \Phi^\natural(\overline{\Delta}) = (I \circ \pi^{n,1})(\Phi^*(\Delta)). \tag{3.57}$$

Part (i) of Corollary 3.22 coincides with the general change of variables formula discovered by Olver and presented as Exercise 5.59 of [55]. To see this explicitly we define, for functions $f$ and $g$ on $J^\infty(U)$, the local differential operator

$$\mathcal{D}_f\colon J^\infty(U) \to \mathcal{F}^1(J^\infty(U))$$

by

$$\mathcal{D}_f(g) = F_\beta(g d_V f)\, \theta^\beta = \sum_{|I|=0}^{k} (-D)_I[g \partial^I_\beta(f)]\, \theta^\beta.$$

Now, with $n = 2$ for simplicity, and $\Phi[x,u] = (y^i, v^\mu_I)$ given locally by

$$y^i = P^i[x,u] \qquad \text{and} \qquad v^\mu_I = Q^\mu_I[x,u],$$

we deduce that

$$\begin{aligned}
\Phi^\sharp\big(\bar\theta^\mu \wedge dy^1 \wedge dy^2\big) &= \pi^{n,1}[d_V Q^\mu \wedge d_H P^1 \wedge d_H P^2 \\
&\quad + d_H Q^\mu \wedge d_V P^1 \wedge d_H P^2 + d_H Q^\mu \wedge d_H P^1 \wedge d_V P^2 \\
&\quad + \{\text{ forms of lower horizontal degree }\}] \\
&= \det \begin{bmatrix} d_V P^1 & \dfrac{dP^1}{dx^1} & \dfrac{dP^1}{dx^2} \\ d_V P^2 & \dfrac{dP^2}{dx^1} & \dfrac{dP^2}{dx^2} \\ d_V Q^\alpha & \dfrac{dQ^\mu}{dx^1} & \dfrac{dQ^\mu}{dx^2} \end{bmatrix} \wedge dx^1 \wedge dx^2.
\end{aligned}$$

Consequently if $\bar\lambda = \bar L[y, v]\,\bar\nu$ is a Lagrangian on $J^\infty(V)$, then

$$\lambda = \Phi^\sharp(\bar\lambda) = (\bar L \circ \Phi)\det(D_j P^i)\,\nu$$

and $E(\lambda) = \Phi^\natural(E(\bar\lambda)$, where

$$\Phi^\natural\bigl(\bar E_\alpha(\bar L)\bar\theta^\alpha \wedge dy^1 \wedge dy^2\bigr) = I\left(\det\begin{bmatrix} d_V P^1 & \dfrac{dP^1}{dx^1} & \dfrac{dP^1}{dx^2} \\ d_V P^2 & \dfrac{dP^2}{dx^1} & \dfrac{dP^2}{dx^2} \\ d_V Q^\alpha & \dfrac{dQ^\alpha}{dx^1} & \dfrac{dQ^\alpha}{dx^2} \end{bmatrix}\bigl(\overline{E}_\alpha \circ \Phi\bigr) \wedge dx^1 \wedge dx^2\right)$$

$$= \left(\det\begin{bmatrix} \mathcal{D}_{P^1} & \dfrac{dP^1}{dx^1} & \dfrac{dP^1}{dx^2} \\ \mathcal{D}_{P^2} & \dfrac{dP^2}{dx^1} & \dfrac{dP^2}{dx^2} \\ \mathcal{D}_{Q^\alpha} & \dfrac{dQ^\alpha}{dx^1} & \dfrac{dQ^\alpha}{dx^2} \end{bmatrix}\bigl(\overline{E}_\alpha \circ \Phi\bigr)\right) \wedge dx^1 \wedge dx^2.$$

This shows that (3.55) coincides with Olver's result.

Example 3.23. Two specific examples illustrate some of the important features of Corollary 3.22. First let $n = m = 1$ and consider the Lagrangian $\bar\lambda = \frac{1}{2}\dot v^2 dy$ and the contact transformation $\Phi$ given by

$$y = x, \qquad v = \dot u, \qquad \dot v = \ddot u, \ldots \quad .$$

The transform of this Lagrangian is $\Phi^\sharp(\bar\lambda) = \frac{1}{2}\ddot u^2 dx$ while the Euler-Lagrange form $E(\bar\lambda) = -\ddot v\,\bar\theta \wedge dy$ transforms under $\Phi^\natural$ to

$$\Phi^\natural\bigl(E(\lambda)\bigr) = I(-\dddot u\,\dot\theta \wedge dx) = u^{(\mathrm{iv})}\theta \wedge dx.$$

This trivial example shows that the order of the source form $E(\lambda)$ will generally increase when pulled back by the map $\Phi^\natural$. It also highlights the importance of treating Euler-Lagrange equations as source forms — under the contact transformation $\Phi$ the equation $\ddot v = 0$, which is described by a variational principle, is mapped to the equation $\dddot u = 0$, which is not variational.

A more interesting illustration of Corollary 3.22 is provided by the contact transformation $\Phi$ discussed in Lychagin [*47*]. Here $n = 2$ and $m = 1$ and, to second order, this transformation $\Phi$ is given by

$$(\bar x, \bar y, v, v_{\bar x}, v_{\bar y}, v_{\bar x\bar x}, v_{\bar x\bar y}, v_{\bar y,\bar y}) = \Phi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}),$$

where

$$\begin{aligned}
\bar{x} &= u_x & v_{\bar{x}} &= -x & v_{\bar{x}\bar{x}} &= -\frac{1}{u_{xx}} \\
\bar{y} &= y & v_{\bar{y}} &= u_y & v_{\bar{x}\bar{y}} &= \frac{u_{xy}}{u_{xx}} \\
v &= u - xu_x & & & v_{\bar{y}\bar{y}} &= u_{yy} - \frac{u_{xy}^2}{u_{xx}}.
\end{aligned}$$

Since $\Phi^*\bar{\theta} = \theta$ and $\Phi^*(d\bar{x}) = \theta_x + u_{xx}dx + u_{yy}dy$, it is easily checked that this transformation pulls the source form representing Laplace's equation, *viz.*,

$$\overline{\Delta} = (v_{\bar{x}\bar{x}} + v_{\bar{y}\bar{y}})\,\bar{\theta} \wedge d\bar{x} \wedge d\bar{y}$$

back to the source form for an elliptic Monge-Ampere equation, *viz.*,

$$\Delta = \Phi^\natural\overline{\Delta} = (u_{xx}u_{yy} - u_{xy}^2 - 1)\,\theta \wedge dx \wedge dy.$$

The Lagrangian $\bar{\lambda} = -\frac{1}{2}(v_{\bar{x}}^2 + v_{\bar{y}}^2)d\bar{x} \wedge d\bar{y}$ for $\overline{\Delta}$ is pulled back to the Lagrangian

$$\begin{aligned}
\lambda = \Phi^\sharp(\bar{\lambda}) &= \pi^{2,0}[\frac{1}{2}(x^2 + u_y^2)\,du_x \wedge dy] \\
&= -\frac{1}{2}(x^2 + u_y^2)u_{xx}\,dx \wedge dy.
\end{aligned}$$

Obviously, this is not the usual Lagrangian for the Monge-Ampere equation but a direct calculation confirms that $E(\lambda) = \Delta$, in accordance with Corollary 3.22. ∎

**C. A Lie Derivative Formula for Functional Forms.** We begin by computing the Lie derivative of a functional form with respect to an evolutionary vector field.

Lemma 3.24. *Let $\omega \in \mathcal{F}^s(J^\infty(E))$ be a functional form. If $Y$ is an evolutionary vector field on $E$, then*

$$\mathcal{L}^\natural_{\operatorname{pr} Y}\,\omega = \delta_V(\operatorname{pr} Y \lrcorner\, \omega) + I(\operatorname{pr} Y \lrcorner\, \delta_V\omega). \tag{3.58}$$

Proof: Since $Y$ is an evolutionary vector field, $\mathcal{L}_{\operatorname{pr} Y}$ preserves the bigrading of forms on $J^\infty(E)$ (see Proposition 1.16) and hence

$$\mathcal{L}^\sharp_{\operatorname{pr} Y}\omega = \mathcal{L}_{\operatorname{pr} Y}\omega = d_V(\operatorname{pr} Y \lrcorner\, \omega) + \operatorname{pr} Y \lrcorner\, d_V\omega. \tag{3.59}$$

Because

$$\begin{aligned}
\operatorname{pr} Y \lrcorner\, d_V\omega &= \operatorname{pr} Y \lrcorner\, \big(\delta_V\omega + d_H\eta\big) \\
&= \operatorname{pr} Y \lrcorner\, \delta_V\omega - d_H(\operatorname{pr} Y \lrcorner\, \eta),
\end{aligned}$$

the application of the interior Euler operator $I$ to (3.59) yields (3.58). ∎

THEOREM 3.25. *Let $\omega \in \mathcal{F}^s(J^\infty(E))$. If $X$ is a generalized vector field on $E$, then*

$$\mathcal{L}^\natural_{\operatorname{pr} X}\,\omega = \delta_V(\operatorname{pr} X_{\text{ev}} \lrcorner\, \omega) + I(\operatorname{pr} X_{\text{ev}} \lrcorner\, \delta_V\omega). \tag{3.60}$$

PROOF: By virtue of the lemma, it suffices to show that

$$\mathcal{L}^\natural_{\operatorname{pr} X}\omega = \mathcal{L}^\natural_{\operatorname{pr} X_{\text{ev}}}\omega.$$

This follows immediately from the decomposition $\operatorname{pr} X = \operatorname{pr} X_{\text{ev}} + \operatorname{tot} X$ and the following formula for Lie differentiation with respect to total vector fields

$$\begin{aligned}\mathcal{L}^\sharp_{\operatorname{tot} X}\,\omega &= \pi^{n,s}[\operatorname{tot} X \lrcorner\, d\,\omega + d\,(\operatorname{tot} X \lrcorner\, \omega)]\\ &= \pi^{n,s}[\operatorname{tot} X \lrcorner\, d_V\omega + d_H(\operatorname{tot} X \lrcorner\, \omega) + d_V(\operatorname{tot} X \lrcorner\, \omega)]\\ &= d_H(\operatorname{tot} X \lrcorner\, \omega).\end{aligned} \tag{3.61}$$

Here we used the fact (see Proposition 1.18) that interior evaluation by $\operatorname{tot} X$ lowers horizontal degree by 1. ∎

COROLLARY 3.26. *Let $\omega \in \mathcal{F}^s(J^\infty(E))$. If $X$ is a projectable vector field on $E$, then*

$$\mathcal{L}_{\operatorname{pr} X}\omega = \delta_V(\operatorname{pr} X_{\text{ev}} \lrcorner\, \omega) + I(\operatorname{pr} X_{\text{ev}} \lrcorner\, \delta_V\omega). \tag{3.62}$$

PROOF: If $X$ is a projectable vector field on $E$, then $\mathcal{L}^\sharp_{\operatorname{pr} X}\omega = \mathcal{L}_{\operatorname{pr} X}\omega$ and, owning to Theorem 2.12, $I$ commutes with $\mathcal{L}_{\operatorname{pr} X}$. Together, these two facts imply that $\mathcal{L}^\natural_{\operatorname{pr} X}\omega = \mathcal{L}_{\operatorname{pr} X}\omega$ so that (3.62) is a consequence of (3.60). ∎

In order to interpret Theorem 3.25 as Noether's theorem, we need several definitions.

DEFINITION 3.27. *Let $\Delta$ be a source form on $J^\infty(E)$. A generalized vector field $X$ on $J^\infty(E)$ is a distinguished, generalized symmetry of $\Delta$ if*

$$\mathcal{L}^\natural_{\operatorname{pr} X}\Delta = 0. \tag{3.63}$$

Distinguished generalized symmetries differ from ordinary generalized symmetries for a source form $\Delta$ in that the Lie derivative of $\Delta$ with respect to the former must vanish identically whereas the Lie derivative of $\Delta$ with respect to the latter need only vanish on solutions of the source equations $\Delta = 0$. The set of all distinguished, generalized symmetries of a given source form define a subalgebra of the Lie algebra of generalized symmetries for the corresponding source equation.

Note that if $X$ is an vector field on $E$, then it is a distinguished symmetry for $\Delta$ if

$$I\big(\mathcal{L}_{\operatorname{pr} X}\Delta\big) = 0.$$

If $X$ is a projectable vector field on $E$, then it is a distinguished symmetry of $\Delta$ if

$$\mathcal{L}_{\operatorname{pr} X}\Delta = 0.$$

Also observe that, by virtue of (3.61), equation (3.63) is equivalent to the condition

$$\mathcal{L}^{\natural}_{\operatorname{pr} X_{\mathrm{ev}}}\Delta = 0,$$

which in turn is equivalent, at least locally, to the existence of a type $(n-1, s)$ form $\eta$ such that

$$\mathcal{L}^{\sharp}_{\operatorname{pr} X_{\mathrm{ev}}}\Delta = d_H\eta. \tag{3.64}$$

If the source form $\Delta$ is an Euler-Lagrange form, say $\Delta = E(\lambda)$, then $X$ is called a generalized *Bessel-Hagen* symmetry or a *divergence* symmetry for $\lambda$ if

$$\mathcal{L}^{\sharp}_{\operatorname{pr} X}\lambda = d_H\eta. \tag{3.65}$$

By virtue of Corollary 3.22 and the fact that $E$ annihilates $d_H$ exact forms this implies

$$\mathcal{L}^{\natural}_{\operatorname{pr} X}\Delta = E\big(\mathcal{L}^{\sharp}_{\operatorname{pr} X}\lambda\big) = 0.$$

Thus the algebra of generalized Bessel-Hagen symmetries for a given Lagrangian is a subalgebra of the algebra of distinguished, generalized symmetries for the Euler-Lagrange source form $\Delta = E(\lambda)$. In view of (3.64), we have equality of these two algebras if we require only that (3.65) hold locally but not if we require (3.65) to hold globally.

Example 3.28. *Distinguished, first order generalized symmetries for geodesic equations.*

Distinguished, generalized symmetries for many equations have been computed in the literature. It is not our intent to survey these results here or to discuss any of the techniques available for computing these symmetries. Nevertheless we will consider the problem of finding distinguished, generalized symmetries for the geodesic equations for a Riemannian metric $ds^2 = g_{ij}du^i \otimes du^j$ on a manifold $F$. This example will, once again, demonstrate the use moving coframes in computing Euler-Lagrange forms and will also serve as as important illustration of Noether's theorem.

The bundle for this example is $\mathbf{R} \times F \to \mathbf{R}$ with local coordinates $(x, u^i) \to (x)$. The Lagrangian form for these equations is[1]

$$\lambda = \frac{1}{2} g_{ij} \dot{u}^i \dot{u}^j \, dx$$

and the Euler-Lagrange form is

$$E(\lambda) = g_{ij} \Delta^j \theta^i \wedge dx,$$

where

$$\Delta^j = \frac{D\dot{u}^j}{Dx} = \ddot{u}^j + \Gamma^j_{hk} \dot{u}^h \dot{u}^k,$$

and $\Gamma^i_{hk}$ are the Christoffel symbols for the metric $g_{ij}$. We derive necessary and sufficient conditions in order that the evolutionary vector field

$$Y = Y^j(u^i, \dot{u}^i) \frac{\partial}{\partial u^j}$$

be a distinguished symmetry. We have assumed, for the sake of simplicity, that $Y$ does not depend upon the independent variable $x$. In order to state the result, let

$$Y_i = g_{ij} Y^j \qquad \text{and} \qquad Y_{ij} = \frac{\partial Y_i}{\partial \dot{u}^j},$$

and let

$$\nabla_j Y_i = \frac{\partial Y_i}{\partial u^j} - \Gamma^h_{ij} Y_h - \Gamma^h_{jk} \dot{u}^k Y_{hi}.$$

Note that

$$\frac{DY_i}{Dx} = \dot{u}^j (\nabla_j Y_i) + Y_{ij} \Delta^j.$$

PROPOSITION 3.29. *The evolutionary vector field $Y$ is a distinguished symmetry for the source form $E(\lambda)$ if and only if*

$$Y_{ij} = Y_{ji}, \tag{3.66a}$$

$$\nabla_j Y_i + \nabla_i Y_j + \dot{u}^h \nabla_h Y_{ij} = 0, \tag{3.66b}$$

*and*

---

[1]For this Lagrangian to yield the geodesic equations, $x$ must be identified with the arclength parameter. A more accurate interpretation of this Lagrangian is that for a free particle on $F$ with kinetic energy $\frac{1}{2} g_{ij} \dot{u}^i \dot{u}^j$.

$$\dot{u}^h \dot{u}^k [\nabla_h \nabla_k Y_i + Y_j R_h{}^j{}_{ki}] = 0. \tag{3.66c}$$

*In particular, if $Y$ is homogeneous in the derivative variables $\dot{u}^j$ of degree $p$, i.e.,*

$$Y_i = A_{ij_1 j_2 \dots j_p}(u)\, \dot{u}^{j_1} \dot{u}^{j_2} \cdots \dot{u}^{j_p},$$

*then $Y$ is a distinguished symmetry for the geodesic equation if and only if $A$ is a symmetric rank $p+1$ Killing tensor, i.e., $A$ is symmetric in all of its indices and*

$$\nabla_{(j_1} A_{j_2 j_3 \dots j_{p+1})} = 0.$$

PROOF: The second part of the proposition follows easily from the first part. Equation (3.66a) implies that $A$ is symmetric, (3.66b) implies that $A$ is a Killing tensor, and (3.66c) holds identically as a consequence of (3.66a) and (3.66b).

To establish the first part of the proposition, we must compute the Euler-Lagrange form for the second order Lagrangian

$$Y \mathbin{\lrcorner} E(\lambda) = Y_i \Delta^i \, dx \tag{3.67}$$

The most efficient way to do this is to introduce a covariant basis for the contact ideal. Let

$$\begin{aligned}
\Theta^i &= \theta^i = dx^i - \dot{u}^i dx, \\
\dot{\Theta}^i &= \frac{D\theta^i}{Dx} = \dot{\theta}^i + \Gamma^i_{jk} \dot{u}^j \theta^k \\
\ddot{\Theta}^i &= \frac{D^2\theta^i}{Dx^2} = \ddot{\theta}^i + 2\Gamma^i_{jk} \dot{u}^j \dot{\theta}^k + \big[\Gamma^i_{jk} \Delta^j - \Gamma^i_{jk} \Gamma^j_{hl} \dot{u}^h \dot{u}^l \\
&\qquad\qquad + \Gamma^i_{jk,l} \dot{u}^l \dot{u}^j + \Gamma^i_{jl} \Gamma^l_{hk} \dot{u}^j \dot{u}^h \big] \theta^k,
\end{aligned}$$

and so on. Let $\pi_i$, $\dot{\pi}_i$, $\ddot{\pi}_i \dots$ be the dual basis for the space of vertical vectors. These vectors are all projectable and, to second order, are given by

$$\begin{aligned}
\ddot{\pi}_i &= \frac{\partial}{\partial \ddot{u}^i}, \\
\dot{\pi}_i &= \frac{\partial}{\partial \dot{u}^i} - 2\Gamma^k_{ij} \dot{u}^j \frac{\partial}{\partial \ddot{u}^k}, \qquad \text{and} \\
\pi_i &= \frac{\partial}{\partial u^i} - \Gamma^k_{ij} \dot{u}^j \frac{\partial}{\partial \dot{u}^k} - \big[\Gamma^k_{ij} \Delta^j - \Gamma^k_{ij} \Gamma^j_{hl} \dot{u}^h \dot{u}^l \\
&\qquad + \Gamma^k_{ij,l} \dot{u}^j \dot{u}^l - \Gamma^k_{jl} \Gamma^l_{ih} \dot{u}^j \dot{u}^h \big] \frac{\partial}{\partial \ddot{u}^k}.
\end{aligned}$$

With respect to this basis, the vertical differential of a second order Lagrangian

$$\lambda = L(u^i, \dot{u}^i, \ddot{u}^i)\, dx$$

is given by

$$d_V \lambda = [\pi_i(L)\, \Theta^i + \dot{\pi}_i(L)\, \dot{\Theta}^i + \ddot{\pi}_i(L)\, \ddot{\Theta}^i] \wedge dx.$$

Therefore the components $E_i$ of the Euler-Lagrange form $E(\lambda) = E_i\, \Theta^i \wedge dx$ are

$$E_i = \pi_i(L) - \frac{D}{Dx}\big[\dot{\pi}_i(L)\big] + \frac{D^2}{Dx^2}\big[\ddot{\pi}(L)\big]. \tag{3.68}$$

For the particular Lagrangian (3.67) we calculate that

$$\ddot{\pi}_i(\Delta^j) = \delta_i^j, \qquad \dot{\pi}_i(\Delta^j) = 0,$$

and

$$\pi_i(\Delta^j) = -\Gamma^j_{ik}\Delta^k + R_h{}^j{}_{ki}\dot{u}^h\dot{u}^k,$$

and also that

$$\ddot{\pi}_i(Y_j) = 0, \qquad \dot{\pi}_i(Y_j) = Y_{ji},$$

and

$$\pi_i(Y_j) = \frac{\partial Y_j}{\partial u^i} - \Gamma^h_{ik}\dot{u}^k Y_{jh}.$$

When these expressions are substituted into (3.68), it is found that

$$\begin{aligned} E_i(Y \lrcorner \Delta) &= (\nabla_j Y_i)\Delta^j + Y_j R_h{}^j{}_{ki}\dot{u}^h\dot{u}^k - \frac{D}{Dx}\big(Y_{ji}\Delta^j\big) + \frac{D^2}{Dx^2}\big(Y_i\big) \\ &= \frac{D}{Dx}\big[(Y_{ij} - Y_{ji})\Delta^j\big] + \big(\nabla_j Y_i + \nabla_i Y_j + \dot{u}^k \nabla_k Y_{ij}\big)\Delta^j \\ &\quad + \big(\nabla_h \nabla_k Y_i + Y_j R_h{}^j{}_{ki}\big)\dot{u}^h\dot{u}^k. \end{aligned}$$

For this to vanish identically, the coefficients of $\dfrac{D\Delta^j}{Dx}$ and $\Delta^j$ must vanish. This yields (3.66a) and (3.66b) respectively. The vanishing of the remaining terms gives (3.66c). ∎

Definition 3.30. *Let $\Delta = P_\alpha[x,u]\,\theta^\alpha \wedge \nu$ be a source form on $J^\infty(E)$. An evolutionary vector field $Y$ on $J^\infty(E)$ is a generator of a local conservation law for $\Delta$ if the type $(n,0)$ form $\operatorname{pr} Y \mathbin{\lrcorner} \Delta = (Y^\alpha P_\alpha)\,\nu$ has vanishing Euler-Lagrange form, i.e.,*

$$E(\operatorname{pr} Y \mathbin{\lrcorner} \Delta) = 0. \tag{3.69}$$

Local exactness of the variational bicomplex will show that if (3.69) holds then there exists, at least locally, a type $(n-1,0)$ form $\rho$ such that

$$d_H\rho = Y \mathbin{\lrcorner} \Delta.$$

On solutions $s\colon M \to E$ of the source equation $\Delta = 0$ the $n-1$ form $\bigl(j^\infty(s)\bigr)^*\rho$ is $d$ closed and therefore $\rho$ is a local conservation law for the source equation $\Delta$. In the special case $n=1$, $\rho$ is a function on $J^\infty(E)$ which is constant on solutions of $\Delta$ and the notion of a conservation law becomes synonymous with that of a first integral.

The following result was established by Takens [*65*] for the case of vector fields on $E$.

Proposition 3.31. *Let $\Delta$ be a source form on $J^\infty(E)$. Then the vector space of generalized vector fields $X$ on $J^\infty(E)$ which are*

(i) *distinguished symmetries of $\Delta$; and*
(ii) *for which the associated evolutionary vector field $X_{\rm ev}$ is a generator of a local conservation law for $\Delta$*

*forms a Lie algebra.*

Proof: Suppose $X$ and $Y$ are distinguished symmetries of $\Delta$ and that $X_{\rm ev}$ and $Y_{\rm ev}$ are generators of local conservation laws for $\Delta$. We must prove that the same is true of $[X,Y]$.

To prove that $[X,Y]$ is a distinguished symmetry, we need only show that

$$\mathcal{L}^\natural_{\operatorname{pr} Y}\mathcal{L}^\natural_{\operatorname{pr} X}\Delta = (I \circ \pi^{n,s})[\mathcal{L}_{\operatorname{pr} Y}\mathcal{L}_{\operatorname{pr} X}\Delta] \tag{3.70}$$

in order to invoke the usual arguments for this kind of result. Equation (3.70) is an easily consequence of the fact that $\mathcal{L}^\natural_{\operatorname{pr} Y}$ commutes with $I$ (Theorem 3.21) and the fact that

$$\pi^{n,s}\bigl[\mathcal{L}_{\operatorname{pr} Y}\bigl(\pi^{n,s}\mathcal{L}_{\operatorname{pr} X}\Delta\bigr)\bigr] = \pi^{n,s}\bigl[\mathcal{L}_{\operatorname{pr} Y}\mathcal{L}_{\operatorname{pr} X}(\Delta)\bigr].$$

This equation can be verified using the argument presented in the proof of Corollary 3.16.

To prove that $[X, Y]_{\rm ev}$ is a generator for a local conservation law for $\Delta$, it suffices to assume only that $X$ is a distinguished symmetry and that $Y_{\rm ev}$ is a generator for a local conservation law. The assumption that $X$ is a distinguished symmetry implies that (3.63) holds. We now apply $\mathcal{L}^{\natural}_{\operatorname{pr} X_{\rm ev}}$ to the identity $E(\operatorname{pr} Y_{\rm ev} \lrcorner\, \Delta) = 0$ to conclude that

$$E\big(\mathcal{L}^{\sharp}_{\operatorname{pr} X_{\rm ev}}(\operatorname{pr} Y_{\rm ev} \lrcorner\, \Delta)\big] = 0. \tag{3.71}$$

We use Proposition 1.21 and (3.63) to compute:

$$\begin{aligned}\mathcal{L}^{\sharp}_{\operatorname{pr} X_{\rm ev}}(\operatorname{pr} Y_{\rm ev} \lrcorner\, \Delta) &= \pi^{n,0}\big[(\mathcal{L}_{\operatorname{pr} X_{\rm ev}} \operatorname{pr} Y_{\rm ev}) \lrcorner\, \Delta + \operatorname{pr} Y_{\rm ev} \lrcorner\, (\mathcal{L}_{\operatorname{pr} X_{\rm ev}} \Delta)\big] \\ &= [\operatorname{pr} X_{\rm ev}, \operatorname{pr} Y_{\rm ev}] \lrcorner\, \Delta + Y_{\rm ev} \lrcorner\, (\mathcal{L}^{\sharp}_{\operatorname{pr} X_{\rm ev}} \Delta)] \\ &= (\operatorname{pr}\,[X, Y]_{\rm ev}) \lrcorner\, \Delta - d_H(\operatorname{pr} Y_{\rm ev} \lrcorner\, \eta).\end{aligned} \tag{3.72}$$

Since

$$\operatorname{pr}\,[X, Y]_{\rm ev} \lrcorner\, \Delta = [X, Y]_{\rm ev} \lrcorner\, \Delta$$

the combination of (3.71) and (3.72) proves that [X,Y] is a generator for a local conservation law for $\Delta$. ∎

With these definitions and results in hand we can state our first version of Noether's theorem as follows. We emphasize that in this version the conservation laws are derived from the symmetries of the source form. No reference is made to the symmetries of the underlying Lagrangian.

THEOREM 3.32. *Let $\Delta$ be a locally variational source form. Then a generalized vector field $X$ is a distinguished symmetry for $\Delta$ if and only if $X_{\rm ev}$ is a generator for a local conservation law for $\Delta$.*

PROOF: If $\delta_V \Delta = 0$, the Lie derivative formula (3.60) reduces to

$$\mathcal{L}^{\natural}_{\operatorname{pr} X} \Delta = E\big(\operatorname{pr} X_{\rm ev} \lrcorner\, \Delta\big). \tag{3.73}$$

∎

Throughout this notes we shall give a number of examples of Noether's theorem which are intended to illustrate aspects of this theorem which are not adequately discussed in the literature. Our first example is meant to debunk the usefulness of Noether's theorem as a technique for finding *all* the conservation laws for a given system of differential equations by first finding all the generalized symmetries. The point to be made here is that calculation of generalized symmetries is as difficult a problem as that of directly determining the conservation laws — indeed, for locally variational equations it is apparent from (3.73) that the two problems are identical.

EXAMPLE 3.33. *First integrals for Liouville metrics in* $\mathbf{R}^3$

Consider, as a case in point, the problem of finding the generalized symmetries and/or first integrals for the geodesic equations — specifically, let us consider the geodesic equations for the Liouville metric on $\mathbf{R}^3$. The Lagrangian is

$$\lambda = \frac{1}{2}[a(u) + b(v) + c(w)][\dot{u}^2 + \dot{v}^2 + \dot{w}^2]\, dx$$

where $a$, $b$ and $c$ are positive functions of a single variable. The Euler-Lagrange form for $\lambda$ is

$$E(\lambda) = (E_u\theta^u + E_v\theta^v + E_w\theta^w) \wedge dx,$$

where

$$E_u = -\frac{d}{dx}\big[(a+b+c)\dot{u}\big]^2 + \frac{1}{2}a'(\dot{u}^2 + \dot{v}^2 + \dot{w}^2),$$

$$E_v = -\frac{d}{dx}\big[(a+b+c)\dot{v}\big]^2 + \frac{1}{2}b'(\dot{u}^2 + \dot{v}^2 + \dot{w}^2),$$

and

$$E_w = -\frac{d}{dx}\big[(a+b+c)\dot{w}\big]^2 + \frac{1}{2}c'(\dot{u}^2 + \dot{v}^2 + \dot{w}^2).$$

In addition to the Hamiltonian function, these source equations also admit the two quadratic first integrals

$$I_1 = \frac{1}{2}[(a+b+c)\dot{u}]^2 - La$$

and

$$I_2 = \frac{1}{2}[(a+b+c)\dot{v}]^2 - Lb.$$

The generators of these two first integrals are the evolutionary vector fields

$$Y_1 = -\dot{u}(b+c)\frac{\partial}{\partial u} + a\dot{v}\frac{\partial}{\partial v} + a\dot{w}\frac{\partial}{\partial w}$$

and

$$Y_2 = b\dot{u}\frac{\partial}{\partial u} - \dot{v}(a+c)\frac{\partial}{\partial v} + b\dot{w}\frac{\partial}{\partial w}.$$

Let $X$ be an arbitrary vector field on $E = \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$, say

$$X = \alpha\frac{\partial}{\partial x} + A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v} + C\frac{\partial}{\partial w}.$$

Since the corresponding evolutionary vector field is

$$X_{\text{ev}} = (A - \alpha\dot{u})\frac{\partial}{\partial u} + (B - \alpha\dot{v})\frac{\partial}{\partial v} + (C - \alpha\dot{w})\frac{\partial}{\partial w}$$

it is apparent that neither $Y_1$ or $Y_2$ is the evolutionary vector field derived from a vector field on $E$. ∎

In other words, in order to derive the first integrals $I_1$ and $I_2$ from Noether's theorem it is necessary to compute the distinguished, generalized symmetries of the source form $E(\lambda)$. We proved directly that the generalized vector field

$$Y = [g^{ij_1} A_{j_1 j_2 \dots j_p} \dot{u}^{j_2} \dot{u}^{j_3} \cdots \dot{u}^{j_p}] \frac{\partial}{\partial u^i}$$

is a distinguished symmetry of the geodesic equation if and only if $A_{j_1 j_2 \dots j_p}$ are the components of a symmetric Killing tensor for the given metric. But this is not a very useful conclusion since it is already well-known, and easily verified, that a homogenous function

$$I = B_{j_1 j_2 \dots j_p} \dot{u}^{j_1} \dot{u}^{j_2} \cdots \dot{u}^{j_p}$$

is a first integral for the geodesic equation if and only if $B_{j_1 j_2 \dots j_p}$ are the components of a symmetric Killing tensor. Thus, either way, one is confronted with the same, difficult problem of finding higher rank Killing tensors. ∎

In the case of source equations for partial differential equations this situation remains the same — for locally variational source forms the determining equations for distinguished symmetries are same as those for generators of local conservation laws. However it often happens, although not always, these determining equations are much more tractable and can often be completely solved. In Chapter Seven we shall compute all the conservation laws for several well-known equations.

In Noether's original paper on variational problems with symmetries, two distinct cases were considered according to whether the Lie algebra of distinguished symmetries for the given source form is finite dimensional or infinite dimensional (in the sense that the generators of the Lie algebra depend upon arbitrary smooth functions). In this latter case, Noether's theorem states that the coefficients of the source form are related by certain differential identities. These identities can be easily derive from our basic Lie derivative formula (3.60). The following well-known example illustrates this point.

Example 3.34. *Noether's Theorem for natural variational principles on Riemannian structures*

Let $E = \mathrm{Sym}^2_+(T^*M)$ be the bundle of symmetric, positive definite rank $(0,2)$ tensors on a manifold $M$. A section of $E$ is a choice of Riemannian metric on $M$. Any local diffeomorphism $\phi_0$ of $M$ lifts to a local diffeomorphism $\phi$ of $E$ which in turn lifts to a local diffeomorphism $\Phi = \mathrm{pr}\,\phi$ of $J^\infty(E)$. A source form

$$\Delta[g] = P^{ij}(x^i, g_{ij},\, g_{ij,h}\,, g_{ij,hk} \dots)\, dg_{ij} \wedge \nu$$

is called a *natural* Riemannian source form if for all local diffeomorphism $\phi_0$,

$$\Delta[g] = \Phi^*\big(\Delta[\Phi(g)]\big).$$

Roughly speaking, a natural Riemannian source form is one whose components $P^{ij}$ are constructed from the metric tensor, the curvature tensor, and covariant derivatives of the curvature tensor by contraction of indices. We shall treat the subject of natural Riemannian tensors in greater detail in Chapter Six.

A natural Riemannian source form is said to be conformally invariant if for all functions $h$ on $M$,

$$\Delta[g] = \Delta[e^h g].$$

Proposition 3.35. *If $\Delta[g] = P^{ij}[g]\, dg_{ij} \wedge \nu$ is a locally variational, natural source form, then it is divergence-free, i.e.,*

$$\nabla_j P^{ij} = 0, \tag{3.74}$$

*where $\nabla_j$ denotes total covariant differentiation with respect to the symmetric metric connection of $g$.*

*Moreover, if $\Delta$ is conformally invariant, then $\Delta$ is trace-free, i.e.,*

$$g_{ij} P^{ij} = 0. \tag{3.75}$$

Proof: Let $X_0 = X^i \dfrac{\partial}{\partial x^i}$ be any vector field on $M$. Then $X_0$ lifts to the vector field

$$X = X^i \frac{\partial}{\partial x^i} - 2X^l{}_{,i}\, g_{jl} \frac{\partial}{\partial g_{ij}} \tag{3.76}$$

on $E$. The corresponding evolutionary vector field is

$$\begin{aligned} X_{\text{ev}} &= -\big(2X^l{}_{,i} g_{lj} + g_{il,j} X^l\big) \frac{\partial}{\partial g_{ij}} = -2\big(\mathcal{L}_{X_0} g_{ij}\big) \frac{\partial}{\partial g_{ij}} \\ &= -2\big(\nabla_j X_i\big) \frac{\partial}{\partial g_{ij}}, \end{aligned}$$

where $X_i = g_{il} X^l$.

The naturality of $\Delta$ implies that $\mathcal{L}_{\operatorname{pr} X} \Delta = 0$ for all vector fields $X$ of the type (3.76). The Lie derivative formula (3.62) therefore reduces to $E(\lambda) = 0$, where $\lambda = \nabla_j \big(X_i P^{ij}\big)\, \nu$. Since the components $P^{ij}$ are those of a tensor density, we can rewrite this Lagrangian as

$$\nabla_j \big(X_i P^{ij}\big) = D_j \big(X_i P^{ij}\big) - X_i \nabla_j P^{ij}$$

in order to conclude that

$$E\big(X_i(\nabla_j P^{ij})\big) = 0.$$

Because the vector field $X_0$ is arbitrary, we can appeal to Corollary 2.9 to deduce that the $\nabla_j P^{ij}$ are constant. There are no non-zero, rank 1 constant natural tensors and hence (3.74) holds.

The proof of (3.75) is similar since the condition of conformal invariance requires that $\mathcal{L}_{\operatorname{pr} Y}\Delta = 0$, where $Y = h(x)g_{ij}\dfrac{\partial}{\partial g_{ij}}$ and $h$ is a arbitrary function on $M$. ∎

Proposition 3.35 is easily extended to include natural tensors which depend upon more than just one tensor field. For example, if $E = \mathrm{Sym}^2_+(T^*M) \times \mathrm{Sym}^p(T^*M)$ with local coordinates $(x^i, g_{ij}, \psi_I)$, $|I| = p$, then a natural source form $\Delta \in \mathcal{F}^1$ is a type $(n,1)$ form of the type

$$\Delta = \big[P^{ij}[g,\psi]\, dg_{ij} + B^I[g,\psi]\, d\psi_I\big] \wedge \nu.$$

In particular, if $\lambda = L[g,\psi]\nu$ is a Lagrangian $n$ form on $J^\infty(E)$ and $\Delta = E(\lambda)$, then the $P^{ij}$ and the $B^I$ are the components of the Euler-Lagrange expressions of $L$ with respect to the variations of $g_{ij}$ and $\psi_I$ respectively. An arbitrary vector field $X_0$ on $M$ again lifts to a vector field X on $E$. The corresponding evolutionary vector field is now given by

$$X_{\mathrm{ev}} = -2\big(\nabla_j X^l g_{il}\big)\frac{\partial}{\partial g_{ij}} - \big(X^i\nabla_i\psi_J + p\psi_{iJ'}\nabla_j X^i\big)\frac{\partial}{\partial \psi_J}.$$

The same arguments used in the proof of Proposition 3.35 now show that if $\Delta$ is a locally variational, natural source form, then

$$\nabla_j(g_{ik}P^{jk}) + p\nabla_j(\psi_{J'i}B^J) - (\nabla_i\psi_J)B^J = 0. \tag{3.77}$$

Two special cases of this identity are noteworthy. First, if $p = 1$, then $\psi = \psi_j dx^j$ is a one form and (3.77) can be re-expressed as

$$\nabla_j\big(g_{ik}P^{jk}\big) + F_{ji}B^j + \psi_i\nabla_j B^j = 0,$$

where $F_{ji} = \psi_{i,j} - \psi_{j,i}$ are the components of $d_H\psi$. If $p = 2$, then (3.77) becomes the identity (2.76) which was needed to simplify the Euler-Lagrange form for geometric variational problems for surfaces in $\mathbf{R}^3$.

Chapter Four

# LOCAL PROPERTIES OF THE VARIATIONAL BICOMPLEX

This chapter is devoted to a detailed analysis of the variational bicomplex for the trivial bundle

$$E : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n.$$

In section A we construct homotopy operators for the augmented variational bicomplex

$$\begin{tikzcd}
 & & |[alias=a]| & & & & |[alias=b]| & |[alias=c]| & \\
 & 0 \arrow[r] & \Omega^{0,3} \arrow[u, "d_V"] & & \cdots & & \Omega^{n,3} \arrow[r, "I"] \arrow[u, "d_V"] & \mathcal{F}^3 \arrow[r] \arrow[u, "\delta_V"] & 0 \\
 & 0 \arrow[r] & \Omega^{0,2} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,2} \arrow[r, "d_H"] & \cdots & \Omega^{n-1,2} \arrow[r, "d_H"] & \Omega^{n,2} \arrow[r, "I"] \arrow[u, "d_V"] & \mathcal{F}^2 \arrow[r] \arrow[u, "\delta_V"] & 0 \\
 & 0 \arrow[r] & \Omega^{0,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots & \Omega^{n-1,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{n,1} \arrow[r, "I"] \arrow[u, "d_V"] & \mathcal{F}^1 \arrow[r] \arrow[u, "\delta_V"] & 0 \\
0 \arrow[r] & \mathbf{R} \arrow[r] & \Omega^{0,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots & \Omega^{n-1,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{n,0} \arrow[u, "d_V"] & & \\
0 \arrow[r] & \mathbf{R} \arrow[r] & \Omega^0_M \arrow[r, "d"] \arrow[u, "(\pi^\infty_M)^*"] & \Omega^1_M \arrow[r, "d"] \arrow[u, "(\pi^\infty_M)^*"] & \cdots & \Omega^{n-1}_M \arrow[r, "d"] \arrow[u, "(\pi^\infty_M)^*"] & \Omega^n_M \arrow[u, "(\pi^\infty_M)^*"] & &
\end{tikzcd} \tag{4.1}$$

This establishes the exactness of (4.1).

The homotopy operator $h_V$ constructed for the vertical complexes $(\Omega^{r,*}, d_V)$ is very similar to the usual homotopy operator used to prove the exactness of the de Rham complex on $\mathbf{R}^n$. The homotopy operator $h_H$ used to prove the exactness of the augmented horizontal complexes $(\Omega^{*,s}, d_H)$ is, for $s \geq 1$, a local *differential*

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operator. Thus, unlike $h_V$, the operator $h_H$ can actually be defined on the space of germs of forms in $\Omega^{r,s}$. The existence of a differential homotopy operator for the interior horizontal rows of the variational bicomplex was first explicitly observed by Tulczyjew [*70*]. Tulczyjew makes extensive use of the Frölicher-Nijenhuis theory of derivations to define $h_H$ recursively. We use the theory of Euler operators developed in Chapter Two to give explicit formulas for $h_H$ and to simplify the proof of the homotopy property. Different proofs of the local exactness of the rows $(\Omega^{*,s}, d_H)$, where $s \geq 1$, have been given by Takens [*66*], Tsujishita [*68*] and Vinogradov [*75*]. These authors use induction on the order of the form $\omega \in \Omega^{r,s}$ to infer local exactness of these complexes from the exactness of either the Kozul complex or the Spencer sequence. Actually, this inductive approach is already evident in Gilkey's paper [*28*] on the classification of the Pontryagin forms.

Homotopy operators are also constructed for the Euler-Lagrange complex $\mathcal{E}^*$ on $J^\infty(E)$:

$$
\begin{aligned}
0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \\
\xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \cdots \quad .
\end{aligned}
\tag{4.2}
$$

In particular, the homotopy operator

$$\mathcal{H}^1 \colon \mathcal{F}^1 \to \Omega^{n,0}$$

coincides with the Volterra-Vainberg ([*76*], [*72*]) formula for constructing a local Lagrangian $\lambda$ for a variationally closed source form $\Delta = P_\alpha[x,u]\,\theta^\alpha \wedge \nu$, *viz.*,

$$\mathcal{H}^1(\Delta) = \Big[\int_0^1 u^\alpha P_\alpha[x,tu]\,dt\Big]\,\nu \tag{4.3}$$

As our first application of the local exactness of the variational bicomplex, we shall reprove a theorem of Cheung [*17*] concerning variationally trivial natural Lagrangians for either plane or space curves. We also show how the horizontal homotopy operators can be used to reproduce Chern's celebrated proof of the generalized Gauss-Bonnet theorem.

The horizontal homotopy operators $h_H$ will be used extensively in subsequent chapters. In this chapter we shall exploit the fact that they preserve functional dependencies on parameters and that they respect the invariance of forms under a certain class of affine transformations on $E$. In Chapter Five, we shall see that close relationship exists between these homotopy operators and the local theory of Poincaré-Cartan forms. In special situations we can use these homotopy operators to explicitly determine the cohomology classes on $J^\infty(E)$ which represent the obstructions to the global exactness of the variational bicomplex. In Chapter Five, we also show how one can modify $h_H$, given a symmetric connection on the base manifold $M$, to arrive at invariantly defined, and therefore global, homotopy operators for the interior rows of the variational bicomplex. This construction, although somewhat complicated, really synthesizes many different properties of the variational bicomplex. In Chapter Six, we again exploit important invariance properties of these homotopy operators as a first step in the calculation of some equivariant cohomology of the variational bicomplex. Finally, in Chapter Seven, we use other elementary properties of these homotopy operators to give a proof of Vinogradov's Two Line Theorem.

Unfortunately, the horizontal homotopy operators $h_H$ suffer from one, very annoying, drawback. If, for instance, $\omega \in \Omega^{r,0}$ is a $d_H$ closed form of order $k$, then we have that $\omega = d_H\eta$, where $\eta = h_H^{r,0}(\omega)$. However, the order of $\eta$ is in general much higher than that of $\omega$ and therefore $\eta$ is not, in this sense, the simplest possible form whose $d_H$ differential equals $\omega$. The same problem occurs with the homotopy operators $\mathcal{H}^s$ for the complex of functional forms. In particular, if $\Delta \in \mathcal{F}^1$ is a locally variational source form, then we can write $\Delta = E(\lambda)$, where $\lambda = \mathcal{H}^1(\Delta)$ is given by (4.3). It is apparent that if $\Delta$ is of order $k$, then so is $\lambda$. For some source forms, such as the one defining the Monge-Ampere equation

$$\Delta = \det(u_{ij})\, \theta \wedge \nu,$$

(4.3) gives a Lagrangian of least possible order. For other source forms, a Lagrangian of order as low as $[k/2]$ may exist. In section B of this chapter, the problem of finding minimal order forms is studied. In fact, by introducing a system of weights for forms with some polynomial dependencies in the derivative variables, we are able to obtain fairly detailed information concerning the structure of these minimal order forms. This, in turn, leads to what is in essence a method of undetermined coefficients for solving either the equation $\omega = d_H\eta$ for $\eta$ or for solving the equation $\Delta = E(\lambda)$ for $\lambda$. This method is an effective and often superior alternative to the direct application of the homotopy formulas. As another application of our system of weights, we completely describe those source forms of order $2k$ which are derivable from a Lagrangian of order $k$. In other words, we characterize the image in $\mathcal{F}^1$ of the space of $k$-th order Lagrangians under the Euler-Lagrange operator. This solves a sharper version of the inverse problem to the calculus of variations. The case $k = 1$ was treated, by different methods, in Anderson and Duchamp [*4*].

We have already observed, on numerous occasions, that if $\omega$ is a type $(r, s)$ form of order $k$, then $d_H\omega$ is in general of form of order $k+1$. This simple fact prevents us from immediately restricting the variational bicomplex to forms defined on any fixed, finite jet bundle. In section C we introduce subspaces

$$\mathcal{J}_k^{r,s} \subset \Omega^{r,s}(J^\infty(E)) \cap \Omega^{r+s}(J^{k+1}(E))$$

with the property that

$$d_H \mathcal{J}_k^{r,s} \in \mathcal{J}_k^{r+1,s} \qquad \text{and} \qquad d_V \mathcal{J}_k^{r,s} \in \mathcal{J}_k^{r,s+1}.$$

Elements of $\mathcal{J}_k^{r,s}(J^\infty(E))$ are characterized by the property that their highest derivative dependency occurs via Jacobian determinants and so, for this reason, we call the bicomplex $(\mathcal{J}_k^{*,*}(J^\infty(E)), d_H, d_V)$ the *Jacobian sub-bicomplex* of the variational bicomplex on $J^\infty(E)$. By using techniques from classical invariant theory and the minimal weight results of section B, we are able to establish the local exactness of the Jacobian sub-bicomplex. As a corollary to the local exactness of the Jacobian sub-bicomplex, we re-establish the structure theorem for variationally trivial $k^{\text{th}}$ order Lagrangians (Anderson and Duchamp [*4*], Olver [*53*]) — such Lagrangians are necessarily polynomial in the derivatives of order $k$ of degree $\leq n$ and, moreover, this polynomial dependence must occur via Jacobian determinants. The problem of obtaining a similar structure theorem is for variationally closed source forms is addressed and a few partial results are obtained. As an application of the latter, we solve the equivariant inverse problem to the calculus of variations for natural differential equations for plane curves.

**A. Local Exactness and the Homotopy Operators for the Variational Bicomplex.** Let $E$ be the trivial bundle $E : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$. Let $\Omega^{r,s} = \Omega^{r,s}(J^\infty(E))$. In this section we shall proof the local exactness of the variational bicomplex by establishing the following three propositions.

Proposition 4.1. *For each $r = 0, 1, 2, \ldots, n$, the vertical complex*

$$0 \longrightarrow \Omega_M^r \xrightarrow{(\pi_M^\infty)^*} \Omega^{r,0} \xrightarrow{d_V} \Omega^{r,1} \xrightarrow{d_V} \Omega^{r,2} \to \cdots \tag{4.4}$$

*is exact.*

Proposition 4.2. *For each $s \geq 1$, the augmented horizontal complex*

$$0 \longrightarrow \Omega^{0,s} \xrightarrow{d_H} \Omega^{1,s} \xrightarrow{d_H} \cdots \Omega^{n-1,s} \xrightarrow{d_H} \Omega^{n,s} \xrightarrow{I} \mathcal{F}^s \longrightarrow 0 \tag{4.5}$$

*is exact.*

PROPOSITION 4.3. *The Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$*

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \tag{4.6}$$
$$\xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \cdots \quad .$$

*is exact.*

PROOF OF PROPOSITION 4.1: The exactness ( in fact, global exactness) of (4.4) at $s = 0$ has already been established in Proposition 1.9.

For $s \geq 1$, the proof of exactness proceeds along the very same lines as the proof, of the local exactness of the de Rham complex as found in, for example, Warner [*78*] (pp. 155-157, §4.18). Let

$$R = u^\alpha \frac{\partial}{\partial u^\alpha} \tag{4.7}$$

be the vertical radial vector field on $E$. Then the prolongation of $R$ is the radial vector field

$$\operatorname{pr} R = u^\alpha \partial_\alpha + u^\alpha_i \partial^i_\alpha + u^\alpha_{ij} \partial^{ij}_\alpha + \cdots$$

on $J^\infty(E)$ and the corresponding flow on $J^\infty(E)$ is the one parameter family of diffeomorphism

$$\Phi_\epsilon[x, u] = [x, e^\epsilon u] = (x^i, e^\epsilon u^\alpha,\ e^\epsilon u^\alpha_i,\ e^\epsilon u^\alpha_{ij}, \dots).$$

Let $\omega$ be a type $(r, s)$ form on $J^\infty(E)$. Then the Lie derivative formula established in Proposition 1.16 gives

$$\begin{aligned} \frac{d}{d\epsilon}\big[\Phi^*_\epsilon\, \omega\big] &= \Phi^*_\epsilon\big[\mathcal{L}_{\operatorname{pr} R}\,, \omega\big] \\ &= d_V\big[\Phi^*_\epsilon(\operatorname{pr} R \lrcorner\, \omega)\big] + \Phi^*_\epsilon\big[\operatorname{pr} R \lrcorner\, d_V\omega\big]. \end{aligned}$$

In this equation we replace $\epsilon$ by $\log t$ and integrate the result from $t = 0$ to $t = 1$ to arrive at

$$\omega = d_V[h^{r,s}_V(\omega)] + h^{r,s+1}_V(d_V\omega), \tag{4.8}$$

where the vertical homotopy operator

$$h^{r,s}_V \colon \Omega^{r,s} \to \Omega^{r,s-1}$$

is defined by

$$h_V^{r,s}(\omega) = \int_0^1 \frac{1}{t} \Phi^*_{\log t}(\operatorname{pr} R \mathbin{\lrcorner} \omega)\, dt. \tag{4.9}$$

Note that the integrand is a actually smooth function at $t = 0$. Indeed, let $\omega[x, tu]$ denote the form obtained by evaluating the coefficients of $\omega$ at the point $[x, tu]$. For instance, if $f$ is a real-valued function on $J^\infty(E)$ and

$$\omega = f[x, u]\, \gamma,$$

where $\gamma$ is the wedge product of $r$ of the horizontal forms $dx^i$ and $s$ of the vertical forms $\theta^\alpha_I$, then

$$\omega[x, tu] = f[x, tu]\, \gamma$$

even though the contact forms $\theta^\alpha_I$ contain an explicit $u^\alpha_I$ dependence. With this convention, the integrand in (4.9) becomes

$$\Bigl(\frac{1}{t} \Phi^*_{\log t}(\operatorname{pr} R \mathbin{\lrcorner} \omega)\Bigr)[x, u] = t^{s-2}\bigl(\operatorname{pr} R \mathbin{\lrcorner} \omega\bigr)[x, tu] = t^{s-1} \operatorname{pr} R \mathbin{\lrcorner} \omega[x, tu]. \tag{4.10}$$

Because $s \geq 1$, this is certainly a smooth function of $t$. ∎

For future use we remark that because $d_H$ commutes with $\Phi^*_s$ and anti-commutes with $\operatorname{pr} R \mathbin{\lrcorner}$, it anti-commutes with $h_V^{r,s}$, *i.e.*,

$$d_H h_V^{r,s}(\omega) = -h_V^{r+1,s}(d_H \omega). \tag{4.11}$$

To prove Proposition 4.2, we need the following identity. Recall that the inner Euler operators $F^I_\alpha$ were defined in Chapter Two by (2.24) and that $D_j$ is the total vector field $D_j = \operatorname{tot} \dfrac{\partial}{\partial x^j}$ as defined by (1.38).

LEMMA 4.4. *Let $\omega \in \Omega^{r,s}$ and set $\omega_j = D_j \mathbin{\lrcorner} \omega$. Then*

$$(|I| + 1) F^{Ij}_\alpha(D_j \mathbin{\lrcorner} d_H \omega) + |I| F^{j(I'}_\alpha(dx^{i)} \wedge \omega_j) = (n - r + |I|) F^I_\alpha(\omega). \tag{4.12}$$

PROOF: From Proposition 2.10, with $I$ replaced by $Ij$ in (2.25a), we find that

$$(|I| + 1) F^{Ij}_\alpha(d_H \omega) = F^I_\alpha(dx^j \wedge \omega) + |I| F^{j(I'}_\alpha(dx^{i)} \wedge \omega).$$

Now inner evaluate this equation with $D_j$ and sum on $j$. By virtue of the formulas

$$D_j \mathbin{\lrcorner} F^I_\alpha(\omega) = -F^I_\alpha(D_j \mathbin{\lrcorner} \omega) \qquad \text{and} \qquad D_j \mathbin{\lrcorner} (dx^i \wedge \omega) = \delta^i_j \omega - dx^i \wedge \omega_j$$

the resulting equation reduces to (4.12), as required. ∎

Proof of Proposition 4.2: For $s \geq 1$, the horizontal homotopy operator

$$h_H^{r,s} \colon \Omega^{r,s} \to \Omega^{r-1,s}$$

is defined by

$$h_H^{r,s}(\omega) = \frac{1}{s} \sum_{|I|=0}^{k-1} \frac{|I|+1}{n-r+|I|+1} D_I\big[\theta^\alpha \wedge F_\alpha^{Ij}(\omega_j)\big]. \tag{4.13}$$

Let $\omega$ be a $k^{\text{th}}$ order form of type $(r,s)$. To verify that

$$h_H^{r+1,s}(d_H\omega) + d_H[h_H^{r,s}(\omega)] = \omega, \tag{4.14}$$

for $s \geq 1$ and $1 \leq r \leq n$, we multiple (4.12) by $\dfrac{1}{s(n-r+|I|)}\,\theta^\alpha$, apply the differential operator $D_I$ and sum on $|I|$. On account of (2.23), we have that

$$s\omega = \sum_{|I|=0}^{k} D_I[\theta^\alpha \wedge F_\alpha^I(\omega)]$$

so that the result of this calculation reduces to (4.14).

Equation (4.14) also holds for $r = 0$ (with the understanding that $\Omega^{-1,s} = 0$) since $D_j \mathbin{\lrcorner} \omega = 0$ for any $\omega \in \Omega^{0,s}$. With $r = n$, $h_H^{n,s}(\omega)$ coincides with the form $\eta$ in (2.35) as given by (2.37) and consequently we can rewrite (2.35) as

$$I(\omega) + d_H[h_H^{n,s}(\omega)] = \omega. \tag{4.15}$$

Together equations (4.14) and (4.15) prove the exactness of the horizontal augmented horizontal complex (4.5). ∎

In the next lemma we use the Lie-Euler operators $E_\alpha^I$ which were introduced in Chapter 2B.

Lemma 4.5. *Let $\omega \in \Omega^{r,0}$ be a horizontal, type $(r,0)$ form and let $Y$ be an evolutionary vector field on $E$. Then, for $r \leq n$,*

$$\mathcal{L}_{\operatorname{pr} Y}\,\omega = I_Y^{r+1}(d_H\omega) + d_H(I_Y^{r+1}(\omega)), \tag{4.16}$$

*where $I_Y^r \colon \Omega^{r,0} \to \Omega^{r-1,0}$ is defined by*

$$I_Y^r(\omega) = \sum_{|I|=0}^{k} \frac{|I|+1}{n-r+|I|+1} D_I\big[Y^\alpha E_\alpha^{Ij}(\omega_j)\big]. \tag{4.17}$$

For $r = n$,

$$\mathcal{L}_{\operatorname{pr} Y}\, \omega = \operatorname{pr} Y \lrcorner\, E(\lambda) + d_H[I^n_Y(\omega)]. \tag{4.18}$$

PROOF: From the defining properties of the Lie-Euler operators and the interior Euler operators, it is easy verified that for a horizontal form $\omega$

$$E^I_\alpha(\omega) = F^I_\alpha(d_V \omega)$$

and hence

$$I^r_Y(\omega) = -\operatorname{pr} Y \lrcorner\, h^{r,1}_V(\omega).$$

Equation (4.16) now follows directly from Proposition 2.8 and (4.14).

To derive (4.18), it suffices to observe that $I^n_Y(\omega)$ coincides with the form $\operatorname{pr} Y \lrcorner\, \sigma$ in (2.17b) in which case (2.17a) becomes (4.18). ∎

Since $I^r_Y(\omega)$ is linear over $\mathbf{R}$ in both $Y$ and $\omega$, since it drops the horizontal degree by one, and since (4.16) is similar to the usual Cartan formula for Lie derivatives, one can think of the $I^r_Y$ as a generalized, *local* inner product operator. We emphasize that on forms of order $k \geq 1$ it is neither $C^\infty(J^\infty(E))$ linear and nor is it invariantly defined. Still, we find (4.16) to be useful in a various situations.

PROOF OF PROPOSITION 4.3: It is possible to prove Proposition 4.3 from Propositions 4.1 and 4.2 using elementary spectral sequence arguments. However, we shall often use the explicit formulas provided by the homotopy operators for the Euler-Lagrange complex (4.6). To define these operators, it is necessary to break the Euler-Lagrange complex into two pieces and to construct different homotopy operators for each piece. The first piece

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \tag{4.19}$$

consists of the spaces of horizontal forms, except for the last term, and the second piece

$$\Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \xrightarrow{\delta_V} \cdots \tag{4.20}$$

consists the spaces of functional forms, except for the first term.

For $\omega \in \Omega^{r,0}$, let $\omega_s = [j^\infty(s)]^*(\omega)$ denote the pullback of $\omega$ to $\Omega^r(M)$ by a section $s\colon M \to E$. Set $\tilde{\omega} = \omega - \omega_s$, where, by a slight abuse of notation, we have identified $\omega_s$ with its image in $\Omega^{r,0}$ under $(\pi^\infty_M)^*$. Notice that $d_H\omega = 0$ implies that $d_M\omega_s = 0$ and that $d_H\tilde{\omega} = 0$. Therefore, if $\omega$ is $d_H$ closed we can infer from the exactness of the de Rham complex $(\Omega^*_M, d_M)$ for the base space $M = \mathbf{R}^n$ that $\omega_s$ is $d_M$ exact and therefore, as a form back on $J^\infty(E)$, $d_H$ exact. Consequently in order to prove the exactness of (4.19) it suffices to prove that the closed forms $\omega \in \Omega^{r,0}$ are $d_H$ exact, modulo the form $\omega_s$.

For $r = 0$, $I_Y^0(\omega) = 0$ and hence, if $f \in \Omega^{0,0}$ satisfies $d_H f = 0$, then

$$\mathcal{L}_{\mathrm{pr}\, Y} f = \mathrm{pr}\, Y \mathbin{\lrcorner} d_V f = 0$$

for all vertical vector fields $Y$. It thus follows that $d_V f = 0$ and so, by Proposition 4.1, $f = f(x)$ is a function of the base variables alone. By our preliminary remarks, $f$ is a therefore a constant and the exactness of (4.19) at $\Omega^{0,0}$ is established.

To define the homotopy operators for the first piece of the Euler-Lagrange complex, let $J^\infty(E) \times J^\infty(E) \to M$ be the product of two copies of the infinite jet bundle over $E$. Local coordinates on $J^\infty(E) \times J^\infty(E)$ are $[x, v, u] = (x^i, v_I^\alpha, u_I^\alpha)$. Define a bundle map

$$\rho_t \colon J^\infty(E) \times J^\infty(E) \to J^\infty(E)$$

by

$$\rho_t[x, v, u] = [x, tu + (1-t)v].$$

When $t = 0$ or $t = 1$, $\rho_t$ is the projection onto the first or the second factor of $J^\infty(E) \times J^\infty(E)$. Set $Y^\alpha = u^\alpha - v^\alpha$ and evaluate (4.16) and (4.18) at the point $\rho_t$. One readily checks that for all functions $f$ on $J^\infty(E)$

$$\big(D_i(f)\big)[\rho_t] = D_i(f[\rho_t])$$

and that for all horizontal forms

$$\frac{d}{dt}\,\omega[\rho_t] = (\mathcal{L}_Y \omega)[\rho_t].$$

Integration of equations (4.16) and (4.18) with respect to $t$ from 0 to 1 now yields

$$\omega[x, u] - \omega[x, v] = h_H^{r+1,0}(d_H \omega) + d_H[h_H^{r,0}(\omega)] \tag{4.21}$$

for $\omega \in \Omega^{r,0}$, and

$$\lambda[x, u] - \lambda[x, v] = \mathcal{H}^1(E(\lambda)) + d_H h_H^{n,0}(\lambda), \tag{4.22}$$

for $\lambda \in \Omega^{n,0}$. The homotopy operators

$$h_H^{r,0} \colon \Omega^{r,0}(J^\infty(E)) \to \Omega^{r-1,0}(J^\infty(E) \times J^\infty(E))$$

and

$$\mathcal{H}^1 \colon \mathcal{F}^1(J^\infty(E)) \to \Omega^{n,0}(J^\infty(E) \times J^\infty(E))$$

are defined by

$$h_H^{r,0}(\omega) = \int_0^1 \sum_{|I|=0}^{k} \frac{|I|+1}{n-r+|I|+1} D_I\big[(u^\alpha - v^\alpha) E_\alpha^{Ij}(\omega_j)[\rho_t]\big]\, dt \tag{4.23}$$

and

$$\mathcal{H}^1(\Delta) = \big[\int_0^1 (u^\alpha - v^\alpha) P_\alpha[\rho_t]\, dt\big]\, \nu \tag{4.24}$$

for $\Delta = P_\alpha\, \theta^\alpha \wedge \nu \in \mathcal{F}^1$. With $v = j^\infty(s)$ for some section $s \colon M \to E$, $\omega[x, v] = \omega_s$ and local exactness now follows by virtue of our earlier remarks.

We now turn to the complex of functional forms (4.20). Let $\omega$ be a type $(n, s)$ form in $\mathcal{F}^s$. The vertical homotopy formula gives

$$\omega = d_V[h_V^{n,s}(\omega)] + h_V^{n,s+1}(d_V\omega). \tag{4.25}$$

The two $\delta_V$ differentials in this equation can be expressed in terms of $d_V$ differentials by

$$d_V\omega = \delta_V\omega + d_H\eta_1$$

and

$$d_V[h_V^{n,s}(\omega)] = \delta_V[h_V^{n,s}(\omega)] + d_H\eta_2.$$

Apply the interior Euler operator $I$ to (4.25). Because $d_H$ anti-commutes with $h_V^{n,s}$ and because $I \circ d_H = 0$ and $I(\omega) = \omega$, we conclude that

$$\omega = d_V\mathcal{H}^s(\omega) + \mathcal{H}^{s+1}(d_V\omega) \tag{4.26}$$

where $\mathcal{H}^s \colon \mathcal{F}^s \to \mathcal{F}^{s-1}$ is defined by

$$\mathcal{H}^s(\omega) = (I \circ h_V^{n,s})(\omega). \tag{4.27}$$

This proves the exactness of the second piece of (4.6). ∎

Several remarks are in order. Firstly, notice that for a first order form $\omega$, (4.23) reduces to

$$h_H^{r,0}(\omega) = \int_0^1 \frac{1}{n-r+1}(u^\alpha - v^\alpha)(\frac{\partial}{\partial u_j^\alpha}\omega_j)[x, \rho_t]\, dt. \tag{4.28}$$

Secondly, we could have also arrived at (4.21) and (4.22) (in the special case $v = 0$) starting from the homotopy operators used in the Propositions 4.1 and 4.2. If $\omega \in \Omega^{r,0}$, then $d_V\omega \in \Omega^{r,1}$ and the horizontal homotopy formula (4.14), as applied to $d_V\omega$ leads to

$$d_V\omega = d_H[h_H^{r,1}(d_V\omega)] + h_H^{r+1,1}(d_H d_V\omega).$$

To this equation, we now apply the vertical homotopy operator $h_V^{r,1}$. Since $h_V^{r,1}$ anti-commutes with $d_H$, this gives rise to the homotopy formula

$$\omega[x,u] - \omega[x,0] = -d_H\big[(h_V^{r-1,1} \circ h_H^{r,1} \circ d_V)(\omega)\big] - (h_V^{r,1} \circ h_H^{r,1} \circ d_V)(d_H\omega). \tag{4.29}$$

A direct calculation shows that

$$h_H^{r,0}\mid_{v=0} = -h_V^{r,1} \circ h_H^{r,1} \circ d_V$$

so that the two formulas (4.21) and (4.29) actually coincide.

Thirdly, in Proposition 3.7 we proved that every type $(n,s)$ functional form $\omega$ can be written uniquely in the form

$$\omega = \theta^\alpha \wedge P_\alpha,$$

where $P_\alpha$ is formally skew-adjoint. We also showed (see (3.12)) that for any vertical vector field $Y = Y^\alpha \dfrac{\partial}{\partial u^\alpha}$

$$\operatorname{pr} Y \mathbin{\lrcorner} \omega = sY^\alpha P_\alpha + d_H\eta.$$

By substituting this equation into (4.10), we find that (4.27) becomes

$$\mathcal{H}^s(\omega) = I\Big(\int_0^1 st^{s-1}u^\alpha P_\alpha[x,tu]\, dt\Big). \tag{4.30}$$

Corollary 4.6. *Let $\omega \in \Omega^{r,0}$ and suppose that $d_H\omega = 0$ if $r \le n$ or that $E(\omega) = 0$ if $r = n$. Let $s_t\colon M \to E$, $0 \le t \le 1$, be a smooth one parameter family of sections of $E$. Then*

$$\omega(j^\infty(s_1)) - \omega(j^\infty(s_0)) = d\,\eta, \tag{4.31a}$$

*where*

$$\eta = \int_0^1 I^r_{Y_t}(\omega)(j^\infty(s_t))\,dt, \tag{4.31b}$$

*and where $Y_t$ is the vertical vector field $Y_t = \dot u^\alpha_t \dfrac{\partial}{\partial u^\alpha}$.*

PROOF: Since $\omega$ is a horizontal form and $j^\infty(s)$ covers the identity on the base space $\mathbf{R}^n$, we have written $\omega(j^\infty(s))$ for the pullback $(j^\infty(s))^*\omega$. This corollary also follows from the Lie derivative formula (4.16):

$$\omega(j^\infty(s_1)) - \omega(j^\infty(s_2)) = \int_0^1 \frac{d}{dt}\,[(j^\infty(s_t))^*\omega]\,dt = \int_0^1 (\mathcal{L}_{\mathrm{pr}\,Y_t}\,\omega)(j^\infty(s_t))\,dt.$$

∎

Let $f[x,u]$ be a smooth function on $J^\infty(E)$. We say that $f$ is homogeneous of degree $p$ if it is homogeneous of degree $p$ in the *fiber* variables $u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots$, *i.e.*,

$$f[x, t^p u] = t^p f[x,u] \qquad \text{for } t > 0.$$

A type $(r,s)$ form $\omega$ is said to be homogeneous of degree $p$ if each of its coefficient functions is homogeneous of degree $p$, *i.e.*,

$$\omega[x, tu] = t^p \omega[x,u].$$

The homotopy operators $h^{r,0}_H$ and $\mathcal{H}^s$ simplify when applied to homogenous forms.

COROLLARY 4.7. *(i) Let $\omega \in \Omega^{r,0}$ and suppose that $\omega$ is homogeneous of degree $p \neq 0$. If $d_H\omega = 0$ for $1 \le r \le n-1$ or if $E(\omega) = 0$ for $r = n$, then $\omega = d_H\eta$, where*

$$\eta = \frac{1}{p}\, I^r_R(\omega), \tag{4.32}$$

*where $I^r_Y$ is the inner product operator (4.17) and where $R$ is the radial vector field (4.7).*

*(ii) Let $\omega = \theta^\alpha \wedge P_\alpha \in \mathcal{F}^s$, where $P(Y) = Y^\alpha P_\alpha$ is a formally skew adjoint operator, and suppose that $\omega$ is homogeneous of degree $p \neq -s$. If $\delta_V\omega = 0$, then $\omega = \delta_V\eta$, where*

$$\eta = I\Big(\frac{s}{p+s}\, u^\alpha P_\alpha\Big). \tag{4.33}$$

PROOF: If $\omega \in \Omega^{r,s}$ is homogeneous of degree $p$, then an elementary calculation shows that

$$\mathcal{L}_{\mathrm{pr}\,R}\,\omega = (p+s)\omega.$$

Part (i) now follows immediately from (4.16) with $Y = R$ while part (ii) can be established by repeating the derivation of (4.26) using with the formula

$$(p+s)\omega = d_V(\operatorname{pr} R \rightharpoonup \omega) + \operatorname{pr} R \rightharpoonup d_V \omega$$

in place of (4.25)

Alternatively, one can use the homogeneity of $\omega$ to explicitly evaluate the integrals in the homotopy formulas (4.23) ( with $v = 0$) and (4.30). ∎

We note that Corollary 4.7 even applies with $p < 0$ — that is, to forms which are singular at the point $u_I^\alpha = 0$. More generally, consider a $d_H$ closed horizontal $r$ form $\omega$ which is not defined at $u_I^\alpha = 0$. Suppose, however, that for $t > 0$

$$\omega[x, tu] = \frac{1}{t^\epsilon}\omega_t[x, u],$$

where $\omega_t$ is a smooth function of $t$. Then

$$E_\alpha^{Ij}(\omega_j)[x, tu] = \frac{1}{t^{\epsilon+1}} E_\alpha^{Ij}((\omega_t)_j)$$

in which case the integrand in the homotopy formula (4.23) (with $v = 0$) may be singular at $t = 0$. If, however, we know that $\omega_t$ and its derivatives with respect to all the variables $u_I^\alpha$ are bounded as $t \to \infty$, then we can integrate (4.16), not from 0 to 1, but rather from 1 to $\infty$ to deduce that

$$\tilde{h}_H^{r+1,0}(d_H\omega) + d_H \tilde{h}_H^{r,0}(\omega) = \omega[x, u] - \omega[x, v] \tag{4.34}$$

where

$$\tilde{h}_H^{r,0}(\omega) = \int_1^\infty \frac{1}{t^{\epsilon+1}} \sum_{|I|=0}^{k} \frac{|I|+1}{n-r+|I|+1} D_I\left[u^\alpha E_\alpha^{Ij}((\omega_t)_j)[x, u_t]\right] dt. \tag{4.35}$$

The point to emphasize here is that neither homotopy operator $h_H^{r,0}$ nor $\tilde{h}_H^{r,0}$ is applicable to forms which are homogeneous of degree zero. Consequently, when we restrict the variational bicomplex on $J^\infty(E)$ to the open set

$$\mathcal{R} = \{\, [x, u] \mid u_I^\alpha \text{ are not all zero} \,\},$$

then the only type $(r, 0)$, homogeneous, $d_H$ closed forms which can represent nontrivial cohomology classes in the Euler-Lagrange complex $\mathcal{E}^*(\mathcal{R})$ are those which are homogeneous of degree zero. Likewise, the only type $(n, s)$, homogeneous, $\delta_V$ closed functional forms which can represent nontrivial cohomology classes in $H^{n+s}(\mathcal{E}^*(J^\infty(E)))$ are those which are homogeneous of degree $-s$.

We now describe some invariance properties of our homotopy operators. While the invariance group under consideration is admittedly small, it is nevertheless large enough to play an important role in applications. We continue to work on the trivial bundle $E : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$. Let $G$ be the direct product of the group of affine transformations of the base (with trivial fiber action) with the group of linear transformations of the fiber (with trivial action on the base). Thus if $\psi \in G$, then there are matrices $A = (a^i_j)$ and $B = (b^\alpha_\beta)$ in $GL(n)$ and $GL(m)$ respectively and a vector $x_0 = (x^i_0)$ in $\mathbf{R}^n$ such that

$$\psi(x, u) = (y, v) = (Ax + x_0, Bu).$$

The prolonged action on $J^\infty(E)$ is easily determined to be $\Psi[x, u] = [y, v]$, where

$$v^\alpha_I = b^\alpha_\beta c^J_I u^\beta_J, \tag{4.36}$$

where $C = (c^j_i) = A^{-1}$ and where $c^J_I = c^{j_1}_{i_1} c^{j_2}_{i_2} \cdots c^{j_k}_{i_k}$.

Proposition 4.8. *Let $\psi \in G$ and let $\Psi \colon J^\infty(E) \to J^\infty(E)$ be the prolongation of $\psi$. The homotopy operators $h^{r,s}_V$, $h^{r,s}_H$ and $\mathcal{H}^s$ all commute with the pullback map $\Psi^*$.*

Proof: It suffices to check that $\Psi^*$ commutes with $h^{r,s}_V$ and $h^{r,s}_H$, for $s \geq 1$, since the remaining operators $h^{r,0}_H$ and $\mathcal{H}^s$ are defined in terms of these and $d_V$ and $I$ (which we already know commute with $\Psi^*$).

To prove that $\Psi^*$ commutes with the vertical homotopy operator, first note that $\psi$ commutes with the flow $\phi_\epsilon$ of the radial vector field $R = u^\alpha \dfrac{\partial}{\partial u^\alpha}$. This implies that $\Psi$ commutes with $\Phi_\epsilon = \operatorname{pr} \phi_\epsilon$,

$$\Psi \circ \Phi_\epsilon = \Phi_\epsilon \circ \Psi, \tag{4.37}$$

and consequently $\operatorname{pr} R$ is preserved by $\Psi_*$, *i.e.*,

$$\Psi_*(\operatorname{pr} R) = \operatorname{pr} R. \tag{4.38}$$

Let $\omega \in \Omega^{r,s}$. Since $\Phi^*_{\log t}(\omega[x, u]) = t^s \omega[x, tu]$, we can also infer from (4.37) that

$$\Psi^*(\omega[x, tv]) = (\Psi^*(\omega))[y, tu]. \tag{4.39}$$

Equations (4.38) and (4.39) imply that

$$\operatorname{pr} R \mathbin{\lrcorner} (\Psi^*(\omega))[y, tu] = \operatorname{pr} R \mathbin{\lrcorner} \Psi^*(\omega)[x, tu] = \Psi^*(\operatorname{pr} R \mathbin{\lrcorner} \omega[x, tv]).$$

This suffices to show that $h_V^{r,s}$ commutes with $\Psi^*$.

We verify that $\Psi^*$ commutes with $h_H^{r,s}$, $s \geq 1$ by calculating the change of variables formula for the inner Euler operators under the transformation (4.36). Let

$$\overline{\omega}[y, v] = (\Psi^{-1})^* \omega[x, u].$$

Let $\overline{F}^I_\alpha$ be the inner Euler operator in $[y, v]$ coordinates, *i.e.*,

$$\overline{F}^I_\alpha(\overline{\omega}) = \sum_{|J|=0}^{k-|I|} (-\overline{D})_J \big(\overline{\partial}^{IJ}_\alpha \lrcorner (\overline{\omega})\big),$$

where $\overline{D}_i = \dfrac{d}{dy^i}$ and $\overline{\partial}^I_\alpha$ is the symmetrized partial derivative with respect to $v^\alpha_I$. Because the matrices $a^j_i$ and $b^\beta_\alpha$ are constant it is a straightforward matter to conclude that

$$\overline{\partial}^I_\alpha = c^I_J b^\beta_\alpha \partial^J_\beta, \qquad \overline{D}_i = a^j_i D_j, \qquad \overline{F}^I_\alpha(\overline{\omega}) = c^I_J b^\beta_\alpha F^J_\beta(\omega)$$

and

$$\overline{D}_I[\bar{\theta}^\alpha \wedge \overline{F}^{Ij}_\alpha(\overline{\omega}_j)] = D_I[\theta^\alpha \wedge F^{Ij}_\alpha(\omega_j)].$$

Since $h_H^{r,s}$ is a weighted sum of these latter expressions, this proves that

$$\overline{h}^{r,s}_H(\overline{\omega}) = h^{r,s}_H(\omega),$$

as required. ∎

Let $\Psi : J^\infty(E) \to J^\infty(E)$ be a smooth map. We say that a type $(r, s)$ form $\omega$ is a relative $\Psi$ invariant with character $\chi$ if

$$\Psi^*(\omega) = \chi\omega.$$

Let $\Omega^{r,s}_\Psi(J^\infty(E))$ be the space of type $(r, s)$ relative $\Psi$ invariant forms with a fixed character.

COROLLARY 4.9. *Let $\psi \in G$ and let $\Psi \colon J^\infty(E) \to J^\infty(E)$ be the prolongation of $\psi$.*

*(i) Except for along the bottom edge ($s = 0$) the relative $\Psi$ invariant, augmented variational bicomplex $(\Omega^{*,*}_\Psi, d_H, d_V)$ is exact.*

*(ii) If $\omega$ is a relative $\Psi$ invariant, $d_H$ closed, type $(r,0)$ form, then there is relative $\Psi$ invariant type $(r-1,0)$ form $\eta$ and an degree $r$ form $\omega_0$ on the base manifold $M$ such that*

$$\omega = d_H\eta + \omega_0.$$

*The form $\omega_0$ may not be invariant under the action of $\psi$ restricted to $M$.*

Example 4.10. It is known that the evolutionary vector field

$$Y = (u_{xxx} + \frac{1}{2}u_x^3)\,\frac{\partial}{\partial u} \tag{4.40}$$

is a distinguished symmetry (see Definition 3.27) for the sine-Gordon equation

$$\Delta = (u_{xt} - \sin u)\, du \wedge dx \wedge dt.$$

Since this source form is variational, the Lagrangian form

$$\begin{aligned}\lambda &= L\, dx \wedge dt = Y \lrcorner\, \Delta \\ &= (u_{xxx} + \frac{1}{2}u_x^3)(u_{xt} - \sin u)\, dx \wedge dt\end{aligned}$$

is variationally trivial and determines the conservation law

$$Y \lrcorner\, \Delta = d_H\eta.$$

The coefficients of the one form

$$\eta = P\, dt - Q\, dx,$$

as given by the homotopy operator $h_H^{2,0}$, are found to be

$$P = \int_0^1 \big\{D_{xx}\big(uE^{xxx}(L)[su]\big) + D_x(uE^{xx}(L)[su]) + uE^x(L)[su]\big\}\, ds$$

and

$$Q = \int_0^1 \big\{D_xE^{xt}(L)[su] + uE^t(L)[su]\big\}\, ds.$$

From the definition (2.15) of the Euler operators and the definition of the symmetrized partial derivative operators (*e.g.*, $\partial^{xt} = \dfrac{1}{2}\dfrac{\partial}{\partial u_{xt}}$. See (1.15)) we calculate that

$$E^{xxx}(L) = u_{xt} - \sin u \qquad\qquad E^{xx}(L) = -3D_x(u_{xt} - \sin u),$$

$$E^x(L) = \frac{3}{2}u_x^2(u_{xt} - \sin u) + 3D_{xx}(u_{xt} - \sin u) - D_t(u_{xxx} + \frac{1}{2}u_x^3),$$

$$E^{xt}(L) = \frac{1}{2}(u_{xxx} + \frac{1}{2}u_x^3) \qquad \text{and} \qquad E^t(L) = -D_x(u_{xxx} + \frac{1}{2}u_x^3).$$

The formulas for the coefficients $P$ and $Q$ of $\eta$ now yield

$$\eta = [-\frac{1}{2}u_x u_{xxt} + \frac{1}{2}u_{xx}u_{xt} - u_{xx}\sin u + \frac{1}{2}u_x^2\cos u]\,dt$$
$$- [-\frac{1}{4}uu_{xxxx} + \frac{1}{4}u_x uxxx + \frac{1}{16}u_x^4 - \frac{3}{16}uu_x^2u_{xx}]\,dx.$$

Observe that this form is of order 4 even though the Lagrangian $\lambda$ is of order 3. One easily checks that $\eta$ differs from the usual expression for the conservation law for the symmetry $Y$, *viz.*,

$$\tilde{\eta} = [u_{xx}u_{xt} - u_{xx}\sin u + \frac{1}{2}u_x^2\cos u]\,dt - [-\frac{1}{2}u_{xx}^2 + \frac{1}{8}u_x^4]\,dx$$

by an exact one form. The results of the next section will enable us to either systematically pass from the form $\eta$ to $\tilde{\eta}$ or to obtain $\tilde{\eta}$ directly via a method of undetermined coefficients. ∎

EXAMPLE 4.11. The need for a practical alternative to the homotopy operators is made even more apparent if one determines, via the Volterra-Vainberg formula, a Lagrangian for the minimal surface equation. The corresponding source form is

$$\Delta = \frac{(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy}}{\sqrt{(1+u_x^2+u_y^2)^3}}\,du\wedge dx\wedge dy \tag{4.41}$$

One computes $\lambda = \mathcal{H}^1(\Delta) = L\,dx\wedge dy$, where

$$\begin{aligned} L &= u\Big[\int_0^1 \frac{t}{\sqrt{(1+t^2(u_x^2+u_y^2))^3}}\,dt\Big](u_{xx}+u_{yy}) \\ &\quad + u\Big[\int_0^1 \frac{t^3}{\sqrt{(1+t^2(u_x^2+u_y^2)^3}}\,dt\Big](u_y^2u_{xx} - 2u_xu_yu_{xy} + u_x^2u_{yy}) \\ &= u\frac{(1+u_y^2+\sqrt{1+u_x^2+u_y^2})u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2+\sqrt{1+u_x^2+u_y^2})u_{yy}}{(\sqrt{1+u_x^2+u_y^2})(1+\sqrt{1+u_x^2+u_y^2})^2}. \end{aligned}$$

This is a far cry from the usual first order Lagrangian

$$\lambda = -\sqrt{1+u_x^2+u_y^2}\,dx\wedge dy.$$

∎

EXAMPLE 4.12. *Variationally trivial, natural Lagrangians for plane and space curves.*

The following proposition, due to Cheung [*17*], determines the one dimensional cohomology $H^1(\mathcal{E}^*)$ for the natural Euler-Lagrange complex for regular plane and space curves (See §2C).

PROPOSITION 4.13. *(i) Let $\lambda = L(\kappa, \dot{\kappa}, \ddot{\kappa}, \dots)\, ds$ be a variationally trivial, natural Lagrangian for plane curves with curvature $\kappa$. Then*

$$\lambda = a\kappa\, ds + d_H f,$$

*where $a$ is a constant and $f = f(\kappa, \dot{\kappa}, \ddot{\kappa}, \dots)$ is a natural function.*

*(ii) Let $\lambda = L(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \dots)\, ds$ be a variationally trivial, natural Lagrangian for space curves with curvature $\kappa$ and torsion $\tau$. Then*

$$\lambda = d_H f,$$

*where $f = f(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \dots)$ is a natural function.*

PROOF: To prove (i), we observe from (2.65) that $E(\lambda) = 0$ if and only if

$$\ddot{E}_\kappa + \kappa^2 E_\kappa + \kappa H = 0. \tag{4.42}$$

Here $E_\kappa$ and $H$ are respectively the Euler-Lagrange expression and the Hamiltonian obtained through the variation of the curvature $\kappa$. It is easy to see that this equation implies that

$$E_\kappa = \text{constant}. \tag{4.43}$$

Indeed, if $E_\kappa$ is not a constant, then there is a largest integer $l \geq 0$ for which $\dfrac{dE_\kappa}{d\kappa^{(l)}} \neq 0$. From the identity $\dot{H} = -\dot{\kappa} E_\kappa$, we can infer that $H$ is of order at most $l-1$ if $l \geq 1$ and order zero if $l = 1$. Consequently, by differentiating (4.42) with respect to $\kappa^{(l+2)}$, it follows that $\dfrac{dE_\kappa}{d\kappa^{(l)}} = 0$. This contradiction establishes (4.43).

Denote the constant in (4.43) by $a$. Then the natural Lagrangian

$$\tilde{L} = L - a\kappa$$

satisfies $E_\kappa(\tilde{L}) = 0$. We now apply the horizontal homotopy operator $h_H^{1,0}$ to $\tilde{L}$ to construct a natural function $f = f(\kappa, \dot{\kappa}, \ddot{\kappa}, \dots)$ such that

$$L = a\kappa + \frac{df}{ds} + b,$$

where $b$ is a constant. Note that because $\tilde{L}$ is independent of the base variable $s$, then so is $f = h_H^{1,0}(\tilde{L})$. Although $E_\kappa(b) = 0$, we have

$$E(b\,ds) = b\kappa\,\Theta^2 \wedge ds.$$

Since $L$ is variationally trivial, this forces $b = 0$. Part(i) is established.

It is clear that there is no natural function $g$ for which $d_H g = a\kappa\,ds$.

We emphasize that we could not have arrived at this result by directly applying the the homotopy operator to $\lambda$, viewed as a Lagrangian on $J^\infty(\mathbf{R}^2 \times \mathbf{R})$ in the variables $(t, x, y, x', y', \ldots)$. In the first place, the singularities in the curvature $\kappa$ at $x' = y' = 0$ prevent us from applying the homotopy operator globally. Secondly, even if $\lambda[x, y]$ is a natural Lagrangian, it may not be true that $h_H^{1,0}(\lambda[x, y])$ is a natural function.

In the case of natural Lagrangians for space curves, we use Proposition 2.16 and a simple generalization of the argument we used in the planar case to conclude that if $\lambda$ is variational trivial then both $E_\kappa$ and $E_\tau$ are constant. The third component $E_3$ of the Euler-Lagrange form $E(\lambda)$ therefore reduces to

$$E_3 = \dot{\tau} E_\kappa - \dot{\kappa} E_\tau.$$

This vanishes if and only if $E_\kappa = E_\tau = 0$. Consequently there is a natural function $f$ and a constant $c$ such that

$$L = \frac{df}{ds} + c.$$

For the same reasons as before, $c = 0$. ∎

Cheung also considers natural Lagrangians for curves on surfaces of constant curvature. In view of Proposition 2.21, it is easily seen that if such a Lagrangian is variationally trivial, then it is the horizontal differential of a natural function.

Example 4.14. *The Gauss-Bonnet-Chern Theorem.*

Let $n = 2m$ be even and let $gl(n)$ be the Lie algebra of $GL(n)$. Define a symmetric multi-linear map

$$P\colon [gl(n)]^m \to \mathbf{R}$$

by

$$P(a, b, \ldots, c) = \frac{1}{n!}\,\varepsilon^{i_1 j_1 i_2 j_2 \cdots i_m j_m} a_{i_1 j_1} b_{i_2 j_2} \cdots c_{i_m j_m}.$$

The Pfaffian of $a \in gl(n)$ is defined to be

$$\mathrm{Pf}(a) = P(a, a, \dots, a).$$

Ordinarily, the Pfaffian is defined only for skew-symmetric matrices but we shall need this extension to all of $gl(n)$. Note that only the skew-symmetric part of the matrix $a$ contributes to $\mathrm{Pf}(a)$; if $b = \frac{1}{2}(a - a^{\mathrm{t}})$ then $\mathrm{Pf}(a) = \mathrm{Pf}(b)$.

Let $M$ be a compact manifold and let $\pi : E \to M$ be the product bundle of the bundle of metrics on $M$ with the bundle of connections on $M$. Let $g = g_{ij}\, dx^i \otimes dx^j$ be a metric on $M$ and let $\omega^i_j = \Gamma^i_{jk}\, dx^k$ be any set of connection one forms for the tangent bundle $TM$. Let

$$\Omega_j{}^i = d\,\omega^i_j - \omega^h_j \wedge \omega^i_h = -\frac{1}{2} K_j{}^i{}_{hk}\, dx^h \wedge dx^k$$

be the curvature two-form and let $\Omega_{ij} = g_{il}\Omega_j{}^l$. We define a Lagrangian $\lambda \in \Omega^{n,0}(J^\infty(E))$ by

$$\lambda[g, \omega] = \frac{1}{\sqrt{g}}\ \mathrm{Pf}(\Omega_{ij}).$$

This Lagrangian is first order in the derivatives of the connection and zeroth order in the derivatives of the metric. Because we wish to compute the Euler-Lagrange form for $\lambda$ by varying $g$ and $\omega$ independently, we have *not* restricted the domain of $\lambda$ to metric connections. Thus the matrix $\Omega_{ij}$ may not be skew-symmetric. It is for this reason that we extended the domain of the Pfaffian. If $\omega$ is the Christoffel connection $\omega_g$ for $g$, then a simple calculation shows that

$$\begin{aligned} \lambda[g, \omega_g] &= \frac{1}{2^m n!} \Big[\frac{1}{\sqrt{g}} \varepsilon^{i_1 j_1 \cdots i_m j_m} \frac{1}{\sqrt{g}} \varepsilon^{h_1 k_1 \cdots h_m k_m}\Big] K_{i_1 j_1 h_1 k_1} \\ &\qquad\qquad \cdots K_{i_m j_m h_m k_m} (\sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n) \\ &= \sqrt{g} K_n\, dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

where $K_n$ is the total curvature of $g$. In other words, $\lambda[g, \omega_g]$ is the integrand in the Gauss-Bonnet-Chern formula.

If $g_1$ and $g_0$ are two metrics on $M$ and $\omega_1$ and $\omega_0$ are Riemannian connections for $g_1$ and $g_0$, then it is well known that $\lambda[g_1, \omega_1] - \lambda[g_0, \omega_0]$ is exact and hence $\int_M \lambda$ is independent of the choice of metric $g$ and Riemannian connection $\omega$. This proves, albeit indirectly, that the Euler-Lagrange form $E(\lambda)[g, \omega]$ must vanish when $\omega$ is a Riemannian connection for $g$. In this example, we shall reverse the order of this argument. We explicitly commute $E(\lambda)$ and show that this source form vanishes whenever $\omega$ is a $g$ compatible connection. Our homotopy formula (4.22) then implies that

$$\lambda[g_1, \omega_1] - \lambda[g_0, \omega_0] = d_H \eta. \tag{4.44}$$

Because $\lambda$ is a first order natural Lagrangian, it is easily seen that $\eta$ is invariantly defined and consequently (4.44) holds globally. Moreover, if $X$ is vector field on $M$, then away from the zero set of $X$ one can construct a metric connection $\omega_0$ for which $\lambda[g_0, \omega_0] = 0$. The explicit formula for $\eta$, as provided by the homotopy operator $h_H^{n,0}$, coincides with that used by Chern in his original proof of the generalized Gauss-Bonnet theorem.

Set

$$[\mathrm{Pf}'(a)](b) = [\frac{d}{dt}[\mathrm{Pf}(a + tb)]|_{t=0} = mP(b, a, \ldots, a) = P^{ij}(a) b_{ij}$$

and

$$[\mathrm{Pf}''(a)](b, b) = [\frac{d^2}{dt^2}\, \mathrm{Pf}(a + tb)]|_{t=0} = P^{ijhk}(a) b_{ij} b_{hk}.$$

Note that

$$P^{ij}(a) = -P^{ji}(a)$$

and that

$$P^{ijhk}(a) = -P^{jihk}(a) = P^{hkij}(a).$$

In the next lemma $D_\omega g_{ih}$ is the covariant exterior derivative of $g_{ih}$ with respect to the connection $\omega$, *viz.*.

$$D_\omega g_{ih} = \nabla_k g_{ih}\, dx^k.$$

LEMMA 4.15. *The Euler-Lagrange form for $\lambda[g, \omega]$ is*

$$E(\lambda) = E_g(\lambda) + E_\omega(\lambda),$$

*where*

$$E_g(\lambda) = \frac{1}{2\sqrt{g}}\, g^{ri} P^{sj}(\Omega) \wedge [\Omega_{ij} + \Omega_{ji}] \wedge d_V g_{rs} \tag{4.45}$$

*and*

$$E_\omega(\lambda) = \frac{1}{\sqrt{g}}\, g_{ri} D_\omega g_{jh} \wedge [P^{iskj}(\Omega) \wedge {\Omega_k}^h + g^{ih} P^{js}(\Omega) - \frac{1}{2} g^{jh} P^{is}(\Omega)] \wedge d_V \omega^r_s. \tag{4.46}$$

*Consequently, the Euler-Lagrange form $E[\lambda)(g,\omega]$ vanishes whenever $\omega$ is any Riemannian connection for $g$.*

PROOF: The Pfaffian is a relative $GL(n)$ invariant in the sense that for any matrix $M = (m^r_s) \in GL(n)$,

$$\mathrm{Pf}(M^{\mathrm{t}} a M) = \det M \ \mathrm{Pf}(a).$$

Differentiate this identity with respect to $m^r_s$ and put $M = I$ to obtain the invariance condition

$$P^{sj}(a) a_{rj} + P^{is}(a) a_{ir} = \delta^s_r \, \mathrm{Pf}(a). \tag{4.47}$$

Again we emphasize that this identity holds for all $a \in gl(n)$. Differentiation of (4.47) with respect to $a_{hk}$ yields

$$P^{sjhk}(a) a_{rj} + P^{ishk}(a) a_{ir} + P^{sk}(a)\delta^h_r + P^{hs}(a)\delta^k_r = \delta^s_r P^{hk}(a). \tag{4.48}$$

These identities will be used to simplify the formulas for the Euler-Lagrange forms.

Since $\lambda[g,\omega]$ is of order zero in the metric $g$, we find that

$$\begin{aligned} E_g(\lambda)[g,\omega] &= \frac{\partial \lambda}{\partial g_{rs}} \, d_V g_{rs}, \\ &= [\frac{\partial}{\partial g_{rs}} (\frac{1}{\sqrt{g}}) \, \mathrm{Pf}(\Omega) + \frac{1}{\sqrt{g}} P^{ij} \wedge \frac{\partial \Omega_{ij}}{\partial g_{rs}}] \wedge d_V g_{rs} \\ &= \frac{1}{\sqrt{g}} [-\frac{1}{2} g^{rs} \, \mathrm{Pf}(\Omega) + P^{rj}(\Omega) \wedge \Omega_j{}^s] \wedge d_V g_{rs}. \end{aligned} \tag{4.49}$$

By using (4.47), it a simple matter to rewrite (4.49) as (4.45).

In a similar fashion, one finds by virtue of the formulas

$$\frac{\partial \Omega_j{}^l}{\partial \Gamma^r_{st}} = (\delta^s_j \omega^l_r - \delta^l_r \omega^s_j) \, dx^t \qquad \text{and} \qquad \frac{\partial \Omega_j{}^l}{\partial \Gamma^s_{st,u}} = \delta^s_j \delta^l_r \, dx^u \wedge dx^t, \tag{4.50}$$

that

$$\begin{aligned} E_\omega(\lambda) &= [\frac{1}{\sqrt{g}} \frac{\partial \, \mathrm{Pf}(\Omega)}{\partial \Gamma^r_{st}} - D_u \big( \frac{1}{\sqrt{g}} \frac{\partial \, \mathrm{Pf}(\Omega)}{\partial \Gamma^r_{st,u}} \big)] \wedge d_V \Gamma^r_{st} \\ &= [-\frac{1}{\sqrt{g}} g_{il} P^{is} \wedge \omega^l_r + \frac{1}{\sqrt{g}} g_{ir} P^{ij} \wedge \omega^s_j + d_H \big( \frac{1}{\sqrt{g}} g_{ir} P^{is} \big)] \wedge d_V \Gamma^r_{st}. \end{aligned} \tag{4.51}$$

To evaluate the last term in (4.51), we observe that

$$d_H(\frac{1}{\sqrt{g}}) = -\frac{1}{2\sqrt{g}} g^{hk} D_\omega g_{hk} - \frac{1}{2\sqrt{g}} \omega^l_l,$$

and

$$d_H(g_{ir}) = D_\omega g_{ir} + g_{il}\omega^l_r + g_{lr}\omega^l_i.$$

Moreover, on account of the second Bianchi identity, we conclude

$$\begin{aligned} d_H P^{is}(\Omega) &= P^{ishk}(\Omega) \wedge d_H \Omega_{hk} \\ &= P^{ishk}[D_\omega g_{kl} \wedge \Omega_h{}^l + \omega^l_k \wedge \Omega_{hl} + \omega^l_h \wedge \Omega_{lk}]. \end{aligned}$$

These last three equations are substituted into (4.51). The result is simplified using the invariance identity (4.48) to arrive at (4.46). ∎

COROLLARY 4.16. *If $\omega_1$ and $\omega_0$ are Riemannian connections for metrics $g_1$ and $g_0$, then*

$$\int_M \lambda[g_1, \omega_1] = \int_M \lambda[g_0, \omega_0]. \tag{4.52}$$

PROOF: Let $\omega_{g_i}$ be the Christoffel connection for $g_i$, $i = 0, 1$. We show that

$$\int_M \lambda[g_1, \omega_{g_1}] = \int_M \lambda[g_0, \omega_{g_0}] \tag{4.53}$$

and

$$\int_M \lambda[g_i, \omega_i] = \int_M \lambda[g_i, \omega_{g_i}] \tag{4.54}$$

for $i = 0, 1$. Together these three equalities prove (4.52).

Let $(g_t, \omega_t)$, $0 \le t \le 1$, be any curve of metrics and connections. Corollary 4.6 implies that

$$\begin{aligned} \lambda[g_1, \omega_1] - \lambda[g_0, \omega_0] &= d\,\eta \\ &\quad + \int_0^1 (\dot{g}_t)_{ij}[E_g(\lambda)]^{ij}[g_t, \omega_t]\, dt + \int_0^1 (\dot{\Gamma})^h_{ij}[E_\omega(\lambda)]^{ij}_h[g_t, \omega_t]\, dt, \end{aligned} \tag{4.55}$$

where, in accordance with (4.28), $\eta$ is the $(n-1)$ form defined by

$$\eta = \int_0^1 (\dot{\Gamma}_t)^h_{ij} D_k \lrcorner \, (\frac{\partial L}{\partial \Gamma^h_{ij,k}})[g_t, \omega_t]\, dt.$$

Since $(\dot{\Gamma}_t)^h_{ij}$ is the infinitesimal difference of connections, it is a tensor of type $(1,2)$. Since $\lambda = L\,\nu$ is a of first order in the derivatives of $\Gamma$, the highest order derivative $\dfrac{\partial L}{\partial \Gamma^h_{ij,k}}$ is a tensor density of type $(3,1)$. Thus $\eta$ is an invariantly defined form and equation (4.55) holds globally.

To establish (4.53), let

$$g_t = tg_1 + (1-t)g_0$$

and let $\omega_t = \omega_{g_t}$ be the Christoffel connection of $g_t$. To establish (4.54), let $g_t = g_i$ and let

$$\omega_t = t\omega_i + (1-t)\omega_{g_i}.$$

According to Proposition 4.15, the Euler-Lagrange form $E(\lambda)[g_t, \omega_t]$ vanishes in either case and so (4.53) and (4.54) follow from (4.55) and an application of Stokes theorem. ∎

Now we turn to the proof of the generalized Gauss-Bonnet theorem. Fix a metric $g$ and a Riemannian connection $\omega$ on $M$. Let $\nabla$ denote covariant differentiation with respect to this connection. Let $X = X^i \dfrac{\partial}{\partial x^i}$ be a unit vector defined on some open set $U$ of $M$. Introduce the type $(1,1)$ tensor-valued one form

$$\begin{aligned} E^i_j &= X^i D_\omega X_j - X_j D_\omega X^i \\ &= (X^i \nabla_l X_j - X_j \nabla_l X^i)\, dx^l. \end{aligned}$$

Because $X$ is a unit vector field and because $\omega$ is a Riemannian connection

$$X^j E^i_j = -D_\omega X^i \qquad \text{and} \qquad g_{ih} E^i_j + g_{ij} E^i_h = 0.$$

These formulas are needed to check some of the statements in the next paragraph.

Define a curve of connection one forms $\omega_t$, $0 \le t \le 1$ by

$$(\omega_t)^i_j = \omega^i_j + (1-t)E^i_j.$$

It is easily verified that each $\omega_t$ is a Riemannian connection for the metric $g$ and that the associated curvature two-form is

$$\begin{aligned} (\Omega_t)_j{}^i &= \Omega_j{}^i + (1-t) D_\omega E^i_j + (1-t)^2 E^i_h \wedge E^h_j \\ &= \Omega_j{}^i + (1-t)[X^i X_l \Omega_j{}^l - X_j X^l \Omega_l{}^i] + (1-t)^2 [D_\omega X^i \wedge D_\omega X_j]. \end{aligned} \tag{4.56}$$

Evidently the skew-symmetric matrix $(\Omega_0)_{ij}$ has $X$ as a zero eigenvalue and so $\mathrm{Pf}(\Omega_0) = 0$. With $g_t = g$, the fundamental variational formula (4.55) reduces to

$$\lambda[g, \omega] = d\,\eta,$$

where

$$\eta = -\int_0^1 E^h_{ij}\, D_k \lrcorner \left(\frac{\partial L}{\partial \Gamma^h_{ij,k}}\right)[g, \omega_t]\, dt. \tag{4.57}$$

It is not difficult to explicitly evaluate the integral in (4.57).[1] Indeed, on account of (4.50), this formula for $\eta$ simplifies to

$$\eta = - - \frac{2m}{\sqrt{g}} \int_0^1 P(\alpha, \Omega_t)\, dt, \tag{4.58}$$

where $\alpha$ is the matrix $(X_i D_\omega X_j)$. Now observe that if $a = X \wedge Y$ and $b = X \wedge Z$, then $\mathrm{Pf}(a, b, \ldots, c) = 0$. Hence the coefficient of $(1-t)$ in the expression (4.56) for $\Omega_t$ does not contribute to the integrand in (4.58) and thus

$$\mathrm{Pf}(\alpha, \Omega_t) = \sum_{r=0}^{m-1} (1-t^2)^r \binom{m-1}{r} Q^r(X, \Omega), \tag{4.59}$$

where

$$Q^r(X, \Omega) = \mathrm{Pf}(\alpha, \underbrace{\beta, \ldots \beta,}_{r \text{ times}} \underbrace{\Omega, \ldots, \Omega}_{m-r+1 \text{ times}}),$$

and $\beta = (D_\omega X_i \wedge D_\omega X_j)$. By substituting (4.59) into (4.58) and evaluating the resulting integral we arrive at Chern's original formula for the generalized Gauss-Bonnet integrand.

PROPOSITION 4.17. *Let $X$ be a unit vector field defined on an open set $U \subset M$. Then , on $U$*

$$\lambda[g, \omega] = d\eta,$$

*where*

[1] Since we are ultimately interested in the form of $\eta$ only when $g$ is flat, this step is, strictly speaking, unnecessary in our derivation of the generalized Gauss-Bonnet theorem.

$$\eta = \sum_{r=0}^{m-1} \frac{1}{\sqrt{g}} c_r Q^r(X, \Omega) \qquad \textit{and} \qquad c_r = -\frac{2^{2r+1}(r!)^2}{(2r+1)!}\binom{m-1}{r}. \tag{4.60}$$

The generalized Gauss-Bonnet theorem now follows from Hopf's formula for the Euler characteristic $\chi(M)$ and the local integral formula for the index of a vector field. Let $X_0$ be a vector field on $M$ with isolated zeros at points $p_1, p_2, \ldots, p_k$. Let $X = X_0/||X_0||$ be the corresponding unit vector field. Then Hopf's formula states that

$$\chi(M) = \sum_{i=0}^{k} \operatorname{index} X(p_i),$$

where (see, *e.g.*, Spivak [*63*](Vol. 1, pp. 373-375 and pp. 606-609))

$$\operatorname{index} X(p_i) = \frac{(m-1)!}{(n-1)!} \frac{1}{2\pi^m} \int\limits_{S_\epsilon(p_i)} \sqrt{g}\varepsilon_{jj_2j_3\ldots j_n} X^j dX^{j_2} \wedge dX^{j_3} \cdots \wedge dX^{j_n},$$

and $S_\epsilon(p_i)$ is a $(n-1)$ dimensional sphere of sufficiently small radius $\epsilon$ around the point $p_i$.

Since $\displaystyle\int_M \sqrt{g}K_n\,\nu$ is independent of $g$, we are free to pick a metric on $M$ which is flat around each zero $p_i$. Let $B_\epsilon(p_i)$ be the ball of radius $\epsilon$ around $p_i$ and let $M_\epsilon = M - \bigcup_{i=1}^k B_\epsilon(p_i)$. Then by Stoke's theorem

$$\int_M \sqrt{g}K_n\,\nu = \lim_{\epsilon\longrightarrow 0} \int_{M_\epsilon} \sqrt{g}K_n\,\nu = -\sum_{i=1}^{k} \lim_{\epsilon\longrightarrow 0} \int\limits_{S_\epsilon(p_i)} \eta. \tag{4.61}$$

Since the metric is flat around $p_i$, only the term with $r = m$ in (4.61) will contribute to $\eta$ and therefore

$$\begin{aligned} \int_M \sqrt{g}K_n\,\nu &= -c_m \sum_{i=1}^{k} \lim_{\epsilon\longrightarrow 0} \int_{S_\epsilon(p_i)} \frac{1}{n!} \sqrt{g}\varepsilon_{jj_2j_3\ldots j_n} X^j dX^{j_2} \wedge dX^{j_3} \cdots \wedge dX^{j_n} \\ &= -\frac{2\pi^m}{n(m-1)!} c_m \sum_{i=1}^{k} \operatorname{index} X(p_i) = \frac{2^n \pi^m m!}{n!} \chi(M). \end{aligned}$$

This is the celebrated Gauss-Bonnet-Chern formula.

Our objective in establishing this result was simply to focus attention on the role that the variational calculus can play in the study of characteristic forms. The first variational formula (4.22) can also be used to reproduce the calculations of Bott [*12*] and Baum and Cheeger [*9*] required to compute the Pontryagin numbers of a compact Riemannian manifold from the zeros of a Killing vector field as well as the calculations of Chern [*14*] needed to compute the Chern numbers of a complex, compact manifold from the residues of meromorphic vector fields.

**B. Minimal Weight Forms.** In this section we introduce a system of well-defined (*i.e.*, invariant under fiber preserving diffeomorphisms) weights for forms on $J^\infty(E)$ which have a polynomial dependence in the fiber variables $u^\alpha_I$ from some order on. These weights describe the distribution of derivatives in these polynomial terms. Given a $d_H$ closed form $\omega \in \Omega^{r,s}$ we construct a form $\eta$ such that $\omega = d_H \eta$ and such that the weights of $\eta$ are as small as possible. Similar results are obtained for the complex $(\mathcal{F}^*, \delta_V)$ of functional forms. In particular, the existence of minimal weight Lagrangians for locally variational source forms is established.

It is convenient to focus on horizontal forms and the complex (4.5). Once the results are established here the generalization to forms of type $(r, s)$, $s \geq 1$, is easily obtained. Again $E$ is the trivial $\mathbf{R}^m$ bundle over $\mathbf{R}^n$.

Let $C^\infty_k = C^\infty(J^k(E))$ be the ring of smooth functions on $J^k(E)$. We let $u_k$ denote all possible $k^{\text{th}}$ order derivatives of $u^\alpha$:

$$u_k \sim u^\alpha_{i_1 i_2 \cdots i_k}.$$

Let

$$\mathcal{P}_{j,k} = C^\infty_j[u_{j+1}, u_{j+2}, \dots, u_k]$$

be the polynomial ring in the variables $u_{j+1}, u_{j+2}, \dots, u_k$ with coefficients in $C^\infty_j$. A function $P \in \mathcal{P}_{j,k}$ is therefore a smooth function on $J^\infty(E)$ of order $k$ which is a sum of monomials

$$M = A\,(u_{j+1})^{a_{j+1}}(u_{j+2})^{a_{j+2}} \cdots (u_k)^{a_k}, \tag{4.62}$$

where $A \in C^\infty_j$, By convention $\mathcal{P}_{k,k} = C^\infty_k$.

Definition 4.18. *Let $\mathcal{P}_{j,k}[t]$ be the polynomial ring in the single variable $t$ with coefficients in $\mathcal{P}_{j,k}$. For each $p = j,\ j+1,\ \dots,\ k-1$, define*

$$\mathcal{W}_p \colon \mathcal{P}_{jk} \to \mathcal{P}_{jk}[t]$$

*by*

$$[\mathcal{W}_p(P)](t) = P(u_{j+1}, u_{j+2}, \dots, u_p, tu_{p+1}, t^2 u_{p+2}, \dots, t^{k-p} u_k).$$

*We call the degree of $\mathcal{W}_p(P)[t]$ as a polynomial in $t$ the $p^{\text{th}}$ weight of $P$ and denote it by $w_p(P)$.*

*For $p \geq k$, we set $\mathcal{W}_p(P)[t] = P$ and $w_p(P) = 0$.*

For example, if $P = u_{xxxy} + \cos(u_x) u_{yy} u_{xxx}$, then $P \in \mathcal{P}_{1,4}$ and

$$\begin{aligned}
\mathcal{W}_1(P)(t) &= t^3 u_{xxxy} + t^3 \cos(u_x) u_{yy} u_{xxx}, \\
\mathcal{W}_2(P)(t) &= t^2 u_{xxxy} + t \cos(u_x) u_{yy} u_{xxx}, \quad \text{and} \\
\mathcal{W}_3(P)(t) &= t u_{xxxy} + \cos(u_x) u_{yy} u_{xxx}.
\end{aligned}$$

The weight $w_p(P)$ counts the total number of derivatives in $P$ in excess of $p$. For example, the weights of the monomial $M = u_{xx}^2 u_{xyz}^3 u_{zzzz} \in \mathcal{P}_{0,4}$ are

$$\begin{aligned}
&w_0(M) = 4 + 9 + 4 = 17, && w_1(M) = 2 + 6 + 3 = 11, \\
&w_2(M) = 3 + 2 = 5, && w_3(M) = 1, \qquad \text{and} \\
&w_4(M) = 0.
\end{aligned}$$

The $p^{\text{th}}$ weight of the monomial (4.62) is

$$w_p(M) = a_{p+1} + 2a_{p+2} + \cdots + (k-p)a_k.$$

The weight $w_p(P)$ of a polynomial $P \in \mathcal{P}_{j,k}$ is the largest $p^{\text{th}}$ weight of its monomials. If $w_p(P) = 0$, then $P$ is independent of all derivatives of order $\geq p+1$ and hence $P \in \mathcal{P}_{j,p}$.

We now prove some elementary properties of these weights. Evidently, $\mathcal{W}_p$ is a ring homomorphism and consequently, for $P$ and $Q$ in $\mathcal{P}_{j,k}$,

$$w_p(P+Q) = \max\{\, w_p(P), w_p(Q) \,\} \quad \text{and} \quad w_p(PQ) = w_p(P) + w_p(Q).$$

Lemma 4.19. *The space of functions $\mathcal{P}_{j,k}(J^\infty(E))$ is an invariantly defined subspace of $C^\infty(J^\infty(E))$ and the weights $w_j, w_{j+1}, \ldots, w_{k-1}$ are numerical invariants. Thus, if $\phi\colon E \to E$ is a fiber-preserving transformation $y = y(x), v = v(x,u)$ whose prolongation $\Phi\colon J^\infty(E) \to J^\infty(E)$ transforms $Q[y,v] \in \mathcal{P}_{j,k}$ into $P[x,u] \in J^k(E)$, then $P \in \mathcal{P}_{j,k}$ and*

$$w_p(P) = w_p(Q) \qquad \textit{for} \quad p = j,\, j+1,\, \ldots,\, k-1.$$

Proof: A simple induction shows that the prolongation $\Phi\colon J^\infty(E) \to J^\infty(E)$ of $\phi$ mapping $[x,u]$ to $[y,v]$ takes the general form

$$\begin{aligned}
v^\alpha_I = &\frac{\partial x^J}{\partial y^I} \frac{\partial v^\alpha}{\partial u^\beta} u^\beta_J \\
&\qquad + \sum A^{\alpha\, K_1\, K_2 \cdots K_l}_{I\, \gamma_1\, \gamma_2 \cdots \gamma_l} u^{\gamma_1}_{K_1} u^{\gamma_2}_{K_2} \cdots u^{\gamma_l}_{K_l},
\end{aligned} \tag{4.63}$$

where $|I| = |J| = q$, where $\dfrac{\partial x^J}{\partial y^I} = \dfrac{\partial x^{j_1}}{\partial y^{i_1}}\dfrac{\partial x^{j_2}}{\partial y^{i_2}}\cdots\dfrac{\partial x^{j_q}}{\partial y^{i_q}}$, where the coefficients $A^{\cdots}_{\cdots}$ are functions of the coordinates $(x, u)$ on $E$ alone, and where the summation ranges over all multi-indices $K_1, K_2, \ldots, K_l$ such that

$$1 \leq |K_1| \leq |K_2| \leq \cdots \leq |K_l| \leq q - 1$$

and

$$|K_1| + |K_2| + \cdots |K_l| \leq q.$$

Let $P = Q \circ \Phi$. Since the right-hand side of equation (4.63) is a polynomial in the derivatives of $u$, it follows that if $Q$ is a polynomial in the derivatives $v_{j+1}, v_{j+2}, \ldots, v_k$, then $P$ is also a polynomial in the derivatives $u_{j+1}, u_{j+2}, \ldots, u_k$. This proves that $\mathcal{P}_{j,k}$ is an invariantly defined subspace of $C^\infty(J^\infty(E))$.

To prove that the weights of $P$ and $Q$ coincide, it suffices to show that the $p^{\text{th}}$ weight of $v^\alpha_I$, $|I| = q$, treated as a polynomial in the derivatives of $u$, equals the $p^{\text{th}}$ weight of $v^\alpha_I$, treated as a polynomial in the derivatives of $v$, *i.e.*, we must show that

$$w_p(v^\alpha_I[x, u]) = \begin{cases} q - p & \text{if} \quad q > p \\ 0 & \text{if} \quad q \leq p. \end{cases}$$

Suppose $q > p$. Denote one of the terms in the summation in (4.63) by

$$M = A(u_1)^{a_1}(u_2)^{a_2}\cdots(u_{q-1})^{a_{q-1}}.$$

The bounds on the lengths of the indices $K_1, K_2, \ldots, K_l$ imply that

$$a_i \geq 0 \qquad \text{and} \qquad a_1 + 2a_2 + \cdots + (q-1)a_{q-1} \leq q. \tag{4.64}$$

From these inequalities one can prove, using the fact that the maximum of a linear function defined on a convex, polygonal region is realized at a vertex, that

$$a_{p+1} + 2a_{p+2} + \cdots + (q - p - 1)a_{q-1} \leq q - p.$$

This shows that the $p^{\text{th}}$ weight of the monomial $M$ is no more that $q - p$. It actually equals $q - p$ since the $p^{\text{th}}$ weight of $u^\alpha_I$, the leading monomial in (4.63), is $q - p$ and the Jacobians $(\dfrac{\partial x^j}{\partial y^i})$ and $(\dfrac{\partial v^\alpha}{\partial u^\beta})$ are non-singular.

The case $q \leq p$ is similar. ∎

LEMMA 4.20. *Let $P \in \mathcal{P}_{j,k}$. If $w_p(P) \neq 0$, then*

$$w_{p+1}(P) \leq w_p(P) - 1.$$

*In particular, if the weights $w_j(P)$, $w_{j+1}(P)$, ..., $w_{k-1}(P)$ are non-zero, then they are a strictly decreasing sequence of integers.*

PROOF: It suffices to check this lemma for monomials

$$M = (u_{j+1})^{a_{j+1}} (u_{j+2})^{a_{j+2}} \cdots (u_k)^{a_k}$$

for which, by definition,

$$w_p(P) = \qquad (k-p)a_k + (k-p-1)a_{k-1} + \cdots + 2a_{p+2} + a_{p+1}$$

and

$$w_{p+1}(P) = (k-p-1)a_k + (k-p-2)a_{k-1} + \cdots + \ \ a_{p+2}$$

The assumption that $w_p(P) \neq 0$ requires that $k > p$ and that one of the exponents $a_l \geq 1$ for $l > p$. Hence

$$w_p(M) - w_{p+1}(M) = a_k + a_{k-1} + \cdots a_{p+1} \geq 1.$$

∎

For each $p \geq 0$, we introduce local, *non*-invariant differential operators

$$(D_p^1)_i = \frac{\partial}{\partial x^i} + \sum_{|J|=0}^{p-1} u^\alpha_{Ji}\, \partial^J_\alpha$$

and

$$(D_p^2)_i = \sum_{|J|=p}^{\infty} u^\alpha_{Ji}\, \partial^J_\alpha.$$

The sum of these two operators is the total derivative operator

$$D_i = (D_p^1)_i + (D_p^2)_i.$$

We shall suppress the indices in these equations and write $D$ for $D_i$ and $\partial^{|J|}$ for $\partial^J_\alpha$.

LEMMA 4.21. *Let $P \in \mathcal{P}_{j,k}$. Then $DP \in \mathcal{P}_{j,k+1}$,*

$$\mathcal{W}_p(DP)(t) = D_p^1[\mathcal{W}_p(P)(t)] + tD_p^2[\mathcal{W}_p(P)(t)], \tag{4.65}$$

*and the weights of $DP$ increase over those of $P$ by one, i.e.,*

$$w_p(DP) = w_p(P) + 1 \qquad \text{for} \quad p = j,\, j+1,\, \ldots,\, k.$$

PROOF: From the chain rule we deduce immediately that

$$\begin{aligned}
\frac{\partial}{\partial x}[\mathcal{W}_p(P)(t)] &= \mathcal{W}_p(\frac{\partial P}{\partial x})(t), \\
\partial^l[\mathcal{W}_p(P)(t)] &= \mathcal{W}_p(\partial^l P)(t) \qquad &&\text{for} \quad l \le p, \text{ and} &&(4.68)\\
\partial^l[\mathcal{W}_p(P)(t)] &= t^{l-p}\mathcal{W}_p(\partial^l P)(t) \qquad &&\text{for} \quad l \ge p+1. &&(4.69)
\end{aligned}$$

We substitute these formulas into the right-hand side of (4.65) to arrive at

$$\begin{aligned}
D_p^1&[\mathcal{W}_p(P)(t)] + tD_p^2[\mathcal{W}_p(P)(t)] \\
&= \mathcal{W}_p(D_p^1 P)(t) + t\sum_{l=p}^{k} u_{l+1}\partial^l[\mathcal{W}_p(P)(t)] \\
&= \mathcal{W}_p(D_p^1 P)(t) + \sum_{l=p}^{k} t^{l+1-p}u_{l+1}\mathcal{W}_p(\partial^l P)(t) \\
&= \mathcal{W}_p(D_p^1 P)(t) + \mathcal{W}_p(D_p^2 P)(t) \\
&= \mathcal{W}_p(DP),
\end{aligned}$$

as required. ∎

LEMMA 4.22. *Let $P \in \mathcal{P}_{j,k}$ and let $E(P)$ be the Euler-Lagrange function computed from $P$. Then $E(P) \in \mathcal{P}_{j,l}$ for some $l \le 2k$ and the weights of $E(P)$ are bounded by those of $P$ according to*

$$w_p\big(E(P)\big) \le w_p(P) + p \qquad \text{for} \quad p = j,\, j+1,\, \ldots,\, k-1\ . \tag{4.71a}$$

*and*

$$w_p\big(E(P)\big) \le 2k - p \qquad \text{for} \quad p = k,\, k+1,\, \ldots,\, l-1. \tag{4.71b}$$

Proof: These inequalities follow from (4.65), (4.68) and (4.69) which, in the case of the Euler-Lagrange operator, leads to

$$\begin{aligned}\mathcal{W}_p\big(E(\lambda)\big)(t) = \sum_{i=0}^{k} &(-D_p^1 - tD_p^2)_i\{\partial^i[\mathcal{W}_p(P)(t)]\} \\ &+ \sum_{i=p+1}^{k} t^{p-i}(-D_p^1 - tD_p^2)_i\{\partial^i[\mathcal{W}_p(P)(t)]\}.\end{aligned} \tag{4.72}$$

For $p = j,\, j+1,\, \ldots,\, k-1$, the degree of the $i^{\text{th}}$ term in the first summation is

$$\deg\big[(-D_p^1 - tD_p^2)_i\{\partial^i[\mathcal{W}_p(P)(t)]\}\big] \le i + w_p(P) \le w_p(P) + p$$

while the degree of the $i^{\text{th}}$ term in the second summation is

$$\deg\big[t^{p-i}(-D_p^1 - tD_p^2)_i\{\partial^i[\mathcal{W}_p(P)(t)]\}\big] \le (p-i) + i + w_p(P) = w_p(P) + p.$$

This establishes (4.71a).

For $p = k$, the second summation in (4.72) is absent. Since $\mathcal{W}_p(P) = P$, the degree of the $i^{\text{th}}$ term in the first summation is $i \le k$. This proves (4.71b) for $p = k$. For $p > k$, (4.71b) follows from Lemma 4.20. ∎

It is a straightforward matter to check that the bounds on the weights of $DP$ and $E(P)$ given in Lemmas 4.21 and 4.22 are sharp.

Let $\Omega^{r,0}_{\mathcal{P}_{j,k}}(J^\infty(E))$ be the space of horizontal $r$ forms on $J^\infty(E)$ with coefficients in $\mathcal{P}_{j,k}(J^\infty(E))$. We extend the definition of $\mathcal{W}_p$ from $\mathcal{P}_{j,k}$ to $\Omega^{r,0}_{\mathcal{P}_{j,k}}$ by the action of $\mathcal{W}_p$ on the coefficients. If $\omega \in \Omega^{r,0}_{\mathcal{P}_{j,k}}$, then the $p^{\text{th}}$ weight $w_p(\omega)$ is the degree of the polynomial $\mathcal{W}_p(\omega)(t)$. The weight of $\omega$ is the largest weight of its coefficients. Lemma 4.21 asserts that $d_H\omega \in \Omega^{r,0}_{\mathcal{P}_{j,k+1}}$ and that

$$w_p(d_H\omega) \le w_p(\omega) + 1 \qquad \text{for } p = j,\, j+1,\, \ldots,\, k.$$

Theorem 4.23. *For $r = 1, 2, \ldots, n$ and $0 \le j < k$, let $\omega \in \Omega^{r,0}_{\mathcal{P}_{j,k}}$ and suppose that $d_H\omega = 0$ if $r < n$ or that $E(\omega) = 0$ if $r = n$. Then there exists a form $\eta \in \Omega^{r-1,0}_{\mathcal{P}_{j,k}}$ with weights*

$$w_p(\eta) = w_p(\omega) - 1 \qquad \text{for} \quad p = j,\, j+1,\, \ldots,\, k-1$$

*such that*

$$\omega = d_H \eta.$$

PROOF: Actually, we must prove slightly more, *viz.*, if $\omega$ depends polynomially ( or smoothly) on additional parameters, then so does $\eta$. We proceed by induction on $r$ so that there are two parts to the proof. First we assume that the theorem is true for $r-1$ forms, where $2 \le r \le n$, and prove it true for $r$ forms. Second, a simple modification of the general argument shows that the theorem holds for one forms.

Let $\omega$ be an closed $r$ form on $J^\infty(E)$ with coefficients in $\mathcal{P}_{j,k}$. These coefficients may depend polynomially or smoothly on other parameters. Using the homotopy operator of §4A we find that there is an horizontal $r-1$ form $\eta$ such that

$$\omega = d_H \eta. \tag{4.73}$$

It is clear from the homotopy formula that $\eta$ is polynomial in derivatives of order $> j$ so that $\eta \in \Omega^{r-1,0}_{\mathcal{P}_{j,l}}$ for some $l > k-1$. It is also clear that $\eta$ will depend parameters in the same fashion as $\omega$.

The induction hypothesis on $r$ is used to establish the following lemma.

LEMMA 4.24. *Fix $p$ in the range $j \le p \le l-1$. Then there is an horizontal $r-1$ form $\eta^* \in \Omega^{r-1,0}_{\mathcal{P}_{j,l}}$ with weights*

$$w_p(\eta^*) = \begin{cases} w_p(\eta) - 1, & \text{if} \quad w_p(\omega) \ge 1 \\ 0, & \text{if} \quad w_p(\omega) = 0 \end{cases}$$

*and*

$$w_q(\eta^*) \le w_q(\eta) \qquad \text{for} \quad q = p+1, \ldots, l-1$$

*and such that*

$$\omega = d_H \eta^*. \tag{4.74}$$

*In other words, for each fixed $p$, the form $\eta$ in (4.73) can be replaced by a form $\eta^*$ where the $p^{\text{th}}$ weight $w_p(\eta^*)$ can be minimized without increasing the weights $w_{p+1}(\eta^*)$, $w_{p+2}(\eta^*)$, $\ldots$ over the initial values $w_{p+1}(\eta)$, $w_{p+2}(\eta)$, $\ldots$ .*

*The form $\eta^*$ will also depend polynomially (or smoothly) on parameters.*

PROOF: The proof of this lemma is based upon arguments first presented in Olver [54].

Let $a = w_p(\omega)$ and let $b = w_p(\eta)$. We can assume that $b \ge a$ or that $b \ge 1$ if $a = 0$ since, otherwise, there is nothing to prove — the lemma holds with $\eta = \eta^*$. To begin, we isolate the terms in $\eta$ of maximum $p^{\text{th}}$ weight by writing

$$\eta = \alpha + \beta, \tag{4.75}$$

where $\mathcal{W}_p(\alpha)(t) = t^b\alpha$ and $w_p(\beta) < b$. The form $\alpha$ can be explicitly computed from the formula

$$\alpha = \frac{1}{b!}\frac{d^b}{dt^b}[\mathcal{W}_p(\eta)(t)] \tag{4.76}$$

This shows that, for each $q = p,\ p+1,\ \ldots,\ l-1$,

$$w_q(\alpha) \leq w_q(\eta)$$

and consequently

$$w_q(\beta) \leq w_q(\eta).$$

Now apply the weight map $\mathcal{W}_p$ to (4.73) and invoke Lemma 4.21 to conclude that

$$\begin{aligned}\mathcal{W}_p(\omega)(t) &= \mathcal{W}_p(d_H\eta)(t)\\ &= D_p^1[\mathcal{W}_p(\eta)(t)] + tD_p^2[\mathcal{W}_p(\eta)(t)]\\ &= t^{b+1}D_p^2\alpha + \{\text{terms of degree at most } b \text{ in } t\}.\end{aligned}$$

Since the left-hand side of this equation is a polynomial in $t$ of degree $a \leq b$, this implies that

$$D_p^2\alpha = 0. \tag{4.77}$$

The key step in the proof is to use the induction hypothesis to analyze this equation. Pick a point $(y, v, v_1, \ldots, v_{p-1})$ in $J^{p-1}(J^\infty(E))$ and define a new horizontal $r-1$ form $\gamma$ with coefficients in $\mathcal{P}_{j,l}$ and depending on $(y, v, v_1, \ldots, v_{p-1})$ as parameters by fixing the dependence of $\alpha$ on $x$, $u$, $u_1$, $\ldots$, $u_{p-1}$ at $x = y$, $u = v$, $u_1 = v_1$, $\ldots$ $u_{p-1} = v_{p-1}$, *i.e.*, set

$$\gamma(x, u, u_1, \ldots, u_l, y, v, v_1, \ldots, v_{p-1}) = \alpha(y, v, v_1, \ldots, v_{p-1}, u_p, u_{p+1} \ldots, u_l). \tag{4.78}$$

Clearly $\gamma \in \Omega^{r-1,0}_{\mathcal{P}_{j,l}}$ and

$$w_q(\gamma) = w_q(\alpha) \tag{4.79}$$

for $q = p,\ p+1,\ \ldots,\ l-1$. Furthermore, $\gamma$ is a polynomial in the parameters $v_{j+1}$, $v_{j+2}$, $\ldots$, $v_{p-1}$ and varies smoothly in the parameters $y$, $v$, $v_1$, $\ldots$, $v_j$.

Equation (4.77) implies that the $r-1$ form $\gamma$ is closed:

$$d_H\gamma = 0.$$

The induction hypothesis now applies to $\gamma$. Consequently there is a horizontal $r-2$ form $\sigma$ such that

$$\gamma = d_H \sigma, \tag{4.80}$$

where

$$\sigma = \sigma(x, u, u_1, \ldots, u_l, y, v, v_1, \ldots, v_{p-1})$$

belongs to $\Omega^{r-2,0}_{\mathcal{P}_{j,l}}$, is a polynomial in $v_{j+1}$, $v_{j+2}$, $\ldots$, $v_{p-1}$ and varies smoothly in $y$, $v$, $v_1$, $\ldots$, $v_j$. The form $\sigma$ also has the appropriate dependency on any additional parameters on which $\gamma$ may depend. Most importantly, the induction hypothesis also insures that $\sigma$ is a minimal weight form, *i.e.*,

$$w_q(\sigma) = w_q(\gamma) - 1 \tag{4.81}$$

for $q = p$, $p+1$, $\ldots$, $l-1$.

Put

$$\tau(x, u, u_1, \ldots, u_l) = \sigma|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots,\ v_{p-1}=u_{p-1})}\,. \tag{4.82}$$

Then, owing to (4.78) and (4.80), we obtain

$$\begin{aligned}
\alpha &= \gamma|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots\ )} = [d_H\sigma]|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots\ )} \\
&= dx \wedge \big[\frac{\partial\sigma}{\partial x} + \frac{\partial\sigma}{\partial u}u_1 + \frac{\partial\sigma}{\partial u_1}u_2 + \frac{\partial\sigma}{\partial u_l}u_{l+1}\,\big]\big|_{(y=x, v=u,\ v_1=u_1,\ \ldots\ )} \\
&= dx \wedge \big[\frac{\partial}{\partial x}(\,\sigma|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots\ )} + \frac{\partial}{\partial u}(\,\sigma|_{(y=x,\ v=u, v_1=u_1,\ \ldots\ )}\,u_1 + \cdots\big] \\
&\qquad - dx \wedge \big[\frac{\partial\sigma}{\partial y} + \frac{\partial\sigma}{\partial v}u_1 + \frac{\partial\sigma}{\partial v_1}u_2 + \cdots\,\big]\big|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots\ )}
\end{aligned}$$

*i.e.*,

$$\alpha = d_H\tau - \mu, \tag{4.83}$$

where

$$\mu = dx \wedge \big[\frac{\partial\sigma}{\partial y} + \frac{\partial\sigma}{\partial v}u_1 + \cdots + \frac{\partial\sigma}{\partial v_{p-1}}u_p + \cdots\,\big]\big|_{(y=x,\ v=u,\ v_1=u_1,\ \ldots,\ v_{p-1}=u_{p-1})}$$

Note that

$$w_q(\mu) \le w_q(\sigma)$$

for $q = p,\ p+1,\ \ldots,\ l-1$, and therefore, in view of (4.79)and (4.81),

$$w_q(\mu) \le w_q(\alpha) - 1. \tag{4.84}$$

On account of (4.75) and (4.83) we can now write

$$\eta = d_H\tau + \eta^*,$$

where

$$\eta^* = \beta - \mu \tag{4.85}$$

and hence, because of (4.73),

$$\omega = d_H\eta^*.$$

The weights of $\eta^*$ are

$$w_p(\eta^*) \le \max\{\, w_p(\beta), w_p(\mu)\,\} = b - 1$$

and, for $q = p+1,\ p+2,\ \ldots,\ l-1$,

$$w_q(\eta^*) \le \max\{\, w_q(\beta), w_q(\mu)\,\} \le w_q(\eta).$$

In summary, if $\omega = d_H\eta$ and $w_p(\eta)$ is larger than the minimum possible $p^{\text{th}}$ weight, then $\eta$ is equivalent to a new form $\eta^*$ with strictly smaller $p^{\text{th}}$ weight and no larger $q^{\text{th}}$ weights, $q = p+1,\ p+2,\ \ldots,\ l-1$. This argument can be repeated until a form $\eta^*$ with minimum $p^{\text{th}}$ weight is obtained. This proves the lemma. ∎

The proof of Theorem 4.23 can now be completed by repeated use of the lemma, first with $p = l-1$, then with $p = l-2$ and so on until $p = j$. At each step, we have

$$\omega = d_H\eta$$

with weights

$$w_q(\eta) = \begin{cases} w_q(\omega) - 1, & \text{or} \\ 0, & \text{if} \quad w_q(\omega) = 0 \end{cases} \tag{4.86}$$

for $q = p,\ p+1,\ \ldots,\ l-1$. Due to Lemma 4.21, these are the minimal weights possible. With $p = j$, this establishes the theorem for $l$ forms, $l \ge 2$.

Finally, to check the case $l = 1$, we observe that $\alpha$, as define by (4.75), is now a function and consequently (4.77) implies directly that

$$\alpha = \alpha(x, u, u_1, \ldots, u_{p-1}).$$

This immediately reduces the $p^{\text{th}}$ weight of the function $\eta$ in (4.75) without changing the higher weights. The lemma therefore holds for one forms. The theorem, for $l = 1$, again follows from the lemma as above. ∎

Let $\omega$ be a $d_H$ closed, horizontal one form. The function $f$ for which $\omega = d_H f$ is unique up to an additive constant and therefore a minimal weight function $f$ can be computed using the horizontal homotopy operator, $f = h_H^{1,0}(\omega)$.

EXAMPLE 4.25. A concrete example helps to clarify the proof of Theorem 4.23 and to illustrate the constructive nature of the argument. Let $E : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ and consider the variational trivial Lagrangian

$$\lambda = 2u_{yyy}u_{xxxx}\, dx \wedge dy. \tag{4.87}$$

The coefficient of $\lambda$ belongs to $\mathcal{P}_{0,4}$ and the weights are

$$w_0(\lambda) = 7, \quad w_1(\lambda) = 5, \quad w_2(\lambda) = 3, \quad \text{and} \quad w_3(\lambda) = 1.$$

According to the theorem, it is possible to write

$$\lambda = d_H \eta,$$

where $\eta \in \Omega^{1,0}_{\mathcal{P}_{0,3}}$ and the weights of $\eta$ are

$$w_0(\eta) = 6, \quad w_1(\lambda) = 4, \quad \text{and,} \quad w_2(\eta) = 2.$$

To construct the form $\eta$, we first use the horizontal homotopy operator (4.32) to write $\lambda = \eta_0$, where

$$\begin{aligned}\eta_0 = (&-uu_{xxxxyy} + u_y u_{xxxxy} - u_{yy}u_{xxxx})\, dx \\ &+ (-uu_{xxxyyy} + u_x u_{xxyyy} - u_{xx}u_{xyyy} + u_{xxx}u_{yyy})\, dy.\end{aligned}$$

The weights of $\eta_0$ are far from minimal:

$$\begin{aligned}&w_0(\eta_0) = 6, \quad w_1(\eta_0) = 5, \quad w_2(\eta_0) = 4, \quad w_3(\eta_0) = 3, \\ &w_4(\eta_0) = 2, \quad w_5(\eta_0) = 1, \quad w_6(\eta) = 0.\end{aligned}$$

To begin, we find a form $\eta_1$, equivalent to $\eta_0$, but with $w_5(\eta_1) = 0$. Write

$$\eta_0 = \alpha_0 + \beta_0,$$

where

$$\alpha_0 = (-uu_{xxxxyy})\, dx + (-uu_{xxxyyy})\, dy$$

consists of all those terms in $\eta$ with $w_5 = 1$ and $\beta$ consists of the remaining terms,

$$\beta_0 = (u_y u_{xxxxy} - u_{yy} u_{xxxx})\, dx + (u_x u_{xxyyy} - u_{xx} u_{xyyy} + u_{xxx} u_{yyy})\, dy.$$

In accordance with (4.78), we let

$$\gamma_0 = (-v u_{xxxxyy})\, dx + (-v u_{xxxyyy})\, dy.$$

This form is $d_H$ closed (bear in mind that $v$ is a independent parameter here and so $d_H v = 0$) and has weights $w_q(\gamma_0) = 6 - q$, $q = 0, \ldots, 6$. A minimal order form ( in this case function) $\sigma_0$ for which $d_H \sigma_0 = \gamma_0$ is

$$\sigma_0 = -v u_{xxxyy}.$$

In accordance with (4.82), put

$$\tau_0 = -u u_{xxxyy}.$$

Then

$$d_H \tau_0 + (u_x u_{xxxyy})\, dx + (u_y u_{xxxyy})\, dy = \alpha_0$$

and so we can replace $\eta_0$ by

$$\begin{aligned} \eta_1 = {} & (u_x u_{xxxyy} + u_y u_{xxxxy} - u_{yy} u_{xxxx})\, dx \\ & + (u_y u_{xxxyy} + u_x u_{xxyyy} - u_{xx} u_{xxyyy} + u_{xxx} u_{yyy})\, dy. \end{aligned}$$

The weights of this form are

$$\begin{aligned} w_1(\eta_1) = 6, \quad & w_1(\eta_1) = 2, \quad w_2(\eta_1) = 3 \\ w_3(\eta_1) = 2, \quad & w_4(\eta_1) = 1, \quad\ \ w_5(\eta) = 0. \end{aligned}$$

We repeat this process again, this time to reduce the weight $w_4$ to zero. We write

$$\eta_1 = \alpha_1 + \beta_1,$$

where

$$\alpha_1 = (u_x u_{xxxyy} + u_y u_{xxxxy})\, dx + (u_y u_{xxxyy} + u_x u_{xxyyy})\, dy.$$

The form

$$\gamma_1 = (v_x u_{xxxyy} + v_y u_{xxxxy})\, dx + (v_y u_{xxxyy} + v_x u_{xxyyy})\, dy$$

is $d_H$ closed and a minimum order function for $\gamma_1$ is provided by

$$\sigma_1 = v_x u_{xxyy} + v_y u_{xxxy}.$$

Since

$$\begin{aligned} \alpha_1 =& d_H(u_x u_{xxyy} + u_y u_{xxxy}) - (u_{xx} u_{xxyy} + u_{xy} u_{xxxy})\,dx \\ &- (u_{xy} u_{xxyy} + u_{yy} u_{xxxy})\,dy \end{aligned}$$

we can replace $\eta_1$ by the form

$$\begin{aligned} \eta_2 = &(-u_{xx} u_{xxyy} - u_{xy} u_{xxxy} - u_{yy} u_{xxxx})\,dx \\ &+ (-u_{xy} u_{xxyy} - u_{xx} u_{xyyy} - u_{yy} u_{xxxy} + u_{xxx} u_{yyy})\,dy. \end{aligned}$$

The weights of this form are

$$\begin{aligned} &w_0(\eta_2) = 6, \quad w_1(\eta_2) = 4, \quad w_2(\eta_2) = 2, \\ &w_3(\eta_2) = 1, \quad w_4(\eta_2) = 0. \end{aligned}$$

Finally we reduce the weight $w_3$ to zero. Put

$$\eta_2 = \alpha_2 + \beta_2,$$

where

$$\begin{aligned} \alpha_2 = &(-u_{xx} u_{xxyy} - u_{xy} u_{xxxy} - u_{yy} u_{xxxx})\,dx \\ &+ (-u_{xy} u_{xxyy} - u_{xx} u_{xyyy} - u_{yy} u_{xxxy})\,dy. \end{aligned}$$

Since

$$\begin{aligned} &d_H(-u_{xx} u_{xyy} - u_{xy} u_{xxy} - u_{yy} u_{xxx}) \\ &\qquad + (2u_{xxx} u_{xyy} + u_{xxy}^2)\,dx + (u_{xxx} u_{yyy} + 2u_{xxy} u_{xyy})\,dy = \alpha_2 \end{aligned}$$

the form $\eta_2$ is equivalent to

$$\eta_3 = (2u_{xxx} u_{xyy} + u_{xxy}^2)\,dx + (2u_{xxy} u_{xyy} + 2u_{xxx} u_{yyy})\,dy.$$

This is a minimal weight form. ∎

Let $G$ be the group of affine-linear, fiber-preserving diffeomorphisms introduced at the end of the previous section. A map $\psi \in G$ if

$$\psi(x, u) = (y, v) = (Ax + x_0, Bu).$$

Let $\Psi$ be the prolongation of $\psi$ to $J^\infty(E)$. A form $\omega \in \Omega^{r,s}$ is said to a relative $\Psi$ invariant with character $\chi$ if $\Psi^*\omega = \chi\omega$.

LEMMA 4.26. *The pullback $\Psi^*$ of the prolongation of any $\psi \in G$ commutes with the weight maps $\mathcal{W}_p$, i.e., if $\omega \in \Omega^{r,0}_{\mathcal{P}_{j,k}}$, then*

$$\mathcal{W}_p\big(\Psi^*(\omega)\big)(t) = \Psi^*\big(\mathcal{W}_p(\omega)(t)\big) \qquad \textit{for} \quad p = j,\, j+1,\, \ldots,\, k-1. \tag{4.88}$$

PROOF: Since $\mathcal{W}_p$ acts on forms by its action on coefficients, it suffices to check the validity of (4.88) for functions. Let $P[y, v] \in \mathcal{P}_{j,k}$. With $\Psi$ given by (4.36), we find that

$$\begin{aligned} \mathcal{W}_p\big(\Psi(P)\big)(t) &= \mathcal{W}_p\big(P(\Psi[x, u])\big)(t) \\ &= \mathcal{W}_p\big(P(Ax + x_0, Bu, BCu_1, \ldots, BC^k u_k)\big)(t) \\ &= P(Ax + x_0, Bu, \ldots, BC^p u_p, BC^p(tu_{p+1}), \ldots, BC^{k-p}(t^{k-p}u_k)) \end{aligned} \tag{4.89}$$

while

$$\begin{aligned} \Psi^*\big(\mathcal{W}_p(P)(t)\big) &= \Psi^*\big(P(y, v, v_1, \ldots, v_p, tv_{p+1}, \ldots, t^{k-p}v_k)\big) \\ &= P(Ax + x_0, Bu, \ldots, BC^p u, t(BC^{p+1}u_{p+1}), \ldots, t^{k-p}(BC^k u_k)). \end{aligned} \tag{4.90}$$

Since scalar multiplication of $u_q$ by $t$ commutes with matrix multiplication by $B$ and $C = A^{-1}$, the arguments in (4.89) and (4.90) coincide. This proves (4.88) for functions. ∎

A simple example shows that Lemma 4.26 does not hold for more general maps $\psi$. Let

$$y = x \quad \text{and} \quad v = \frac{1}{2}u^2 + x$$

Then the second prolongation of this map is

$$v_y = uu_x + 1 \quad \text{and} \quad v_{yy} = uu_{xx} + u_x^2$$

so that

$$\Psi^*\big(\mathcal{W}_1(v_{yy})(t)\big) = \Psi^*(tv_{yy}) = t(uu_{xx} + u_x^2)$$

while

$$\mathcal{W}_1\big(\Psi^*(v_{yy})\big) = \mathcal{W}_1(uu_{xx} + u_x^2) = tuu_{xx} + u_x^2.$$

In fact, it is not too difficult to prove that if $\psi\colon E \to E$ is a map for which $\Psi$ commutes with $\mathcal{W}_p$ for all $p$, then $\psi$ is a affine linear transformation on both the base and the fiber, *i.e.*,

$$\psi(x, u) = (Ax + x_0, Bu + u_0).$$

COROLLARY 4.27. *Let $\psi \in G$ and let $\Psi$ be the prolongation of $\psi$ to $J^\infty(E)$. Suppose, in addition to the hypothesis of Theorem 4.23, that $\omega$ is a relative $\Psi$ invariant with character $\chi$. Then there exists a minimal weight form $\eta$ which is also a relative $\Psi$ invariant with character $\chi$ and a form $\rho$ on the base space $M$ such that*

$$\omega = d_H\eta + \rho.$$

PROOF: It suffices to check that the various forms introduced in the proof of Theorem 4.23 are $\Psi$ invariant. By Corollary 4.9, there is a relative $\Psi$ invariant form $\eta$ and a form $\rho$ on $M$ such that

$$\omega = d_H\eta + \rho.$$

From Lemma 4.26 and (4.76), we can deduce that the form $\alpha$ in the decomposition (4.75) is a relative $\Psi$ invariant. Consequently, the form $\beta$ in the decomposition (4.75) is also a relative invariant.

We now extend the action of the group $G$ to the space of parameters and let $G$ act on $(y, v, v_1, \ldots, v_{p-1})$ in the obvious fashion, *viz.*,

$$\Psi(y, v, v_1, \ldots, v_{p-1}) = (Ay + x_0, Bv, BCv, \ldots, BC^{p-1}v_{p-1}).$$

With this definition of the group action, it is easily checked that the form $\gamma$, as defined by (4.78), is a relative invariant. The induction hypothesis implies that the form $\sigma$ is a relative invariant. We then check that $\tau$, as defined by (4.82), is a relative invariant in which case it follows immediately that $\eta^*$ is a relative invariant.

To check the validity of the corollary for one forms, we simply observe that the minimal weight form $\eta$ coincides with the form obtained from the horizontal homotopy which, as we have already seen, is a relative $\Psi$ invariant. ∎

EXAMPLE 4.28. Corollary 4.27 is particularly useful in the case of scaling transformations. Consider the variationally trivial Lagrangian

$$\lambda = (u_{xxx} + \frac{1}{2}u_x^3)(u_{xt} - \sin u)\, dx \wedge dt$$

which arises from the distinguished, generalized symmetry (4.40) of the sine-Gordon equation. In the previous section we used the horizontal homotopy operator to find the conservation law $\eta$ associated to this symmetry. Now we use Corollary 4.27. The weights of the Lagrangian $\lambda$ are

$$w_0(\lambda) = 5, \quad w_1(\lambda) = 3, \quad w_2(\lambda) = 1$$

and hence the weights of a minimal order form $\eta = P\, dt - Q\, dx$ are

$$w_0(\eta) = 4, \quad w_1(\eta) = 2, \quad w_2(\eta) = 0.$$

From this weight information we can infer that the coefficients $P$ and $Q$ are linear combinations, with coefficients that are functions of $(x, t, u)$, of the following terms

$$\begin{array}{cccccc}
u_{xx}^2 & u_{xt}^2 & u_{tt}^2 & u_{xx}u_{xt} & u_{xx}u_{tt} & u_{xt}u_{tt} \\
u_{xx}u_x^2 & u_{xx}u_xu_y & u_{xx}u_t^2 & u_{xx}u_t & u_{xx}u_x & \\
u_{xt}u_x^2 & u_{xt}u_xu_y & u_{xt}u_t^2 & u_{xt}u_t & u_{xt}u_x & \\
u_{tt}u_x^2 & u_{tt}u_xu_y & u_{tt}u_t^2 & u_{tt}u_t & u_{tt}u_x & \\
u_x^4 & u_x^3u_t & u_x^2u_t^2 & u_xu_t^3 & u_t^4 & \\
u_x^3 & & u_x^2u_t & u_xu_t^2 & u_t^3 & \\
u_x^2 & u_xu_t & u_t^2 & u_x & u_t & 1.
\end{array}$$

Obviously, the application of the method of undetermined coefficients at this point would be, at best, unwieldy.

To shorten the above list of terms, we observe that under the transformation

$$x \to \frac{1}{\epsilon}x, \quad \text{and} \quad t \to \epsilon t$$

the two form $\lambda$ transforms as a relative invariant with character $\chi = \epsilon^3$. We can therefore assume that $\eta$ is a relative invariant with the same character. This implies that $P$ is a relative invariant with character $\chi = \epsilon^2$, and that that $Q$ is a relative invariant with character $\chi = \epsilon^4$. The possibilities for these coefficients are therefore narrowed to

$$\begin{aligned} P = p_1 u_{xx}u_{xt} + p_2 u_{xx}u_xu_t + p_3 u_{xt}u_x^2 + p_4 u_{xx} \\ + p_5 u_x^3 + p_6 u_x^2, \end{aligned}$$

and

$$Q = q_1 u_{xx}^2 + q_2 u_{xx} u_x^2 + q_3 u_x^4.$$

Furthermore, because $\lambda$ is translationally invariant in $x$ and $t$ we assume that the coefficients $p_i$ and $q_i$ are functions of $u$ alone. It is now a relatively straightforward matter to substitute into the equation $d_H \eta = \lambda$ and determine that

$$\begin{aligned} P &= u_{xx} u_{tt} - \sin u\, u_{xx} + \frac{1}{2} \cos u\, u_x^2, \qquad \text{and} \\ Q &= -\frac{1}{2} u_{xx}^2 + \frac{1}{8} u_x^4. \end{aligned}$$

This is the usual form of the conservation law for the Sine-Gordon equation associated to to generalized symmetry (4.40). ∎

We now turn to the counterpart of Theorem 4.23 for locally variational source forms. Let $\Delta \in \mathcal{F}^1$ be a source form with coefficients in $\mathcal{P}_{j,k}$, $0 \le j < k$, and with weight $w_{k-1}(\Delta) \ge 0$. Since the weights $w_j(\Delta)$, $w_{j+1}(\Delta)$, ..., $w_{k-1}(\Delta)$, $w_k(\Delta) = 0$ are, by Lemma 4.20, strictly decreasing there exists a *first* integer $l$ such that

$$w_l(\Delta) \le l. \tag{4.91}$$

It is not difficult to see, again by Lemma 4.20, that $l$ lies in the range $[\frac{k}{2}] \le l \le k$. For example, if the coefficient of $\Delta$ is $u_2 u_3^2$ then $l = 2$. If the coefficient of $\Delta$ is $u_3^3$, then $l = 3$.

Theorem 4.29. *Let $\Delta$ be a locally variational source form with coefficients in $\mathcal{P}_{jk}$ and let $l$ be defined by (4.91). Then there is a Lagrangian $\lambda$ for $\Delta$ with coefficient in $\mathcal{P}_{j,l}$ and with weights*

$$w_p(\lambda) = w_p(\Delta) - p$$

*for $p = j, j+1, \ldots, l-1$.*

Proof: The argument is essentially a repetition of that used in Theorem 4.23. The homotopy $\mathcal{H}^1$ provides us with a Lagrangian $\lambda \in \Omega^{n,0}_{\mathcal{P}_{j,k}}$ for $\Delta$. Now fix $p$, $j \le p \le k-1$. We construct an equivalent Lagrangian $\lambda^*$ such that

$$w_p(\lambda^*) = \begin{cases} w_p(\Delta) - p & \text{if} \quad j \le p \le l-1, \text{ or} \\ 0 & \text{if} \quad l \le p \le k-1, \end{cases}$$

and

$$w_q(\lambda^*) \le w_q(\lambda) \qquad \text{for } q = p+1, p+2, \ldots, k-1.$$

The theorem can then be proved by downward induction on $p$, starting with $p = k-1$ and ending with $p = j$.

Let $w_p(\Delta) = a$ and $w_p(\lambda) = b$. If $p$ is in the range $j \leq p \leq l-1$, we suppose that $b > a - p$; if $p$ is in the range $l \leq p \leq k-1$, we suppose that $b > 0$. Because $w_l(\Delta) \leq l$, it follows that for $p$ in this latter range,

$$p \geq l \geq w_l(\Delta) \geq w_p(\Delta) = a$$

Hence, regardless of the value of $p$, our suppositions lead to the inequality

$$b + p > a.$$

Decompose $\lambda$ into the sum

$$\lambda = \alpha + \beta, \tag{4.92}$$

where $\mathcal{W}_p(\alpha)(t) = t^b\alpha$ and $\mathcal{W}(\beta)(t)$ is a polynomial in $t$ of degree $< b$. Now evaluate $\mathcal{W}_p(\Delta)(t)$ using (4.72) to arrive at

$$\begin{aligned}\mathcal{W}_p(\Delta)(t) &= \mathcal{W}_p\bigl(E(\lambda)\bigr)\\ &= \sum_{i=0}^{p}(-D_p^1 - tD_p^2)_i\{\partial^i[t^b\alpha + \mathcal{W}_p(\beta)(t)]\}\\ &\qquad + \sum_{i=p+1}^{k} t^{p-i}(-D_p^1 - tD_p^2)_i\{\partial^i[t^b\alpha + \mathcal{W}_p(\beta)(t)]\}\\ &= t^{b+p}\bigl[(-D_p^2)_{p+1}\partial^{p+1}\alpha + (-D_p^2)_{p+2}\partial^{p+2}\alpha + \cdots + (-D_p^2)_k\partial^k\alpha\bigr]\\ &\qquad + \{\,\text{terms of degree} < b+p \text{ in } t\}.\end{aligned}$$

Since $\mathcal{W}_p(\Delta)(t)$ is a polynomial in $t$ of degree $a < b + p$, this implies that

$$(-D_p^2)_{p+1}(\partial^{p+1}\alpha) + (-D_p^2)_{p+2}(\partial^{p+2}\alpha) + \cdots + (-D_p^2)_k(\partial^k\alpha) = 0.$$

Define the $n$ form $\gamma$ as in the proof of Theorem 4.23 by fixing the dependency of $\alpha$ on $(x, u, u_1, \ldots, u_{p-1})$ at $(y, v, v_1, \ldots, v_{p-1})$. This last equation gives rise to

$$E(\gamma) = 0.$$

By Theorem 4.23, there is a minimal order form $\sigma$ such that $\gamma = d_H\sigma$. We can now continue with precisely the same arguments used in the proof of Theorem 4.23, starting from (4.80), to complete the proof of Theorem 4.29 . ∎

A complete characterization of the image of the Euler-Lagrange operator on finite order jet bundles is now possible. Let

$$\Omega_k^{n,0} = \Omega^{n,0} \cap \Omega^n(J^k(E))$$

and let

$$\mathcal{F}_l^1 = \mathcal{F}^1 \cap \Omega^{n+1}(J^l(E)).$$

COROLLARY 4.30. *Let $\mathcal{E}_k$ be the image of $\Omega_k^{n,0}$ in $\mathcal{F}_{2k}^1$ under the Euler-Lagrange operator,*

$$\mathcal{E}_k = \{\Delta \in \mathcal{F}_{2k}^1 \,|\, \Delta = E(\lambda) \textit{ for some} \lambda \in \Omega_k^{n,0}\};$$

*let $\mathcal{V}_{2k}$ be the space of variationally closed source forms in $\mathcal{F}_{2k}^1$,*

$$\mathcal{V}_{2k} = \{\, \Delta \in \mathcal{F}_{2k}^1 \,|\, \delta_V \Delta = 0 \,\};$$

*and let*

$$\mathcal{Q}_k = \{\Delta \in \mathcal{F}_{\mathcal{P}_{k,2k}}^1 \,|\, w_k(\Delta) \le k \,\}.$$

*Then*

$$\mathcal{E}_k = \mathcal{V}_{2k} \cap \mathcal{Q}_k.$$

PROOF: If $\Delta \in \mathcal{E}_k$, then $\Delta$ is certainly variationally closed and, by Lemma 4.22, $w_k(\Delta) \le k$. Conversely, if $\Delta \in \mathcal{V}_{2k} \cap \mathcal{Q}_k$ then Theorem 4.29 may be invoked ( with $j = k$, $k$ replaced by $2k$ and $l = k$) to conclude that $\Delta \in \mathcal{E}_k$. ∎

EXAMPLE 4.31. It again seems appropriate to illustrate the algorithm described in the proof of Theorem 4.29 for constructing minimal order Lagrangians. Consider the source form on $E = \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ given by

$$\Delta = (2u_{yy}u_{xxxy} - 4u_{xy}u_{xxyy} + 2u_{xx}u_{xyyy} + u_{xxx}u_{yyy} - u_{xxy}u_{xyy})\,\theta \wedge dx \wedge dy. \quad (4.93)$$

It is not difficult to check that $\Delta$ satisfies the Helmholtz conditions. The coefficient of $\Delta$ belongs to $\mathcal{P}_{0,4}$ and the weights are

$$w_0(\Delta) = 6, \quad w_1(\Delta) = 4, \quad w_2(\Delta) = 2, \quad \text{and} \quad w_3(\Delta) = 1.$$

In this example $l = 2$ and, according to the theorem, there is a Lagrangian $\lambda$ with coefficients in $\mathcal{P}_{0,2}$ and weights $w_0(\lambda) = 6$ and $w_1(\lambda) = 3$.

To find this minimal weight Lagrangian, we start with the Lagrangian furnished by the homotopy $\mathcal{H}^1$,*viz.*,

$$\lambda_0 = \frac{1}{3}u(2u_{yy}u_{xxxy} - 4u_{xy}u_{xxyy} + 2u_{xx}u_{xyyy} + u_{xxx}u_{yyy} - u_{xxy}u_{xyy})\, dx \wedge dy.$$

The weights of this Lagrangian are the same as those of $\Delta$ and therefore the first step in our algorithm is to reduce $w_3$ to zero. In accordance with (4.92), we split $\lambda$ into the sum

$$\lambda_0 = \alpha_0 + \beta_0$$

where

$$\alpha_0 = \frac{1}{3}u(2u_{yy}u_{xxxy} - 4u_{xy}u_{xxyy} + 2u_{xx}u_{xyyy})\, dx \wedge dy.$$

The form

$$\gamma_0 = \frac{1}{3}v(2v_{yy}u_{xxxy} - 4v_{xy}u_{xxyy} + 2v_{xx}u_{xyyy})\, dx \wedge dy$$

is variationally trivial. The weights are $w_q(\gamma_0) = 4 - q$ for $q = 0, 1, \ldots 4$. A minimal weight one form $\sigma_0$ for which $d_H\sigma_0 = \gamma_0$ is

$$\sigma_0 = \frac{2}{3}(vv_{xy}u_{xxy} - vv_{yy}u_{xxx})\, dx + \frac{2}{3}(vv_{xx}u_{yyy} - vv_{xy}u_{xyy})\, dy.$$

Let $\tau_0$ be the form obtained from $\sigma_0$ by setting $v = u$,$v_x = u_x$, $v_{xx} = u_{xx}$, $\ldots$. Then

$$\begin{aligned} d_H\tau_0 - &\frac{2}{3}(u_xu_{xx}u_{yyy} + u_{xxx}u_{yyy} - u_xu_{xy}u_{xyy} - uu_{xxy}u_{xyy})\, dx \wedge dy \\ &- \frac{2}{3}(-u_yu_{xy}u_{xxy} - uu_{xyy}u_{xxy} + u_yu_{yy}u_{xxx} + uu_{yyy}u_{xxx})\, dx \wedge dy = \alpha_0 \end{aligned}$$

and hence $\lambda_0$ is equivalent to the Lagrangian

$$\begin{aligned} \lambda_1 = [&\frac{2}{3}(u_yu_{xy}u_{xxy} - u_yu_{yy}u_{xxx} - u_xu_{xx}u_{yyy} + u_xu_{xy}u_{xyy}) \\ &- u(u_{xxx}u_{yyy} - u_{xxy}u_{xyy})]\, dx \wedge dy. \end{aligned}$$

The weights of this Lagrangian are

$$w_0(\lambda_1) = 6, \quad w_1(\lambda_1) = 4, \quad w_2(\lambda) = 2.$$

We repeat this process to reduce $w_2$. For the next step we find that

$$\alpha_1 = -u(u_{xxx}u_{yyy} - u_{xxy}u_{xyy})\,dx \wedge dy,$$
$$\gamma_1 = -v(u_{xxx}u_{yyy} - u_{xxy}u_{xyy})\,dx \wedge dy, \qquad \text{and}$$
$$\sigma_1 = -v(u_{xx}u_{xyy})\,dx - v(u_{xx}u_{yyy})dy,$$

which yields the equivalent Lagrangian

$$\lambda_2 = \frac{1}{3}(-3u_y u_{xx} u_{xyy} + 2u_y u_{xy} u_{xxy} - 2u_y u_{yy} u_{xxx} + u_x u_{xx} u_{yyy} + 2u_x u_{xy} u_{xyy})\,dx \wedge dy.$$

The weight $w_2$ has been reduce by 1 to 1. One more iteration will be needed to reduce the weight $w_2$ to zero.

This time $\alpha_2 = \lambda_2$,

$$\gamma_2 = \frac{1}{3}(-3v_y u_{xx} u_{xyy} + 2v_y u_{xy} u_{xxy} - 2v_y u_{yy} u_{xxx} + v_x u_{xx} u_{yyy} + 2v_x u_{xy} u_{xyy})\,dx \wedge dy,$$
$$\sigma_2 = \frac{1}{3}(v_y u_{xx} u_{xy} - v_x u_{xx} u_{yy} - \frac{3}{2} v_x u_{xy}^2)\,dx + \frac{1}{3}(-2v_y u_{xx} u_{yy} + \frac{3}{2} v_y u_{xy}^2 - v_x u_{xy} u_{yy})\,dy$$

and the resulting minimal order Lagrangian is

$$\lambda_3 = u_{xy}(u_{xx}u_{yy} - u_{xy}^2)\,dx \wedge dy. \tag{4.94}$$

∎

COROLLARY 4.32. *Let $\psi \in G$ and let $\Psi$ be its prolongation to $J^\infty(E)$. Suppose, in addition to the hypotheses of Theorem 4.29, that $\Delta$ is a relative $\Psi$ invariant with character $\chi$. Then there is a minimal weight, relative $\Psi$ invariant Lagrangian $\lambda$ for $\Delta$.*

PROOF: All the forms introduced in the proof of Theorem 4.29 are relative $\Psi$ invariants. ∎

EXAMPLE 4.33. As an alternative to the algorithm described in the proof of Theorem 4.29, we now use Corollary 4.32 to find a minimal weight Lagrangian for the source form (4.93). This corollary asserts that there is a Lagrangian $\lambda \in \Omega^{2,0}_{\mathcal{P}_{0,2}}$ with weights $w_0(\lambda) = 6$ and $w_1(\lambda) = 3$. Since $\Delta$ is homogenous in $u$ of degree 3, we can assume that $\lambda$ is also homogenous of degree 3. Therefore $\lambda$ is a sum of terms of the type

$$u^a u_1^b u_2^c$$

where $a \geq 0$, $b \geq 0$ and $c \geq 0$ and

$$a + b + c = 3, \quad b + 2c \leq 6 \quad \text{and} \quad c \leq 3.$$

Since $\Delta$ is translationally invariant in the $x$ and $y$ directions, the coefficients of these terms are constants. Under the transformation $x \to \epsilon x$ and $y \to \epsilon y$, $\Delta$ is a relative invariant with character $\chi = \epsilon^{-4}$. For the same to be true of $\lambda$, we must have

$$b + 2c = 6.$$

These equalities and inequalities force $a = 0$, $b = 0$ and $c = 3$. The two form $\lambda$ is therefore a constant linear combination of the terms

$$\begin{aligned}
&u_{xx} \cdot \{\, mu_{xx}^2, \quad u_{xy}^2, \quad u_{yy}^2, \quad u_{xx}u_{xy}, \quad u_{xx}u_{yy}, \quad u_{xy}u_{yy} \,\} \\
&u_{xy} \cdot \{\, u_{xx}^2, \quad u_{xy}^2, \quad u_{yy}^2, \quad u_{xx}u_{xy}, \quad u_{xx}u_{yy}, \quad u_{xy}u_{yy} \,\} \\
&u_{yy} \cdot \{\, u_{xx}^2, \quad u_{xy}^2, \quad u_{yy}^2, \quad u_{xx}u_{xy}, \quad u_{xx}u_{yy}, \quad u_{xy}u_{yy} \,\}.
\end{aligned}$$

Finally we observe that $\Delta$ is a relative invariant under the transformation $x \to \epsilon_1 x$ and $y \to \epsilon_2 y$ with character $\chi = \epsilon_1^{-2}\epsilon_2^{-2}$. This reduces the form of the trial Lagrangian to

$$\lambda = (au_{xx}u_{xy}u_{yy} + bu_{xy}^3)\, dx \wedge dy.$$

From the equation $E(\lambda) = \Delta$, it follows that $a = 1$ and $b = -1$. This agrees with the Lagrangian (4.94) obtained by our previous method. ∎

EXAMPLE 4.34. The source form for the Monge-Ampere equation is

$$\Delta = (u_{xx}u_{yy} - u_{xy}^2)\, dx \wedge dy.$$

The weights of this form are $w_0 = 4$, $w_1 = 2$ and $w_2 = 0$. Arguments similar to those of the previous example lead to the minimal weight Lagrangian

$$\lambda = -\frac{1}{6}(u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx})\, dx \wedge dy.$$

∎

EXAMPLE 4.35. We return to the minimal surface equation whose source form is (4.41). Here the coefficient of $\Delta$ belongs to $\mathcal{P}_{1,2}$ and the first weight is $w_1(\Delta) = 1$. The corollary implies that there is a first order Lagrangian

$$\lambda = L(x, y, u, u_x, u_y)\, dx \wedge dy$$

for $\Delta$. Since $\Delta$ is translationally invariant in the $x$ and $y$ directions, we can assume that $L$ is independent of $x$ and $y$. Since $\Delta$ is rotationally invariant in the $x$–$y$ plane, we can assume that

$$L = L(u, \rho), \qquad \text{where} \quad \rho = u_x^2 + u_y^2. \tag{4.95}$$

The equation $\Delta = E(\lambda)$ leads to a system of four equations involving

$$\frac{\partial L}{\partial u}, \quad \frac{\partial L}{\partial \rho}, \quad \frac{\partial^2 L}{\partial u \partial \rho}, \quad \frac{\partial L}{\partial^2 \rho}$$

whose solution is

$$\frac{\partial L}{\partial u} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \rho} = -\frac{1}{\sqrt{1+\rho}}.$$

The point to be made by this example is that our general theory insures, *a priori*, that this overdetermined system is consistent and that a Lagrangian of the form (4.95) can be found. Of course, these latter two equations integrate to give the usual Lagrangian for the minimal surface equation. ∎

EXAMPLE 4.36. In this example $M = \mathbf{R}^3$ and the fiber is the space of metrics $g = (g_{ij})$ on $M$. We consider the Cotton tensor

$$C = \varepsilon^{ihk} g^{jl} \nabla_k R_{hl}\, dg_{ij} \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where $R_{hl}$ are the components of the Ricci tensor of $g_{ij}$ and $\nabla_k$ denotes partial covariant differentiation with respect to $x^k$. The Cotton tensor vanishes if and only if the metric $g$ is conformally flat. (Recall that the Weyl tensor vanishes identically when $n = 3$.) The Cotton tensor was first derived from a variational principle by Chern and Simons [*16*] . At about the same time Horndeski [*35*], apparently unaware of the Chern-Simons paper, explicitly verified that $C$ satisfied the Helmholtz conditions but he was unable to explicitly find a Lagrangian for $C$. The techniques afforded by Corollary 4.27 enable us to explicitly construct a Lagrangian, in fact a special form of the Chern-Simons Lagrangian, without too much difficulty.

Since the curvature tensor is linear in the second derivatives of the metric and quadratic in the first derivatives, the coefficients of the Cotton tensor belong to $\mathcal{P}_{0,3}$ The weights of $C$ are

$$w_0(C) = 3, \quad w_1(C) = 2, \quad w_2(C) = 1.$$

In addition, $C$ is a natural Riemannian tensor and, as such, is invariant under the change of variables

$$y^h = a^h_i x^i + x^h_0 \quad \text{and} \quad \overline{g}_{hk} = b^i_h b^j_k g_{ij},$$

where $(a) \in \mathbf{GL}(3)$ and $(b) = (a)^{-1}$. Observe that this is a subgroup $H$ of $G$.

Our theory implies that there is an $H$ invariant Lagrangian $\lambda \in \Omega^{3,0}_{\mathcal{P}_{0,2}}$ with weights

$$w_0(\lambda) = 3 \quad \text{and} \quad w_1(\lambda) = 1$$

whose Euler-Lagrange form is $C$. The most general Lagrangian with these weights is

$$\begin{aligned} \lambda = \big[ & A^{abcd,ijh} g_{ab,cd} g_{ij,h} + A^{abcd} g_{ab,cd} + B^{abc,ijh,rst} g_{ab,c} g_{ij,h} g_{rs,t} \\ & + B^{abc,ijh} g_{ab,c} g_{ij,h} + B^{ab,c} g_{ab,c} + B \big] \, dx^1 \wedge dx^2 \wedge dx^3, \end{aligned}$$

where the various coefficients $A^{\cdots}$ and $B^{\cdots}$ are functions of the metric alone. The $\mathbf{GL}(3)$ invariance of $C$ implies that these coefficients are natural Riemannian tensors densities. Using Weyl's classification theorem, we have that each one of these tensors is a constant coefficient, linear combination of tensor densities formed from the permutation symbol $\epsilon^{rst}$ and the inverse of the metric $g^{rs}$. We immediately conclude that the tensors with an even number of indices, *viz.*, $A^{abcd}$, $B^{abc,ijh}$ and $B$, are zero. Moreover, because $B^{ab,c}$ must be a multiple of $\varepsilon^{abc}$, this term also vanishes.

At this point it becomes convenient to replace the first derivatives of the metric by a combination of the Christoffel symbols, *viz.*,

$$g_{ij,h} = g_{pj} \Gamma^p_{ih} + g_{ip} \Gamma^p_{jh}$$

and the second derivatives of the metric by a combination of derivatives of the Christoffel symbols and terms quadratic in the Christoffel symbols. Accordingly, we can rewrite $\lambda$ in the form

$$\lambda = \big[ A^{abc,ij}_{\ \ d \ \ h} \, \Gamma^d_{ab,c} \Gamma^h_{ij} + B^{ab,ij,rs}_{\ c \ \ h \ \ t} \, \Gamma^c_{ab} \Gamma^h_{ij} \Gamma^t_{rs} \big] \, dx^1 \wedge dx^2 \wedge dx^3. \tag{4.96}$$

We can now proceed in one of two ways. It is possible to directly compute the Euler-Lagrange form of $\lambda$, where the coefficients $A$ and $B$ as taken to be unknown functions of the metric $g_{ij}$. By matching the coefficients of the various derivatives of the connection in the equation $C = E(\lambda)$, one can determine the $A$ and $B$. If one adopts this approach, it is helpful to observe that $A^{abc,ij}_{\ \ d\ \ h}$ can be taken to be skew-symmetric under the interchange of the indices $(a, b, d)$ with $(i, j, h)$ since the symmetric part can be recast as a divergence plus a term cubic in the connection.

Alternatively, we can use Weyl's theorem to explicitly describe the the most general form of the coefficients $A$ and $B$ in terms of arbitrary constants. We then determine these constants from the equation $C = E(\lambda)$. However, of the many possibilities for $A$ the form of coefficient for the second derivatives of the connection in $C$ suggests that we try an $A$ which is a sum of tensors of the form

$$\varepsilon^{xxx}\,\delta^x_y\,\delta^x_y,$$

where the upper indices $x$ are chosen from the set $\{\,i, j, a, b, c\,\}$ and the lower indices $y$ are chosen from $\{\,d, h\,\}$. Because the Christoffel symbols which are contracted against $A^{\because}_{\because}$ are symmetric in the indices $ab$ and $ij$, the only possibilities for the first term of $\lambda$ of this form are

$$\begin{aligned} A^{abc,ij}_{\ \ d\ \ h}\,\Gamma^d_{ab,c}\Gamma^h_{ij} &= \bigl[a_1\varepsilon^{aci}\delta^b_d\delta^j_h + a_2\varepsilon^{aci}\delta^b_h\delta^j_d\bigr]\Gamma^d_{ab,c}\Gamma^h_{ij} \\ &= a\varepsilon^{aci}\Gamma^b_{ab,c}\Gamma^j_{ij} + b\varepsilon^{aci}\Gamma^j_{ah,c}\Gamma^h_{ij}\,, \end{aligned}$$

where $a$ and $b$ are constants. Since

$$\Gamma^b_{ab} = \frac{\partial}{\partial x^a}(\log\sqrt{g})$$

the coefficient of $a$ vanishes. Likewise, we try a $B^{ab,ij,rs}_{\ c\ \ h\ \ t}$ which is a sum of terms of the form

$$\varepsilon^{xxx}\,\delta^x_y\,\delta^x_y\,\delta^x_y.$$

A calculation of moderate length shows that the only possible nonzero second term in (4.96) is

$$c\varepsilon^{air}\Gamma^j_{ab}\Gamma^s_{ij}\Gamma^a_{rs},$$

where $c$ is a constant.

Thus, the final form of our trial Lagrangian is

$$\lambda = \bigl[b\varepsilon^{aci}\Gamma^j_{ah,c}\Gamma^h_{ij} + c\varepsilon^{air}\Gamma^j_{ab}\Gamma^s_{ij}\Gamma^a_{rs}\bigr]\,dx^1\wedge dx^2\wedge dx^3.$$

By repeatedly using the formula

$$d_V \Gamma^k_{ij} = g^{kl}[\nabla_j \theta_{il} + \nabla_i \theta_{jl} - \nabla_l \theta_{ij}],$$

where $\theta_{ij} = d_V g_{ij}$ we can compute $E(\lambda)$ and conclude that $E(\lambda) = C$ for

$$b = \frac{1}{2} \quad \text{and} \quad c = \frac{1}{3}.$$

∎

EXAMPLE 4.37. Let $M = \mathbf{R}^n$ and let $E$ be the bundle of metrics over $M$. Consider the Einstein tensor

$$G = \sqrt{g}(R^{ij} - \frac{1}{2} g^{ij})\, dg_{ij} \wedge \nu.$$

Since $G$ is homogeneous of degree $\dfrac{n}{2} - 1$, the homotopy formula (4.33) leads to the standard Lagrangian

$$\lambda_0 = -\sqrt{g} R\, \nu.$$

On the one hand this is a natural Lagrangian — it is invariant under all orientation preserving diffeomorphisms of $M$. On the other hand, $\lambda_0$ is a second order Lagrangian and is therefore not a minimal weight Lagrangian. The minimal weight Lagrangian is the first order Lagrangian

$$\lambda_1 = \sqrt{g}[g^{jh} \Gamma^i_{hk} \Gamma^k_{ij} - g^{ij} \Gamma^h_{ij} \Gamma^k_{hk}]\, \nu,$$

which, however, is not natural. Thus, a minimal order Lagrangian may not be a Lagrangian with the largest possible symmetry group. Indeed, the remarks following Lemma 4.26 suggest that Corollaries 4.27 and 4.32 are sharp in the sense that the group $G$ is the "largest" group for which one is assured the existence of invariant minimal weight Lagrangians. ∎

It is simple matter to extend these minimal weight results to forms in $\Omega^{r,s}$, where $s \geq 1$. Given a type $(r, s)$ form $\omega$, we interior evaluate with vertical vector fields $Y_i = Y^\alpha_i \dfrac{\partial}{\partial u^\alpha}$, $i = 1, 2, \ldots, s$ in order to pull $\omega$ down to the horizontal $r$ form

$$\tilde{\omega} = \operatorname{pr} Y_s \mathbin{\lrcorner} \operatorname{pr} Y_{s-1} \mathbin{\lrcorner} \cdots \operatorname{pr} Y_1 \mathbin{\lrcorner} \omega$$

If $r < n$ and $\omega$ is $d_H$ closed, then $\tilde{\omega}$ is $d_H$ closed. If $r = n$ and $I(\omega) = 0$, then by Lemma 3.2 $E(\tilde{\omega}) = 0$. We treat $\tilde{\omega}$ not as a form on $J^\infty(E)$ but rather as a form on

$J^\infty(E \times T_V(E)^s)$ — in other words we treat the components $Y_i^\alpha$ as new dependent variables. The $p^{\text{th}}$ weight of $\omega$ is defined to be the $p^{\text{th}}$ weight of $\tilde{\omega}$, where we now include in our derivative count the derivatives of the $Y_i^\alpha$. For instance, the weights of the type $(2,1)$ form

$$\omega = u_x u_{xxx}\, \theta \wedge \theta_{xx} \wedge dx$$

are

$$w_0(\omega) = 6, \quad w_1(\omega) = 3, \quad w_2(\omega) = 1.$$

The existence of minimal weight forms now follows from our basic result, Theorem 4.23.

**C. The Jacobian Subcomplex.** Let $E \xrightarrow{\pi} M$ be an arbitrary fibered manifold. All of our considerations thus far have been based upon the variational bicomplex $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ over the infinite jet bundle of $E$. In this section we introduce a subcomplex $(\mathcal{J}_k^{*,*}, d_H, d_V)$ which is defined over the finite dimensional jet bundle $J^{k+1}(E)$ in the sense that

$$\mathcal{J}_k^{r,s} \subset \Omega^{r+s}(J^{r+1}(E)). \tag{4.97}$$

To begin, we define the subring of forms $\Omega_k^*$,

$$\Omega_k^* \subset \Omega^*(J^{k+1}(E)) \subset \Omega^*(J^\infty(E)),$$

by

$$\Omega_k^* = \{\, \text{the } d_V \text{ closure of } \Omega^*(J^k(E))\,\}.$$

This ring is generated by the functions $f \in C^\infty(J^k(E))$, the horizontal differentials $dx^i$, and the contact forms $\theta_I^\alpha$ of order $|I| \le k$. Let

$$\Omega_k^{r,s} = \Omega^{r,s}(J^\infty(E)) \cap \Omega_k^*.$$

Evidently, the double complex $(\Omega_k^{*,*})$ is $d_V$ closed but it is not $d_H$ closed — for $f \in \Omega_k^{0,0} = C^\infty(J^k(E))$, $d_H f \in C^\infty(J^{k+1}(E))$ and $d_H \theta_I^\alpha = -\theta_{Ij}^\alpha \wedge dx^j$, *i.e.*,

$$d_H \Omega_k^{*,*} \subset \Omega_{k+1}^{*,*}.$$

Suppose, however, that we further restrict our attention to forms in $\Omega_k^{r,s}$ for the which the $k^{\text{th}}$ order derivatives $u^\alpha_{i_1 i_2 \dots i_k}$ and the $k^{\text{th}}$ order contact forms $\theta^\alpha_{j_1 j_2 \dots j_k}$ occur only through expressions of the form

$$J = d_H u_{I_1}^{\alpha_1} \wedge d_H u_{I_2}^{\alpha_2} \cdots \wedge d_H u_{I_p}^{\alpha_p} \wedge d_H \theta_{J_1}^{\beta_1} \wedge d_H \theta_{J_2}^{\beta_2} \cdots \wedge d_H \theta_{J_q}^{\beta_q}, \tag{4.98}$$

where $|I_l| = |J_l| = k-1$ and $p+q \le r$. The sub-bicomplex consisting of these forms is now both $d_H$ and $d_V$ closed. If we write the horizontal form $J$ as

$$J = \frac{1}{(p+q)!}\, J_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q}\, dx^{i_1} \wedge dx^{i_2} \cdots dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \cdots dx^{j_q}$$

then the coefficient $J_{\ldots}$ is the $(p+q)$ dimensional total derivative Jacobian

$$J_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} = \frac{D(u^{\alpha_1}_{I_1}, u^{\alpha_2}_{I_2}, \ldots, u^{\alpha_p}_{I_p}, \theta^{\beta_1}_{J_1}, \theta^{\beta_2}_{J_2}, \ldots, \theta^{\beta_q}_{J_q})}{D(x^{i_1}, x^{i_2} \ldots, x^{i_p}, x^{j_1}, x^{j_2}, \ldots x^{j_q})}$$

of the quantities $u^{\alpha_1}_{I_1}$, …, $\theta^{\beta_q}_{J_q}$ with respect to the variables $x^{i_1}$, …, $x^{j_q}$. For this reason, we call the subspace of all such type $(r,s)$ forms the space of $k^{\text{th}}$ order *Jacobian forms* $\mathcal{J}^{r,s}_k$.

Let $E\colon \mathbf{R}^3 \times \mathbf{R}^2 \to \mathbf{R}^3$ with coordinates $(x,y,z,u,v) \to (x,y,z)$. Examples of Jacobian forms include

$$\omega_1 = d_H u = u_x\, dx + u_y\, dy + u_z\, dz \in \mathcal{J}^{1,0}_1,$$

$$\omega_2 = d_H v_{xx} = v_{xxx}\, dx + v_{xxy}\, dy + v_{xxz}\, dz \in \mathcal{J}^{1,0}_3,$$

$$\begin{aligned} \omega_3 &= d_H u_x \wedge dH u_y \\ &= \begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix} dx \wedge dy + \begin{vmatrix} u_{xy} & u_{xz} \\ u_{yx} & u_{yz} \end{vmatrix} dy \wedge dz + \begin{vmatrix} u_{xx} & u_{xz} \\ u_{yx} & u_{yz} \end{vmatrix} dx \wedge dz, \end{aligned}$$

and

$$\begin{aligned} \omega_4 &= d_H u_x \wedge d_H v_x \wedge dz + d_H u_x \wedge d_H v_y \wedge v_z \\ &= \left\{ \begin{vmatrix} u_{xx} & u_{xy} \\ v_{xx} & v_{xy} \end{vmatrix} + \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ v_{xy} & v_{yy} & v_{yz} \\ v_{xz} & v_{yz} & v_{zz} \end{vmatrix} \right\} dx \wedge dy \wedge dz. \end{aligned}$$

Here $\omega_3$ and $\omega_4$ belong to $\mathcal{J}^{2,0}_2$ and $\mathcal{J}^{3,0}_2$ respectively. We emphasize that the first term in $\omega_3$, *viz.*,

$$\begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix} dx \wedge dy$$

is not a Jacobian form on $J^\infty(E)$ in its own right but it would be if the base space were $\mathbf{R}^2$. The vertical derivative of any of these forms is also a Jacobian form. For example,

$$\begin{aligned} \omega_5 &= \omega_2 \wedge d_V \omega_2 \wedge dz \\ &= \begin{vmatrix} v_{xxy} & v_{xxz} \\ \theta^v_{xxy} & \theta^v_{xxz} \end{vmatrix} dx \wedge dy \wedge dz \end{aligned}$$

belongs to $\mathcal{J}_3^{3,1}$. We also remark that if $\omega \in \mathcal{J}_k^{r,s}$ and $f$ is a $C^\infty$ function on $J^{k-1}(E)$, then $f\omega \in \mathcal{J}_k^{r,s}$.

We can describe the Jacobian subcomplex $(\mathcal{J}_k^{*,*}, d_H, d_V)$ intrinsically as follows. Recall that $\pi^{r,s}\colon \Omega^{r+s}(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$ is the projection map. Note that if $\omega \in \Omega^r(J^k(E))$, then $\pi^{r,0}(\omega) \in \Omega^{r,0}_{k+1}$.

Lemma 4.38. *Let*

$$\widetilde{\Omega}^*_k = \{ \text{ the } d_V \text{ closure of } \pi^{*,0}[\Omega^*(J^{k-1}(E))] \ \}$$

*as a $C^\infty(J^{k-1}(E))$ module. Then the space of $k^{\text{th}}$ order, type $(r,s)$ Jacobian forms is*

$$\mathcal{J}_k^{r,s} = \Omega^{r,s} \cap \widetilde{\Omega}^*_k. \tag{4.99}$$

Proof: Observe that $\Omega^*(J^{k-1}(E))$ is generated by functions $f \in C^\infty(J^{k-1}(E)$, and by the forms $dx^i$ and $du^\alpha_I$, $|I| \leq k-1$. Hence $\pi^{*,0}[\Omega^*(J^{k-1}(E))]$ is generated by the functions $f \in C^\infty(J^{k-1}(E))$ and by the forms $dx^i$ and $d_H u^\alpha_I = u^\alpha_{Ij} dx^j$, where $|I| = k-1$. The $d_V$ closure of $\pi^{*,0}[\Omega^*(J^{k-1}(E))]$ is generated by the functions $f \in C^\infty(J^{k-1}(E))$, and by the forms $dx^i$, $\theta^\alpha_J$ for $|J| \leq k-1$, and $d_H u^\alpha_I$ and $d_H \theta^\alpha_I$ for $|I| = k-1$. This proves (4.99). ∎

Since $\pi^{r,0} \circ d = d_H$ acting on horizontal forms, it follows that $\pi^{*,0}[\Omega^*(J^{k-1}(E))]$ is $d_H$ closed and therefore $\tilde{\Omega}^*_k$ is both $d_H$ and $d_V$ closed. Thus $(\mathcal{J}_k^{r,s}, d_H, d_V)$ is indeed a sub-bicomplex of the variational bicomplex on $J^\infty(E)$.

The property of being $d_H$ closed provides another intrinsic characterization of the space of Jacobian forms.

Theorem 4.39. *For $r < n$, $\mathcal{J}_k^{r,s}$ consists of those froms whose order is not increased by $d_H$, i.e.,*

$$\mathcal{J}_k^{r,s} = \{ \omega \in \Omega_k^{r,s} \mid d_H \omega \in \Omega_k^{r+1,s} \}.$$

Outline of Proof: We first establish the theorem for horizontal forms. Let

$$\omega = A_I[x, u^{(k)}]\, dx^I$$

be a type $(r,0)$ form of order $k$. If $d_H \omega = 0$, then the dual tensor

$$B = \frac{1}{|I|!}\, [\varepsilon^{IJ} A_I]\, \frac{\partial}{\partial x^I} \tag{4.100}$$

is a skew-symmetric rank $p = n - r$ tensor which is divergence-free, *i.e.*,

$$D_j B^{j j_2 j_3 \dots j_p} = 0. \tag{4.101}$$

Note that (4.100) can be inverted to give

$$\omega = \frac{1}{|J|!}\,[\varepsilon_{IJ} B^J]\, dx^I. \tag{4.102}$$

Since $B$ is of order $k$, we may equate the coefficients of the $(k+1)^{\text{st}}$ order derivatives in (4.101) to zero. This yields

$$\partial_\alpha^{(i_1 i_2 \ldots i_k} B^{i_{k+1})j_2\ldots j_p} = 0. \tag{4.103}$$

The theorem follows from a careful analysis of this symmetry condition. Specifically, we first prove that $B$ is a polynomial of degree $m \le n-p = r$. Accordingly, we may write

$$B^J = \sum_{\substack{l=0\\|I_h|=k}}^{r} B^{J\,I_1\,I_2\,\cdots I_l}_{\;\;\alpha_1\,\alpha_2\cdots\alpha_l} u^{\alpha_1}_{I_1} u^{\alpha_2}_{I_2} \ldots u^{\alpha_l}_{I_l}, \tag{4.104}$$

where the coefficients $B^{\cdots}_{\cdots}$ are functions of order $k-1$. We then show, again on account of (4.103), that there are $(k-1)$ order functions $Q^{\cdots}_{\cdots}$ such that $B^{\cdots}_{\cdots}$ can be expressed in the form

$$B^{j_1\ldots j_p\,I_1\,\cdots I_l}_{\;\;\alpha_1\cdots\alpha_l} = \operatorname{sym} I_1 \cdots \operatorname{sym} I_l\, \varepsilon^{j_1\ldots j_p i_1\ldots i_l k_1\ldots k_t} Q_{k_1\ldots k_t\,\alpha_1\ldots\alpha_l}^{\;\;\;\;\;\;\;\;\;\;\;I_1'\,\ldots I_l'}, \tag{4.105}$$

where $t = r-l$ and $I_h = i_h I_h'$. When (4.104) and (4.105) are substituted into (4.102), it is found that

$$\omega = \sum_{\substack{l=0\\|I_h'|=k-1}}^{r} Q_{k_1\ldots k_t\,\alpha_1\ldots\alpha_l}^{\;\;\;\;\;\;\;\;\;\;\;I_1'\,\ldots I_l'}\, d_H u^{\alpha_1}_{I_1'} \wedge \cdots \wedge d_H u^{\alpha_l}_{I_l'} \wedge dx^{k_1} \wedge \cdots \wedge dx^{k_p}.$$

This proves that $\omega \in \mathcal{J}^{r,0}_k$.

To prove the theorem for type $(r,s)$ forms, $s \ge 1$, let $\omega \in \Omega^{r,s}_k$ and suppose $d_H\omega = 0$. Then for arbitrary vertical vector fields $Y_1$, $Y_2$, …,$Y_s$, the form

$$\tilde\omega = \operatorname{pr} Y_s \,\lrcorner\, \operatorname{pr} Y_{s-1} \,\lrcorner\, \cdots \,\lrcorner\, \operatorname{pr} Y_1 \,\lrcorner\, \omega$$

is a $d_H$ closed horizontal form which is of order $k$ in both the derivatives of $u^\alpha$ and in the derivatives of the the components $Y^\alpha_i$ of $Y_i$. By what we have just proved, this $k^{\text{th}}$ order dependence must occur via the Jacobians

$$\widetilde{J} = J_1 \wedge J_2,$$

where

$$J_1 = d_H u^{\alpha_1}_{I_1} \wedge d_H u^{\alpha_2}_{I_2} \cdots \wedge d_H u^{\alpha_p}_{I_p},$$

and

$$J_2 = d_H (Y_{h_1})^{\beta_1}_{J_1} \wedge d_H (Y_{h_2})^{\beta_2}_{J_2} \cdots \wedge d_H (Y_{h_q})^{\beta_q}_{J_q},$$

and where the length of each multi-index $|I_l| = |J_l| = k - 1$. This shows that the $k^{\text{th}}$ order derivatives $u^\alpha_I$ and the $k^{\text{th}}$ order contact forms $\theta^\alpha_I$ present in $\omega$ must occur via the Jacobians (4.98) and proves the theorem for $s \geq 1$. Actually, to make this last statement more precise, observe that $\tilde{\omega}$ is alternating in the variables $Y_1$, $Y_2$, $\dots$, $Y_s$ so that the Jacobian $J_2$ must occur in $\tilde{\omega}$ as a term in the alternating sum

$$\begin{aligned}\tilde{J}_2 &= \sum_{\sigma \in \mathcal{S}_q} d_H (Y_{h_{\sigma(1)}})^{\beta_1}_{J_1} \wedge d_H (Y_{h_{\sigma(2)}})^{\beta_2}_{J_2} \wedge \cdots \wedge d_H (Y_{h_{\sigma(q)}})^{\beta_q}_{J_q} \\ &= \pm \operatorname{pr} Y_{h_1} \lrcorner \operatorname{pr} Y_{h_2} \cdots \lrcorner \operatorname{pr} Y_{h_q} \lrcorner \{ d_H \theta^{\beta_1}_{J_1} \wedge d_H \theta^{\beta_2}_{J_2} \wedge \dots d_H \theta^{\beta_q}_{J_q} \}.\end{aligned}$$

∎

To prove that (4.103) implies (4.104) and (4.105) some results from multi-linear algebra are needed. Let $V = \mathbf{R}^n$ and let $S^q(V)$ be the vector space of symmetric, rank $q$ tensors on $V$. If $X \in V$, we let $\mathcal{X}$ be the $q$-tuple $\mathcal{X} = (X, X, \dots, X)$. By polarization, the values of a tensor $T \in S^q(V)$ are uniquely determined by the values of $T(\mathcal{X})$ for all $X \in V$. More generally, if

$$T \in S^Q(V) = S^{q_1}(V) \otimes S^{q_2}(V) \otimes \cdots \otimes S^{q_m}(V)$$

and, for $X^j$ in $V$, we let $\mathcal{X}^j$ be the $q_j$-tuple $\mathcal{X}^j = (X^j, X^j, \dots, X^j)$, then the values of $T$ are uniquely determined by the values of

$$T(\mathcal{X}^1; \mathcal{X}^2; \dots; \mathcal{X}^m).$$

Here, and in the sequel, we use a semicolon to separate the arguments belonging to the different factors of $S^Q(V)$.

Definition 4.40. *A tensor $T \in S^Q(V)$ is said to have symmetry property $\mathcal{A}$ if*

$$T(\mathcal{X}^1; \mathcal{X}^2; \dots; \mathcal{X}^m) = 0 \tag{4.106}$$

*whenever the vectors $X^1$, $X^2$, $\dots$ $X^m$ are linearly dependent.*

The next lemma will be used to prove that $l$-fold derivatives of $B$ with respect to the variables $u^\alpha_I$, $|I| = k$ have property $\mathcal{A}$.

LEMMA 4.41. *Suppose that $q_1 = 1$ and that*

$$T(X^1; \mathcal{X}^2; \dots; \mathcal{X}^m) = 0 \tag{4.107}$$

*whenever $X^1 = X^i$ for each $i = 2, 3, \dots, m$. Then $T$ has property $\mathcal{A}$.*

PROOF: In order to establish (4.106), we consider two cases.
Case 1. $X^1$ is a linear combination of $\{ X^2, X^3, \dots, X^m \}$.
Case 2. For some $j \geq 2$, $X^j$ is a linear combination of

$$\{ X^1, \dots, X^{j-1}, X^{j+1}, \dots, X^m \}.$$

Equation (4.106) follows immediately from (4.107) in Case 1. If $X^1 = \sum_{j=2}^{m} c_j X^j$ then, owing to the linearity of $T$ in its first argument,

$$T(X^1; \mathcal{X}^2; \dots; \mathcal{X}^m) = \sum_{j=2}^{l} c_j T(X^j; \mathcal{X}^2; \dots; \mathcal{X}^m) = 0.$$

For Case 2, let us suppose, merely for the sake of notational simplicity, that $j = 2$. Then, again using the multi-linearity of $T$, we find that

$$T(X^1; \mathcal{X}^2; \dots; \mathcal{X}^m) = \sum_{\mathcal{Y}} c_{\mathcal{Y}} T(X^1; \mathcal{Y}; \mathcal{X}^3; \dots; \mathcal{X}^m),$$

where the $c_{\mathcal{Y}}$ are constants and the sum ranges over all $q_2$-tuples

$$\mathcal{Y} = (Y^1, Y^2, \dots, Y^{q_2}) \qquad \text{where each} \quad Y^k \in \{ X^1, X^3, \dots, X^m \}.$$

To complete the proof of the lemma we must prove that

$$T(X^1; \mathcal{Y}; \mathcal{X}^3; \dots; \mathcal{X}^m) = 0. \tag{4.108}$$

Let $\mathcal{Y}^k$ be the $q_2$-tuple obtained from $\mathcal{Y}$ by replacing $Y^k$ by $X^1$. The symmetry condition (4.107) implies that

$$T(X^1; \mathcal{Y}; \mathcal{X}^3; \dots; \mathcal{X}^l) + \sum_{k=1}^{q_2} T(Y^k; \mathcal{Y}^k; \mathcal{X}^3; \dots; \mathcal{X}^m) = 0. \tag{4.109}$$

If $Y^k = X^1$, then obviously

$$T(Y^k; \mathcal{Y}^k; \mathcal{X}^3; \dots; \mathcal{X}^m) = T(X^1; \mathcal{Y}; \mathcal{X}^2; \dots; \mathcal{X}^m),$$

while if $Y^k = X^i$, $i = 3, \dots, m$, then because of property $\mathcal{A}$,

$$T(Y^k; \mathcal{Y}^k; \mathcal{X}^3; \dots; \mathcal{X}^m) = T(X^i; \mathcal{Y}^k; \mathcal{X}^3; \dots; \mathcal{X}^m) = 0.$$

In short, (4.109) reduces to a multiple of (4.108) and the lemma is established. ∎

PROOF OF (4.104): Given the horizontal form $\omega$ define a tensor

$$(\partial^l B) \in \underbrace{S^1 \otimes \cdots \otimes S^1}_{p \text{ copies}} \otimes \underbrace{S^k \otimes \dots S^k}_{l \text{ copies}}$$

the components of which are the $l^{\text{th}}$ order derivatives of $B$ with respect to $u^\alpha_I$, $|I| = k$, at any point fixed in $J^\infty(E)$. For example, with $l = 2$,

$$\begin{aligned}(\partial^2 B)&(X^1; X^2; \dots; X^p; \mathcal{X}^{p+1}; \mathcal{X}^{p+2}) \\ &= X^1_{j_1} X^2_{j_2} \cdots X^p_{j_p} X^{p+1}_{i_1} \cdots X^{p+1}_{i_k} X^{p+2}_{h_1} \cdots X^{p+2}_{h_k} (\partial^{i_1 \dots i_k}_\alpha \partial^{h_1 \dots h_k}_\beta B^{j_1 j_2 \dots j_p}).\end{aligned}$$

Let $X^1$, $X^2$, …, $X^{p+l} \in V$ and consider the value

$$b = (\partial^l B)(X^1; X^2; \dots X^p; \mathcal{X}^{p+1}; \dots; \mathcal{X}^{p+l}).$$

Because $(\partial^l B)$ is skew-symmetric in its first $p$ arguments, $b = 0$ if $X^1 = X^j$, for $j = 2, \dots, p$. Moreover, the symmetry condition (4.103) implies that $b = 0$ if $X^1 = X^j$, $j = p+1, \dots, p+l$. Thus $\partial^l B$ satisfies the hypothesis of Lemma 4.41 and hence $\partial^l B$ has property $\mathcal{A}$. Since the vectors $X^1$, $X^2$, …, $X^{p+l}$ are always linearly dependent if $p + l = n - r + l > n$, this proves that $\partial^l B$ vanishes identically if $l > r$. This establishes (4.104). ∎

To prove (4.105), we need the structure theorem for tensors with property $\mathcal{A}$. To this end, it is convenient to temporarily adopt a slightly different viewpoint. Let $M_{m,n}$ be the ring of real $m \times n$ matrices and let $R[X]$ be the polynomial ring in the $m \times n$ matrix of indeterminants $X = (x^i_j)$. Denote the rows of $X$ by $X^i = (x^i_1, x^i_2, \dots, x^i_n)$. Then for each tensor $T \in S^Q(V)$, we can construct a polynomial $\hat{T} \in R[X]$ by setting

$$\hat{T}(x^i_j) = T(\mathcal{X}^1; \mathcal{X}^2; \dots; \mathcal{X}^m).$$

Note that $\hat{T}$ is homogenous in the variables $X^i$ of degree $q_i$.

Now let $I_r$ be the ideal in $R[X]$ generated by all the $r \times r$ minors of $X$ and let $V_r$ be the variety in $M_{m,n}$ which vanishes on $I_r$, *i.e.*,

$$V_r = \{\, A \in M_{m,n} \mid P(A) = 0 \text{ for all } P \in I_r \}.$$

THEOREM 4.42. *If $Q \in R[X]$ and $Q(A) = 0$ for all $A \in V_r$, then $Q \in I_r$.*

De Concini, Eisenbud and Procesi [*20*] prove this theorem using the Straightening Theorem for Young tableaus. Another proof, based the Cappelli identity from classical invariant theory, can be found in Anderson [3]. Actually we need a version of this theorem which accounts for dependencies on parameters — if $Q_t \in R[X]$ depends smoothly on a parameter $t$ and $Q_t(A) = 0$ for all $t$ and all $A \in V_r$, then

$$Q_t(X) = \sum_k M^k(X) P^k_t(X)$$

where each $M^k \in I_r$ and each $P^k_t \in R[X]$ depends smoothly on $t$. However, owing to the constructive nature of the above proofs, this is immediate.

Proof of (4.105): It is now easy to complete the derivation of (4.105). If $T \in S^Q$ has property $\mathcal{A}$, then the polynomial

$$\hat{T}(X) = T(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m)$$

vanishes whenever $\operatorname{rank}(X) < m$. Theorem 4.42 implies that $\hat{T} \in I_m$. Since the $m \times m$ minors of $X$ can be expressed in the form

$$M^{k_1 \ldots k_t}(X) = \varepsilon^{k_1 \ldots k_t j_1 \ldots j_m} X^1_{j_1} \ldots X^m_{j_m}, \tag{4.110}$$

this shows that $T$ can be expressed in the form

$$T(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m) = M^{k_1 \ldots k_t}(X^1, X^2, \ldots, X^m) Q_{k_1 \ldots k_t}(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m). \tag{4.111}$$

Here each $Q_{\ldots} \in S^{q_1 - 1} \otimes S^{q_2 - 1} \otimes S^{q_m - 1}$. With $T = \partial^l B$, (4.111) becomes (4.105), as required. ∎

Corollary 4.43. *Let $E \xrightarrow{\pi} M$ be the trivial $\mathbf{R}^m$ bundle over $\mathbf{R}^n$. For each $k = 1, 2, \ldots,$ the Jacobian bicomplex $(\mathcal{J}^{*,*}_k, d_H, d_V)$ is exact.*

Proof: Let $\omega \in \mathcal{J}^{r,s}_k$. Then $\omega$ is a polynomial in the $u^\alpha_I$, $\theta^\alpha_I$, where $|I| = k$, of degree $\leq r$, *i.e.*, the highest weights of $\omega$ are $w_{k-1}(\omega) \leq r$ and $w_k(\omega) = 0$. By the minimal weight results of the previous section, there is form $\eta \in \Omega^{r-1,s}_k$ such that $d_H \eta = \omega$. Since the order of $\eta$ is the same as that of $d_H \eta$, we can infer from Theorem 4.39 that $\eta \in \mathcal{J}^{r-1,s}_k$. This proves the exactness of the rows.

Let $\omega \in \mathcal{J}^{r,s}_k$. Since $\operatorname{pr} R \lrcorner\, \theta^\alpha_{Ij} dx^j = u^\alpha_{Ij} dx^j$, it is apparent that $h^{r,s}_V(\omega) \in \mathcal{J}^{r,s-1}_k$. This observation suffices to prove the exactness of each column. Alternatively, for $r < n$, it suffices to recall that $d_H$ anti-commutes with $h_V$ and to note that $h^{r,s}_V \colon \Omega^{r,s}_k \to \Omega^{r,s-1}_k$ so that

$$d_H h^{r,s}_V(\omega) = -h^{r+1,s}_V(d_H \omega) \in \Omega^{r+1,s-1}_k.$$

Owing to Theorem 4.39, this shows that $h^{r,s}_V(\omega) \in \mathcal{J}^{r,s-1}_k$. ∎

COROLLARY 4.44. *Let $E \xrightarrow{\pi} M$ be the trivial $\mathbf{R}^m$ bundle over $\mathbf{R}^n$. If $\omega \in \Omega_k^{n,s}$ and $E(\omega) = 0$ if $s = 0$ or $I(\omega) = 0$ if $s \geq 1$, then $\omega \in \mathcal{J}_k^{n,s}$ and $\omega = d_H \eta$ for $\eta \in \mathcal{J}_k^{r-1,s}$.*

PROOF: Since $\omega$ is of order $k$, our minimal order results show that $\omega = d_H \eta$, where $\eta$ is also of order $k$. This implies that $\eta \in \mathcal{J}_k^{n-1,s}$ and so $\omega \in \mathcal{J}_k^{n,s}$. ∎

COROLLARY 4.45. *For $r < n$, let $\omega \in \Omega_l^{r,s}$ and suppose that $d_H \omega \in \Omega_k^{r+1,s}$, where $k \leq l$. Then there are forms $\tilde{\omega} \in \mathcal{J}_k^{r,s}$ and $\eta \in \mathcal{J}_l^{r-1,s}$ such that*

$$\omega = \tilde{\omega} + d_H \eta. \tag{4.112}$$

PROOF: Let $\rho = d_H \omega \in \Omega_k^{r+1,s}$. Then $\rho$ is closed in the appropriate sense, *i.e.*, $d_H \rho = 0$ if $r \leq n-2$, or $I(\rho) = 0$ if $r = n-1$ and $s \geq 1$, or $E(\rho) = 0$ if $r = n-1$ and $s = 0$. In any case, we can conclude that there is a form $\tilde{\omega} \in \mathcal{J}_k^{r,s}$ such that $d_H \tilde{\omega} = \rho$, or

$$d_H(\omega - \tilde{\omega}) = 0.$$

Since $\omega - \tilde{\omega} \in \Omega_l^{r,s}$, there must be a form $\eta \in \mathcal{J}_l^{r-1,s}$ such that (4.112) holds. ∎

EXAMPLE 4.46. *Local Exactness of the Gauss-Bonnet Lagrangian.*

Let $E$ be the trivial bundle $\mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^2$ with coordinates $(x, y, R) \to (x, y)$, where $R = (u, v, w)$ is the position vector in $\mathbf{R}^3$. We restrict our attention to the open set $\mathcal{R} \subset J^\infty(E)$ defined by

$$\mathcal{R} = \{\, (x, y, R, R_x, R_y, \dots) \mid R_x \times R_y \neq 0 \,\}.$$

Sections of $E$ are the graphs of regular parametrized surfaces in $\mathbf{R}^3$. In this example we do not restrict to the bicomplex on $\mathcal{R}$ of forms which are parameter invariant, *i.e.*, invariant under diffeomorphism of the base $\mathbf{R}^2$.

Let

$$E = \langle R_x, R_x \rangle, \qquad F = \langle R_x, R_y \rangle, \qquad G = \langle R_y, R_y \rangle$$

be the components of the first fundamental form. Let $D = \sqrt{EG - F^2}$ and let $K$ be the Gaussian curvature. The Gauss-Bonnet integrand is the second order Lagrangian $\lambda \in \Omega^{2,0}(\mathcal{R})$, where $\lambda = L\, dx \wedge dy$ and $L = KD$. Struik [*64*](p. 112) gives the following explicit formula for $L$:

$$L = \frac{1}{D^3} \cdot$$
$$\left\{ \begin{vmatrix} \langle R_{xx}, R_{yy} \rangle & \langle R_{xx}, R_x \rangle & \langle R_{xx}, R_y \rangle \\ \langle R_x, R_{yy} \rangle & E & F \\ \langle R_y, R_{yy} \rangle & F & G \end{vmatrix} - \begin{vmatrix} \langle R_{xy}, R_{xy} \rangle & \langle R_{xy}, R_x \rangle & \langle R_{xy}, R_y \rangle \\ \langle R_{xy}, R_x \rangle & E & F \\ \langle R_{xy}, R_y \rangle & F & G \end{vmatrix} \right\}.$$

This enables us to rewrite $\lambda$ in the form

$$\begin{aligned}\lambda =&\frac{1}{D}\langle d_H R_x \wedge d_H R_y\rangle \\ &+\frac{1}{D^3}\big[-E\langle R_y, d_H R_x\rangle \wedge \langle R_y, d_H R_y\rangle + F\langle R_x, d_H R_x\rangle \wedge \langle R_y, d_H R_y\rangle \\ &\qquad + F\langle R_y, d_H R_x\rangle \wedge \langle R_x, d_H R_y\rangle - G\langle R_x, d_H R_x\rangle \wedge \langle R_x, d_H R_y\rangle\big]\end{aligned}$$

The Lagrangian $\lambda$ is variational trivial. To find a one form $\eta \in \Omega^{1,0}(\mathcal{R})$ such that

$$\lambda = d_H \eta \tag{4.113}$$

we first observe that $\lambda$ has the following properties:

(i) $\lambda \in \Omega^{2,0}_{\mathcal{P}_{1,2}}$ and $w_1(\lambda) = 2$;
(ii) $\lambda \in \mathcal{J}^{2,0}_2$;
(iii) $\lambda$ is invariant under the group $\mathbf{SO(3)}$ acting on the fiber; and
(iv) $\lambda$ is invariant under the group of translations in the base.

By our minimal weight results of §4B, we can assume that $\eta \in \Omega^{1,0}_{\mathcal{P}_{1,2}}$, that $w_1(\eta) = 1$, and that $\eta$ is both $\mathbf{SO(3)}$ and translational invariant. By property (ii) and Theorem 4.39, we know that $\eta \in \mathcal{J}^{1,0}_2$ and that the second derivative dependencies of $\eta$ must occur via the degree 1 Jacobians $d_H R_x$ and $d_H R_y$. Thus $\eta$ is of the form

$$\begin{aligned}\eta =&a\,\langle R_x, d_H R_x\rangle + b\,\langle R_y, d_H R_x\rangle + e\,\langle N, d_H R_x\rangle \\ &c\,\langle R_x, d_H R_y\rangle + d\,\langle R_y, d_H R_y\rangle + f\,\langle N, d_H R_y\rangle + \eta_0,\end{aligned}$$

where $N$ is the unit normal vector, where the coefficients $a, b, \ldots, f$ are, *a priori*, $\mathbf{SO(3)}$ invariant functions of $R$, $R_x$ and $R_y$, and where $\eta_0$ is a first order one form. From the form of $\lambda$, it seems reasonable to suppose that $e = f = 0$ and that the coefficients $a, \ldots, d$ are independent of $R$. This implies that these coefficients can now be considered to be functions of $E$, $F$, and $G$. With the $R$ dependence eliminated from $a, \ldots, d$, there is no way that the first order form $\eta_0$ can contribute to the solution of the equation (4.113) and therefore we assume that $\eta_0 = 0$.

We now compute $d_H\eta$ and match coefficients in (4.113) to arrive at the following system of equations:

$$\begin{aligned} c - b = \frac{1}{D}, \qquad & -2\frac{\partial a}{\partial G} + 2\frac{\partial d}{\partial E} = \frac{F}{D^3}, \qquad & 2\frac{\partial c}{\partial G} = \frac{\partial d}{\partial F}, \\ \frac{\partial a}{\partial F} - 2\frac{\partial c}{\partial E} = \frac{G}{D^3}, \qquad & \frac{\partial b}{\partial G} - \frac{\partial c}{\partial F} = \frac{E}{D^3}, \qquad & 2\frac{\partial b}{\partial E} = \frac{\partial a}{\partial F}. \end{aligned}$$

This system has many solutions, one of which is

$$a = \frac{F}{ED}, \quad b = -\frac{1}{D}, \quad c = 0, \quad d = 0.$$

This gives

$$\eta = \frac{\langle R_x R_y \rangle \langle R_x, d_H R_x \rangle - \langle R_x, R_x \rangle \langle R_y, d_H R_x \rangle}{\langle R_x, R_x \rangle D}.$$

For this choice of $\eta$, Struik [*64*](p. 114) attributes the formula $d_H \eta = KD\, dx \wedge dy$ to Liouville. ∎

In analogy with Corollary 4.45, let us now consider Lagrangians $\lambda \in \Omega_l^{n,0}$ whose Euler-Lagrange form $\Delta = E(\lambda) \in \mathcal{F}_k^1$, where $k \leq 2l$. Since the Lagrangian $\tilde{\lambda} = \mathcal{H}^1(\Delta)$ is of order $k$ and $E(\tilde{\lambda}) = \Delta$, we can conclude that

$$\lambda = \tilde{\lambda} + d_H \eta,$$

where $\eta \in \mathcal{J}_l^{n-1,0}$. The problem now becomes that of characterizing the functional dependencies of Lagrangians of order $k$ whose Euler-Lagrange forms are also of order $k$. In view of the Volterra-Vainberg homotopy operator $\mathcal{H}^1$, this is tantamount to the problem of classifying the functional dependencies of source forms of order $k$ which satisfy the Helmholtz conditions. This is the problem to which we now turn.

Suppose that $\Delta \in \mathcal{F}_k^1$ is a source form of order $k \geq 1$ and that $\delta_V \Delta = 0$. Since $d_V \omega \in \Omega_k^{n,2}$, the condition $I(d_V \Delta) = 0$ implies that $d_V \Delta \in \mathcal{J}_k^{n,2}$ and hence $\Delta \in \mathcal{J}_k^{n,1}$. Therefore, *if $\Delta$ is a locally variational source form of order $k$, then it must be a polynomial in the $k^{\text{th}}$ order derivatives of degree $\leq n$.* This, however, is no means a full characterization of the functional dependence of a locally variational source form. Considerably more structure is imposed by the Helmholtz equations.

To begin to uncover this structure, let $\Delta = P_\alpha[x, u^{(k)}]\, \theta^\alpha \wedge \nu$ be a locally variational source form of order $k$. By differentiating the Helmholtz equation

$$\partial_\beta^{j_2 j_3 \ldots j_k} P_\alpha = (-1)^k \left[\partial_\alpha^{j_2 j_3 \ldots j_k} P_\beta - k D_j \partial_\alpha^{j j_2 j_3 \ldots j_k} P_\beta\right] \tag{4.114}$$

with respect to $u^\gamma_{i_1 i_2 \ldots i_{k+1}}$ we deduce that

$$\partial_\gamma^{(i_1 i_2 \ldots i_k} \partial_\beta^{i_{k+1}) j_2 j_3 \ldots j_k} P_\alpha = 0. \tag{4.115}$$

Although this symmetry condition appears to be the same as (4.103), it is in fact a much stronger condition because, in this instance, (4.115) is symmetric in the indices $j_2 j_3 \cdots j_k$.

Definition 4.47. *Let $T \in S^{q_1} \otimes S^{q_2} \cdots \otimes S^{q_m}$. The tensor $T$ has symmetry property $\mathcal{B}$ if for each $q_i$-tuple $\mathcal{Y} = (Y^1, Y^2, \dots, Y^{q_i})$, $i = 1, 2, \dots, m$*

$$T(\mathcal{X}^1; \dots \mathcal{X}^{i-1}; \mathcal{Y}; \mathcal{X}^{i+1}; \dots; \mathcal{X}^m) = 0 \qquad \text{if some} \quad Y^j = X^i,\ j \neq i.$$

Equation (4.115) implies that $\partial^m P$ enjoys property $\mathcal{B}$. To analyze this symmetry property, it is convenient to assume that the factors of the tensor product $S^Q$ are ordered according to length, *i.e.*, $q_1 \le q_2 \le \cdots \le q_m$.

Lemma 4.48. *If $T \in S^Q$ has property $\mathcal{B}$, then $T$ vanishes whenever any $q_j + 1$ of its first $q_1 + q_2 \cdots + q_j$ arguments coincide.*

Proof: Suppose a vector $X$ occurs $q_j + 1$ times amongst the first $q_1 + q_2 \cdots + q_j$ arguments of $T$, say

$$t = T(X, X, \dots; X, X, \dots; \dots; X, \dots; \dots)$$

By repeated using the symmetry property $\mathcal{B}$, $t$ can be expressed as a sum of terms, where all the $X$ arguments have been gathered together into a single factor $S^{q_l}$, $l \le j$ with at least one additional $X$ amongst the arguments of $X$, *i.e.*,

$$t = \sum T(\dots; X, \dots; \underbrace{X, X, \dots, X}_{l^{\text{th}} \text{ factor}}; \dots).$$

By property $\mathcal{B}$, this vanishes.

For example, if $T \in S^2 \otimes S^4 \otimes S^4$, then

$$\begin{aligned} &T(X, W; X, X, U, V; X, X, Y, Z) \\ &\qquad = -\frac{1}{3}\big[T(X, W; X, Y, U, V; X, X, X, Z) + T(X, W; X, Z, U, V; X, X, X, Y)\big] \\ &\qquad = \frac{1}{6} T(X, W; Z, Y, U, V; X, X, X, X) = 0. \qquad \blacksquare \end{aligned}$$

Let $A = (a^i) \in V$. Define

$$\begin{aligned} (\nabla_A^j T)(\mathcal{X}^1; \mathcal{X}^2; \dots; \mathcal{X}^m) &= a^i \frac{\partial}{\partial X_i^j} \left[T(\mathcal{X}^1; \mathcal{X}^2; \dots; \mathcal{X}^m)\right] \\ &= q_j T(\mathcal{X}^1; \dots; \mathcal{X}^{j-1}; \mathcal{Y}; \mathcal{X}^{j+1}; \dots; \mathcal{X}^m), \end{aligned}$$

where $\mathcal{Y} = (X^j, X^j, \dots, X^j, A)$.

Lemma 4.49. *Suppose $T \in S^Q$ has property $\mathcal{B}$. Fix $p \le q_1 - 1$ and let $A^1$, $A^2$, $\ldots$, $A^p \in V$. Then*

$$(\nabla^{j_1}_{A_1}\nabla^{j_2}_{A_2}\ldots\nabla^{j_p}_{A_p}T)(\mathcal{X}^1;\mathcal{X}^2;\ldots;\mathcal{X}^m) = 0, \tag{4.116}$$

*whenever $\{X^1, X^2, \ldots, X^m\}$ are linearly dependent.*

Proof: The left-hand side of (4.116) is a sum of terms of the form

$$T(\underbrace{X^1, X^1, \ldots}_{r_1};\underbrace{X^2, X^2, \ldots}_{r_2};\ldots;\underbrace{X^l, X^l, \ldots}_{r_l};\ldots;\underbrace{X^m, X^m, \ldots}_{r_m}), \tag{4.117}$$

where each $r_i \ge 1$. and $\sum r_i = \sum q_i - p$.

Now suppose that $X^l$ is a linear combination of the forms $X^1$, $\ldots$, $X^{l-1}$, $X^{l+1}$, $\ldots$, $X^m$. Then (4.117) becomes a sum of terms of the form

$$T(\underbrace{X^1, X^1, \ldots}_{r_1};\underbrace{X^2, X^2, \ldots}_{r_2};\ldots;\overbrace{\underbrace{X^1, X^1, \ldots}_{s_1},\underbrace{X^2, X^2, \ldots}_{s_2},\ldots,\underbrace{X^m, X^m, \ldots}_{s_m}}^{l^{\text{th}}\text{ factor}};\ldots;\underbrace{X^m, X^m, \ldots}_{r_m}), \tag{4.118}$$

where $s_l = 0$ and $\displaystyle\sum_{j=1}^m s_j = r_l$. I claim that for some $j \ne l$,

$$s_j + r_j > q_j. \tag{4.119}$$

Suppose the contrary. Then for all $j \ne l$, $s_j + r_j \le q_j$ and therefore

$$\begin{aligned} p = \sum_{j=1}^m (q_j - r_j) = q_l - r_l + \sum_{\substack{j=1\\ j\ne l}}^m (q_j - r_j) \ge q_l - r_l + \sum_{j=1}^m s_l \\ = q_l - r_l + r_l = q_l \ge q_1. \end{aligned}$$

This contradicts the hypothesis that $p \le q_1 - 1$ and proves (4.119). Lemma 4.48 now shows that the expression in (4.118) is zero. ∎

We revert once again to the algebraic viewpoint introduced earlier. Let $I_r^p$ be the $p^{\text{th}}$ power of the the ideal $I_r$ in $R[X]$ — each polynomial $P \in I_r^p$ is a finite sum

$$P = \sum_k M^k Q^k,$$

where each $Q^k \in R[X]$ and each $M^k$ is a $p$-fold product of $r \times r$ minors. If $P \in R[X]$, let $Df$ denote the matrix of partial derivatives of $P$ with respect to the variables $x^i_j$. The $p^{\text{th}}$ *symbolic* power of $I^r$ is

$$I_r^{(p)} = \{\, P \in R[X] \mid P,\, DP,\, \ldots,\, D^{p-1}P \in I_r \,\}.$$

THEOREM 4.50. *If $r = m$, then the $p^{\text{th}}$ power and the $p^{\text{th}}$ symbolic power of the ideal $I_r$ coincide,*

$$I_r^p = I_r^{(p)}.$$

The proof of this theorem can also be found in [*20*] or [*3*].

We use this theorem to complete the analysis of symmetry property $\mathcal{B}$ as follows. Let $T \in S^Q$ be a tensor with property $\mathcal{B}$ and let

$$\hat{T}(x^i_j) = T(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m).$$

Lemma 4.49 shows that $\hat{T} \in I_m^{(q_1)}$ and thus, on account of Theorem 4.50, $\hat{T} \in I_m^{q_1}$. This implies that

$$T(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m) = \sum M^{K_1} M^{K_2} \cdots M^{K_{q_1}} Q_{K_1 K_2 \ldots K_{q_1}}(\mathcal{X}^1; \mathcal{X}^2; \ldots; \mathcal{X}^m), \tag{4.120}$$

where each $M^{K_i}$ is an $r \times r$ minor of $X$, as defined by (4.110). Since

$$\partial^m P \in S^k \otimes S^k \cdots \otimes S^k \qquad (m \text{ factors})$$

has property $\mathcal{B}$, $\partial^m P$ must be of the form (4.120) with $q_1 = k$. To convert this result into a statement describing the functional dependences of $\Delta$ on its $k^{\text{th}}$ order derivatives, it is helpful to introduce Olver's [*53*] notion of hyperjacobians.

DEFINITION 4.51. *A hyperjacobian $J_q(u^1, u^2, \ldots, u^p)$ of degree $p$ and order $q$ is a multi-linear, alternating sum of the $q^{\text{th}}$ order derivatives of the $p$ functions $u^1$, $u^2$, $\ldots$,$u^p$. Specifically, fix multi-indices $K_1$, $K_2$, $\ldots$, $K_q$ of length $|K_i| = n - p$. Then the hyperjacobians of degree $p$ and order $q$ are*

$$J_q^{K_1 K_2 \ldots K_q}(u^1, u^2, \ldots, u^p) = \varepsilon^{IK_1} \varepsilon^{JK_2} \cdots \varepsilon^{TK_q}\, u^1_{i_1 j_1 \ldots t_1} u^2_{i_2 j_2 \ldots t_2} \cdots u^p_{i_p j_p \ldots t_p}, \tag{4.121}$$

*where $I = i_1 i_2 \ldots i_p$, $J = j_1 j_2 \ldots j_p$, $\ldots$, $T = t_1 t_2 \ldots t_p$ are a set of $q$ multi-indices of length $p$.*

For example, when $q = 1$,

$$J_1^K(u^1, u^2, \ldots, u^p) = \varepsilon^{K i_1 i_2 \ldots i_p} u^1_{i_1} u^2_{i_2} \ldots u^p_{i_p}$$

is just the ordinary Jacobian of $u^1$, $u^2$, ... $u^p$ with respect to the $p$ variables $\{ x^i \,|\, i \notin K\}$. For $p = n = 2$,

$$J_1(u,v) = u_x v_y - u_y v_x,$$
$$J_2(u,v) = u_{xx} v_{yy} - 2u_{xy} v_{xy} + u_{yy} v_{xx},$$
$$J_3(u,v) = u_{xxx} v_{yyy} - 3u_{xxy} v_{xyy} + 3u_{xyy} v_{xyy} - u_{yyy} v_{xxx},$$

and so on. If $p = n$ and $u^1 = u^2 = \cdots = u^p = u$, then $J_q(u, u, \ldots, u)$ is the generalized determinant of the rank $q$ symmetric form $\partial^q u$.

We combine (4.120) and (4.121) to arrive the following characterization of the $k^{\text{th}}$ order derivative dependencies of a locally variational source form of order $k$.

PROPOSITION 4.52. *Let $\Delta$ be a locally variational source form of order $k$. Then the components of $\Delta$ are linear combinations over $C^\infty(J^{k-1}(E))$ of the $k^{\text{th}}$ order hyperjacobians of the dependent variables $u^\alpha$ of degree 0,1, ...,n.*

EXAMPLE 4.53. Let $J_n(u_k, u_k \ldots, u_k)$ be the $k^{\text{th}}$, $k$ even, order hyperjacobian of degree $n$. For instance, with $n = 2$ and $k = 4$ we have

$$\begin{aligned} J_2(u_4, u_4) &= \varepsilon^{i_1 i_2} \varepsilon^{j_1 j_2} \varepsilon^{h_1 h_2} \varepsilon^{k_1 k_2} u_{i_1 j_1 h_1 k_1} u_{i_2 j_2 h_2 k_2} \\ &= 2(u_{xxxx} u_{yyyy} - 4u_{xxxy} u_{xyyy} + 3u^2_{xxyy}). \end{aligned} \tag{4.122}$$

Since $D_j \partial^{j j_2 j_3 \ldots j_k} J_n = 0$, it follows that the source form

$$\Delta = J_n \, \theta \wedge \nu, \tag{4.123}$$

which depends exclusively on $k^{\text{th}}$ order derivatives, satisfies the Helmholtz conditions. Since $\Delta$ is homogenous of degree $n$, the Lagrangian $\lambda = \mathcal{H}^1(\Delta)$ is given by (4.33), viz

$$\lambda = \frac{1}{n+1} u J_n(u_k, u_k, \ldots, u_k) \, \nu$$

The equation $E(\lambda) = \Delta$ is equivalent to

$$J_n = \frac{1}{n} D_I(u \partial^I J_n), \qquad |I| = k$$

which makes explicit the fact (Olver [*52*], [*53*]) that $J_n$ is a $k$-fold divergence. ∎

Since the proof of Proposition 4.52 depends only on the highest derivative terms of the single Helmholtz equation (4.114), it should not be too surprising to find that this proposition fails in general to give a sharp description of the possible functional dependencies of a locally variational source form. In this regard, we have the following conjecture.

CONJECTURE 4.54. *Let $\Delta \in \mathcal{F}^1_k$ be a locally variational source form of order $k = 2l$ if $k$ is even, or $k = 2l - 1$ if $k$ is odd. Then $\Delta \in \mathcal{F}^1_{\mathcal{P}_{l,k}}$, i.e., $\Delta$ is a polynomial in derivatives of order $> l$, and the weights of $\Delta$ are bounded by*

$$w_p(\Delta) \leq (k-p)n \qquad \text{for } p = l,\, l+1,\, \ldots,\, k-1. \tag{4.124}$$

The source form (4.123) shows that the bounds (4.124) are sharp. It is undoubtedly true that terms in $\Delta$ of weight $w_p > k - p$ must occur via hyperjacobians of some type but I have been unable to even formulate a good conjecture along these lines.

Although many authors have, in recent years, rederived the Helmholtz equations it is disappointing to find that none have attempted to uncover the general structure of locally variational differential equations. Historically, this has not always been the case. Before the introduction of the Volterra-Vainberg formula, the local sufficiency of the Helmholtz conditions was established though a detailed analysis of these equations. This approach is still a valuable one and should not be dismissed for its inefficiencies. For example, in the case of second order ordinary differential equations, the direct analysis of the Helmholz equations enables one to correctly guess the possible global obstructions to the solution of the inverse problem. As the final example of this chapter will show, the detailed structure of locally variational source forms is needed to solve an equivariant version of the inverse problem where the Volterra-Vainberg formula fails to furnish us with a Lagrangian with the sought after symmetries.

The next three examples establish special cases of the above conjecture.

EXAMPLE 4.55. *Structure of locally variational, ordinary differential equations.*

In the case of ODE, that is, when $E : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$, the above conjecture completely characterizes the functional dependencies of locally variational source forms. Let

$$\Delta = P_\alpha(x, u^\beta, \dot{u}^\beta, \ldots, \overset{(k)}{u}{}^\alpha)\, \theta^\beta \wedge dx$$

be a locally variational source form of order $k$. We know that there is a Lagrangian $\lambda = L[x, u^{(k)}]$ of order $k$ for $\Delta$. Since $E_\alpha(L) = P_\alpha$ is only of order $k$, we must have that

$$E^{(2k)}_\beta(E_\alpha(L)) = \frac{\partial^2 L}{\partial \overset{(k)}{u}{}^\alpha \partial \overset{(k)}{u}{}^\beta} = 0$$

and so

$$L = A_\alpha \overset{(k)}{u}{}^\alpha + \tilde{L},$$

where $A_\alpha$ and $\tilde{L}$ are of order $k-1$. From the equation

$$E^{2k-1}_\beta(E_\alpha(L)) = \frac{\partial^2 L}{\partial \overset{(h)}{u}{}^\alpha \partial \overset{(k)}{u}{}^\beta} - \frac{\partial^2 L}{\partial \overset{(h)}{u}{}^\beta \partial \overset{(k)}{u}{}^\alpha} = 0,$$

where $h = k-1$, we can deduce that

$$A_\alpha = \frac{\partial f}{\partial \overset{(h)}{u}{}^\alpha}$$

for some function $f$ of order $k-1$. Since

$$\frac{df}{dx} = \frac{\partial f}{\partial \overset{(h)}{u}{}^\alpha} \overset{(k)}{u}{}^\alpha + \{\,\text{terms of order } k-1\},$$

this shows that the original Lagrangian $L$ is equivalent to a Lagrangian of order $k-1$.

We can repeat this argument to further reduce the order of the Lagrangian for $\Delta$. If $k = 2l$ or $k = 2l-1$, this reduction stops when the order of the Lagrangian is $l$; moreover, if $k = 2l-1$, then the Lagrangian is linear in the derivatives of order $l$. When taken in conjecture with Lemma 4.22, this proves the following proposition.

Proposition 4.56. *Let $E : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ and suppose that $\Delta$ is a source form for a system of $k^{\text{th}}$ order ordinary differential equations on $E$. Let $k = 2l$, or $2l-1$ according to whether $k$ is even or odd. Then $\Delta$ is a polynomial in derivatives of order $> l$ with weights*

$$w_p(\Delta) \le k - p, \qquad j = l,\, l+1,\, \ldots,\, k-1.$$

*In particular, $\Delta$ is linear in derivatives of order $k$, i.e.,*

$$P_\alpha = A_{\alpha\beta} \overset{(k)}{u}{}^\alpha + \tilde{P}_\alpha[x, u^{(k-1)}],$$

*where the coefficients $A_{\alpha\beta}$ are functions of order $k-l$.*

Now consider the case of scalar partial differential equations.

Example 4.57. *Structure of locally variational, second order scalar equations.*

For second order scalar equations, Proposition 4.52 completely characterizes the functional form of locally variational source forms. Indeed, if $B(u, u_i)$ is any smooth function on $J^1(E)$ and $J_n = J_n(u_2, u_2, \ldots u_2)$ is the second order hyperjacobian of degree $n$ (in this case $J_n$ is, apart from a numerical factor, the Hessian $\det u_{ij}$), then it is not difficult to verify that

$$P(u, u_i, u_{ij}) = [(n+1)B + (u_i \partial^i B)]J_n + (\partial_u B) u_i u_j \partial^{ij} J_n$$

satisfies the Helmholtz equations

$$\partial^r P = D_s \partial^{rs} P.$$

With $B = \int_0^1 t^n A(u, tu_i)\, dt$, $P$ becomes

$$P = AJ_n + (\partial_u B) u_i u_j \partial^{ij} J_n.$$

This proves that the coefficient $A = A(u, u_i)$ of the highest weight term $J_2$ can be arbitrarily specified. No further constraints on the possible functional dependencies of the the coefficients of $\Delta$ are imposed by the Helmholtz conditions. ∎

We know that there are no third order scalar equation solutions to the Helmholtz equations so that following example describes the next simplest, non-trivial case to consider.

EXAMPLE 4.58. *Structure of locally variational, fourth order scalar equations in two independent variables.*

Let $E \colon \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$. Let

$$\Delta = P(x^i, u, u_i, u_{ij} u_{ijh}, u_{ijhk})\, \theta \wedge dx \wedge dy$$

be a locally variational source form. Our starting point is Proposition 4.52 which, in this case, asserts that

$$P = AJ + B^{ijhk} u_{ijhk} + C,$$

where $J = \frac{1}{2} J_4(u_4, u_4)$ is the $4^{\text{th}}$ order hyperjacobian of degree two given by (4.122) and where $A$, $B^{\cdots}$, and $C$ are third order functions. The problem to be addressed here is the extent to which the coefficient $A$ of the highest weight term $J$ is arbitrary.

The derivative of $P$ with respect to $u_I$, $|I| = q$ will be denoted by $\partial^q P \in S^q$, for example

$$(\partial^3 \partial^4 P)(X, X, X; Y, Y, Y, Y) = X_{i_1} X_{i_2} X_{i_3} Y_{j_1} Y_{j_2} Y_{j_3} Y_{j_4} \partial^{i_1 i_2 i_3} \partial^{j_1 j_2 j_3 j_4} P.$$

The tensor $\partial^4 \partial^4 P = \partial^4 \partial^4 J \in S^4 \otimes S^4$ is given by

$$(\partial^4 \partial^4 P) = A \det{}^4(X, Y). \tag{4.125}$$

This tensor has symmetry property $\mathcal{B}$. If $R \in S^q$, then its total divergence belongs to $S^{q-1}$ and shall be denoted by

$$R(X, \dots, X, \nabla) = X_{i_1} X_{i_2} \dots X_{i_q - 1} (D_j R^{i_1 i_2 \dots i_{p-1} j}).$$

With these notational conventions the first Helmholtz condition on $\Delta$ can be expressed as

$$(\partial^3 P)(X, X, X) = 2(\partial^4 P)(\nabla, X, X, X). \tag{4.126}$$

We shall make use of the higher Euler operators

$$E^{(p)}(Z,Z,\ldots,Z)=Z_{i_1}Z_{i_2}\cdots Z_{i_p}E^{i_1 i_1\ldots i_p},$$

defined by (2.15) and the commutation rule (2.18), *viz.*, if $R\in S^q$, then

$$E^p(Z,Z,\ldots Z)T(\nabla,X,\ldots,X)=E^{p-1}(Z,Z,\ldots)T(Z,X,\ldots X). \tag{4.127}$$

We first show that $A$ is of second order. We apply apply $E^4(Y,Y,Y,Y)$ to (4.126), use the commutation rule (4.127) and the symmetry property $\mathcal{B}$ to deduce that

$$\begin{aligned}&(\partial^3\partial^4P)(XXX;YYYY)\\ &\qquad=2[(\partial^3\partial^4P)(YYY;YXXX)+(\partial^4\partial^4P)(XXX\nabla;YYYY)].\end{aligned} \tag{4.128}$$

Here, and in what follows, it is convenient to suppress the commas that separate the arguments within a single factor for a tensor $T\in S^{q_1}\otimes S^{q_2}\otimes\cdots$. By replacing one of the vectors $X$ in (4.128) by the vector $Y$, we find that

$$(\partial^3\partial^4P)(XXY;YYYY)=2(\partial^3\partial^4P)(YYY;YYXX).$$

This, in turn, leads to

$$(\partial^3\partial^4P)(XYY;YYYY)=0. \tag{4.129}$$

Since

$$(\partial^3\partial^4\partial^4P)(XXX;YYYY;ZZZZ)=(\partial^3A)(XXX)\det{}^4(Y,Z)$$

we can conclude that

$$(\partial^3\partial^4\partial^4T)(XYY;YYYY;ZZZZ)=(\partial^3A)(XYY)\det{}^4(Y,Z)=0$$

which proves that

$$\partial^3A=0. \tag{4.130}$$

Next we show that $B^{\cdots}$ is at most quadratic in the third order derivatives. By virtue of Theorem 4.50, equation (4.129) implies that there exists a tensor $E\in S^1\otimes S^2$ for which

$$(\partial^3\partial^4P)(XXX;YYYY)=E(X;YY)\det{}^2(X,Y).$$

From the integrability condition

$$(\partial^3\partial^3\partial^4P)(XXX;ZZZ;YYYY)=(\partial^3\partial^3\partial^4P)(ZZZ;XXX;YYYY)$$

we can prove that there is a symmetric tensor $F \in S^2$ for which

$$(\partial^3\partial^3\partial^4 P)(XXX; ZZZ; YYYY) = F(XZ)\det{}^2(X,Y)\det{}^2(Z,Y). \tag{4.131}$$

The application of $E^3(ZZZ)$ to (4.128) gives rise to

$$\begin{aligned}(\partial^3\partial^3\partial^4 P)&(XXX; ZZZ; YYYY) \\ &= 2[(\partial^3\partial^3\partial^4 P)(YYY; ZZZ; XXXY) + (\partial^2\partial^4\partial^4 P)(ZZ; ZXXX; YYYYY)]\end{aligned}$$

Into this equation we substitute from (4.125) and (4.131) to conclude that

$$F = 2\partial^2 A. \tag{4.132}$$

Since $A$ is of second order, the same must be true of $F$ and hence

$$\partial^3\partial^3\partial^3\partial^4 P = 0.$$

By applying the third Euler operator three times to (4.126), we can also prove that $C$ is at most quartic in third order derivatives.

At this point we have verified the foregoing conjecture in the case of fourth order scalar equations; $\Delta$ is polynomial in the third and fourth derivatives of $u$ with weights $w_2 \leq 4$ and $w_3 \leq 2$.

PROPOSITION 4.59. *Let $\Delta = P[x, u^{(4)}]\,\theta \wedge dx \wedge dy$ be a locally variational source form in two independent and one dependent variables. Then*

$$P = AJ + B^{ijhk}u_{ijhk} + C, \tag{4.133}$$

*where $J = \frac{1}{2}J_4(u_4, u_4)$. The coefficient $A$ is a function of order 2; the coefficients $B^{\cdots}$ are polynomials of degree two in the third derivatives of $u$ and $C$ is a polynomial of degree at most four in the third derivatives of $u$.*

To complete this example, we must exhibit a Lagrangian whose Euler-Lagrange expression has $AJ$ as its leading term. This would show that the coefficient $A$ in (4.133) is arbitrary and not subject to any further constraints. I have not been able to find such a Lagrangian. However, if we suppose that $A$ is a function of $u_{xx}$ alone, then the Lagrangian

$$\begin{aligned}L = u[&(u_{xxxx}u_{yyyy} - 4u_{xxxy}u_{xyyy} + 3u_{xxyy}^2)A + \\ &(u_{xyy}^2u_{xxxx} - 4u_{xyy}u_{xxy}u_{xxxy} + (2u_{xyy}u_{xxx} + 4u_{xxy}^2)u_{xxyy} \\ &- 4u_{xxx}u_{xxy}u_{xyyy} + u_{xxx}^2u_{yyyy})A' \\ &+ (u_{xxx}^2u_{xyy}^2 - 2u_{xxx}u_{xyy}u_{xxy}^2 + u_{xxy}^4)A''].\end{aligned}$$

does have the property that $AJ$ is the leading term of $E(L)$. This at least shows that $A$ need not be a polynomial in the second derivatives of $u$.

EXAMPLE 4.60. *The inverse problem to the calculus of variations for natural differential equations in the plane.*

Let

$$\lambda = L(\kappa, \dot{\kappa}, \ddot{\kappa}, \dots)\, ds \tag{4.134}$$

be a natural Lagrangian for curves in the $(x, y)$ plane with curvature $\kappa$. In Section 2C, we computed the Euler-Lagrange form for $\lambda$ to be

$$E(\lambda) = [\ddot{E}_\kappa(L) + \kappa^2 E_\kappa(L) + \kappa H(L)]\, \Theta^2 \wedge ds, \tag{4.135}$$

where the contact form $\Theta^2$ is defined by

$$\Theta^2 = -\dot{v}\theta^u + \dot{u}\theta^v. \tag{4.136}$$

In this example, we solve the inverse problem for natural differential equations for plane curves. Specifically, the problem now at hand is to determine when a natural source form

$$\Delta = P(\kappa, \dot{\kappa}, \dots, \kappa^{(p)})\, \Theta^2 \wedge ds \tag{4.137}$$

is the Euler-Lagrange form obtained from a natural Lagrangian (4.134) through the variation of the curve in the $(x, y)$ plane with curvature $\kappa$. The Helmholtz conditions (3.16), as they now stand, are in expressed in terms of the original variables $(t, x, y, x', y', \dots)$ and accordingly are of little use. The first step in the solution to this inverse problem is then to re-derive these conditions in terms of the Lie-Euler operators ( or partial derivatives) with respect to the variables $(\kappa, \dot{\kappa}, \dots)$. We shall then use our structure theorem for locally variational ODE to integrate the Helmholtz equations and thereby determine the obstructions to the construction of natural Lagrangians.

Since $\delta_V(\Delta) = I(d_V\Delta)$ the first step in the determination of the Helmholtz conditions is the evaluation of $d_V\Delta$. By repeating the calculations presented in Section 2D, it is not difficult to show that

$$d_V\Delta = \Theta^2 \wedge [\ddot{E}_\kappa(P\,\Theta^2) + \kappa^2 E_\kappa(P\,\Theta^2) + H(P\,\Theta^2)] \wedge ds + d_H\eta.$$

Here $E_\kappa(P\,\Theta^2)$ and $H(P\,\Theta^2)$ are the Euler-Lagrange expression and Hamiltonian for the contact form $P\,\Theta^2$ as formally defined in the usual fashion but with the understanding that

$$\frac{\partial}{\partial\kappa^{(p)}}(P\,\Theta^2) = \frac{\partial P}{\partial\kappa^{(p)}}\,\Theta^2.$$

For example, if $P = \ddot{\kappa}$ then

$$E_\kappa(\ddot{\kappa}\,\Theta^2) = \ddot{\Theta}^2 \quad \text{and} \quad H(\ddot{\kappa}\,\Theta^2) = -\dot{\kappa}\,\dot{\Theta}^2$$

and

$$d_V\Delta = \Theta^2 \wedge [\overset{(iv)}{\Theta}{}^2 + \kappa^2\ddot{\Theta}^2 - \kappa\dot{\kappa}\,\dot{\Theta}^2] \wedge ds. \tag{4.138}$$

The next step is to expand the operators $E_\kappa(P\,\Theta^2)$ and $H(P\,\Theta^2)$ so as to express $d_V\Delta$ in the form

$$d_V\Delta = \Theta^2 \wedge [Q^{(p+2)}\, D_{(p+2)}\Theta^2 + Q^{(p+1)}\, D_{(p+1)}\Theta^2 + \cdots + Q^2\,\ddot{\Theta}^2 + Q^1\,\dot{\Theta}^2]\, ds, \tag{4.139}$$

where the coefficients $Q^i = Q^i(P)$ are various total differential combinations of the Lie-Euler operators $E_\kappa^{(j)}(P)$. Explicit formulas for the $Q^i(P)$ can be obtained by using the product rule (Proposition 2.8) for the Euler-Lagrange operator to expand $E_\kappa(P\,\Theta^2)$ and by the using the identity

$$\frac{d}{ds}H(P\,\Theta^2) = -\dot{\kappa}E_\kappa(P\,\Theta^2) - P\,\Theta^2$$

to determine the expansion of $H(P\,\Theta^2)$. Given the coefficients $Q^{(i)}(P)$, the Helmholtz conditions can be explicitly obtained by applying the interior Euler operator

$$I = \sum_{j=0}^{p+2} \Theta^2 \wedge [\frac{d^j}{ds^j}(\frac{\partial}{\partial \overset{(j)}{\Theta}{}^2} \lrcorner\ )]$$

to (4.139) and setting the result to zero. As in the derivation of (3.12), this leads to the system of equations

$$(-1)^k Q^{(k)}(P) = \sum_{j=0}^{p+2-k} (-1)^j \binom{j+k}{j} \frac{d^j}{ds^j}[Q^{(j+k)}(P)].$$

As we observed in Proposition 3.12, there is considerable redundancy in this system.

For example, if $P = \ddot{\kappa}$, then (4.138) shows that

$$Q^{(4)} = 1, \quad Q^{(3)} = 0, \quad Q^{(2)} = \kappa^2, \quad \text{and} \quad Q^{(1)} = -\kappa\dot{\kappa}.$$

The first Helmholtz condition

$$Q^{(3)} = 2\dot{Q}^{(4)}$$

holds, but the second one

$$Q^{(1)} = \dot{Q}^{(2)} - \dddot{Q}^{(4)}$$

does not and hence the source form

$$\Delta = \ddot{\kappa}\,\Theta^2 \wedge ds$$

is not locally variational.

Proposition 4.61. *Let $\Delta = P[\kappa]\,\Theta^2 \wedge ds$ be a natural, locally variational source form for plane curves with curvature $\kappa$. Let $\Delta_0 = \Theta^2 \wedge ds$. Then there is a natural Lagrangian $\lambda = L[\kappa]\,ds$ and a constant $a$ such that*

$$\Delta = E(\lambda) + a\Delta_0. \tag{4.140}$$

*The decomposition (4.140) is unique in that there is no natural Lagrangian whose Euler-Lagrange form is $\Delta_0$.*

Proof: Let us assume that $\Delta$ is of order $p$ in the derivatives of $\kappa$. Since

$$Q^{(p+2)}(P) = \frac{\partial P}{\partial \kappa^{(p)}},$$

the first Helmholtz condition

$$(-1)^p Q^{(p+2)} = Q^{(p+2)}$$

is an identity if $p$ is even and implies that $P$ is of order $p-1$ if $p$ is odd. Thus, without loss of generality, we can assume that $p = 2q$ is even.

Since $\Delta$ is locally variational, we know that there is a Lagrangian

$$\lambda_0 = L_0(x, u, v, u', v', \dots)\,ds,$$

defined at least locally in the neighborhood of each point $[t, x, y]$, for which

$$E(\lambda_0) = [E_u\,\theta^u + E_v\,\theta^v] \wedge ds = \Delta$$

In view of (4.136), this implies that

$$E_u = -\dot{v}P \quad \text{and} \quad E_v = \dot{u}P. \tag{4.141}$$

Now we use the structure theorem, Proposition 4.56, for locally variational ODE. Since $P$ is of order $2q$ in the derivatives of $\kappa$, the components $E_u$ and $E_v$ are of order $2q+2$ in the derivatives of $u$ and $v$. They are necessarily linear in these top derivatives, *i.e.*,

$$\begin{aligned} E_u &= au^{(2q+2)} + bv^{(2q+2)} + \{\,\text{lower order terms}\,\} \\ E_v &= bu^{(2q+2)} + cv^{(2q+2)} + \{\,\text{lower order terms}\,\} \end{aligned} \tag{4.142}$$

and, moreover, the coefficients $a$, $b$ and $c$ are of order at most $q+1$. A comparison of (4.141) with (4.142) shows that $P$ must have the form

$$P = M\kappa^{(2q)} + \{\,\text{lower order terms}\,\}$$

where, and this is the crucial point, $M$ is of order at most $q-1$ in the derivatives of $\kappa$.

Now consider the Lagrangian

$$\lambda_1 = L_1(\kappa, \dot{\kappa}, \ldots, \kappa^{(l-1)})\, ds,$$

where the Lagrange function $L_1$ is any solution to the equation

$$\frac{\partial^2 L}{\partial \kappa^{(l-1)} \partial \kappa^{(l-1)}} = M(\kappa, \dot{\kappa}, \ldots, \kappa^{(l-1)}).$$

From (4.135) it follows that

$$E(\lambda_1) = [M \kappa^{(2q)} + \{\text{ lower order terms }\}]\, \Theta^2 \wedge ds$$

and therefore

$$\Delta = E(\lambda_1) + \Delta_1,$$

where $\Delta_1$ is of order no more that $2q-1$. In fact, because $\Delta_1$ is locally variational, its order is no more that $2q-2$.

An easy induction argument now proves that

$$\Delta = E(\lambda) + N(\kappa)\, \Theta^2 \wedge ds, \tag{4.143}$$

where $\lambda$ is a natural Lagrangian and $N$ is of order zero. For the source form $\tilde{\Delta} = N\, \Theta^2 \wedge ds$, one computes that

$$d_V \tilde{\Delta} = \Theta^2 \wedge [\frac{d^2}{ds^2}(N'\, \Theta^2) + \kappa N'\, \Theta^2 + \kappa N\, \Theta^2] \wedge ds$$

so that $Q^{(2)}(N) = N'$ and $Q^{(1)}(N) = 2\dfrac{d}{ds}N'$. The Helmholtz equation $Q^{(1)} = \dot{Q}^{(2)}$ implies that

$$\frac{d}{ds}N' = 0$$

and therefore

$$N = b\kappa + a,$$

where $a$ and $b$ are constants. Since the source form $b\kappa\, \Theta^2 \wedge ds$ is the Euler-Lagrange form for the natural Lagrangian $b\, ds$, (4.143) can be re-expressed as (4.140).

The same arguments used in the proof of Proposition 4.13 can be repeated here to prove that $\Delta_0$ is not the Euler- Lagrange form of a natural Lagrangian. ∎

Consider the source form $\Delta$ whose source equation is the natural equation for circles of a fixed radius, *viz.*,

$$\Delta = (\kappa - a)\,\Theta^2 \wedge ds.$$

This source form is locally variational and, in fact, is the Euler-Lagrange form for the Lagrangian

$$\lambda = [1 - \frac{a}{2}(-x\dot{y} + y\dot{x})]\,ds.$$

However, according to Proposition 4.61, $\Delta$ is not the Euler-Lagrange form of a natural Lagrangian.

Chapter Five

# GLOBAL PROPERTIES OF THE VARIATIONAL BICOMPLEX

In this chapter we explore some of the global aspects of the variational bicomplex on the infinite jet bundle $J^\infty(E)$ of the fibered manifold $\pi : E \to M$. We begin in section A by proving that the interior rows of the augmented variational bicomplex

$$
\begin{aligned}
0 \longrightarrow \Omega^{0,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,s}(J^\infty(E)) \xrightarrow{d_H} \cdots \\
\xrightarrow{d_H} \Omega^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}^s(J^\infty(E)) \longrightarrow 0,
\end{aligned} \tag{5.1}
$$

with $s \geq 1$, are globally exact. This is a fundamental property of the variational bicomplex and is an essential part of our variational calculus. We use this result to prove the global direct sum decomposition

$$
\Omega^{n,s}(J^\infty(E)) = \mathcal{B}^{n,s}(J^\infty(E)) \oplus \mathcal{F}^s(J^\infty(E)),
$$

for $s \geq 1$, where $\mathcal{B}^{n,s}(J^\infty(E)) = d_H\,[\Omega^{n-1,s}(J^\infty(E))]$ and $\mathcal{F}^s(J^\infty(E))$ is the subspace of type $(n,s)$ functional forms. This decomposition leads immediately to the global first variational formula — for any Lagrangian $\lambda \in \Omega^{n,0}(J^\infty(E))$, there is a type $(n-1,1)$ form $\eta$ such that

$$
d_V\lambda = E(\lambda) + d_H\eta.
$$

The exactness of (5.1) also implies, by standard homological algebra arguments, that the cohomology of the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$:

$$
\begin{aligned}
0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,0}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,0}(J^\infty(E)) \cdots \\
\xrightarrow{d_H} \Omega^{n,0}(J^\infty(E)) \xrightarrow{E} \mathcal{F}^1(J^\infty(E)) \xrightarrow{\delta_V} \mathcal{F}^2(J^\infty(E)) \xrightarrow{\delta_V} \cdots
\end{aligned} \tag{5.2}
$$

is isomorphic to the de Rham cohomology of $J^\infty(E)$, *i.e.*,

$$
H^*(\mathcal{E}^*(J^\infty(E))) \cong H^*(\Omega^*(J^\infty(E))). \tag{5.3}
$$

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We actually show that the projection map

$$\Psi^p : \Omega^p(J^\infty(E)) \to \mathcal{E}^p(J^\infty(E))$$

defined by

$$\Psi^p(\omega) = \begin{cases} \pi^{p,0}(\omega), & \text{for } p \leq n, \quad \text{and} \\ I \circ \pi^{n,s}(\omega), & \text{if } p = n + s \text{ and } s \geq 1. \end{cases}$$

is a cochain map which induces this isomorphism. Numerous examples illustrate this result. Let $E : \mathbf{R}^n \times F \to \mathbf{R}^n$ and $G$ be the group of translations on $\mathbf{R}^n$. As an application of the isomorphism (5.3), we compute the cohomology of the $G$ invariant Euler-Lagrange complex $\mathcal{E}^*_G(J^\infty(E)))$. This generalizes the work of Tulczyjew [*71*].

In section B we analyze the vertical cohomology $H^{*,*}_V(\Omega^{*,*}(J^\infty(E)), d_V)$ of the variational bicomplex. We first prove, as a consequence of the homotopy invariance of the vertical cohomology, that

$$H^{r,s}_V(\Omega^{*,*}(J^\infty(E))) = 0 \qquad \text{if} \quad s > m,$$

where $\dim E = m + n$ and $\dim M = n$. To proceed further, we suppose that $\pi : E \to M$ is a fiber bundle with $m$ dimensional fiber $F$. We also suppose that there are $p$ forms on $E$, for $p = 1, 2, \ldots, m$, which form a basis for the $p^{\text{th}}$ de Rham cohomology of each fiber of $E$. Because these forms need not themselves be closed on $E$, this assumption holds for a variety of bundles of practical interest. We prove that

$$H^{r,s}_V(\Omega^{*,*}(J^\infty(E))) \cong \Omega^r(M) \otimes H^s(F).$$

Thus, under the above hypothesis, the vertical cohomology of the variational bicomplex agrees with the $E_1$ term of the Serre spectral sequence for the bundle $E$.

The material in section C is motivated, in part, by the observation that for first order, single integral Lagrangians

$$\lambda = L(x, u^\alpha, \dot{u}^\alpha)\, dx$$

the well-known Poincaré-Cartan form

$$\Phi_{\text{P.C.}}(\lambda) = \lambda + \frac{\partial L}{\partial \dot{u}^\alpha}\, \theta^\alpha = (L - \frac{\partial L}{u^\alpha})\, dx + \frac{\partial L}{\partial \dot{u}^\alpha}\, du^\alpha$$

induces an isomorphism $H^1(\mathcal{E}^*(J^\infty(E))) \to H^1(\Omega^*(J^\infty(E)))$. (In fact, $\Phi_{\text{P.C.}}$ is the inverse to $\Psi^1$.) We try to generalize this property of the Poincaré-Cartan form to more general Lagrangians as well as to other terms in the Euler-Lagrange complex by finding maps

$$\Phi^p \colon \mathcal{E}^p(J^\infty(E)) \to \Omega^p(J^\infty(E))$$

which will induce the isomorphism (5.3). We are specifically interested in maps $\Phi^p$ which, like the Poincaré-Cartan map $\Phi_{\text{P.C.}}$, are natural (or universal) differential operators but we are forced to conclude, albeit tentatively, that such maps exist in general only when the base space is 1 dimensional. Our basic variational calculus on $J^\infty(E)$ is enhanced by these maps when $\dim M = 1$. This conclusion also underscores, once again, the deep differences between the geometric analysis of single and multiple integral variational problems and of ordinary and partial differential equations.

Let $\nabla$ be a symmetric, linear connection on the base manifold $M$ of the fibered manifold $\pi \colon E \to M$. In section D we use such a connection to construct another set of homotopy operators $h^{r,s}_{\triangledown}$ for the augmented horizontal complexes (5.1) for $s \geq 1$. These operators are defined by local formulas on the adapted coordinate charts of $E$ but in such a fashion that the invariance of these operators under change of coordinates is manifest. Accordingly, these invariant homotopy operators have a number of applications. First, since the base manifold $M$ always admits a symmetric, linear connection $\nabla$, we can always construct these invariant homotopy operators for any fibered manifold $\pi \colon E \to M$. Their invariance under change of coordinates insures that the $h^{r,s}_{\triangledown}$ patch together to give *global* homotopy operators

$$h^{r,s}_{\triangledown} \colon \Omega^{r,s}(J^\infty(E)) \to \Omega^{r-1,s}(J^\infty(E)). \tag{5.4}$$

This immediately furnishes us with another, quite different, proof of the global exactness of the horizontal complexes (5.1). These operators can also be used to construct cochain maps

$$\Phi_{\triangledown} \colon \mathcal{E}^*(J^\infty(E)) \to \Omega^*(J^\infty(E))$$

from the Euler-Lagrange complex to the de Rham complex on $J^\infty(E)$ which induce the isomorphism (5.2). Finally, and perhaps most significantly, the local, invariant character of the homotopy operators $h^{r,s}_{\triangledown}$ provide us with an effective means of studying the equivariant cohomology of the variational bicomplex over certain tensor bundles. As a simple application in this direction, we establish the exactness (*i.e.*, triviality) of the Taub conservation law in general relativity. Other essential applications of these operators will be found in the Chapter Six.

A special case of the invariant homotopy operators (5.4) can be found, at least implicitly, in Gilkey's paper [*28*] on smooth invariants of Riemannian metrics. In the study of divergence-free, natural tensors, Anderson [*2*] explicitly constructed similar invariant homotopy operators, although only for second order forms. Our work here also generalizes the work of Ferraris [*24*] Kolar [*41*], and Masqué [*49*] who use connections in a similar fashion to define global Lepage equivalents and Poincaré-Cartan forms. Ideas from all these papers, as well as from §3.A, are needed here to carry out our construction of the invariant homotopy operators $h^{r,s}_{\nabla}$. Although this construction is rather complicated, the mere fact that these invariant homotopy operators exist really embodies many of the salient global and equivariant properties of the variational bicomplex.

**A. The Horizontal Cohomology of the Variational Bicomplex.** We begin by proving that the interior horizontal rows of the augmented variational bicomplex are exact.

THEOREM 5.1. *Let $\pi\colon E \to M$ be a fibered manifold. Then, for each $s \geq 1$, the augmented horizontal complex*

$$\begin{aligned} 0 \longrightarrow \Omega^{0,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,s}(J^\infty(E)) \xrightarrow{d_H} \cdots \\ \xrightarrow{d_H} \Omega^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}^s(J^\infty(E)) \longrightarrow 0 \end{aligned} \tag{5.5}$$

*is exact.*

PROOF: The exactness of (5.5) at $\Omega^{r,s}(J^\infty(E))$ is established by using a standard partition of unity argument together with induction on $r$. For small values of $r$, say $r = 1$ or $r = 2$, the induction step is simple enough to present directly. This we shall do. For larger values of $r$, the induction argument remains basically unchanged but some additional machinery, as provided by the generalized Mayer-Vietoris sequence, is needed to complete all the details.

Let $\mathcal{U} = \{\, U_\alpha \,\}_{\alpha \in J}$ be a countable open cover of $E$ consisting of adapted coordinate neighborhoods

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\ \simeq\ } & \mathbf{R}^n \times \mathbf{R}^m \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ U^0_\alpha & \xrightarrow{\ \simeq\ } & \mathbf{R}^n. \end{array}$$

The index set $J$ is assumed to be ordered. No other assumptions are made here concerning the nature of the cover $\mathcal{U}$. We do not require that this cover be finite nor is it necessary to suppose that $\mathcal{U}$ is a good cover in the sense that all non-empty intersections diffeomorphic to $\mathbf{R}^{n+m}$ . By Proposition 4.2, we are assured that the augmented horizontal complex $(\Omega^{*,s}(J^\infty(U_\alpha), d_H)$ is exact.

The global exactness of (5.5) at $\Omega^{0,s}$ is a trivial consequence of the local exactness at $\Omega^{0,s}$. Therefore, let us take $\omega \in \Omega^{1,s}(J^\infty(E))$ and suppose that that $d_H\omega = 0$. Each restriction of $\omega_\alpha = \omega_{|\, J^\infty(U_\alpha)}$ of $\omega$ is $d_H$ closed and hence, by local exactness, there are forms $\eta_\alpha \in \Omega^{0,s}(J^\infty(U_\alpha))$ such that

$$\omega_\alpha = d_H \eta_\alpha. \tag{5.6}$$

On non-empty double intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $\omega_\alpha = \omega_\beta$ and therefore

$$d_H(\eta_\beta - \eta_\alpha) = 0. \tag{5.7}$$

The exactness of (5.5) on the infinite jet bundle $J^\infty(U_{\alpha\beta})$ at $\Omega^{0,s}(J^\infty(U_{\alpha\beta}))$ now implies that $\eta_\alpha = \eta_\beta$. This proves that the forms $\eta_\alpha$ must be the restriction to $U_\alpha$ of a global form $\eta \in \Omega^{0,s}(J^\infty(E))$. Equation (5.6) shows that $\omega = d_H\eta$. This establishes the exactness of (5.5) at $\Omega^{1,s}(J^\infty(E))$.

To prove the exactness of (5.5) at $\Omega^{2,s}(J^\infty(E))$, we now assume horizontal exactness of the variational bicomplex at $\Omega^{1,s}(J^\infty(E'))$, where $\pi' \colon E' \to M'$ is any fibered manifold. Let $\omega \in \Omega^{2,s}(J^\infty(E))$ be a $d_H$ closed form. Then, as before, equations (5.6) and (5.7) hold but now the forms $\eta_\alpha \in \Omega^{1,s}(J^\infty(U_\alpha))$. By assumption, the variational bicomplex over the fibered manifold $U_{\alpha\beta}$ is horizontally exact at $\Omega^{1,s}(J^\infty(U_{\alpha\beta}))$ and therefore (5.7) implies that there are type $(0,s)$ forms $\sigma_{\alpha\beta}$ on $J^\infty(U_{\alpha\beta})$ such that

$$\eta_\beta - \eta_\alpha = d_H \sigma_{\alpha\beta}. \tag{5.8}$$

We emphasize that local exactness cannot be used to justify (5.8) since the double intersections $U_{\alpha\beta}$ will not, in general, be adapted coordinated neighborhoods for $E$. We are free to suppose that $\sigma_{\beta\alpha} = -\sigma_{\alpha\beta}$. The next step is to use the forms $\sigma_{\alpha\beta}$ to modify the $\eta_\alpha$ in such a way that (i) (5.6) remains valid, and (ii) that these modified forms agree on double intersections and are therefore the restriction of a global form.

On non-empty triple intersections $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, (5.8) gives rise to

$$d_H(\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta}) = 0 \tag{5.9}$$

and hence, because the forms $\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta}$ are of type $(0,s)$,

$$\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} = 0. \tag{5.10}$$

Let $\{ f_\gamma \}$ be a partition of unity on $E$ subordinate to the cover $\mathcal{U}$. Define new type $(1, s)$ forms $\tilde{\eta}_\alpha$ on $J^\infty(U_\alpha)$ by

$$\tilde{\eta}_\alpha = d_H \big[ \sum_{\gamma \in J} f_\gamma \, \sigma_{\gamma\alpha} \big]. \tag{5.11}$$

Then, on account of (5.10), we find that on any double intersection $U_\alpha \cap U_\beta$,

$$\begin{aligned}
\tilde{\eta}_\alpha - \tilde{\eta}_\beta &= d_H \big[ \sum_{\gamma \in J} f_\gamma (\sigma_{\gamma\alpha} - \sigma_{\gamma\beta} \big] \\
&= -d_H \big[ \sum_{\gamma \in J} (f_\gamma) \sigma_{\alpha\beta} \big] = -d_H \sigma_{\alpha\beta} \\
&= \eta_\alpha - \eta_\beta .
\end{aligned}$$

This proves that the type $(1, s)$ forms $\tau_\alpha = \eta_\alpha - \tilde{\eta}_\alpha$ coincide on all double intersections, *i.e.*, $\tau_\alpha = \tau_\beta$ on $U_{\alpha\beta}$, and are therefore the restriction of a global, type $(1, s)$, form $\tau$. Since $\tilde{\eta}_\alpha$ is a $d_H$ exact form on $J^\infty(U_\alpha)$,

$$d_H(\tau_{\mid U_\alpha}) = d_H \eta_\alpha = \omega_{\mid U_\alpha}$$

and so $\omega = d_H \tau$ is exact.

To repeat this proof of exactness for (5.5) at $\Omega^{r,s}(J^\infty(E))$, $r > 2$, it is necessary to formalize each of the above individual steps. To begin, denote the non-empty $(p+1)$-fold intersections of the cover $\mathcal{U}$ by

$$U_{\alpha_0 \alpha_1 \cdots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p} .$$

Let $K^{p,r,s}$ be the Cartesian product

$$K^{p,r,s} = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} \Omega^{r,s}(J^\infty(U_{\alpha_0 \alpha_1 \cdots \alpha_p})).$$

An element $\omega$ of $K^{p,r,s}$ is an ordered tuple of type $(r, s)$ forms $\omega_{\alpha_0 \alpha_1 \cdots \alpha_p}$ defined on $J^\infty(U_{\alpha_0 \alpha_1 \cdots \alpha_p})$. Let

$$r \colon \Omega^{r,s}(J^\infty(E)) \to \prod_{\alpha \in J} \Omega^{r,s}(J^\infty(U_\alpha)) = K^{0,r,s}$$

be the restriction map

$$(r(\omega))_\alpha = \omega_{\mid U_\alpha},$$

and let

$$\delta\colon K^{p,r,s} \to K^{p+1,r,s}$$

be the difference map

$$(\delta\omega)_{\alpha_0\alpha_1\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i \omega_{\alpha_0\alpha_1\cdots\,\widehat{\alpha_i}\,\cdots\alpha_{p+1}}. \tag{5.12}$$

Clearly, $\delta \circ r = 0$ and a standard calculation shows that $\delta^2 = 0$. The complex

$$0 \longrightarrow \Omega^{r,s}(J^\infty(E)) \xrightarrow{r} K^{0,r,s} \xrightarrow{\delta} K^{1,r,s} \xrightarrow{\delta} K^{2,r,s} \xrightarrow{\delta} \cdots$$

is called the *generalized Mayer-Vietoris sequence* of type $(r,s)$ forms on $J^\infty(E)$ with respect to the cover $\mathcal{U}$.

It is not difficult to prove that this sequence is exact. Indeed, if $\omega \in K^{0,r,s}$ satisfies $\delta\omega = 0$, then the components $\omega_\alpha \in \Omega^{r,s}(J^\infty(U_\alpha))$ of $\omega$ all agree on double intersections $U_\alpha \cap U_\beta$ and therefore the $\omega_\alpha$ are the restrictions to $U_\alpha$ of a global type $(r,s)$ form on $J^\infty(E)$. Furthermore, the operator

$$\mathcal{K}\colon K^{p,r,s} \to K^{p-1,r,s}$$

defined by

$$[\mathcal{K}(\omega)]_{\alpha_0\alpha_1\cdots\alpha_{p-1}} = \sum_{\gamma\in J} f_\gamma \omega_{\gamma\alpha_0\alpha_1\cdots\alpha_{p-1}},$$

where $\{ f_\gamma \}$ is the partition of unity subordinate to $\mathcal{U}$, is a homotopy operator for the Mayer-Vietoris sequence, *i.e.*, for any $\omega \in K^{p,r,s}$,

$$\mathcal{K}(\delta\omega) + \delta\mathcal{K}(\omega) = \omega. \tag{5.13}$$

In terms of the Mayer-Vietoris sequence, the forms $\eta$ and $\sigma$ used in the foregoing proof of exactness of (5.5) at $\Omega^{2,s}$ belong to $K^{0,1,s}$ and $K^{1,0,s}$. Equations (5.8), (5.10), and (5.11) become $\delta\eta = d_H\sigma$, $\delta\sigma = 0$, and $\tilde{\eta} = d_H[\mathcal{K}(\sigma)]$.

Evidently, the horizontal differential $d_H$ maps $K^{p,r,s}$ to $K^{p,r+1,s}$ and commutes with both $r$ and $\delta$. This gives us the double complex

$$
\begin{tikzcd}
 & {} & {} & & & \\
0 \arrow[r] & \Omega^{3,s}(J^\infty(E)) \arrow[u] \arrow[r, "r"] & K^{0,3,s} \arrow[u] \arrow[r] & {} & & \\
0 \arrow[r] & \Omega^{2,s}(J^\infty(E)) \arrow[u, "d_H"] \arrow[r, "r"] & K^{0,2,s} \arrow[u, "d_H"] \arrow[r, "\delta"] & K^{1,2,s} \arrow[u] \arrow[r] & {} & \\
0 \arrow[r] & \Omega^{1,s}(J^\infty(E)) \arrow[u, "d_H"] \arrow[r, "r"] & K^{0,1,s} \arrow[u, "d_H"] \arrow[r, "\delta"] & K^{1,1,s} \arrow[u, "d_H"] \arrow[r, "\delta"] & K^{2,1,s} \arrow[u] \arrow[r] & {} \\
0 \arrow[r] & \Omega^{0,s}(J^\infty(E)) \arrow[u, "d_H"] \arrow[r, "r"] & K^{0,0,s} \arrow[u, "d_H"] \arrow[r, "\delta"] & K^{1,0,s} \arrow[u, "d_H"] \arrow[r, "\delta"] & K^{2,0,s} \arrow[u, "d_H"] \arrow[r] & {} \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
.
$$

We are now prepared to complete the proof of the theorem by induction on $r$. The induction hypothesis asserts that the variational bicomplex for any fibered manifold $E' \to M'$ is horizontally exact at $\Omega^{p,s}(J^\infty(E'))$ for all $p \leq r$, where $r \leq n-1$ and $s \geq 1$. This implies, in particular, that the interior rows of the variational bicomplex on each $(p+1)$-fold intersection $U_{\alpha_0\alpha_1\ldots\alpha_p}$ are exact up to and including the $r^{\text{th}}$ column. Let $\omega \in \Omega^{r+1,s}(J^\infty(E))$ and suppose that $d_H\omega = 0$ if $r+1 < n$ or $I(\omega) = 0$ if $r+1 = n$. We prove that $\omega$ is $d_H$ exact.

To begin, we use the local horizontal exactness of the variational bicomplex to deduce that $r(\omega) \in K^{0,r+1,s}$ is $d_H$ exact, *i.e.*,

$$r(\omega) = d_H\eta^0,$$

where $\eta^0 \in K^{0,r,s}$. Since $\delta \circ r = 0$ and $d_H$ commutes with $\delta$ this, in turn, implies that

$$d_H(\delta\eta^0) = 0. \tag{5.14}$$

By the induction hypothesis, $d_H$ closed forms in $K^{1,r,s}$ are $d_H$ exact and therefore

$$\delta\eta^0 = d_H\eta^1, \tag{5.15}$$

where $\eta^1 \in K^{1,r-1,s}$. This argument can be iterated to obtain a sequence of forms $\eta^p \in K^{p,r-p,s}$, for $p = 1, 2, \ldots, r-1$ which satisfy

$$\begin{aligned} &\delta\eta^p = d_H \eta^{p+1} \qquad \text{for} \quad p = 0, 1, \ldots, r-1, \quad \text{and} \\ &\delta\eta^r = 0. \end{aligned} \tag{5.16}$$

These equations can be represented schematically as

$$\begin{tikzcd} \omega \arrow[r, "r"] & \star & & & \\ & \eta^0 \arrow[u, "d_H"'] \arrow[r, "\delta"] & \star & & \\ & & \eta^1 \arrow[u, "d_H"'] \arrow[r, "\delta"] & \star & \\ & & & \eta^2 \arrow[u, "d_H"'] & \cdots . \end{tikzcd}$$

Now we use the partition of unity $\{ f_\alpha \}$ and the homotopy operator $\mathcal{K}$ to define another sequence of forms $\tau^p \in K^{p,r-p,s}$ for $p = r, r-1, \ldots, 1, 0$ by

$$\tau^r = \eta^r$$

and

$$\tau^p = \eta^p - d_H(\mathcal{K}\tau^{p+1}) \qquad \text{for} \quad p = r-1, r-2, \ldots, 0.$$

On account of (5.13) and (5.16), these forms satisfy

$$\begin{aligned} \delta\tau^p &= \delta\eta^p - d_H[\delta(\mathcal{K}\tau^{p+1})] \\ &= d_H\eta^{p+1} - d_H[\tau^{p+1} - \mathcal{K}(\delta\tau^{p+1})] \\ &= d_H\mathcal{K}(\delta\tau^{p+1}). \end{aligned}$$

Because $\delta\tau^r = 0$, this shows that the forms $\tau^p$ are all $\delta$ closed. In particular, since $\tau^0 \in K^{0,r,s}$, $\tau^0$ is the restriction of a global type $(r-1, s)$ form $\tau$ on $J^\infty(E)$. Since

$$\begin{aligned} r(d_H\tau) = d_H(r(\tau)) &= d_H(\tau^0) \\ &= d_H\eta^0 = r(\omega) \end{aligned}$$

we must have that $d_H\tau = \omega$ on all of $J^\infty(E)$. Therefore, $\omega$ is exact. This proves that the horizontal complex (5.5) is exact at $\Omega^{r,s}(J^\infty(E))$ and completes the induction proof. ∎

In Chapter Two we introduced the spaces of functional forms $\mathcal{F}^s(J^\infty(E))$ as subspaces of $\Omega^{n,s}(J^\infty(E))$ and proved the local direct sum decomposition

$$\Omega^{n,s} = d_H\Omega^{n-1,s} \oplus \mathcal{F}^s.$$

Our first corollary to Theorem 5.1 asserts that this decomposition holds globally. This is a fundamental result.

COROLLARY 5.2. *Let $\mathcal{B}^{n,s}(J^\infty(E)) = d_H\big[\Omega^{n-1,s}(J^\infty(E))\big]$ be the space of $d_H$ exact, type $(n,s)$ forms. For each $s \geq 1$, the space of type $(n,s)$ forms on $J^\infty(E)$ admits the global, direct sum decomposition*

$$\Omega^{n,s}(J^\infty(E)) = \mathcal{B}^{n,s}(J^\infty(E)) \oplus \mathcal{F}^s(J^\infty(E)).$$

PROOF: If $\omega \in \Omega^{n,s}(J^\infty(E))$, then

$$\omega = I(\omega) + (1 - I)(\omega).$$

By definition, we have $I(\omega) \in \mathcal{F}^s(J^\infty(E))$ and, because $I^2 = I$, the type $(n,s)$ form $\beta = (1 - I)(\omega)$ satisfies $I(\beta) = 0$. Theorem 5.1 implies that $\beta = d_H\sigma$ for some $\sigma \in \Omega^{n-1,s}(J^\infty(E))$ and hence

$$\omega = I(\omega) + d_H(\sigma).$$

That this is a direct sum decomposition follows from the fact that $I$ is a projection operator which satisfies $I \circ d_H = 0$. ∎

In Chapter Two, we also derived the local, first variational formula for the calculus of variations, *viz.* (2.17). We emphasized that this analysis is insufficient to establish the global validity of this formula because, in general, the type $(n-1,1)$ form $\sigma$, defined by the local formula (2.17b), does not transform properly under fiber-preserving change of variables. The global validity of the first variational formula follows easily from Theorem 5.1, or more precisely, from Corollary 5.2.

COROLLARY 5.3. *There exists a global first variational formula. Specifically, given a Lagrangian $\lambda$ on $J^\infty(E)$, there is a type $(n-1,1)$ form $\sigma$ on $J^\infty(E)$ such that*

$$d_V\lambda = E(\lambda) + d_H\sigma. \tag{5.17}$$

*If $\dim M = 1$, then for a given Lagrangian $\lambda$, the type $(0,1)$ form $\sigma$ is unique.*

*If $X$ is any generalized vector field on $E$ and $X_{\mathrm{ev}}$ its associated evolutionary vector field, then there is a type $(n-1,0)$ form $\eta$ on $J^\infty(E)$ such that*

$$\mathcal{L}^\sharp_{\mathrm{pr}\,X}\lambda = X_{\mathrm{ev}} \mathbin{\lrcorner} E(\lambda) + d_H\eta. \tag{5.18}$$

PROOF: Since $I(d_V\lambda) = E(\lambda)$, the application of the previous corollary to the type $(n,1)$ form $d_V\lambda$ yields (5.17). If $n=1$, $\sigma$ is of type $(0,1)$. If $\tilde\sigma$ also satisfies (5.17), then $d_H(\sigma - \tilde\sigma) = 0$ and therefore $\sigma = \tilde\sigma$.

To prove (5.18), we write, in accordance with Proposition 1.20,

$$\mathrm{pr}\,X = \mathrm{pr}\,X_{\mathrm{ev}} + \mathrm{tot}\,X \tag{5.19}$$

and compute

$$\mathcal{L}^\sharp_{\mathrm{pr}\,X}\lambda = \pi^{n,0}[\mathcal{L}_{\mathrm{pr}\,X_{\mathrm{ev}}}\lambda + \mathcal{L}_{\mathrm{tot}\,X}\lambda].$$

Since $X_{\mathrm{ev}}$ is an evolutionary vector field we find, using (5.17) and Proposition 1.16, that

$$\begin{aligned}
\pi^{n,0}\,[\mathcal{L}_{\mathrm{pr}\,X_{\mathrm{ev}}}\lambda] &= \pi^{n,0}[\mathrm{pr}\,X_{\mathrm{ev}} \mathbin{\lrcorner} d_V\lambda] \\
&= \pi^{n,0}\,[\mathrm{pr}\,X_{\mathrm{ev}} \mathbin{\lrcorner} E(\lambda) + \mathrm{pr}\,X_{\mathrm{ev}} \mathbin{\lrcorner} d_H\sigma] \\
&= X_{\mathrm{ev}} \mathbin{\lrcorner} E(\lambda) - d_H[\pi^{n-1,0}(\mathrm{pr}\,X_{\mathrm{ev}} \mathbin{\lrcorner} \sigma)].
\end{aligned}$$

Moreover, since interior evaluation by a total vector field lowers horizontal degree by one,

$$\begin{aligned}
\pi^{n,0}\,[\mathcal{L}_{\mathrm{tot}\,X}\lambda] &= \pi^{n,0}[d(\mathrm{tot}\,X \mathbin{\lrcorner} \lambda) + \mathrm{tot}\,X \mathbin{\lrcorner} d_V\lambda] \\
&= d_H(\mathrm{tot}\,X \mathbin{\lrcorner} \lambda).
\end{aligned}$$

Consequently (5.18) holds with

$$\eta = -\pi^{n-1,0}\,[\mathrm{pr}\,X_{\mathrm{ev}} \mathbin{\lrcorner} \sigma] + \mathrm{tot}\,X \mathbin{\lrcorner} \lambda.$$

∎

Equation (5.18) furnishes us with a global version of Noether's Theorem. Note that in this version the generalized vector field $X$ is a symmetry of the Lagrangian $\lambda$ whereas in Theorem 3.13, $X$ is a symmetry of the source form $\Delta = E(\lambda)$.

Corollary 5.4. *Let $\lambda$ be a Lagrangian on $J^\infty(E)$. Then every global, generalized symmetry $X$ of $\lambda$ generates a global conservation law for the Euler-Lagrange equations $E(\lambda)$.*

Proof: If $\mathcal{L}^\sharp_{\mathrm{pr}\,X}\lambda = 0$, then (5.18) reduces to

$$-X_{\mathrm{ev}} \lrcorner\, E(\lambda) = d_H \eta$$

so that $\eta$ is the global conservation law with generator $-X_{\mathrm{ev}}$. ∎

Let $\Lambda$ be an $n$ form on $J^\infty(E)$. The form $\Lambda$ is called *Lepagean* if, for all $\pi^\infty_E$ vertical vector fields $Y$ on $J^\infty(E)$,

$$\pi^{n,0}[Y \lrcorner\, d\Lambda] = 0. \tag{5.20}$$

Given a Lagrangian $\lambda \in \Omega^{n,0}(J^\infty(E))$, an $n$ form $\Lambda$ on $J^\infty(E)$ is called a Lepagean equivalent if $\Lambda$ is a Lepagean form and

$$\pi^{n,0}\Lambda = \lambda. \tag{5.21}$$

This latter condition implies that for any local section $s\colon U \to E$,

$$[j^\infty(s)]^*\Lambda = [j^\infty(s)]^*\lambda,$$

and therefore $\Lambda$ and $\lambda$ determine the same fundamental integral. For example, if $\dim M = 1$ and

$$\lambda = L(x, u^\alpha, \dot u^\alpha)dx$$

is a first order Lagrangian, then the Poincaré-Cartan form

$$\Lambda = \lambda + \frac{\partial L}{\partial \dot u^\alpha}\,\theta^\alpha$$

is easily seen to be a Lepagean equivalent to $\lambda$. Thus, the notion of Lepagean equivalents furnishes us with one possible means by which the classical Poincaré-Cartan form in mechanics can be generalized to arbitrary variational problems. The theory of Lepage equivalents and their role in the calculus of variations have been extensively. The review article by Krupka [*44*] describes this work. The global existence of Lepage equivalents can be established by various means; see, for example, [*34*], [*41*], [*49*], and [*48*].

Corollary 5.5. *Every Lagrangian $\lambda$ admits a Lepagean equivalent $\Lambda$. If $n = 1$, $\Lambda$ is unique.*

Proof: First observe that if $Y$ is a $\pi_E^\infty$ vertical vector field on $J^\infty(E)$ and $\omega$ is a type $(r, s)$ form on $J^\infty(E)$, then $Y \lrcorner\, \omega$ is of type $(r, s-1)$. If $\omega$ is of type $(n, 1)$ and $Y \lrcorner\, \omega = 0$ for all $\pi_E^\infty$ vertical vector fields, then $\omega \in \mathcal{F}^1(J^\infty(E))$.

Every $n$ form $\Lambda$ on $J^\infty(E)$ can be written uniquely in the form

$$\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \cdots ,$$

where $\Lambda_i$ is of type $(n-i, i)$. The condition (5.21) evidently requires that $\Lambda_0 = \lambda$ so that, by the first variational formula (5.17),

$$\begin{aligned} d\Lambda &= (d_V\lambda + d_H\Lambda_1) + (d_V\Lambda_1 + d_H\Lambda_2) + \cdots \\ &= (E(\lambda) + d_H(\sigma + \Lambda_1)) + \xi_2 + \xi_3 + \cdots , \end{aligned} \tag{5.22}$$

where $\xi_i$ is of type $(n-i, i)$. Hence, if $Y$ is any $\pi_E^\infty$ vertical vector field,

$$\pi^{n,0}[Y \lrcorner\, d\Lambda] = Y \lrcorner\, d_H(\Lambda_1 + \sigma)$$

and thus, by virtue of our earlier remark, $\Lambda$ is Lepagean if and only if

$$d_H(\Lambda_1 + \sigma) \in \mathcal{F}^1(J^\infty(E)).$$

But by Corollary 5.2 this is possible if and only if

$$d_H(\Lambda_1 + \sigma) = 0. \tag{5.23}$$

This proves that the $n$ form

$$\Lambda = \lambda - \sigma$$

is a Lepage equivalent for $\lambda$. When $n = 1$, $\Lambda_1 + \sigma$ is of type $(0, 1)$ and (5.23) holds if and only if $\Lambda_1 = -\sigma$. The uniqueness of the Lepage equivalent $\Lambda$ now follows from that of $\sigma$. ∎

Two further remarks concerning Lepagean equivalents are in order. First, if $\Lambda$ is any Lepage equivalent for $\lambda$ and if $Y$ is any $\pi_M^\infty$ vertical vector field, then (5.22) and (5.23) show that

$$Y \lrcorner\, d\,\Lambda = Y \lrcorner\, E(\lambda) + (Y \lrcorner\, \xi_2) + (Y \lrcorner\, \xi_3) + \cdots .$$

Thus, a local section $s\colon U \to E$ is a solution to the Euler-Lagrange equations for $\lambda$ if and only if

$$[j^\infty(s)]^*(Y \mathbin{\lrcorner}\, d\,\Lambda) = 0$$

for all $\pi^\infty_M$ vector fields $Y$. Second, observe that we used the first variational formula (5.17) to establish the existence of Lepage equivalents. This can easily turned around — that is, given a Lepage equivalent, one immediately derives the first variational formula. These two remarks illustrate an important point in Krupka's work — that a general, differential geometric treatment of the calculus of variations can be based entirely on the theory of Lepage equivalents.

Let $\beta$ be a closed $p$ form on a manifold $M$ and let $X$ be a vector field on $M$. Since Lie differentiation commutes with $d$, $\mathcal{L}_X\beta$ is also closed. Moreover, by Cartan's formula,

$$\mathcal{L}_X\beta = d(X \mathbin{\lrcorner}\, \beta) + X \mathbin{\lrcorner}\, d\beta = d(X \mathbin{\lrcorner}\, \beta)$$

it follows that $\mathcal{L}_X\beta$ is exact. Furthermore, $\mathcal{L}_X\beta$ is *naturally* exact since the $p-1$ form $\alpha = X \mathbin{\lrcorner}\, \beta$ is a natural form (or concomitant) constructed solely from $X$ and $\beta$. The next corollary describes analogous results for forms on the variational bicomplex.

Corollary 5.6. *Let $\omega \in \Omega^{r,s}(J^\infty(E))$.*

*(i) Suppose $\omega$ is $d_V$ closed. Let $X$ be a vector field on $J^\infty(E)$ which is $\pi^\infty_M$ related to a vector field $X_0$ on $M$. Then $\mathcal{L}^\sharp_X\,\omega$ is $d_V$ closed but not, in general, $d_V$ exact. If, however, $X$ is $\pi^\infty_M$ vertical, then $\mathcal{L}^\sharp_X\omega$ is naturally exact:*

$$\mathcal{L}^\sharp_X\,\omega = d_V(X \mathbin{\lrcorner}\, \omega). \tag{5.24}$$

*(ii) Suppose $\omega$ is $d_H$ closed. Let $X$ be a generalized vector field on $E$. Then $\mathcal{L}^\sharp_{\mathrm{pr}\,X}\omega$ is $d_H$ exact but not naturally so.*

Proof: (i) That $\mathcal{L}^\sharp_X\omega$ is $d_V$ closed is simply a restatement of Proposition 3.17. To show that $\mathcal{L}^\sharp_X$ need not be $d_V$ exact, consider $E : \mathbf{R} \times S^1 \to \mathbf{R}$, let $\omega$ be the type $(1,1)$ form

$$\omega = dx \wedge \theta = dx \wedge du, \qquad \text{and let} \qquad X = x\,\frac{\partial}{\partial x}.$$

Then the form $\mathcal{L}^\sharp_X\,\omega = \omega$ is $d_V$ closed but not $d_V$ exact.

To prove (5.24), we simple note that if $X$ is $\pi^\infty_M$ vertical, then $X \mathbin{\lrcorner}\, \omega$ is of type $(r, s-1)$ and hence

$$\begin{aligned}\mathcal{L}^\sharp_X\omega &= \pi^{r,s}[d_H(X \mathbin{\lrcorner}\, \omega) + d_V(X \mathbin{\lrcorner}\, \omega) + X \mathbin{\lrcorner}\, d_H\omega] \\ &= d_V(X \mathbin{\lrcorner}\, \omega),\end{aligned}$$

as required.

(ii) Again Proposition 3.17 shows that $\mathcal{L}^\sharp_{\operatorname{pr} X}\omega$ is $d_H$ closed. In view of (5.19), it suffices to evaluate $\mathcal{L}^\sharp_{\operatorname{pr} X_{\mathrm{ev}}}\omega$ and $\mathcal{L}^\sharp_{\operatorname{tot} X}\omega$ separately. Since $\operatorname{tot} X \lrcorner\, \omega$ is of type $(r-1, s)$, the same calculation as above gives

$$\begin{aligned}\mathcal{L}^\sharp_{\operatorname{tot} X}\omega &= \pi^{r,s}[d_H(\operatorname{tot} X \lrcorner\, \omega) + d_V(\operatorname{tot} X \lrcorner\, \omega) + \operatorname{tot} X \lrcorner\, d_V\omega] \\ &= d_H(\operatorname{tot} X \lrcorner\, \omega).\end{aligned}$$

Now consider the evolutionary vector field $Y = X_{\mathrm{ev}}$. If $s \geq 1$, then by Theorem 5.1, $\omega$ is $d_H$ exact, say $\omega = d_H\eta$, and

$$\mathcal{L}^\sharp_{\operatorname{pr} Y}\omega = d_H(\mathcal{L}^\sharp_{\operatorname{pr} Y}\eta).$$

If $s = 0$, then $d_V\omega$ is a $d_H$ closed form of type $(r, 1)$ and so $d_V\omega = d_H\eta$. We now use Proposition 1.16 to conclude that

$$\begin{aligned}\mathcal{L}^\sharp_{\operatorname{pr} Y}\omega &= d_V(\operatorname{pr} Y \lrcorner\, \omega) + \operatorname{pr} Y \lrcorner\, d_V\omega \\ &= \operatorname{pr} Y \lrcorner\, d_H\eta = -d_H(\operatorname{pr} Y \lrcorner\, \eta).\end{aligned}$$

This proves (ii). Observe that in this latter case $\mathcal{L}^\sharp_{\operatorname{pr} Y}\omega$ is not naturally exact since $\operatorname{pr} Y \lrcorner\, \eta$ is not a natural concomitant of $X$ and $\omega$ — the partition of unity used in the proof of Theorem 5.1 is needed to construct the form $\eta$ from $\omega$. ∎

In Chapter Four, we introduced a system of invariantly defined weights for forms in $\Omega^{r,s}_{\mathcal{P}_{j,k}}(J^\infty(E))$. Recall that these are forms in $\Omega^{r,s}_k(J^\infty(E))$ whose coefficients are polynomial in the derivatives of the independent variables of order $j+1, j+2, \ldots, k$. We proved in Theorem 4.23 that if $\omega \in \Omega^{r,s}_{\mathcal{P}_{j,k}}$ is $d_H$ closed then, locally, $\omega = d_H\eta$ where $\eta$ is a minimal weight form. We also introduced the Jacobian subcomplex $(\mathcal{J}^{*,*}_k, d_H, d_V)$ of the variational bicomplex. Corollary 4.43 asserts that the Jacobian subcomplex is locally exact. Now if $f$ is any function on $E$, then the weights of $\omega$ and $f\omega$ are the same at points where $f$ is non-zero; also if $\omega \in \mathcal{J}^{r,s}_k(J^\infty(E))$, then $f\omega \in \mathcal{J}^{r,s}_k(J^\infty(E))$. Consequently, the Mayer-Vietoris argument used to prove Theorem 5.1 can be repeated, without change, to prove the existence of global minimal weight forms and to prove the global exactness of the interior rows of the Jacobian subcomplex.

Corollary 5.7. *(i) For $s \geq 1$, let $\omega \in \Omega^{r,s}_{\mathcal{P}_{j,k}}(J^\infty(E))$ and suppose that $d_H\omega = 0$ if $r < n$ or $I(\omega) = 0$ if $r = n$. Then there is a form $\eta \in \Omega^{r-1,s}_{\mathcal{P}_{j,k}}(J^\infty(E))$ with weights*

$$w_p(\eta) = w_p(\omega) - 1$$

*for $p = j, j+1, \ldots, k-1$ such that $\omega = d_H \eta$.*

*(ii) The horizontal Jacobian subcomplex*

$$0 \longrightarrow \mathcal{J}_k^{0,s}(J^\infty(E)) \xrightarrow{d_H} \mathcal{J}_k^{1,s}(J^\infty(E)) \cdots \xrightarrow{d_H} \mathcal{J}_k^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}^s(J^\infty(E))$$

*is exact. In addition, if $\omega \in \mathcal{J}_k^{n,s}(J^\infty(E))$ and $I(\omega) = 0$, then there is a form $\eta \in \mathcal{J}_k^{n-1,s}$ such that $\omega = d_H \eta$.*

Theorem 5.1 also enables us to compute the cohomology of the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$. We shall need the following lemma.

LEMMA 5.8. *Let $\gamma$ be a $d$ closed $p$ form on $J^\infty(E)$. If $p \leq n$ and $\pi^{p,0}(\gamma) = 0$ or if $p = n+s$ and $(I \circ \pi^{n,s})(\gamma) = 0$, then $\gamma$ is $d$ exact.*

PROOF: For $p \leq n$, write

$$\gamma = -\gamma_1 + \gamma_2 - \cdots + (-1)^p \gamma_p,$$

where $\gamma_i$ is of type $(p-i, i)$. Since $\gamma$ is $d$ closed, these forms satisfy

$$d_H \gamma_1 = 0, \quad d_V \gamma_i = d_H \gamma_{i+1} \quad \text{for } i = 1, 2, \ldots, p, \quad \text{and} \quad d_V \gamma_p = 0.$$

On account of Theorem 5.1, these equations imply that there are type $(p-i-1, i)$ forms $\rho_i$ on $J^\infty(E)$ such that

$$\gamma_1 = d_H \rho_1, \quad \gamma_{i+1} = -d_V \rho_i + d_H \rho_{i+1} \quad \text{for } i = 1, 2, \ldots, p-2, \quad \text{and} \quad d_V \rho_{p-1} = 0.$$

It now follows that

$$d\,(\rho_1 - \rho_2 + \cdots + (-1)^{p-1} \rho_{p-1}) = \gamma,$$

which proves that $\gamma$ is $d$ exact.

For $p = n+s$, the proof is similar except that now the condition $(I \circ \pi^{n,s})(\gamma) = 0$ implies, by Theorem 5.1, that $\pi^{n,s}(\gamma)$, the type $(n,s)$ component of $\gamma$, is $d_H$ exact. ∎

THEOREM 5.9. *The cohomology of the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$*

$$\begin{aligned} 0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0}(J^\infty(E)) &\xrightarrow{d_H} \Omega^{1,0}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,0}(J^\infty(E)) \cdots \\ &\xrightarrow{d_H} \Omega^{n,0}(J^\infty(E)) \xrightarrow{E} \mathcal{F}^1(J^\infty(E)) \xrightarrow{\delta_V} \mathcal{F}^2(J^\infty(E)) \xrightarrow{\delta_V} \cdots \end{aligned} \tag{5.25}$$

*is isomorphic to the de Rham cohomology of the total space $E$, that is*

$$H^p(\Omega^{*,0}(J^\infty(E)), d_H) \cong H^p(\Omega^*(E), d), \tag{5.26a}$$

*for $p \leq n$, and*

$$H^s(\mathcal{F}^*(J^\infty(E)), \delta_V) \cong H^p(\Omega^*(E), d), \tag{5.26b}$$

*for $p = n + s$ and $s \geq 1$.*

Proof: In view of Theorem 5.1, this theorem is a standard, elementary result in homological algebra which is established by diagram chasing. We present the details of this chase because they are important in their own right as part of the variational calculus.

We begin with the observation that the projection map $\pi^\infty_E \colon J^\infty(E) \to E$ is a homotopy equivalence ( see Lemma (5.25)) and therefore the de Rham cohomology of $J^\infty(E)$ is isomorphic to that of $E$. Consequently the theorem can be established by constructing an isomorphism from $H^*\big(\Omega^*(J^\infty(E)), d\big)$, the de Rham cohomology of $J^\infty(E)$, to the cohomology of the Euler-Lagrange complex $H^*\big(\mathcal{E}^*(J^\infty(E))\big)$.

Since the projection map $\pi^{r,s} \colon \Omega^{r+s}(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$ satisfies

$$\begin{aligned} \pi^{r+1,0} \circ d &= d_H \circ \pi^{r,0}, &\qquad &\text{for } r \leq n-1 \\ I \circ \pi^{n,1} \circ d &= E \circ \pi^{n,0}, &\qquad &\text{and} \\ I \circ \pi^{n,s+1} \circ d &= \delta_V \circ I \circ \pi^{n,s}, &\qquad &\text{for } s \geq 1 \end{aligned}$$

the map

$$\Psi \colon \Omega^*(J^\infty(E)) \to \mathcal{E}^*(J^\infty(E))$$

defined, for $\omega \in \Omega^p(J^\infty(E))$, by

$$\Psi(\omega) = \begin{cases} \pi^{p,0}(\omega), & \text{for } p \leq n \quad \text{and} \\ I \circ \pi^{n,s}(\omega), & \text{if } p = n+s \text{ and } s \geq 1. \end{cases}$$

is a cochain map from the de Rham complex on $J^\infty(E)$ to the Euler-Lagrange complex on $J^\infty(E)$. Note that if $\tilde{\omega}$ is a $p = n + s$ form on $E$ then $\omega = (\pi^\infty_E)^*(\tilde{\omega})$ is a $p$ form on $J^\infty(E)$ whose projection $\pi^{n,s}(\omega)$ already lies in $\mathcal{F}^s(J^\infty(E))$, *i.e.*,

$$\Psi(\omega) = \pi^{n,s}(\omega). \tag{5.27}$$

The the induced map in cohomology will be denoted by

$$\Psi^* \colon H^p\big(\Omega^*(J^\infty(E))\big) \to H^p\big(\mathcal{E}^*(J^\infty(E))\big).$$

We prove that $\Psi^*$ is an isomorphism in cohomology by constructing the inverse map

$$\Phi\colon H^p\big(\mathcal{E}^*(J^\infty(E))\big) \to H^p\big(\Omega^*(J^\infty(E))\big). \tag{5.28}$$

To define $\Phi$, it is convenient to consider separately the two pieces of the complex $\mathcal{E}^*(J^\infty(E))$, the first piece being the horizontal edge

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1.$$

Let $[\omega] \in H^p\big(\mathcal{E}^*(J^\infty(E))\big)$ for $p \leq n$. Put $\omega_0 = \omega \in \Omega^{p,0}(J^\infty(E))$. Then the type $(p,1)$ form $\sigma_0 = d_V\omega$ satisfies $d_H(\sigma_0) = -d_V d_H\omega_0 = 0$ if $p < n$ and $I(\sigma_0) = E(\omega) = 0$ if $p = n$. By Theorem 5.1, the form $\sigma_0$ is $d_H$ exact. Let $d_H\omega_1 = \sigma_0$.

Now repeat this process to obtain a sequence of forms $\omega_i \in \Omega^{p-i,i}(J^\infty(E))$ and $\sigma_i \in \Omega^{p-i,i+1}(J^\infty(E))$ satisfying

$$d_H\omega_i = \sigma_{i-1} \qquad \text{and} \qquad \sigma_i = d_V\omega_i \qquad \text{for } i = 1,\, 2, \ldots, p, \tag{5.29}$$

that is,

$$\begin{tikzcd}
\cdots & \sigma_2 & & \\
& \omega_2 \arrow[u, "d_V"] \arrow[r, "d_H"] & \sigma_1 & \\
& & \omega_1 \arrow[u, "d_V"] \arrow[r, "d_H"] & \sigma_0 \\
& & & \omega_0. \arrow[u, "d_V"]
\end{tikzcd}$$

Since $\sigma_p$ is a type $(0, p+1)$ form, the equation $d_H\sigma_p = 0$ implies that $\sigma_p = 0$, *i.e.*, $d_V\omega_p = 0$. We define the form $\beta \in \Omega^p(J^\infty(E))$ by

$$\beta = \omega_0 - \omega_1 + \omega_2 - \cdots + (-1)^p\omega_p. \tag{5.30}$$

A simple calculation, based upon (5.29), shows that $d\beta = 0$. Moreover, it is not difficult to check that the cohomology class $[\beta] \in H^p\big(J^\infty(E))\big)$ is independent of the choice of the representative $\omega_0$ for the class $[\omega]$ and independent of the choices taken for the $\omega_i$ in (5.29). The map $\varPhi$ is defined by

$$\varPhi([\omega]) = [\beta].$$

Note that while $\Psi$ is defined on forms, $\varPhi$ is only well-defined in cohomology. In §5C and §5D, we shall consider circumstances under which there are maps defined on forms which induce the map $\varPhi$ in cohomology.

Evidently, for $[\omega] \in H^p\bigl(\Omega^{*,0}(J^\infty(E))\bigr)$, we have

$$\Psi^* \circ \varPhi([\omega]) = \Psi^*([\beta]) = [\omega].$$

Accordingly, it remains to show that $\varPhi \circ \Psi^*$ is the identity map on $H^p\bigl(\Omega^*(J^\infty(E))\bigr)$. To this end, let $\alpha \in \Omega^p(J^\infty(E))$ be a $d$ closed form. Decompose $\alpha$ by type into the sum

$$\alpha = \alpha_0 - \alpha_1 + \alpha_2 - \cdots + (-1)^p \alpha_p,$$

where $\alpha_i$ is of type $(p-i, i)$. Since $d\alpha = 0$, these forms satisfy

$$d_H \alpha_0 = 0, \quad d_V \alpha_i = d_H \alpha_{i+1} \quad \text{for } i = 0, 1, \ldots, p-1, \quad \text{and} \quad d_V \alpha_p = 0.$$

Let $\omega$ be the $d_H$ closed, type $(p, 0)$ form $\alpha_0$ and define the $p$ form $\beta$ by (5.30). Then

$$\varPhi \circ \Psi^*([\alpha]) = \varPhi([\omega]) = [\beta]$$

and hence, to complete the proof of the theorem, we must show that $\alpha$ and $\beta$ define the same cohomology class on $J^\infty(E)$. But, since the type $(p, 0)$ components of $\alpha$ and $\beta$ coincide, the difference $\gamma = \beta - \alpha$ satisfies the hypothesis of Lemma 5.8 and is therefore $d$ exact. This proves that $[\beta] = [\alpha]$, as required.

The proof of (5.26) for the case $p = n + s$ is similar — the inverse map $\varPhi$ can be defined exactly as above since, for $\omega \in \mathcal{F}^s(J^\infty(E))$, the condition $\delta_V \omega = 0$ implies that $d_V \omega = d_H \omega_1$ for some type $(n-1, s+1)$ form $\omega_1$. ∎

The explicit nature of the isomorphism from $H^*(\Omega^*(E))$ to $H^*\bigl(\mathcal{E}^*(J^\infty(E))\bigr)$ should be emphasized — if $\omega \in \Omega^p(E)$ represents a nontrivial cohomology class on $E$ and if we identify $\omega$ with its pullback to $J^\infty(E)$ via $\pi^\infty_E$, then the projection

$$\Psi(\omega) = \begin{cases} \pi^{p,0}(\omega), & \text{if } p \leq n \\ \pi^{n,s}(\omega), & \text{if } p = n + s,\ s \geq 1 \end{cases}$$

is a nontrivial cohomology class in the Euler-Lagrange complex. This observation enables us to readily construct examples of variationally trivial Lagrangians which are not globally exact and examples of locally variational source forms which are not the Euler-Lagrange forms of global Lagrangians.

EXAMPLE 5.10. Let $E$ be the product bundle

$$E : S^2 \times S^1 \times S^1 \to S^2.$$

Now $H^2(E)$ is the two dimensional vector space generated by $\nu$, the volume form on $S^2$, and by $\mu = du \wedge dv$, where $du$ and $dv$ are the standard angular one forms on the fiber $S^1 \times S^2$. Thus $H^2(\mathcal{E}^*)$ is generated by the two Lagrangians

$$\lambda_1 = \pi^{2,0}(\nu) = \nu$$

and

$$\begin{aligned}\lambda_2 &= \pi^{2,0}(du \wedge dv)\\ &= \pi^{2,0}[(\theta^u + u_x\, dx + u_y\, dy) \wedge (\theta^v + v_x\, dx + v_y\, dy)]\\ &= (u_x v_y - u_y v_x)\, dx \wedge dy.\end{aligned}$$

Here $(x, y)$ are arbitrary coordinates on the base manifold $S^2$. Theorem 5.9 states that every variationally trivial Lagrangian $\lambda$ on $J^\infty(E)$ can be expressed uniquely in the form

$$\lambda = d_H\eta + c_1\lambda_1 + c_2\lambda_2,$$

where $\eta \in \Omega^{1,0}(J^\infty(E))$ and $c_1$ and $c_2$ are constants.

The Lagrangian $\lambda_2$ has a particularly interesting property. Let

$$s \colon S^2 \to E$$

be a section of $E$. Then the pullback of $\lambda_2$ by $s$ to $S^2$ is

$$j^\infty(s)^*\lambda_2 = s^*(du \wedge dv) = \alpha \wedge \beta, \tag{5.31}$$

where $\alpha = s^*(du)$ and $\beta = s^*(dv)$. Since $H^1(S^2) = 0$, and $\alpha$ is closed on $S^2$ it follows that $\alpha$ is exact; say $\alpha = df$, where $f$ is a real-valued function on $S^2$. Since $\beta$ is closed on $S^2$, we can rewrite (5.31) as

$$j^\infty(s)^*(\lambda_2) = d(f\beta). \tag{5.32}$$

Thus, on every section of $E$, the Lagrangian $\lambda_2$ pulls back to an exact form on the base manifold every though $\lambda_2$ itself, as a Lagrangian in the variational bicomplex is not exact. The point here is that the one form $f\beta$ in (5.32) and in particular the function $f$, can not be computed at any given point from the knowledge of the jet of $s$ at that point. In general then, *the condition that $\omega \in \Omega^{r,0}(J^\infty(E))$ be exact on all sections of $E$ is a necessary condition for $\omega$ to be $d_H$ exact but it is not a sufficient condition.*

Since $H^3(E)$ is spanned by the two 3 forms $du \wedge \nu$ and $dv \wedge \nu$, the source forms

$$\pi^{2,1}(du \wedge \nu) = \theta^u \wedge \nu \quad \text{and} \quad \pi^{2,1}(dv \wedge \nu) = \theta^v \wedge \nu$$

are locally variational ( local Lagrangians are $-u\,\nu$ and $-v\,\nu$, respectively) but not globally variational. Admittedly, these source forms don't determine very reasonable source equations but bear in mind that these forms are simply representatives of the cohomology of the Euler-Lagrange complex at $\mathcal{F}^1(J^\infty(E))$. For example, let $g$ be a Riemannian metric on $S^2$ and let $\Delta_g$ be the real-valued Laplacian on $S^2$. Then the source form

$$\Delta = (\Delta_g u + a)\theta^u \wedge \nu + (\Delta_g v + b)\theta^v \wedge \nu,$$

where $a$ and $b$ are constants, is always locally variational. The source form $\Delta$ is globally variational if and only if the constants $a$ and $b$ vanish; a global Lagrangian being

$$\lambda = \frac{1}{2}(g^{ij}u_i u_j + g^{ij}v_i v_j)\sqrt{g}dx^1 \wedge dx^2.$$

∎

Example 5.11. Now consider the bundle

$$E \colon M \times S^1 \times S^1 \to M,$$

where $M = \mathbf{R}^3 - \{\,0\,\}$. The total space $E$ is homotopic to that of the previous example; the de Rham cohomology ring is generated by the angular one forms $du$ and $dv$ and the two form

$$\sigma = \frac{z\,dx \wedge dy - y\,dx \wedge dz + x\,dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}} = \frac{1}{r^3}\,R \lrcorner\, \nu,$$

where $\nu = dx \wedge dy \wedge dz$, $R$ is the radial vector field $R = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ and $r^2 = R \cdot R$. However, because we have changed the dimension of the base manifold from 2 to 3, the interpretations of these cohomology classes on $E$ as cohomology classes in the variational bicomplex has changed. The obstructions to writing a variationally trivial Lagrangian as a $d_H$ exact form have now shifted to $H^3(E)$ and, likewise, the obstructions to finding global Lagrangians for locally variational source forms have shifted to $H^4(E)$. The generators of the cohomology of the Euler-Lagrange complex at $\Omega^{3,0}(J^\infty(E))$ are therefore

$$\begin{aligned}\lambda_1 &= \pi^{3,0}(du \wedge \sigma) = (u_x\, dx + u_y\, dy + u_z\, dz) \wedge \sigma \\ &= \frac{1}{r^3}(R \cdot \nabla u)\, \nu \end{aligned} \tag{5.33a}$$

and

$$\begin{aligned}\lambda_2 &= \pi^{3,0}(dv \wedge \sigma) = (v_x\, dx + v_y\, dy + v_z\, dz) \wedge \sigma \\ &= \frac{1}{r^3}(R \cdot \nabla v)\, \nu, \end{aligned} \tag{5.33b}$$

while the single generator of the cohomology at $\mathcal{F}^1$ is

$$\begin{aligned}\Delta &= \pi^{3,1}(du \wedge dv \wedge \sigma) \\ &= \frac{1}{r^3}[(R \cdot \nabla u)\, \theta^u - (R \cdot \nabla v)\theta^v] \wedge\, \nu. \end{aligned} \tag{5.34}$$

This source form merits further discussion. The source equations defined by $\Delta$ are

$$xu_x + yu_y + zu_z = 0 \quad \text{and} \quad xv_x + yv_y + zv_z = 0. \tag{5.35}$$

These equations are now defined on all of $\mathbf{R}^3$; however, by dropping the factor of $\dfrac{1}{r^3}$, these equations are no longer the components of a locally variational source form. It is natural to ask if (5.35) is equivalent to some system of locally variational equations defined on all of $\mathbf{R}^3$. Let

$$\tilde{\Delta} = \Delta_1\, \theta^u \wedge \nu + \Delta_2\, \theta^v \wedge \nu,$$

where

$$\begin{aligned}\Delta_1 &= A(R \cdot \nabla u) + B(R \cdot \nabla v), \\ \Delta_2 &= C(R \cdot \nabla u) + D(R \cdot \nabla v), \end{aligned} \tag{5.36}$$

and $A$, $B$, $C$, and $D$ are functions on $\mathbf{R}^3 \times S^1 \times S^1$ with $AD - BC \neq 0$. This latter condition insures that the system of equations $\Delta_1 = 0$ and $\Delta_2 = 0$ is equivalent to (5.35). We now substitute (5.36) into the Helmholtz conditions (see (3.16))

$$\frac{\partial \Delta_1}{\partial u_i} = 0, \quad \frac{\partial \Delta_1}{\partial v_i} = -\frac{\partial \Delta_2}{\partial u_i}, \quad \frac{\partial \Delta_2}{\partial v_i} = 0,$$

and

$$\frac{\partial \Delta_1}{\partial v} = \frac{\partial \Delta_2}{\partial u} - D_i(\frac{\partial \Delta_2}{\partial u_i})$$

to deduce that $A = D = 0$, $B = -C$, and

$$x^i \, \frac{\partial B}{\partial x^i} + 3B = 0. \tag{5.37}$$

The source form (5.36) simplifies to

$$\tilde{\Delta} = B[(R \cdot \nabla v)\, \theta^u - (R \cdot \nabla u)\, \theta^v] \wedge \nu.$$

Since the only smooth solution to (5.37), defined on all of $\mathbf{R}^3 \times S^1 \times S^1$, is $B = 0$, we conclude that the system (5.35) is not globally equivalent to a system of locally variational equations. Of course, $B = \dfrac{1}{r^3}$ is the solution to (5.37) on $E$ from which we recover the the original source form (5.34).

We now examine the possibility of using Noether's theorem, in the form of Theorem 3.32, to find global conservation laws for $\Delta$. Let

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Then it is easily verified that $\mathcal{L}_{\mathrm{pr}\, X} \Delta = 0$ and therefore $X$ is a distinguished symmetry for $\Delta$. By Theorem 3.32, the Lagrangian $\lambda = X_{\mathrm{ev}} \mathbin{\lrcorner} \Delta$ is variationally trivial and so, at least locally, we obtain a conservation law $\lambda = d_H \eta$. However, in this instance, the cohomology class of $\lambda$, *viz.* $[\lambda] \in H^3(\mathcal{E}^*)$, vanishes and consequently the symmetry $X$ gives rise to a global conservation law. Indeed, a straightforward calculation shows that

$$X_{\mathrm{ev}} = -(X \cdot \nabla u) \frac{\partial}{\partial u} - (X \cdot \nabla v) \frac{\partial}{\partial v}$$

and that

$$\begin{aligned}
\lambda &= \frac{1}{r^3} [(R \cdot \nabla u)(X \cdot \nabla v) - (R \cdot \nabla u)(X \cdot \nabla v)]\, \nu \\
&= \frac{(R \times X)}{r^3} \cdot [\nabla u \times \nabla v]\, \nu \\
&= \frac{1}{r^3} [-xz\, dx - yz\, dy + (x^2 + y^2)\, dz] \wedge d_H u \wedge d_H v \\
&= d_H(\frac{z}{r} d_H u \wedge d_H v).
\end{aligned}$$

Thus the conservation law for $\Delta$ generated by the rotational symmetry $X$ is

$$\eta = \frac{z}{r}\, d_H u \wedge d_H v.$$

The form $\eta$ is defined on all on $J^\infty(E)$.

Consider next the vector field

$$Y = \frac{\partial}{\partial u}.$$

This vector field is also a distinguished symmetry of $\Delta$ and therefore the Lagrangian $\lambda = Y_{\text{ev}} \mathbin{\lrcorner} \Delta$ is variationally trivial. But in this case

$$Y_{\text{ev}} \mathbin{\lrcorner} \Delta = \frac{1}{r^3}(R \cdot \nabla u)\, \nu = \lambda_1,$$

where $\lambda_1$ is defined by (5.33a). Since $[\lambda_1] \in H^3(\mathcal{E}^*)$ is not zero, the global, distinguished symmetry $Y$ generates a local conservation law for $\Delta$ but not a global one. This example highlights an important aspect of Noether's Theorem as we have formulated it. *Given a locally variational source form, a global conservation law can sometimes be constructed from a distinguished symmetry even in the absence of a global variational principle. The obstructions to constructing global conservation laws and global Lagrangians lie in different cohomology groups, viz.,* $H^n(E)$ *and* $H^{n+1}(E)$ *respectively.* ∎

Example 5.12. Now take $M = (\mathbf{R}^3 - \{\, 0 \,\}) \times \mathbf{R}$ and let

$$E : M \times S^1 \times S^1 \to M.$$

Again $E$ is homotopic to the total spaces in the previous two examples but now, because $\dim M = 4$ and $H^5(E) = 0$, there are no obstructions to the construction of global variational principles. ∎

Example 5.13. Let $E : \mathbf{R}^2 \times F \to \mathbf{R}^2$, where

$$F = \{\, (R_1, R_2) \mid R_1, R_2 \in \mathbf{R}^3 \text{ and } R_1 \neq R_2 \,\}.$$

A section $s$ of $E$ of the special form

$$s(x, y) = (x, y, R_1(x), R_2(y)) \tag{5.38}$$

defines a pair of smooth, non-intersecting space curves. By using coordinates

$$S = R_2 - R_1 \quad \text{and} \quad T = R_2 + R_1,$$

on the fiber $F$, we see immediately that the 8 dimensional manifold $E$ is homotopy equivalent to $\mathbf{R}^3 - \{\,0\,\}$. Thus the de Rham cohomology of $E$ is generated by the single two form

$$\alpha = \frac{1}{4\pi}\frac{-s^3\,ds^1 \wedge ds^2 + s^2\,ds^1 \wedge ds^3 - s^1\,ds^2 \wedge ds^3}{[(s^1)^2 + (s^2)^2 + (s^3)^2]^{3/2}},$$

where $(s^1, s^2, s^3)$ are the components of $S$. The projection $\lambda = \pi^{2,0}(\alpha)$ is a first order, variationally trivial Lagrangian on $E$. If $\gamma_1$ and $\gamma_2$ are smooth, regular non-intersecting closed curves parametrized by maps $R_1, R_2 \colon I \to \mathbf{R}^3$ and if $s$ is the section (5.38) then

$$L(\gamma_1, \gamma_2) = \iint\limits_{I\times I} (j^1(s))^*(\lambda) = \frac{1}{4\pi}\iint\limits_{I\times I} \frac{S\cdot(\dot R_1 \times \dot R_2)}{||S||^3}\,dx \wedge dy$$

is the linking number of the two space curves $\gamma_1$ and $\gamma_2$ (see, for example, Dubrovin, Fomenko, and Novikov [*22*]). The fact that $L(\gamma_1, \gamma_2)$ is a deformation invariant of the pair of curves $\gamma_1$ and $\gamma_2$ (through smooth, non-intersecting deformations) is a consequence of the first variational formula established in Corollary 4.6. ∎

EXAMPLE 5.14. Let $M = \mathbf{R}$, $F = \mathbf{R}^2 - \{\,0\,\}$, and let $E : M \times F \to M$. It is easy to check that the source form

$$\Delta = [\ddot u - a(u,v)]\,du \wedge dx + [\ddot v - b(u,v)]\,dv \wedge dx \tag{5.39}$$

is locally variational if and only if

$$\frac{\partial a}{\partial v} = \frac{\partial b}{\partial u}\,.$$

Since $H^2(E) = 0$, all locally variational source forms are globally variational and indeed, a global Lagrangian $\lambda$ for $\Delta$ is given by

$$\lambda = [-\frac{1}{2}\dot u^2 - \frac{1}{2}\dot v^2 + x(a\dot u + b\dot v)]\,dx.$$

Note that although $\Delta$ is autonomous (considering the base coordinate $x$ to be time), the Lagrangian $\lambda$ explicitly contains the independent variable $x$. As our next theorem will show, $\Delta$ admits an autonomous Lagrangian if and only if the one form

$$\rho = a\,du + b\,dv$$

is exact on $\mathbf{R}^2 - \{\,0\,\}$. ∎

Example 5.15. *Translational invariant variational principles for translational invariant differential equations.*

Let $F$ be any $M$ dimensional manifold and let $E\colon \mathbf{R}^n \times F \to \mathbf{R}^n$. Let $x = (x^i)$ denote Cartesian coordinates on $\mathbf{R}^n$. For the purposes of this example, we restrict the admissible class of transformations on $E$ to those induced by maps on the fiber, that is, a map $\phi\colon E \to E$ is a admissible if $\phi(x,u) = (x, f(u))$, where $f\colon F \to F$. Let $G$ denote the group of all translations on the base space $\mathbf{R}^n$ and let

$$\Omega_G^{r,s}(J^\infty(E)) = \{\, \omega \in \Omega^{r,s}(J^\infty(E)) \,|\, \omega \text{ is } G \text{ invariant}\}.$$

A type $(r,s)$ form $\omega$ is $G$-invariant if and only if its coefficients do not depend explicitly on the independent variables $x^i$. The edge complex

$$\begin{aligned} 0 \longrightarrow \mathbf{R} \longrightarrow & \Omega_G^{0,0}(J^\infty(E)) \xrightarrow{d_H} \Omega_G^{1,0}(J^\infty(E)) \xrightarrow{d_H} \cdots \\ & \xrightarrow{d_H} \Omega_G^{n-1,0}(J^\infty(E)) \xrightarrow{d_H} \Omega_G^{n,0}(J^\infty(E)) \xrightarrow{E} \mathcal{F}_G^1(J^\infty(E)) \xrightarrow{\delta_V}, \end{aligned}$$

is called the translational invariant Euler-Lagrange complex and is denoted by $\mathcal{E}_G^*(J^\infty(E))$.

The following theorem generalizes the work of Tulczyjew [*71*].

Theorem 5.16. *The cohomology of the translational invariant Euler-Lagrange complex on $E\colon \mathbf{R}^n \times F \to \mathbf{R}^n$ is*

$$H^*\big(\mathcal{E}_G^*(J^\infty(E))\big) = H^*(T^n \times F, d),$$

*where $T^n = S^1 \times S^1 \times \cdots \times S^1$ is the $n$-torus. In particular, if $n = 1$, then*

$$H^p\big(\mathcal{E}_G^*(J^\infty(E))\big) = H^{p-1}(F) \oplus H^p(F).$$

Proof: Our proof is based upon Theorem 5.9. We begin with two observations. Firstly, let $E'\colon T^n \times F \to T^n$ and let $\phi\colon E \to E'$ be the bundle map induced from the standard covering map from $\mathbf{R}^n \to T^n$. The group $G$ also acts on $E'$. By Theorem 3.15 and Proposition 3.18, the prolongation of $\phi$,

$$\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(E'),$$

induces a cochain map

$$(\operatorname{pr}\phi)^*\colon \mathcal{E}^*(J^\infty(E')) \to \mathcal{E}^*(J^\infty(E)).$$

Not all forms in $\mathcal{E}^*(J^\infty(E'))$ pullback by $(\operatorname{pr}\phi)^*$ to $G$ invariant forms on $J^\infty(E)$. However, elementary covering space arguments show that for each $\omega \in \mathcal{E}^*_G(J^\infty(E))$, there is a unique form $\omega' \in \mathcal{E}^*(J^\infty(E'))$ such that

$$\omega = (\operatorname{pr}\phi)^*(\omega').$$

Secondly we know, by virtue of the Kunneth formula, that the de Rham cohomology of $E' = T^n \times F$ has representatives obtained by wedging the standard angular forms $dx^i$ on $T^n$ with forms representing the cohomology classes of $F$. Let $\{\alpha'_1, \alpha'_2, \dots, \alpha'_q\}$ be a collection of $p$ forms of this type which represent a basis for $H^p(T^n \times F)$. We identify the forms $\alpha'_i$ with their pullbacks to $J^\infty(E')$ via $\pi^\infty_{E'}$. Then, by Theorem 5.9, the forms

$$\beta'_i = \begin{cases} \pi^{p,0}(\alpha'_i) & \text{if } p \le n \\ \pi^{n,p-n}(\alpha'_i) & \text{if } p \ge n+1 \end{cases}$$

on $J^\infty(E')$ represent a basis for $H^p\big(\mathcal{E}^*(J^\infty(E'))\big)$. Because each of the forms $\alpha'_i$ is $G$ invariant, the forms

$$\beta_i = (\operatorname{pr}\phi)^*(\beta'_i)$$

are $G$ invariant and therefore $\beta_i \in \mathcal{E}^p_G(J^\infty(E))$.

We complete the proof of the theorem by proving that the forms $\beta_i$ represent a basis for $H^p\big(\mathcal{E}^*_G(J^\infty(E))\big)$. Since the forms $\alpha'_i$ are $d$ closed on $T^n \times F$, the forms $\beta_i$ are closed in $\mathcal{E}^*_G(J^\infty(E))$. The forms $\beta_i$ represent independent cohomology classes in $\mathcal{E}^*_G(J^\infty(E))$ — if a constant linear combination $\sum a_i\beta_i$ is exact on $\mathcal{E}^*_G(J^\infty(E))$, then $\sum a_i\beta'_i$ is exact on $\mathcal{E}^*(J^\infty(E'))$ and therefore the constants $a_i$ vanish.

It remains to prove that the representatives $\beta_i$ span $H^p\big(\mathcal{E}^*_G(J^\infty(E))\big)$. Let $\omega$ be any closed form in $\mathcal{E}^p_G(J^\infty(E))$. Then the form $\omega' \in \mathcal{E}^p(J^\infty(E'))$, defined by $\omega = (\operatorname{pr}\phi)^*(\omega')$, is closed ($\operatorname{pr}\phi$ is a local diffeomorphism) and hence

$$\omega' = d_H\eta' + \sum a_i\beta'_i. \tag{5.40}$$

(Here we are assuming that $p \le n$. If $p \ge n+1$, then the differential $d_H$ in this equation is replaced by $\delta_V$.) The form $\eta' \in \mathcal{E}^{p-1}(J^\infty(E'))$ may not be $G$ invariant and consequently $\eta'$ may not lift to a $G$ invariant form on $J^\infty(E)$. However, if we average (5.40) over $T^n$ by integrating, then we can replace the form $\eta'$ in (5.40) by a $G$ invariant form $\tau'$. Let $\tau = (\operatorname{pr}\phi)^*(\tau') \in \mathcal{E}^{p-1}_G(J^\infty(E))$. Then the pullback of (5.40) to $J^\infty(E)$ gives

$$\omega = d_H\tau + \sum a_i\beta_i.$$

This proves that the $\beta_i$ span $H^p\big(\mathcal{E}^*_G(J^\infty(E))\big)$.

In integrating (5.40), we used the fact that because $\omega'$ and $\beta'_i$ are $G$ invariant,

$$\int\limits_{y\in T^n} (\operatorname{pr} L_y)^*\omega' = \omega' \quad \text{and} \quad \int\limits_{y\in T^n} (\operatorname{pr} L_y)^*\beta'_i = \beta'_i,$$

where $L_y\colon E' \to E'$ is the prolongation of left multiplication (translation) by $y \in T^n$ to $J^\infty(E')$, *i.e.*,

$$L_y([e^x, u]) = [e^{x+y}, u].$$

We also used the fact that because the differentials $d_H$ and $\delta_V$ of the Euler-Lagrange complex commute with $(\operatorname{pr} L_y)^*$, they commute with integration over $T^n$.

Another proof for the special case $\dim M = 1$ will be given in the next section. ∎

Example 5.17. For locally variational, autonomous ordinary differential equations ($n = 1$), the obstructions to finding autonomous Lagrangians lie in $H^1(F) \oplus H^2(F)$. For example, consider the locally variational source form on $E\colon \mathbf{R} \times F \to \mathbf{R}$, where $F = \mathbf{R}^2 - \{0\}$, as given by (5.39). Since

$$\begin{aligned} d_V\Delta &= (\ddot\theta^u - \frac{\partial a}{\partial v}\,\theta^v)\,\theta^u \wedge dx + (\ddot\theta^v - \frac{\partial b}{\partial v}\,\theta^u)\,\theta^v \wedge dx \\ &= d_H(\dot\theta^u \wedge \theta^u + \dot\theta^v \wedge \theta^v), \end{aligned}$$

the form

$$\begin{aligned} \beta &= \Delta - (\dot\theta^u \wedge \theta^u + \dot\theta^v \wedge \theta^v) \\ &= d\,(-\dot u\,du + \frac{1}{2}\dot u^2\,dx - \dot v\,dv + \frac{1}{2}v^2\,dx) - (a\,du + b\,dv) \wedge dx \end{aligned}$$

is a $d$ closed, $G$ invariant two form on $J^\infty(E)$. In accordance with the proof of Theorem 5.16, we pull $\beta$ back to the form $\beta' \in \Omega^2(J^\infty(E'))$, where $E' = S^1 \times F$. Since $\beta'$ is cohomologous on $J^\infty(E')$ to the two form

$$\gamma' = -(a\,du + b\,dv) \wedge dx,$$

we deduce that $\Delta$ admits a global, autonomous variational principle if and only if $\gamma'$ is $d$ exact on $J^\infty(E')$. But, as a form on $E'$, $\gamma'$ is exact if and only if the one form

$$\rho = -(a\,du + b\,dv)$$

on the fiber $F$ is exact. ∎

We now consider some examples where the cohomology of the variational bicomplex arises, not from the topology of $E$, but rather from open restrictions on the derivatives of local sections of $E$. Let $J_0^k$ be any *open* submanifold of $J^k(E)$. For example, to study regular plane curves we take $E : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ and restrict the one jets on $E$ to the open set

$$J_0^1 = \{\, (x, u, v, \dot u, \dot v) \mid \dot u^2 + \dot v^2 \neq 0 \,\}.$$

In this example the projection map $\pi_E^1 \colon J_0^1 \to E$ is still surjective — in general we need not impose this condition on the set $J_0^k$. Given an open set $J_0^k \subset J^k(E)$, define

$$\mathcal{R} = (\pi_k^\infty)^{-1}(J_0^k). \tag{5.41}$$

Theorems 5.1 and 5.9 immediately generalize to the variational bicomplex restricted to the open set $\mathcal{R} \subset J^\infty(E)$.

Theorem 5.18. *Let $\mathcal{R}$ be an open set in $J^\infty(E)$ of the type (5.41).*

*(i) For $s \geq 1$, the augmented horizontal complex*

$$\begin{aligned} 0 \longrightarrow \Omega^{0,s}(\mathcal{R}) \xrightarrow{d_H} \Omega^{1,s}(\mathcal{R}) \xrightarrow{d_H} \Omega^{2,s}(\mathcal{R}) \xrightarrow{d_H} \cdots \\ \xrightarrow{d_H} \Omega^{n,s}(\mathcal{R}) \xrightarrow{I} \mathcal{F}^s(\mathcal{R}) \longrightarrow 0 \end{aligned} \tag{5.42}$$

*is exact.*

*(ii) The cohomology of the Euler-Lagrange complex $\mathcal{E}^*(\mathcal{R})$ is isomorphic to the cohomology of the de Rham complex of $J_0^k$.*

Proof: For every point $q \in J_0^k$, there is an open neighborhood $V_q^k$ of $q$ such that the augmented horizontal complex on $V_q^\infty = (\pi_k^\infty)^{-1}(V_q^k)$ is exact. Let $\mathcal{V} = \{\, V_\alpha \,\}$ be a cover of $\mathcal{R}$ by such neighborhoods and let $\{\, F_\alpha \,\}$ be a partition of unity on $\mathcal{R}$ subordinate to this cover.

The proof of exactness of (5.42) now follows that of Theorem 5.1 with the cover $\mathcal{V}$ on $\mathcal{R}$ used in place of the cover $\mathcal{U}$ on $E$.

The same homological algebra used to prove Theorem 5.9 proves (ii). ∎

Example 5.19. Let $E : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ and let

$$J_0^1 = \{\, (x, u, v, \dot u, \dot v) \mid \dot u^2 + \dot v^2 \neq 0 \,\}.$$

Then the cohomology of the Euler-Lagrange complex on $\mathcal{R} = (\pi_1^\infty)^{-1}(J_0^1)$ is isomorphic to the de Rham cohomology of $J_0^1$. Since the latter is generated by

$$\alpha = \frac{\dot u\, d\dot v - \dot v\, d\dot u}{\dot u^2 + \dot v^2},$$

the former is generated by

$$\omega = \pi^{1,0}(\alpha) = \frac{\dot u d_H \dot v - \dot v d_H \dot u}{\dot u^2 + \dot v^2} = \frac{\dot u \ddot v - \dot v \ddot u}{\dot u^2 + \dot v^2}\, dx.$$

The integral of this cohomology class around a closed curve $\gamma$ is the rotation index of $\gamma$.

In this particular instance, $\omega$ happens to be invariant under the group $G$ of isometries of the fiber $\mathbf{R}^2$ and under arbitrary oriented diffeomorphisms of the base $\mathbf{R}$. Thus $\omega$ also defines a cohomology class in $\mathcal{E}^*_G(\mathcal{R})$. In fact, by Proposition 4.13, we know that $\omega$ generates the only cohomology class in $H^1(\mathcal{E}^*_G(\mathcal{R}))$ and therefore

$$H^1(\mathcal{E}^*(\mathcal{R})) = H^1(\mathcal{E}^*_G(\mathcal{R})).$$

It must, however, be emphasized that this equality is purely coincidental — by Theorem 5.18

$$H^2(\mathcal{E}^*(\mathcal{R})) = 0,$$

while by Proposition 4.61,

$$H^2(\mathcal{E}^*_G(\mathcal{R})) = \mathbf{R}$$

is the one dimensional vector space generated by $\Theta^2 \wedge ds$. ∎

Example 5.20. Now let $E : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$ and let

$$J^1_0 = \{\, (x, R, \dot R) \mid \rho^2 = \langle \dot R, \dot R\rangle = \dot u^2 + \dot v^2 + \dot w^2 \neq 0 \,\}.$$

The single generator for the de Rham cohomology of $J^1_0$ is the two form

$$\alpha = \frac{1}{\rho^2}(\dot u\, d\dot v \wedge d\dot w - \dot v\, d\dot u \wedge d\dot w + \dot w\, d\dot u \wedge d\dot v).$$

The projection of $\alpha$ to a form in the Euler-Lagrange complex gives rise to the source form

$$\begin{aligned}
\Delta &= I(\pi^{1,1}\alpha)) \\
&= I\big(\frac{1}{\rho^3}[(\dot w \ddot v - \dot v \ddot w)\,\dot\theta^u + (\dot u \ddot w - \dot w \ddot u)\,\dot\theta^v + (\dot v \ddot u - \dot u \ddot v)\,\dot\theta^w] \wedge dx\big) \\
&= \big\{\frac{d}{dx}[\frac{1}{\rho^3}(\dot v \ddot w - \dot w \ddot v)]\,\theta^u + \frac{d}{dx}[\frac{1}{\rho^3}(\dot w \ddot u - \dot u \ddot w)]\,\theta^v + \frac{d}{dx}[\frac{1}{\rho^3}(\dot u \ddot v - \dot v \ddot u)]\,\theta^w\big\} \wedge dx.
\end{aligned}$$

Therefore the system of third order ordinary differential equations

$$\frac{d}{dx}[\frac{1}{\rho^3}(\dot{R}\times\ddot{R})]=0 \tag{5.43}$$

is locally, but not globally, variational.

Note that

$$\frac{1}{\rho^3}\dot{R}\times\ddot{R}=\kappa B, \tag{5.44}$$

where $\kappa$ is the curvature and $B$ the unit binormal. If the initial conditions at $x=x_0$ for the system of equations (5.43) are such that $(\dot{R}\times\ddot{R})(x_0)=0$, then the solutions are straight lines. If $(\dot{R}\times\ddot{R})(x_0)\neq 0$, then the solutions are circles.

Now replace the open set $J_0^1$ by

$$J_0^2=\{\,(x,R,\dot{R},\ddot{R})\mid \dot{R}\times\ddot{R}\neq 0\,\}.$$

The open set $J_0^2\subset J^2(E)$ has the same homotopy type as the Steifel manifold of 2 frames in $\mathbf{R}^3$ which, in turn, is homotopy equivalent to the special orthogonal group $\mathbf{SO(3)}$. Since $H^2(\mathbf{SO(3)})=0$, the locally variational source form (5.43) must now admit a global Lagrangian.

A Lagrangian for (5.43) is easily found by using the moving frame formalism developed in §2D. By virtue of (5.44), the Frenet formula (2.45) and (2.46), we can re-write the source form $\Delta$ as

$$\Delta=\frac{d}{ds}[\kappa B]\cdot\theta\wedge\sigma=[-\kappa\tau\Theta^2+\dot{\kappa}\Theta^3]\wedge\sigma.$$

*Assume that* $\Delta$ admits a natural Lagrangian

$$\lambda=L(\kappa,\tau,\dot{\kappa},\dot{\tau},\dots)\,\sigma.$$

(We have not yet computed the 2 dimensional equivariant cohomology of the Euler-Lagrange complex for space curves so there could be obstructions here and $\Delta$ might not admit a natural Lagrangian.) Then by (2.64) we must have that

$$-\kappa\tau=\kappa H+(\kappa^2-\tau^2)E_\kappa+\ddot{E}_\kappa+2\kappa\tau E_\tau+[\frac{\kappa\dot{\tau}-2\tau\dot{\kappa}}{\kappa^2}]\dot{E}_\tau+2\frac{\tau}{\kappa}\ddot{E}_\tau,$$

and

$$-\dot{\kappa}=\dot{\tau}E_\kappa+\tau\dot{E}_\kappa-\dot{\kappa}E_\tau+[\frac{\tau^2\kappa^2-\kappa^4-2\dot{\kappa}^2+\kappa\ddot{\kappa}}{\kappa^3}]\dot{E}_\tau+2\frac{\dot{\kappa}}{\kappa^2}\ddot{E}_\tau-\frac{1}{\kappa}\dddot{E}_\tau.$$

By inspection we see that a Lagrangian $\lambda$ which satisfies these equations is

$$\lambda=-\tau\,\sigma=-\frac{\dot{R}\cdot(\ddot{R}\times\dddot{R})}{||\dot{R}\times\ddot{R}||}\,||\dot{R}||\,dx.$$

∎

EXAMPLE 5.21. Let $U$ be an open domain in $\mathbf{R}^2$, let $E : U \times \mathbf{R}^2 \to U$ and let

$$J_0^1 = \{\, (x, y, u, v, u_x, u_y, v_x, v_y) \mid \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} > 0 \,\}.$$

Sections of $E$ whose one jets lie in $J_0^1$ are orientation preserving local diffeomorphisms from $\mathbf{R}^2$ to $\mathbf{R}^2$. The open submanifold $\mathcal{R} = (\pi_1^\infty)^{-1}(J_0^1)$ has the same homotopy type as $\mathbf{Gl}^+(2, \mathbf{R})$ which is homotopy equivalent to the circle $S^1$. In fact, if we set

$$\begin{aligned} u_x &= r + s & u_y &= p - q \\ v_x &= p + q & v_y &= r - s \end{aligned}$$

then

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = r^2 + q^2 - s^2 - p^2,$$

and consequently we can conclude that a generator for the de Rham cohomology of $\mathcal{R}$ is given by

$$\begin{aligned} \alpha &= \frac{r\,dq - q\,dr}{r^2 + q^2} \\ &= \frac{(u_x + v_y)\,d(v_x - u_y) - (v_x - u_y)\,d(u_x + v_y)}{(u_x + v_y)^2 + (v_x - u_y)^2}. \end{aligned}$$

Therefore, for $p > 1$, $H^p(\mathcal{E}^*(\mathcal{R})) = 0$ while for $p = 1$ the cohomology is represented by the one form

$$\omega = \frac{(u_x + v_y)\,d_H(v_x - u_y) - (v_x - u_y)\,d_H(u_x + v_y)}{(u_x + v_y)^2 + (v_x - u_y)^2}.$$

This form has the following geometric interpretation.

PROPOSITION 5.22. *Let $\phi\colon U \to \mathbf{R}^2$ be a local diffeomorphism and let $\operatorname{pr}\phi$ be the prolongation of $\phi$ to $\mathcal{R}$. Let $\gamma$ be any smooth closed curve in $U$ and let $\tilde\gamma = \phi \circ \gamma$ be the image curve under $\phi$. Then*

$$\frac{1}{2\pi} \int_\gamma (\operatorname{pr}\phi)^*(\omega) = [\,\textit{rotation index of } \tilde\gamma\,] - [\,\textit{rotation index of } \gamma\,].$$

PROOF: Let $\gamma(t) = (x(t), y(t))$, let $\tilde{\gamma}(t) = ((u(t), v(t))$ where $u(t) = u(x(t), y(t))$ and $v(t) = v(x(t), y(t))$, and let

$$\sigma = (\operatorname{pr}\phi)^*(\omega) - \Bigl(\frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2}\Bigr)\, dt + \Bigl(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\Bigr)\, dt.$$

We prove the proposition by showing that $\sigma$ is a $d$ exact one form. We know that on its domain of definition, the function

$$f(t) = \Bigl(\arctan\Bigl(\frac{v_x - u_y}{u_x + v_y}\Bigr)\Bigr)(x(t), y(t)) - \arctan\Bigl(\frac{\dot{v}}{\dot{u}}\Bigr)(t) + \arctan\Bigl(\frac{\dot{y}}{\dot{x}}\Bigr)(t)$$

satisfies

$$d\, f = \sigma.$$

That $f$ can be defined globally follows from the identity

$$\tan f = \frac{(v_x - u_y)(\dot{u}\dot{x} + \dot{v}\dot{y}) - (u_x + v_y)(\dot{v}\dot{x} - \dot{y}\dot{u})}{(v_x - u_y)(\dot{v}\dot{x} - \dot{y}\dot{u}) + (u_x + v_y)(\dot{u}\dot{x} + \dot{v}\dot{y})}$$

and the fact that the denominator $Q$ of this rational function is the positive definite quadratic form

$$Q = [\,\dot{x} \quad \dot{y}\,] \begin{bmatrix} u_x^2 + u_y^2 + (u_x v_y - v_x u_y) & u_x v_y - v_x u_y \\ u_x v_y - v_x u_y & v_x^2 + v_y^2 + (u_x v_y - v_x u_y) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}.$$

∎

Observe that when $\phi\colon \mathbf{R}^2 - \{\,0\,\} \to \mathbf{R}^2$ is complex analytic,

$$\frac{1}{2\pi} \int_\gamma (\operatorname{pr}\phi)^*(\omega) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\phi''\, dz}{\phi'}.$$

Consequently, if $\gamma$ surrounds the origin and $m_{\mathrm{z}}$ and $m_{\mathrm{p}}$ denote the multiplicities of the zeros and poles of $\phi'$ at 0, then by the argument principle (see, *e.g.*, Conway [*18*])

$$\frac{1}{2\pi} \int_\gamma (\operatorname{pr}\phi)^*(\omega) = m_{\mathrm{z}} - m_{\mathrm{p}}$$

For example, under the maps $z \to e^z$, $z \to z^n$ or $z \to z^{-n}$ the rotation index of the curve $\gamma$ is left unchanged, is increased by $n$, or is decreased by $n$ respectively. ∎

EXAMPLE 5.23. Let $E : \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^3$ and let

$$J_0^1 = \{\, (x, y, R, R_x, R_y) \mid R_x \times R_y \neq 0 \,\}.$$

A section of $E$ whose one jet lies in $J_0^1$ defines a regularly parametrized surface in $\mathbf{R}^3$. The Gauss-Bonnet integrand

$$\lambda = K\, dA$$

is a closed, type $(0, 2)$ form on $\mathcal{R}$ but in view of the Liouville formula derived in Example 4.46, $\lambda$ is $d_H$ exact and so does *not* determine a non-trivial cohomology class in $\mathcal{E}^*(J_0^*)$. Indeed, since $J_0^1$ is homotopy equivalent to the rotation group $\mathbf{SO}(3)$, we deduce that

$$H^p(\mathcal{E}^*(\mathcal{R})) = \begin{cases} 0, & \text{if } p > 0 \text{ and } p \neq 3, \text{ and} \\ \mathbf{R}, & \text{if } p = 3. \end{cases}$$

If $\{\, E_1, E_2, E_3 \,\}$ is an orthonormal frame on $\mathbf{R}^3$ with

$$E_3 = \frac{R_x \times R_y}{||R_x \times R_y||},$$

and $E_1$ and $E_2$ depending smoothly on the jets of $R$, and if

$$\alpha_{ij} = \langle E_i, dE_j \rangle$$

denote the Mauer-Cartan forms on $\mathbf{SO}(3)$, then a representative of this class is the source form

$$\Delta = I \circ \pi^{2,1}(\alpha_{12} \wedge \alpha_{13} \wedge \alpha_{23}).$$

I have been unable to attach any special geometric significance to the partial differential equations defined by this source form.

If $G$ is the group of isometries of the fiber $\mathbf{R}^3$ and orientation preserving diffeomorphism of the base $\mathbf{R}^2$, then one can prove that the Gauss-Bonnet integrand represents the sole equivariant cohomology class in $H^2(\mathcal{E}^*_G(\mathcal{R}))$. ∎

EXAMPLE 5.24. There are differential-topological invariants which do not seem to arise as cohomology classes in an appropriate Euler-Lagrange complex. Perhaps the most famous of these is the Hopf invariant for smooth maps $\phi\colon S^3 \to S^2$. If $\nu$ is the volume form on $S^2$, then $\phi^*(\nu)$ is an closed two form on $S^3$ and therefore there is a one form $\alpha$ on $S^2$ such that $d\alpha = \phi^*(\omega)$. The Hopf invariant is

$$H(\phi) = \int_{S^3} \alpha \wedge d\alpha.$$

One cannot expect the integrand $\lambda = \alpha \wedge d\alpha$ to arise as a cohomology class in the Euler-Lagrange complex on $E : S^3 \times S^2 \to S^3$ since $\lambda$ cannot be computed pointwise from the jets of $\phi$.

Another invariant to be mentioned in this regard is the self-linking number of a smooth, regular curve $\gamma$ imbedded in $\mathbf{R}^3$ (Pohl, [*58*]). One ought not to expect this invariant to arise as a cohomology class in the variational bicomplex since the condition that $\gamma$ be non-self intersecting is a global condition which cannot be expressed as an open restriction on the jets of $\gamma$. ∎

**B. The Vertical Cohomology of the Variational Bicomplex.** In this section we compute the cohomology of the vertical complexes $(\Omega^{r,*}(J^\infty(E)), d_V)$.

Let $\alpha$ be a $p$ form on the base manifold $M$. We shall identify $\alpha$ with its pullback by $\pi^\infty_M$ to $J^\infty(E)$. Then $d_V\alpha = 0$ and hence, if $\omega$ represents a cohomology class in $H^{r,s}_V\big(\Omega^{*,*}(J^\infty(E))\big)$, then $\alpha \wedge \omega$ represents a class in $H^{r+p,s}_V$. In particular, *the vertical cohomology spaces $H^{r,s}_V\big(\Omega^{*,*}(J^\infty(E))\big)$ are modules over the ring $C^\infty(M)$ of smooth functions on $M$.*

Now consider two fibered manifolds

$$\pi\colon E \to M \quad \text{and} \quad \pi'\colon E' \to M'$$

and maps

$$\Phi, \Psi\colon J^\infty(E) \to J^\infty(E').$$

We suppose that $\Phi$ and $\Psi$ cover maps $\phi_0$ and $\psi_0$ from $M$ to $M'$. According the Theorem 3.15, this condition is both necessary and sufficient for the projected pull-back maps $\Phi^\sharp$ and $\Psi^\sharp$ to define cochain maps

$$\Phi^\sharp, \Psi^\sharp\colon \big(\Omega^{r,*}(J^\infty(E')\big), d_V) \to \big(\Omega^{r,*}(J^\infty(E)), d_V\big),$$

for $r = 0,\, 1,\, 2,\, \ldots, n$. We begin this section by showing that if $\Phi$ and $\Psi$ cover the same map, *i.e.*, if $\phi_0 = \psi_0$, and are homotopic in the sense below, then the cochain maps $\Phi^\sharp$ and $\Psi^\sharp$ induce the same map in vertical cohomology.

DEFINITION 5.25. *Two maps $\Phi$ and $\Psi$ from $J^\infty(E)$ to $J^\infty(E')$ which cover the same map $h\colon M \to M'$, that is*

$$\begin{array}{ccc} J^\infty(E) & \xrightarrow{\Phi,\Psi} & J^\infty(E') \\ {\scriptstyle \pi^\infty_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi^\infty_{M'}} \\ M & \xrightarrow{h} & M', \end{array}$$

*are homotopic if there is an open interval $I \supset [0,1]$ and a smooth map $H$ such that*

$$\begin{array}{ccc} J^\infty(E)\times I & \xrightarrow{H} & J^\infty(E') \\ {\scriptstyle \pi^\infty_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi^\infty_{M'}} \\ M & \xrightarrow{h} & M', \end{array}$$

*and*

$$H(q,0) = \Phi(q) \quad \textit{and} \quad H(q,1) = \Psi(q)$$

*for all $q \in J^\infty(E)$.*

As in Proposition 1.1, the map $H$ is smooth if and only if for each $k = 0, 1, 2, \ldots,$ there exists an integer $m_k$ and a smooth map

$$H_k^{m_k}\colon J^{m_k}(E)\times I \to J^k(E')$$

such that $\pi'^\infty_k \circ H = H_k^{m_k} \circ \pi^\infty_{m_k}$.

PROPOSITION 5.26. *The projection map $\pi^\infty_E\colon J^\infty(E) \to E$ is a homotopy equivalence over the identity on $M$.*

PROOF: It suffices to show that if $\sigma\colon E \to J^\infty(E)$ is any fixed section, then the map

$$\Phi = \sigma \circ \pi^\infty_E\colon J^\infty(E) \to J^\infty(E)$$

is smoothly homotopic to the identity on $J^\infty(E)$. Let $\mathcal{U} = \{\, U_\gamma \,\}$ be a cover of $E$ by adopted coordinate neighborhoods and let $\{\, f_\gamma \,\}$ be a partition of unity on $E$ subordinate to $\mathcal{U}$. With

$$\sigma_{\,|\,U_\gamma}(x^i, u^\alpha) = (x^i, u^\alpha, \sigma^\alpha_I(x,u)),$$

define, for $t \in I$,

$$H_t^\gamma \colon J^\infty(U_\gamma) \to J^\infty(U_\gamma)$$

by

$$H_t^\gamma[x,u] = (x^i, u^\alpha, t\sigma_I^\alpha(x,u) + (1-t)u_I^\alpha).$$

The maps $H_t^\gamma$ are projectable and hence smooth. When $t = 0$, $H_t^\gamma$ is the identity on $J^\infty(U)$ and when $t = 1$, $H_t^\gamma$ is the restriction of $\Phi$ to $J^\infty(U)$. The required homotopy

$$H_t \colon J^\infty(E) \to J^\infty(E)$$

can then be defined by

$$H_t = \sum_\gamma f_\gamma H_t^\gamma. \tag{5.45}$$

∎

Theorem 5.27. *Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be two fibered manifolds and let $\Phi$ and $\Psi$ be two smooth maps,*

$$\Phi, \Psi \colon J^\infty(E) \to J^\infty(E'),$$

*which cover the same map from $M$ to $M'$. If $\Phi$ and $\Psi$ are homotopic, then the projected pullback maps $\Phi^\sharp$ and $\Psi^\sharp$ define the same map in vertical cohomology.*

*In fact, there are homotopy operators*

$$\mathcal{H}_V^{r,s} \colon \Omega^{r,s}(J^\infty(E')) \to \Omega^{r,s-1}(J^\infty(E)),$$

*depending on $\Phi$ and $\Psi$, and such that for any $\omega' \in \Omega^{r,s}(J^\infty(E'))$*

$$\Psi^\sharp(\omega') - \Phi^\sharp(\omega') = \mathcal{H}_V^{r,s+1}(d_V\omega') + d_V\mathcal{H}_V^{r,s}(\omega'). \tag{5.46}$$

Proof: Given the homotopy $H$ from $\Phi$ to $\Psi$, there is a standard homotopy operator

$$\mathcal{K}^p \colon \Omega^p(J^\infty(E')) \to \Omega^{p-1}(J^\infty(E))$$

for the de Rham complex on $J^\infty(E)$, *i.e.*, for $\omega' \in \Omega^p(J^\infty(E'))$,

$$\Psi^*(\omega') - \Phi^*(\omega') = d\,\mathcal{K}^p(\omega') + \mathcal{K}^{p+1}(d\,\omega'). \tag{5.47}$$

In particular, we have that

$$\pi^{r+1,s}(\mathcal{K}(\omega')) = 0.$$

Then, given $\omega' \in \Omega^{r,s}(J^\infty(E'))$, where $r + s = p$, we show that

$$\mathcal{K}^p(\omega') \in \Omega^{r,s-1} \oplus \Omega^{r+1,s-2} \oplus \cdots . \tag{5.48}$$

By applying the projection map $\pi^{r,s}$ to (5.47), we immediately arrive at (5.46), with

$$\mathcal{H}_V^{r,s} = \pi^{r,s-1} \circ \mathcal{K}^{r+s}.$$

The definition of $\mathcal{K}^p$ and proof of (5.47) are identical to that given for differential forms on finite dimensional manifolds by Spivak [*63*](Vol. 1, pp. 304–306). We need the formula for $\mathcal{K}^p$ in order to verify (5.48). To this end, we first define the inclusion map

$$i_t \colon J^\infty(E) \to J^\infty(E) \times I$$

by

$$i_t(j^\infty(s)) = (j^\infty(s), t)$$

for all $t \in I$. If $\omega \in \Omega^p(J^\infty(E) \times I)$, then there are unique forms $\omega_1 \in \Omega^p(J^\infty(E) \times I)$ and $\omega_2 \in \Omega^{p-1}(J^\infty(E) \times I)$ such that

$$\omega = \omega_1 + dt \wedge \omega_2$$

and

$$\frac{\partial}{\partial t} \lrcorner\, \omega_1 = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \lrcorner\, \omega_2 = 0.$$

With

$$\mathcal{I}^p \colon \Omega^p(J^\infty(E) \times I) \to \Omega^{p-1}(J^\infty(E))$$

defined by

$$\mathcal{I}^p(\omega) = \int_0^1 i_t^*(\omega_2)\, dt$$

it is not difficult to show that

$$i_1^*(\omega) - i_0^*(\omega) = d\,\mathcal{I}^p(\omega) + \mathcal{I}^{p+1}(d\,\omega).$$

Consequently, for $\omega' \in \Omega^{r,s}(J^\infty(E'))$,

$$\begin{aligned}\Psi^*(\omega') - \Phi^*(\omega') &= (H \circ i_1)^*(\omega') - (H \circ i_0)^*(\omega') \\ &= (i_1)^*(H^*(\omega')) - (i_0)^*(H^*(\omega')) \\ &= d\,[(\mathcal{I}^p \circ H^*)(\omega')] + (\mathcal{I}^{p+1} \circ H^*)(d\,\omega').\end{aligned}$$

This establishes the homotopy formula (5.47), where

$$\mathcal{K}^p(\omega') = \mathcal{I}^p \circ H^*(\omega') = \int_0^1 i_t^*[\frac{\partial}{\partial t} \mathbin{\lrcorner} H^*(\omega')]\, dt.$$

In order to prove (5.48), we must show that

$$\mathcal{K}^p(\omega')(X_1, X_2, \ldots, X_{p-1}) = 0 \tag{5.49}$$

whenever any $s$ of the tangent vectors $X_1$, $X_2, \ldots, X_{p-1}$ to $J^\infty(E)$ are $\pi^\infty_M$ vertical. Since

$$\begin{aligned}\mathcal{K}^p(\omega')(X_1, \ldots, X_{p-1}) &= \int_0^1 H^*(\omega')\big(\frac{\partial}{\partial t},\, (i_t)_*X_1,\, \ldots,\, (i_t)_*X_{p-1}\big)\, dt \\ &= \int_0^1 \omega'\big(H_*(\frac{\partial}{\partial t}), Z_1, Z_2, \ldots, Z_{p-1}\big)\, dt,\end{aligned} \tag{5.50}$$

where $Z_j = (H \circ i_t)_* X_j$, equation (5.49) follows from the following observations. First, let $f\colon M' \to \mathbf{R}$ be any smooth function. Then, because $h$ does not depend upon the homotopy parameter $t$,

$$[(\pi^\infty_{M'})_*(H_*\frac{\partial}{\partial t})](f) = \frac{\partial}{\partial t}(f \circ h \circ \pi^\infty_M) = 0. \tag{5.51}$$

This shows that $H_*\big(\dfrac{\partial}{\partial t}\big)$ is a $\pi^\infty_{M'}$ vertical vector on $J^\infty(E')$. Thus, because $\omega'$ is of type $(r,s)$, the integrand in (5.50) will vanish whenever any $s$ of the vector fields $Z_1$, $Z_2$, $\ldots$, $Z_{p-1}$ vanish. But, because $H$ covers the map $h\colon M \to M'$,

$$(\pi^\infty_{M'})_*(Z_j) = (\pi^\infty_{M'})_*\big((H \circ i_t)_* X_j\big) = h_*\big((\pi^\infty_M)_* X_j\big)$$

and therefore $Z_j$ is $\pi^\infty_{M'}$ vertical whenever $X_j$ is $\pi^\infty_M$ vertical. ∎

A simple example shows that the hypothesis

$$\phi_0 = \psi_0,$$

used in proving (5.51), is essential to the validity of Theorem 5.27. Let $E$ be the product $\mathbf{R}^1 \times S^1 \to \mathbf{R}^1$, let $\phi\colon E \to E$ be the identity map and let $\psi\colon E \to E$ be the map $\psi(x,u) = (x+3, u)$. Let $\Phi = \operatorname{pr}\phi$ and $\Psi = \operatorname{pr}\psi$ be the prolongations of these maps to $J^\infty(E)$. Evidently $\psi$ and $\phi$ are homotopic and therefore $\Phi$ and $\Psi$ are homotopic although not in the sense of Definition 5.25. The maps $\Phi^\sharp$ and $\Psi^\sharp$ are not the same in cohomology. Indeed, let $f\colon \mathbf{R} \to \mathbf{R}$ be a smooth function with $\operatorname{supp} f \subset (-1,1)$ and with $f(0) = 1$. The form $\omega = f(x)\, d\theta$, where $\theta$ is the zeroth order contact one form on $E$, defines a non-trivial cohomology class in $H^{0,1}_V$. The difference

$$\eta = \Psi^\sharp(\omega) - \Phi^\sharp(\omega) = [f(x+3) - f(x)]\, \theta$$

is not $d_V$ exact ( at $x = 0$, $\eta = -\theta = -du$) and so the maps $\Phi^\sharp$ and $\Psi^\sharp$ are not the same in cohomology.

Corollary 5.28. *Let $\pi\colon E \to M$ be a fibered manifold of dimension $m+n$, where $n = \dim M$. Then, for all $s > m$,*

$$H^{r,s}_V\big(\Omega^{*,*}(J^\infty(E))\big) = 0.$$

Proof: Observe that if $\alpha$ is any $p$ form on $E$, then

$$\pi^{r,s}[(\pi^\infty_E)^*(\alpha)] = 0$$

whenever $s > m$ and $r + s = p$. Indeed, since $\alpha$ is locally a linear combination of forms containing at most $m$ wedge products of the fiber differentials $du^\alpha$,

$$(\pi^\infty_E)^*(\alpha) \in \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E))$$

is a linear combination of at most $m$ wedge products of the contact forms $\theta^\alpha$ ( and no higher order contact forms $\theta^\alpha_I, |I| \geq 1$). Alternatively, it is easily seen that

$$[(\pi^\infty_E)^*(\alpha)](X_1, X_2, \ldots, X_p) = \alpha\big((\pi^\infty_E)_* X_1, (\pi^\infty_E)_* X_2, \ldots, (\pi^\infty_E)_* X_p\big)$$

vanishes, for dimensional reasons, whenever any $s > m$ of the vectors $X_i$ on $J^\infty(E)$ are $\pi^\infty_M$ vertical.

Now let $\sigma\colon E \to J^\infty(E)$ be any section and define $\Phi\colon J^\infty(E) \to J^\infty(E)$ by $\Phi = \sigma \circ \pi^\infty_E$. Then, with $\Psi$ the identity on $J^\infty(E)$ and $H$ the homotopy (5.45), the homotopy formula (5.46) becomes

$$\omega - \pi^{r,s}[(\pi^\infty_E)^*(\sigma^*(\omega))] = d_V \mathcal{H}^{r,s}_V(\omega) + \mathcal{H}^{r,s+1}_V(d_V\omega).$$

Since $\sigma^*(\omega)$ is a form on $E$ this shows, in view of the above observation, that if $\omega$ is $d_V$ closed and $s > m$, then $\omega$ is $d_V$ exact. ∎

Without imposing additional structure on the fibered manifold $\pi\colon E \to M$, it seems unlikely that much more can be said in general about the vertical cohomology of the variational bicomplex on $J^\infty(E)$. Accordingly, we now suppose that $\pi\colon E \to M$ is a fiber bundle with $m$ dimensional fiber $F$ and structure group $G$. In addition, we shall assume that $F$ admits a *finite* cover $\mathcal{V} = \{\, V_\sigma \}$ with the property that each $V_\sigma$, as well as all non-empty intersections

$$V_{\sigma_0\sigma_1\dots\sigma_p} = V_{\sigma_1} \cap V_{\sigma_2} \cap \cdots \cap V_{\sigma_p}$$

are diffeomorphic to $\mathbf{R}^m$. Such a cover is called a *good* cover. Good covers exist whenever $F$ is compact. The existence of a finite good cover for the fiber $F$ insures that the de Rham cohomology of $F$ is finite dimensional. (See Bott and Tu [*13*] p. 43).

Definition 5.29. *Let $\pi\colon E \to M$ be a fiber bundle with fiber $F$. A collection of $s$ forms $\{\, \beta^1, \beta^2, \dots, \beta^d \,\}$ on $E$ are said to freely generate the $s$ dimensional de Rham cohomology of each fiber of $E$ if for every point $x \in M$, the restriction of these forms to the fiber $F_x = \pi^{-1}(x)$ (i.e., the pullback of these forms by the inclusion map $i_x\colon \pi^{-1}(x) \to E$) are closed forms on $F$ whose cohomology classes $[i_x^*(\beta^i)]$, $i = 1$, 2, ..., $d$, form a basis for the vector space $H^s(F_x)$.*

Two aspects of this definition should be emphasized. First, the forms $\beta^i$ need not be closed on E — in other words, the forms $\beta^i$ do *not* have to satisfy the hypothesis of the well-known Leray-Hirsch theorem. Secondly, the cohomology classes $[i_x^*(\beta^i)]$ in $H^s(F_x)$ must be independent at every point $x \in M$. It is always possible to construct forms on $E$ which freely generate the cohomology of any single fiber but such forms are of little use to us here.

Many fiber bundles admit forms which freely generate the cohomology of each fiber.

Example 5.30. Let $E : M \times F \to M$ and let $p\colon E \to F$ be the projection onto the fiber $F$. If $\gamma^1, \gamma^2, \dots, \gamma^d$ are closed $s$ forms on $F$ whose cohomology classes $[\gamma^i]$ form a basis for $H^s(F)$, then $\beta^i = p^*(\gamma^i)$ are forms on $E$ which freely generate the cohomology of each fiber. In this instance, the forms $\beta^i$ are closed on $E$. ∎

Example 5.31. Let $\pi\colon E \to M$ be an oriented sphere bundle with fiber $F = S^m$. According to Bott and Tu [*13*] (pp. 116–122), it is always possible to construct an $m$ form $\beta$ on $E$, called a *global angular form*, with the following properties

(i) $\beta_{|\,F_x}$ is a non-zero multiple of the volume form on $S^m$. Therefore $\beta_{|\,F_x}$ generates the top dimensional cohomology of the fiber.

(ii) There is an $m$ form $\chi$ on the base space $M$ such that

$$d\beta = -\pi^*\chi.$$

The form $\chi$ is closed on $M$ and the cohomology class $[\chi] \in H^{m+1}(M)$ is called the Euler class of the sphere bundle $E$. The bundle $E$ admits a closed angular form if and only if the Euler class of $E$ vanishes. ∎

EXAMPLE 5.32. Let $G$ be a connected Lie group, let $H$ be a closed subgroup of $G$ and let $M$ be the homogeneous space $G/H$. Then $\pi\colon G \to M$ is a fiber bundle with fiber $H$. Suppose $H$ is compact and semi-simple. Let $\gamma^1, \gamma^2, \dots, \gamma^d$ be a basis for the vector space of harmonic $s$ forms on $H$. These forms are closed and left invariant and generate the $s$ dimensional cohomology of $H$. Define left invariant forms $\beta^i$ on $G$ by

$$\beta^i(g) = (L^*_{g^{-1}}(\gamma^i(e))(g),$$

where $L_g\colon G \to G$ is left multiplication by $g \in G$ and $e$ is the identity of $H$. Let $h \in H$ and let $gh \in G$ denote a point in the fiber over $[g] \in M$. Since

$$\beta^i{}_{\,|\,\pi^{-1}[g]}(h) = \beta^i(gh) = (L_{g^{-1}})^*(\gamma^i(h)),$$

it follows that the forms $\beta^i$ freely generate the cohomology of each fiber. ∎

EXAMPLE 5.33. Let $\pi\colon P \to M$ be a principal fiber bundle with fiber $G$. We assume that $G$ is a compact, connected Lie group. Let $\gamma$ be a connection one form on $P$ and let $\Omega$ be the associated curvature two form. Let $I$ be an $\operatorname{ad} G$ invariant polynomial on the Lie algebra $\mathbf{g}$ of $G$, homogenous of degree $l$. Then, as is well-known (see, *e.g.*, Chern [*15*]) the characteristic form

$$\Xi = I(\Omega, \Omega, \dots, \Omega)$$

is exact on $P$. Indeed, with

$$\Omega_t = t\Omega + \frac{1}{2}(t - t^2)[\gamma, \gamma]$$

and

$$\beta = l \int_0^1 I[\gamma, \Omega_t, \dots, \Omega_t]\, dt,$$

it follows from the Bianchi identity for $\Omega$ and the invariance identity for $I$ that

$$d\beta = \Xi.$$

Since the curvature forms $\Omega$ are horizontal, their restriction to each fiber vanishes and therefore $\beta_{|\,\pi^{-1}(x)}$ is a closed, $2l-1$ form on each fiber $G_x = \pi^{-1}(x)$ of $P$.

Let $X_a, a = 1, 2, \ldots, m$ be a basis for $\mathbf{g}$. Then we can write

$$\gamma_{|\,G_x} = \sum_{a=1}^{m} X_a \omega^a,$$

where the one forms $\omega^a$ are the Maurer-Cartan forms on $G_x$ associated to the left-invariant vector fields $\widetilde{X}_a = (L_g)_*(X_a)$ on $G$. This proves that

$$\beta_{|\,G_x} = c_l I(\omega, [\omega, \omega], \ldots, [\omega, \omega])$$

where

$$c_l = l \int_0^1 (t - t^2)^{l-1}\, dt \neq 0.$$

Finally, as the polynomial $I$ ranges over a a generating set for the ring of $\operatorname{ad} G$ invariant polynomials on $\mathbf{g}$, the forms $\beta_{|\,G_x}$ will generate a basis for the cohomology ring $H^*(\Omega^*(G_x))$. ∎

Example 5.34. If $\pi \colon E \to M$ is a fiber bundle over a simply connected base manifold $M$ then, by a monodromy argument, there are forms on $E$ which will freely generate the cohomology of each fiber. Let $\mathcal{U} = \{\, U_\alpha, \}_{\alpha \in J}$ be a good cover of $M$. Let $U_\alpha$ and $U_\beta$ belong to $\mathcal{U}$. Because $U_\alpha$, $U_\beta$ and $U_\alpha \cap U_\beta$ are contractible, the bundles $\pi^{-1}(U_\alpha)$, $\pi^{-1}(U_\beta)$ and $\pi^{-1}(U_\alpha \cap U_\beta)$ are all trivial bundles and the restriction maps $\rho^\alpha_{\alpha\beta}$ and $\rho^\beta_{\alpha\beta}$ from $\pi^{-1}(U_\alpha)$ to $\pi^{-1}(U_\alpha \cap U_\beta)$ and from $\pi^{-1}(U_\beta)$ to $\pi^{-1}(U_\alpha \cap U_\beta)$ induce isomorphisms

$$(\rho^\alpha_{\alpha\beta})^* \colon H^p(\pi^{-1}(U_\alpha)) \to H^p(\pi^{-1}(U_\alpha \cap U_\beta))$$

and

$$(\rho^\beta_{\alpha\beta})^* \colon H^p(\pi^{-1}(U_\beta)) \to H^p(\pi^{-1}(U_\alpha \cap U_\beta)).$$

We now define an isomorphism

$$\phi^\alpha_\beta \colon H^p(\pi^{-1}(U_\alpha)) \to H^p(\pi^{-1}(U_\beta))$$

by

$$\phi^\alpha_\beta = ((\rho^\beta_{\alpha\beta})^*)^{-1} \circ (\rho^\alpha_{\alpha\beta})^*.$$

The map $\phi^{\alpha}_{\beta}$ is only defined in cohomology — it is not induced from a map from $\pi^{-1}(U_\beta)$ to $\pi^{-1}(U_\alpha)$.

Now pick a fixed open set $U_0$ in $\mathcal{U}$ and define, for each $\alpha \in J$, an isomorphism

$$\Upsilon_\alpha \colon H^p(\pi^{-1}(U_0)) \to H^p(\pi^{-1}(U_\alpha))$$

by

$$\Upsilon_\alpha = \phi^{\alpha_k}_{\alpha} \circ \cdots \circ \phi^{\alpha_1}_{\alpha_2} \circ \phi^{0}_{\alpha_1},$$

where $U_0 U_{\alpha_1} U_{\alpha_2} \cdots U_{\alpha_k} U_\alpha$ is a "chain" from $U_0$ to $U_\alpha$ (more precisely, $0\alpha_1\alpha_2 \cdots \alpha_k \alpha$ is an edge path in $N(\mathcal{U})$, the nerve of the cover $\mathcal{U}$). Under the hypothesis that $M$ is simply connected, one can prove that $\Upsilon_\alpha$ is well-defined, that is, independent of the choices $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}$. This, for us, is the essence of the monodromy theorem presented in Bott and Tu [*13*](pp. 141–152).

Finally, let $\beta^i$, $i = 1,, 2, \ldots, d$ be a set of closed $s$ forms on $F$ which determine a basis for $H^s(F)$. We pull these forms back to $\pi^{-1}(U_0)$ using a local trivialization of $E$ on $U_0$ to obtain a basis $\beta^i_0$ for $H^s(\pi^{-1}(U_0))$. For each $\alpha \in J$, pick forms $\beta^i_\alpha$ on $\pi^{-1}(U_\alpha)$ such that

$$[\beta^i_\alpha] = \Upsilon_\alpha[\beta^i_0].$$

Because $\Upsilon_{\alpha'} = \phi^{\alpha}_{\alpha'} \circ \Upsilon_\alpha$, it follows that if $x \in U_\alpha \cap U_{\alpha'}$ then

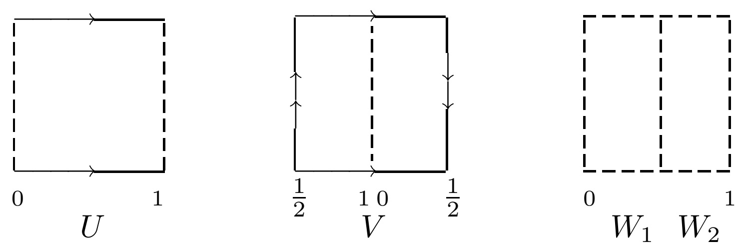
$$[(\beta^i_{\alpha'})_{\,|\, F_x}] = [(\beta^i_{\alpha})_{\,|\, F_x}].$$

Thus, if $f_\alpha$ is a partition of unity on $M$ subordinate to the cover $\{\, U_\alpha \,\}$, then

$$\beta^i = \sum_\gamma f_\gamma \beta^i_\gamma$$

are globally defined $s$ forms on $E$ which freely generate the cohomology of each fiber. ∎

EXAMPLE 5.35. The Klein bottle $\pi \colon K \to S^1$ is perhaps the simplest example of a fiber bundle for which the cohomology of the fibers cannot be freely generated by forms on the total space. To show this, we represent the Klein bottle as the unit square with the sides appropriately identified and we let $K = U \cup V$ be the standard trivialization of $K$, where $U = (0,1) \times S^1 \to U_0 = (0,1)$ has coordinates $(x,u) \to x$ and where $V = (0,1) \times S^1 \to V_0 = (0,1)$ has coordinates $(y,v) \to y$:




The intersection $U \cap V$ of these two coordinate charts consists of two disjoint open sets $W_1$ and $W_2$. On $W_1$, the two sets of coordinates are related by

$$y = x + \tfrac{1}{2} \quad \text{for } 0 < x < \tfrac{1}{2} \quad \text{and} \quad v = u$$

while, on $W_2$, the change of coordinates is

$$y = x - \tfrac{1}{2} \quad \text{for } \tfrac{1}{2} < x < 1 \quad \text{and} \quad v = 1 - u.$$

Now let $\alpha$ be any one form on $K$. We prove that there is a point $q \in V_0$ such that $\alpha_{\,|\,\pi^{-1}(q)}$ is exact thereby showing that cohomology of the fibers of $K$ cannot be freely generated by a form on $K$.

Without a loss in generality, we can suppose that at one point $p$ on the base space $S^1$, say $p \in U_0$ with $x(p) = \frac{1}{2}$, $\alpha_{\,|\,\pi^{-1}(p)}$ is the standard generator of the fiber cohomology, *i.e.*, $\alpha_{\,|\,\pi^{-1}(p)} = du$. Write

$$\alpha_{\,|\,U} = a(x,u)\,dx + b(x,u)\,du$$

and

$$\alpha_{\,|\,V} = c(x,u)\,dy + d(y,v)\,dv.$$

Then, in view of the above change of variables formula, the functions $b$ and $d$ are related by

$$b(x,u) = d(x + \tfrac{1}{2}, u) \qquad \text{for} \quad 0 < x < \tfrac{1}{2}$$

and

$$-\,b(x,u) = d(x - \tfrac{1}{2}, 1 - u) \qquad \text{for} \quad \tfrac{1}{2} < x < 1.$$

We have assumed that $b(\frac{1}{2}, u) = 1$ and therefore, for all sufficiently small $\epsilon > 0$,

$$d(1-\epsilon, u) > 0 \qquad \text{and} \qquad d(\epsilon, u) < 0.$$

This implies that

$$\int_0^1 d(1-\epsilon, u)\, du > 0 \qquad \text{and} \qquad \int_0^1 d(\epsilon, u) < 0.$$

and consequently, for some $q \in (\epsilon, 1-\epsilon) \subset V_0$

$$\int_0^1 d(q, u)\, du = 0.$$

This condition is both necessary and sufficient for the one form $\alpha$ to be exact on the fiber $\pi^{-1}(q)$. ∎

Our next theorem shows that if there are forms on $E$ which freely generate the fiber cohomology, then the vertical cohomology of the variational bicomplex agrees with the $E_1$ term of the Serre spectral sequence for the bundle $E$.

THEOREM 5.36. *Let $\pi \colon E \to M$ be a fiber bundle with fiber $F$. If there are forms on $E$ which freely generate the cohomology of each fiber, then*

$$H^{r,s}_V\bigl(\Omega^{*,*}(J^\infty(E))\bigr) \cong \Omega^r(M) \otimes H^s(\Omega^*(F)). \tag{5.52}$$

*Specifically, let $\beta^1$, $\beta^2, \dots, \beta^d$ be degree $s$ forms on $E$ which freely generate the $s$ dimensional cohomology of each fiber. Then the forms*

$$\alpha^i = \pi^{0,s}[(\pi^\infty_E)^*(\beta^i)] \tag{5.53}$$

*are type $(0,s)$, $d_V$ closed forms on $J^\infty(E)$. If $\omega \in \Omega^{r,s}(J^\infty(E))$ is any $d_V$ closed form, then there are forms $\xi_i \in \Omega^r(M)$, $i = 1, 2, \dots, d$ and a type $(r, s-1)$ form $\eta$ on $J^\infty(E)$ such that*

$$\omega = \sum_{i=1}^d \xi_i \wedge \alpha^i + d_V \eta. \tag{5.54}$$

*The forms $\xi_i$ are unique in that the form $\omega$ is $d_V$ exact if and only if the forms $\xi_i$ all vanish.*

PROOF: The proof of this theorem consists of two steps. First, we use a generalized Mayer-Vietoris sequence to prove the theorem for product bundles $E \colon M \times F \to M$, where the base space $M$ is diffeomorphic to $\mathbf{R}^n$. We then use a partition of unity argument to prove the theorem for bundles with arbitrary base manifolds. Since the second step is the easier of the two, we dispose of it first.

STEP 2. Suppose that the theorem is true for all bundles of the form $\mathbf{R}^n \times F \to \mathbf{R}^n$. Let $\pi\colon E \to M$ be any fiber bundle and let $\mathcal{U} = \{\, U_\sigma \,\}$ be a cover of $M$ by local trivializations, *i.e.*, each $U_\sigma$ is diffeomorphic to $\mathbf{R}^n$ and $\pi^{-1}(U_\sigma) \cong U_\sigma \times F$. Since the restrictions $\beta^i_\sigma = \beta^i_{\,|\,U_\sigma}$ of the forms $\beta^i$ freely generate the cohomology of each fiber in $\pi^{-1}(U_\sigma)$, the theorem applies, by assumption, to each fiber bundle $\pi^{-1}(U_\sigma)$ over $U_\sigma$

Let $\omega \in \Omega^{r,s}(J^\infty(E))$ be a $d_V$ closed form. Then the restriction $\omega_\sigma$ of $\omega$ to $J^\infty(\pi^{-1}(U_\sigma))$ can be expressed in the form

$$\omega_\sigma = \sum_{i=1}^{d} \xi_{i,\sigma} \alpha^i_\sigma + d_V \eta_\sigma, \tag{5.55}$$

where $\xi_{i,\sigma}$ are $r$ forms on $U_\sigma$ and $\eta_\sigma$ is a type $(r, s-1)$ form on $J^\infty(\pi^{-1}(U_\sigma))$. Consequently, if $\{\, f_\sigma \,\}$ is a partition of unity on $M$ subordinate to the cover $\mathcal{U}$, then

$$\omega = \sum_\sigma f_\sigma \omega_\sigma = \xi_i \wedge \alpha^i + d_V \eta, \tag{5.56}$$

where $\xi_i = \sum_\sigma f_\sigma \xi_{i,\sigma}$ and $\eta = \sum_\sigma f_\sigma \eta_\sigma$. (Multiplication by the functions $f_\sigma$ commutes with $d_V$ because the $f_\sigma$ are functions on $M$). As we shall see, (5.55) always holds for some choice of forms $\alpha^i_\sigma$ on $J^\infty(\pi^{-1}(U_\sigma))$ but this partition of unity argument fails unless the forms $\alpha^i_\sigma$ are known to be the restrictions of global forms.

The uniqueness of the forms $\xi_{i,\sigma}$ implies, on overlapping trivializations $U_\sigma$ and $U_\tau$, that

$$\xi_{i,\sigma} = \xi_{i,\tau} \qquad \text{on} \quad U_\sigma \cap U_\tau.$$

Hence the forms $\xi_{i,\sigma}$ are already the restrictions of global forms $\xi_i$ on $M$. This proves the uniqueness of the forms $\xi_i$ in (5.54).

STEP 1. Now let $E\colon M \times F \to M$, where $M = \mathbf{R}^n$ and let $\mathcal{V} = \{\, V_\sigma\}$ be a good cover for $F$. Let $\{\, f_\sigma\}$ now denote a partition of unity on $F$ subordinate to the cover $\mathcal{V}$. To prove the theorem for the bundle $E$, we show that the same combinatorics of the cover $\mathcal{V}$ which determines $H^*(\Omega^*(F))$ also determines $H^{*,*}_V(\Omega^{*,*}(J^\infty(E))$.

Since $\mathcal{V}$ is a good cover, each element $V_\sigma$ and each non-empty intersection

$$V_{\sigma_0\sigma_1\cdots\sigma_p} = V_{\sigma_0} \cap V_{\sigma_1} \cdots \cap V_{\sigma_p}$$

is diffeomorphic to $\mathbf{R}^m$. Consider the bundles

$$E_{\sigma_0\sigma_1\cdots\sigma_p} : M \times V_{\sigma_0\sigma_1\cdots\sigma_p} \to M.$$

Each $E_{\sigma_0\sigma_1\ldots\sigma_p}$ is a trivial bundle over $M = \mathbf{R}^n$ with fiber $F = \mathbf{R}^m$ and consequently we can introduce the variational bicomplex $\Omega^{*,*}(J^\infty(E_{\sigma_0\sigma_1\ldots\sigma_p}))$. Set

$$K^{p,r,s} = \prod_{\sigma_0<\sigma_1<\sigma\cdots<\sigma_p} \Omega^{r,s}(J^\infty(E_{\sigma_0\sigma_1\ldots\sigma_p})).$$

An element of $K^{p,r,s}$ is a $p$ cochain $\omega$ on the cover $\mathcal{V}$ whose "components" $\omega_{\sigma_0\sigma_1\ldots\sigma_p}$ are type $(r,s)$ forms on $J^\infty(E_{\sigma_0\sigma_1\ldots\sigma_p})$. Let

$$r \colon \Omega^{r,s}(J^\infty(E)) \to K^{0,r,s}$$

be the restriction map and let

$$\delta \colon K^{p,r,s} \to K^{p+1,r,s}$$

be the difference map as defined by (5.12). The vertical differential $d_V$ and the difference map $\delta$ commute. The kernel of the differential

$$d_V \colon K^{p,r,0} \to K^{p,r,1}$$

is, by Proposition 1.9, a $p$ cochain on $\mathcal{V}$ with values in $\Omega^r(M)$, that is, an element of

$$C^p = C^p(\mathcal{V}, \Omega^r(M)).$$

In summary, given the good cover $\mathcal{V}$ of the fiber $F$, we can construct the following double complex:

$$\begin{tikzcd}
 & {} & {} & & & \\
0 \arrow[r] & \Omega^{r,3}(J^\infty(E)) \arrow[u] \arrow[r, "r"] & K^{0,r,3} \arrow[u] \arrow[r] & {} & & \\
0 \arrow[r] & \Omega^{r,2}(J^\infty(E)) \arrow[u, "d_V"] \arrow[r, "r"] & K^{0,r,2} \arrow[u, "d_V"] \arrow[r, "\delta"] & K^{1,r,2} \arrow[u] \arrow[r] & {} & \\
0 \arrow[r] & \Omega^{r,1}(J^\infty(E)) \arrow[u, "d_V"] \arrow[r, "r"] & K^{0,r,1} \arrow[u, "d_V"] \arrow[r, "\delta"] & K^{1,r,1} \arrow[u, "d_V"] \arrow[r, "\delta"] & K^{2,r,1} \arrow[u] \arrow[r] & {} \\
0 \arrow[r] & \Omega^{r,0}(J^\infty(E)) \arrow[u, "d_V"] \arrow[r, "r"] & K^{0,r,0} \arrow[u, "d_V"] \arrow[r, "\delta"] & K^{1,r,0} \arrow[u, "d_V"] \arrow[r, "\delta"] & K^{2,r,0} \arrow[u, "d_V"] \arrow[r] & {} \\
0 \arrow[r] & \Omega^r(M) \arrow[u, "(\pi^\infty_M)^*"] \arrow[r, "r"] & C^0 \arrow[u, "(\pi^\infty_M)^*"] \arrow[r, "\delta"] & C^1 \arrow[u, "(\pi^\infty_M)^*"] \arrow[r, "\delta"] & C^2 \arrow[u, "(\pi^\infty_M)^*"] \arrow[r] & {} .
\end{tikzcd} \tag{5.57}$$

By Proposition 4.1, the interior columns of this double complex are exact. The operator

$$\mathcal{K}\colon K^{p,r,s} \to K^{p-1,r,s}$$

defined by

$$[\mathcal{K}(\omega]_{\sigma_0\sigma_1\cdots\sigma_{p-1}} = \sum_{\sigma} f_\sigma\, \omega_{\sigma\sigma_0\sigma_1\cdots\sigma_{p-1}}$$

is a homotopy operator for the interior rows of (5.57). Consequently, the cohomology of each edge complex of (5.57) is isomorphic to the cohomology of the total complex with differential $\delta + d_V^{\pm}$ where, for $\omega \in K^{p,r,s}$, $d_V^{\pm}(\omega) = (-1)^p d_V\omega$. This implies that the edge complexes are isomorphic to each other, *i.e.*,

$$H_V^{r,s}\big(\Omega^{*,*}(J^\infty(E))\big) \cong H^s\big(C^*(\mathcal{V}, \Omega^r(M))\big). \tag{5.58}$$

In fact, it follows from the general collocation formula (Bott and Tu [*13*](pp. 102–105) that the isomorphism (5.58) is induced by the map

$$\Psi_{J^\infty(E)}\colon C^s(\mathcal{V}, \Omega^r(M)) \to \Omega^{r,s}(J^\infty(E))$$

defined by

$$\Psi_{J^\infty(E)}(\gamma) = (-1)^s (d_V^{\pm} \circ \mathcal{K})^s(\gamma),$$

where $\gamma$ is an $s$ cochain on $\mathcal{V}$ with values in $\Omega^r(M)$. ( The subscript $J^\infty(E)$ attached to $\Psi$ distinguishes it from another similar map to be introduced momentarily.) Note that if the components of $\gamma$ are the $r$ forms $\gamma_{\sigma_0\sigma_1\cdots\sigma_p}$ on $M$, then $\Psi_{J^\infty(E)}(\gamma)$ is the global form on $J^\infty(E)$ whose restriction to $J^\infty(E_{\sigma_0})$ is

$$[\Psi_{J^\infty(E)}(\gamma)]_{\sigma_0} = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_s} d_V f_{\sigma_1} \wedge d_V f_{\sigma_2} \wedge \cdots \wedge d_V f_{\sigma_s} \wedge \gamma_{\sigma_0\sigma_1\cdots\sigma_s}. \tag{5.59}$$

In Bott and Tu [*13*](pp. 89–104), this Mayer-Vietoris argument is applied in the same way to show that for any manifold $F$ with good cover $\mathcal{V}$, the de Rham cohomology of $F$ can be computed from the combinatorics of the cover by the isomorphism

$$H^s(\Omega^*(F)) \cong H^s(C^*(\mathcal{V}, \mathbf{R})). \tag{5.60}$$

Since the cover $\mathcal{V}$ is finite, $H^s(C^*(\mathcal{V}, \mathbf{R}))$ is a finite dimensional real vector space and accordingly we can endow it with the structure of a free, finite dimensional $C^\infty(M)$ module. Since $\Omega^r(M)$ is also a free, finite dimensional $C^\infty(M)$ module, we have that

$$H^s\big(C^*(\mathcal{V}, \Omega^r(M))\big) \cong H^s(C^*(\mathcal{V}, \mathbf{R})) \otimes \Omega^r(M).$$

This isomorphism, together with (5.58) and (5.60), proves (5.52).

It remains to verify (5.54). We first observe that if we identify a $s-1$ form $\eta$ on $F$ with its pullback to $E = M \times F$, then

$$[\pi^{0,s} \circ (\pi_E^\infty)^*](d_F\eta) = d_V[(\pi^{0,s-1} \circ (\pi_E^\infty)^*)\eta]. \tag{5.61}$$

Secondly, the isomorphism (5.60) is induced by the map

$$\Psi_F \colon C^s(\mathcal{V}, \mathbf{R}) \to \Omega^s(F)$$

defined by

$$\begin{aligned}\Psi_F(\gamma) &= (-1)^s(d_F^\pm \circ \mathcal{K})^s(\gamma) \\ &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_s} d_F f_{\sigma_1} \wedge d_F f_{\sigma_2} \wedge \cdots \wedge d_F f_{\sigma_s} \wedge \gamma_{\sigma_0\sigma_1\cdots\sigma_s}.\end{aligned} \tag{5.62}$$

Extend $\Psi_F$ by linearity to a $C^\infty(M)$ module map

$$\Psi_F \colon C^s(\mathcal{V}, \Omega^r(M)) \to \Omega^s(F) \otimes \Omega^r(M).$$

A comparison of (5.59) and (5.62) shows that

$$[\pi^{r,s} \circ (\pi_E^\infty)^*] \circ \Psi_F = \Psi_{J^\infty(E)}. \tag{5.63}$$

Pick $s$ forms $\beta^1, \beta^2, \ldots, \beta^d$ on $F$ which generate the $s$ dimensional cohomology $H^s(F)$. Let $\gamma^i$ be corresponding $s$ dimensional cochains in $C^s(\mathcal{V}, \mathbf{R})$, *i.e.*,

$$\Psi_F(\gamma^i) = \beta^i + d_F\tau^i, \tag{5.64}$$

where each $\tau^i$ is a $s-1$ form on $F$. The cochains $\gamma^i$ form a basis for $H^s(C^*(\mathcal{V}, \mathbf{R}))$ as a real vector space. They also form a basis for $H^s\big(C^*(\mathcal{V}, \Omega^r(M))\big)$ as a $\Omega^r(M)$ module. In view of (5.53) and (5.63), (5.64) implies that for any $r$ form $\xi$ on $M$,

$$\Psi_{J^\infty(E)}(\gamma^i\xi) = \alpha^i \wedge \xi + d_V(\eta_1^i \wedge \xi), \tag{5.65}$$

where $\eta_1^i = \pi^{0,s-1}((\pi_E^\infty)^*(\tau^i))$. Finally, let $\omega$ be any type $(r,s)$, $d_V$ closed form on $J^\infty(E)$. The isomorphism (5.58) proves that there are $r$ forms $\xi_i$ on $M$ such that

$$\omega = \sum_{i=1}^d \big[\Psi_{J^\infty(E)}(\xi_i\gamma^i)\big] + d_V\eta_2.$$

On account of (5.65), this yields (5.54) with

$$\eta = \eta_2 + \sum_{i=1}^d \tilde{\eta}_1^i \wedge \xi_i,$$

as required ∎

In Chapter Four we introduced an invariant system of weights for forms on $J^\infty(E)$ with polynomial dependencies in the derivative variables. We proved that if $\omega$ is either $d_V$ or $d_H$ closed then there exist, at least locally, a minimal weight form $\eta$ such that $d_V\eta = \omega$ or $d_H\eta = \omega$. In the previous section we showed that there are no obstructions to the global construction of minimal weight forms for the interior rows of the variational bicomplex. Now we prove that there are no further obstructions, other than those identified in Theorem 5.36, to the construction of minimal weight forms on the vertical complexes of the variational bicomplex. With this result in hand it will be a simple matter to obtain global minimal weight results for the Euler-Lagrange complex.

PROPOSITION 5.37. *Let $\omega \in \Omega^{r,s}_{\mathcal{P}_{j,k}}(J^\infty(E))$, $s > 1$, and suppose that $\omega$ is $d_V$ exact. Then there is a form $\eta \in \Omega^{r,s-1}_{\mathcal{P}_{j,k}}(J^\infty(E))$ such that $\eta$ has the same weights as $\omega$ and*

$$\omega = d_V\eta.$$

PROOF: We first remark that because the weights $w_p$ satisfy

$$w_p(\eta) \geq w_p(d_V\eta),$$

a form $\eta$ with the same weights as $\omega$ and satisfying $d_V\eta = \omega$ is necessarily a minimal weight form for $\omega$. Secondly, if $f$ is any function on $M$ then the weights of $f\eta$ are no more than those of $\eta$ ( at points where $f$ is nonzero, the weights coincide). Hence, just as in the proof of Theorem 5.36, it suffices to prove this proposition for product bundles $E = M \times F \to M$, where $M = \mathbf{R}^m$.

Let $\mathcal{V}$ be a good cover of $F$ and let $\{\, K^{p,r,s}, \delta, d_V \,\}$ be the double complex (5.57). Define a sequence of forms $\omega^0$, $\omega^1, \ldots, \omega^s$ and $\eta^0$, $\eta^1, \ldots, \eta^{s-1}$, where $\omega^i \in K^{i,r,s-i}$ and $\eta^i \in K^{i,r,s-i-1}$ by $r(\omega) = \omega^0$,

$$d_V\eta^i = \omega^i \qquad \text{and} \qquad \delta\eta^i = \omega^{i+1} \qquad s = 0, 1, 2, \ldots, s-1.$$

By virtue of the local existence of minimal weight forms, we can assume that that each $\eta^i$ is a minimal weight form for $\omega^i$. This implies that the weights of each $\eta^i$ coincide with that of $\omega$. Since $d_V\omega^s = 0$, the components of $\omega^s$ are the pullbacks of forms on $M$, *i.e.*, $\omega \in C^s(\mathcal{V}, \Omega^r(M))$. Because $\omega$ is $d_V$ exact, $\omega^s$ must be $\delta$ exact and hence there is a cochain $\gamma^{s-1} \in C^{s-1}(\mathcal{V}, \Omega^r(M))$ such that $\delta\gamma^{s-1} = \omega^s$. Define another sequence of forms $\tau^{s-1}$, $\tau^{s-2}, \ldots, \tau^0$, where $\tau^i \in K^{i,r,s-i}$ by

$$\tau^{s-1} = \eta^{s-1} - \gamma^{s-1} \qquad \text{and} \qquad \tau^i = \eta^i - d_V\mathcal{K}\tau^{i+1}. \qquad i = s-2, s-3, \ldots, 0.$$

Here $\mathcal{K}$ is the homotopy operator for the generalized Mayer-Vietoris sequence. Since $\tau^{s-1}$ is $\delta$ exact, it follows that each $\tau^i$ is $\delta$ exact. In particular, $\tau^0$ is the restriction of a global form $\tau$ satisfying $d_V\tau = \omega$. Finally, since weights are not increased by either the homotopy operator $\mathcal{K}$ or by $d_V$ the weights of $\tau^i$ are those of $\omega^i$ and therefore $\tau$ is a minimal weight form for $\omega$. ∎

The next theorem completes our analysis of minimal weight forms in the variational bicomplex.

THEOREM 5.38.

*(i) Let $\omega \in \Omega^{r,0}_{\mathcal{P}_{j,k}}(J^\infty(E))$. If $\omega$ is exact, then $\omega = d_H\eta$, where $\eta$ is a minimal weight form.*

*(ii) Let $\Delta$ be a source form in $\mathcal{F}^1_{\mathcal{P}_{j,k}}(J^\infty(E))$. If $\Delta$ is the Euler-Lagrange form for some Lagrangian on $J^\infty(E)$, then $\Delta = E(\lambda)$, where $\lambda$ is a minimal order Lagrangian.*

PROOF: The proof of this theorem involves a simple modification of the proof of Theorem 5.9. To prove (i), we first invoke Corollary 5.7 and Proposition 5.37 to pick minimal order forms $\omega_i$ in (5.29). Secondly, the form $\beta$ defined by (5.30) is $d$ exact because $\omega$ is $d_H$ exact. By appealing once again to Corollary 5.7 and Proposition 5.37, a minimal order form $\alpha$ on $J^\infty(E)$ can be constructed such that $d\alpha = \beta$. The proof of (ii) is similar. ∎

COROLLARY 5.39. *Let $\Delta \in \mathcal{F}^1(J^\infty(E))$ be a source form of order $2k$. Then $\Delta$ is the Euler-Lagrange form of a Lagrangian of order $k$ if and only if*

(i) $\delta_V\Delta = 0$, *i.e., $\Delta$ is locally variational;*

(ii) $\Delta \in \mathcal{F}^1_{\mathcal{P}_{k,2k}}(J^\infty(E))$ *and* $w_k(\Delta) \leq k$; *and*

(iii) *the cohomology class* $[\Delta] \in H^{n+1}(J^\infty(E))$ *vanishes.*

Our next goal is to compute the cohomology of the complex

$$\{H^{*,s}_V\bigl(\Omega^{*,*}(J^\infty(E))\bigr), d_H\}.$$

This is the so-called $E_2$ term of the (second) spectral sequence for the variational bicomplex, defined by

$$E^{r,s}_2(J^\infty(E)) = \frac{\ker\{d_H\colon H^{r,s}_V(\Omega^{*,*}(J^\infty(E))) \to H^{r,s+1}_V(\Omega^{*,*}(J^\infty(E)))\}}{\operatorname{im}\{d_H\colon H^{r,s-1}_V(\Omega^{*,*}(J^\infty(E))) \to H^{r,s}_V(\Omega^{*,*}(J^\infty(E)))\}}.$$

If we let

$$\mathcal{Z}^{r,s}_2(J^\infty(E)) = \{\,\omega \in \Omega^{r,s}(J^\infty(E)) \mid d_V\omega = 0,\ d_H\omega = d_V\beta\,\}$$

and

$$\mathcal{B}_2^{r,s}(J^\infty(E)) = \{\, \omega \in \Omega^{r,s}(J^\infty(E)) \,|\, \omega = d_H\beta - d_V\gamma,\, d_V\beta = 0 \,\}.$$

then it is easy to see that

$$E_2^{r,s}(J^\infty(E)) \cong \frac{\mathcal{Z}^{r,s}(J^\infty(E))}{\mathcal{B}^{r,s}(J^\infty(E))}. \tag{5.66}$$

LEMMA 5.40. *Let $s \geq 1$ and suppose that $\beta$ is an $s$ form on $\pi\colon E \to M$ with the property that*

$$d\beta = \pi^*\chi \tag{5.67}$$

*for some $(s+1)$ form $\chi$ on $M$. Then the pullback of $\beta$ to $\Omega^{0,s}(J^\infty(E))$, viz.,*

$$\alpha = \pi^{0,s}\big((\pi_E^\infty)^*(\beta)\big)$$

*is a $d_V$ closed form which belongs to $\mathcal{Z}_2^{0,s}(J^\infty(E))$.*

PROOF: Apply $(\pi_E^\infty)^*$ to (5.67) to arrive at

$$d\,[(\pi_E^\infty)^*(\beta)] = (\pi_M^\infty)^*(\chi). \tag{5.68}$$

Since $(\pi_E^\infty)^*(\beta)$ is an $s$ form on $J^\infty(E)$, it can be decomposed into the sum

$$(\pi_E^\infty)^*(\beta) = \alpha + \beta_1 + \beta_2 + \cdots + \beta_s,$$

where $\beta_i$ is a form on $J^\infty(E)$ of type $(i, s-i)$. Equation (5.68) becomes

$$d_V\alpha + (d_H\alpha + d_V\beta_1) + (d_H\beta_1 + d_V\beta_2) + \cdots = (\pi_M^\infty)^*(\chi).$$

Because $(\pi_M^\infty)^*(\chi)$ is of type $(s+1, 0)$, the $(0, s+1)$ and $(1, s)$ components of this equation are $d_V\alpha = 0$ and $d_H\alpha + d_V\beta_1 = 0$. This proves that $\alpha \in \mathcal{Z}_2^{0,s}$, as required. ∎

THEOREM 5.41. *Let $\pi\colon E \to M$ be a fiber bundle with fiber $F$. Suppose, for $s \geq 1$, that there are $d$ type $(0,s)$ forms $\{\alpha^1, \alpha^2, \ldots, \alpha^d\}$ on $J^\infty(E)$ which*

(i) *freely generate the vertical cohomology $H_V^{*,s}$ on $J^\infty(E)$; and*

(ii) *belong to $\mathcal{Z}_2^{0,s}(J^\infty(E))$.*

*Then*

$$E_2^{r,s}(J^\infty(E)) \cong H^r(M) \otimes H^s(F). \tag{5.69}$$

PROOF: This is strictly an exercise in homological algebra based upon Theorem 5.36 and (5.66). Let $\omega \in \mathcal{Z}_2^{r,s}(J^\infty(E))$. Since $\omega$ is $d_V$ closed, there are $d$ degree $r$ forms $\xi_i$ on $M$ and a type $(r, s-1)$ form $\eta$ on $J^\infty(E)$ such that

$$\omega = \sum_{i=1}^{d} \xi_i \wedge \alpha^i + d_V \eta. \tag{5.70}$$

There is also a type $(r+1, s-1)$ form $\gamma_1$ for which

$$d_H \omega = d_V \gamma_1. \tag{5.71}$$

Since the generators $\alpha^i$ belong to $\mathcal{Z}_2^{0,s}$, there are type $(1, s-1)$ forms $\gamma_2^i$ such that

$$d_H \alpha^i = d_V \gamma_2^i. \tag{5.72}$$

Now apply the horizontal differential $d_H$ to (5.70) and substitute from (5.71) and (5.72) to arrive at

$$\sum_{i=1}^{d} d_H \xi_i \wedge \alpha^i = d_V \gamma_3.$$

This implies, by the uniqueness statement in Theorem 5.36, that $d_H \xi_i = 0$. Thus, the forms $\xi_i$ are closed on $M$.

It is an easy exercise to verify that map

$$\Psi\colon E_2^{r,s}(J^\infty(E)) \to H^r(M) \otimes H^s(F)$$

defined by

$$\Psi([\omega]) = \sum_{i=1}^{d} [\xi_i] \otimes [\alpha^i]$$

is both well-defined and an isomorphism. ∎

Our earlier remarks show that Theorem 5.41 applies if $\pi\colon E \to M$ is either an oriented sphere bundle or a principal fiber bundle with compact fiber $G$. Theorem 5.41 also applies whenever the base manifold $M$ is simply connected.

PROPOSITION 5.42. *Let $s \geq 1$ and suppose $\{\alpha^1, \alpha^2, \ldots, \alpha^d\}$ are $d_V$ closed, type $(0, s)$ forms on $J^\infty(E)$ which freely generate the $s$ dimensional vertical cohomology on $J^\infty(E)$. If $M$ is simply-connected, then there are forms $\{\tilde\alpha^1, \tilde\alpha^2, \ldots, \tilde\alpha^d\}$ in $\mathcal{Z}_2^{0,s}(J^\infty(E))$ which also freely generate the $s$ dimensional vertical cohomology on $J^\infty(E)$.*

PROOF: Since the forms $\alpha^i$ are $d_V$ closed, the type $(1, s)$ forms $d_H\alpha^i$ are also $d_V$ closed. Because the forms $\alpha^i$ freely generate the vertical cohomology of $J^\infty(E)$, there are one forms $\xi^i_j$ on $M$ and $(1, s-1)$ forms $\eta^i$ such that

$$d_H\alpha^i = \sum_{j=1}^{d} \xi^i_j \wedge \alpha^j + d_V\eta^i.$$

To this equation we apply $d_H$ to conclude that

$$\sum_{j=1}^{d}(d\xi^i_j - \sum_{k=1}^{d} \xi^i_k \wedge \xi^k_j) \wedge \alpha^j = d_V\sigma^i. \tag{5.73}$$

In this equation we have identified $d\xi^i_j$ with $d_H\xi^i_j$. On account of Theorem 5.36, equation (5.73) implies that

$$d\xi^i_j = \sum_{k=1}^{d} \xi^i_k \wedge \xi^k_j. \tag{5.74}$$

To prove the proposition, it suffices to construct functions $f^i_j$ on $M$, for $i, j = 1$, $2, \ldots, d$, such that $\det(f^i_j) \neq 0$ and such that the forms

$$\tilde\alpha^i = \sum_{j=1}^{d} f^i_j \alpha^j$$

belong to $\mathcal{Z}_2^{0,s}$. Since the $f^i_j$ are functions on $M$, $d_V\tilde\alpha = 0$. Since

$$d_H\tilde\alpha^i = \sum_{j=1}^{d}(d\, f^i_j + \sum_{k=1}^{d} f^i_k\, \xi^k_j) \wedge \alpha^j + d_V(\sum_{k=1}^{d} f^i_k \eta^k))$$

the forms $\tilde\alpha^i$ belong to $\mathcal{Z}_2^{0,s}$ if and only if

$$d\, f^i_j + \sum_{k=1}^{d} f^i_k\, \xi^k_j = 0. \tag{5.75}$$

It is a simple matter to check that the integrability conditions for (5.75) are (5.74). Fix a point $x_0 \in M$. Then there is a unique solution $f^i_j$ to (5.75), defined on all of $M$ with $f^i_j(x_0) = \delta^i_j$. Moreover, $\det(f^i_j) \neq 0$ on all of $M$.

This is easily proved from first principles. Let $\gamma\colon [0,1] \to M$ be a smooth curve with $\gamma(0) = x_0$ and $\gamma(1) = x$. Let $y = (y^i_j)\colon [0,1] \to \mathbf{R}^{d^2}$ be the unique solution to the system of ordinary differential equations

$$\dot{y}^i_j + \sum_{k=1}^{d} y^i_k \gamma^*(\xi^k_j) = 0 \qquad y^i_j(0) = \delta^i_j.$$

Because this is a linear system of equations the solution exists over the entire interval $[0,1]$. The integrability conditions (5.74) and the fact that $M$ is simply connected enable one to prove that $y^i_j(1)$ is independent of the path $\gamma$. See T. Y. Thomas [*67*]. The solution to (5.75) is then $f^i_j(x) = y^i_j(1)$. Finally, along $\gamma$, $\det(y^i_j)$ satisfies a differential equation which insures that this determinant does not vanish. ∎

In many instances, knowledge of this $E_2$ term of the variational bicomplex is sufficient to compute, by spectral sequence methods, the de Rham cohomology

$$H^*(\Omega^*(J^\infty(E)), d) \cong H^*(\Omega(E), d).$$

For examples, see McCleary [*50*].

**C. Generalized Poincaré-Cartan Forms and Natural Differential Operators on the Variational Bicomplex.** In Theorem 5.9, we established the existence of a vector space isomorphism

$$\Phi\colon H^*(\mathcal{E}^*(J^\infty(E))) \to H^*(\Omega^*(J^\infty(E))). \tag{5.76}$$

Despite the importance of this result, its direct applications are somewhat limited because the map $\Phi$, as things now stand, is rather difficult to evaluate. First, as we pointed out in the proof of Theorem 5.9, the map $\Phi$ is only well-defined in cohomology; it is not defined as the induced map of a cochain map on the underlying complexes of forms. Second, to compute

$$\Phi([\omega_0]) = \omega_0 - \omega_1 + \omega_2 - \cdots \tag{5.77a}$$

it is necessary to solve successively the equations

$$\sigma_i = d_V\omega_i \quad \text{and} \quad d_H\omega_{i+1} = \sigma_i \tag{5.77b}$$

for $i = 0, 1, 2, \ldots$, a task which generally is not easily accomplished. (Indeed, throughout all the examples in §5A we assiduously avoided using the map $\Phi$ in favor of its inverse $\Psi$, the projection map from $\Omega^*$ to $\mathcal{E}^*$.) This raises the problem of whether or not there is in fact a cochain map

$$\Phi\colon \mathcal{E}^*(J^\infty(E)) \to \Omega^*(J^\infty(E)) \tag{5.78}$$

which will induce the isomorphism $\Phi$ in cohomology.

This problem is motivated by two different considerations. First, given such a map $\Phi$, one can use it to immediately translate questions concerning the Euler-Lagrange complex to questions on the de Rham complex of $J^\infty(E)$. Such a map, in effect, completely geometrizing the formal aspects of the calculus of variations. Consider, as an illustration, the autonomous inverse problem to the calculus of variations which we solved in Example 5.15. Here, for reasons which will become apparent later, we restrict our attention to the autonomous inverse problem for ordinary differential equations. Let $E\colon \mathbf{R} \times F \to \mathbf{R}$ and let $\Delta$ be an autonomous, locally variational source form on $J^\infty(E)$. Then, with the map (5.78) in hand, we have that $\Phi(\Delta)$ is a two form on $J^\infty(E)$ and, as such, can be decomposed uniquely into the form

$$\Phi(\Delta) = \Phi_2(\Delta) + \Phi_1(\Delta) \wedge dx,$$

where $\Phi_2(\Delta)$ is a two form satisfying $\dfrac{d}{dx} \lrcorner\, \Phi_2(\Delta) = 0$ and $\Phi_1(\Delta)$ is a one form. Because $\Delta$ is locally variational and because $\Phi$ is a cochain map, $\Phi(\Delta)$ is a $d$ closed form on $J^\infty(E)$. This, in turn, implies that

$$d\,\Phi_2(\Delta) = 0 \quad \text{and} \quad d\,\Phi_1(\Delta) = 0.$$

Thus every autonomous, locally variational source form $\Delta$ determines a pair of cohomology classes

$$([\Phi_1(\Delta)],\, [\Phi_2(\Delta)]) \,\in\, H^1(J^\infty(E)) \oplus H^2(J^\infty(E)).$$

Now suppose that there is an autonomous Lagrangian $\lambda$ for $\Delta$. Then

$$\Phi(\lambda) = \Phi_1(\lambda) + \Phi_0(\lambda) \wedge dx$$

and, because $d(\Phi(\lambda)) = \Phi(E(\lambda))$, we must have that

$$\Phi_2(\Delta) = d\,(\Phi_1(\lambda)) \quad \text{and} \quad \Phi_1(\Delta) = d\,(\Phi_0(\lambda)).$$

Therefore, if $\Delta$ admits an autonomous Lagrangian, the forms $\Phi_2(\Delta)$ and $\Phi_1(\Delta)$ must be exact. This proves, consistent with the results of Theorem 5.16, that the obstructions to the solution to the autonomous inverse problem to the calculus of variations lie in $H^1(J^\infty(E)) \oplus H^2(J^\infty(E))$. Note that we have tacitly assumed that the forms $\Phi_1(\Delta)$ and $\Phi_2(\Delta)$ are themselves autonomous differential forms. We shall address this assumption momentarily.

The second, and I think more compelling, reason for seeking cochain maps (5.78) comes from the observation that the Poincaré-Cartan form in mechanics determines just such a map. For the first order Lagrangian

$$\lambda = L(x, u^\alpha, \dot{u}^\alpha)\, dx$$

the associated Poincaré-Cartan form is

$$\Phi_{\text{P.C.}}(\lambda) = \lambda + \frac{\partial L}{\partial \dot{u}^\alpha}\, \theta^\alpha. \tag{5.79}$$

A simple calculation shows that $\Phi_{\text{P.C.}}(\lambda)$ is $d$ closed if and only if $E(\lambda) = 0$ and, moreover, that $\lambda$ is a global total derivative if and only if $\Phi_{\text{P.C.}}$ is exact. For example, on $\mathbf{R} \times S^1 \to \mathbf{R}$, the Lagrangian $\lambda = \dot{u}\, dx$ is not exact because the the one form $\Phi_{\text{P.C.}}(\lambda) = du$ is not exact on $\mathbf{R} \times S^1$. The Poincaré-Cartan form $\Phi_{\text{P.C.}}(\lambda)$ has two other properties which contribute to its importance. It is computable locally from $\lambda$ in that $\Phi_{\text{P.C.}}(\lambda)(j^\infty(s))$ is depends smoothly on only the *jets of the Lagrangian* $L$ at $j^\infty(s)$. Furthermore, the Poincaré-Cartan form is invariant under all local, fiber-preserving diffeomorphisms of $E$.

Because of the ubiquitous role that the Poincaré-Cartan form plays in mechanics, it is important to seek appropriate generalization of this form for general variational principles. One approach to this problem is via the theory of Lepagean equivalents which we briefly touched upon in §A of this chapter. The Poincaré-Cartan form in mechanics also emerges immediately from the application of the Cartan method of equivalence (Gardner [*26*]) and this provides a second means by which generalizations of (5.79) can be obtained. Here we propose to generalize (5.79) by characterizing cochain maps (5.78). This approach insures that the our generalized Poincaré-Cartan form $\Phi(\lambda)$ will change by an exact form whenever the Lagrangian is modified by a divergence.

Accordingly, in an attempt to emulate all of the above properties of the Poincaré-Cartan form, we look for maps

$$\Phi^p : \mathcal{E}^p(J^\infty(E)) \to \Omega^p(J^\infty(E)) \tag{5.80}$$

which are natural differential operators in the sense that

(P1) for any $\omega \in \mathcal{E}^p$, $\Phi^p(\omega)(j^\infty(s))$ is a smooth function of the coefficients of $\omega$ and their derivatives to some finite order evaluated at $j^\infty(s)$; and

(P2) for all local diffeomorphisms $\phi\colon E \to E$ with prolongation $\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(E)$
$$\Phi^p\bigl((\operatorname{pr}\phi)^*\omega\bigr) = (\operatorname{pr}\phi)^*\bigl(\Phi(\omega)\bigr). \tag{5.81}$$

Properties P1 and P2 insure that $\Phi^p(\omega)$ is given by some local formula in the coefficients of $\omega$ which patches together under change of coordinates on $E$ to define a global form on $J^\infty(E)$. Equation (5.81) also implies that $\Phi(\omega)$ inherits the symmetries of $\omega$. For example, in our foregoing discussion of the autonomous source form $\Delta$, we are assured that the coefficients of the forms $\Phi_1(\Delta)$ and $\Phi_2(\Delta)$ contain no explicit $x$ dependence. The infinitesimal version of (5.80) is the Lie derivative commutation formula
$$\Phi^p(\mathcal{L}_{\operatorname{pr} X}\omega) = \mathcal{L}_{\operatorname{pr} X}\Phi^p(\omega),$$
where $X$ is any projectable vector field on $E$. This naturality property is *not* in general enjoyed by the Lepagean forms constructed in §5A. The maps $\Phi^p$ are also required to satisfy

(P3) $\Psi \circ \Phi^p =$ identity on $\mathcal{E}^p(J^\infty(E))$; and

(P4) $d\bigl(\Phi^p(\omega)\bigr) = 0$ whenever $\omega$ is closed in $\mathcal{E}^*(J^\infty(E))$.

The condition P4 is weaker than the cochain condition
$$d\Phi^p(\omega) = \Phi^{p+1}(\delta\omega) \tag{5.82}$$
but, because of property P3 and the fact that $\Psi$ induces an isomorphism in cohomology, P4 is still sufficient to insure that $\Phi^p$ induces a well-defined map in cohomology. Indeed, since $\Psi\bigl(\Phi^p(\omega)\bigr) = \omega$, it follows that $\Phi^p(\omega)$ must be exact whenever $\omega$ is exact. Condition P4 has the advantage that it decouples the problem of constructing the maps $\Phi^p$ for different values of $p$ whereas (5.82) represents an equation for $\Phi^{p+1}$, given that $\Phi^p$ has already been found. Actually, with regards to the specific goal of finding maps on forms which will induce the isomorphism (5.76), it suffices to find maps $\Phi^p$ whose domain need not be all of $\mathcal{E}^p(J^\infty(E))$ but rather any subspace of $\mathcal{E}^p(J^\infty(E))$ which contains all the closed forms.

We are able to construct maps satisfying P1–P4 when the dimension of the base manifold $M$ is $n = 1$ and we conjecture that such maps do not exist when $n \geq 2$. For $n \geq 2$, partial results are possible if we restrict the domain of the maps $\Phi^p$ to first order forms.

Our construction of maps $\Phi^p$ satisfying properties P1–P4 depends upon the following observation. The horizontal homotopy operators $h_H^{r,s}$ introduced in Chapter 4A are local differential operators for $s \geq 1$. Suppose that these operators are natural differential operators. We could then define maps $\Phi^p$ with all the prescribed properties by using equations (5.77), where the homotopy operators are used to solve the second of (5.77b) for the forms $\omega_{i+1}$ in terms of $\sigma_i$, *i.e.*, given $\omega_0$ we inductively define the forms $\omega_{i+1}$ by

$$\omega_{i+1} = h_H^{r,s}(d_V \omega_i) \tag{5.83a}$$

and compute

$$\Phi^p(\omega) = \omega_0 - \omega_1 + \omega_2 - \cdots . \tag{5.83b}$$

Accordingly, we are lead to the problem of finding conditions under which the horizontal homotopy operators satisfy

$$h_H^{r,s} \circ (\operatorname{pr}\phi)^* = (\operatorname{pr}\phi)^* \circ h_H^{r,s}$$

for all local diffeomorphisms $\phi$ of $E$. Again, this naturality condition implies that

$$h_H^{r,s} \circ \mathcal{L}_{\operatorname{pr} X} = \mathcal{L}_{\operatorname{pr} X} \circ h_H^{r,s} \tag{5.84}$$

for all projectable vector fields $X$ on $E$.

To describe circumstances where (5.84) holds, we define subspaces

$$\mathcal{W}^{r,s}(J^\infty(E)) \subset \Omega^{r,s}(J^\infty(E))$$

as follows. A form $\omega \in \Omega^{r,s}(J^\infty(E))$ belongs to $\mathcal{W}^{r,s}(J^\infty(E))$ if for any vector field $X$ on $J^\infty(E)$ such that $(\pi_2^\infty)_*(X) = 0$,

$$\text{(W1)} \quad X \,\lrcorner\, \omega = 0, \qquad \text{and} \qquad \text{(W2)} \quad X \,\lrcorner\, d_H\omega = 0.$$

In local coordinates these two conditions become

$$\partial_\alpha^I \,\lrcorner\, \omega = 0 \qquad \text{for} \quad |I| > 3, \tag{5.85}$$

and

$$(n - r + 2)\partial_\alpha^{hk} \,\lrcorner\, \omega + dx^h \wedge \partial_\alpha^{ki} \,\lrcorner\, \omega_i + dx^k \wedge \partial_\alpha^{hi} \,\lrcorner\, \omega_i = 0, \tag{5.86}$$

where $\omega_i = D_i \lrcorner\, \omega$. This latter equation follows easily from W1 and Proposition 2.10 with $I = ihk$. Condition W1 simply states that $\omega$ belongs to the ideal generated by the forms $dx^i$, $\theta^\alpha$, $\theta^\alpha_i$ and $\theta^\alpha_{ij}$. No contact forms of order 3 or higher can occur although the coefficients of $\omega$ may depend upon higher order derivatives. Note that condition W2 holds automatically when $\omega$ lies in the ideal generated by the forms $dx^i$, $\theta^\alpha$ and $\theta^\alpha_i$ and also when $d_H\omega = 0$.

Conditions W1 and W2 are "generically" necessary conditions for $h^{r,s}_H$ to act naturally. To see this, consider first the type $(2,1)$ form

$$\omega = \theta_{xxx}\, dx \wedge dy = d_H(-\theta_x \wedge dy), \tag{5.87}$$

which does not satisfy W1, and the vector field

$$X = x^2 \frac{\partial}{\partial x}.$$

Then a series of elementary calculations show that

$$h^{2,1}_H(\omega) = -\theta_{xx} \wedge dy$$

and

$$\mathcal{L}_{\mathrm{pr}\, X}\big(h^{2,1}_H(\omega)\big) - h^{2,1}_H\big(\mathcal{L}_{\mathrm{pr}\, X}\omega\big) = \theta_x \wedge dx + \theta_y \wedge dy.$$

This shows that with $n = 2$, $h^{2,1}_H$ does not in general act naturally — even on the subspace of $d_H$ closed forms. Next, let $X$ be an vector field on a fixed chart $(x, u, U)$ of $E$ which is of the form

$$X = a^i(x) \frac{\partial}{\partial x^i}.$$

Suppose $\omega \in \Omega^{r,s}(J^\infty(U))$ satisfies W1. Then a lengthy calculation shows that

$$\mathcal{L}_{\mathrm{pr}\, X}\big(h^{r,s}_H(\omega)\big) - h^{r,s}_H\big(\mathcal{L}_{\mathrm{pr}\, X}\omega\big) = \frac{1}{s(n-r+1)(n-r+2)}\, \sigma, \tag{5.88}$$

where

$$\sigma = (D_{hk}a^p)\, \theta^\alpha \wedge \{D_p \lrcorner\, [(n-r+2)\partial^{\,hk}_{\,\alpha} \lrcorner\, \omega + 2dx^{(h} \wedge \partial^{\,k)i}_{\,\alpha} \lrcorner\, \omega_i]\}.$$

Consequently, if $h^{r,s}_H$ is to act naturally, $\sigma$ must vanish. For $s = 1$ this implies W2.

The next proposition (part(ii)) establishes the sufficiency of the conditions W1 and W2.

PROPOSITION 5.43.

*(i) Let* $\mathcal{Z}_H^{r,s}(J^\infty(U)) \subset \Omega^{r,s}(J^\infty(U))$ *be the subspace of* $d_H$ *closed forms. Then, for* $s \geq 1$,

$$h_H^{1,s} \colon \mathcal{Z}_H^{1,s}(J^\infty(U)) \to \Omega^{0,s}(J^\infty(U))$$

*is a natural differential operator.*

*(ii) For* $s \geq 1$, *the maps*

$$h_H^{r,s} \colon \mathcal{W}^{r,s}(J^\infty(U)) \to \Omega^{r-1,s}(J^\infty(U))$$

*are natural differential operators.*

PROOF: (i) Recall that if $\eta_1$ and $\eta_2$ are two type $(0, s)$ forms on $J^\infty(U)$, where $s \geq 1$, then $d_H\eta_1 = d_H\eta_2$ implies that $\eta_1 = \eta_2$. Let $f \colon U \to V$ be a local (fiber-preserving) diffeomorphism between two coordinate charts $U$ and $V$ of $E$ and let $F = \operatorname{pr} f$. Since

$$d_H\bigl(F^* h_H^{1,s}(\omega)\bigr) = F^*\bigl(d_H h_H^{1,s}(\omega)\bigr) = F^*(\omega) = d_H\bigl(h_H^{1,s}(F^*(\omega)\bigr)$$

we conclude that

$$F^*\bigl(h_H^{1,s}(\omega)\bigr) = h_H^{1,s}\bigl(F^*(\omega)\bigr)$$

for all $\omega \in \mathcal{Z}^{1,s}(J^\infty(U))$, as required.

(ii) Given $\omega \in \mathcal{W}^{r,s}(J^\infty(U))$, define a second order total differential operator

$$P \colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s-1}(J^\infty(E))$$

on the space of evolutionary vector fields by

$$P(Y) = \operatorname{pr} Y \mathbin{\lrcorner} \omega.$$

Because of conditions W1 and W2, it is a simple matter, using (2.23), to write this operator in the form

$$P(Y) = Q(Y) + d_H[R(Y)], \tag{5.89}$$

where $Q$ and $R$ are the first order operators defined by

$$Q(Y) = Y^\alpha F_\alpha(\omega) + D_i[Y^\alpha\bigl(F_\alpha^i(\omega) - \frac{1}{n-r+1}\, dx^i \wedge F_\alpha^j(\omega_j)\bigr)],$$

and

$$R(Y) = \frac{1}{n-r+1} Y^\alpha F^j_\alpha(\omega_j) + \frac{2}{n-r+2} D_i[Y^\alpha F^{ij}_\alpha(\omega_j)]. \tag{5.90}$$

Note that both $Q$ and $R$ are trace-free (see Proposition 2.4).

The decomposition (5.89) of $P$ into the sum of two first order trace-free operators is unique. Indeed, suppose that $\widetilde{Q}$ and $\widetilde{R}$ are two first order, trace-free operators, *i.e.*,

$$\widetilde{Q}(Y) = Y^\alpha A_\alpha + D_i(Y^\alpha B^i_\alpha), \qquad D_i \lrcorner\, B^i_\alpha = 0$$

and

$$\widetilde{R}(Y) = Y^\alpha C_\alpha + D_i(Y^\alpha D^i_\alpha), \qquad D_i \lrcorner\, D^i_\alpha = 0$$

and that

$$\widetilde{Q}(Y) + d_H \widetilde{R}(Y) = 0 \tag{5.91}$$

for all evolutionary vector fields $Y$. By setting the coefficient of $Y^\alpha_{ij}$ in (5.91) to zero, it is found that

$$dx^i \wedge D^j_\alpha + dx^j \wedge D^i_\alpha = 0.$$

Because the coefficients $D^i_\alpha$ is trace-free, the inner evaluation of the equation with the total vector field $D_i$ yields $D^i_\alpha = 0$. The coefficient of $Y^\alpha_i$ in (5.91) now reduces to

$$B^i_\alpha + dx^i \wedge C_\alpha = 0,$$

from which it follows, again by inner evaluation with $D_i$, that $C_\alpha = 0$. This proves that $\widetilde{R} = 0$ and therefore $\widetilde{Q} = 0$. This suffices to prove the uniqueness of the decomposition (5.89).

This also suffices to prove that for $\omega \in \mathcal{W}^{r,s}(J^\infty(E))$ the operators $Q(Y)$ and $R(Y)$ are invariantly defined, *e.g.*, $F^*[R(F_*(Y))] = R(Y)$. Finally, with $Y$ replaced by

$$Y = \theta^\alpha \otimes \frac{\partial}{\partial u^\alpha}\,,$$

we find, on comparing (4.13) with (5.90), that

$$h^{r,s}_H(\omega) = R(Y)$$

and the invariance, or naturality, of the horizontal homotopy operators is established.

Of course, one could also prove (ii) by a direct change of variables calculation. ∎

We now use Proposition 5.43 to establish the naturality of the map (5.83) in two special cases.

EXAMPLE 5.44. Here we take $\dim M = 1$. In this case the augmented variational bicomplex collapses to the three column double complex

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{I} & \mathcal{F}^2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} & & \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{I} & \mathcal{F}^1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \\
\mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0}. & & & &
\end{array}$$

By Proposition 5.43(i), the homotopy operators

$$h_H^{1,s} \colon \Omega^{1,s}(J^\infty(E)) \to \Omega^{0,s}(J^\infty(E))$$

are all natural differential operators and hence the maps

$$\Phi^p \colon \Omega^{1,p}(J^\infty(E)) \to \Omega^{p+1}(J^\infty(E))$$

defined by

$$\Phi^p(\omega) = \omega - h_H^{1,p+1}(d_V \omega)$$

are natural. In this case the maps $\Phi^p$ also determine a cochain map $\Phi$ from the Euler-Lagrange complex to the de Rham complex.

PROPOSITION 5.45. *For any $\lambda \in \Omega^{1,0}(J^\infty(E))$ and any $\omega \in \mathcal{F}^p(J^\infty(E))$, $\Phi$ satisfies*

$$d\,(\Phi(\lambda)) = \Phi(E(\lambda)) \quad \textit{and} \quad d\,(\Phi(\omega)) = \Phi(\delta_V(\omega)). \tag{5.92}$$

PROOF: We first note that the two type $(0, p+2)$ forms

$$d_V h_H^{1,p+1}(d_V \omega) \quad \text{and} \quad h_H^{1,p+2}(d_V I d_V \omega)$$

have the same horizontal exterior derivative and are therefore equal. To prove this we use the identity (4.15), *i.e.*,

$$\eta = I(\eta) + d_H h_H^{1,s}(\eta),$$

first with $\eta = d_V\omega$ and then with $\eta = d_V I d_V \omega$ to find that

$$d_H d_V h_H^{1,p+1}(d_V\omega) = -d_V[d_V\omega - I(d_V\omega)] = d_V I d_V \omega,$$

and (recall that $\delta_V = I \circ d_V$)

$$d_H h_H^{1,p+2}(d_V I d_V \omega) = d_V I d_V \omega - \delta_V^2(\omega) = d_V I d_V \omega.$$

Consequently, we deduce that

$$\begin{aligned} d\,[\Phi(\omega)] &= d_V\omega - d_H h_H^{1,p+1}(d_V\omega) - d_V h^{1,p+1}(d_V\omega) \\ &= d_V\omega - (d_V\omega - I d_V\omega) - h_H^{1,p+2}(d_V I d_V \omega) \\ &= \delta_V\omega - h_H^{1,p+2}(d_V \delta_V \omega) \\ &= \Phi^{p+1}(\delta_V\omega), \end{aligned}$$

as required. The proof of the first equation in (5.92) is the same. ∎

COROLLARY 5.46. *Let $E\colon \mathbf{R} \times F \to \mathbf{R}$ and let $G$ be the group of translations on the base space $\mathbf{R}$. Then*

$$H^*(\mathcal{E}_G^*(J^\infty(E))) \cong H^1(F) \oplus H^2(F).$$

PROOF: The proof of this result was outlined in our introductory remarks at the beginning of this section. ∎

For a $k^{\text{th}}$ order single integral Lagrangian $\lambda = L[x, u^{(k)}]\,dx$ in $\Omega^{1,0}(J^\infty(E))$ it readily follows from the formula (4.13) for $h_H^{1,1}$ that

$$\Phi(\lambda) = \lambda + \sum_{j=0}^{k} P_\alpha^{(j)}\,\theta_{(j)}^{\alpha},$$

where

$$P_\alpha^{(j)} = \sum_{l=0}^{k-j} (-1)^l \frac{d^l}{dx^l}\Big(\frac{\partial L}{\partial u_{j+l+1}^{\alpha}}\Big).$$

This is the standard Poincaré-Cartan form for higher order, single integral problems in the calculus of variations. See, *e.g.*, Krupka [*44*] or Hsu [*37*].

For a source form $\Delta \in \Omega^{1,1}(J^\infty(E))$ described locally by

$$\Delta = A_\alpha[x,u]\, du^\alpha \wedge dx,$$

we obtain what one might call *the Poincaré-Cartan two form* $\Phi(\Delta)$ *for ordinary differential equations.* For a third source form, $\Phi(\Delta)$ is explicitly given by

$$\begin{aligned}\Phi(\Delta) = \Delta + \frac{1}{2}\Big[\frac{\partial A_\alpha}{\partial \dot u^\beta} - \frac{d}{dx}\frac{\partial A_\alpha}{\partial \ddot u^\beta} + \frac{d^2}{dx^2}\frac{\partial A_\alpha}{\partial \dddot u^\beta}\Big]\, \theta^\alpha \wedge \theta^\beta \\ + \frac{1}{2}\Big[\Big(\frac{\partial A_\alpha}{\partial \ddot u^\beta} + \frac{\partial A_\beta}{\partial \ddot u^\alpha}\Big) - \frac{d}{dx}\Big(\frac{\partial A_\alpha}{\partial \dddot u^\beta} + 2\frac{\partial A_\beta}{\partial \dddot u^\alpha}\Big)\Big]\, \theta^\alpha \wedge \dot\theta^\beta \\ + \frac{1}{2}\Big[\frac{\partial A_\alpha}{\partial \dddot u^\beta} - \frac{\partial A_\beta}{\partial \dddot u^\alpha}\Big]\, \theta^\alpha \wedge \ddot\theta^\beta - \frac{1}{2}\Big[\frac{\partial A_\alpha}{\partial \dddot u^\beta}\Big]\dot\theta^\alpha \wedge \dot\theta^\beta.\end{aligned}$$

We emphasize that this is an invariant defined two form for any third order source form $\Delta$ which is $d$ closed if and only if $\Delta$ is locally variational. ∎

Example 5.47. Let $F$ be any open domain in $\mathbf{R}^m$ and let $E\colon \mathbf{R} \times F \to \mathbf{R}$. Let $\eta_{\alpha\beta} = \operatorname{diag}[\pm 1, \pm 1, \dots, \pm 1]$ and let

$$P_\alpha = \eta_{\alpha\beta}\ddot u^\beta - A_{\alpha\beta}\dot u^\beta - B_\alpha,$$

where the coefficients $A_{\alpha\beta} = -A_{\beta\alpha}$ and $B_\alpha$ are smooth functions on $F$. For instance, when $m = 4$, when $\eta_{\alpha\beta} = \operatorname{diag}[1,1,1,-1]$, and when $B_\alpha = 0$, the source equations $P_\alpha = 0$ become the Lorentz source equations in electrodynamics. The Poincaré-Cartan two form for the second order source form $\Delta = P_\alpha\, \theta^\alpha \wedge dx$ is

$$\begin{aligned}\Phi(\Delta) &= \Delta + \frac{1}{2}\Big[\frac{\partial P_\alpha}{\partial \dot u^\beta} - \frac{d}{dx}\frac{\partial P_\alpha}{\partial \ddot u^\beta}\Big]\, \theta^\alpha \wedge \theta^\beta \\ &\quad + \frac{1}{2}\Big[\frac{\partial P_\alpha}{\partial \ddot u^\beta} + \frac{\partial P_\beta}{\partial \ddot u^\alpha}\Big]\, \theta^\alpha \wedge \dot\theta^\beta \\ &= \alpha + \beta \wedge dx + d\gamma,\end{aligned}$$

where

$$\alpha = -\frac{1}{2}A_{\alpha\beta}\, du^\alpha \wedge du^\beta, \qquad \beta = -B_\alpha\, du^\alpha,$$

and $\eta = \eta_{\alpha\beta}\ddot u^\beta\, du^\alpha$. Thus, in the notation introduced at the beginning of this section, $\Phi_1(\Delta) = \beta$ and $\Phi_2(\Delta) = \alpha + d\gamma$. Therefore, $\Delta$ admits a global, autonomous

Lagrangian if and only if $\alpha$ and $\beta$ are exact on $F$. In fact, if $\alpha = d\,f$ and $\beta = d\eta$, then

$$\Phi(\Delta) = d\,(f\,dx + \eta + \gamma)$$

and an autonomous Lagrangian for $\Delta$ is given by

$$\lambda = \pi^{1,0}(f\,dx + \eta + \gamma).$$

∎

EXAMPLE 5.48. Let $\Omega_1^{r,s}(J^\infty(E))$ denote the space of type $(r,s)$ forms which lie in the ideal generated by $dx^i$, $\theta^\alpha$, and $\theta_i^\alpha$ with coefficients which depend only upon the first order variables $(x^i, u^\alpha, u_i^\alpha)$. As we remarked in Chapter 4C, this is a well-defined subspace of $\Omega^{r,s}(J^\infty(E))$ and

$$d_V \colon \Omega_1^{r,s} \to \Omega_1^{r,s+1}.$$

Forms in $\Omega_1^{r,s}$ all satisfy conditions W1 and W2 and therefore belong to the subspace $\mathcal{W}^{r,s}(J^\infty(E))$. By Proposition 5.43 (ii)

$$h_H^{r,s} \colon \Omega_1^{r,s} \to \Omega_1^{r-1,s}$$

is a natural differential operator and so the map

$$\Phi^r \colon \Omega_1^{r,0} \to \Omega^r(J^\infty(E)),$$

as defined by (5.83), is natural. If we bear in mind that, for $\omega \in \Omega_1^{r,0}$, the partial derivative $\dfrac{\partial \omega}{\partial u_i^\alpha}$ transforms tensorial under coordinate transformations, then this naturality is manifest in the explicit formula

$$\Phi(\omega) = \sum_{k=0}^{r} c_k\, \theta^{\alpha_1} \wedge \theta^{\alpha_2} \cdots \wedge \theta^{\alpha_k}\, \partial_{\alpha_1}^{i_1} \partial_{\alpha_2}^{i_2} \cdots \partial_{\alpha_k}^{i_k}\, \omega_{i_1 i_2 \dots i_k} \tag{5.93}$$

where $c_k = \dfrac{1}{(k!)^2 \binom{n-r+k}{k}}$ and $\omega_{i_1 i_2 \dots i_k} = D_{i_k} \lrcorner\, D_{i_{k-1}} \cdots \lrcorner\, D_{i_1} \lrcorner\, \omega$.

For $\lambda \in \Omega^{n,0}(J^\infty(E))$, the form $\Phi^n(\lambda)$ coincides with the generalized Poincaré-Cartan form introduced independently by Rund [*61*] and Betounes [*10*]. Betounes' result, that $E(\lambda) = 0$ if and only if $d\,[\Phi(\omega)] = 0$ is clear from our general construction. Because of the naturality of this construction it is also immediate that symmetries of $\lambda$ are also symmetries of $\Phi^n(\lambda)$. We remark that the generalized Poincaré-Cartan form $\Phi^n(\lambda)$ is different from that introduced by Goldschmidt and Sternburg [*30*] in their study of the Hamilton-Jacobi theory for first order multiple integral problems. Their form retains only the terms in $\Phi^n(\lambda)$ which are linear in the contact forms. It is a Lepagean equivalent for $\lambda$ but it need not be closed when $\lambda$ is variationally trivial. ∎

The two cases presented here may well be the only instances where mapping $\Phi^p$ satisfying properties (P1)–(P4) are possible. Specifically, we have the following conjectures.

CONJECTURE 5.49. *Let $k > 2$ and suppose*

$$\Phi^*\colon \mathcal{E}^*_k(J^\infty(E)) \to \Omega^*(J^\infty(E))$$

*is a natural differential cochain map defined on the entire Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$. The $n = 1$ and $\Phi^*$ is the cochain map described in Example 5.44.*

CONJECTURE 5.50. *Suppose that*

$$\Phi^r : \Omega^{r,0}_k(J^\infty(E)) \to \Omega^r(J^\infty(E))$$

*satisfies properties P1–P4. Then either $r = 0$ and $\Phi^0$ is the identity map or else $k = 1$ and $\Phi^r$ is the map presented in Example 5.48.*

EXAMPLE 5.51. Under additional hypothesis, it is possible to find isolated situations where maps satisfying properties (P1)–(P4) exist. Again, these depend upon showing that if $d_V\omega_i \in \mathcal{W}^{r,s+1}$ , then with $\omega_{i+1}$ defined by (5.83a), $d_V\omega_{i+1} \in \mathcal{W}^{r-1,s+2}$ so that by Proposition 5.43 the map (5.83) is natural. This is the case, for example, on quasi-linear, second order variationally closed source forms

$$\Delta = [A^{ij}_{\alpha\beta} u^\alpha_{ij} + B_\alpha]\, \theta^\alpha \wedge \nu$$

where the coefficients $A^{ij}_{\alpha\beta}$ and $B_\alpha$ are first-order functions. Equation (5.83) becomes

$$\Phi^{n+1}(\Delta) = \sum_{k=0}^{n} \Delta_k$$

with

$$\begin{aligned}\Delta_k = {}&\\ &\frac{1}{(k+1)!\,k!}[\partial^{\,i_1}_{\,\alpha_1}\partial^{\,i_2}_{\,\alpha_2}\cdots\partial^{\,i_k}_{\,\alpha_k}\Delta_\alpha]\,\theta^\alpha\wedge\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_k}\wedge\nu_{i_1i_2\dots i_k}\\ &-\frac{2k}{(k+1)!\,k!}[\partial^{\,hi_1}_{\,\alpha_1}\partial^{\,i_2}_{\,\alpha_2}\cdots\partial^{\,i_k}_{\,\alpha_k}\Delta_\alpha]\,\theta^\alpha_h\wedge\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_k}\wedge\nu_{i_1i_2\dots i_k}\end{aligned}$$

We stress that this form is not invariantly defined unless $\Delta$ is locally variational. This map satisfies

$$\Phi^{n+1}(E(\lambda)) = d\,\Phi^n(\lambda)$$

for $\lambda \in \Omega^{n,0}_1$ and $\Phi^n$ defined by (5.93). ∎

Conjectures 5.49 and 5.50 have been verified for small, specific values of $r$ and $k$. Their proof in general may simply be more tedious than difficult. These conjectures are part of the more general program of characterizing all natural, linear differential operators on the variational bicomplex. This project is more complicated that the analogous problem solved by Palais [57] for the standard de Rham complex on finite dimensional manifolds. There are two reasons for this. Firstly, in the present context, naturality refers not to all local diffeomorphisms of the underlying space $J^\infty(E)$ but only to those local diffeomorphisms induced by maps on $E$. This precludes the use of the normal form arguments developed by Palais. Secondly, a really satisfactory characterization of natural differential operators on the variational bicomplex ought to describe for the various exceptional natural operators that arise when the domain of definition is restricted to $\Omega_k^{r,s}$ for small values for $k$. But, at this time, this seems to be akin to the opening of Pandora's box.

One characterization of natural differential operators on the variational bicomplex is given the following.

PROPOSITION 5.52. *The only* $\mathbf{R}$ *linear, natural differential operator*

$$\Phi\colon \Omega^{n,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$$

*is a constant multiple of the Euler-Lagrange operator.*

SKETCH OF PROOF: The symbols of natural differential operators are natural tensors. Natural tensors are readily classified by classical invariant theory and it not difficult to show that the only natural tensor which can arise as the symbol of a natural differential operator from $\Omega^{n,0}$ to $\mathcal{F}^1$ is a multiple of the symbol of the Euler-Lagrange operator. ∎

Another problem along these lines is the characterization of natural Lepagean equivalents.

PROPOSITION 5.53. *Let*

$$\Phi\colon \Omega_k^{n,0}(J^\infty(E)) \to \Omega^n(J^\infty(E))$$

*be a natural differential operator such that for any* $\lambda \in \Omega_k^{n,0}$, $\Phi(\lambda)$ *is a Lepagean equivalent to* $\lambda$. *Then* $k \leq 2$ *and*

$$\begin{aligned} \Phi(\lambda) &= \lambda - h_H^{n,1}(d_V\lambda) \\ &= L\,\nu + [\frac{\partial L}{\partial u^\alpha_j} - D_i\frac{\partial L}{\partial u^\alpha_{ij}}]\,\theta^\alpha \wedge \nu_j + [\frac{\partial L}{\partial u^\alpha_{ij}}]\,\theta^\alpha_i \wedge \nu_j. \end{aligned} \tag{5.94}$$

*In particular, there are no natural Lepagean equivalents for third order Lagrangians.*

Sketch of Proof: The naturality of (5.94) follows from Proposition 5.43 (ii) and the fact that if $\lambda$ is any second order Lagrangian, then $d_V\lambda \in \mathcal{W}^{n,1}$.

Suppose that

$$\Phi\colon \Omega_3^{n,0}(J^\infty(E)) \to \Omega^n(J^\infty(E))$$

is a natural Lepagean equivalent. Then

$$(\pi^{n-1,1}\circ\Phi)\colon \Omega_3^{n,0}(J^\infty(E)) \to \Omega^{n-1,1}(J^\infty(E))$$

is a natural differential operator. It is not hard to show that the only such operator invariant under general affine linear changes of coordinates and compatible with the Lepagean condition (5.20) and (5.21) is a constant multiple of

$$\begin{aligned}(\pi^{n-1,1}\circ\Phi)(\lambda) &= -h_H^{n-1,1}(d_V\lambda)\\ &= [\frac{\partial L}{\partial u_k^\alpha} - D_i(\frac{\partial L}{\partial u_{ik}^\alpha}) + D_{ij}(\frac{\partial L}{\partial u_{ijk}^\alpha})]\,\theta^\alpha\wedge\nu_k\\ &\quad + [\frac{\partial L}{\partial u_{ik}^\alpha} - D_j(\frac{\partial L}{\partial u_{ijk}^\alpha})]\,\theta_i^\alpha\wedge\nu_k + [\frac{\partial L}{\partial u_{ijk}^\alpha}]\,\theta_{ij}^\alpha\wedge\nu_k.\end{aligned}$$

But in view of (5.88) this map is not invariant under all (fiber-preserving) changes of variables and so no natural Lepagean form exists. ∎

This work leaves open the even more general problem of finding non-linear natural operators on, in particular, the space of $k^{\text{th}}$ order Lagrangians $\Omega_k^{n,0}(J^\infty(E))$. One such natural differential operator, defined for *non-zero* first order Lagrangians

$$\lambda = L(x^i, u^\alpha, u_i^\alpha)\,\nu$$

is

$$\Phi(\lambda) = \frac{1}{L^{n-1}}\,\sigma^1\wedge\sigma^2\wedge\cdots\wedge\sigma^n,$$

where

$$\sigma^i = L\,dx^i - \frac{\partial L}{\partial u_i^\alpha}\,\theta^\alpha.$$

This form was first introduced by Rund [*61*] as the generalization of the Poincaré-Cartan form appropriate for Carathéodory's approach to the Hamilton-Jacobi theory for first order multiple integral problems. While it is not closed on variationally trivial Lagrangians, it does have the important geometric property of being a decomposable $n$ form. This form also arises directly though the application of the Cartan equivalence algorithm, see Garnder [*27*]. In fact, Cartan's method provides a systematic way for characterizing the invariants (under any prescribed transformation group) of a given class (*i.e.*, with order and number of dependent and independent variables prescribed) of Lagrangians. For example, in solving the equivalence problem defined for second order Lagrangians in the plane, *i.e.*, for Lagrangians of the type

$$\lambda = L(x, u, \dot{u}, \ddot{u})\, dx,$$

Kamran and Olver [*39*] discovered the new natural differential operator

$$\mathcal{D}(\lambda) = D_x(L\frac{\partial L}{\partial \ddot{u}}) - L\frac{\partial L}{\partial \dot{u}}\frac{\partial L}{\partial \ddot{u}}.$$

**D. Invariant Horizontal Homotopy Operators on Manifolds with Symmetric Connections.** Let $\nabla$ be a symmetric, linear connection on the base manifold $M$ of the fibered manifold $\pi\colon E \to M$. We first show that the connection $\nabla$ induces a process of covariant total differentiation of certain tensor-valued, type $(r, s)$ forms on $J^\infty(E)$. This construction is, in itself, noteworthy since one might have anticipated that additional geometric structures (such as connection on the bundle of vertical vectors on $E$) would be required.

Let $T^{p,q}(J^\infty(E))$ be the bundle of type $(p, q)$ tensors on $J^\infty(E)$. Let $\sigma = j^\infty(s)$ be a point in $J^\infty(E)$. We call a tensor $T_\sigma \in T^{p,q}$ *horizontal* if

$$T(\alpha_1, \alpha_2, \dots, \alpha_p, X_1, X_2, \dots, X_q) = 0$$

whenever either,

(i) one of the covectors $\alpha_i$, $1 \le i \le p$, belongs to the ideal $\mathcal{C}$ of contact forms at $\sigma$, or

(ii) one of the vectors $X_j$, $1 \le j \le q$, is a $\pi^\infty_M$ vertical vector at $\sigma$.

In local coordinates $[x, u] = (x^i, u^\alpha, u^\alpha_i, \dots)$ around $\sigma$, $T$ assumes the form

$$T_\sigma = T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}\, D_{i_1} \otimes D_{i_2} \cdots \otimes D_{i_p} \otimes dx^{j_1} \otimes dx^{j_2} \cdots \otimes dx^{j_q}, \tag{5.95}$$

where, as usual, $D_i$ denotes the total vector field

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots .$$

Denote the bundle of horizontal type $(p,q)$ tensors on $J^\infty(E)$ by $T_H^{p,q}(J^\infty(E))$. Sections of this bundle are locally of the form (5.95), where the coefficients $T^{\cdots}_{\cdots}$ are smooth functions of the variables $[x,u]$.

Let $T^{p,q}(M)$ be the bundle of type $(p,q)$ tensors on $M$. Then $T_H^{p,q}(J^\infty(E))$ can be identified with the induced bundle

$$\begin{array}{ccc} T_H^{p,q}(J^\infty(E)) & \xrightarrow{\hat\pi_M^\infty} & T^{p,q}(M) \\ \big\downarrow & & \big\downarrow \\ J^\infty(E) & \xrightarrow[\pi_M^\infty]{} & M\,. \end{array}$$

If $T_\sigma$ is a type $(p,q)$ horizontal tensor at $\sigma = j^\infty(s)(x)$, then $\hat\pi_M^\infty(T_\sigma)$ is the type $(p,q)$ tensor on $M$ at $x$ defined by

$$\hat\pi_M^\infty(T_\sigma)(\alpha_1,\alpha_2,\ldots,\alpha_p,X_1,X_2,\ldots,X_q) = (T_\sigma)(\tilde\alpha_1,\tilde\alpha_2,\ldots,\tilde\alpha_p,\widetilde X_1,\widetilde X_2,\ldots,\widetilde X_q)$$

for covectors $\alpha_i$ and vectors $X_j$ at $x\in M$, where $\tilde\alpha_i = (\pi_M^\infty)^*(\alpha_i)$ and $\widetilde X_j = \operatorname{tot} X_j$. Let $\pi\colon U\to U_0$ be an adapted coordinate chart on $E$. If $T$ is a horizontal type $(p,q)$ tensor field on $J^\infty(U)$ and $s\colon U_0\to U$ is a local section of $E$, then $\hat\pi_M^\infty[T(j^\infty(s))]$ is a type $(p,q)$ tensor field on $U_0$.

Denote by $S_H^{p,0}(J^\infty(E))$ the bundle of horizontal, symmetric type $(p,0)$ tensors on $J^\infty(E)$.

The connection $\nabla$ on $M$ induces a process of total covariant differentiation on sections of $T_H^{p,q}(J^\infty(E))$. If $X$ is a generalized vector field on $M$ and $T$ is a horizontal, type $(p,q)$ tensor field on $J^\infty(E)$ then at a point $\sigma = j^\infty(s)$, where $s$ is a local section on $E$,

$$(\nabla_{\operatorname{tot} X}T)(j^\infty(s)) = \nabla_{X_0}\{\hat\pi_M^\infty[T(j^\infty(s))]\},$$

where $X_0(x) = X(j^\infty(s)(x))$. In coordinates, if $X = X^h\dfrac{\partial}{\partial x^h}$ is a generalized vector field on $M$, then (see (1.38))

$$\operatorname{tot} X = X^h D_h$$

is the associated total vector field on $J^\infty(E)$. If $T$ is, say, a type $(1,1)$ horizontal tensor field on $J^\infty(E)$ with components

$$T = T^i_j\, D_i\otimes dx^j,$$

then the components of $\nabla_{\mathrm{tot}\,X} T$ are

$$\nabla_{\mathrm{tot}\,X} T = X^h [D_h T^i_j + \Gamma^i_{lh} T^l_j - \Gamma^l_{jh} T^i_l]\, D_i \otimes dx^j.$$

A routine change of coordinates calculation, based on the fact that our coordinate transformations

$$y^j = y^j(x^i) \quad \text{and} \quad v^\beta = v^\beta(x^i, u^\alpha)$$

are projectable and therefore satisfy

$$D_h[\frac{\partial y^j}{\partial x^i}] = \frac{\partial^2 y^i}{\partial x^i \partial x^h}$$

directly verifies the tensorial character of $\nabla_{\mathrm{tot}\,X} T$. Total covariant differentiation with respect to the coordinate vector field $\dfrac{\partial}{\partial x^h}$ will be denoted by $\nabla_h$. If $T^i_j$ are the components of a type $(1,1)$ horizontal tensor field, then $\nabla_h T^i_j$ are the components of a type $(1,2)$ horizontal tensor field on $J^\infty(U)$.

Now consider a $p$ form $\Xi$ on $J^\infty(E)$ which takes its values in $T^{p,q}_H$, *i.e.*, $\Xi$ is a section of $\Lambda^p(J^\infty(E)) \otimes T^{p,q}_H(J^\infty(E))$. If $\Xi$ is a $p$ form of type $(r,s)$, then for all evolutionary vector fields $Y_1$, $Y_2$, $\ldots$, $Y_s$ on $J^\infty(E)$

$$\widetilde{\Xi} = \Xi\,(\mathrm{pr}\,Y_1, \mathrm{pr}\,Y_2, \ldots, \mathrm{pr}\,Y_s)$$

is a type $(p, q+r)$ horizontal tensor field on $J^\infty(E)$. We define the total covariant derivative of $\Xi$ to be the type $(r,s)$ form with values in $T^{p,q}_H$ as given by

$$(\nabla_{\mathrm{tot}\,X}\,\Xi)(\mathrm{pr}\,Y_1, \mathrm{pr}\,Y_2, \ldots, \mathrm{pr}\,Y_s) = \nabla_{\mathrm{tot}\,X}[\,\widetilde{\Xi}\,]. \tag{5.96}$$

For example, if $\Xi$ is a type $(r,s)$ form with values in $T^{1,0}_H$ which is given locally by

$$\Xi = T^i_j \otimes D_i \otimes dx^j,$$

where each component $T^i_j$ is a type $(r,s)$ form, then

$$\nabla_{\mathrm{tot}\,X}\,\Xi = X^h[D_h T^i_j + \Gamma^i_{lh} T^l_j - \Gamma^l_{jh} T^i_l - \gamma^l_h \wedge (D_l \mathbin{\lrcorner} T^i_j)] \otimes D_i \otimes dx^j.$$

Here $\gamma^l_h$ are the connection one forms

$$\gamma^l_h = \Gamma^l_{hk}\, dx^k.$$

This formula follows from the definition (5.96) and the fact (see Proposition 1.16) that total differentiation commutes with inner evaluation by the prolongation of arbitrary evolutionary vector fields, *i.e.*,

$$D_h(\operatorname{pr} Y \lrcorner\, \Xi) = \operatorname{pr} Y \lrcorner\, (D_h\, \Xi).$$

We write

$$\nabla_{\text{tot}\,X}\, \Xi = X^h[\nabla_h T^i_j] \otimes D_i \otimes dx^j$$

and observe that $\nabla_h T^i_j$ are the components of a type $(r, s)$ form with values in $T^{1,2}_H(J^\infty(E))$.

We remark that if $\Xi = T^i \otimes D_i$ is a vector-valued type $(n, s)$ form (where $\dim M = n$), then

$$\gamma^l_h \wedge (D_l \lrcorner\, T^i) = \Gamma^l_{hl} T^i$$

and hence

$$\nabla_h T^h = D_h T^h. \tag{5.97}$$

Because the connection $\nabla$ is assumed to be symmetric, it also follows that for $\omega \in \Omega^{r,s}(J^\infty(E))$,

$$d_H\omega = dx^h \wedge D_h\omega = dx^h \wedge \nabla_h\omega = d^\nabla_H\omega, \tag{5.98}$$

that is, *the covariant horizontal exterior derivative* $d^\nabla_H$ *on forms coincides with the ordinary horizontal exterior derivative.* Note too that

$$\nabla_h(dx^h \wedge \omega) = dx^h \wedge (\nabla_h\omega).$$

In this equation, $dx^h \wedge \omega$ is properly viewed as the components of a a vector-valued type $(r + 1, s)$ form.

Next we use the connection $\nabla$ to construct invariant counterparts of the Lie-Euler operators $E^I_\alpha$ and interior evaluation operators $F^I_\alpha$ introduced in Chapter Two. The construction of these operators is based upon the systematic replacement of total derivatives $D_h$ by total covariant derivatives $\nabla_h$.

Let $P$ be a total differential operator as defined in §2.A, *i.e.*,

$$P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$$

where, for an evolutionary vector field $Y = Y^\alpha \dfrac{\partial}{\partial u^\alpha}$,

$$P(Y) = \sum_{|I|=0}^{k} (D_I Y^\alpha) P^I_\alpha.$$

According to Proposition 2.1, $P(Y)$ may be rewritten in the form

$$P(Y) = \sum_{|I|=0}^{k} D_I[Q^I(Y)], \tag{5.99}$$

where

$$Q^I(Y) = Y^\alpha Q^I_\alpha = Y^\alpha \Big[\sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|J|}(-D)_J P^{IJ}_\alpha\Big].$$

Recall that the $Q^I(Y)$ are assumed to be symmetric in the indices $I$. Because the operator $P$ is of order $k$, the $Q^{i_1 i_2\cdots i_k}(Y)$ are the components of a type $(r,s)$ form with values in $T^{k,0}_H$ or, to be more specific, with values in $S^{k,0}_H$. This type $(r,s)$ form is the symbol of $P$. Because of the tensorial character of the symbol, we may successively replace all the total derivatives in the expression $D_{i_1 i_2\dots i_k} Q^{i_1 i_2\dots i_k}$ by total covariant derivatives and write

$$D_{i_1 i_2\dots i_k}[Q^{i_1 i_2\dots i_k}(Y)] = \nabla_{i_1 i_2\dots i_k}[Q^{i_1 i_2\dots i_k}(Y)] + \sum_{|I|=0}^{k-1} D_I[\overset{(k)}{\Gamma}{}^I(Y)], \tag{5.100}$$

where each $\overset{(k)}{\Gamma}{}^I$, $|I| = 0,\, 1,\, 2,\, \dots,\, k-1$, is a sum of products of $Q^{i_1 i_2\cdots i_k}(Y)$ with the connection coefficients $\Gamma^i_{jk}$ and their partial derivatives to order $k-1$.

For example, with $k=2$, a straightforward calculation shows that

$$D_{ij}[Q^{ij}(Y)] = \nabla_{ij}[Q^{ij}(Y)] + D_i[\overset{(2)}{\Gamma}{}^i(Y)] + \overset{(2)}{\Gamma}(Y),$$

where

$$\overset{(2)}{\Gamma}{}^i(Y) = -\Gamma^i_{ja} Q^{aj}(Y) + 2\,\Gamma^j_{ja} Q^{ai}(Y) + 2\,\gamma^p_j \wedge Q^{ij}_p(Y),$$

and

$$\begin{aligned}\overset{(2)}{\Gamma}(Y) = {} & \Gamma^i_{ia,j} Q^{aj}(Y) - \gamma^p_{i,j} \wedge Q^{ij}_p(Y) - \Gamma^i_{ia}\Gamma^a_{jb} Q^{bj}(Y) - \Gamma^i_{ia}\Gamma^j_{jb} Q^{ab}(Y) \\ & + 2\,\Gamma^i_{ia}\gamma^p_j \wedge Q^{aj}_p(Y) + \Gamma^i_{ja}\gamma^p_i \wedge Q^{aj}_p(Y) - \Gamma^q_{jp}\gamma^p_i \wedge Q^{ij}_q(Y) \\ & - \gamma^p_i \wedge \gamma^q_j \wedge Q^{ij}_{qp}(Y).\end{aligned}$$

Here, and the sequel, we write

$$Q^I_p(Y) = D_p \lrcorner\, Q^I(Y) \quad \text{and} \quad Q^I_{qp}(Y) = D_q \lrcorner\, Q^I_p(Y).$$

When (5.100) is substituted into (5.99), it is found that

$$P(Y) = \nabla_{i_1 i_2 \cdots i_k}[Q^{i_1 i_2 \cdots i_k}(Y)] + \widetilde{P}(Y),$$

where

$$\widetilde{P}(Y) = \sum_{|I|=0}^{k-1} D_I[Q^I(Y) + \overset{(k)}{\Gamma}{}^I(Y)].$$

Since both $P(Y)$ and $\nabla_{i_1 i_2 \cdots i_k}[Q^{i_1 i_2 \cdots i_k}(Y)]$ are invariantly defined total differential operators, the same must be true of $\widetilde{P}(Y)$. Note that $\widetilde{P}$ is of order $k-1$. We now repeat this process of successively replacing the total derivatives of the symbol by total covariant derivatives of the symbol plus a lower order operator until the original operator $P$ is expressed in the form

$$P(Y) = \sum_{|I|=0}^{k} \nabla_I[Q_{\nabla}{}^I(Y)]. \tag{5.101}$$

Each coefficient $Q_{\nabla}{}^I(Y) = Y^\alpha Q_{\nabla\alpha}^{\ I}$ represents the components of a type $(r,s)$ form $Q_{\nabla}^{(l)}(Y)$, $l = |I|$, with values in $S_H^{l,0}(J^\infty(E))$. We pause to formally record this result.

PROPOSITION 5.54. *Let $\nabla$ be a symmetric connection on $M$ and let*

$$P\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$$

*be a $k^{\text{th}}$ order total differential operator. Then, for each $l = 0, 1, 2, \ldots, k$, there exists a unique, zeroth order map*

$$\overset{(l)}{Q}_{\nabla}\colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E)) \otimes S_H^{l,0}(J^\infty(E))$$

*with components*

$$\overset{(l)}{Q}_{\nabla}(Y) = Y^\alpha Q_{\nabla\alpha}^{\ i_1 i_2 \ldots i_l}\, D_{i_1} \otimes D_{i_2} \cdots \otimes D_{i_l}$$

*such that $P(Y)$ takes the form (5.101).*

We call $\overset{(l)}{Q}_{\nabla}$ the $l^{\text{th}}$ invariant symbol of $P$ with respect to the connection $\nabla$. For a second order operator

$$P(Y) = Q(Y) + D_i[Q^i(Y)] + D_{ij}[Q^{ij}(Y)],$$

one can show that

$$
\begin{aligned}
Q_{\nabla}{}^{ij}(Y) &= Q^{ij}(Y), \\
Q_{\nabla}{}^{i}(Y) &= Q^{i}(Y) - \Gamma^{i}_{ja} Q^{aj}(Y) - 2\,\Gamma^{j}_{ja} Q^{ia}(Y) + 2\,\gamma^{p}_{j} \wedge Q^{ij}_{p}(Y), \quad \text{and} \\
Q_{\nabla}(Y) &= Q(Y) - \Gamma^{i}_{ib} Q^{b}(Y) + \gamma^{q}_{i} \wedge Q^{i}_{q}(Y) - \Gamma^{i}_{ia,j} Q^{aj}(Y) \\
&\quad + \gamma^{p}_{i,j} \wedge Q^{ij}_{p}(Y) + \Gamma^{i}_{ib}\Gamma^{j}_{ja} Q^{ab}(Y) - 2\,\Gamma^{j}_{ja}\gamma^{q}_{i} \wedge Q^{ia}_{q}(Y) \\
&\quad + \Gamma^{p}_{jq}\gamma^{q}_{i} \wedge Q^{ij}_{p}(Y) + \gamma^{p}_{i} \wedge \gamma^{q}_{j} \wedge Q^{ij}_{qp}(Y).
\end{aligned}
\tag{5.102}
$$

Needless-to-say, these formulas for the invariant symbols $\overset{(l)}{Q}_{\nabla}$ of $P$ become increasing complex as the order of $P$ increases. However, owing largely to (5.97), some simplifications arise when $r = n$. Indeed, for $r = n$ and $k = 3$ we find that

$$
\begin{aligned}
Q_{\nabla}{}^{ijh}(Y) &= Q^{ijh}(Y), \\
Q_{\nabla}{}^{ij}(Y) &= Q^{ij}(Y) - \frac{3}{2}\bigl(\Gamma^{i}_{ah} Q^{jah}(Y) + \Gamma^{j}_{ah} Q^{iah}(Y)\bigr), \\
Q_{\nabla}{}^{i}(Y) &= Q^{i}(Y) - \Gamma^{i}_{aj} Q^{aj}(Y) + (\Gamma^{i}_{ja,h} + \Gamma^{i}_{bj}\Gamma^{b}_{ah}) Q^{ajh}(Y), \quad \text{and} \\
Q_{\nabla}(Y) &= Q(Y).
\end{aligned}
\tag{5.103}
$$

This last formula is of particular interest. It shows that the zeroth order invariant symbol $\overset{(0)}{Q}_{\nabla}$ coincides with the Euler operator $E(P)(Y) = Q(Y)$ of $P$ ( see Definition 2.3) and is therefore independent of the connection $\nabla$. This is true generally.

PROPOSITION 5.55. *Let $P \colon \mathcal{E}v(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$ be a total differential operator, where $n = \dim M$. Then the Euler operator $E(P)$ of $P$ and the zeroth order invariant symbol $\overset{(0)}{Q}_{\nabla}$ of $P$ with respect to any connection $\nabla$ coincide.*

PROOF: Equation (5.101) implies that

$$
P(Y) = \overset{(0)}{Q}_{\nabla}(Y) + \nabla_i[T^i(Y)], \tag{5.104}
$$

where

$$
T^i(Y) = \sum_{|I|=0}^{k-1} \nabla_I[Q_{\nabla}{}^{iI}(Y)].
$$

Since each component $T^i(Y)$ is a type $(n,s)$ form, it follows that

$$T^i(Y) = dx^i \wedge [D_j \lrcorner\, T^j(Y)] = dx^i \lrcorner\, \widetilde{T}(Y),$$

where $\widetilde{T}(Y)$ is a type $(n-1,s)$ form. Owing to (5.98), (5.104) can be rewritten as

$$P(Y) = \overset{(0)}{Q}_{\nabla}(Y) + d_H[\widetilde{T}(Y)].$$

The proposition now follows from the uniqueness of this decomposition of $P$ as established in Proposition 2.2. ∎

We continue, in the spirit of §2A, by applying Propositions 5.54 and 5.55 first to the operator

$$P(Y) = \mathcal{L}_{\mathrm{pr}\, Y}\lambda,$$

where $\lambda \in \Omega^{n,0}(J^\infty(E))$ is a Lagrangian on $J^\infty(E)$, and then to the operator

$$P(Y) = \mathrm{pr}\, Y \lrcorner\, \omega,$$

where $\omega \in \Omega^{n,s}(J^\infty(E))$. In the first instance, we deduce that

$$\mathcal{L}_{\mathrm{pr}\, Y}\lambda = Y \lrcorner\, E(\lambda) + d_H\eta, \tag{5.105}$$

where $E(\lambda)$ is the Euler-Lagrange form of $\lambda$ and

$$\eta = \sum_{|I|=0}^{k-1} \nabla_I[Y^\alpha E_{\nabla\alpha}^{Ij}(D_j \lrcorner\, \lambda)]. \tag{5.106}$$

The tensors

$$(\overset{(l)}{E}_{\nabla})(\lambda) = E_{\nabla\alpha}^{\,i_1 i_2\cdots i_l}(\lambda) \otimes \theta^\alpha \otimes D_{i_1} \otimes D_{i_2} \otimes \cdots \otimes D_{i_l}$$

are called the invariant Lie-Euler operators of $\lambda$ with respect to the connection $\nabla$. To third order, these operators are given by (5.103) with $Q^I(Y) = Y^\alpha E^I_\alpha(\lambda)$, where $E^I_\alpha(\lambda)$ are the ordinary Lie-Euler operators defined by (2.15). The form $\eta$ is manifestly invariantly defined. Thus, (5.105) immediately establishes the existence of a global first variational formula for the calculus of variations (see also Corollary 5.3).

In the second instance, Proposition 5.54 leads to the decomposition

$$\mathrm{pr}\, Y \lrcorner\, \omega = \sum_{|I|=0}^{k} \nabla_I[Y^\alpha F_{\nabla\alpha}^{I}(\omega)].$$

The tensors

$$\overset{(l)}{F}_{\nabla} = F_{\nabla\alpha}^{\;i_1 i_2 \cdots i_l}(\omega) \otimes \theta^\alpha \otimes D_{i_1} \otimes D_{i_2} \otimes \cdots \otimes D_{i_l}$$

are the invariant interior Euler operators of $\omega$ with respect to the connection $\nabla$. Proposition 5.55 now implies that $F_{\nabla\alpha}(\omega) = F_\alpha(\omega)$ and hence the interior Euler operator (2.14) coincides with

$$I(\omega) = \frac{1}{s}\,\theta^\alpha \wedge F_{\nabla\alpha}(\omega).$$

Consequently, for $\omega \in \Omega^{n,s}(J^\infty(E))$, it follows that

$$\omega = \frac{1}{s}\sum_{|I|=0}^{k} D_I[\theta^\alpha \wedge F_\alpha^I(\omega)] = \frac{1}{s}\sum_{|I|=0}^{k} \nabla_I[\theta^\alpha \wedge F_{\nabla\alpha}^{\;I}(\omega)],$$

and therefore

$$\omega = I(\omega) + d_H[h_\nabla^{n,s}(\omega)],$$

where

$$h_\nabla^{n,s}(\omega) = \frac{1}{s}\sum_{|I|=0}^{k-1} \nabla_I\{D_j \lrcorner\, [\theta^\alpha \wedge F_{\nabla\alpha}^{\;Ij}(\omega)]\}.$$

This is the sought after invariantly defined horizontal homotopy operator on $\Omega^{n,s}(J^\infty(E))$. Note that this operator is obtained, at least formally, from its local "non-invariant" counterpart $h_H^{r,s}$ (see (4.13)) by replacing the interior product operators $F_\alpha^{Ij}$ by their invariant counterparts $F_{\nabla\alpha}^{\;Ij}$ and by replacing the total derivatives $D_I$ by total covariant derivatives $\nabla_I$. This suggests that for $r \le n$, $h_\nabla^{r,s}$ might be similarly defined. In fact, the formulas for $h_\nabla^{r,s}$ will contain the invariant analogue of our previous homotopy operator $h_H^{r,s}$ although additional terms, involving the curvature tensor of the connection $\nabla$, must be introduced to compensate for the fact that repeated covariant derivatives do not commute.

Theorem 5.56. *Let $\nabla$ be a symmetric connection on $M$. Let $\pi\colon U \to U_0$ be an adapted coordinate neighborhood of $E$. Then, for $1 \le r \le n$ and $s \ge 1$, there exists invariantly defined operators*

$$h_\nabla^{r,s}\colon \Omega^{r,s}(J^\infty(U)) \to \Omega^{r-1,s}(J^\infty(U))$$

*such that, for $1 \le r \le n-1$ and $\omega \in \Omega^{r,s}(J^\infty(U))$,*

$$\omega = d_H[h_\nabla^{r,s}(\omega)] + h_\nabla^{r+1,s}(d_H\omega), \tag{5.107a}$$

*while, for $\omega \in \Omega^{n,s}(J^\infty(U))$,*

$$\omega = d_H[h_\nabla^{n,s}(\omega)] + I(\omega). \tag{5.107b}$$

Before presenting the proof of Theorem 5.56, we give an alternative proof of Proposition 4.1 wherein our original, non-invariant homotopy operators $h_H^{r,s}$ were introduced. This alternative proof is inductive in nature and, although somewhat more complicated than the one already given in §4A, it serves to motivate (and actually simplify) the approach required to construct the invariant homotopy operators $h_\nabla^{r,s}$.

For $s \geq 1$, let $\mathcal{D}^{r,s}(J^\infty(E))$ be the vector space of all total differential operators

$$P \colon \mathcal{E}v(J^\infty(E)) \to \Omega^{r,s-1}(J^\infty(E)).$$

For $P \in \mathcal{D}^{r,s}(J^\infty(E))$, define $DP \in \mathcal{D}^{r+1,s}(J^\infty(E))$ by the rule

$$(DP)(Y) = -d_H[P(Y)]. \tag{5.108}$$

If $P$ is a total differential operator of order $k$, then $DP$ is a total differential operator of order no larger than $k+1$.

Given a $k^{\text{th}}$ order form $\omega \in \Omega^{r,s}(J^\infty(E))$, we define the associated $k^{\text{th}}$ order operator $P_\omega \in \mathcal{D}^{r,s}(J^\infty(E))$ by

$$P_\omega(Y) = \operatorname{pr} Y \,\lrcorner\, \omega.$$

The sign convention adopted in (5.108) is such that, in accordance with (1.35),

$$DP_\omega = P_\tau \qquad \text{where} \quad \tau = d_H\omega. \tag{5.109}$$

Conversely, given an operator $P \in \mathcal{D}^{r,s}(J^\infty(E))$, we can define a form $\omega_P \in \Omega^{r,s}(J^\infty(E))$ by

$$\omega_P(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \operatorname{pr} Y_s) = \frac{1}{s}\sum_{i=1}^{s}(-1)^{i+1} P(Y_i)(\operatorname{pr} Y_1, \operatorname{pr} Y_2, \ldots, \widehat{\operatorname{pr} Y^i}, \ldots, \operatorname{pr} Y_s),$$

where $Y_1$, $Y_2$, $\ldots$, $Y_s$ are evolutionary vector fields. It is readily checked that

$$\omega = \omega_P \qquad \text{where} \quad P = P_\omega \tag{5.111a}$$

and

$$d_H(\omega_P) = \omega_R \qquad \text{where} \quad R = DP. \tag{5.111b}$$

Because $D(DP) = 0$ the vector spaces $\mathcal{D}^{r,s}(J^\infty(E))$, together with the maps $D : \mathcal{D}^{r,s}(J^\infty(E)) \to \mathcal{D}^{r+1,s}(J^\infty(E))$, form a differential complex. Let $\pi\colon U \to U_0$ be an adapted coordinate neighborhood of $E$. We prove this complex is locally exact by constructing homotopy operators

$$J^{r,s}\colon \mathcal{D}^{r,s}(J^\infty(U)) \to \mathcal{D}^{r-1,s}(J^\infty(U)), \tag{5.112a}$$

so that, for all $P \in \mathcal{D}^{r,s}(J^\infty(U))$,

$$P = d_H[J^{r,s}(P)] + J^{r+1,s}(d_H P). \tag{5.112b}$$

With this result in hand, homotopy operators for the interior horizontal complexes $(\Omega^{*,s}(J^\infty(U)), d_H)$ are easily obtained by setting, for $\omega \in \Omega^{r,s}(J^\infty(U))$,

$$\tilde{h}^{r,s}_H(\omega) = \omega_R \quad \text{where} \quad R = J^{r,s}(P_\omega).$$

From equations (5.109) and (5.111), it is a simple matter to check that

$$\omega = d_H[\tilde{h}^{r,s}_H(\omega)] + \tilde{h}^{r+1,s}_H(d_H\omega),$$

as required. While not for *a priori* reasons, it turns out that $\tilde{h}^{r,s}_H = h^{r,s}_H$.

We define the maps $J^{r,s}$ by induction on the order of the operators $P$. Let $\mathcal{D}^{r,s}_k(J^\infty(E))$ be the subspace of $\mathcal{D}^{r,s}(J^\infty(E))$ consisting of all total differential operators of order $k$. We construct maps

$$J^{r,s}_k\colon \mathcal{D}^{r,s}_k(U) \to \mathcal{D}^{r-1,s}_{k-1}(U) \tag{5.113a}$$

such that, for $k \geq 1$ and all $P_k \in \mathcal{D}^{r,s}_k(J^\infty(U))$,

$$P_k = D[J^{r,s}_k(P_k)] + J^{r+1,s}_{k+1}(DP_k). \tag{5.113b}$$

In addition, these maps have the property that for $P_k \in \mathcal{D}^{r,s}_k$ and $l \geq k$,

$$J^{r,s}_l(P_k) = J^{r,s}_k(P_k).$$

Thus, (5.112) can be derived from (5.113) by taking inverse limits.

To begin the inductive process, let $P_2 \in \mathcal{D}^{r,s}_2(J^\infty(U))$ be given by

$$P_2(Y) = Q(Y) + D_i[Q^i(Y)] + D_{ij}[Q^{ij}(Y)].$$

We define

$$[J_2^{r,s}(P_2)](Y) = -\frac{1}{n-r+1}\, Q_i^i(Y) - \frac{2}{n-r+2}\, D_j[Q_i^{ij}(Y)], \tag{5.114}$$

where $Q_j^I(Y) = D_j \lrcorner\, Q^I(Y)$. Implicit in this definition of $J_2^{r,s}$ is the definition of $J_1^{r,s}$ — if $P_1 \in \mathcal{D}_1^{r,s}(J^\infty(U))$ is given by

$$P_1(Y) = S(Y) + D_i[S^i(Y)],$$

then

$$[J_1^{r,s}(P)](Y) = -\frac{1}{n-r+1}\, S_i^i(Y).$$

Now suppose that $DP_1 = P_2$. Then

$$Q^{ij}(Y) = \frac{1}{2}[dx^i \wedge S^j(Y) + dx^j \wedge S^i(Y)]$$

$$Q^i(Y) = -dx^i \wedge S(Y) \quad \text{and} \quad Q(Y) = 0,$$

and a straightforward calculation, identical to that given in Lemma 4.5 and the original proof of Proposition 4.2, leads to

$$P_1 = D[J_1^{r,s}(P_1)] + J_2^{r+1,s}(DP_1).$$

This proves (5.113) for $k = 1$.

Now decompose each $P_k \in \mathcal{D}_k^{r,s}(J^\infty(U))$ into the sum

$$P_k(Y) = \sum_{|I|=0}^{k} D_I[Q^I(Y)] = \sigma_{P_k}(Y) + \rho_{P_k}(Y),$$

where

$$\sigma_{P_k}(Y) = D_I[Q^I(Y)] \qquad \text{for} \quad |I| = k,$$

and

$$\rho_{P_k}(Y) = \sum_{|I|=0}^{k-1} D_I[Q^I(Y)].$$

Note that $\rho_{P_k}$ is an operator of order $k-1$. We define $J_k^{r,s}$ in terms of $J_{k-1}^{r,s}$, for $k = 2, 3, \ldots$, by

$$[J_k^{r,s}(P_k)](Y) = [J_{k-1}^{r,s}(\rho_{P_k})](Y) - \frac{k}{n-r+k}\, D_{I'}[Q_j^{jI'}(Y)],$$

where $|I'| = k - 1$. Fix $k \geq 2$. We assume that $J^{r,s}_{k-1}$ and $J^{r,s}_{k}$ satisfy (5.113b) with $k$ replaced by $k-1$.

To prove that $J^{r,s}_{k}$ and $J^{r,s}_{k+1}$ satisfy (5.113b), decompose the two total differential operators $P_k$ and $R_{k'} = DP_k$, $k' = k+1$ into the sums

$$P_k(Y) = \sigma_{P_k}(Y) + \rho_{P_k}(Y)$$

and

$$(DP_k)(Y) = \sigma_{R_{k'}}(Y) + \rho_{R_{k'}}(Y)$$

where, just like $\sigma_{P_k}$, $\sigma_{R_{k'}}$ is of the form

$$\sigma_{R_{k'}}(Y) = D_I[S^I(Y)] = D_I[Y^\alpha S^I_\alpha] \qquad \text{for} \quad |I| = k+1,$$

and where $\rho_{R_{k'}}$ of order $k' - 1 = k$. It is easily seen that

$$(D\rho_{P_k})(Y) = \rho_{R_{k'}}(Y) \quad \text{and} \quad (D\sigma_{P_k})(Y) = \sigma_{R_{k'}}(Y). \tag{5.115}$$

This latter equation implies that

$$S^I(Y) = -dx^{(i} \wedge Q^{I')}(Y) \qquad \text{for} \quad |I| = k+1.$$

The induction hypothesis implies that

$$\rho_{P_k}(Y) = D[J^{r-1,s}_{k-1}(\rho_{P_k})](Y) + (J^{r,s}_{k}(D\rho_{P_k}))(Y), \tag{5.116}$$

while a calculation identical to that given in the proof of Proposition 4.2 shows that, for $I = k$,

$$\begin{aligned} \sigma_{P_k}(Y) &= D_I[Q^I(Y)] \\ &= d_H\left\{\frac{k}{n-r+k} D_{I'}[Q^{jI'}_j(Y)]\right\} - \left\{\frac{k+1}{n-r+k} D_I[S^{jI}_j(Y)\right\}. \end{aligned} \tag{5.117}$$

Now add (5.116) to (5.117). The left-hand side of this sum equals $P_k(Y)$. The sum of the first terms on the right is $\big(D[J^{r,s}_{k}(P_k)]\big)(Y)$ and the sum of the second terms is $[J^{r+1,s}_{k+1}(DP_k)](Y)$. This proves (5.113).

Proof of Theorem 5.56: Since we have already established (5.107b), we restrict our considerations to the case $1 \le r \le n-1$. We proceed just as above by constructing operators

$$J^{r,s}_{\nabla\, k}\colon \mathcal{D}^{r,s}_k(J^\infty(U)) \to \mathcal{D}^{r-1,s}_{k-1}(J^\infty(U))$$

such that, for all $P_k \in \mathcal{D}^{r,s}_k(J^\infty(U))$,

$$P_k = D[J^{r,s}_{\nabla\, k}(P_k)] + J^{r+1,s}_{\nabla\, k+1}(DP_k). \tag{5.118}$$

Then, just as above, the required homotopy operator $h^{r,s}_{\nabla,k}$ is defined by

$$h^{r,s}_{\nabla,k}(\omega) = \omega_R, \quad \text{where } R = J^{r,s}_{\nabla\, k}(P_\omega).$$

The maps $J^{r,s}_{\nabla\, k}$ are defined inductively. In fact, the complicated nature of these homotopy operators really precludes the possibility of giving explicit general formula. For $k=2$ and $P_2 \in \mathcal{D}^{r,s}_2(J^\infty(U))$ given by

$$\begin{aligned} P_2(Y) &= Q(Y) + D_i[Q^i(Y)] + D_{ij}[Q^{ij}(Y)] \\ &= Q_\nabla(Y) + \nabla_i[Q_\nabla{}^i(Y)] + \nabla_{ij}[Q_\nabla{}^{ij}(Y)], \end{aligned}$$

we set

$$[J^{r,s}_{\nabla\, 2}(P_2)](Y) = -\frac{1}{n-r+1}[D_i \lrcorner\, Q_\nabla{}^i(Y)] - \frac{2}{n-r+2}\nabla_j[D_i \lrcorner\, Q_\nabla{}^{ij}(Y)].$$

Clearly, $J^{r,s}_{\nabla\, 2}$ is invariantly defined. The invariant symbols $Q_\nabla{}^i(Y)$ and $Q_\nabla{}^{ij}(Y)$ of $P_2$ are given explicitly in terms of the coefficients $Q^i(Y)$ and $Q^{ij}(Y)$ by (5.102). By virtue of these equations, a straightforward calculation shows that

$$\begin{aligned} [J^{r,s}_{\nabla\, 2}(P_2)](Y) &= [J^{r,s}_2(P_2)](Y) \\ &\quad - \frac{1}{n-r+1}\bigl\{\Gamma^l_{ij}Q^{ij}_l(Y) - \frac{2}{n-r+2}[\Gamma^l_{li}Q^{ij}_j(Y) + \gamma^l_i \wedge Q^{ij}_{lj}(Y)]\bigr\}, \end{aligned} \tag{5.119}$$

where $J^{r,s}_2$ is the non-invariant homotopy operator defined by (5.116). Consequently, if $P_1 \in \mathcal{D}^{r,s}_1(J^\infty(U))$ is a first order total differential operator, then

$$J^{r,s}_{\nabla\, 1}(P_1) = J^{r,s}_1(P_1).$$

Moreover, if the second order operator $P_2$ is $D$ closed then

$$dx^h \wedge Q^{ij}(Y) + dx^j \wedge Q^{hi}(Y) + dx^i \wedge Q^{jh}(Y) = 0$$

and therefore

$$(n-r+2)Q^{ij}(Y) = dx^i \wedge Q_h^{hj}(Y) + dx^j \wedge Q_h^{hi}(Y).$$

Interior evaluation of this equation by $D_l$ and multiplication of the result by $\Gamma^l_{ij}$ shows that the terms in braces in (5.119) vanish. This proves that if $DP_2 = 0$, then

$$J^{r,s}_{\nabla\,2}(P_2) = J_2^{r,s}(P_2).$$

We remark that this result is consistent with the findings of the previous section (see Proposition 5.43). In particular, for a first order operator $P_1$ we deduce that

$$J^{r,s}_{\nabla\,2}(DP_1) = J_2^{r,s}(DP_1).$$

Owing to (5.113), we therefore conclude that

$$\begin{aligned} D[J^{r,s}_{\nabla\,1}(P_1)] + J^{r,s}_{\nabla\,2}(DP_1) &= D[J_1^{r,s}(P_1)] + D[J_2^{r,s}(DP_1)] \\ &= P_1. \end{aligned}$$

This proves (5.118) for $k = 1$.

Before defining $J^{r,s}_{\nabla\,k}$ for $k = 3, 4, \ldots$, it is necessary to introduce certain auxiliary operators. Let $P_k \in \mathcal{D}_k^{r,s}(J^\infty(U))$ be given by

$$P_k(Y) = \sum_{|I|=0}^{k} \nabla_I[Q_\nabla{}^I(Y)].$$

Define

$$\sigma_{P_k}(Y) = \nabla_I[\sigma_{P_k}{}^I(Y)],$$

where $k = |I|$ and $\sigma_{P_k}{}^I(Y) = Q_\nabla{}^I(Y)$, and

$$\rho_{P_k}(Y) = \sum_{|I|=0}^{k-1} \nabla_I[Q_\nabla{}^I(Y)].$$

Clearly both $\sigma_{P_k}$ and $\rho_{P_k}$ are invariantly defined total differential operators and

$$P_k(Y) = \sigma_{P_k}(Y) + \rho_{P_k}(Y).$$

Both of these operators depend on the connection $\nabla$ although, in view of (5.100),

$$\sigma_{P_k}{}^I(Y) = Q^I(Y) \qquad \text{for} \quad |I| = k$$

is independent of $\nabla$. The operator $\rho_{P_k}$ is of order $k-1$. Next set

$$\mu_{P_k}(Y) = \nabla_j \nabla_I [dx^j \wedge \sigma_{P_k}{}^I(Y) - dx^{(j} \wedge \sigma_{P_k}{}^{I)}(Y)] \quad \text{for} \quad |I| = k \tag{5.120}$$

and

$$\tau_{P_k}(Y) = \nabla_j \nabla_{I'} [dx^j \wedge \sigma_{P_k i}{}^{I' i}(Y) - dx^{(j} \wedge \sigma_{P_k i}{}^{I')i}(Y)], \quad \text{for} \quad |I'| = k-1. \tag{5.121}$$

The operators $\mu_{P_k}$ and $\tau_{P_k}$ are also invariantly defined. Were it the case that the total covariant derivatives with respect to $\nabla$ commuted, then $\mu_{P_k}$ and $\tau_{P_k}$ would vanish identically. Thus we can interpret the presence of these operators in our subsequent formulas as the correction terms needed to compensate for the non-commutativity of covariant derivatives.

By virtue of the Ricci identities, it is possible to prove that $\tau_{P_k}$ is of order $k-2$. Consider, for example, the case $k=3$. If, for the sake of notational convenience we write

$$T^{jhk}(Y) = dx^j \wedge \sigma_{P_3 i}^{hki}(Y) = dx^j \wedge Q_i^{hki},$$

we find that

$$\begin{aligned} \tau_{P_3}(Y) &= \nabla_j \nabla_h \nabla_k [T^{jhk}(Y) - \frac{1}{3}\big(T^{jhk}(Y) + T^{hkj}(Y) + T^{kjh}(Y)\big)] \\ &= \nabla_j \nabla_h \nabla_k [\frac{2}{3}\big(T^{jhk}(Y) - T^{hkj}(Y)\big) + \frac{1}{3}\big(T^{hkj}(Y) - T^{kjh}(Y)\big)] \\ &= \frac{2}{3}[(\nabla_j \nabla_h - \nabla_h \nabla_j)(\nabla_k T^{jhk}(Y))] + \frac{1}{3}\nabla_j [(\nabla_h \nabla_k - \nabla_k \nabla_h) T^{hkj}(Y)]. \end{aligned} \tag{5.122}$$

At this point it is apparent that $\tau_{P_3}$ is indeed a first order operator. However, for a subsequent calculation, we need to simplify this last equation. To this end, let us temporarily assume that the connection $\nabla$ is the Riemannian connection for some metric $g$ on $M$. This allows us to utilize the maximum possible symmetries of the curvature tensor. We denote the curvature tensor of the connection $\nabla$ by $R_l{}^j{}_{hk}$ where, for any vector field $X^l$ on $M$,

$$R_l{}^j{}_{hk} X^l = \nabla_h \nabla_k X^j - \nabla_k \nabla_h X^j.$$

The Ricci tensor is $R_{hk} = R_h{}^l{}_{kl}$ and $\Omega_l{}^j = \dfrac{1}{2} R_l{}^j{}_{hk}\, dx^h \wedge dx^k$ is the curvature 2 form. We now apply the Ricci identities to each of the two expressions in brackets in (5.122). When the first set of terms is "integrated by parts", we arrive at

$$\begin{aligned} \tau_{P_3}(Y) &= \nabla_j [R_{hk}\, dx^h \wedge Q_i^{kji}(Y) + \frac{1}{3} R_l{}^j{}_{hk}\, dx^h \wedge Q_i^{kli}(Y) - \frac{1}{2}\Omega_h{}^l \wedge Q_{li}^{hji}(Y)] \\ &\quad - \frac{2}{3}(\nabla_j R_{hk})\, dx^h \wedge Q_i^{kji}(Y) - \frac{1}{3}(\nabla_j \Omega_h{}^l) \wedge Q_{li}^{hji}(Y). \end{aligned} \tag{5.123}$$

Finally, in way of preparation, we compute $D\tau_{P_k}$:

$$\begin{aligned}[D\tau_{P_k}](Y) &= -d_H[\tau_{P_k}(Y)] \\ &= -d_H\big\{d_H\big[\nabla_{I'}[\sigma_{P_k}{}_i^{I'i}(Y)]\big] - \nabla_j\nabla_{I'}dx^{(j} \wedge \sigma_{P_k}{}_i^{I')i}(Y)\big\} \\ &= \nabla_j\nabla_I[dx^j \wedge dx^{(i} \wedge \sigma_{P_k}{}_h^{I')h}(Y)]. \end{aligned} \tag{5.124}$$

For each $k = 3, 4, \ldots,$ define $J^{r,s}_{\nabla\, k}$ as follows. Let $P_k \in \mathcal{D}^{r,s}_k(J^\infty(U))$ and let $\sigma_{P_k} \in \mathcal{D}^{r,s}_k(J^\infty(U))$ and $\rho_{P_k}, \tau_{P_k} \in \mathcal{D}^{r,s}_{k-1}(J^\infty(U))$ be the associated operators as defined above. Set

$$\begin{aligned}&[J^{r,s}_{\nabla\, k}(P_k)](Y) \\ &\quad = -\frac{k}{n-r+k}\big\{\nabla_I[\sigma_{P_k}{}_i^{Ii}(Y)] + \big(J^{r,s}_{\nabla\, k-1}(\tau_{P_k})\big)(Y)\big\} + J^{r,s}_{\nabla\, k-1}(\rho_{P_k})(Y),\end{aligned} \tag{5.125}$$

where $|I| = k$. To prove (5.118), let $P_k \in \mathcal{D}^{r,s}_k(J^\infty(U))$. Decompose $P_k$ and $R_{k'} = DP_k$, where $k' = k+1$, into the sums

$$P_k(Y) = \sigma_{P_k}(Y) + \rho_{P_k}(Y)$$

and

$$R_{k'}(Y) = \sigma_{R_{k'}}(Y) + \rho_{R_{k'}}(Y).$$

Since

$$\begin{aligned}(DP_k)(Y) = -d_H[P_k(Y)] &= (D\rho_{P_k})(Y) - D_j\{dx^j \wedge \nabla_I[\sigma_{P_k}{}^I(Y)]\} \\ &= [(D\rho_{P_k}) - \mu_{P_k}](Y) - \nabla_{jI}[dx^{(j} \wedge \sigma_{P_k}{}^{I)}(Y)]\end{aligned}$$

it follows that

$$\sigma_{R_{k'}}{}^{iI}(Y) = -dx^{(i} \wedge \sigma_{P_k}{}^{I)}(Y), \quad \text{for} \quad k = |I|, \tag{5.126}$$

and

$$\rho_{R_{k'}} = D\rho_{P_k} - \mu_{P_k}. \tag{5.127}$$

(Compare with (5.115).) From (5.126), it is found that

$$\frac{k+1}{n-r+k}\sigma_{R_{k'}}{}_i^{Ii}(Y) = -\sigma_{P_k}{}^I(Y) + \frac{k}{n-r+k}dx^{(i} \wedge \sigma_{P_k}{}_j^{I)j}(Y). \tag{5.128}$$

By virtue of (5.124), substitution of this formula into (5.121) leads to

$$\frac{k+1}{n-r+k}\tau_{R_{k'}}(Y) = -\mu_{P_k}(Y) + \frac{k}{n-r+k}[D\tau_{P_k}](Y). \tag{5.129}$$

We now substitute these last three equations, (5.127), (5.128) and (5.129) into the definition (5.125) of $J^{r+1,s}_{\nabla\, k+1}$, as applied to the operator $R' = DP$. The two terms involving $\mu_{P_k}(Y)$ cancel to yield

$$\begin{aligned}[J^{r+1,s}_{\nabla\, k+1}(DP_k)](Y) &= \nabla_I[\sigma_{P_k}{}^I(Y)] + [J^{r,s}_{\nabla\, k}(D\rho_{P_k})](Y) \\ &\quad - \frac{k}{n-r+k}\big\{J^{r+1,s}_{\nabla\, k}(D\tau_{P_k})](Y) + \nabla_I[dx^{(i} \wedge \sigma_{P_k j}{}^{I)j}(Y)]\big\}.\end{aligned} \tag{5.130}$$

Furthermore, from the definition of $J^{r,s}_{\nabla\, k}(P_k)$, we find that

$$\begin{aligned}[D\big(J^{r,s}_{\nabla\, k}(P_k)\big)](Y) = -&\frac{k}{n-r+k}\big\{-\nabla_j\nabla_{I'}[dx^j \wedge \sigma_{P_k i}{}^{I'i}(Y)] + [DJ^{r,s}_{\nabla\, k-1}(\tau_{P_k})](Y)\big\} \\ &+ [DJ^{r,s}_{\nabla\, k-1}(\rho_{P_k})](Y).\end{aligned} \tag{5.131}$$

Equations (5.130) and (5.131) are now added together. The terms in braces vanish on account of the induction hypothesis, as applied to the operator $\tau_{P_k}$. By applying the induction hypothesis again, this time to the operator $\rho_{P_k}$, we deduce that

$$\begin{aligned}&[J^{r+1,s}_{\nabla\, k+1}(DP_k)](Y) + [DJ^{r,s}_{\nabla\, k}(P_k)](Y) \\ &\qquad = \nabla_I[\sigma_{P_k}{}^I(Y)] + [J^{r+1,s}_{\nabla\, k}(D\rho_{P_k})](Y) + [DJ^{r,s}_{\nabla\, k-1}(\rho_{P_k})](Y) \\ &\qquad = \nabla_I[\sigma_{P_k}{}^I(Y)] + \rho_{P_k}(Y) = P_k(Y).\end{aligned}$$

This establishes the homotopy formula (5.118) and completes the proof of Theorem 5.56. ∎

EXAMPLE 5.57. If $P_3 \in \mathcal{D}^{r,s}_3(J^\infty(U))$ is a third order operator, then it follows from (5.123), and (5.125) that

$$\begin{aligned}&[J^{r,s}_{\nabla\, 3}(P_3)](Y) \\ &\quad = -\frac{3}{n-r+3}\nabla_{hk}[Q_{\nabla j}{}^{hkj}(Y)] - \frac{2}{n-p+2}\nabla_h[Q_{\nabla j}{}^{hj}(Y)] - \frac{1}{n-r+1}Q_{\nabla j}{}^{j}(Y) \\ &\qquad + \frac{1}{n-r+1}\big\{\frac{2}{n-r+3}[R_{hk}Q_{\nabla j}{}^{hkj}(Y) - (R_h{}^p{}_{kl}dx^l) \wedge Q_{\nabla pj}{}^{hkj}(Y)]\big\}.\end{aligned} \tag{5.132}$$

The first three terms in this formula are precisely the invariant analogs of the three terms defining original homotopy $J_3^{r,s}(P_3)$. The last two terms are the correction terms needed to compensate for the non-commutativity of the total covariant derivatives. As we shall see, this rather complicated formula simplifies rather dramatically when $P_3$ is $D$ closed. ∎

COROLLARY 5.58. *The augmented interior rows of the variational bicomplex, viz.*

$$0 \longrightarrow \Omega^{0,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,s}(J^\infty(E)) \xrightarrow{d_H} \cdots$$
$$\xrightarrow{d_H} \Omega^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}^s(J^\infty(E)) \longrightarrow 0$$

*for* $s \geq 1$, *are exact.*

PROOF: Because of their invariance under change of coordinates, the local invariant homotopy operators

$$h_\nabla^{r,s} \colon \Omega^{r,s}(J^\infty(U)) \to \Omega^{r-1,s}(J^\infty(U))$$

patch together to determine global homotopy operators

$$h_\nabla^{r,s} \colon \Omega^{r,s}(J^\infty(E)) \to \Omega^{r-1,s}(J^\infty(E)).$$

∎

The invariant homotopy operators $J_\nabla^{r,s}$ constructed in the the proof of Theorem 5.56 have an addition property which actually uniquely characterizes them. Let $P_k \in \mathcal{D}_k^{r,s}(J^\infty(E))$ be a $k^{\text{th}}$ order total differential operator, say

$$P(Y) = \sum_{|I|=0}^{k} \nabla_I [Q_\nabla{}^I(Y)].$$

We say that $P$ is *trace-free with respect to the connection* $\nabla$ if all of its invariant symbols $Q_\nabla{}^I$ are trace- free, *i.e.*,

$$D_i \lrcorner\, Q_\nabla{}^{I'i}(Y) = 0, \quad \text{for} \quad |I'| = 0, 1, \ldots, k-1.$$

Since $J_{\nabla\,2}^{r,s}(P_2)$ is a trace-free operator ( whether or not $P_2$ itself is trace-free), it is easily verified, by induction, that $J_{\nabla\,k}^{r,s}(P)$ is trace-free for any $k$. Consequently, the arguments presented in Proposition 2.4 can now be extended to arbitrary order total differential operators. Thus, if $P, \widetilde{P} \in \mathcal{D}^{r,s}(J^\infty(E))$ are two trace-free differential operators, with $r < n$, and $DP = D\widetilde{P}$, then $P = \widetilde{P}$.

COROLLARY 5.59. *Suppose that $P \in \mathcal{D}^{r,s}_k(J^\infty(E))$, where $r < n$, and that*

$$DP = 0.$$

*Then there is a unique operator $R_\triangledown \in \mathcal{D}^{r-1,s}_{k-1}(J^\infty(E))$ which is trace-free with respect to the connection $\nabla$ and such that*

$$DR_\triangledown = P.$$

PROOF: In fact, $R$ must coincide with $J^{r,s}_\triangledown(P)$. ∎

EXAMPLE 5.60. Corollary 5.59 can now be used to construct directly local, invariant solutions $R$ to the equation $DR = P$ and thereby circumvent the direct use of the operators $J^{r,s}_\triangledown$. For example, suppose that

$$P_3(Y) = Q(Y) + D_i[Q^i(Y)] + D_{ij}[Q^{ij}(Y)] + D_{ijh}[Q^{ijh}(Y)].$$

According to the corollary, we may take

$$\begin{aligned} R_2(Y) &= S_\triangledown(Y) + \nabla_i[S_\triangledown{}^i(Y)] + \nabla_{ij}[S_\triangledown{}^{ij}] \\ &= S(Y) + D_i[S^i(Y)] + D_{ij}[S^{ij}(Y)]. \end{aligned}$$

The equation $DR_2 = P_3$ implies that

$$-dx^{(h} \wedge S^{ij)} = Q^{ijh}, \qquad -dx^{(h} \wedge S^{i)} = Q^{ij}, \tag{5.134a}$$

$$dx^i \wedge S = Q^i, \quad \text{and} \qquad 0 = Q, \tag{5.134b}$$

while the trace-free conditions on $R_2$ imply, in view of the formulas (5.102) for the invariant symbols of $R$, that

$$S^{ij}_j = 0 \qquad \text{and} \qquad S^i_i + \Gamma^p_{ij} S^{ij}_p = 0. \tag{5.134c}$$

Equations (5.134) can now be solved uniquely for the coefficients $S$, $S^i$ and $S^{ij}$ in terms of the coefficients $Q^I$, the result being that

$$R_2 = J^{r,s}_3(Q) + D\widetilde{Q}, \tag{5.135}$$

where

$$\widetilde{Q}(Y) = \frac{3}{(n-r+2)(n-r+3)} \Gamma^p_{ij} Q^{ijh}_{ph}(Y).$$

Thus, for a third order $D$ closed operator $P_3$, $D\widetilde{Q}$ is the single correction term that must be added to $J^{r,s}_3(P_3)$ to obtain an invariantly defined operator. One can check directly, by a rather tedious calculation, that the two formulas for $R$, *viz.*, (5.132) and (5.135) coincide — not identically but rather by virtue of the condition $DP_3 = 0$. ∎

EXAMPLE 5.61. Similar trace-free conditions were introduced by Kolar [*41*] in his construction of global Lepage equivalents. For example, if $\lambda \in \Omega^{n,0}(J^\infty(E))$ is a third order Lagrangian, then it readily follows from (5.135) (with $r = n$ and $P = P_{(d_V\lambda)}$) that

$$h_\nabla^{n,1}(d_V\lambda) = \theta^\alpha \wedge E^i_\alpha(\lambda_i) + D_j[\theta^\alpha \wedge E^{ij}_\alpha(\lambda_i)] + D_{jh}[\theta^\alpha \wedge E^{ijh}_\alpha(\lambda_i)] + \frac{1}{2} d_H[\Gamma^p_{ij}\,\theta^\alpha \wedge \partial^{ijh}_\alpha(\lambda_{ph})].$$

The global Lepage equivalent

$$\Theta(\lambda) = \lambda + h_\nabla^{n,1}(d_V\lambda)$$

therefore coincides with that defined by Kolar [*41*], at least for third order Lagrangians. ∎

COROLLARY 5.62. *Let $\nabla$ be a connection on $M$. Then there exists a cochain map*

$$\Phi_\nabla \colon \mathcal{E}^*(J^\infty(E)) \to \Omega^*(J^\infty(E))$$

*from the Euler-Lagrange complex on $J^\infty(E)$ to the de Rham complex on $J^\infty(E)$ which induces an isomorphism in cohomology.*

*If $\nabla$ and $\widetilde{\nabla}$ are two connections on $M$, then $\Phi_\nabla$ and $\Phi_{\widetilde{\nabla}}$ induce the same isomorphism.*

PROOF: In accordance with the spectral sequence argument used in the proof of Theorem 5.9, the required map is easily constructed from the iterates of the map $h_\nabla^{r,s} \circ d_V$. For instance, if $r \leq n$ and $\omega \in \mathcal{E}^r(J^\infty(E)) = \Omega^{r,0}(J^\infty(E))$, then

$$\Phi_\nabla(\omega) = \sum_{k=0}^{r} (-1)^r \omega_r$$

where $\omega_0 = \omega$ and, for $k = 1, 2, \ldots, r$, $\omega_k = h_\nabla^{r+1-k,k}(d_V\omega_{k-1})$.

The inverse isomorphism $\Psi^*$ from the de Rham cohomology on $J^\infty(E)$ to the cohomology of the Euler-Lagrange complex on $J^\infty(E)$ is induced by the projection maps $\Psi^* = \pi^{r,0}$ and $\Psi^* = I \circ \pi^{n,s}$. Since $\Psi^*$ is the inverse to both $\Phi_\nabla$ and $\Phi_{\widetilde{\nabla}}$, these maps must coincide in cohomology. ∎

One final remark concerning our invariant-theoretical constructions is in order. As we have seen, any total differential operator $P \in \mathcal{D}^{r,s}(J^\infty(E))$ may be described in a variety of ways, *viz.*,

$$P(Y) = \sum_{|I|=0}^{k} (D_I Y^\alpha) P_\alpha^I = \sum_{|I|=0}^{k} D_I [Y^\alpha Q_\alpha^I] = \sum_{|I|=0}^{k} \nabla_I [Y^\alpha Q_{\nabla\alpha}^{\ I}]. \tag{5.136}$$

Each of these representations is unique in the sense that $P = 0$ if and only if the coefficients $P_\alpha^I$, $Q_\alpha^I$ and $Q_{\nabla\alpha}^{\ I}$ in each representation vanish. Suppose now that, in addition to the linear connection $\nabla$ on $M$ we have a connection $\nabla'$ on the bundle $\mathcal{E}v \to J^\infty(E)$ of evolutionary vector fields. Specifically, $\nabla'$ assigns to each total vector field $Z = \operatorname{tot} X$, where $X$ is a generalized vector field on $M$, and to each evolutionary vector field $Y$, an evolutionary vector field $\nabla'_Z Y$. By definition $\nabla'_Z Y$ is linear (over $C^\infty$ functions on $J^\infty(E)$) in the argument $Z$ and a derivation in the argument $Y$. In local coordinates $[x, u]$ on $J^\infty(U)$, $\nabla'$ is given by

$$\nabla'_Z Y = X^i [D_i Y^\alpha + \Gamma^\alpha_{\beta i} Y^\beta] \frac{\partial}{\partial u^\alpha}.$$

The connection coefficients $\Gamma^\alpha_{\beta i}$, defined by

$$\nabla_{D_i} (\frac{\partial}{\partial u^\alpha}) = \Gamma^\alpha_{\beta i} \frac{\partial}{\partial u^\alpha},$$

are the components of an evolutionary vector field and therefore, as such, are functions on $J^\infty(U)$. Under the change of coordinates $v^\beta = v^\beta(x^i, u^\alpha)$ and $y^j = y^j(x^i)$ these connection coefficients transform according to

$$\Gamma^\alpha_{\beta i} = \frac{\partial v^\gamma}{\partial u^\beta} \frac{\partial u^\alpha}{\partial v^\delta} \frac{\partial y^j}{\partial x^i} \overline{\Gamma}^\delta_{\gamma j} + [D_i (\frac{\partial v^\delta}{\partial u^\beta})] \frac{\partial u^\alpha}{\partial v^\delta}.$$

Let $T_V^{(p',q')}(J^\infty(E)) \to J^\infty(E)$ denote the bundle of type $(p', q')$ tensors associated to the vector bundle $\mathcal{E}v(J^\infty(E)) \to J^\infty(E)$ of evolutionary vector fields. The total covariant derivative of "mixed tensor fields", *i.e.*, sections of $T_H^{p,q} \otimes T_V^{p',q'}$, can then be constructed using both connections $\nabla$ and $\nabla'$. For example, if $T$ is a mixed tensor of types $(p, q) = (1, 0)$ and $(p', q') = (0, 1)$ then, in components,

$$\nabla'_j T_i^\alpha = D_j T_i^\alpha + \Gamma^\alpha_{\beta j} T_i^\beta - \Gamma^l_{ij} T_l^\alpha.$$

We have allowed ourselves an abuse of notation here by writing $\nabla'_j$ to denote a process of covariant differentiation which depends upon both connection $\nabla$ and $\nabla'$;

we retain the notation $\nabla_j$ for covariant differentiation involving only the connection $\nabla$.

The total derivatives $D_I Y^\alpha$ in the first representation in (5.136) can now be systematically replaced by total covariant derivatives $\nabla'_I Y^\alpha$. This leads to another expression for $P$ of the form

$$P(Y) = \sum_{|I|=0}^{k} (\nabla'_I Y^\alpha) P_{\nabla'}{}^{I}_{\alpha},$$

where the coefficients $P^I_{\nabla'}$ depend upon both connections $\nabla$ and $\nabla'$. Upon repeated integration by parts, this can be recast into the form

$$P(Y) = \sum_{|I|=0}^{k} \nabla_I [Y^\alpha Q_{\nabla'}{}^{I}_{\alpha}], \tag{5.137a}$$

where, just as in the proof of Proposition (2.1),

$$Q_{\nabla'}{}^{I}_{\alpha} = \sum_{|J|=0|}^{k-|I|} \binom{|I|+|J|}{|I|} (-1)^{|J|} \nabla'_J (P_{\nabla'}{}^{IJ}_{\alpha}). \tag{5.137b}$$

The connection coefficients $\Gamma^\beta_{\alpha i}$ enter into in the formula for $Q_{\nabla'}{}^{I}_{\alpha}$ twice — first in the formulas for the $P_{\nabla'}{}^{I}_{\alpha}$ in terms of the original coefficients $P^I_\alpha$ and then in the calculation of the covariant derivatives $\nabla'_J$. However, a comparison of (5.137b) and the last representation in (5.136) shows that

$$Q_{\nabla}{}^{I}_{\alpha} = Q_{\nabla'}{}^{I}_{\alpha}$$

*i.e.*, all the terms in the coefficient $Q_{\nabla'}{}^{I}_{\alpha}$ involving the connection coefficients $\Gamma^\beta_{\alpha i}$ must vanish! A similar situation is described by Masqué [*49*] who used a pair of connections to construct global Poincaré-Cartan forms but concluded that the forms so constructed are actually independent of one of the connections.

In the special case where the total differential operator $P \in \mathcal{D}^{r,0}(J^\infty(E))$ is defined by Lie differentiation, *i.e.*, $P(Y) = \mathcal{L}_{\operatorname{pr} Y}\omega$, we have that $P^I_\alpha = \partial^I_\alpha(\omega)$. The operators $P_{\nabla'}(\omega)$ are the so-called *tensorial partial derivatives* of $\omega$ with respect to $u^\alpha_I$ and the connections $\nabla$ and $\nabla'$. The concept of tensorial partial differentiation was first introduced by Rund[*60*] and subsequently developed by du Plessis [*23*], Wainwright [*77*] and Horndeski [*36*]. We denote these tensorial derivatives by ${\nabla'}^I_\alpha(\omega)$ — they are defined by

$$\mathcal{L}_{\operatorname{pr} Y}\omega = \sum_{|I|=0}^{k} (\nabla'_I Y^\alpha) {\nabla'}^I_\alpha(\omega).$$

For second order forms $\omega$ we find, after some straightforward calculation, that

$$\begin{aligned}
&{\nabla'}^{ij}_{\alpha}(\omega) = \partial^{ij}_{\alpha}\,\omega \\
&{\nabla'}^{i}_{\alpha}(\omega) = \partial^{i}_{\alpha}\,\omega - 2\Gamma^{\beta}_{\alpha j}\partial^{ij}_{\beta}\,\omega + \Gamma^{i}_{jh}\partial^{jh}_{\alpha}\,\omega, \quad \text{and} \\
&{\nabla'}_{\alpha}(\omega) = \partial_{\alpha}\,\omega - \Gamma^{\beta}_{\alpha i}\partial^{i}_{\beta}\,\omega + \Gamma^{\beta}_{\alpha i}\Gamma^{\gamma}_{\beta j}\partial^{ij}_{\gamma}\,\omega - \Gamma^{\beta}_{\alpha i,j}\partial^{ij}_{\beta}\,\omega.
\end{aligned}$$

The operators $Q^{\ I}_{\nabla\alpha} = Q^{\ I}_{\nabla'\alpha}$ become the *tensorial Lie-Euler operators*

$$E^{\ I}_{\nabla'\alpha}(\omega) = \sum_{|J|=0}^{k-|I|} \tbinom{|I|+|J|}{|I|}(-1)^{|J|}\nabla'_J[{\nabla'}^{IJ}_{\alpha}(\omega)].$$

These first appeared, albeit in a somewhat specialized context, in Horndeski's [*35*] analysis of variational principles on Riemannian structures. Note that when $r = n$ and $|I| = 0$, we obtain the following manifestly invariant expression for the components of the Euler-Lagrange form

$$E_{\alpha}(\lambda) = \sum_{|I|=0}^{k}(-1)^{|I|}\nabla'_I[{\nabla'}^{I}_{\alpha}(\lambda)].$$

Example 5.63. *Exactness of the Taub conservation law in general relativity.*

Let $g = g_{ij}\,dx^i \otimes dx^j$ be a metric ( of any signature) on an $n$ dimensional manifold $M$. Let $R_i{}^j{}_{hk}$ be the curvature tensor of $g$, $R_{ih} = R_i{}^k{}_{hk}$ the Ricci tensor and $R = g^{ij}R_{ij}$ the curvature scalar. We denote covariant differentiation defined in terms of the Christoffel symbols of $g$ by either $\nabla_j$ or $|j$.

The Euler-Lagrange equations derived from the second order Lagrangian

$$\lambda[g] = \sqrt{g}\,R\,\nu$$

are the vacuum Einstein field equations

$$G^{ij} = 0 \quad \text{where} \quad G^{ij} = R^{ij} - \frac{1}{2}g^{ij}R. \tag{5.138}$$

Because of the general coordinate invariance of $\lambda$, Noether's Theorem (see Example 3.34) implies that the Einstein tensor $G^{ij}$ is divergence-free, *i.e.*,

$$\nabla_j G^{ij} = 0.$$

Consequently, if $X$ is any vector field on $M$, then the divergence of the vector density

$$S = [\sqrt{g}\, X_j G^{ij}]\,\frac{\partial}{\partial x^i}$$

is given by

$$\operatorname{div} S = D_i S^i = \nabla_i S^i = \sqrt{g}\,(\nabla_i X_j) G^{ij}. \tag{5.139}$$

Therefore, if $X$ is a Killing vector field of $g$, *i.e.*,

$$\mathcal{L}_X\, g_{ij} = \nabla_j X_i + \nabla_i X_j = 0,$$

then $S$ is divergence-free. This is a trivial conservation law in the sense that it holds independent of the fields equations (5.138).

Now let $y = y_{ij}\, dx^i \otimes dx^j$ be any symmetric, type $(0,2)$ tensor field, let $Y = y_{ij}\,\dfrac{\partial}{g_{ij}}$ and let

$$T(Y) = \mathcal{L}_{\operatorname{pr} Y} S. \tag{5.140}$$

The Lie derivative of (5.139) with respect to $\operatorname{pr} Y$ gives

$$\operatorname{div}\,[T(Y)] = (\nabla_i X_j)\,\mathcal{L}_{\operatorname{pr} Y}\big(\sqrt{g}\, G^{ij}\big) + \big(\mathcal{L}_{\operatorname{pr} Y}(\nabla_i X_j)\big)\sqrt{g}\, G^{ij}.$$

Thus $T(Y)$ is a divergence-free vector if

(i) $g$ is a solution to the Einstein field equations, and

(ii) $X$ is a Killing vector field for the metric $g$.

The conservation law $T(Y)$ is called the *Taub conservation law* for the Einstein field equations.

To obtain an explicit expression for $T(Y)$, we first compute

$$\begin{aligned}
\mathcal{L}_{\operatorname{pr} Y} R_r{}^t{}_{su} &= \nabla_u[\mathcal{L}_{\operatorname{pr} Y}\Gamma^t_{rs}] - \nabla_s[\mathcal{L}_{\operatorname{pr} Y}\Gamma^t_{ru}] \\
&= \frac{1}{2} g^{tp}[y_{rp|s} + y_{sp|r} - y_{rs|p}]_{|u} - \frac{1}{2} g^{tp}[y_{rp|u} + y_{up|r} - y_{ru|p}]_{|s} \\
&= \frac{1}{2}[y^t_{s|ru} - y^t_{u|rs} - y_{rs}{}^{|t}{}_{|u} + y_{ru}{}^{|t}{}_{|s} - y^t_q R_r{}^q{}_{us} - y_{rq} R^{tq}{}_{us}].
\end{aligned}$$

Here $y^{|t}_{rs} = g^{tu} y_{rs|u}$. Let $y = g^{ij} y_{ij}$. Then, because (5.139) implies that $R_{ij} = 0$ and $R = 0$, it is not difficult to show, under the hypothesis that $g$ is a solution to the Einstein field equations, that

$$\begin{aligned}
T^j &= \sqrt{g}\, X^s g^{jt}[\mathcal{L}_{\operatorname{pr} Y} G_{rs}] \\
&= \frac{1}{2}\sqrt{g}\, g^{jt} X^s [g^{ab} y_{sa|tb} - y_{|ts} - g^{ab} y_{ts|ab} + g^{ab} y_{ta|bs} \\
&\qquad - g_{ts} h^{ab}{}_{|ab} + g_{ts} g^{ab} y_{|ab} - y^a_b R_t{}^b{}_{as}].
\end{aligned} \tag{5.141}$$

The vector density $T^j(Y)$ is an example of a *natural tensor* — it is constructed from the metric $g_{ij}$, the curvature tensor $R_i{}^j{}_{hk}$, the tensor $y_{ij}$ and its covariant derivatives and the vector field $X^s$ by the natural operations of forming tensor products and taking contractions of indices. More formally, let

$$E = S^{0,2}_+(M) \oplus S^{0,2}(M) \oplus TM \to M$$

be the direct sum of the bundle $S^{0,2}_+$ of metrics, the bundle $S^{0,2}$ of symmetric type $(0,2)$ tensors, and the tangent bundle of $M$. Then $T$ is a weight 1, horizontal type $(1,0)$ tensor on $J^\infty(E)$ which is invariant under the induced action of the group of orientation preserving diffeomorphism of $M$. We apply our invariant homotopy operator to prove that $T$ is naturally exact. In the present context (of vector densities and divergences) this means that there is a natural type $(2,0)$ skew-symmetric tensor density

$$P = P^{ij}[g,h,X]\,\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

on $J^\infty(E)$ such that

$$\operatorname{div} P = [\nabla_j P^{ij}]\,\frac{\partial}{\partial x^i} = T. \tag{5.142}$$

To begin, we first recall the relationship between the divergence operator and the exterior derivative. Let

$$T = T^{i_1 i_2 \cdots i_p}\,\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}}$$

be a skew-symmetric, type $(p,0)$ horizontal tensor density on $J^\infty(E)$. Define, for $q = n - p$, a type $(q,0)$ form $T^\flat$ by

$$T^\flat = \frac{1}{p!\,q!}\,\epsilon_{i_1\ldots i_p j_1 \ldots j_q} T^{i_1\ldots i_p}\,dx^{j_1} \wedge \ldots dx^{j_q}$$

Conversely, given a type $(q,0)$ form

$$\rho = A_{j_1 j_2 \ldots j_q}\,dx^{j_1} \wedge dx^{j_2} \wedge \ldots dx^{j_q}$$

we define a horizontal type $(p,0)$ skew-symmetric tensor density $\rho^\sharp$ by

$$\rho^\sharp = \epsilon^{i_1\ldots i_p j_1 \ldots j_q} A_{j_1\ldots j_q}\,\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}}.$$

A standard calculation shows that $(T^\flat)^\sharp = T$. In addition, if $T$ is a vector density and $P$ is a horizontal, type $(2,0)$ skew-symmetry tensor density, then $\operatorname{div} T = 0$ if and only if $d_H T^\flat = 0$ and $\operatorname{div} P = T$ if and only if $d_H(P^\flat) = T^\flat$.

In light of (5.140), the Taub conservation law can be viewed as a vector density valued total differential operator and so, in accordance with 5.54, we can write

$$T(Y) = \big[Q_\triangledown^{\,j}(Y) + \nabla_h[Q_\triangledown^{\,hj}(Y)] + \nabla_{hk}[Q_\triangledown^{\,hkj}(Y)\big]\, \frac{\partial}{\partial x^j}$$

where $Q_\triangledown^{\,Ij}(Y) = y_{ab} Q_\triangledown^{\,abIj}$. The components $Q_\triangledown^{\,Ij}$ are symmetric in the indices $I$, but need not have any further symmetries. From the explicit formula (5.141), it is not difficult to deduce that

$$Q_\triangledown^{\,kj}(Y) = -2\nabla_h[Q_\triangledown^{\,hkj}(Y)] \tag{5.143a}$$

and

$$\begin{aligned} Q_\triangledown^{\,hkj} = \frac{1}{4}[&g^{jh}X^l y_l^k + g^{jk}X^l y_l^h - g^{hj}X^k y - g^{jk}X^h y - 2g^{hk}X^l y_l^j \\ &+ X^k y^{hj} + X^h y^{jk} - 2X^j y^{hk} + 2X^j g^{hk} y]. \end{aligned} \tag{5.143b}$$

Since $T(Y)$ is divergence-free, $T^\flat(Y)$ is a $d_H$ closed and therefore, by (5.112),

$$T^\flat(Y) = d_H[J_{\triangledown\,2}^{n-1,1}(T^\flat)(Y)] \tag{5.144}$$

where $J_{\triangledown\,2}^{n-1,1}$ is given by (5.114). The key point to make here is that $\nabla$ be now taken to be the metric connection of $g$. The coordinate invariance of $J_{\triangledown\,2}^{n-1,1}$ implies, because $T^\flat(Y)$ is natural form on $J^\infty(E)$, that $J_{\triangledown\,2}^{n-1,1}(T^\flat)(Y)$ is also a natural form. When (5.144) is rewritten back in terms of $T$ (by applying $\sharp$), it is found that

$$T(Y) = \operatorname{div}[P(Y)],$$

where the components of $P$ are given by

$$P^{jk}(Y) = \frac{1}{2}[Q_\triangledown^{\,jk}(Y) - Q_\triangledown^{\,kj}(Y)] + \frac{2}{3}\nabla_h[Q_\triangledown^{\,hjk}(Y) - Q_\triangledown^{\,hkj}(Y)].$$

Into this equation we substitute from (5.143) to arrive at, after some simplification,

$$\begin{aligned} P^{jk} = [&(X^j y^{|k} - X^k y^{|j}) - X^{j|k} y + (X^l y_l^{k|j} - X^l y_l^{j|k}) \\ &+ (X^k y^{j|l}{}_{|l} - X^j y^{kl}{}_{|l}) + (X^{l|k} h_l^j - X^{l|j} h_l^k)]. \end{aligned} \tag{5.145}$$

This agrees with the result presented in Arms and Anderson [*6*] where (5.142) was verified using (5.145) by direct calculation.

The triviality of the Taub conservation law plays a crucial role in the analysis by Fisher, Marsden and Moncreif [*25*] of the linearization stability of solutions to the Einstein field equations.

With our invariant homotopy operators in hand, we immediately arrive at the following generalization of the above result. Let $\lambda = L[g]\,\nu$ be any natural Lagrangian in the metric $g$, and let

$$S = \left[X_i E^{ij}(L)[g]\right] \frac{\partial}{\partial x^j}.$$

The generalized Taub conservation law

$$T = \mathcal{L}_{\mathrm{pr}\,Y} S$$

is a natural tensor. If $g$ is any solution to the Euler-Lagrange equations

$$E^{ij}(L)[g] = 0$$

and if $X$ is a Killing vector field of $g$, then $T$ is the divergence of a type $(2,0)$ skew-symmetric natural tensor.

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