(METTIX) Optimal Constant for Generalized Diagonal Update Method

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Target Matrix Equation

Target Matrix Equation (1/3)

Original Target Matrix Equation

Consider the following quadratic matrix equation

$$Q_1(X) = AX^2 + BX + C = 0 \tag{1}$$

where

- $A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal elements,
- $B \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix,
- $C \in \mathbb{R}^{n\times n}$ is an M-matrix such that $B^{-1}C \geq 0$.

($*$) The inequality on the third condition is natural entrywise inequality.
($**$) (Z-matrix) $M \in \mathbb{R}^{n\times n}$ is called Z-matrix if all its off-diagonal elements are nonpositive, that is, $A = \alpha I - P$ for some $P \geq 0$.
($***$) (M-matrix) $A = \alpha I - P$ for some $P \geq 0$ is called M-matrix if $\alpha \geq \rho(P)$, when $\rho(P)$ denotes the spectral radius of P.

Target Matrix Equation (2/3)

Remark (Motivation) [1]

The assumed M-matrices on coefficient matrices are motivated by a **quadratic eigenvalue problem (QEP)** arising from an overdamped vibrating system [2, 3].

Simplified Target Matrix Equation

We consider the simplified equation

$$Q_2(X) = X^2 + BX + C = 0 \tag{2}$$

where

- $B \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix,
- $C \in \mathbb{R}^{n\times n}$ is an M-matrix such that $B^{-1}C \geq 0$.

Target Matrix Equation (3/3)

Question 1

What solution (solvent) do you want?

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Target Matrix Equation (3/3)

Question 1

What solution (solvent) do you want?

Answer. We will find **primary solvent** X^* which is the **maximal nonpositive solvent** of (2).

Question 2

Why do you need the primary solvent?

Answer. Guo and Lancaster [4] showed that the QEP in an overdamped vibrating system can be solved effectively by a solvent approach. In particular, the n largest nonpositive eigenvalues can be derived by the primary solvent [1].

Previous Methods

Previous Methods (1/6)

Method 1 - Yu (2011) [1]

- Condition: $B - C - I$ is a nonsingular M-matrix
- Method:

$$\begin{cases} X_0 = 0 \in \mathbb{R}^{n\times n} \\ X_{k+1} = \mathcal{F}_i(X_k), \quad k = 0, 1, 2, \cdots \end{cases}$$

where

$$\mathcal{F}_1(X) = -(B + X)^{-1}C,$$
$$\mathcal{F}_2(X) = -B^{-1}(X^2 + C).$$

Previous Methods (2/6)

Method 2 - Kim (2016) [5]

- Condition: $B - C - 2I$ is a nonsingular M-matrix
- Method:

$$\begin{cases} X_0 = 0 \in \mathbb{R}^{n\times n} \\ X_{k+1} = \mathcal{F}_i(X_k), \quad k = 0, 1, 2, \cdots \end{cases}$$

  where

$$\mathcal{F}_3(X) = -(B + X - \delta_X I)^{-1}(C + \delta_X X),$$
$$\mathcal{F}_4(X) = -(B - 2\delta_X I)^{-1}(X^2 + 2\delta_X X + C),$$

  for $\delta_X = \min\{\, 1, \min\{\, |\mathrm{diag}(X)| \,\}\,\}$.

- Advantage: faster than the method 1.

(∗) This method is called **diagonal update method**.

Previous Methods (3/6)

Question 3

Note that there are much more examples which don't satisfy the condition that $B - C - 2I$ is a nonsingular M-matrix. Can the condition be **weaken**, in order to use the idea of **the diagonal update method** for solving **more** examples faster?

Previous Methods (3/6)

Question 3

Note that there are much more examples which don't satisfy the condition that $B - C - 2I$ is a nonsingular M-matrix. Can the condition be **weaken**, in order to use the idea of **the diagonal update method** for solving **more** examples faster?

Answer. Yes, we generalized the diagonal update method!

Example 1

$$A = I,$$

$$B = \epsilon \begin{bmatrix} 20 & -10 & & & & \\ -10 & 30 & -10 & & & \\ & -10 & 30 & -10 & & \\ & & -10 & \ddots & \ddots & \\ & & & \ddots & 30 & -10 \\ & & & & -10 & 20 \end{bmatrix},$$

$$C = \begin{bmatrix} 15 & -5 & & & & \\ -5 & 15 & -5 & & & \\ & -5 & 15 & -5 & & \\ & & -5 & \ddots & \ddots & \\ & & & \ddots & 15 & -5 \\ & & & & -5 & 15 \end{bmatrix}$$

($*$) If $\epsilon \leq 0.9603$, then $B - C - 2I$ is **not** a nonsingular M-matrix.

Remark (Controlling the Size of the Damping Term) [6]

The equation at Example 1 with the form

$$AX^2 + \epsilon BX + C = 0$$

can be considered to solve a quadratic eigenvalue problem with real parameter ϵ which is introduced to control the size of the damping term B.

Previous Methods (6/6)

Example 2 [7]

$$A = C = I, \qquad B = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}.$$

(∗) For all size n of these square matrices, $B - C - I$ is a nonsingular M-matrix, but $B - C - 2I$ is not.

Generalized Diagonal Update Method

Generalized Diagonal Update Method (1/3)

Lemma [8]

For a Z-matrix A, the followings are equivalent:

(i) A is a nonsingular M-matrix.

(ii) A^{-1} is nonnegative.

(iii) $Av > 0$ for some vector $v > 0$.

(iv) All eigenvalues of A have positive real parts.

Generalized Diagonal Update Method (2/3)

Remark (Optimal Constant)

The original condition that $B - C - I$ is a nonsingular M-matrix derives a more general condition that

$$B - C - \gamma I \text{ is a nonsingular } M\text{-matrix}$$

for some γ.
Indeed, we can take the **optimal constant**

$$\gamma^* = \min\{\ \text{real}\ (\text{eig}\ (B - C)\ ),\ 2\ \}.$$

Generalized Diagonal Update Method (3/3)

Generalized Diagonal Update Method

- Condition: $B - C - I$ is a nonsingular M-matrix
- Method: $X_0 = 0 \in \mathbb{R}^{n\times n}$,

  (i) $X_{k+1} = \mathcal{G}_\gamma(X_k), \;\; k = 0, 1, 2, \cdots$, where

$$\mathcal{G}_\gamma(X) = -(B + X - (\gamma - 1)\delta_X I)^{-1}(C + (\gamma - 1)\delta_X X),$$

  (ii) $X_{k+1} = \mathcal{H}_\gamma(X_k), \;\; k = 0, 1, 2, \cdots$, where

$$\mathcal{H}_\gamma(X) = -(B - \gamma\delta_X I)^{-1}(X^2 + \gamma\delta_X X + C),$$

  for $\delta_X = \min\{\, 1, \min\{\, |\text{diag}(X)|\,\}\,\}$ and $1 \le \gamma < \gamma^*$.
- Advantage: **faster** than method 1 and **more available** than method 2.

Numerical Experiments

Numerical Experiments (1/3)

Algorithms

We used following algorithms:

$$\begin{cases} X_0 = 0, \ \delta_X = \min\{1, \min\{|\mathrm{diag}(X)|\}\}, \\ \gamma = \min\{\mathrm{real}(\mathrm{eig}(B - C)), 2\} - 0.0001 \\ X_{i+1} = -(B + X_i)^{-1}C, & \text{(BI1)} \\ X_{i+1} = -(B + X_i - (\gamma - 1)\delta_{X_i} I)^{-1}(C + (\gamma - 1)\delta_{X_i} X_i). & \text{(BI1-OC)} \end{cases}$$

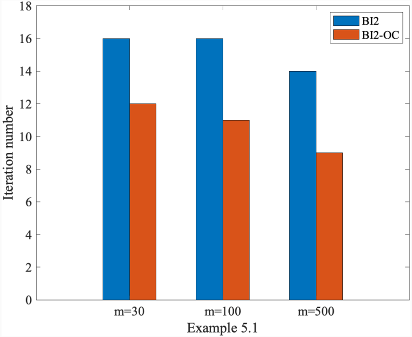
$$\begin{cases} X_0 = 0, \ \ \delta_X = \min\{1, \min\{|\mathrm{diag}(X)|\}\}, \\ \gamma = \min\{\mathrm{real}(\mathrm{eig}(B - C)), 2\} - 0.0001 \\ X_{i+1} = -B^{-1}(X_i^2 + C), & \text{(BI2)} \\ X_{i+1} = -(B - \gamma\delta_{X_i} I)^{-1}(X_i^2 + \gamma\delta_{X_i} X_i + C). & \text{(BI2-OC)} \end{cases}$$

Numerical Experiments (2/3)

For Example 1,

TABLE 5.1. Numerical results for Example 5.1 with $\epsilon = 0.95, \gamma = 1.8683$

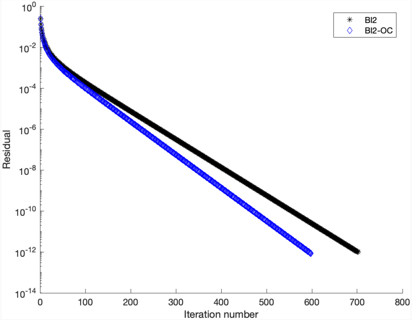
Iteration methods	$m = 30$		$m = 100$		$m = 500$	
	Residual	Time	Residual	Time	Residual	Time
BI1	9.73E-14	0.00528	9.30E-13	0.01326	9.31E-13	0.16213
BI1-OC	1.26E-13	0.00151	1.25E-13	0.00882	2.32E-12	0.12486
BI2	4.05E-13	0.00350	4.06E-13	0.01760	2.70E-12	0.27946
BI2-OC	4.75E-14	0.00193	5.95E-13	0.00726	1.69E-13	0.16732



For Example 2,

TABLE 5.2. Numerical results for Example 5.1

Iteration methods	$m = 30$			$m = 100$		
	It(s)	Residual	Time	It(s)	Residual	Time
BI1	130	1.01E-13	0.01375	369	3.67E-13	0.12687
BI1-OC	130	9.33E-14	0.00853	369	3.65E-13	0.09148
BI2	240	1.97E-13	0.02851	702	7.46E-13	0.22766
BI2-OC	203	1.60E-13	0.01041	598	6.37E-13	0.20155



Conclusion

Conclusion (1/1)

Strategy to Use (Generalized) Diagonal Update Method

Assume that $B - C - I$ is a nonsingular M-matrix.

(1) If we don't know that $B - C - 2I$ is not a nonsingular M-matrix or we have that $B - C - 2I$ is not a nonsingular M-matrix, then use the generalized diagonal update method with γ^*.

(2) If we also have that $B - C - 2I$ is a nonsingular M-matrix, then use the original diagonal update method.

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Thank you!