

eScholarship

Combinatorial Theory

Title

Rowmotion on (m) -Tamari and biCambrian lattices

Permalink

<https://escholarship.org/uc/item/5x945586>

Journal

Combinatorial Theory, 4(1)

ISSN

2766-1334

Authors

Defant, Colin

Lin, James

Publication Date

2024

DOI

10.5070/C64163852

Supplemental Material

<https://escholarship.org/uc/item/5x945586#supplemental>

Copyright Information

Copyright 2024 by the author(s). This work is made available under the terms of a Creative Commons Attribution License, available at

<https://creativecommons.org/licenses/by/4.0/>

Peer reviewed

ROWMOTION ON m -TAMARI AND BICAMBRIAN LATTICES

Colin Defant^{*†1} and James Lin^{†‡2}

¹*Department of Mathematics, Harvard University, Cambridge, MA 02139, U.S.A.*

colindefant@gmail.com

²*Department of Mathematics and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.*

jameslin@mit.edu

Submitted: Sep 13, 2022; Accepted: Mar 1, 2024; Published: Jun 30, 2025

© The authors. Released under the CC BY license (International 4.0).

Abstract. Thomas and Williams conjectured that rowmotion acting on the rational (a, b) -Tamari lattice has order $a + b - 1$. We construct an equivariant bijection that proves this conjecture when $a \equiv 1 \pmod{b}$; in fact, we determine the entire orbit structure of rowmotion in this case, showing that it exhibits the cyclic sieving phenomenon. We additionally show that the down-degree statistic is homomesic for this action. In a different vein, we consider the action of rowmotion on Barnard and Reading's biCambrian lattices. Settling a different conjecture of Thomas and Williams, we prove that if c is a bipartite Coxeter element of a coincidental-type Coxeter group W , then the orbit structure of rowmotion on the c -biCambrian lattice is the same as the orbit structure of rowmotion on the lattice of order ideals of the doubled root poset of type W .

Keywords. Rowmotion, m -Tamari lattice, biCambrian lattice, cyclic sieving, homomesy, homometry

Mathematics Subject Classifications. 05E18, 06B10, 06D75

1. Introduction

The growing field of *dynamical algebraic combinatorics* seeks to understand dynamical systems arising from operators defined on objects in algebraic combinatorics. One of the most vigorously-studied operators in this area is *rowmotion*, which is an operator Row that acts on the set $\mathcal{J}(P)$ of order ideals of a finite poset P .

*Supported by the National Science Foundation under Award No. DGE-1656466 and Award No. 2201907, by a Fannie and John Hertz Foundation Fellowship, and by a Benjamin Peirce Fellowship at Harvard University.

†Supported by the MIT CYAN Mathematics Undergraduate Activities Fund.

‡This research was conducted at the University of Minnesota Duluth Mathematics REU and was supported in part by NSF-DMS Grant 1949884 and NSA Grant H98230-20-1-0009.

More precisely, $\text{Row}: \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ is defined by¹

$$\text{Row}(I) = \{x \in P : x \not\geq y \text{ for all } y \in \max(I)\},$$

where $\max(I)$ is the set of maximal elements of I . When ordered by inclusion, the order ideals of P form a lattice; in fact, Birkhoff's Fundamental Theorem of Finite Distributive Lattices [Bir37] states that a finite lattice is distributive if and only if it is isomorphic to the lattice of order ideals of a finite poset. Thus, one can view rowmotion as an operator defined on a distributive lattice.

Another very influential operator is *Kreweras complementation*, which Kreweras introduced in 1972 in its most basic form in [Kre72] as a natural anti-automorphism of the lattice of non-crossing set partitions of the set $[n] = \{1, \dots, n\}$. More generally, if one chooses a Coxeter element c of a finite irreducible Coxeter group W , then one can consider the *noncrossing partition lattice* $\text{NC}(W, c)$, which is the interval $[e, c]$ in the absolute order on W . In this setting, *Kreweras complementation* is the operator that sends w to $w^{-1}c$. Associated to the pair (W, c) is a *Cambrian lattice*, which is a lattice on the set of c -sortable elements of W [Rea06]. Using Reading's bijection between $\text{NC}(W, c)$ and the set of c -sortable elements of W [Rea07], one can view Kreweras complementation as an operator on a Cambrian lattice.

In 2006, Thomas [Tho06] introduced *trim lattices* as generalizations of distributive lattices that are not necessarily graded. Another natural family of lattices that generalize distributive lattices is the family of *semidistributive lattices*. All distributive lattices and finite Cambrian lattices are both trim and semidistributive. Barnard [Bar19] gave a natural definition of rowmotion on semidistributive lattices, and Thomas and Williams [TW19] gave a natural definition of rowmotion on trim lattices; in recent work, the first author and Williams provided a simultaneous generalization of both of these definitions to the even broader class of *semidistributive lattices* [DW23]. These definitions generalize that of classical rowmotion on distributive lattices and that of Kreweras complementation on finite Cambrian lattices.

Prototypical examples of Cambrian lattices are provided by the *Tamari lattices*, which Tamari originally introduced in 1962 [Tam62]. These lattices, which are often defined on Dyck paths, have grown in prominence to become fundamental in algebraic combinatorics [MHPS12]. In 2012, Bergeron and Préville-Ratelle [BPR12] defined generalizations of Tamari lattices called *m-Tamari lattices* in order to state conjectural combinatorial interpretations of the dimensions of certain spaces arising from the study of trivariate diagonal harmonics. We denote the n -th m -Tamari lattice by $\text{Tam}_n(m)$. These lattices have now received a great deal of attention (see [BMCPR13, BMFPR11, CP15] and the references therein).

The m -Tamari lattices are special examples of *rational Tamari lattices*. Each rational Tamari lattice is trim and semidistributive, so it comes equipped with a natural rowmotion operator. Thomas and Williams conjectured a formula for the order of rowmotion on rational Tamari lattices; one of our main results settles this conjecture for m -Tamari lattices by showing that the order of $\text{Row}: \text{Tam}_n(m) \rightarrow \text{Tam}_n(m)$ is $(m+1)n$. In fact, we will determine the entire orbit structure of rowmotion on $\text{Tam}_n(m)$, showing that it exhibits the cyclic sieving phenomenon

¹Several authors define rowmotion to be the inverse of the operator that we have defined. Our definition agrees with the conventions used in [Bar19, DW23, TW19].

with respect to the polynomial $\text{Cat}_n^{(m)}(q) = \frac{1}{[(m+1)n+1]_q} \left[\begin{matrix} (m+1)n+1 \\ n \end{matrix} \right]_q$. Moreover, we will conjecture that rowmotion on an arbitrary rational Tamari lattice exhibits a similar cyclic sieving phenomenon.

A popular and ubiquitous notion in dynamical algebraic combinatorics is that of *homomesy*, which occurs when a statistic on a set of objects has the same average along every orbit of an operator. The *down-degree statistic* on a poset P is the function $\text{ddeg}: P \rightarrow \mathbb{R}$ given by $\text{ddeg}(x) = |\{y \in P : y \triangleleft x\}|$. The articles [AST13, PR15, RW15] prove that ddeg is homomesic for rowmotion on certain distributive lattices, and the article [Hop19] establishes the homomesy of ddeg for rowmotion on certain semidistributive lattices. As a byproduct of our methods, we will show that this story extends into the realm of m -Tamari lattices. Namely, we will prove that the down-degree statistic on $\text{Tam}_n(m)$ is homomesic for rowmotion with average $m(n-1)/(m+1)$. We will also conjecture that ddeg on the rational Tamari lattice $\text{Tam}(a, b)$ is homomesic for rowmotion with average $(a-1)(b-1)/(a+b-1)$.

In a different direction, we will consider rowmotion operators arising from Coxeter-biCatalan combinatorics, which Barnard and Reading introduced in [BR18]. Each finite irreducible Coxeter group of coincidental type has an associated *doubled root poset* that is minuscule, and the orbit structure of rowmotion acting on the set of order ideals of that doubled root poset is already understood. In the same article, Barnard and Reading introduced the *biCambrian lattice* associated to a pair (W, c) , where W is a finite irreducible Coxeter group and c is a Coxeter element of W . Each biCambrian lattice is semidistributive and, consequently, comes equipped with a rowmotion operator. Thomas and Williams [TW19] conjectured that if W is a finite Coxeter group of coincidental type and c is a bipartite Coxeter element of W , then the orbit structure of rowmotion on the biCambrian lattice associated to (W, c) is the same as the orbit structure of rowmotion acting on the lattice of order ideals of the doubled root poset of W . Our second main result is a proof of this conjecture.

In Section 2, we provide background information concerning rowmotion, cyclic sieving, and homomesy. We will only need to define rowmotion for semidistributive lattices (and not trim lattices) because all of the specific lattices that we consider belong to this family. Section 3 defines ν -Tamari lattices, of which rational Tamari lattices are specific examples. In Section 4, we give an explicit description of rowmotion on ν -Tamari lattices. In Section 5, we specialize our attention to m -Tamari lattices, producing an equivariant bijection that transfers from rowmotion on $\text{Tam}_n(m)$ to cyclic rotation on Armstrong's set of $(m+1)$ -shuffle noncrossing partitions of $[(m+1)n]$. This bijection allows us to prove the aforementioned cyclic sieving phenomenon and homomesy for rowmotion on $\text{Tam}_n(m)$. In Section 6, we shift gears and gather notions from Barnard and Reading's Coxeter-biCatalan combinatorics. Namely, we discuss lattice congruences, doubled root posets, and biCambrian lattices. Section 7 is devoted to understanding rowmotion on biCambrian lattices of type A with respect to bipartite Coxeter elements; we produce an equivariant bijection demonstrating that the orbit structure of rowmotion on such a biCambrian lattice is the same as the orbit structure of rowmotion on the lattice of order ideals of the associated doubled root poset of type A (in this case, the doubled root poset is a product of two chains of the same length). In Section 8, we prove the analogous result in type B (in this case, the doubled root poset is a shifted staircase). The remaining cases of the conjecture of Thomas and Williams are for types H_3 and $I_2(m)$; Section 9 handles these remaining (not

so difficult) cases. Finally, Section 10 mentions some ideas for future research. This includes generalizing our results in Section 5 on the cyclic sieving phenomenon and homomesy for rowmotion on $\text{Tam}_n(m)$ to all rational Tamari lattices $\text{Tam}(a, b)$. We also include two conjectures about the behavior of the down-degree statistic on the orbits of rowmotion on the even larger class of ν -Tamari lattices. The first conjecture describes a phenomenon that we call *asymptotic homomesy*; the second concerns the *homometry phenomenon*, which was introduced recently by Elizalde, Plante, Roby, and Sagan [EPRS23].

2. Background

2.1. Basic notation and terminology

Given positive integers a and b , we write $[a, b]$ for the set $\{a, \dots, b\}$ of integers weakly between a and b , where we make the convention that this set is empty if $a > b$. We also write $[b]$ instead of $[1, b]$. Given a finite set X and an invertible map $f: X \rightarrow X$, we define the *order* of f to be the smallest positive integer ω such that $f^\omega(x) = x$ for all $x \in X$. When we refer to the *orbit structure* of f , we mean the multiset of the sizes of its orbits.

2.2. Posets and lattices

We assume basic familiarity with the theory of posets, as discussed in [Sta99, Chapter 3]. All posets throughout this article are assumed to be finite and connected (meaning their Hasse diagrams are finite connected graphs). Suppose P is a poset. For $x, y \in P$, we say y *covers* x and write $x \lessdot y$ to mean that $x < y$ and there does not exist $z \in P$ with $x < z < y$. We say P is *ranked* if there exists a function $\text{rk}: P \rightarrow \mathbb{Z}_{\geq 0}$ such that $\text{rk}(y) = \text{rk}(x) + 1$ whenever $x \lessdot y$ and such that there exists $x_0 \in P$ with $\text{rk}(x_0) = 0$. Note that the rank function rk is unique if it exists (because P is connected).

A *lattice* is a poset L such that any two elements $x, y \in L$ have a unique greatest lower bound, called their *meet* and denoted $x \wedge y$, and a unique least upper bound, called their *join* and denoted $x \vee y$. Given a subset X of a finite lattice, we write $\bigwedge X$ for the meet of X (i.e., the greatest lower bound of all elements of X).

An element j in a lattice L is called *join-irreducible* if it covers exactly 1 element of L . Similarly, an element $m \in L$ is called *meet-irreducible* if it is covered by exactly 1 element of L .

2.3. Rowmotion on semidistributive lattices

Let P be a finite poset, and let $\mathcal{J}(P)$ be the set of order ideals of P . As discussed in the introduction, rowmotion is the operator $\text{Row}: \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ defined by declaring $\text{Row}(I)$ to be the complement in P of the order filter generated by the maximal elements of I . The origins of rowmotion date back to the works of Duchet [Duc74], Brouwer [Bro75], and Brouwer–Schriever [BS74]. It was studied further in the 1990’s by Deza and Fukada [DF90], by Cameron and Fon-der-Flass [CFdF95], and by Fon-der-Flass [FdF93]. Panyushev [Pan09] revived interest in rowmotion via several conjectures about root posets that were later settled by Armstrong,

Stump, and Thomas [AST13]. We refer to [SW12, TW19] for more detailed historical accounts of rowmotion.

A lattice L is called *semidistributive* if the implications

$$x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y \quad \text{and} \quad x \wedge y = x \wedge z \implies x \wedge (y \vee z) = x \wedge y$$

hold for all $x, y, z \in L$. An equivalent definition says that L is semidistributive if and only if for all $x, y \in L$ with $x \leq y$, the set $\{z \in L : z \wedge y = x\}$ has a unique maximal element and the set $\{z \in L : z \vee x = y\}$ has a unique minimal element. Every distributive lattice (i.e., every lattice of order ideals of a finite poset) is semidistributive. We refer the reader to [Bar19, TW19, DW23] for the facts about semidistributive lattices that we discuss next.

Suppose L is a semidistributive lattice. Let \mathcal{J} and \mathcal{M} be the set of join-irreducible elements of L and the set of meet-irreducible elements of L , respectively. For each $j \in \mathcal{J}$, let j_* be the unique element covered by j ; for each $m \in \mathcal{M}$, let m^* be the unique element that covers m . There is a bijection $\kappa: \mathcal{J} \rightarrow \mathcal{M}$ given by $\kappa(j) = \max\{z \in L : z \wedge j = j_*\}$ for all $j \in \mathcal{J}$; its inverse is given by $\kappa^{-1}(m) = \min\{z \in L : z \vee m = m^*\}$ for all $m \in \mathcal{M}$.

Lemma 2.1. *Let L be a semidistributive lattice, and let $\kappa: \mathcal{J} \rightarrow \mathcal{M}$ be the bijection defined above. We have $\kappa(j) \wedge j = j_*$ and $\kappa(j) \vee j = (\kappa(j))^*$ for all $j \in \mathcal{J}$. Furthermore, for each cover relation $x \triangleleft y$ in L , there is a unique $j_{xy} \in \mathcal{J}$ such that $j_{xy} \leq y$ and $\kappa(j_{xy}) \geq x$.*

The first statement in Lemma 2.1 actually characterizes the bijection κ . For each $z \in L$, Lemma 2.1 allows us to define the *downward label set* and the *upward label set* of z by

$$\mathcal{D}(z) = \{j_{xz} : x \triangleleft z\} \quad \text{and} \quad \mathcal{U}(z) = \{j_{zy} : z \triangleleft y\}.$$

The set $\mathcal{D}(z)$ is also called the *canonical join representation* of z , while $\kappa(\mathcal{U}(z))$ is called the *canonical meet representation* of z .

It turns out that for each $x \in L$, there is a unique element of L whose upward label set is equal to the downward label set of x . We define the *rowmotion* operator on L to be the bijective operator $\text{Row}: L \rightarrow L$ given by

$$\mathcal{U}(\text{Row}(x)) = \mathcal{D}(x).$$

When L is the lattice of order ideals of a finite poset, this definition agrees with the classical definition of rowmotion given above. This more general definition is originally due to Barnard [Bar19], who denoted rowmotion by $\bar{\kappa}$ because it extends the map κ . Indeed, one can show that $\text{Row}(j) = \kappa(j)$ for every $j \in \mathcal{J}$.

Given a lattice L with minimum element $\hat{0}$, the *pop-stack sorting operator* on L is the operator $\text{Pop}: L \rightarrow L$ defined by $\text{Pop}(x) = \bigwedge\{y \in L : y \triangleleft x\}$ for $x \neq \hat{0}$ and by $\text{Pop}(\hat{0}) = \hat{0}$. This operator was introduced in [Def22], and it was found to have strong connections with rowmotion in [DW23] in the case when L is semidistributive. In particular, if L is semidistributive, then the pop-stack sorting operator affords a useful alternative description of rowmotion.

Proposition 2.2 ([DW23, Section 9]). *If L is a semidistributive lattice and $x \in L$, then $\text{Row}(x)$ is the unique maximal element of the set $\{z \in L : z \wedge x = \text{Pop}(x)\}$.*

2.4. The cyclic sieving phenomenon

The cyclic sieving phenomenon, which Reiner, Stanton, and White introduced in [RSW04], occurs when the orbit structure of a finite dynamical system is determined by the evaluations of a specific relatively nice generating function at roots of unity. To make this more precise, let X be a finite set, and let $f: X \rightarrow X$ be an invertible map of order ω (i.e., ω is the smallest positive integer such that $f^\omega(x) = x$ for all $x \in X$). Let $F(q) \in \mathbb{C}[q]$ be a polynomial in the variable q . We say the triple $(X, f, F(q))$ *exhibits the cyclic sieving phenomenon* if for all integers d , the number of elements of X fixed by f^d is $F(e^{2\pi id/\omega})$. Notice that this forces $F(1)$ to be the cardinality of X . The cyclic sieving phenomenon has been observed in many settings, including the relevant setting of noncrossing partitions in [BR16].

2.5. Homomesy and homometry

Suppose X is a finite set and $f: X \rightarrow X$ is an invertible function. A *statistic* on X is simply a real-valued function on X . We say a statistic $\text{stat}: X \rightarrow \mathbb{R}$ is *homomesic* for f if there exists a constant c such that

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} \text{stat}(x) = c$$

for every orbit \mathcal{O} of f . Propp and Roby coined the term *homomesy* in [PR15]. This phenomenon has now become one of the central focuses of dynamical algebraic combinatorics because of its surprising ubiquity and its ability to encapsulate interesting structural aspects of combinatorial dynamical systems. We refer the reader to [Rob16] for a very accessible survey of homomesy.

The very recent article [EPRS23] introduces a new generalization of homomesy called *homometry*. We say a statistic $\text{stat}: X \rightarrow \mathbb{R}$ is *homometric* for f if

$$\sum_{x \in \mathcal{O}} \text{stat}(x) = \sum_{x' \in \mathcal{O}'} \text{stat}(x')$$

whenever \mathcal{O} and \mathcal{O}' are orbits of f of the same cardinality. Note that homomesy implies homometry, but not the other way around. In Section 10, we will conjecture that the down-degree statistic is homometric for rowmotion on an arbitrary ν -Tamari lattice.

3. ν -Tamari Lattices

3.1. Lattice paths

For us, a *lattice path* is a finite path in the plane that starts at the origin and uses unit north steps and unit east steps. We denote north and east steps by N and E, respectively, thereby identifying a lattice path with a finite word over the alphabet $\{N, E\}$. We use exponents to denote concatenation of a word with itself; for instance, $(NE^3)^2 = NEEENE$. Given a lattice path ν , let $\text{Tam}(\nu)$ be the set of lattice paths lying weakly above ν that have the same endpoints as ν . For example, if $\nu = \text{ENNEEEENNE}$ is the lattice path shown in black in Figure 3.1, then the path $\mu = \text{NENENEENE}$ shown in red is in $\text{Tam}(\nu)$.

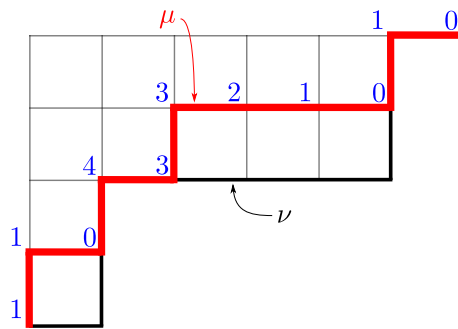


Figure 3.1: The lattice path $\mu = \text{NENENEEENE}$ in the ν -Tamari lattice $\text{Tam}(\nu)$, where $\nu = \text{ENNEEEENNE}$. Each lattice point p is labeled with its horizontal distance.

Suppose $\mu \in \text{Tam}(\nu)$. Given a lattice point p on μ , define the *horizontal distance* of p to be the maximum number of east steps that can be taken, starting at p , before crossing ν ; we illustrate this definition in Figure 3.1. Now suppose p is a lattice point on μ that is preceded by an east step and followed by a north step in μ . Let p' be the first lattice point on μ that appears after p and has the same horizontal distance as p . If we let $D_{[p,p']}$ be the subpath of μ that starts at p and ends at p' , then we can write $\mu = X E D_{[p,p']} Y$ for some lattice paths X and Y . Let $\mu' = X D_{[p,p']} E Y$. Then $\mu' \in \text{Tam}(\nu)$. In this case, we write $\mu < \mu'$. See Figure 3.2.

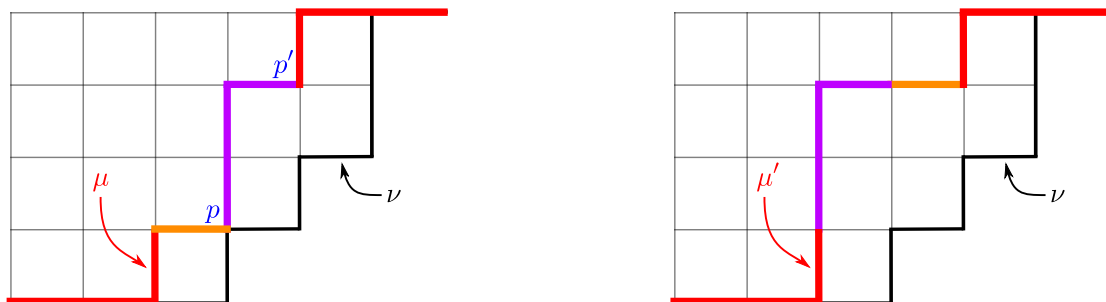


Figure 3.2: The lattice path μ on the left is covered by the lattice path μ' on the right in $\text{Tam}(\nu)$.

The relations $\mu < \mu'$ described in the previous paragraph form the cover relations of a partial order \leq on $\text{Tam}(\nu)$. Préville-Ratelle and Viennot [PRV17] introduced this partial order and proved that it is a lattice. In fact, they proved that $\text{Tam}(\nu)$ is an interval of a Tamari lattice, and it is well known that Tamari lattices are semidistributive. Because intervals of semidistributive lattices are semidistributive, it follows that $\text{Tam}(\nu)$ is semidistributive. This means that we have a rowmotion operator $\text{Row}: \text{Tam}(\nu) \rightarrow \text{Tam}(\nu)$.

When $\nu = (\text{NE}^m)^n$, the lattice $\text{Tam}(\nu)$ is precisely the n -th *m-Tamari lattice*, which we denote by $\text{Tam}_n(m)$. When $\nu = (\text{NE})^n$, $\text{Tam}(\nu)$ is the classical n -th *Tamari lattice*. Given relatively prime positive integers a and b , let $\text{Tam}(a, b)$ be the set of lattice paths from $(0, 0)$ to (a, b) that lie strictly above the line $y = (b/a)x$ except at their endpoints. We have $\text{Tam}(a, b) = \text{Tam}(\nu_{a,b})$ for an appropriately-chosen lattice path $\nu_{a,b}$, so we can view $\text{Tam}(a, b)$

as a lattice. When $a = mn + 1$ and $b = n$, we have $\nu_{a,b} = (\mathbb{N} E^m)^n E$, and it is straightforward to check that $\text{Tam}(a, b)$ is isomorphic to $\text{Tam}_n(m)$. Thomas and Williams stated the following conjecture.

Conjecture 3.1 ([TW19]). If a and b are relatively prime positive integers, then the rowmotion operator $\text{Row}: \text{Tam}(a, b) \rightarrow \text{Tam}(a, b)$ has order $a + b - 1$.

In Theorem 5.13, we will determine the exact orbit structure of rowmotion on $\text{Tam}_n(m)$, thereby settling and extending Conjecture 3.1 in the case where $a = mn + 1$ and $b = n$. We will also formulate a stronger version of Conjecture 3.1 for arbitrary coprime a and b in Conjecture 5.15.

3.2. ν -bracket vectors

There is a very useful alternative description of ν -Tamari lattices due to Ceballos, Padrol, and Sarmiento [CPS20]. In what follows, we make the convention that vectors are 0-indexed. For example, we would say that the number 8 appears in positions 0 and 4 in the vector $(8, 3, 6, 5, 8, 6)$. Our convention is that b_i always denotes the entry in position i of a vector \mathbf{b} .

Definition 3.2. Fix a lattice path ν that starts at $(0, 0)$ and ends at $(\ell - n, n)$. Let us write $\mathbf{b}(\nu) = (b_0(\nu), \dots, b_\ell(\nu))$ for the vector obtained by reading the heights (i.e., y -coordinates) of the lattice points on ν in the order they appear in ν . For $0 \leq k \leq n$, let f_k be the maximum index such that $b_{f_k}(\nu) = k$. We call f_0, \dots, f_n the *fixed positions* of ν . A *ν -bracket vector* is an integer vector $\mathbf{b} = (b_0, \dots, b_\ell)$ such that:

- (I) $b_{f_k} = k$ for all $k \in [0, n]$;
- (II) $b_i(\nu) \leq b_i \leq n$ for all $i \in [0, \ell]$;
- (III) if $b_i = k$, then $b_j \leq k$ for all $j \in [i + 1, f_k]$.

Remark 3.3. It is often useful to note that condition (III) in Definition 3.2 can be replaced with the condition that \mathbf{b} avoids the pattern 121; this condition means that there do not exist indices $i_1 < i_2 < i_3$ such that $b_{i_1} = b_{i_3} < b_{i_2}$.

Remark 3.4. If \mathbf{b} is a ν -bracket vector, then $b_{f_{k+1}} \geq b_{f_{k+2}} \geq \dots \geq b_{f_{k+1}}$ for each $k \in [-1, n-1]$, where we make the convention $f_{-1} = -1$.

Define the *componentwise* partial order \leq on the set of ν -bracket vectors by the condition that $\mathbf{b} \leq \mathbf{b}'$ if $b_i \leq b'_i$ for all $i \in [0, \ell]$. It is straightforward to check that the componentwise minimum $\min(\mathbf{b}, \mathbf{b}') = (\min\{b_0, b'_0\}, \dots, \min\{b_\ell, b'_\ell\})$ of two ν -bracket vectors \mathbf{b} and \mathbf{b}' is also a ν -bracket vector. In fact, the set of ν -bracket vectors forms a lattice in which the meet operation is given by the componentwise minimum: $\mathbf{b} \wedge \mathbf{b}' = \min(\mathbf{b}, \mathbf{b}')$. Ceballos, Padrol, and Sarmiento [CPS20] showed that this lattice is isomorphic to $\text{Tam}(\nu)$ (using an explicit bijection from the set of ν -bracket vectors to $\text{Tam}(\nu)$). Therefore, by abuse of notation, we will henceforth write $\text{Tam}(\nu)$ to refer to the lattice of ν -bracket vectors with the componentwise order. Similarly, $\text{Tam}_n(m)$ will denote the lattice of ν -bracket vectors when $\nu = (\mathbb{N} E^m)^n$. This convention should not lead to any confusion since we will no longer be dealing with lattice paths. The Hasse diagram of $\text{Tam}_3(2)$ appears in Figure 4.1.

4. Describing Rowmotion on ν -Tamari Lattices

Let us fix a lattice path ν that starts at $(0, 0)$ and ends at $(\ell - n, n)$. Let f_0, \dots, f_n be the fixed positions of ν . As before, $\text{Tam}(\nu)$ denotes the ν -Tamari lattice, which we view as the lattice of ν -bracket vectors under the componentwise order. Recall that there is a rowmotion operator $\text{Row} : \text{Tam}(\nu) \rightarrow \text{Tam}(\nu)$ because $\text{Tam}(\nu)$ is semidistributive.

Recall that b_i always denotes the entry in position i of \mathbf{b} . Let $\mathbf{b} \leftarrow_i t$ be the vector obtained from \mathbf{b} by replacing the entry in position i with t . For instance,

$$(0, 1, 1, 2, 2, 3, 4, 5) \leftarrow_3 5 = (0, 1, 1, 5, 2, 3, 4, 5).$$

Recall that for $k \in [0, n]$, every ν -bracket vector has the entry k in position f_k . For $\mathbf{b} \in \text{Tam}(\nu)$ and $0 \leq i \leq \ell$, we are going to define integers $\zeta_i(\mathbf{b})$ and $\eta_i(\mathbf{b})$; the primary motivation for these definitions is that they will allow us to describe rowmotion on $\text{Tam}(\nu)$ via a step-by-step procedure. The i -th step in this procedure will either do nothing or replace the entry in a ν -bracket vector \mathbf{b} with $\zeta_i(\mathbf{b})$ or $\eta_i(\mathbf{b})$.

For $\mathbf{b} \in \text{Tam}(\nu)$ and $0 \leq i \leq \ell$, let

$$\zeta_i(\mathbf{b}) = \min\{\mathbf{b}_j : j \in [0, i - 1] \text{ and } \mathbf{b}_j \geq \mathbf{b}_i\},$$

with the convention that $\zeta_i(\mathbf{b}) = n$ if $\{\mathbf{b}_j : j \in [0, i - 1] \text{ and } \mathbf{b}_j \geq \mathbf{b}_i\} = \emptyset$. Let

$$\Delta(\mathbf{b}) = \{i \in [0, \ell - 1] : \mathbf{b}_i > \mathbf{b}_{i+1}\}.$$

Suppose $i \in \Delta(\mathbf{b})$, and let $k = \mathbf{b}_{i+1}$. We have $b_i(\nu) = b_{i+1}(\nu) \leq k \leq \mathbf{b}_i - 1$, and it follows from condition (III) in Definition 3.2 that $\mathbf{b}_j \leq k$ for all $j \in [i + 1, f_k]$. Therefore, we can define

$$\eta_i(\mathbf{b}) = \max\{h \in [b_i(\nu), \mathbf{b}_i - 1] : \mathbf{b}_j \leq h \text{ for all } j \in [i + 1, f_h]\}.$$

On the other hand, if $i' \in [0, \ell] \setminus \Delta(\mathbf{b})$, then we let $\eta_{i'}(\mathbf{b}) = \mathbf{b}_{i'}$.

In the following proposition, we write Pop for the pop-stack sorting operator on $\text{Tam}(\nu)$ (see Section 2.3).

Proposition 4.1 ([Def22, Proposition 4.4]). *For each $\mathbf{b} \in \text{Tam}(\nu)$, we have*

$$\text{Pop}(\mathbf{b}) = (\eta_0(\mathbf{b}), \dots, \eta_\ell(\mathbf{b})).$$

The following theorem provides a description of rowmotion on $\text{Tam}(\nu)$; in order to state it, we define maps $\Theta_r : \text{Tam}(\nu) \rightarrow \text{Tam}(\nu)$ according to the following rules:

1. If $r \in \{f_0, \dots, f_n\}$, then $\Theta_r(\mathbf{b}) = \mathbf{b}$ for all $\mathbf{b} \in \text{Tam}(\nu)$.
2. If $r \in \Delta(\mathbf{b})$, then $\Theta_r(\mathbf{b}) = \mathbf{b} \leftarrow_r \eta_r(\mathbf{b})$.
3. If $r \notin \{f_0, \dots, f_n\}$ and $r \notin \Delta(\mathbf{b})$, then $\Theta_r(\mathbf{b}) = \mathbf{b} \leftarrow_r \zeta_r(\mathbf{b})$.

It is straightforward to check that Θ_r does in fact send ν -bracket vectors to ν -bracket vectors. Indeed, (2) says that if b_r is followed by a descent, then it is decreased by at least 1, and the amount by which it is decreased is as small as possible for $\Theta_r(b)$ to be a ν -bracket vector. On the other hand, (3) says that if b_r is not followed by a descent (and r is not a fixed position), then it is increased as much as possible for $\Theta_r(b)$ to be a ν -bracket vector.

Theorem 4.2. *For $b \in \text{Tam}(\nu)$, we have*

$$\text{Row}(b) = (\Theta_\ell \circ \cdots \circ \Theta_1 \circ \Theta_0)(b).$$

The computation of rowmotion in Theorem 4.2 is similar to the computation in *slow motion* that Thomas and Williams introduced for trim lattices in [TW19]. However, our step-by-step procedure is slightly different because we have added additional steps that always “do nothing” (applying Θ_r when r is a fixed position) and we have combined some of the steps from the description in [TW19] (thereby “speeding up” parts of the “slow motion”).

Example 4.3. Let $\nu = (\mathbb{N}E^2)^3$ so that $\text{Tam}(\nu) = \text{Tam}_3(2)$. Let us compute the image of the ν -bracket vector $b = (0, 2, 2, 1, 2, 2, 2, 3, 3, 3)$ under rowmotion using Theorem 4.2. To ease notation, let

$$b^{(\alpha)} = (\Theta_\alpha \circ \cdots \circ \Theta_0)(b).$$

First, $b^{(0)} = \Theta_0(b) = b$ since $0 = f_0$. Next, note that $1 \notin \Delta(b^{(0)})$ and $\zeta_1(b^{(0)}) = n = 3$. Thus,

$$b^{(1)} = \Theta_1(b^{(0)}) = b^{(0)} \leftarrow_1 \zeta_1(b^{(0)}) = b^{(0)} \leftarrow_1 3 = (0, 3, 2, 1, 2, 2, 2, 3, 3, 3).$$

Since $2 \in \Delta(b^{(1)})$ and $\eta_2(b^{(1)}) = 1$, we have

$$b^{(2)} = \Theta_2(b^{(1)}) = b^{(1)} \leftarrow_2 \eta_2(b^{(1)}) = b^{(1)} \leftarrow_2 1 = (0, 3, 1, 1, 2, 2, 2, 3, 3, 3).$$

Now $3 = f_1$, so $b^{(3)} = \Theta_3(b^{(2)}) = b^{(2)}$. Since $4 \notin \Delta(b^{(3)})$ and $\zeta_4(b^{(3)}) = 3$, we have

$$b^{(4)} = \Theta_4(b^{(3)}) = b^{(3)} \leftarrow_4 \zeta_4(b^{(3)}) = b^{(3)} \leftarrow_4 3 = (0, 3, 1, 1, 3, 2, 2, 3, 3, 3).$$

Since $5 \notin \Delta(b^{(4)})$ and $\zeta_5(b^{(4)}) = 3$, we have

$$b^{(5)} = \Theta_5(b^{(4)}) = b^{(4)} \leftarrow_5 \zeta_5(b^{(4)}) = b^{(4)} \leftarrow_5 3 = (0, 3, 1, 1, 3, 3, 2, 3, 3, 3).$$

Then $6 = f_2$, so $b^{(6)} = \Theta_6(b^{(5)}) = b^{(5)}$. Since $7 \notin \Delta(b^{(6)})$, we have

$$b^{(7)} = b^{(6)} \leftarrow_7 \zeta_7(b^{(6)}) = b^{(6)} \leftarrow_7 3 = b^{(6)} = b^{(5)}.$$

Since $8 \notin \Delta(b^{(7)})$, we have

$$b^{(8)} = b^{(7)} \leftarrow_8 \zeta_8(b^{(7)}) = b^{(7)} \leftarrow_8 3 = b^{(7)} = b^{(5)}.$$

Finally, $9 = f_3$, so

$$\text{Row}(b) = b^{(9)} = \Theta_9(b^{(8)}) = b^{(8)} = (0, 3, 1, 1, 3, 3, 2, 3, 3, 3).$$

This fact is illustrated by the arrow from $(0, 2, 2, 1, 2, 2, 2, 3, 3, 3)$ to $(0, 3, 1, 1, 3, 3, 2, 3, 3, 3)$ in Figure 4.1.

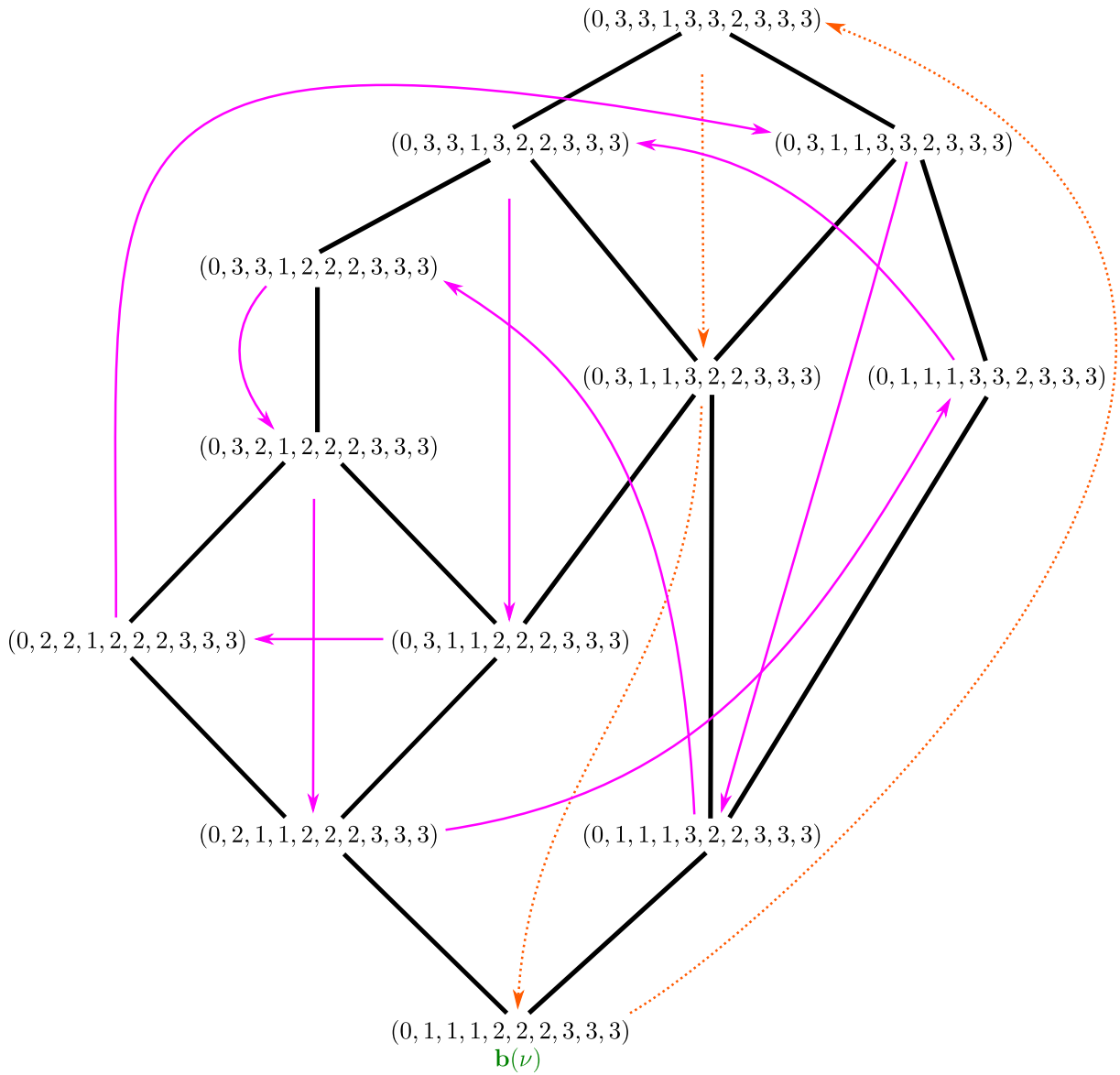


Figure 4.1: Rowmotion on $\text{Tam}_3(2)$. The solid pink and dotted orange arrows indicate the action of rowmotion. Notice that there is one orbit of size 9 (solid pink) and one orbit of size 3 (dotted orange).

Proof of Theorem 4.2. Fix $\mathbf{b} \in \text{Tam}(\nu)$, and let $\text{Row}(\mathbf{b}) = \widehat{\mathbf{b}} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_\ell)$. For each $\alpha \in [0, \ell]$, let $\mathbf{b}^{(\alpha)} = (\Theta_\alpha \circ \dots \circ \Theta_0)(\mathbf{b})$; let us also write $\mathbf{b}^{(-1)} = \mathbf{b}$. We claim that $\mathbf{b}^{(\alpha)} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_\alpha, \mathbf{b}_{\alpha+1}, \dots, \mathbf{b}_\ell)$ for all $\alpha \in [-1, \ell]$; once we prove this claim, setting $\alpha = \ell$ will yield the desired result.

We will prove the desired claim by induction on α ; it is certainly true when $\alpha = -1$, so we may assume that $\alpha \in [0, \ell]$ and that $\mathbf{b}^{(\alpha-1)} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_{\alpha-1}, \mathbf{b}_\alpha, \dots, \mathbf{b}_\ell)$. Since Θ_α can only change the entry in position α of a ν -bracket vector, we just need to show that the entry in position α of the vector $\Theta_\alpha(\mathbf{b}^{(\alpha-1)}) = \mathbf{b}^{(\alpha)}$ is $\widehat{\mathbf{b}}_\alpha$. If $\alpha = f_k$ for some $k \in [0, n]$, then every ν -bracket vector has the entry k in position α , so $\mathbf{b}^{(\alpha)} = \widehat{\mathbf{b}}_\alpha = k$, as desired.

Now suppose $\alpha \in \Delta(\mathbf{b})$. In this case, the definition of Θ_α ensures that $\mathbf{b}^{(\alpha)} = \eta_\alpha(\mathbf{b}^{(\alpha-1)})$. Since $\mathbf{b}^{(\alpha-1)}$ agrees with \mathbf{b} in positions $\alpha, \alpha + 1, \dots, \ell$, we have $\eta_\alpha(\mathbf{b}^{(\alpha-1)}) = \eta_\alpha(\mathbf{b})$. Note that $\eta_\alpha(\mathbf{b}) < \mathbf{b}_\alpha$. Proposition 4.1 tells us that $\eta_\alpha(\mathbf{b})$ is the entry in position α of $\text{Pop}(\mathbf{b})$, and Proposition 2.2 tells us that $\text{Pop}(\mathbf{b}) = \text{Row}(\mathbf{b}) \wedge \mathbf{b} = \widehat{\mathbf{b}} \wedge \mathbf{b}$. Since the meet of two ν -bracket vectors is their componentwise minimum, this shows that $\widehat{\mathbf{b}}_\alpha = \eta_\alpha(\mathbf{b}) = \mathbf{b}^{(\alpha)}$.

Finally, assume that $\alpha \notin \{f_0, \dots, f_n\}$ and $\alpha \notin \Delta(\mathbf{b})$. In this case, the definition of Θ_α ensures that $\mathbf{b}^{(\alpha)} = \zeta_\alpha(\mathbf{b}^{(\alpha-1)})$ is the minimum element of $\{\widehat{\mathbf{b}}_j : j \in [0, \alpha - 1] \text{ and } \widehat{\mathbf{b}}_j \geq \mathbf{b}_\alpha\}$ (or is n if this set is empty). Let $k = \zeta_\alpha(\mathbf{b}^{(\alpha-1)})$; we want to prove that $\widehat{\mathbf{b}}_\alpha = k$.

Suppose by way of contradiction that $\widehat{\mathbf{b}}_\alpha > k$. This implies that $k \neq n$, so there exists $j \in [0, \alpha - 1]$ such that $\widehat{\mathbf{b}}_j = k \geq \mathbf{b}_\alpha$. Then $f_k \geq f_{\mathbf{b}_\alpha} > \alpha$ (the last inequality is strict because $\alpha \notin \{f_0, \dots, f_n\}$). Since $\widehat{\mathbf{b}}_\alpha > k$, the entries $\widehat{\mathbf{b}}_j, \widehat{\mathbf{b}}_\alpha, \widehat{\mathbf{b}}_{f_k}$ form a 121-pattern in $\widehat{\mathbf{b}}$, contradicting Remark 3.3. This shows that $\widehat{\mathbf{b}}_\alpha \leq k$.

Now suppose by way of contradiction that $\widehat{\mathbf{b}}_\alpha < k$. Let $\mathbf{b}' = \widehat{\mathbf{b}} \leftarrow_\alpha k$. We claim that \mathbf{b}' is a ν -bracket vector. It is straightforward to check that \mathbf{b}' satisfies the first two conditions in Definition 3.2, so (by Remark 3.3) we just need to check that \mathbf{b}' does not contain a 121-pattern. Assume instead that \mathbf{b}' contains a 121-pattern in positions $i_1 < i_2 < i_3$. Let $k' = \mathbf{b}'_{i_3}$, and note that $i_3 \leq f_{k'}$ and $\mathbf{b}'_{f_{k'}} = k'$. Therefore, \mathbf{b}' contains a 121-pattern in positions $i_1 < i_2 < f_{k'}$. Since \mathbf{b}' agrees with the ν -bracket vector $\widehat{\mathbf{b}}$ except in position α , one of $i_1, i_2, f_{k'}$ must be α (otherwise, $\widehat{\mathbf{b}}$ would contain a 121-pattern). We have assumed that $\alpha \notin \{f_0, \dots, f_n\}$, so either $\alpha = i_1$ or $\alpha = i_2$. If $\alpha = i_1$, then $k = \mathbf{b}'_\alpha < \mathbf{b}'_{i_2} \leq n$, so (by the definition of k) there exists $j \in [0, \alpha - 1]$ such that $\widehat{\mathbf{b}}_j = k = \widehat{\mathbf{b}}_\alpha$. In this case, $\widehat{\mathbf{b}}$ has a 121-pattern in positions $j < i_2 < f_{k'}$, which is a contradiction. On the other hand, if $\alpha = i_2$, then the ν -bracket vector

$$\mathbf{b}^{(\alpha)} = \Theta_\alpha(\mathbf{b}^{(\alpha-1)}) = \mathbf{b}^{(\alpha-1)} \leftarrow_\alpha k$$

agrees with \mathbf{b}' in positions i_1, i_2 , and $f_{k'}$. However, this means that $\mathbf{b}^{(\alpha)}$ contains a 121-pattern in positions $i_1 < i_2 < f_{k'}$, which is impossible. This proves our claim that \mathbf{b}' is a ν -bracket vector. Now, $\min\{\mathbf{b}'_\alpha, \mathbf{b}_\alpha\} = \min\{k, \mathbf{b}_\alpha\} = \mathbf{b}_\alpha = \eta_\alpha(\mathbf{b})$, where the last equality comes from the fact that $\alpha \notin \Delta(\mathbf{b})$. For $i \in [0, \ell] \setminus \{\alpha\}$, we can use Propositions 2.2 and 4.1 to see that

$$\min\{\mathbf{b}'_i, \mathbf{b}_i\} = \min\{\widehat{\mathbf{b}}_i, \mathbf{b}_i\} = (\text{Pop}(\mathbf{b}))_i = \eta_i(\mathbf{b}).$$

This proves that $\mathbf{b}' \wedge \mathbf{b} = (\eta_0(\mathbf{b}), \dots, \eta_\ell(\mathbf{b})) = \text{Pop}(\mathbf{b})$ (using Proposition 4.1 for the last equality), which contradicts Proposition 2.2 because $\mathbf{b}' > \widehat{\mathbf{b}}$. \square

The following corollary is immediate from the proof of Theorem 4.2.

Corollary 4.4. *If $\mathbf{b} \in \text{Tam}(\nu)$ and $i \in \Delta(\mathbf{b})$, then $(\text{Row}(\mathbf{b}))_i = \eta_i(\mathbf{b}) < \mathbf{b}_i$.*

Before we proceed, let us record a simple lemma that will be useful later.

Lemma 4.5. *Suppose $\mathbf{b} \in \text{Tam}(\nu)$ and $i \in \Delta(\mathbf{b})$, and let $z = \eta_i(\mathbf{b})$. We have $\mathbf{b}_{f_z+1} = \mathbf{b}_i$.*

Proof. Let $k = f_z + 1$ and $h = \mathbf{b}_k$. Note that $h \geq b_k(\nu) = z + 1$. The definition of $\eta_i(\mathbf{b})$ tells us that $\mathbf{b}_j \leq z$ for all $j \in [i + 1, f_z]$. We have $\mathbf{b}_{f_z+1} = h$, and condition (III) in Definition 3.2 implies that $\mathbf{b}_j \leq h$ for all $j \in [f_z + 2, f_h]$. Thus, $\mathbf{b}_j \leq h$ for all $j \in [i + 1, f_h]$. It follows from the maximality in the definition of $\eta_i(\mathbf{b})$ that h cannot satisfy $z < h < \mathbf{b}_i$. We know that $z < h$, so $h \geq \mathbf{b}_i$. If we had $h > \mathbf{b}_i$, then the entries of \mathbf{b} in positions i, k, f_{b_i} would form a 121 pattern, contradicting Remark 3.3. \square

5. m -Tamari Lattices

In this section, we fix positive integers m and n and focus our attention on the m -Tamari lattice $\text{Tam}_n(m)$, which is the lattice $\text{Tam}(\nu)$ for $\nu = (\mathbb{N} E^m)^n$. It is a classical result [DM47] that the cardinality of $\text{Tam}_n(m)$ is the *Fuss–Catalan number*

$$\text{Cat}_n^{(m)} = \frac{1}{(m + 1)n + 1} \binom{(m + 1)n + 1}{n}.$$

Let $[k]_q! = [k]_q[k - 1]_q \cdots [1]_q$, where $[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{k-1}$ is the q -analogue of the integer k . The *q -binomial coefficient* $\begin{bmatrix} k \\ r \end{bmatrix}_q$ is defined to be $\frac{[k]_q!}{[r]_q![k - r]_q!}$. There is a natural q -analogue of the Fuss–Catalan number $\text{Cat}_n^{(m)}$ given by

$$\text{Cat}_n^{(m)}(q) = \frac{1}{[(m + 1)n + 1]_q} \begin{bmatrix} (m + 1)n + 1 \\ n \end{bmatrix}_q.$$

We are going to completely describe the orbit structure of $\text{Row}: \text{Tam}_n(m) \rightarrow \text{Tam}_n(m)$ by showing that the triple $(\text{Tam}_n(m), \text{Row}, \text{Cat}_n^{(m)}(q))$ exhibits the cyclic sieving phenomenon (see Section 2.4).

Consider a partition ρ of a totally ordered set X . We say two distinct blocks B_1 and B_2 in ρ form a *crossing* if there exist $i_1, j_1 \in B_1$ and $i_2, j_2 \in B_2$ such that either $i_1 < i_2 < j_1 < j_2$ or $i_1 > i_2 > j_1 > j_2$. We say ρ is *noncrossing* if no two distinct blocks in ρ form a crossing. Let $\text{NC}(k)$ be the set of noncrossing partitions of $[k]$.

We are going to make use of the *Kreweras complementation* map $\text{Krew}: \text{NC}(k) \rightarrow \text{NC}(k)$. Given a noncrossing partition ρ of the set $[k]$, let $K'(\rho)$ be the coarsest partition of the set $\{1', \dots, k'\}$ such that $\rho \cup K'(\rho)$ is a noncrossing partition of the totally ordered set $\{1 < 1' < 2 < 2' < \cdots < k < k'\}$. Then let $\text{Krew}(\rho)$ be the noncrossing partition of $[k]$ obtained from $K'(\rho)$ by removing the primes from the elements of $\{1', \dots, k'\}$. For example, if

$$\rho = \{\{1, 2, 3\}, \{4, 8, 12\}, \{5, 6, 7\}, \{9, 10, 11\}\} \in \text{NC}(12)$$

is as shown in pink in Figure 5.1, then $K'(\rho)$ is the partition shown in orange in Figure 5.1, so

$$\text{Krew}(\rho) = \{\{1\}, \{2\}, \{3, 12\}, \{4, 7\}, \{5\}, \{6\}, \{8, 11\}, \{9\}, \{10\}\} \in \text{NC}(12).$$

Kreweras complementation is a bijection.

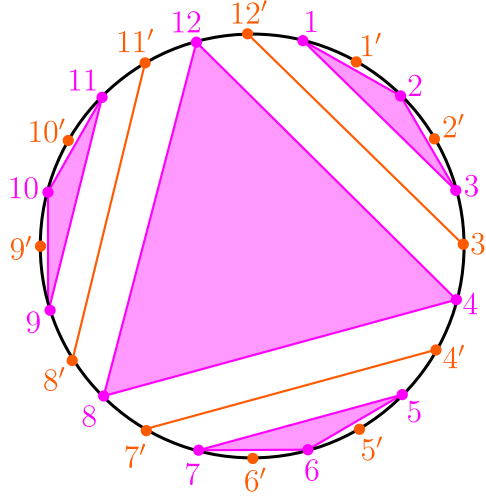


Figure 5.1: A noncrossing partition $\rho \in \text{NC}(12)$ (pink) together with $K'(\rho)$ (orange).

Following [BR16] we say a partition $\rho \in \text{NC}((m+1)n)$ is $(m+1)$ -homogeneous if each of its blocks has size $m+1$ (such partitions are sometimes called $(m+1)$ -equal). Let $\text{HNC}^{(m)}(n)$ denote the set of $(m+1)$ -homogeneous noncrossing partitions of $[(m+1)n]$. Following [Arm09], we say $\rho \in \text{NC}((m+1)n)$ is $(m+1)$ -shuffle if it has the property that $i \equiv j \pmod{m+1}$ whenever i and j are in the same block of ρ . Let $\text{ShNC}^{(m)}(n)$ denote the set of $(m+1)$ -shuffle noncrossing partitions of $[(m+1)n]$ that have exactly $mn+1$ blocks. One can readily check (see also [Arm09, Section 4.3]) that

$$\text{Krew}(\text{HNC}^{(m)}(n)) = \text{ShNC}^{(m)}(n) \quad \text{and} \quad \text{Krew}(\text{ShNC}^{(m)}(n)) = \text{HNC}^{(m)}(n).$$

For example, the partition ρ shown in pink in Figure 5.1 is 3-homogeneous, while its Kreweras complement $\text{Krew}(\rho)$ is 3-shuffle.

There is a natural *rotation* operator $\text{Rot}: \text{NC}((m+1)n) \rightarrow \text{NC}((m+1)n)$ that acts by cyclically rotating the blocks of a partition one space to the left. More precisely, for $\rho \in \text{NC}((m+1)n)$, two integers i and j lie in the same block of ρ if and only if $i-1$ and $j-1$ lie in the same block of $\text{Rot}(\rho)$, where the integers are taken modulo $(m+1)n$. It is well known that $\text{Rot} = \text{Krew}^2$. Notice that Rot preserves each of the sets $\text{HNC}^{(m)}(n)$ and $\text{ShNC}^{(m)}(n)$.

Bodnar and Rhoades [BR16] proved that the triple $(\text{HNC}^{(m)}(n), \text{Rot}, \text{Cat}_n^{(m)}(q))$ exhibits the cyclic sieving phenomenon. Since Krew is a bijection from $\text{HNC}^{(m)}(n)$ to $\text{ShNC}^{(m)}(n)$ satisfying

$$\text{Rot} \circ \text{Krew} = \text{Krew} \circ \text{Rot},$$

it follows that

$$(\text{ShNC}^{(m)}(n), \text{Rot}, \text{Cat}_n^{(m)}(q)) \text{ exhibits the cyclic sieving phenomenon.} \tag{5.1}$$

Therefore, in order to prove the main result of this section, it suffices to construct a bijection $\Psi: \text{Tam}_n(m) \rightarrow \text{ShNC}^{(m)}(n)$ satisfying $\Psi \circ \text{Row} = \text{Rot} \circ \Psi$.

Now let $\mathbf{b} \in \text{Tam}_n(m)$ be a ν -bracket vector, where $\nu = (\text{NE}^m)^n$. The fixed positions f_0, \dots, f_n of ν are given by $f_k = (m + 1)k$, so $\mathbf{b}_{(m+1)k} = k$ for all $k \in [0, n]$. Furthermore, $\mathbf{b}_{(m+1)s+t}(\nu) = s + 1$ for all $s \in [0, n - 1]$ and $t \in [m + 1]$. Recall from Section 4 that $\Delta(\mathbf{b})$ is the set of indices $i \in [0, (m + 1)n - 1]$ such that $\mathbf{b}_i > \mathbf{b}_{i+1}$; it follows from Definition 3.2 that none of the fixed positions $(m + 1)k$ belong to $\Delta(\mathbf{b})$. Let us say an index $i \in [(m + 1)n]$ is *k-initial* in \mathbf{b} if $\mathbf{b}_i = k$ and $\mathbf{b}_j \neq k$ for all $j \in [0, i - 1]$.

If ρ is a partition of the set $[(m + 1)n]$ and $i \in [(m + 1)n]$ is an integer that is not maximal in its block in ρ , we define the *successor* of i in ρ to be the smallest integer that lies in the same block as i and is greater than i . We are going to define the set partition $\Psi(\mathbf{b})$ by specifying successors of various integers. To begin, we define a function $\varphi_{\mathbf{b}}: [(m + 1)n] \rightarrow [(m + 1)n] \cup \{\emptyset\}$. (To motivate this definition, the reader may wish to look ahead at Lemma 5.3 to see how $\varphi_{\mathbf{b}}$ will be used in defining $\Psi(\mathbf{b})$.) Suppose $i \in [(m + 1)n]$; there are three cases to consider.

Case 1 If $i < (m + 1)n$ is a multiple of $m + 1$ and $i + 1$ is k -initial in \mathbf{b} for some k , then we define $\varphi_{\mathbf{b}}(i) = f_k = (m + 1)k$.

Case 2 Suppose $i \in \Delta(\mathbf{b})$ (in particular, $i \not\equiv 0 \pmod{m + 1}$). In this case, we define $\varphi_{\mathbf{b}}(i)$ to be the smallest integer $j \in [(m + 1)n]$ satisfying $j > i$, $j \equiv i \pmod{m + 1}$, and $\mathbf{b}_i = \mathbf{b}_j$. To see that such an integer exists, let us write $i = (m + 1)s + t$, where $0 \leq s \leq n - 1$ and $1 \leq t \leq m$. If we let $k = \mathbf{b}_{(m+1)s+t}$, then (by the definition of $\Delta(\mathbf{b})$) we have $k > \mathbf{b}_{(m+1)s+t+1} \geq \mathbf{b}_{(m+1)s+t+1}(\nu) = s + 1$. Then $(m + 1)s + t < (m + 1)(k - 1) + t < (m + 1)k = f_k$, so it follows from condition (III) in Definition 3.2 that $\mathbf{b}_{(m+1)(k-1)+t} \leq k$. We also have $\mathbf{b}_{(m+1)(k-1)+t} \geq \mathbf{b}_{(m+1)(k-1)+t}(\nu) = k$, so $\mathbf{b}_{(m+1)(k-1)+t} = k$. This shows that $(m + 1)(k - 1) + t$ satisfies the conditions defining j (though it might not be the smallest integer with these properties).

Case 3 If we are not in Case 1 or Case 2, then we simply define $\varphi_{\mathbf{b}}(i) = \emptyset$.

Example 5.1. Suppose $m = 2$, $n = 4$, and $\mathbf{b} = (0, 4, 1, 1, 3, 3, 2, 3, 3, 3, 4, 4, 4)$. Then $\varphi_{\mathbf{b}}(1) = 10$, $\varphi_{\mathbf{b}}(3) = 9$, $\varphi_{\mathbf{b}}(5) = 8$, and $\varphi_{\mathbf{b}}(i) = \emptyset$ for all $i \in [12] \setminus \{1, 3, 5\}$.

Lemma 5.2. *The map $\varphi_{\mathbf{b}}: [(m + 1)n] \rightarrow [(m + 1)n] \cup \{\emptyset\}$ is injective on $[(m + 1)n] \setminus \varphi_{\mathbf{b}}^{-1}(\emptyset)$.*

Proof. Suppose that $i, i' \in [(m + 1)n] \setminus \varphi_{\mathbf{b}}^{-1}(\emptyset)$ are such that $\varphi_{\mathbf{b}}(i) = \varphi_{\mathbf{b}}(i')$. Without loss of generality, assume $i \leq i'$. It follows from the definition of $\varphi_{\mathbf{b}}$ that $i \equiv \varphi_{\mathbf{b}}(i) \equiv i' \pmod{m + 1}$. If $i \equiv i' \equiv 0 \pmod{m + 1}$ (i.e., i and i' both fall into Case 1 in the definition of $\varphi_{\mathbf{b}}$), then $\varphi_{\mathbf{b}}(i) = \varphi_{\mathbf{b}}(i') = (m + 1)k$ for some k . In this case, $i + 1$ and $i' + 1$ are both k -initial in \mathbf{b} , so we must have $i = i'$. If $i \equiv i' \not\equiv 0 \pmod{m + 1}$ (i.e., i and i' both fall into Case 2 in the definition of $\varphi_{\mathbf{b}}$), then $\mathbf{b}_i = \mathbf{b}_{i'} = \mathbf{b}_{\varphi_{\mathbf{b}}(i)}$ and $i \leq i' < \varphi_{\mathbf{b}}(i)$, so the minimality in the definition of $\varphi_{\mathbf{b}}(i)$ forces us to have $i = i'$. □

Lemma 5.3. *Let $\mathbf{b} \in \text{Tam}_n(m)$. There is a unique set partition $\Psi(\mathbf{b})$ of $[(m+1)n]$ such that the following conditions hold for all $i \in [(m+1)n]$:*

- *We have $\varphi_{\mathbf{b}}(i) = \emptyset$ if and only if i is the largest element in its block in $\Psi(\mathbf{b})$.*
- *If $\varphi_{\mathbf{b}}(i) \neq \emptyset$, then $\varphi_{\mathbf{b}}(i)$ is the successor of i in $\Psi(\mathbf{b})$.*

Proof. Let $\Psi(\mathbf{b})$ be the finest partition of $[(m+1)n]$ such that i and $\varphi_{\mathbf{b}}(i)$ are in the same block for every $i \in [(m+1)n] \setminus \varphi_{\mathbf{b}}^{-1}(\emptyset)$. Choose a block B of $\Psi(\mathbf{b})$, and let $p = \min B$. Let h be the smallest positive integer such that $\varphi_{\mathbf{b}}^h(p) = \emptyset$. The definition of $\varphi_{\mathbf{b}}$ guarantees that $\varphi_{\mathbf{b}}(j) > j$ for all $j \in [(m+1)n] \setminus \varphi_{\mathbf{b}}^{-1}(\emptyset)$. It follows that $B = \{p < \varphi_{\mathbf{b}}(p) < \varphi_{\mathbf{b}}^2(p) < \cdots < \varphi_{\mathbf{b}}^{h-1}(p)\}$. This proves that the largest element of B is in $\varphi_{\mathbf{b}}^{-1}(\emptyset)$ and that $\varphi_{\mathbf{b}}(i)$ is the successor of i for every non-maximal element i of B . We have proven the existence of the partition $\Psi(\mathbf{b})$ with the desired properties; the uniqueness is immediate. \square

The preceding lemma provides us with a map Ψ that sends ν -bracket vectors in $\text{Tam}_n(m)$ to set partitions of $[(m+1)n]$.

Example 5.4. Let $m = 2$, $n = 4$, and $\mathbf{b} = (0, 4, 1, 1, 3, 3, 2, 3, 3, 3, 4, 4, 4)$; we computed the function $\varphi_{\mathbf{b}}$ in Example 5.1. The set partition described in Lemma 5.3 is

$$\Psi(\mathbf{b}) = \{\{1, 10\}, \{2\}, \{3, 9\}, \{4\}, \{5, 8\}, \{6\}, \{7\}, \{11\}, \{12\}\}.$$

Lemma 5.5. *If $\mathbf{b} \in \text{Tam}_n(m)$, then $\Psi(\mathbf{b})$ has exactly $mn + 1$ blocks.*

Proof. According to the description of $\Psi(\mathbf{b})$ in Lemma 5.3, our goal is to show that

$$|\varphi_{\mathbf{b}}^{-1}(\emptyset)| = mn + 1.$$

For each $k \in [n]$, let g_k be the smallest integer such that $\mathbf{b}_{g_k} = k$ (i.e., g_k is the unique integer that is k -initial in \mathbf{b}). Let $a \in [n]$ be the integer such that $g_a = 1$. We will prove that

$$\varphi_{\mathbf{b}}^{-1}(\emptyset) = [(m+1)n] \setminus \{g_k - 1 : k \in [n] \setminus \{a\}\}.$$

To this end, we first observe that $(m+1)n \in \varphi_{\mathbf{b}}^{-1}(\emptyset)$.

Fix $k \in [n]$, and let r be the number of occurrences of k in \mathbf{b} . Let $i_1 < \cdots < i_r$ be the indices such that $\mathbf{b}_{i_1} = \cdots = \mathbf{b}_{i_r} = k$; thus, $i_1 = g_k$. Consider $t \in [2, r]$. If $i_t \equiv 1 \pmod{m+1}$, then $\varphi_{\mathbf{b}}(i_t - 1) = \emptyset$ because i_t is not k -initial in \mathbf{b} . If $i_t \not\equiv 1 \pmod{m+1}$, then the observation that $\mathbf{b}_{i_{t-1}} = \mathbf{b}_{i_t}$ implies (via Remark 3.3) that $\mathbf{b}_{i_{t-1}} \leq \mathbf{b}_{i_t}$, so $i_t - 1 \notin \Delta(\mathbf{b})$. In either case, $i_t - 1 \in \varphi_{\mathbf{b}}^{-1}(\emptyset)$. We now want to show that $i_1 - 1 \notin \varphi_{\mathbf{b}}^{-1}(\emptyset)$. This is obvious if $k = a$ (i.e., if $i_1 - 1 = 0$), so we may assume $k \neq a$. If $i_1 \equiv 1 \pmod{m+1}$, then $\varphi_{\mathbf{b}}(i_1 - 1) = f_k$ because i_1 is k -initial in \mathbf{b} , so $i_1 - 1 \notin \varphi_{\mathbf{b}}^{-1}(\emptyset)$. If $i_1 \not\equiv 1 \pmod{m+1}$, then we can use Remark 3.4 to see that $\mathbf{b}_{i_1-1} \geq \mathbf{b}_{i_1}$. In this case, we must actually have $\mathbf{b}_{i_1-1} > \mathbf{b}_{i_1}$ since i_1 is k -initial in \mathbf{b} , so $i_1 - 1 \in \Delta(\mathbf{b})$. This shows that $i_1 - 1 \notin \varphi_{\mathbf{b}}^{-1}(\emptyset)$ in this case as well. \square

Proposition 5.6. *If $\mathbf{b} \in \text{Tam}_n(m)$, then $\Psi(\mathbf{b}) \in \text{ShNC}^{(m)}(n)$.*

Proof. The definition of φ_b ensures that $i \equiv \varphi_b(i) \pmod{m+1}$ for all $i \in [(m+1)n] \setminus \varphi_b^{-1}(\emptyset)$. Hence, it follows from the description of $\Psi(b)$ given in Lemma 5.3 that any two numbers in the same block of $\Psi(b)$ must be congruent modulo $m+1$. We also know by Lemma 5.5 that $\Psi(b)$ has exactly $mn+1$ blocks. We are left to prove that $\Psi(b)$ is noncrossing.

Suppose by way of contradiction that $\Psi(b)$ is not noncrossing. This implies that there are indices $i_1 < i_2 < j_1 < j_2$ such that j_1 is the successor of i_1 and j_2 is the successor of i_2 . Then $j_1 = \varphi_b(i_1)$ and $j_2 = \varphi_b(i_2)$ by the description of $\Psi(b)$ in Lemma 5.3.

Let us first consider the case in which neither i_1 nor i_2 is divisible by $m+1$. This implies that $i_1, i_2 \in \Delta(b)$ and that $b_{i_1} = b_{j_1}$ and $b_{i_2} = b_{j_2}$. If we had $b_{i_2} > b_{i_1}$, then $b_{i_1}, b_{i_2}, b_{j_1}$ would form a 121-pattern in b , contradicting Remark 3.3. If we had $b_{i_2} < b_{i_1}$, then $b_{i_2}, b_{j_1}, b_{j_2}$ would form a 121-pattern in b , again contradicting Remark 3.3. Therefore, it must actually be the case that $b_{i_1} = b_{i_2} = b_{j_1} = b_{j_2}$. Let us write

$$\begin{aligned} i_1 &= (m+1)s_1 + t_1, & i_2 &= (m+1)s_2 + t_2, \\ j_1 &= (m+1)s'_1 + t_1, & j_2 &= (m+1)s'_2 + t_2 \end{aligned}$$

where $0 \leq s_1 \leq s_2 \leq s'_1 \leq s'_2 \leq n-1$ and $t_1, t_2 \in [m]$. Because $i_1 \in \Delta(b)$, we have $b_{(m+1)s_1+t_1} > b_{(m+1)s_1+t_1+1} \geq b_{(m+1)s_1+r}$ for all $r \in [t_1+1, m]$ by Remark 3.4. This implies that $s_1 < s_2$. Similarly, the fact that $i_2 \in \Delta(b)$ implies that $s_2 < s'_1$. If $t_1 \leq t_2$, then we can use Remark 3.3 to see that $b_{(m+1)s_2+t_1} \leq b_{(m+1)s_2+t_2}$ (since $b_{(m+1)s_1+t_1}, b_{(m+1)s_2+t_1}, b_{(m+1)s_2+t_2}$ cannot form a 121-pattern), and we can use Remark 3.4 to see that $b_{(m+1)s_2+t_1} \geq b_{(m+1)s_2+t_2}$. However, this means that $b_{(m+1)s_2+t_1} = b_{i_1}$, which contradicts the minimality in the definition of $\varphi_b(i_1)$ because $(m+1)s_2+t_1 < j_1 = \varphi_b(i_1)$. On the other hand, if $t_2 < t_1$, then we can use Remark 3.3 to see that $b_{(m+1)s'_1+t_2} \leq b_{(m+1)s'_1+t_1}$ (since $b_{(m+1)s_2+t_2}, b_{(m+1)s'_1+t_2}, b_{(m+1)s'_1+t_1}$ cannot form a 121-pattern), and we can use Remark 3.4 to see that $b_{(m+1)s'_1+t_2} \geq b_{(m+1)s'_1+t_1}$. However, this means that $b_{(m+1)s'_1+t_2} = b_{i_2}$, which contradicts the minimality in the definition of $\varphi_b(i_2)$ because $(m+1)s'_1+t_2 < j_2 = \varphi_b(i_2)$. This yields the desired contradiction when neither i_1 nor i_2 is divisible by $m+1$.

We now consider the case where i_1 is divisible by $m+1$. We have $j_1 = \varphi_b(i_1) = f_k$, where $k = b_{i_1+1}$. We know by Definition 3.2 that $b_{j_2} \geq b_{j_2}(\nu) > k$ because $j_2 > f_k$. Condition (III) in Definition 3.2 tells us that $b_{i_2} \leq k$ since $i_2 \in [i_1+1, f_k]$. This shows that $b_{i_2} \neq b_{j_2} = b_{\varphi_b(i_2)}$, so it follows from the definition of φ_b that i_2 is divisible by $m+1$. Thus, $i_1 \equiv i_2 \equiv 0 \pmod{m+1}$. We have $i_1+1 < i_2+1 < j_1$ and $b_{i_1+1} = b_{j_1} = k$, so Remark 3.3 tells us that $b_{i_2+1} \leq k$. We have $i_2+1 < j_1 < j_2$ and $b_{i_2+1} = b_{j_2}$, so Remark 3.3 tells us that $k = b_{j_1} \leq b_{i_2+1}$. Therefore, $k = b_{i_2+1} = b_{j_2}$. However, this is impossible because $j_2 > j_1 = f_k$.

Finally, consider the case where i_2 is divisible by $m+1$ and i_1 is not. Let $k' = b_{i_2+1}$. Then i_2+1 is k' -initial in b , and $k' = b_{i_2+1} = b_{\varphi_b(i_2)} = b_{j_2}$. Since $i_2+1 \leq j_1 < j_2$, it follows from Remark 3.3 that $b_{j_1} \leq b_{i_2+1} = k'$. Because i_1 is not divisible by $m+1$, we must have $b_{i_1} = b_{\varphi_b(i_1)} = b_{j_1}$. Since $i_1 < i_2+1 \leq j_1$, Remark 3.3 tells us that $k' = b_{i_2+1} \leq b_{j_1}$. This shows that $b_{i_1} = b_{j_1} = k'$, which is a contradiction because i_2+1 is k' -initial in b . \square

Our next goal is to show that the map $\Psi: \text{Tam}_n(m) \rightarrow \text{ShNC}^{(m)}(n)$ is a bijection. To do so, we require the following simple lemma about noncrossing partitions.

Lemma 5.7. *If $\rho = \{B_1, \dots, B_r\}$ and $\rho' = \{B'_1, \dots, B'_r\}$ are noncrossing partitions of a totally ordered set X such that*

$$\{\min B_1, \dots, \min B_r\} = \{\min B'_1, \dots, \min B'_r\}$$

and

$$\{\max B_1, \dots, \max B_r\} = \{\max B'_1, \dots, \max B'_r\},$$

then $\rho = \rho'$.

Proof. The statement is trivial if $r = 1$, so we may assume $r \geq 2$ and proceed by induction on r . We may assume that $X = [k]$ for some positive integer k . There exist $u \in \{\min B_1, \dots, \min B_r\}$ and $v \in \{\max B_1, \dots, \max B_r\}$ such that the set $[u + 1, v - 1]$ is disjoint from both

$$\{\min B_1, \dots, \min B_r\} \quad \text{and} \quad \{\max B_1, \dots, \max B_r\}.$$

We must have $[u, v] = B_i = B'_j$ for some $i, j \in [r]$. If we remove B_i from ρ and B'_j from ρ' , we obtain noncrossing partitions of $[k] \setminus [u, v]$. We can apply the inductive hypothesis to find that these noncrossing partitions are equal, so $\rho = \rho'$. \square

Proposition 5.8. *The map $\Psi: \text{Tam}_n(m) \rightarrow \text{ShNC}^{(m)}(n)$ constructed above is a bijection.*

Proof. We already know by [Arm09, Section 4.3] and [DM47] that

$$|\text{Tam}_n(m)| = |\text{ShNC}^{(m)}(n)| = \text{Cat}_n^{(m)},$$

so it suffices to prove that Ψ is injective. Suppose $\mathbf{b}, \mathbf{b}' \in \text{Tam}_n(m)$ are such that $\Psi(\mathbf{b}) = \Psi(\mathbf{b}')$. For each $k \in [n]$, let B_k (respectively, B'_k) be the set of indices $i \in [(m + 1)n]$ such that $\mathbf{b}_i = k$ (respectively, $\mathbf{b}'_i = k$). Let $\rho = \{B_1, \dots, B_n\}$ and $\rho' = \{B'_1, \dots, B'_n\}$. It follows from Remark 3.3 that ρ and ρ' are noncrossing partitions of $[(m + 1)n]$. Moreover, $\max B_k = \max B'_k = (m + 1)k$ for all $k \in [n]$, so $\{\max B_1, \dots, \max B_n\} = \{\max B'_1, \dots, \max B'_n\}$. Observe that 1 is in both $\{\min B_1, \dots, \min B_n\}$ and $\{\min B'_1, \dots, \min B'_n\}$. Now suppose $t \in [2, (m + 1)n]$. We saw in the proof of Lemma 5.5 that $t \in \{\min B_1, \dots, \min B_n\}$ if and only if $t + 1$ has a successor in $\Psi(\mathbf{b})$ (equivalently, $\varphi_{\mathbf{b}}(t + 1) \neq \emptyset$). Similarly, $t \in \{\min B'_1, \dots, \min B'_n\}$ if and only if $t + 1$ has a successor in $\Psi(\mathbf{b}')$. Since $\Psi(\mathbf{b}) = \Psi(\mathbf{b}')$, this shows that

$$\{\min B_1, \dots, \min B_n\} = \{\min B'_1, \dots, \min B'_n\}.$$

Lemma 5.7 tells us that $\rho = \rho'$, from which it follows that $\mathbf{b} = \mathbf{b}'$. \square

In order to complete the proof of the main result of this section, we still need to show that $\Psi \circ \text{Row} = \text{Rot} \circ \Psi$. Let $\ell = (m + 1)n$, and recall the definition of the maps $\Theta_0, \dots, \Theta_\ell$ from Section 4. In the following lemmas, we will find it convenient to write

$$\mathbf{b}^{(\alpha)} = (\Theta_\alpha \circ \dots \circ \Theta_0)(\mathbf{b});$$

observe that $\mathbf{b}^{(\ell)} = \text{Row}(\mathbf{b})$ by Theorem 4.2. Also, each of the vectors $\mathbf{b}^{(\alpha)}$ is a ν -bracket vector since each of the maps Θ_j sends ν -bracket vectors to ν -bracket vectors.

Lemma 5.9. *Let $\mathbf{b} \in \text{Tam}_n(m)$. Suppose $i, j \in [(m+1)n]$ are such that $i \equiv j \equiv 1 \pmod{m+1}$ and $\varphi_{\mathbf{b}}(i) = j$. If $j = (m+1)u + 1$, then i is u -initial in $\text{Row}(\mathbf{b})$.*

Proof. Let us write $\text{Row}(\mathbf{b}) = \widehat{\mathbf{b}} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_\ell)$. Let $i = (m+1)s + 1$. Since $\varphi_{\mathbf{b}}(i) = j \neq \emptyset$, we must have $i \in \Delta(\mathbf{b})$. Therefore, Corollary 4.4 tells us that $\widehat{\mathbf{b}}_i = z < \mathbf{b}_i$, where $z = \eta_i(\mathbf{b})$. In order to show that i is u -initial in $\widehat{\mathbf{b}}$, we need to show that $z = u$ and that $z \notin \{\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_{i-1}\}$.

The definition of $\eta_i(\mathbf{b})$ ensures that $\mathbf{b}_r \leq z$ for all $i \leq r \leq f_z$. Since $\mathbf{b}_j = \mathbf{b}_i > z$, this implies that $j > f_z$. But $j = f_u + 1$, so $u \geq z$. Suppose by way of contradiction that $u > z$. Then $i < f_z + 1 < j$, and Lemma 4.5 tells us that $\mathbf{b}_{f_z+1} = \mathbf{b}_i$. This contradicts the minimality in the definition of $\varphi_{\mathbf{b}}(i)$, so we conclude that $u = z$.

Suppose $\widehat{\mathbf{b}}_r = z$ for some $r \in [0, i-1]$. It follows from Theorem 4.2 that $\mathbf{b}_r^{(i-1)} = \widehat{\mathbf{b}}_r$, so $\mathbf{b}_r^{(i-1)} = z = u$ and $\mathbf{b}_i^{(i-1)} = \mathbf{b}_i = \mathbf{b}_j \geq \mathbf{b}_j(\nu) = u+1$; however, this contradicts condition (III) in Definition 3.2 (applied to $\mathbf{b}^{(i-1)}$) because $r+1 \leq i \leq f_u$. \square

Lemma 5.10. *Let $\mathbf{b} \in \text{Tam}_n(m)$, and suppose $i, j \in [2, (m+1)n]$ are such that $\varphi_{\mathbf{b}}(i) = j$. Then $\varphi_{\text{Row}(\mathbf{b})}(i-1) = j-1$.*

Proof. Let $\text{Row}(\mathbf{b}) = \widehat{\mathbf{b}} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_\ell)$. It follows from the definition of $\varphi_{\mathbf{b}}$ that $i \equiv j \pmod{m+1}$ and $i < j$. If $i \equiv j \equiv 1 \pmod{m+1}$, then Lemma 5.9 tells us that i is u -initial in $\widehat{\mathbf{b}}$, where u is such that $j = (m+1)u + 1 = f_u + 1$. In this case, it follows from the definition of $\varphi_{\widehat{\mathbf{b}}}$ that $\varphi_{\widehat{\mathbf{b}}}(i-1) = j-1$, as desired.

In the remainder of the proof, we may assume $i \equiv j \not\equiv 1 \pmod{m+1}$. Keep in mind that

$$\mathbf{b}^{(\alpha)} = (\widehat{\mathbf{b}}_0, \dots, \widehat{\mathbf{b}}_\alpha, \mathbf{b}_{\alpha+1}, \dots, \mathbf{b}_\ell)$$

for all $\alpha \in [0, \ell]$ by Theorem 4.2 (and the fact that each map Θ_j can only change the entry in position j of a ν -bracket vector). We now consider two cases.

Case 1 Assume $i \equiv j \not\equiv 0, 1 \pmod{m+1}$. Let $x = \mathbf{b}_i$ and $y = \widehat{\mathbf{b}}_{i-1}$. The definition of $\varphi_{\mathbf{b}}$ implies that $\mathbf{b}_j = x$. Invoking Remark 3.3, we find that

$$\mathbf{b}_r \leq x \quad \text{for all } r \in [i, j]. \tag{5.2}$$

We have $y = \mathbf{b}_{i-1}^{(i-1)} \geq \mathbf{b}_i^{(i-1)} = \mathbf{b}_i = x$ by Remark 3.4.

We are going to prove by induction on α that

$$\widehat{\mathbf{b}}_\alpha \notin [x, y-1] \quad \text{for all } \alpha \in [0, j-1]. \tag{5.3}$$

If $\alpha \leq i-2$, then this follows from condition (III) in Definition 3.2 since, otherwise, we would have $\alpha+1 \leq i-1 \leq f_{\widehat{\mathbf{b}}_\alpha}$ and $\widehat{\mathbf{b}}_{i-1} = y > \widehat{\mathbf{b}}_\alpha$. We also know that (5.3) is true when $\alpha = i-1$ since $\widehat{\mathbf{b}}_{i-1} = y$. Now suppose $i \leq \alpha \leq j-1$. If $\alpha = f_k = (m+1)k$ for some k , then $f_k < j < f_x$, so $\widehat{\mathbf{b}}_\alpha = k < x$. If $\alpha \in \Delta(\mathbf{b}^{(\alpha-1)})$, then we can use Theorem 4.2, the definition of Θ_α , and (5.2) to see that

$$\widehat{\mathbf{b}}_\alpha = (\Theta_\alpha(\mathbf{b}^{(\alpha-1)}))_\alpha = \eta_\alpha(\mathbf{b}^{(\alpha-1)}) < \mathbf{b}_\alpha^{(\alpha-1)} = \mathbf{b}_\alpha \leq x.$$

Finally, suppose α is not divisible by $m + 1$ (i.e., is not of the form f_k) and $\alpha \notin \Delta(\mathbf{b}^{(\alpha-1)})$. According to Theorem 4.2 and the definition of Θ_α , we have

$$\widehat{\mathbf{b}}_\alpha = (\Theta_\alpha(\mathbf{b}^{(\alpha-1)}))_\alpha = \zeta_\alpha(\mathbf{b}^{(\alpha-1)}) = \min\{\mathbf{b}_r^{(\alpha-1)} : 0 \leq r \leq \alpha - 1 \text{ and } \mathbf{b}_r^{(\alpha-1)} \geq \mathbf{b}_\alpha^{(\alpha-1)}\}.$$

Our induction hypothesis implies that $\zeta_\alpha(\mathbf{b}^{(\alpha-1)}) \notin [x, y - 1]$. This proves (5.3).

We have $i \in \Delta(\mathbf{b})$ because $\varphi_{\mathbf{b}}(i) \neq \emptyset$. It follows from Corollary 4.4 that $\widehat{\mathbf{b}}_i = \eta_i(\mathbf{b}) < \mathbf{b}_i = x$. Now, $\widehat{\mathbf{b}}_{i-1} = y \geq x > \eta_i(\mathbf{b}) = \widehat{\mathbf{b}}_i$, so $i - 1 \in \Delta(\widehat{\mathbf{b}})$. This means that $\widehat{\varphi}_{\widehat{\mathbf{b}}}(i - 1)$ is the smallest integer p such that $p > i - 1$, $p \equiv i - 1 \pmod{m + 1}$, and $\widehat{\mathbf{b}}_p = y$; our goal is to show that $\widehat{\varphi}_{\widehat{\mathbf{b}}}(i - 1) = j - 1$. By combining (5.2) with Remark 3.4 and the fact that $\mathbf{b}_j = x$, we find that $\mathbf{b}_{j-1} = x$. Therefore, $\mathbf{b}_{j-1}^{(j-2)} = \mathbf{b}_j^{(j-2)} = x$; it follows that $j - 1 \notin \Delta(\mathbf{b}^{(j-2)})$. Theorem 4.2 tells us that

$$\begin{aligned} \widehat{\mathbf{b}}_{j-1} &= (\Theta_{j-1}(\mathbf{b}^{(j-2)}))_{j-1} = \zeta_{j-1}(\mathbf{b}^{(j-2)}) = \min\{\mathbf{b}_r^{(j-2)} : 0 \leq r \leq j - 2 \text{ and } \mathbf{b}_r^{(j-2)} \geq x\} \\ &= \min\{\widehat{\mathbf{b}}_r : 0 \leq r \leq j - 2 \text{ and } \widehat{\mathbf{b}}_r \geq x\}, \end{aligned}$$

so it follows from (5.3) and the fact that $\widehat{\mathbf{b}}_{i-1} = y$ that $\widehat{\mathbf{b}}_{j-1} = y$. Consequently, we are left to show that

$$\widehat{\mathbf{b}}_r \neq y \text{ for all } r \in [i, j - 2] \text{ satisfying } r \equiv i - 1 \pmod{m + 1}. \quad (5.4)$$

Let us write $i = (m + 1)s + t$ and $j = (m + 1)s' + t$, where $0 \leq s < s' \leq n - 1$ and $2 \leq t \leq m$. Let $z_1 = \eta_i(\mathbf{b}) = \widehat{\mathbf{b}}_i$; note that $z_1 < x$. Condition (III) in Definition 3.2 tells us that $\widehat{\mathbf{b}}_r \leq z_1 < x$ for all $r \in [i + 1, (m + 1)z_1]$. In particular,

$$\widehat{\mathbf{b}}_{(m+1)v+t-1} < x \leq y \text{ for all } v \in [s + 1, z_1 - 1]. \quad (5.5)$$

If $z_1 \geq s'$, then this proves (5.4). Now suppose $z_1 < s'$. We can use Lemma 4.5 to conclude that $\mathbf{b}_{(m+1)z_1+1} = x$. By the definition of $\varphi_{\mathbf{b}}(i)$, we have $\mathbf{b}_{(m+1)z_1+t} \neq x$, so (by Remark 3.4) there must be some $1 \leq u_1 < t$ such that

$$\mathbf{b}_{(m+1)z_1+u_1} = x > \mathbf{b}_{(m+1)z_1+u_1+1}.$$

Then $(m + 1)z_1 + u_1 \in \Delta(\mathbf{b})$, so it follows from Corollary 4.4 that $\widehat{\mathbf{b}}_{(m+1)z_1+u_1} = \eta_{(m+1)z_1+u_1}(\mathbf{b})$. Let $z_2 = \eta_{(m+1)z_1+u_1}(\mathbf{b}) = \widehat{\mathbf{b}}_{(m+1)z_1+u_1}$; note that $z_2 < x$. Condition (III) in Definition 3.2 tells us that $\widehat{\mathbf{b}}_r \leq z_2 < x$ for all $r \in [(m + 1)z_1 + u_1, (m + 1)z_2]$. In particular,

$$\widehat{\mathbf{b}}_{(m+1)v+t-1} < x \leq y \text{ for all } v \in [z_1, z_2 - 1]. \quad (5.6)$$

If $z_2 \geq s'$, then (5.5) and (5.6) together complete the proof of (5.4), so we may assume $z_2 < s'$. An argument similar to the one just given shows that there exists $1 \leq u_2 < t$ such that

$$\mathbf{b}_{(m+1)z_2+u_2} = x > \mathbf{b}_{(m+1)z_2+u_2+1}.$$

Then $(m + 1)z_2 + u_2 \in \Delta(\mathbf{b})$, so it follows from Corollary 4.4 that $\widehat{\mathbf{b}}_{(m+1)z_2+u_2} = \eta_{(m+1)z_2+u_2}(\mathbf{b})$. We can then define $z_3 = \eta_{(m+1)z_2+u_2}(\mathbf{b})$ and note that $z_3 < x$. Also, $\widehat{\mathbf{b}}_r \leq z_3 < x$ for all $r \in [(m + 1)z_2 + u_2, (m + 1)z_3]$, so $\widehat{\mathbf{b}}_{(m+1)v+t-1} < x \leq y$ for all $v \in [z_2, z_3 - 1]$. Continuing in this fashion, we construct numbers z_1, z_2, \dots until eventually obtaining some z_h that is at least s' , at which point we obtain a proof of (5.4).

Case 2 Assume $i \equiv j \equiv 0 \pmod{m+1}$. Let $i = (m+1)z_1 = f_{z_1}$ and $j = (m+1)x = f_x$, and note that $z_1 < x$. The definition of φ_b implies that $i + 1$ is x -initial in b ; in particular, $b_{i+1} = x$. Let $y = \widehat{b}_{i-1}$. Condition (III) in Definition 3.2 tells us that

$$b_r \leq x \quad \text{for all } r \in [i + 1, j]. \tag{5.7}$$

In order to prove that $\varphi_{\widehat{b}}(i - 1) = j - 1$, we first want to show that $i - 1 \in \Delta(\widehat{b})$, meaning $y > \widehat{b}_i = z_1$. We will actually show that $y \geq x$. Since $y \geq z_1$ (by Remark 3.4), we just need to prove that $y \notin [z_1, x - 1]$. In fact, we will prove by induction on α the stronger assertion that

$$\widehat{b}_\alpha \notin [z_1, x - 1] \quad \text{for all } \alpha \in [0, i - 1]. \tag{5.8}$$

First, suppose $b_\alpha \geq x$. Since $i + 1$ is x -initial in b , we must actually have $b_\alpha \geq x + 1$. Notice that $b_r^{(\alpha-1)} = b_r \leq x$ for all $r \in [i + 2, f_x]$ by condition (III) in Definition 3.2. Also, $b_{i+1}^{(\alpha-1)} = b_{i+1} = x$ and $b_i^{(\alpha-1)} = b_i = z_1 < x$. This shows that $b_r^{(\alpha-1)} \leq x$ for all $r \in [i, f_x]$. If, in addition, we have $\alpha \in \Delta(b^{(\alpha-1)})$ and $b_r^{(\alpha-1)} \leq x$ for all $r \in [\alpha + 1, i - 1]$, then it follows from Theorem 4.2 and the definition of $\eta_\alpha(b^{(\alpha-1)})$ that

$$\widehat{b}_\alpha = (\Theta_\alpha(b^{(\alpha-1)}))_\alpha = \eta_\alpha(b^{(\alpha-1)}) \geq x.$$

On the other hand, if there exists $r \in [\alpha + 1, i - 1]$ such that $b_r^{(\alpha-1)} > x$, then Remark 3.3 tells us that $\widehat{b}_\alpha \notin [z_1, x - 1]$ since, otherwise, the entries in positions $\alpha, r, f_{\widehat{b}_\alpha}$ would form a 121-pattern in $b^{(\alpha)}$ (since $b_\alpha^{(\alpha)} = \widehat{b}_\alpha$ and $b_r^{(\alpha)} = b_r^{(\alpha-1)}$). Finally, if $\alpha \notin \Delta(b^{(\alpha-1)})$, then

$$\begin{aligned} \widehat{b}_\alpha &= (\Theta_\alpha(b^{(\alpha-1)}))_\alpha = \zeta_\alpha(b^{(\alpha-1)}) = \min\{b_r^{(\alpha-1)} : 0 \leq r \leq \alpha - 1 \text{ and } b_r^{(\alpha-1)} \geq b_\alpha^{(\alpha-1)}\} \\ &= \min\{\widehat{b}_r : 0 \leq r \leq \alpha - 1 \text{ and } \widehat{b}_r \geq b_\alpha^{(\alpha-1)}\}, \end{aligned}$$

so it follows by induction on α that $\widehat{b}_\alpha \notin [x, y - 1]$ in this case as well. This proves (5.8), so we conclude that $y \geq x$ and $i - 1 \in \Delta(\widehat{b})$.

We are now going to prove by induction on α that

$$\widehat{b}_\alpha \notin [x, y - 1] \quad \text{for all } \alpha \in [0, j - 1]. \tag{5.9}$$

This is certainly true if $0 \leq \alpha \leq i - 2$ since, otherwise, we would have $\alpha + 1 \leq i - 1 \leq f_{\widehat{b}_\alpha}$ and $\widehat{b}_{i-1} = y > \widehat{b}_\alpha$ (violating condition (III) in Definition 3.2). We also know that (5.9) is true when $\alpha = i - 1$ or $\alpha = i$ since $\widehat{b}_{i-1} = y$ and $\widehat{b}_i = \widehat{b}_{f_{z_1}} = z_1 < x$. Now suppose $i + 1 \leq \alpha \leq j - 1$. If $\alpha = f_k = (m + 1)k$ for some k , then $f_k < j = f_x$, so $\widehat{b}_\alpha = k < x$. If $\alpha \in \Delta(b^{(\alpha-1)})$, then we can use Theorem 4.2, the definition of Θ_α , and (5.7) to see that

$$\widehat{b}_\alpha = (\Theta_\alpha(b^{(\alpha-1)}))_\alpha = \eta_\alpha(b^{(\alpha-1)}) < b_\alpha^{(\alpha-1)} = b_\alpha \leq x.$$

Finally, suppose α is not divisible by $m + 1$ (i.e., is not of the form f_k) and $\alpha \notin \Delta(b^{(\alpha-1)})$. According to Theorem 4.2 and the definition of Θ_α , we have

$$\widehat{b}_\alpha = (\Theta_\alpha(b^{(\alpha-1)}))_\alpha = \zeta_\alpha(b^{(\alpha-1)}) = \min\{b_r^{(\alpha-1)} : 0 \leq r \leq \alpha - 1 \text{ and } b_r^{(\alpha-1)} \geq b_\alpha^{(\alpha-1)}\}.$$

Our induction hypothesis implies that $\zeta_\alpha(\mathbf{b}^{(\alpha-1)}) \notin [x, y-1]$. This proves (5.9).

We saw above that $i-1 \in \Delta(\widehat{\mathbf{b}})$, so $\varphi_{\widehat{\mathbf{b}}}(i-1)$ is the smallest integer p such that $p > i-1$, $p \equiv -1 \pmod{m+1}$, and $\widehat{\mathbf{b}}_p = y$; our goal is to show that $\varphi_{\widehat{\mathbf{b}}}(i-1) = j-1$. By combining (5.8) with Remark 3.4 and the fact that $\mathbf{b}_j = x$, we find that $\mathbf{b}_{j-1} = x$. Since $\mathbf{b}_{j-1}^{(j-2)} = \mathbf{b}_j^{(j-2)} = x$, we have $j-1 \notin \Delta(\mathbf{b}^{(j-2)})$. Theorem 4.2 tells us that

$$\begin{aligned} \widehat{\mathbf{b}}_{j-1} &= (\Theta_{j-1}(\mathbf{b}^{(j-2)}))_{j-1} = \zeta_{j-1}(\mathbf{b}^{(j-2)}) = \min\{\mathbf{b}_r^{(j-2)} : 0 \leq r \leq j-2 \text{ and } \mathbf{b}_r^{(j-2)} \geq x\} \\ &= \min\{\widehat{\mathbf{b}}_r : 0 \leq r \leq j-2 \text{ and } \widehat{\mathbf{b}}_r \geq x\}, \end{aligned}$$

so it follows from (5.9) and the fact that $\widehat{\mathbf{b}}_{i-1} = y$ that $\widehat{\mathbf{b}}_{j-1} = y$. Therefore, we are left to show that $\widehat{\mathbf{b}}_{(m+1)v+m} \neq y$ for all $v \in [z_1, x-2]$. This is vacuously true if $z_1 = x-1$, so we may assume $z_1 < x-1$.

Recall that $\mathbf{b}_{(m+1)z_1+1} = \mathbf{b}_{i+1} = x$. Let u_1 be the largest element of $[m]$ with the property that $\mathbf{b}_{(m+1)z_1+u_1} = x$. Then $(m+1)z_1 + u_1 \in \Delta(\mathbf{b})$ (by Remark 3.4), so it follows from Corollary 4.4 that

$$\widehat{\mathbf{b}}_{(m+1)z_1+u_1} = \eta_{(m+1)z_1+u_1}(\mathbf{b}).$$

Let $z_2 = \eta_{(m+1)z_1+u_1}(\mathbf{b}) = \widehat{\mathbf{b}}_{(m+1)z_1+u_1}$; note that $z_2 < x$ since $\mathbf{b}_{(m+1)z_1+u_1} \leq x$ by (5.7). Condition (III) in Definition 3.2 tells us that $\widehat{\mathbf{b}}_r \leq z_2 < x$ for all $r \in [(m+1)z_1 + u_1, (m+1)z_2]$. In particular, $\widehat{\mathbf{b}}_{(m+1)v+m} < x$ for all $v \in [z_1, z_2-1]$. Let u_2 be the largest element of $[m]$ such that $\mathbf{b}_{(m+1)z_2+u_2} = x$. If $z_2 < x-1$, then $(m+1)z_2 + u_2 \in \Delta(\mathbf{b})$, so it follows from Corollary 4.4 that $\widehat{\mathbf{b}}_{(m+1)z_2+u_2} = \eta_{(m+1)z_2+u_2}(\mathbf{b})$. We can then define $z_3 = \eta_{(m+1)z_2+u_2}(\mathbf{b})$ and note that $z_3 < x$ since $\mathbf{b}_{(m+1)z_2+u_2} \leq x$ by (5.7). Also, $\widehat{\mathbf{b}}_r \leq z_3 < x$ for all $r \in [(m+1)z_2 + u_2, (m+1)z_3]$, so $\widehat{\mathbf{b}}_{(m+1)v+m} < x$ for all $v \in [z_2, z_3-1]$. Continuing in this fashion, we construct numbers z_1, z_2, \dots until eventually obtaining some z_h that is equal to $x-1$. This argument shows that $\widehat{\mathbf{b}}_{(m+1)v+m} < x \leq y$ for all $v \in [z_1, x-2]$, which proves the desired result in this case. \square

Lemma 5.11. *Let $\mathbf{b} \in \text{Tam}_n(m)$. Let N be the largest element of the block of $\Psi(\mathbf{b})$ containing 1, and assume $N > 1$ (i.e., 1 is not in a singleton block). Then $\varphi_{\text{Row}(\mathbf{b})}(N-1) = (m+1)n$.*

Proof. Let $\widehat{\mathbf{b}} = \text{Row}(\mathbf{b})$. Let $i_1 < \dots < i_p$ be the elements of the block of $\Psi(\mathbf{b})$ containing 1. Then $i_1 = 1$ and $i_p = N$. The description of $\Psi(\mathbf{b})$ given in Lemma 5.3 tells us that $\varphi_{\widehat{\mathbf{b}}}(i_t) = i_{t+1}$ for all $t \in [p-1]$. It follows from the definition of $\varphi_{\widehat{\mathbf{b}}}$ that $1 = i_1 \equiv \dots \equiv i_p \pmod{m+1}$. Let us write $i_t = (m+1)u_t + 1$ for all $t \in [p]$. Since $N-1 \equiv 0 \pmod{m+1}$, we can refer to the definition of $\varphi_{\widehat{\mathbf{b}}}$ to see that our goal is to show that N is n -initial in $\widehat{\mathbf{b}}$.

Consider $t \in [p-1]$. It follows from Lemma 5.9 that i_t is u_{t+1} -initial in $\text{Row}(\mathbf{b})$. Because $f_{u_{t+1}} = (m+1)u_{t+1} = i_{t+1} - 1$, it follows from condition (III) in Definition 3.2 that $\widehat{\mathbf{b}}_r \leq u_{t+1}$ for all $r \in [i_t, i_{t+1} - 1]$. This shows that $\widehat{\mathbf{b}}_r \leq u_p$ for all $r \in [0, N-1]$. In particular, $\widehat{\mathbf{b}}_r \neq n$ for all $r \in [0, N-1]$. Therefore, we will be done if we can prove that $\widehat{\mathbf{b}}_N = n$.

Because N is the largest element of its block in $\Psi(\mathbf{b})$, it is in $\varphi_{\widehat{\mathbf{b}}}^{-1}(\emptyset)$ by Lemma 5.3. This means that $N \notin \Delta(\mathbf{b})$. Since $\mathbf{b}_N^{(N-1)} = \mathbf{b}_N$ and $\mathbf{b}_{N+1}^{(N-1)} = \mathbf{b}_{N+1}$, we also have $N \notin \Delta(\mathbf{b}^{(N-1)})$. By Theorem 4.2 and the definition of Θ_N , we have $\widehat{\mathbf{b}}_N = (\Theta_N(\mathbf{b}^{(N-1)}))_N = \zeta_N(\mathbf{b}^{(N-1)})$. Recall that

our convention is that $\zeta_N(\mathbf{b}^{(N-1)}) = n$ if the set $\{\mathbf{b}_r^{(N-1)} : 0 \leq r \leq N-1 \text{ and } \mathbf{b}_r^{(N-1)} \geq \mathbf{b}_N^{(N-1)}\}$ is empty. We know that $\mathbf{b}_r^{(N-1)} = \widehat{\mathbf{b}}_r \leq u_p$ for all $r \in [0, N-1]$. On the other hand, we have $\mathbf{b}_N^{(N-1)} \geq b_N(\nu) = u_p + 1$. This shows that $\{\mathbf{b}_r^{(N-1)} : 0 \leq r \leq N-1 \text{ and } \mathbf{b}_r^{(N-1)} \geq \mathbf{b}_N^{(N-1)}\}$ is indeed empty, so $\widehat{\mathbf{b}}_N = \zeta_N(\mathbf{b}^{(N-1)}) = n$. \square

We are now finally in a position to prove the main result of this section.

Theorem 5.12. *The map $\Psi: \text{Tam}_n(m) \rightarrow \text{ShNC}^{(m)}(n)$ is a bijection satisfying*

$$\Psi \circ \text{Row} = \text{Rot} \circ \Psi.$$

Proof. We already know by Proposition 5.8 that Ψ is a bijection. Consider a ν -bracket vector $\mathbf{b} \in \text{Tam}_n(m)$; our goal is to show that $\Psi(\text{Row}(\mathbf{b})) = \text{Rot}(\Psi(\mathbf{b}))$. We will make use of Lemma 5.5 (or rather, its proof), which tells us that

$$|\varphi_{\mathbf{b}}^{-1}(\emptyset)| = |\varphi_{\text{Row}(\mathbf{b})}^{-1}(\emptyset)| = mn + 1. \tag{5.10}$$

Let N be the largest element of the block of $\Psi(\mathbf{b})$ containing 1. Let \mathcal{X} be the set of pairs (α, α') such that $\alpha, \alpha' \in [(m+1)n]$ and $\varphi_{\text{Row}(\mathbf{b})}(\alpha) = \alpha'$. Note that $|\mathcal{X}| = (m+1)n - |\varphi_{\text{Row}(\mathbf{b})}^{-1}(\emptyset)| = n - 1$ by (5.10).

Suppose $N = 1$. Let \mathcal{Y} be the set of pairs of the form $(i-1, j-1)$ such that $i, j \in [2, (m+1)n]$ and $\varphi_{\mathbf{b}}(i) = j$. Since $N = 1$, we have $\varphi_{\mathbf{b}}(1) = \emptyset$, so $|\mathcal{Y}| = (m+1)n - |\varphi_{\mathbf{b}}^{-1}(\emptyset)| = n - 1$ by (5.10). Lemma 5.10 tells us that $\mathcal{Y} \subseteq \mathcal{X}$, so we must have $\mathcal{X} = \mathcal{Y}$. This implies that $\Psi(\text{Row}(\mathbf{b})) = \text{Rot}(\Psi(\mathbf{b}))$, as desired.

Now suppose $N > 1$. Let \mathcal{Z}' be the set of pairs of the form $(i-1, j-1)$ such that $i, j \in [2, (m+1)n]$ and $\varphi_{\mathbf{b}}(i) = j$. Let $\mathcal{Z} = \mathcal{Z}' \cup \{(N, (m+1)n+1)\}$. Since $N > 1$, we have $\varphi_{\mathbf{b}}(1) \neq \emptyset$, so $|\mathcal{Z}'| = |([(m+1)n] \setminus \varphi_{\mathbf{b}}^{-1}(\emptyset)) \setminus \{1\}| = n - 2$ by (5.10). This shows that $|\mathcal{Z}| = n - 1 = |\mathcal{X}|$. Lemmas 5.10 and 5.11 tell us that $\mathcal{Z} \subseteq \mathcal{X}$, so we must have $\mathcal{X} = \mathcal{Z}$. This implies that $\Psi(\text{Row}(\mathbf{b})) = \text{Rot}(\Psi(\mathbf{b}))$, as desired. \square

The following result settles Thomas and Williams’s Conjecture 3.1 when $a \equiv 1 \pmod{b}$.

Theorem 5.13. *The order of $\text{Row}: \text{Tam}_n(m) \rightarrow \text{Tam}_n(m)$ is $(m+1)n$. Moreover, the triple $(\text{Tam}_n(m), \text{Row}, \text{Cat}_n^{(m)}(q))$ exhibits the cyclic sieving phenomenon.*

Proof. The first sentence is immediate from Theorem 5.12 because the map

$$\text{Rot}: \text{ShNC}^{(m)}(n) \rightarrow \text{ShNC}^{(m)}(n)$$

clearly has order $(m+1)n$. The second sentence is a consequence of Theorem 5.12 and (5.1). \square

Our final result of this section exhibits an instance of the homomesy phenomenon (see Section 2.5). Recall from the introduction that the *down-degree statistic* on a poset P is the function $\text{ddeg}: P \rightarrow \mathbb{R}$ defined by $\text{ddeg}(x) = |\{y \in P : y \leq x\}|$.

Theorem 5.14. *The down-degree statistic on $\text{Tam}_n(m)$ is homomesic for rowmotion with average*

$$\frac{m(n-1)}{m+1}.$$

Proof. Suppose $\mathbf{b} \in \text{Tam}_n(m)$. It follows from [Def22, Proposition 4.4] that $\text{ddeg}(\mathbf{b}) = |\Delta(\mathbf{b})|$. Referring to the description of $\Psi(\mathbf{b})$ in Lemma 5.3 and the definition of $\varphi_{\mathbf{b}}$, we find that $\Delta(\mathbf{b})$ is the set of indices $i \in [(m+1)n]$ such that i is not maximal in its block of $\Psi(\mathbf{b})$ and $i \not\equiv 0 \pmod{m+1}$. For $\rho \in \text{ShNC}^{(m)}(n)$, let $Q(\rho)$ be the number of indices $i \in [(m+1)n]$ such that i is maximal in its block in ρ and $i \not\equiv 0 \pmod{m+1}$. Then $\text{ddeg}(\mathbf{b}) = |\Delta(\mathbf{b})| = mn - Q(\Psi(\mathbf{b}))$. Invoking Theorem 5.12, we find that the desired result will follow if we can prove that Q is homomesic for $\text{Rot}: \text{ShNC}^{(m)}(n) \rightarrow \text{ShNC}^{(m)}(n)$ with average $mn - m(n-1)/(m+1)$.

For a set $X \subseteq [(m+1)n]$, let $f(X) = 1$ if $\max X \not\equiv 0 \pmod{m+1}$, and let $f(X) = 0$ otherwise. Fix $\rho \in \text{ShNC}^{(m)}(n)$. Because $\text{Rot}: \text{ShNC}^{(m)}(n) \rightarrow \text{ShNC}^{(m)}(n)$ has order $(m+1)n$, the average of Q along the orbit of Rot containing ρ is

$$\frac{1}{(m+1)n} \sum_{k=1}^{(m+1)n} Q(\text{Rot}^k(\rho));$$

our goal is to show that this quantity is $mn - m(n-1)/(m+1)$. For each block $B \in \rho$ and each $k \geq 1$, let $B-k$ be the block of $\text{Rot}^k(\rho)$ whose elements are obtained by subtracting k from the elements of B and reducing modulo $(m+1)n$. Since all of the elements of B are congruent modulo $m+1$ (by the definition of $\text{ShNC}^{(m)}(n)$), we have $\sum_{k=1}^{(m+1)n} f(B-k) = mn$. Thus,

$$\begin{aligned} \frac{1}{(m+1)n} \sum_{k=1}^{(m+1)n} Q(\text{Rot}^k(\rho)) &= \frac{1}{(m+1)n} \sum_{k=1}^{(m+1)n} \sum_{X \in \text{Rot}^k(\rho)} f(X) \\ &= \frac{1}{(m+1)n} \sum_{k=1}^{(m+1)n} \sum_{B \in \rho} f(B-k) \\ &= \frac{1}{(m+1)n} \sum_{B \in \rho} \sum_{k=1}^{(m+1)n} f(B-k) \\ &= \frac{1}{(m+1)n} \sum_{B \in \rho} mn = \frac{1}{(m+1)n} mn(mn+1), \end{aligned}$$

where the last equality follows from the fact that ρ has $mn+1$ blocks (by the definition of $\text{ShNC}^{(m)}(n)$). Finally, note that $\frac{1}{(m+1)n} mn(mn+1) = mn - m(n-1)/(m+1)$. \square

To end our discussion of ν -Tamari lattices, we state two conjectures about rowmotion on rational Tamari lattices. The first conjecture strengthens Conjecture 3.1 and Theorem 5.13, while the second strengthens Theorem 5.14. Recall the definition of $\text{Tam}(a, b)$ from Section 3.1.

Conjecture 5.15. Let a and b be relatively prime positive integers, and let

$$\text{Cat}^{(a,b)}(q) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ b \end{bmatrix}_q.$$

The triple $(\text{Tam}(a, b), \text{Row}, \text{Cat}^{(a,b)}(q))$ exhibits the cyclic sieving phenomenon.

Conjecture 5.16. Let a and b be relatively prime positive integers. The down-degree statistic on $\text{Tam}(a, b)$ is homomesic for rowmotion with average

$$\frac{(a - 1)(b - 1)}{a + b - 1}.$$

6. BiCambrian Lattices and Doubled Root Posets

In the next several sections, we turn our attention to rowmotion on biCambrian lattices. We begin with some notation and terminology surrounding the main result that we seek to prove.

6.1. Lattice congruences

Let L be a lattice. A *lattice congruence* of L is an equivalence relation \equiv on L that respects meets and joins. More precisely, this means that if $x_1, x_2, y_1, y_2 \in L$ satisfy $x_1 \equiv x_2$ and $y_1 \equiv y_2$, then $(x_1 \wedge y_1) \equiv (x_2 \wedge y_2)$ and $(x_1 \vee y_1) \equiv (x_2 \vee y_2)$.

Each equivalence class C of a lattice congruence \equiv is an interval in L , so it has a unique minimal element C_{\min} and a unique maximal element C_{\max} . We obtain projection operators $\pi_{\downarrow}, \pi^{\uparrow} : L \rightarrow L$ by letting $\pi_{\downarrow}(x) = C_{\min}$ and $\pi^{\uparrow}(x) = C_{\max}$ whenever x is in the equivalence class C . Thus, $\pi_{\downarrow}(L)$ is the set of minimal elements of equivalence classes, while $\pi^{\uparrow}(L)$ is the set of maximal elements of equivalence classes. We consider $\pi_{\downarrow}(L)$ and $\pi^{\uparrow}(L)$ as subposets of L .

The lattice congruence \equiv gives rise to a quotient lattice L/\equiv whose elements are the equivalence classes of \equiv . The order relation on L/\equiv is defined so that $C \leq C'$ if and only if there exist $x \in C$ and $x' \in C'$ such that $x \leq x'$. It is straightforward to check that $C \leq C'$ in L/\equiv if and only if $C_{\min} \leq C'_{\min}$ in L if and only if $C_{\max} \leq C'_{\max}$ in L . Thus, $\pi_{\downarrow}(L)$ and $\pi^{\uparrow}(L)$ are lattices that are naturally isomorphic to L/\equiv .

6.2. Doubled root posets

Let Φ be a finite root system with a corresponding Coxeter group W (see [BB05, Hum80] for relevant background). Let Φ^+ be a system of positive roots for Φ . If Φ is crystallographic, then there is a natural partial order on Φ^+ obtained by saying $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combination of positive roots. Endowed with this partial order, Φ^+ is called the *root poset* of Φ . The minimal elements of Φ^+ are the *simple roots*.

Barnard and Reading [BR18] introduced *Coxeter–biCatalan combinatorics* as a variant of Coxeter–Catalan combinatorics motivated by several fascinating enumerative phenomena. Given a root poset Φ^+ , we obtain the associated *doubled root poset* by taking a copy of Φ^+ and a copy of the dual of Φ^+ and identifying corresponding simple roots. For example, the doubled root posets of types A_5 and B_3 are shown on the left in Figure 6.1. In general, the doubled root poset of type A_n is isomorphic to the product of two chains $[n] \times [n]$, and the doubled root poset of type B_n is a shifted staircase with $n(2n - 1)$ elements (this is the quotient of $[n] \times [n]$ modulo the action of the map $(i, j) \mapsto (j, i)$).

Although the root systems of types H_3 and $I_2(m)$ are not (in general) crystallographic, Armstrong gave a definition of what their root posets should be [Arm09, Section 5.4.1] (see also [CS15]). Using this definition, we obtain the doubled root posets of types H_3 and $I_2(6)$ as shown on the right in Figure 6.1. The doubled root poset of type $I_2(m)$ is the obvious analogue of the doubled root poset of type $I_2(6)$ with $2m - 2$ elements.

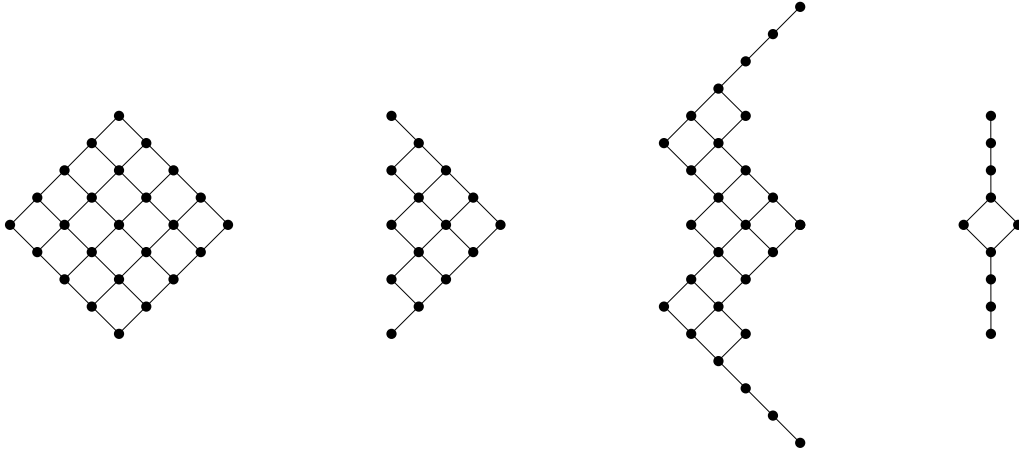


Figure 6.1: Doubled root posets of types A_5 , B_3 , H_3 , and $I_2(6)$ (from left to right).

The root systems of *coincidental type* are those of type A , B , H_3 , or $I_2(m)$. It turns out that the doubled root poset of a root system of coincidental type is always a *minuscule poset*; such posets arise naturally as the weight lattices of certain representations of simple complex Lie algebras. The orbit structures of rowmotion on the lattices of order ideals of minuscule posets are well understood. For type- A minuscule posets (i.e., *rectangle posets*), this follows from a theorem of Stanley [Sta09] and Theorem 1.1(b) in the article [RSW04] by Reiner, Stanton, and White. Type- B minuscule posets (i.e., *shifted staircase posets*) were first worked out (with essentially the same argument used for minuscule posets of type A) by Striker and Williams [SW12, Corollary 6.3]. Rush and Shi [RS13] gave a uniform treatment of all minuscule posets, proving the following theorem. Note that minuscule posets are ranked (see Section 2.2 for the definition of the rank function rk).

Theorem 6.1 ([RS13]). *Let P be a minuscule poset with rank function $\text{rk}: P \rightarrow \mathbb{Z}_{\geq 0}$. The triple $(\mathcal{J}(P), \text{Row}, F_P(q))$ exhibits the cyclic sieving phenomenon, where*

$$F_P(q) = \prod_{x \in P} \frac{1 - q^{\text{rk}(x)+2}}{1 - q^{\text{rk}(x)+1}}.$$

6.3. BiCambrian lattices

In what follows, whenever we refer to the *weak order* on a finite Coxeter group, we mean the right weak order; it is well known that the weak order is a semidistributive lattice. Given a finite irreducible Coxeter group W and a Coxeter element c of W , there is an associated lattice congruence

on the weak order of W called the c -Cambrian congruence [Rea06]. Since c^{-1} is also a Coxeter element, one can also consider the c^{-1} -Cambrian congruence. The *c -biCambrian congruence* is the lattice congruence on the weak order of W obtained by taking the common refinement of the c -Cambrian and c^{-1} -Cambrian congruences. The *c -biCambrian lattice* is the quotient of the weak order modulo the c -biCambrian congruence. Being a quotient of the weak order, the c -biCambrian lattice is semidistributive, so it comes equipped with a rowmotion operator.

Let Φ be a finite root system of coincidental type with a corresponding Coxeter group W and corresponding doubled root poset P . Because the Coxeter diagram of W is a tree, it has a bipartition $X \sqcup Y$. Let $c_+ = \prod_{s \in X} s$ and $c_- = \prod_{s \in Y} s$. Then $c = c_+c_-$ is called a *bipartite Coxeter element*; its inverse is c_-c_+ , which is also bipartite. Our goal in the next three sections is to prove the following theorem, which Thomas and Williams stated as a conjecture in [TW19].

Theorem 6.2. *Let W be a Coxeter group of coincidental type, and let c be a bipartite Coxeter element of W . Let P be the doubled root poset of type W . The orbit structure of rowmotion on the c -biCambrian lattice is the same as the orbit structure of rowmotion on $\mathcal{J}(P)$.*

Let W , c and P be as in Theorem 6.2, and let $F_P(q)$ be the polynomial defined in Theorem 6.1. Theorems 6.1 and 6.2 imply that rowmotion acting on the c -biCambrian lattice exhibits the cyclic sieving phenomenon with respect to $F_P(q)$.

7. Type A BiCambrian Lattices

Our goal in this section is to prove Theorem 6.2 in type A . The Coxeter group of type A_n is isomorphic to the symmetric group S_{n+1} , whose elements are the permutations of the set $[n + 1]$. We view permutations both as bijections from $[n + 1]$ to itself and as words in one-line notation. Let s_i denote the transposition that swaps i and $i + 1$. An index $i \in [n]$ is called a *descent* (respectively, *ascent*) of a permutation w if $w(i) > w(i + 1)$ (respectively, $w(i) < w(i + 1)$); abusing notation, we will often say that $w(i) > w(i + 1)$ is the descent (respectively, that $w(i) < w(i + 1)$ is the ascent). For example, we would say that $6 > 3$ is a descent of 416352, while $3 < 5$ is an ascent of 416352. A permutation in S_{n+1} is join-irreducible (respectively, meet-irreducible) in the weak order if and only if it has exactly 1 descent (respectively, ascent).

An *inversion* of a permutation $w \in S_{n+1}$ is a pair (i, j) such that $1 \leq i < j \leq n + 1$ and $w^{-1}(i) > w^{-1}(j)$. For $w, w' \in S_{n+1}$, we have $w \leq w'$ if and only if every inversion of w is also an inversion of w' .

Let $c = c_+c_-$, where $c_+ = \prod_{i \in [n] \text{ even}} s_i$ and $c_- = \prod_{i \in [n] \text{ odd}} s_i$. Throughout this section, we let \equiv denote the c -biCambrian congruence on S_{n+1} . We will work with the following explicit combinatorial description of \equiv . Suppose $w \in S_{n+1}$ has a descent $a > b$. If there exist integers k, m of different parities such that $b < k < m < a$ and such that k and m appear on the same side of a and b in the one-line notation of w , then we can swap a and b to produce a new permutation w' such that $w' \leq w$ in the weak order and such that $w \equiv w'$. We call this operation that swaps a and b a *valid swap*. We also use the term *valid swap* to refer to the inverse operation that swaps a and b to go from w' to w (in this case, k and m still have opposite parities and lie on the same side of a and b in w'). It is straightforward to check that a valid swap can be performed

on a permutation v by swapping two numbers a and b with $a < b$ if and only if the following hold:

- The entries a and b form a descent or an ascent in v .
- There exists an integer k such that $a < k < k + 1 < b$ and such that k and $k + 1$ appear on the same side of a and b in v .

The *c -biCambrian congruence* \equiv is defined by saying $v \equiv v'$ if and only if there is a sequence of valid swaps that transforms v into v' ; see Figure 7.1. This equivalence relation is in fact a lattice congruence, so we can consider the associated projection operators π_{\downarrow} and π^{\uparrow} defined in Section 6.1. The *c -biCambrian lattice* is the lattice quotient of the weak order on S_{n+1} modulo \equiv . As discussed in Section 6.1, the c -biCambrian lattice is isomorphic to both $\pi_{\downarrow}(S_{n+1})$ and $\pi^{\uparrow}(S_{n+1})$. Note that a permutation w is in $\pi_{\downarrow}(S_{n+1})$ if and only if there does not exist a descent $b > a$ of w for which we can perform a valid swap. Similarly, w is in $\pi^{\uparrow}(S_{n+1})$ if and only if there does not exist an ascent $a < b$ of w for which we can perform a valid swap.

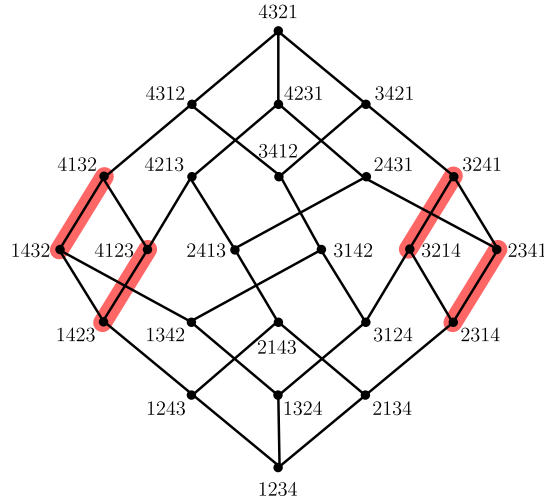


Figure 7.1: There are 20 c -biCambrian congruence classes in S_4 ; 16 of them are singletons while the other 4 (shown in red) have cardinality 2.

Consider a permutation w . Suppose a and b are integers that appear consecutively in the one-line notation of w and satisfy $a < b$. Then either $b > a$ is a descent of w or $a < b$ is an ascent of w . We say this descent or ascent is *right-even* (respectively, *left-even*) if a and b appear to the right (respectively, left) of all even numbers in $[a + 1, b - 1]$ and to the left (respectively, right) of all odd numbers in $[a + 1, b - 1]$ in w . Note that $b - a = 1$ if and only if this descent or ascent is both right-even and left-even. It follows from the combinatorial description of the c -biCambrian congruence that $w \in \pi_{\downarrow}(S_{n+1})$ if and only if every descent of w is right-even or left-even (or both).

Suppose $w \in \pi_{\downarrow}(S_{n+1})$ has k descents $b_1 > a_1, \dots, b_k > a_k$; we will define two k -element subsets $S_{\downarrow}(w), T_{\downarrow}(w) \subseteq [n]$. If the descent $b_i > a_i$ is right-even, then we put a_i into the set $S_{\downarrow}(w)$

and $b_i - 1$ into the set $T_{\downarrow}(w)$; if the descent $b_i > a_i$ is left-even, we put $b_i - 1$ into $S_{\downarrow}(w)$ and a_i into $T_{\downarrow}(w)$. For example, if $w = 896753421 \in \pi_{\downarrow}(S_9)$, then $S_{\downarrow}(w) = \{1, 3, 4, 5, 6\}$ and $T_{\downarrow}(w) = \{1, 2, 3, 6, 8\}$. It is straightforward to check that no number will be added twice into one of these two sets from two different descents—thus, $(S_{\downarrow}(w), T_{\downarrow}(w))$ is indeed a pair of k -element subsets of $[n]$. Let Ω_n denote the set of all pairs (S, T) such that $S, T \subseteq [n]$ and $|S| = |T|$. Barnard and Reading proved the following result.

Theorem 7.1 ([BR18, Theorem 3.8]). *The map $\xi_{\downarrow}: \pi_{\downarrow}(S_{n+1}) \rightarrow \Omega_n$ given by*

$$\xi_{\downarrow}(w) = (S_{\downarrow}(w), T_{\downarrow}(w))$$

is a bijection.

Our general strategy for proving Theorem 6.2 for A_n will be to use the bijection ξ_{\downarrow} to transfer the operator $\text{Row}: \pi_{\downarrow}(S_{n+1}) \rightarrow \pi_{\downarrow}(S_{n+1})$ to an operator $\chi: \Omega_n \rightarrow \Omega_n$ that is much easier to analyze (see Theorem 7.13). However, the main challenge is to prove that the map ξ_{\downarrow} has the properties we need; this will occupy most of our focus in this section.

Before we proceed, let us record an analogue of Theorem 7.1 that concerns $\pi^{\uparrow}(S_{n+1})$. Given $w \in \pi^{\uparrow}(S_{n+1})$ with k ascents $a_1 < b_1, \dots, a_k < b_k$, we define two k -element subsets

$$S^{\uparrow}(w), T^{\uparrow}(w) \subseteq [n].$$

If the ascent $a_i < b_i$ is right-even, we put a_i into the set $S^{\uparrow}(w)$ and $b_i - 1$ into the set $T^{\uparrow}(w)$; if the ascent $a_i < b_i$ is left-even, we put $b_i - 1$ into $S^{\uparrow}(w)$ and a_i into $T^{\uparrow}(w)$. The next theorem follows from Theorem 7.1 by considering duals, so we will attribute it to [BR18] even though it is not stated in this exact form in that article.

Theorem 7.2 ([BR18]). *The map $\xi^{\uparrow}: \pi^{\uparrow}(S_{n+1}) \rightarrow \Omega_n$ given by $\xi^{\uparrow}(w) = (S^{\uparrow}(w), T^{\uparrow}(w))$ is a bijection.*

The next proposition tells us that an element $w \in \pi_{\downarrow}(S_{n+1})$ is join-irreducible if and only if $|S_{\downarrow}(w)| = |T_{\downarrow}(w)| = 1$. The first statement is an immediate consequence of Proposition 4.11 (and the paragraph immediately after) in [BR18], while the second statement follows from the dual version of that result.

Proposition 7.3 ([BR18, Proposition 4.11]). *A permutation in $\pi_{\downarrow}(S_{n+1})$ is join-irreducible in $\pi_{\downarrow}(S_{n+1})$ if and only if it has exactly 1 descent (i.e., it is join-irreducible in the weak order on S_{n+1}). A permutation in $\pi^{\uparrow}(S_{n+1})$ is meet-irreducible in $\pi^{\uparrow}(S_{n+1})$ if and only if it has exactly 1 ascent (i.e., it is meet-irreducible in the weak order on S_{n+1}).*

Suppose $w \in S_{n+1}$, and let $b > a$ be a descent of w . Let $\lambda(w, b > a)$ be the unique permutation in S_{n+1} whose only descent is $b > a$ and such that the set of numbers in $[a + 1, b - 1]$ appearing to the left of b in $\lambda(w, b > a)$ is the same as the set of numbers in $[a + 1, b - 1]$ appearing to the left of b in w . For example, $\lambda(31746285, 6 > 2) = 13462578$. Note that if $b > a$ is right-even (respectively, left-even) in w , then $b > a$ is also right-even (respectively, left-even) in $\lambda(w, b > a)$. It is well known that if $b_1 > a_1, \dots, b_k > a_k$ are the descents of w ,

then $\{\lambda(w, b_1 > a_1), \dots, \lambda(w, b_k > a_k)\}$ is the canonical join representation of w in the weak order (see Section 2.3). Proposition 4.11 (and the paragraph immediately after it) in [BR18] tells us that if $w \in \pi_\downarrow(S_{n+1})$, then the canonical join representation of w in $\pi_\downarrow(S_{n+1})$ is the same as the canonical join representation of w in S_{n+1} . We record this in the following proposition.

Proposition 7.4. *Let $w \in \pi_\downarrow(S_{n+1})$. Let $b_1 > a_1, \dots, b_k > a_k$ be the descents of w . Then $\{\lambda(w, b_1 > a_1), \dots, \lambda(w, b_k > a_k)\}$ is the canonical join representation of w in $\pi_\downarrow(S_{n+1})$.*

Note that if $w \in \pi_\downarrow(S_{n+1})$ has descents $b_1 > a_1, \dots, b_k > a_k$, then

$$S_\downarrow(w) = \bigcup_{i \in [k]} S_\downarrow(\lambda(w, b_i > a_i)) \quad \text{and} \quad T_\downarrow(w) = \bigcup_{i \in [k]} T_\downarrow(\lambda(w, b_i > a_i)). \quad (7.1)$$

If $w \in S_{n+1}$ has an ascent $a < b$, then we let $\tau(w, a < b)$ be the unique permutation in S_{n+1} whose only ascent is $a < b$ and such that the set of numbers in $[a + 1, b - 1]$ appearing to the left of a in $\tau(w, a < b)$ is the same as the set of numbers in $[a + 1, b - 1]$ appearing to the left of a in w . We have the following analogue of Proposition 7.4.

Proposition 7.5. *Let $w \in \pi^\uparrow(S_{n+1})$. Let $a_1 < b_1, \dots, a_k < b_k$ be the ascents of w . Then $\{\tau(w, a_1 < b_1), \dots, \tau(w, a_k < b_k)\}$ is the canonical meet representation of w in $\pi^\uparrow(S_{n+1})$.*

Moreover, if $w \in \pi^\uparrow(S_{n+1})$ has ascents $a_1 < b_1, \dots, a_k < b_k$, then

$$S^\uparrow(w) = \bigcup_{i \in [k]} S^\uparrow(\tau(w, a_i < b_i)) \quad \text{and} \quad T^\uparrow(w) = \bigcup_{i \in [k]} T^\uparrow(\tau(w, a_i < b_i)). \quad (7.2)$$

Proposition 7.6. *Let j be the join-irreducible element of $\pi_\downarrow(S_{n+1})$ with right-even (respectively, left-even) descent $b > a$. Let m be the meet-irreducible element of $\pi^\uparrow(S_{n+1})$ with right-even (respectively, left-even) ascent $a < b$. Then $m = \pi^\uparrow(\text{Row}(j))$, where rowmotion is computed in $\pi_\downarrow(S_{n+1})$.*

Proof. For $x \in S_{n+1}$, let $[x]_{\equiv}$ be the c -biCambrian congruence class containing x ; thus, the equivalence classes of the form $[x]_{\equiv}$ are the elements of the c -biCambrian lattice S_{n+1}/\equiv . Let j_* (respectively, m^*) be the unique element of S_{n+1} that is covered by j (respectively, covers m) in the weak order. One can check that $j \wedge m = j_*$ and $j \vee m = m^*$, where the join and meet are taken in the weak order. This implies that $[j]_{\equiv} \wedge [m]_{\equiv} = [j_*]_{\equiv}$ and $[j]_{\equiv} \vee [m]_{\equiv} = [m^*]_{\equiv}$, where the join and meet are taken in the c -biCambrian lattice. It follows from Lemma 2.1 that $\kappa([j]_{\equiv}) = [m]_{\equiv}$. This proves that $\text{Row}([j]_{\equiv}) = [m]_{\equiv}$ in S_{n+1}/\equiv (recall from Section 2.3 that Row is just κ when restricted to join-irreducible elements). It follows that in the lattice $\pi_\downarrow(S_{n+1})$, rowmotion sends j to the minimum element of $[m]_{\equiv}$. This yields the desired result. \square

We still need to establish a few more technical lemmas before we can transfer the action of rowmotion from $\pi_\downarrow(S_{n+1})$ to Ω_n via the map ξ_\downarrow .

Lemma 7.7. *Suppose that $w, w' \in S_{n+1}$ are in the same c -biCambrian congruence class and that $d_1, d_2 \in [n + 1]$ satisfy $1 \leq |d_1 - d_2| \leq 2$. Then d_1 appears to the left of d_2 in w if and only if d_1 appears to the left of d_2 in w' .*

Proof. This is an immediate consequence of the combinatorial description of the c -biCambrian congruence since there is no valid swap of d_1 and d_2 . \square

In the following lemma, let us write $\text{Inv}(w)|_{[a,b]}$ for the set of inversions (x, y) of a permutation w that satisfy $a = x < y \leq b$ or $a \leq x < y = b$.

Lemma 7.8. *Let $w \in \pi_{\downarrow}(S_{n+1})$, and let w' be a permutation in the same c -biCambrian congruence class as w . If w has a descent $b > a$, then $\text{Inv}(w)|_{[a,b]} = \text{Inv}(w')|_{[a,b]}$. Analogously, if $\pi^{\uparrow}(w)$ has an ascent $d < e$, then $\text{Inv}(\pi^{\uparrow}(w))|_{[d,e]} = \text{Inv}(w')|_{[d,e]}$.*

Proof. We will prove only the first statement since the proof of the second statement is similar. We will also assume that $b > a$ is a right-even descent of w ; the case in which it is left-even is similar. Since $w \leq w'$, we know that $\text{Inv}(w)|_{[a,b]} \subseteq \text{Inv}(w')|_{[a,b]}$. Assume by way of contradiction that $\text{Inv}(w)|_{[a,b]} \neq \text{Inv}(w')|_{[a,b]}$.

Referring to the combinatorial description of the c -biCambrian congruence, we see that there is a chain $w = v_0 \leq v_1 \leq \dots \leq v_m = w'$ in the weak order whose elements are all in the same c -biCambrian congruence class as w and such that each v_i is obtained from v_{i-1} by applying a valid swap. Let j be the smallest index such that $\text{Inv}(v_{j-1})|_{[a,b]} \neq \text{Inv}(v_j)|_{[a,b]}$. Then $\text{Inv}(v_{j-1})|_{[a,b]} = \text{Inv}(w)|_{[a,b]}$. There is some $z \in [a + 1, b - 1]$ such that the valid swap used to move from v_{j-1} to v_j either swaps a and z or swaps z and b ; we will assume that it swaps a and z since the other case is similar. Because this swap is valid, there exists some $k \in [a + 1, z - 1]$ such that k and $k + 1$ appear on the same side of a in v_{j-1} . However, since $\text{Inv}(v_{j-1})|_{[a,b]} = \text{Inv}(w)|_{[a,b]}$, this implies that k and $k + 1$ appear on the same side of a in w . This contradicts the fact that $b > a$ is a right-even descent of w . \square

Lemma 7.9. *Let b be an even (respectively, odd) integer. Let $w \in \pi_{\downarrow}(S_{n+1})$ have a right-even (respectively, left-even) descent $b > a$. Then $\pi^{\uparrow}(w)$ cannot have a right-even (respectively, left-even) ascent of the form $b < d$.*

Proof. Assume by way of contradiction that b is even, w has a right-even descent $b > a$, and $\pi^{\uparrow}(w)$ has a right-even ascent $b < d$; the proof of the other case is similar. Let C be the c -biCambrian congruence class containing w . Because (b, d) is not an inversion of $\pi^{\uparrow}(w)$, it is not an inversion of any element of C . There must be some permutations $v, v' \in C$ such that v' covers v in the weak order, $a < d$ is an ascent of v , and v' is obtained from v by performing a valid swap that swaps a and d . However, it follows from Lemma 7.8 that a and d are to the right of all even elements of $[a + 1, d - 1]$ (including b) and to the left of all odd elements of $[a + 1, d - 1]$ in v . This means that swapping a and d to reach v' is actually not valid, which is a contradiction. \square

Lemma 7.10. *Let $b \leq n - 1$ be an integer. Suppose that either $w \in \pi_{\downarrow}(S_{n+1})$ has an ascent $a < b$ or b is the first integer in w . Suppose also that w has a descent $b + 1 > d$ such that $b + 1$ appears to the right of b . Then either $\pi^{\uparrow}(w)$ has an ascent of the form $e < b + 2$, or $b + 1$ is to the left of $b + 2$ in $\pi^{\uparrow}(w)$ and all integers appearing between $b + 1$ and $b + 2$ in $\pi^{\uparrow}(w)$ are larger than $b + 2$.*

Proof. First we prove that all integers to the left of b in w are smaller than b . Assume for contradiction that there is an integer to the left of b that is larger than b , and let x be the rightmost such integer. There is a descent $x > y$, where y is less than b (a and y may coincide). Both b and $b + 1$ are to the right of x and y , so the descent $x > y$ is not right-even or left-even; this contradicts the fact that $w \in \pi_{\downarrow}(S_{n+1})$.

Now, Lemma 7.7 tells us that b appears to the left of both $b + 1$ and $b + 2$ in $\pi^{\uparrow}(w)$. Because $(d, b + 1)$ is an inversion of w , it is also an inversion of $\pi^{\uparrow}(w)$ (since $w \leq \pi^{\uparrow}(w)$); that is, $b + 1$ appears to the left of d in $\pi^{\uparrow}(w)$. If the lemma we are trying to prove is false, then we must have one of the following cases:

- **Case 1:** $b + 2$ lies to the right of b and to the left of $b + 1$ in $\pi^{\uparrow}(w)$, and $\pi^{\uparrow}(w)$ has a descent of the form $d > b + 2$.
- **Case 2:** $b + 2$ lies to the right of $b + 1$ in $\pi^{\uparrow}(w)$, and there is an integer e between $b + 1$ and $b + 2$ in $\pi^{\uparrow}(w)$ that is smaller than b .

In Case 1, let x_1 be the rightmost integer to the left of $b + 2$ in $\pi^{\uparrow}(w)$ that is less than $b + 1$. Then there is an ascent $x_1 < y_1$ in $\pi^{\uparrow}(w)$ with y_1 larger than $b + 2$. But then, since $b + 1$ and $b + 2$ are both to the right of y_1 , we can perform a valid swap by swapping x_1 and y_1 ; this contradicts the maximality of $\pi^{\uparrow}(w)$.

In Case 2, let x_2 be the rightmost integer to the left of $b + 2$ in $\pi^{\uparrow}(w)$ that is less than b . Then x_2 is to the right of $b + 1$, and there is an ascent $x_2 < y_2$ in $\pi^{\uparrow}(w)$. But then, since b and $b + 1$ are both to the left of x_2 , we can perform a valid swap by swapping x_2 and y_2 ; this contradicts the maximality of $\pi^{\uparrow}(w)$. \square

Lemma 7.11. *Let $w \in \pi_{\downarrow}(S_{n+1})$ have an ascent $a < b$, and suppose there exists some integer smaller than a appearing to the right of b in w . Then $\pi^{\uparrow}(w)$ must have an ascent of the form $a < d$.*

Proof. Take x to be the leftmost integer to the right of b in w that is smaller than a . Then there must be a descent $y > x$ in w , where y is larger than a (b and y may coincide). By Lemma 7.8, a must appear to the left of y in $\pi^{\uparrow}(w)$. We will reach $\pi^{\uparrow}(w)$ from w by performing a particular sequence of valid swaps, each of which swaps the two integers in an ascent and results in a permutation in the same c -biCambrian congruence class. Throughout the sequence of valid swaps, let M denote the consecutive subsequence beginning at a and ending at y .

We claim that whenever we swap an ascent $e < a$, we can choose to swap e with every integer in M from left to right until it is to the right of y before moving on to perform other swaps. This is because we maintain throughout the swapping process that all the integers in M are at least a (except during the process of moving e through the block M from left to right). Indeed, we only add to the set of integers in M when we swap y with an integer that is larger than y and to its right (which adds an integer larger than a to M) or when we swap a with an integer that is smaller than a and to its left (which we immediately move through the subsequence M until it exits to the right of y). When we move an integer e smaller than a through M , each valid swap results in a permutation in the same c -biCambrian congruence class. This is because the first swap of the ascent $e < a$ is valid, so there are integers k, m of opposite parities on the same side of the

ascent satisfying $e < k < m < a$. Since M consists of integers at least a , this means that k and m are not in M —thus, we can choose these two integers to see that any swap of e with an integer in M is valid. It follows that $\pi^\uparrow(w)$ must have only integers larger than a to the right of a and to the left of y , which implies that $\pi^\uparrow(w)$ has an ascent of the form $a < d$. \square

We are now in a position to state and prove our critical theorem that will allow us to describe rowmotion on type- A biCambrian lattices. Recall that s_i denotes the simple transposition in S_{n+1} that swaps i and $i + 1$; when $i \in [n - 1]$, we can also view s_i as a transposition in S_n . Recall also that we defined a bipartite Coxeter element $c = c_+c_-$ in S_{n+1} . In the following theorem, we will need to use the analogues of c_+ and c_- in S_n instead of S_{n+1} . To avoid confusion, we denote these elements by $c'_+ = \prod_{i \in [n-1] \text{ even}} s_i$ and $c'_- = \prod_{i \in [n-1] \text{ odd}} s_i$. For $\sigma \in S_n$ and $X \subseteq [n]$, we use the notation $\sigma X = \{\sigma(x) : x \in X\}$.

Theorem 7.12. *If $w \in \pi_\downarrow(S_{n+1})$, then*

$$T^\uparrow(\pi^\uparrow(w)) = [n] \setminus c'_+S_\downarrow(w), \quad S^\uparrow(\pi^\uparrow(w)) = [n] \setminus c'_-T_\downarrow(w),$$

$$T_\downarrow(w) = [n] \setminus c'_-S^\uparrow(\pi^\uparrow(w)), \quad \text{and} \quad S_\downarrow(w) = [n] \setminus c'_+T^\uparrow(\pi^\uparrow(w)).$$

Proof. The latter two equalities follow immediately from the first two. We will only prove the first equality since the proof of the second is analogous. It suffices to prove each of the following statements:

- Exactly one of the following holds: $1 \in S_\downarrow(w)$ or $1 \in T^\uparrow(\pi^\uparrow(w))$.
- For even $k \in [2, n - 1]$, exactly one of the following holds: $k \in S_\downarrow(w)$ or $k + 1 \in T^\uparrow(\pi^\uparrow(w))$.
- For even $k \in [2, n - 1]$, exactly one of the following holds: $k + 1 \in S_\downarrow(w)$ or $k \in T^\uparrow(\pi^\uparrow(w))$.
- If n is even, then exactly one of the following holds: $n \in S_\downarrow(w)$ or $n \in T^\uparrow(\pi^\uparrow(w))$.

All of these statements have similar proofs, so we will only prove the second statement.

First, let us assume by way of contradiction that $k \in S_\downarrow(\pi_\downarrow(w))$ and $k + 1 \in T^\uparrow(\pi^\uparrow(w))$. The condition $k \in S_\downarrow(w)$ is equivalent to one of the following mutually exclusive cases being true for w :

- **Case 1A:** There is a right-even descent $b_1 > k$ for some $b_1 \geq k + 2$.
- **Case 1B:** There is a left-even descent $k + 1 > a_1$ for some a_1 .

The condition $k + 1 \in T^\uparrow(\pi^\uparrow(w))$ is equivalent to one of the following mutually exclusive cases being true for $\pi^\uparrow(w)$:

- **Case 2A:** There is a right-even ascent $a_2 < k + 2$ for some $a_2 \leq k$.
- **Case 2B:** There is a left-even ascent $k + 1 < b_2$ for some b_2 .

Recall that we have assumed k is even. In Case 1A, the integers $k+2, k, k+1$ lie in that order in w , and in Case 1B, the integers $k+1, k$ lie in that order in w . In Case 2A, the integers $k, k+2, k+1$ lie in that order in $\pi^\uparrow(w)$, and in Case 2B, the integers $k+1, k+2$ lie in that order in $\pi^\uparrow(w)$. If Case 1A holds, then it follows from Lemma 7.7 that neither Case 2A nor Case 2B can hold. If Case 1B holds, then Lemma 7.7 tells us that Case 2A cannot hold, and Lemma 7.9 tells us that Case 2B cannot hold. This exhausts all possibilities and yields our desired contradiction.

Now we prove that we cannot have both $k \notin S_\downarrow(w)$ and $k+1 \notin T^\uparrow(\pi^\uparrow(w))$ for even $k \in [2, n-1]$, which will complete the proof. Assume the contrary. The condition $k \notin S_\downarrow(w)$ is equivalent to one of the following two mutually exclusive cases being true for w :

- **Case 3A:** There is a left-even descent $b_3 > k$ for some $b_3 \geq k+2$.
- **Case 3B:** Either there is an ascent $a_3 < k$ for some a_3 , or the first integer is k .

and one of the following two mutually exclusive cases being true in w :

- **Case 3C:** There is a right-even descent $k+1 > a_4$ for some $a_4 \leq k-1$.
- **Case 3D:** Either there is an ascent $k+1 < b_4$ for some b_4 , or the last integer is $k+1$.

Also, the condition $k+1 \notin T^\uparrow(\pi^\uparrow(w))$ is equivalent to one of the following two mutually exclusive cases being true for $\pi^\uparrow(w)$:

- **Case 4A:** There is a right-even ascent $k+1 < b_5$ for some $b_5 \geq k+3$.
- **Case 4B:** Either there is a descent $k+1 > a_5$ for some a_5 , or the last integer is $k+1$.

and one of the following two mutually exclusive cases being true for $\pi^\uparrow(w)$:

- **Case 4C:** There is a left-even ascent $a_6 < k+2$ for some $a_6 \leq k$.
- **Case 4D:** Either there is a descent $b_6 > k+2$ for some b_6 , or the first integer is $k+2$.

In Case 3A, the integer $k+1$ is to the left of k and $k+2$ in w , and in Case 3C, the integers $k, k+1, k-1$ lie in that order in w . In Case 4A, the integers $k+2, k+1, k+3$ lie in that order in $\pi^\uparrow(w)$, and in Case 4C, the integer $k+1$ is to the left of k and $k+2$ in $\pi^\uparrow(w)$. So it is not possible for either Cases 3A or 4C to hold at the same time as either Cases 3C or 4A.

If Cases 3D and 4B both hold, then by Lemma 7.11, we must have that all integers to the right of $k+1$ in w are larger than $k+1$. But then Case 3A cannot also hold. An analogous argument implies that all integers to the right of $k+1$ in $\pi^\uparrow(w)$ are smaller than $k+1$, so Case 4C also cannot hold. To summarize, if Cases 3D and 4B both hold, then neither Case 3A nor Case 4C can hold.

Now, assume Cases 3B and 3C both hold. By Lemma 7.10, we know that either $\pi^\uparrow(w)$ has an ascent of the form $e < k+2$, or $k+1$ is to the left of $k+2$ and all integers between $k+1$ and $k+2$ are larger than $k+2$. If we additionally assume that Case 4D holds, the former possibility is ruled out. We also know that either Case 4A or Case 4B must hold, but both are contradicted by

| Case 3 | Case 4 | Reason | Case 3 | Case 4 | Reason | Case 3 | Case 4 | Reason | Case 3 | Case 4 | Reason |
|--------|--------|--------|--------|--------|------------|--------|--------|------------|--------|--------|----------------|
| AC | AC | 3A, 3C | AD | AC | 3A, 4A | BC | AC | 4A, 4C | BD | AC | 4A, 4C |
| AC | AD | 3A, 3C | AD | AD | 3A, 4A | BC | AD | 3B, 3C, 4D | BD | AD | 3B, 4A, 4D |
| AC | BC | 3A, 3C | AD | BC | 3A, 3D, 4B | BC | BC | 3C, 4C | BD | BC | 3D, 4B, 4C |
| AC | BD | 3A, 3C | AD | BD | 3A, 3D, 4B | BC | BD | 3B, 3C, 4D | BD | BD | 3B, 3D, 4B, 4D |

Table 7.1: The many cases in the proof of Theorem 7.12.

the latter possibility. That is, it is not possible for Cases 3B, 3C, and 4D to all hold. Analogously, we can prove that Cases 3B, 4A, and 4D cannot all hold simultaneously.

Finally, consider when Cases 3B, 3D, 4B, and 4D all hold. By considering Cases 3D and 4B, from Lemma 7.11 we have that k is to the left of $k + 1$ in w . Analogously, we have that $k + 2$ is to the left of $k + 1$ in $\pi^\uparrow(w)$. Because a valid swap can only swap numbers that differ by at least 3, we know that the numbers in the set $\{k, k + 1, k + 2\}$ appear in the same order in w as in $\pi^\uparrow(w)$; this order is either $k, k + 2, k + 1$ or $k + 2, k, k + 1$. Without loss of generality, assume the order is $k, k + 2, k + 1$; we will show that Case 4D provides a contradiction (when the order is $k + 2, k, k + 1$, Case 3B provides an analogous contradiction). Then in $\pi^\uparrow(w)$, there is a descent $b_6 > k + 2$ to the right of k and to the left of $k + 1$. Let x be the rightmost integer to the left of $k + 2$ that is less than $k + 1$. Then $\pi^\uparrow(w)$ has an ascent $x < y$ with y larger than $k + 2$. But then $k + 1$ and $k + 2$ both lie to the right of x and y , so the ascent $x < y$ is not right-even or left even. This contradicts the fact that $\pi^\uparrow(w) \in \pi^\uparrow(S_{n+1})$. Hence, it is not possible for Cases 3B, 3D, 4B, and 4D to all hold at once.

We have exhausted all $2^4 = 16$ possible scenarios, as shown in Table 7.1. For an example of how to read this table, consider the entries in the first three columns in the first row. The entries in the first two columns assert that Cases 3A, 3C, 4A, and 4C cannot all hold simultaneously, and the entry in the third column asserts that the reason for this is that (as we have already established) Cases 3A and 3C cannot hold simultaneously.

Thus, we conclude that exactly one of $k \in S_\downarrow(\pi_\downarrow(w))$ and $k + 1 \in T^\uparrow(\pi^\uparrow(w))$ holds. \square

Recall that we write Ω_n for the set of all pairs (S, T) such that $S, T \subseteq [n]$ and $|S| = |T|$. Recall also the bijection $\xi_\downarrow: \pi_\downarrow(S_{n+1}) \rightarrow \Omega_n$ from Theorem 7.1. Define the operator $\chi: \Omega_n \rightarrow \Omega_n$ by $\chi(S, T) = ([n] \setminus c'_+T, [n] \setminus c'_-S)$.

Theorem 7.13. For $w \in \pi_\downarrow(S_{n+1})$, we have $\xi_\downarrow(\text{Row}(w)) = \chi(\xi_\downarrow(w))$.

Proof. Let $b_1 > a_1, \dots, b_k > a_k$ be the descents of w . By Proposition 7.4, the canonical join representation of w in $\pi_\downarrow(S_{n+1})$ is $\mathcal{D}(w) = \{\lambda(w, b_i > a_i) : 1 \leq i \leq k\}$. As discussed in Section 2.3, the canonical meet representation of $\text{Row}(w)$ in $\pi_\downarrow(S_{n+1})$ is

$$\kappa(\mathcal{U}(\text{Row}(w))) = \kappa(\mathcal{D}(w)) = \text{Row}(\mathcal{D}(w)) = \{\text{Row}(\lambda(w, b_i > a_i)) : 1 \leq i \leq k\}.$$

Consequently, the canonical meet representation of $\pi^\uparrow(\text{Row}(w))$ in $\pi^\uparrow(S_{n+1})$ is

$$\{\pi^\uparrow(\text{Row}(\lambda(w, b_i > a_i))) : 1 \leq i \leq k\}.$$

Theorem 7.12 and (7.2) tell us that

$$S_\downarrow(\text{Row}(w)) = [n] \setminus c'_+T^\uparrow(\pi^\uparrow(\text{Row}(w))) = [n] \setminus \bigcup_{i \in [k]} c'_+T^\uparrow(\pi^\uparrow(\text{Row}(\lambda(w, b_i > a_i)))).$$

For each $i \in [k]$, it follows from Proposition 7.6 and the definitions of T_\downarrow and T^\uparrow that

$$T^\uparrow(\pi^\uparrow(\text{Row}(\lambda(w, b_i > a_i)))) = T_\downarrow(\lambda(w, b_i > a_i)).$$

Hence,

$$S_\downarrow(\text{Row}(w)) = [n] \setminus \bigcup_{i \in [k]} c'_+ T_\downarrow(\lambda(w, b_i > a_i)) = [n] \setminus c'_+ T_\downarrow(w),$$

where we have used (7.1). A completely analogous argument shows that $T_\downarrow(\text{Row}(w)) = [n] \setminus c'_- S_\downarrow(w)$ as well. Therefore,

$$\xi_\downarrow(\text{Row}(w)) = (S_\downarrow(\text{Row}(w)), T_\downarrow(\text{Row}(w))) = ([n] \setminus c'_+ T_\downarrow(w), [n] \setminus c'_- S_\downarrow(w)) = \chi(\xi_\downarrow(w)). \quad \square$$

The previous theorem shows that the operator $\chi: \Omega_n \rightarrow \Omega_n$ has the same orbit structure as rowmotion on the c -biCambrian lattice. To complete the proof of Theorem 6.2 in type A , we need to relate χ to rowmotion on $\mathcal{J}([n] \times [n])$, where we recall that $[n] \times [n]$ is the doubled root poset of type A_n .

The elements of the rectangle poset $[n] \times [n]$ are ordered pairs (x, y) with $x, y \in [n]$. Let $\{0, 1\}_n^{2n}$ denote the set of words in $\{0, 1\}^{2n}$ that have exactly n occurrences of 0 and exactly n occurrences of 1. Given an order ideal I of $[n] \times [n]$, let $\max(I)$ denote the set of maximal elements of I . If $\max(I) = \{(x_1, y_1), \dots, (x_k, y_k)\}$, then the *Stanley–Thomas word* of I is the word $\text{ST}(I) = u_1 \cdots u_n v_1 \cdots v_n \in \{0, 1\}_n^{2n}$ defined by

$$u_j = \begin{cases} 1, & \text{if } j \in \{x_1, \dots, x_k\} \\ 0, & \text{if } j \notin \{x_1, \dots, x_k\} \end{cases} \quad \text{and} \quad v_j = \begin{cases} 0, & \text{if } j \in \{y_1, \dots, y_k\} \\ 1, & \text{if } j \notin \{y_1, \dots, y_k\}. \end{cases}$$

For example, if I is the order ideal of $[5] \times [5]$ such that $\max(I) = \{(2, 3), (4, 2), (5, 1)\}$, then $\text{ST}(I) = 0101100011$.

Let $\text{cyc}: \{0, 1\}_n^{2n} \rightarrow \{0, 1\}_n^{2n}$ denote the cyclic shift operator defined by

$$\text{cyc}(z_1 z_2 \cdots z_{2n}) = z_2 \cdots z_{2n} z_1.$$

The following fundamental result, which was first described by Stanley in [Sta09] and later discussed further by Propp and Roby in [PR15], relates rowmotion on $\mathcal{J}([n] \times [n])$ to this cyclic shift operator.

Theorem 7.14 ([PR15, Proposition 26]). *The map $\text{ST}: \mathcal{J}([n] \times [n]) \rightarrow \{0, 1\}_n^{2n}$ is a bijection such that $\text{ST}(\text{Row}(I)) = \text{cyc}(\text{ST}(I))$ for all $I \in \mathcal{J}([n] \times [n])$.*

We can now prove Theorem 6.2 in type A .

Proposition 7.15. *The orbit structure of the operator $\text{Row}: \pi_\downarrow(S_{n+1}) \rightarrow \pi_\downarrow(S_{n+1})$ is the same as that of $\text{Row}: \mathcal{J}([n] \times [n]) \rightarrow \mathcal{J}([n] \times [n])$.*

Proof. Theorem 7.13 tells us that $\text{Row}: \pi_\downarrow(S_{n+1}) \rightarrow \pi_\downarrow(S_{n+1})$ has the same orbit structure as $\chi: \Omega_n \rightarrow \Omega_n$, and Theorem 7.14 tells us that $\text{Row}: \mathcal{J}([n] \times [n]) \rightarrow \mathcal{J}([n] \times [n])$ has

the same orbit structure as $\text{cyc}: \{0, 1\}_n^{2n} \rightarrow \{0, 1\}_n^{2n}$. Therefore, it suffices to find a bijection $G: \Omega_n \rightarrow \{0, 1\}_n^{2n}$ such that $\text{cyc} \circ G = G \circ \chi$.

Given $X \subseteq [n]$ and $i \in [n]$, let $X\langle i \rangle = 0$ if $i \notin X$, and let $X\langle i \rangle = 1$ if $i \in X$. Note that if $\sigma \in S_n$, then $(\sigma X)\langle i \rangle = X\langle \sigma^{-1}(i) \rangle$. Let (p_1, \dots, p_n) be the n -tuple obtained by listing the odd elements of $[n]$ in increasing order and then listing the even elements of $[n]$ in decreasing order. For example, if $n = 7$, then $(p_1, \dots, p_7) = (1, 3, 5, 7, 6, 4, 2)$. This n -tuple is chosen so that $c'_+c'_-p_j = p_{j+1}$ for all $j \in [n]$ (with indices taken modulo n). Given $(S, T) \in \Omega_n$, let

$$G(S, T) = T^*\langle p_n \rangle S\langle p_n \rangle T^*\langle p_{n-1} \rangle S\langle p_{n-1} \rangle \cdots T^*\langle p_1 \rangle S\langle p_1 \rangle,$$

where $T^* = [n] \setminus c'_-T$. We claim that this defines our desired bijection $G: \Omega_n \rightarrow \{0, 1\}_n^{2n}$.

We first need to check that $G(S, T)$ is actually in $\{0, 1\}_n^{2n}$, meaning that it has exactly n occurrences of 0 and exactly n occurrences of 1. This is immediate from the definition of G since $|T^*| = n - |T| = n - |S|$ (by the definition of Ω_n). We can easily recover T from T^* (since $T = [n] \setminus c'_-T^*$), so G is evidently injective. We have $|\Omega_n| = |\{0, 1\}_n^{2n}| = \binom{2n}{n}$, so G must be a bijection.

We are left to prove that $\text{cyc} \circ G = G \circ \chi$. Consider $(S, T) \in \Omega_n$, and let $i \in [n]$. For $x \in \{0, 1\}_n^{2n}$ and $k \in [2n]$, let x_k denote the k -th letter in x , where indices are taken modulo $2n$ (so $x_{2n+1} = x_1$ and $x_0 = x_{2n}$). We have

$$\text{cyc}(G(S, T))_{2i} = G(S, T)_{2i+1} = T^*\langle p_{n-i} \rangle = ([n] \setminus c'_-T)\langle p_{n-i} \rangle = 1 - (c'_-T)\langle p_{n-i} \rangle.$$

Since c'_- and c'_+ are involutions and $c'_-(p_{n-i}) = c'_+(c'_+c'_-(p_{n-i})) = c'_+(p_{n-i+1})$, we have

$$\begin{aligned} 1 - (c'_-T)\langle p_{n-i} \rangle &= 1 - T\langle c'_-(p_{n-i}) \rangle \\ &= 1 - T\langle c'_+(p_{n-i+1}) \rangle \\ &= 1 - (c'_+T)\langle p_{n-i+1} \rangle \\ &= ([n] \setminus c'_+T)\langle p_{n-i+1} \rangle \\ &= G(\chi(S, T))_{2i}. \end{aligned}$$

Thus, $\text{cyc}(G(S, T))$ and $G(\chi(S, T))$ have the same $(2i)$ -th letter. We also have

$$\begin{aligned} \text{cyc}(G(S, T))_{2i-1} &= G(S, T)_{2i} \\ &= S\langle p_{n-i+1} \rangle = S^{**}\langle p_{n-i+1} \rangle \\ &= G([n] \setminus c'_+T, S^*)_{2i-1} \\ &= G(\chi(S, T))_{2n-i}, \end{aligned}$$

so $\text{cyc}(G(S, T))$ and $G(\chi(S, T))$ have the same $(2i + 1)$ -th letter as well. We conclude that $\text{cyc}(G(S, T)) = G(\chi(S, T))$. □

Example 7.16. Suppose $n = 7$. Let $S = \{2, 3, 6\}$ and $T = \{1, 2, 4\}$. Then $(S, T) \in \Omega_7$. We have $T^* = [7] \setminus c'_-T = \{4, 5, 6, 7\}$, so

$$G(S, T) = T^*\langle 2 \rangle S\langle 2 \rangle T^*\langle 4 \rangle S\langle 4 \rangle T^*\langle 6 \rangle S\langle 6 \rangle T^*\langle 7 \rangle S\langle 7 \rangle T^*\langle 5 \rangle S\langle 5 \rangle T^*\langle 3 \rangle S\langle 3 \rangle T^*\langle 1 \rangle S\langle 1 \rangle$$

$$= 01101110100100.$$

Thus, $\text{cyc}(G(S, T)) = 11011101001000$. Now, $\chi(S, T) = ([7] \setminus c'_+T, [7] \setminus c'_-S) = (\tilde{S}, \tilde{T})$, where $\tilde{S} = \{2, 4, 6, 7\}$ and $\tilde{T} = \{2, 3, 6, 7\}$. Then $\tilde{T}^* = [7] \setminus c'_-\tilde{T} = \{2, 3, 6\} = S$, so

$$\begin{aligned} G(\chi(S, T)) &= \tilde{T}^* \langle 2 \rangle \tilde{S} \langle 2 \rangle \tilde{T}^* \langle 4 \rangle \tilde{S} \langle 4 \rangle \tilde{T}^* \langle 6 \rangle \tilde{S} \langle 6 \rangle \tilde{T}^* \langle 7 \rangle \tilde{S} \langle 7 \rangle \tilde{T}^* \langle 5 \rangle \tilde{S} \langle 5 \rangle \tilde{T}^* \langle 3 \rangle \tilde{S} \langle 3 \rangle \tilde{T}^* \langle 1 \rangle \tilde{S} \langle 1 \rangle \\ &= 11011101001000 = \text{cyc}(G(S, T)). \end{aligned}$$

8. Type B BiCambrian Lattices

In this section, we prove Theorem 6.2 in type B . The general strategy is to deduce the result for type B from the results derived in the preceding section for type A .

Let $w_0 = (2n)(2n-1)\cdots 1$ denote the longest element of the Coxeter group S_{2n} . There is a group automorphism $\beta: S_{2n} \rightarrow S_{2n}$ given by $\beta(x) = w_0 x w_0$. Explicitly, $\beta(x)(i) = 2n+1-x(2n+1-i)$ for all $1 \leq i \leq 2n$. The Coxeter group B_n is the n -th hyperoctahedral group, which can be realized as the subgroup of S_{2n} consisting of all $x \in S_{2n}$ such that $\beta(x) = x$.

Let s_i denote the simple transposition in S_{2n} that swaps i and $i+1$. The simple generators of B_n are s_1^B, \dots, s_n^B , where $s_i^B = s_i s_{2n-i}$ for $1 \leq i \leq n-1$ and $s_n^B = s_n$. Let $c = c_+ c_-$ be the bipartite Coxeter element of S_{2n} , where $c_+ = \prod_{i \in [2n-1] \text{ even}} s_i$ and $c_- = \prod_{i \in [2n-1] \text{ odd}} s_i$. We can also write $c_+ = \prod_{i \in [n] \text{ even}} s_i^B$ and $c_- = \prod_{i \in [n] \text{ odd}} s_i^B$, showing that c can be viewed as a bipartite Coxeter element of B_n . Our goal is to understand rowmotion on the associated type- B c -biCambrian lattice.

As in the preceding section, we write $\pi_\downarrow: S_{2n} \rightarrow S_{2n}$ for the operator that sends a permutation to the minimal element of its type- A c -biCambrian congruence class. We write

$$\pi_\downarrow^B: B_n \rightarrow B_n$$

for the operator that sends an element of B_n to the minimal element of its type- B c -biCambrian congruence class.

It is well known that β is a lattice automorphism of the weak order on S_{2n} . It follows that the weak order on B_n is a sublattice of the weak order on S_{2n} . Recall that the c -biCambrian lattice of type B is isomorphic to the set $\pi_\downarrow^B(B_n)$ under the weak order. It is straightforward to check that β restricts to a bijection from $\pi_\downarrow(S_{2n})$ to itself. Since β is an automorphism of the weak order on S_{2n} , it follows that $\beta: \pi_\downarrow(S_{2n}) \rightarrow \pi_\downarrow(S_{2n})$ is a lattice automorphism. According to [BR18, Proposition 3.15], we have

$$\pi_\downarrow^B(B_n) = B_n \cap \pi_\downarrow(S_{2n}). \quad (8.1)$$

Therefore, the set of fixed points of $\beta: \pi_\downarrow(S_{2n}) \rightarrow \pi_\downarrow(S_{2n})$ is $\pi_\downarrow^B(B_n)$; this readily implies that $\pi_\downarrow^B(B_n)$ is a sublattice of $\pi_\downarrow(S_{2n})$. Moreover, in the proof of the type- B case of [BR18, Theorem 2.12], Barnard and Reading explained that the type- B c -biCambrian congruence is simply the restriction of the type- A c -biCambrian congruence to B_n . This means that if $y \in B_n$,

then y and $\pi_{\downarrow}^B(y)$ are in the same type- A c -biCambrian congruence class, so $\pi_{\downarrow}(\pi_{\downarrow}^B(y)) = \pi_{\downarrow}(y)$. But (8.1) tells us that $\pi_{\downarrow}^B(y) \in \pi_{\downarrow}(S_{2n})$, so $\pi_{\downarrow}(\pi_{\downarrow}^B(y)) = \pi_{\downarrow}^B(y)$. This proves that

$$\pi_{\downarrow}(y) = \pi_{\downarrow}^B(y) \text{ for all } y \in B_n. \tag{8.2}$$

The Coxeter element c is fixed by the automorphism β . As discussed in the proof of the type- B case of [BR18, Theorem 2.12], this implies that β preserves the type- A c -biCambrian congruence on S_{2n} . In other words,

$$\pi_{\downarrow}(\beta(x)) = \beta(\pi_{\downarrow}(x)) \text{ for all } x \in S_{2n}. \tag{8.3}$$

In what follows, let us write \wedge , \wedge_A , and \wedge_B for the meet operations in the weak order on S_{2n} , in $\pi_{\downarrow}(S_{2n})$, and in $\pi_{\downarrow}^B(B_n)$, respectively.

Lemma 8.1. *If $y \in S_{2n}$, then $y \wedge \beta(y) \in B_n$, and $\pi_{\downarrow}^B(y \wedge \beta(y)) = \pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y))$.*

Proof. Since β is an involutive lattice automorphism of the weak order on S_{2n} , we have

$$\beta(y \wedge \beta(y)) = \beta(y) \wedge \beta^2(y) = y \wedge \beta(y).$$

This proves that $y \wedge \beta(y) \in B_n$.

We have seen that β is a lattice automorphism of $\pi_{\downarrow}(S_{2n})$. Therefore, it follows from (8.3) that

$$\begin{aligned} \beta(\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y))) &= \beta(\pi_{\downarrow}(y)) \wedge_A \beta(\pi_{\downarrow}(\beta(y))) \\ &= \pi_{\downarrow}(\beta(y)) \wedge_A \pi_{\downarrow}(\beta^2(y)) \\ &= \pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)). \end{aligned}$$

This proves that $\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)) \in B_n$. Since $\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)) \in \pi_{\downarrow}(S_{2n})$, it follows from (8.1) that $\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)) \in \pi_{\downarrow}^B(B_n)$. Because $\pi_{\downarrow}(y) \leq y$ and $\pi_{\downarrow}(\beta(y)) \leq \beta(y)$, we have

$$\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)) \leq y \wedge \beta(y).$$

The map π_{\downarrow}^B is order-preserving, so

$$\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y)) = \pi_{\downarrow}^B(\pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y))) \leq \pi_{\downarrow}^B(y \wedge \beta(y)).$$

To prove the reverse inequality, observe that

$$\pi_{\downarrow}^B(y \wedge \beta(y)) = \pi_{\downarrow}(y \wedge \beta(y)) \leq \pi_{\downarrow}(y) \quad \text{and} \quad \pi_{\downarrow}^B(y \wedge \beta(y)) = \pi_{\downarrow}(y \wedge \beta(y)) \leq \pi_{\downarrow}(\beta(y));$$

this implies that $\pi_{\downarrow}^B(y \wedge \beta(y)) \leq \pi_{\downarrow}(y) \wedge_A \pi_{\downarrow}(\beta(y))$. □

We are going to make use of the pop-stack sorting operator defined in Section 2.3. To avoid confusion, let us write Pop_A for the pop-stack sorting operator on $\pi_{\downarrow}(S_{2n})$ and Pop_B for the pop-stack sorting operator on $\pi_{\downarrow}^B(B_n)$.

Proposition 8.2. *If $x \in \pi_{\downarrow}^B(B_n)$, then $\text{Pop}_A(x) = \text{Pop}_B(x)$.*

Proof. If x is the identity permutation, then $\text{Pop}_A(x) = \text{Pop}_B(x) = x$. Now assume x is not the identity permutation. Let y_1, \dots, y_k be the elements covered by x in the weak order on S_{2n} . The elements of B_n covered by x are the elements of the form $y_i \wedge \beta(y_i)$. The elements covered by x in $\pi_\downarrow(S_{2n})$ are $\pi_\downarrow(y_1), \dots, \pi_\downarrow(y_k)$, and the elements covered by x in $\pi_\downarrow^B(B_n)$ are those of the form $\pi_\downarrow^B(y_i \wedge \beta(y_i))$. Using Lemma 8.1 and the fact that $\pi_\downarrow^B(B_n)$ is a sublattice of $\pi_\downarrow(S_{2n})$, we find that

$$\text{Pop}_B(x) = \bigwedge_B \{ \pi_\downarrow^B(y_i \wedge \beta(y_i)) : 1 \leq i \leq k \} = \bigwedge_A \{ \pi_\downarrow(y_i) \wedge_A \pi_\downarrow(\beta(y_i)) : 1 \leq i \leq k \}.$$

Because $x \in B_n$, we have $\beta(y_i) \in \{y_1, \dots, y_k\}$ for each i . Thus,

$$\bigwedge_A \{ \pi_\downarrow(y_i) \wedge_A \pi_\downarrow(\beta(y_i)) : 1 \leq i \leq k \} = \bigwedge_A \{ \pi_\downarrow(y_i) : 1 \leq i \leq k \} = \text{Pop}_A(x). \quad \square$$

In what follows, we write Row_B for rowmotion on the lattice $\pi_\downarrow^B(B_n)$, and we write Row_A for rowmotion on $\pi_\downarrow(S_{2n})$.

Proposition 8.3. *If $x \in \pi_\downarrow^B(B_n)$, then $\text{Row}_A(x) = \text{Row}_B(x)$.*

Proof. Since β is a lattice automorphism of $\pi_\downarrow(S_{2n})$, it commutes with Row_A . Consequently, $\text{Row}_A(x) = \text{Row}_A(\beta(x)) = \beta(\text{Row}_A(x))$. This shows that $\text{Row}_A(x) \in B_n$. We also know that $\text{Row}_A(x) \in \pi_\downarrow(S_{2n})$, so (8.1) tells us that $\text{Row}_A(x) \in \pi_\downarrow^B(B_n)$. Since $\pi_\downarrow^B(B_n)$ is a sublattice of $\pi_\downarrow(S_{2n})$, we can use Propositions 2.2 and 8.2 to see that

$$\text{Row}_A(x) \wedge_B x = \text{Row}_A(x) \wedge_A x = \text{Pop}_A(x) = \text{Pop}_B(x).$$

Proposition 2.2 tells us that $\text{Row}_B(x)$ is the unique maximal element of the set

$$\{ z \in \pi_\downarrow^B(B_n) : z \wedge_B x = \text{Pop}_B(x) \}.$$

We have shown that $\text{Row}_A(x)$ is in this set, so $\text{Row}_A(x) \leq \text{Row}_B(x)$.

On the other hand, we have $\text{Row}_B(x) \in \pi_\downarrow(S_{2n})$, and we can use Propositions 2.2 and 8.2 once again to see that $\text{Row}_B(x) \wedge_A x = \text{Row}_B(x) \wedge_B x = \text{Pop}_B(x) = \text{Pop}_A(x)$. This shows that $\text{Row}_B(x)$ is an element of the set $\{ z \in \pi_\downarrow(S_{2n}) : z \wedge_A x = \text{Pop}_A(x) \}$. The unique maximal element of this set is $\text{Row}_A(x)$, so $\text{Row}_B(x) \leq \text{Row}_A(x)$. \square

Recall that the doubled root posets of types A_{2n-1} and B_n are the rectangle poset $[2n-1] \times [2n-1]$ and the shifted staircase with $n(2n-1)$ elements, respectively (see Figure 6.1). Let SS_n denote the shifted staircase with $n(2n-1)$ elements. We can view SS_n as the subposet of $[2n-1] \times [2n-1]$ consisting of all elements (x, y) with $x \leq y$. There is a natural embedding

$$\iota: \mathcal{J}(\text{SS}_n) \rightarrow \mathcal{J}([2n-1] \times [2n-1])$$

given by $\iota(I) = \{(x, y), (y, x) : (x, y) \in I\}$. It is straightforward to check that ι commutes with rowmotion. In other words, $\text{Row}(\iota(I)) = \iota(\text{Row}(I))$ for all $I \in \mathcal{J}(\text{SS}_n)$, where the first rowmotion is on the lattice $\mathcal{J}([2n-1] \times [2n-1])$ and the second is on $\mathcal{J}(\text{SS}_n)$. Thus, the orbit structure of rowmotion on SS_n is the same as the orbit structure of rowmotion on the

sublattice $\iota(\mathcal{J}(\text{SS}_n))$ of $\mathcal{J}([2n - 1] \times [2n - 1])$; note that this sublattice consists of all order ideals $I \in \mathcal{J}([2n - 1] \times [2n - 1])$ that are symmetric in the sense that $(x, y) \in I$ if and only if $(y, x) \in I$.

We can now prove Theorem 6.2 in type B .

Proposition 8.4. *The orbit structure of the operator $\text{Row}_B: \pi_{\downarrow}^B(B_n) \rightarrow \pi_{\downarrow}^B(B_n)$ is the same as that of $\text{Row}: \mathcal{J}(\text{SS}_n) \rightarrow \mathcal{J}(\text{SS}_n)$.*

Proof. As discussed above, rowmotion on $\mathcal{J}(\text{SS}_n)$ has the same orbit structure as rowmotion on the sublattice $\iota(\mathcal{J}(\text{SS}_n))$ of $\mathcal{J}([2n - 1] \times [2n - 1])$. Recall from Theorem 7.14 that we have a map ST that sends each order ideal $I \in \mathcal{J}([2n - 1] \times [2n - 1])$ to its Stanley–Thomas word $\text{ST}(I) \in \{0, 1\}_{2n-1}^{2(2n-1)}$ and satisfies $\text{ST}(\text{Row}(I)) = \text{cyc}(\text{ST}(I))$. Let Ξ_{2n-1} be the subset of $\{0, 1\}_{2n-1}^{2(2n-1)}$ consisting of all words $u_1 \cdots u_{2(2n-1)}$ such that $u_i = 1 - u_{i+2n-1}$ for all $1 \leq i \leq 2(2n - 1)$, where indices are taken modulo $2(2n - 1)$. It is straightforward to check that Ξ_{2n-1} is the image of the set $\iota(\mathcal{J}(\text{SS}_n))$ under ST . Therefore, it suffices to show that the orbit structure of $\text{Row}_B: \pi_{\downarrow}^B(B_n) \rightarrow \pi_{\downarrow}^B(B_n)$ is the same as that of $\text{cyc}: \Xi_{2n-1} \rightarrow \Xi_{2n-1}$.

Theorem 7.13 and the proof of Proposition 7.15 provide bijections $\xi_{\downarrow}: \pi_{\downarrow}(S_{2n}) \rightarrow \Omega_{2n-1}$ and $G: \Omega_{2n-1} \rightarrow \{0, 1\}_{2n-1}^{2(2n-1)}$ such that $G(\xi_{\downarrow}(\text{Row}_A(x))) = \text{cyc}(G(\xi_{\downarrow}(x)))$ for all $x \in \pi_{\downarrow}(S_{2n})$. We will show that the image of $\pi_{\downarrow}^B(B_n)$ under $G \circ \xi_{\downarrow}$ is Ξ_{2n-1} . Proposition 8.3 tells us that the map $\text{Row}_B: \pi_{\downarrow}^B(B_n) \rightarrow \pi_{\downarrow}^B(B_n)$ is just the restriction of Row_A to $\pi_{\downarrow}^B(B_n)$, so this will complete the proof.

Given a set $S \subseteq [2n - 1]$, let $2n - S = \{2n - s : s \in S\}$. The proof of [BR18, Theorem 3.18] asserts (using different notation) that the image of $\pi_{\downarrow}^B(B_n)$ under ξ_{\downarrow} is the set of all pairs $(S, 2n - S)$ such that $S \subseteq [2n - 1]$ (this is also straightforward to verify directly). Therefore, we must check that G maps the set of such pairs bijectively onto Ξ_{2n-1} . The number of pairs $(S, 2n - S)$ with $S \subseteq [2n - 1]$ is 2^{2n-1} , which is also the size of Ξ_{2n-1} ; since G is injective, we just need to show that $G(S, 2n - S) \in \Xi_{2n-1}$ for all $S \subseteq [2n - 1]$.

As in the proof of Proposition 7.15, we let (p_1, \dots, p_{2n-1}) be the tuple obtained by listing the odd elements of $[2n - 1]$ in increasing order and then listing the even elements of $[2n - 1]$ in decreasing order. Let $c'_- = \prod_{i \in [2n-2]_{\text{odd}}} s_i$. Given $T \subseteq [2n - 1]$, let $T^* = [2n - 1] \setminus c'_- T$. Recalling the definition of the map G , we find that

$$G(S, 2n - S) = (2n - S)^* \langle p_{2n-1} \rangle S \langle p_{2n-1} \rangle (2n - S)^* \langle p_{2n-2} \rangle S \langle p_{2n-2} \rangle \cdots (2n - S)^* \langle p_1 \rangle S \langle p_1 \rangle.$$

To show that this word is in Ξ_{2n-1} , it suffices to check that $(2n - S)^* \langle p_i \rangle = 1 - S \langle p_{i-n+1} \rangle$ for all $1 \leq i \leq 2n - 1$ (with indices taken modulo $2n - 1$). This is straightforward to verify directly. \square

9. Completing the Proof of Theorem 6.2

In Propositions 7.15 and 8.4, we resolved Theorem 6.2 for biCambrian lattices of types A and B . The purpose of this brief section is to handle the remaining cases: types H_3 and $I_2(m)$. We have checked by computer that the conjecture holds in type H_3 , so we only need to prove that it holds for $I_2(m)$.

The Coxeter group $I_2(m)$ is the dihedral group of order $2m$. Let s_1 and s_2 denote the simple generators of $I_2(m)$. Let $[s_1, s_2]_k$ denote the alternating product $s_1 s_2 s_1 \cdots$ of length k that starts with s_1 , and let $[s_2, s_1]_k$ denote the alternating product $s_2 s_1 s_2 \cdots$ of length k that starts with s_2 . The weak order on $I_2(m)$ consists of the two disjoint $(m-1)$ -chains

$$[s_1, s_2]_1 \triangleleft [s_1, s_2]_2 \triangleleft \cdots \triangleleft [s_1, s_2]_{m-1} \quad \text{and} \quad [s_2, s_1]_1 \triangleleft [s_2, s_1]_2 \triangleleft \cdots \triangleleft [s_2, s_1]_{m-1}$$

together with the identity element e below both chains and the longest element w_0 above both chains.

Consider the bipartite Coxeter element $c = s_1 s_2$. The c -biCambrian lattice is actually equal to the weak order on $I_2(m)$. (For instance, see [BR18, Theorem 1.4], which asserts that the size of the c -biCambrian lattice is $2m = |I_2(m)|$.) For $2 \leq k \leq m-1$, we have $\text{Row}([s_1, s_2]_k) = [s_1, s_2]_{k-1}$ and $\text{Row}([s_2, s_1]_k) = [s_2, s_1]_{k-1}$. Furthermore, $\text{Row}(s_1) = [s_2, s_1]_{m-1}$, $\text{Row}(s_2) = [s_1, s_2]_{m-1}$, $\text{Row}(e) = w_0$, and $\text{Row}(w_0) = e$. This shows that rowmotion acting on the c -biCambrian lattice has one orbit of size 2 and one orbit of size $2m-2$. See Figure 9.1.

The doubled root poset of type $I_2(m)$ consists of two chains $x_1 \triangleleft \cdots \triangleleft x_{m-2}$ and $y_1 \triangleleft \cdots \triangleleft y_{m-2}$ together with two incomparable elements z_1 and z_2 satisfying $x_{m-2} \triangleleft z_1 \triangleleft y_1$ and $x_{m-2} \triangleleft z_2 \triangleleft y_1$. It is straightforward to check that rowmotion acting on the lattice of order ideals of this doubled root poset has one orbit of size 2 and one orbit of size $2m-2$. We illustrate this for $m=5$ in Figure 9.1. This proves that Theorem 6.2 holds in type $I_2(m)$, which completes the proof of the full theorem.

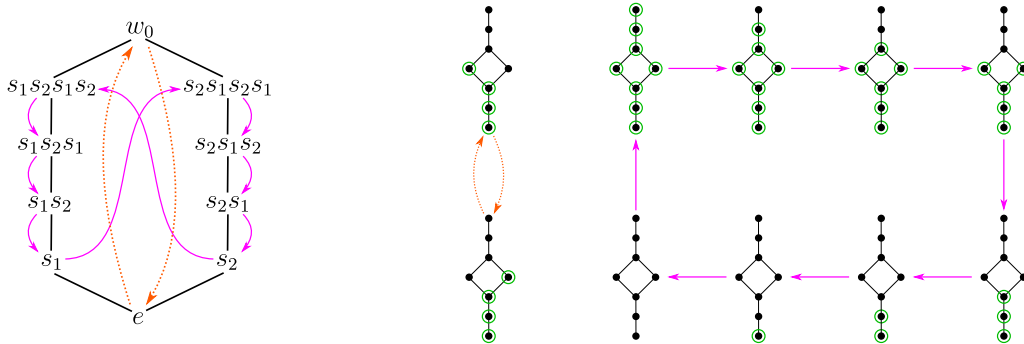


Figure 9.1: On the left is the action of rowmotion on the type- $I_2(5)$ c -biCambrian lattice; this lattice is the same as the weak order on $I_2(5)$. On the right is the action of rowmotion on $\mathcal{J}(P)$, where P is the doubled root poset of type $I_2(5)$. Each order ideal in $\mathcal{J}(P)$ is represented by elements that are circled in green.

10. Future Directions

It would be interesting to extend our results about rowmotion on m -Tamari lattices to the setting of the rational Tamari lattices $\text{Tam}(a, b)$. In particular, recall Conjecture 5.15 and

Conjecture 5.16, which state that $(\text{Tam}(a, b), \text{Row}, \text{Cat}^{(a,b)}(q))$ exhibits the cyclic sieving phenomenon and that the down-degree statistic on $\text{Tam}(a, b)$ is homomesic for rowmotion with average $(a - 1)(b - 1)/(a + b - 1)$.

In general, the down-degree statistic on a ν -Tamari lattice need not be homomesic for rowmotion. However, data suggests that it is “almost” homomesic in the sense that the averages of ddeg on the orbits are all very close to each other. For example, if $\nu = \text{NE}^3 \text{NE}^2 \text{NE}^6 \text{N}$, then rowmotion has orbits of sizes 7, 7, 12, 18, 18, 18, 80; the average values of ddeg along these orbits are (approximately) 2.286, 2.286, 2.25, 2.278, 2.278, 2.278, 2.263, respectively. It would be very interesting to have some explanation of why these values are so close. The following conjecture can be seen as predicting a new generalization of the homomesy phenomenon that we call *asymptotic homomesy*. At the moment, this notion is fairly vague, and we do not wish to give a formal definition of it.

Conjecture 10.1. Let ν_0 be a lattice path that has a north steps and b east steps. For each integer $t \geq 1$, let ν_0^t be the lattice path obtained by concatenating ν_0 (viewed as a word over $\{N, E\}$) with itself t times. Let $\text{Orb}_{\text{Row}}(\text{Tam}(\nu_0^t))$ be the set of orbits of $\text{Row}: \text{Tam}(\nu_0^t) \rightarrow \text{Tam}(\nu_0^t)$. We have

$$\lim_{t \rightarrow \infty} \max_{\mathcal{O} \in \text{Orb}_{\text{Row}}(\text{Tam}(\nu_0^t))} \left| \frac{1}{t} \cdot \frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} \text{ddeg}(x) - \frac{ab}{a+b} \right| = 0.$$

Recall from Section 2.5 that a statistic on a set X is called *homometric* for a function $f: X \rightarrow X$ if it always has the same sum over f -orbits of the same cardinality.

Conjecture 10.2. Let ν be a lattice path. The down-degree statistic $\text{ddeg}: \text{Tam}(\nu) \rightarrow \mathbb{R}$ is homometric for the rowmotion operator $\text{Row}: \text{Tam}(\nu) \rightarrow \text{Tam}(\nu)$.

Finally, we remark that it would be nice to have a more conceptual, type-independent proof of Theorem 6.2.

Acknowledgements

We thank Joe Gallian, Bruce Sagan, and Nathan Williams for helpful comments. We thank the anonymous referees for reading carefully through our paper and making several useful comments.

References

[Arm09] D. Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202(949), 2009. doi:10.1090/s0065-9266-09-00565-1.

[AST13] D. Armstrong, C. Stump, and H. Thomas. A uniform bijection between nonnesting and noncrossing partitions. *Trans. Amer. Math. Soc.*, 365(8), 2013. doi:10.1090/s0002-9947-2013-05729-7.

- [Bar19] E. Barnard. The canonical join complex. *Electron. J. Combin.*, 26(1), 2019. doi:[10.37236/7866](https://doi.org/10.37236/7866).
- [BB05] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, 2005.
- [Bir37] G. Birkhoff. Ring of sets. *Duke Math. J.*, 3:443–454, 1937.
- [BMCPR13] M. Bousquet-Mélou, G. Chapuy, and L.-F. Préville-Ratelle. The representation of the symmetric group on m -Tamari intervals. *Adv. Math.*, 247:309–342, 2013. doi:[10.1016/j.aim.2013.07.014](https://doi.org/10.1016/j.aim.2013.07.014).
- [BMFPR11] M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the m -Tamari lattices. *Electron. J. Combin.*, 18(2), 2011. doi:[10.37236/2027](https://doi.org/10.37236/2027).
- [BPR12] F. Bergeron and L.-F. Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. *J. Comb.*, 3(3):317–341, 2012. doi:[10.4310/joc.2012.v3.n3.a4](https://doi.org/10.4310/joc.2012.v3.n3.a4).
- [BR16] M. Bodnar and B. Rhoades. Cyclic sieving and rational Catalan theory. *Electron. J. Combin.*, 23(2), 2016. doi:[10.37236/5681](https://doi.org/10.37236/5681).
- [BR18] E. Barnard and N. Reading. Coxeter-biCatalan combinatorics. *J. Algebraic Combin.*, 47:241–300, 2018. doi:[10.1007/s10801-017-0775-1](https://doi.org/10.1007/s10801-017-0775-1).
- [Bro75] A. Brouwer. On dual pairs of antichains. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, pages 1–8, 1975.
- [BS74] A. Brouwer and A. Schrijver. On the period of an operator, defined on antichains. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, 1974.
- [CFdF95] P. J. Cameron and D. G. Fon-der Flaass. Orbits of antichains revisited. *European J. Combin.*, 16(6):545–554, 1995. doi:[10.1016/0195-6698\(95\)90036-5](https://doi.org/10.1016/0195-6698(95)90036-5).
- [CP15] G. Châtel and V. Pons. Counting smaller elements in the Tamari and m -Tamari lattices. *J. Combin. Theory Ser. A*, 134:58–97, 2015. doi:[10.1016/j.jcta.2015.03.004](https://doi.org/10.1016/j.jcta.2015.03.004).
- [CPS20] C. Ceballos, A. Padrol, and C. Sarmiento. The ν -Tamari lattice via ν -trees, ν -bracket vectors, and subword complexes. *Electron. J. Combin.*, 27(1), 2020. doi:[10.37236/8000](https://doi.org/10.37236/8000).
- [CS15] M. Cuntz and C. Stump. On root posets for noncrystallographic root systems. *Math. Comp.*, 84(291):485–503, 2015. doi:[10.1090/s0025-5718-2014-02841-x](https://doi.org/10.1090/s0025-5718-2014-02841-x).
- [Def22] C. Defant. Meeting covered elements in ν -Tamari lattices. *Adv. Appl. Math.*, 134, 2022. doi:[10.1016/j.aam.2021.102303](https://doi.org/10.1016/j.aam.2021.102303).
- [DF90] Michel Marie Deza and Komei Fukuda. Loops of clutters. In *Coding Theory and Design Theory*, pages 72–92, New York, NY, 1990. Springer New York. doi:[10.1007/978-1-4613-8994-1_7](https://doi.org/10.1007/978-1-4613-8994-1_7).
- [DM47] A. Dvoretzky and Th. Motzkin. A problem of arrangements. *Duke Math. J.*, 14(2):305–313, 1947. doi:[10.1215/s0012-7094-47-01423-3](https://doi.org/10.1215/s0012-7094-47-01423-3).

- [Duc74] P. Duchet. Sur les hypergraphes invariants. *Discrete Math.*, 8(3):269–280, 1974. doi:[10.1016/0012-365x\(74\)90139-3](https://doi.org/10.1016/0012-365x(74)90139-3).
- [DW23] C. Defant and N. Williams. Semidistributive lattices. *Forum. Math. Sigma*, 11, 2023. doi:[10.1017/fms.2023.46](https://doi.org/10.1017/fms.2023.46).
- [EPRS23] Sergi Elizalde, Matthew Plante, Tom Roby, and Bruce E. Sagan. Rowmotion on fences. *Algebraic Combinatorics*, 6(1):17–36, 2023. doi:[10.5802/alco.256](https://doi.org/10.5802/alco.256).
- [FdF93] D. G. Fon-der Flaass. Orbits of antichains in ranked posets. *European J. Combin.*, 14(1):17–22, 1993. doi:[10.1006/eujc.1993.1003](https://doi.org/10.1006/eujc.1993.1003).
- [Hop19] S. Hopkins. The CDE property for skew vexillary permutations. *J. Combin. Theory Ser. A*, 168:164–218, 2019. doi:[10.1016/j.jcta.2019.06.005](https://doi.org/10.1016/j.jcta.2019.06.005).
- [Hum80] J. E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge University Press, 1980.
- [Kre72] G. Kreweras. Sur les partitions non croisées d’un cycle. *Discrete Math.*, 1(4):333–350, 1972. doi:[10.1016/0012-365x\(72\)90041-6](https://doi.org/10.1016/0012-365x(72)90041-6).
- [MHPS12] F. Müller-Hoissen, J. M. Pallo, and J. Stasheff. *Associahedra, Tamari lattices, and related structures*, volume 299 of *Progress in Mathematics*. Birkhäuser, 2012.
- [Pan09] D. I. Panyushev. On orbits of antichains of positive roots. *European J. Combin.*, 30(2):586–594, 2009. doi:[10.1016/j.ejc.2008.03.009](https://doi.org/10.1016/j.ejc.2008.03.009).
- [PR15] J. Propp and T. Roby. Homomesy in products of two chains. *Electron. J. Combin.*, 22(3), 2015. doi:[10.37236/3579](https://doi.org/10.37236/3579).
- [PRV17] L.-F. Préville-Ratelle and X. Viennot. The enumeration of generalized Tamari intervals. *Trans. Amer. Math. Soc.*, 369(7):5219–5239, 2017. doi:[10.1090/tran/7004](https://doi.org/10.1090/tran/7004).
- [Rea06] N. Reading. Cambrian lattices. *Adv. Math.*, 205(2):313–353, 2006. doi:[10.1016/j.aim.2005.07.010](https://doi.org/10.1016/j.aim.2005.07.010).
- [Rea07] N. Reading. Clusters, Coxeter-sortable elements and noncrossing partitions. *Trans. Amer. Math. Soc.*, 359(12):5931–5958, 2007. doi:[10.1090/s0002-9947-07-04319-x](https://doi.org/10.1090/s0002-9947-07-04319-x).
- [Rob16] T. Roby. Dynamical algebraic combinatorics and the homomesy phenomenon. In *Recent trends in combinatorics*, volume 159 of *IMA Math. Vol. Appl.*, pages 619–652. Springer, 2016. doi:[10.1007/978-3-319-24298-9_25](https://doi.org/10.1007/978-3-319-24298-9_25).
- [RS13] D. B. Rush and X. Shi. On orbits of order ideals of minuscule posets. *J. Algebraic Combin.*, 37:545–569, 2013. doi:[10.1007/s10801-012-0380-2](https://doi.org/10.1007/s10801-012-0380-2).
- [RSW04] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *Journal of Combinatorial Theory, Series A*, 108(1):17–50, 2004. doi:[10.1016/j.jcta.2004.04.009](https://doi.org/10.1016/j.jcta.2004.04.009).
- [RW15] D. B. Rush and K. Wang. On orbits of order ideals of minuscule posets II: homomesy. 2015. [arXiv:1509.08047](https://arxiv.org/abs/1509.08047).

- [Sta99] R. P. Stanley. *Enumerative Combinatorics, vol. 2*. Cambridge University Press, 1999.
- [Sta09] R. P. Stanley. Promotion and evacuation. *Electron. J. Combin.*, 16(2), 2009. doi:[10.37236/75](https://doi.org/10.37236/75).
- [SW12] J. Striker and N. Williams. Promotion and rowmotion. *European J. Combin.*, 33(8):1919–1942, 2012. doi:[10.1016/j.ejc.2012.05.003](https://doi.org/10.1016/j.ejc.2012.05.003).
- [Tam62] D. Tamari. The algebra of bracketings and their enumeration. *Nieuw Archief voor Wiskunde*, 10:131–146, 1962.
- [Tho06] H. Thomas. An analogue of distributivity for ungraded lattices. *Order*, 23:249–269, 2006. doi:[10.1007/s11083-006-9046-9](https://doi.org/10.1007/s11083-006-9046-9).
- [TW19] H. Thomas and N. Williams. Rowmotion in slow motion. *Proc. Lond. Math. Soc.*, 119(5):1149–1178, 2019. doi:[10.1112/plms.12251](https://doi.org/10.1112/plms.12251).