CYCLES AND CUTS IN SUPERSINGULAR *L*-ISOGENY GRAPHS

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ABSTRACT. Supersingular elliptic curve isogeny graphs underlie isogeny-based cryptography. For isogenies of a single prime degree ℓ , their structure has been investigated graph-theoretically. We generalise the notion of ℓ -isogeny graphs to *L*-isogeny graphs (studied in the prime field case by Delfs and Galbraith), where *L* is a set of small primes dictating the allowed isogeny degrees in the graph. We analyse the graph-theoretic structure of *L*-isogeny graphs. Our approaches may be put into two categories: cycles and graph cuts.

On the topic of cycles, we provide: a count for the number of non-backtracking cycles in the L-isogeny graph using traces of Brandt matrices; an efficiently computable estimate based on this approach; and a third ideal-theoretic count for a certain subclass of L-isogeny cycles. We provide code to compute each of these three counts.

On the topic of graph cuts, we compare several algorithms to compute graph cuts which minimise a measure called the *edge expansion*, outlining a cryptographic motivation for doing so. Our results show that a *greedy neighbour* algorithm out-performs standard spectral algorithms for computing optimal graph cuts. We provide code and study explicit examples.

Furthermore, we describe several directions of active and future research.

1. INTRODUCTION

Isogeny-based cryptography has spurred significant research into supersingular elliptic curve isogeny graphs, making research advancements on the underlying structures increasingly critical as cryptographic protocols evolve. Supersingular elliptic curve ℓ -isogeny graphs are Ramanujan [Piz80] and individual isogeny steps in the graph have a low computational cost, making these graphs desirable for cryptographic applications. The first supersingular isogeny-based cryptographic protocol [CLG09] was only developed quite recently, and mathematicians have only been studying these graphs with cryptographic applications in mind for the past two decades.

In this work, we consider supersingular elliptic curve *L*-isogeny graphs over \mathbb{F}_p where $L = \{\ell_1, \ldots, \ell_r\}$ is a collection of allowed (prime) isogeny degrees. This work is motivated by [DG16] who studied such graphs over \mathbb{F}_p . In isogeny-based cryptographic protocols where multiple degrees are allowed, an *L*-isogeny graph is the structure underlying the security of the protocol. These graphs are $(\ell_1 + \cdots + \ell_r + r)$ -regular, but no longer satisfy the Ramanujan property. Having more edges means these graphs have more *cycles*. Cycles in ℓ -isogeny graphs have been wellstudied [BCNE⁺19, EHL⁺20, FIK⁺23, Orv24]. A graph cycle starting at a vertex corresponding to the elliptic curve *E* gives an endomorphism of *E*. Computing the endomorphism ring of a supersingular elliptic curve is an active area of research and a question which underlies almost all of isogeny-based cryptography.

1.1. **Our contributions.** In this work, we study the supersingular elliptic curve L-isogeny graph, a generalisation of the supersingular elliptic curve ℓ -isogeny graph where edges correspond to isogenies of degree ℓ_i for multiple primes $\ell_i \in L$. In this graph, we:

- provide explicit counts of cycles in Section 4: one count using the theory of Brandt matrices in Section 4.1 and the other count using the theory of embeddings in quaternion algebras in Section 4.2. These two sections count slightly different classes of cycles.
- study edge expansion in the *L*-isogeny graph using methodology inspired by Fiedler cuts in Section 5.

We provide code in SageMath [S⁺25] at: https://github.com/jtcc2/cyclesand-cuts. This includes implementations of all our counting arguments, clustering algorithms, figures, and examples.

1.2. **Related work.** While this work was in preparation, the authors were made aware of $[KKA^{+}24]$, which also studies cycles in *L*-isogeny graphs. The authors $[KKA^{+}24]$ take a computational approach with the goal of computing endomorphism rings using cycles found in *L*-isogeny graphs, applying an approach similar to $[EHL^{+}20]$. Their study of cycles restricts to a very special class, aligning with the work of $[EHL^{+}20]$. Our work is a complementary extension: we consider the general class of isogeny cycles and provide explicit cycle counts.

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2. Preliminaries

2.1. Supersingular elliptic curves. We recall some standard facts about supersingular elliptic curves. For more detail, see [Sil94, Voi21]. An elliptic curve over a finite field \mathbb{F}_q is supersingular if and only if the geometric endomorphism ring of E, $\operatorname{End}(E) := \operatorname{End}_{\mathbb{F}_q}(E)$, is a maximal order in a quaternion algebra. The classical Deuring correspondence [Deu41] gives a categorical equivalence through which we can interpret isogenies of elliptic curves as left ideals of maximal orders in a quaternion algebra. To a given left ideal I of $\operatorname{End}(E)$, we associate an isogeny φ_I with kernel given by the scheme-theoretic intersection

$$\ker \varphi_I = \bigcap_{\alpha \in I} \ker \alpha.$$

When I is a principal ideal, φ_I is an endomorphism. If two left ideals I, J of End(E) are in the same ideal class, the codomains of φ_I and φ_J are isomorphic. We almost have a group action: However, the set of left ideal classes of End(E) is not a group.

If E is a supersingular elliptic curve defined over a prime field \mathbb{F}_p , the \mathbb{F}_p -part of its endomorphism ring $\operatorname{End}_{\mathbb{F}_p}(E)$ is an imaginary quadratic order. In this case, we have a true action of the class group of ideals: The invertible ideals of the imaginary quadratic order $\operatorname{End}_{\mathbb{F}_p}(E)$ act freely and transitively on the set of supersingular elliptic curves with endomorphism ring isomorphic to $\operatorname{End}_{\mathbb{F}_p}(E)$.

The \mathbb{F}_p -automorphism groups of supersingular elliptic curves are generally $\{[\pm 1]\}$. For precisely two $\overline{\mathbb{F}}_p$ -isomorphism classes of elliptic curves are these automorphism groups larger: j = 0,1728 [Sil09]. Curves E with j(E) = 0 or 1728 are said to have extra automorphisms.

2.2. *l*-isogeny graphs.

Definition 1 (ℓ -isogeny graphs). The ℓ -isogeny graph, denoted $\mathcal{G}(p, \ell)$ is the graph with vertices given by isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$, with edges given by degree- ℓ isogenies, up to post-composing with an automorphism.

A priori, an isogeny has a unique dual so we should be able to identify the edge corresponding to an isogeny with the edge corresponding to its dual and create an undirected graph. However, the definition of the edges in Definition 1 opens the door to some issues for vertices with automorphism groups larger than $[\pm 1]$: If $\varphi_1 : E_{1728} \to E$ and $\varphi_2 : E_{1728} \to E$ have the same domain with *j*-invariant 1728 and the same codomain, it may happen that $\hat{\varphi}_1 = [i] \circ \hat{\varphi}_2$. In this case, there are two edges from the vertex E_{1728} to the vertex E, but only one edge from E to E_{1728} . If we wish to create an undirected graph, we sacrifice the regularity of the graph. The issue can only arise at a few vertices, so given a fixed ℓ , this problem can be avoided completely by choosing p according to certain congruence conditions.

A cycle in the ℓ -isogeny graph given by explicit rational equations can be composed to an endomorphism. However, if we only specify kernels and one of the vertices in the cycle is an isomorphism class with extra automorphisms, there is no canonical way of identifying this cycle with an endomorphism: Composing with an extra automorphism may completely change the discriminant of the endomorphism obtained. There are finitely many extra automorphisms, so the set of possibilities is finite - the issue is identifying them. To handle this issue, the notion of *arbitrary assignment* was developed in [ACL⁺24].

Definition 2 (Arbitrary assignment, [ACL⁺24, Definition 3.3]). Given an ℓ -isogeny graph $\mathcal{G}(p,\ell)$, an arbitrary assignment is a choice (up to sign) of equivalence class representative $\pm \varphi$ for every edge [φ].

For edges yielding isogenies where the codomain *j*-invariants are not equal to 0 or 1728, an arbitrary assignment is automatic and does not involve any choice. When $p \equiv 1 \pmod{12}$, such an assignment is not necessary as [-1] commutes with every isogeny, so this can always be a post-composition, which is already an equivalence we place on edges.

2.3. L-isogeny graphs.



FIGURE 1. The supersingular $\{2,3\}$ -isogeny graph over $\overline{\mathbb{F}}_{61}$, with vertices labelled by *j*-invariant in \mathbb{F}_{61^2} . Solid lines are 2-isogenies, dashed lines are 3-isogenies, and $\alpha, \overline{\alpha}$ denote conjugate *j*-invariants in $\mathbb{F}_{61^2} \setminus \mathbb{F}_{61}$. Loops may only be traversed in one direction, while all other edges are undirected and may be traversed in either direction.

Notation 3. Let $p, \ell_1, \ldots, \ell_r$ be distinct primes. Let $L := \{\ell_1, \ldots, \ell_r\}$. For convenience and without loss of generality, suppose $\ell_1 < \ell_2 < \cdots < \ell_r$. In what follows, $e_i \in \mathbb{Z}_{>0}$ for all $i = 1, \ldots, r$.

In [Gha22], the author studies two isogeny graphs $\mathcal{G}(p, \ell_1), \mathcal{G}(p, \ell_2)$ simultaneously. We now consider a generalisation to an arbitrary number of isogeny graphs $\mathcal{G}(p, \ell_i)$, for $i = 1, \ldots, r$.

Definition 4 (*L*-isogeny graph). The *L*-isogeny graph, denoted $\mathcal{G}(p,L)$ is the graph with vertices given by isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$, with edges given by degree ℓ_i -isogenies (up to post-composing with an automorphism) for $i = 1, \ldots, r$.

As defined, *L*-isogeny graphs are directed graphs. As in the ℓ -isogeny case described in Section 2.2, they are undirected for appropriate congruence conditions on *p*. See Figure 1 for the $\{2, 3\}$ -isogeny graph of supersingular elliptic curves over $\overline{\mathbb{F}}_{61}$.

2.4. Brandt matrices. Let \mathcal{B}_p denote the (unique up to isomorphism) quaternion algebra over \mathbb{Q} ramified at p and ∞ . Two left ideals I, J in \mathcal{B}_p are equivalent if there exists $\beta \in \mathcal{B}_p^{\times}$ such that $J = I\beta$. The class set $cl(\mathfrak{O}) = \{I_1, I_2, \ldots\}$, for a maximal order \mathfrak{O} in \mathcal{B}_p , is the set of representatives of equivalence classes of left \mathfrak{O} -ideals. We let $I_1 = \mathfrak{O}$ by convention. The class group is finite and its cardinality n is called the *class number* of \mathcal{B}_p (it is the same for every maximal order \mathfrak{O} in \mathcal{B}_p). Let $O_R(I)$ denote the right order of the ideal I. The set $\Gamma_i := O_R(I_i)^{\times}/\mathbb{Z}^{\times}$ is finite, as it is a discrete subgroup of the compact Lie group $(\mathcal{B}_p \otimes \mathbb{R})^{\times}/\mathbb{R}^{\times} \cong SO_3(\mathbb{R})$. Let w_i be its cardinality. See [Voi21].

Following [Gro87], we will introduce theta series as well as Brandt matrices. Let the inverse ideal of I_i be defined as $I_i^{-1} := \{\beta \in \mathcal{B}_p : \beta I_i \subseteq I_i\}$, and let $M_{ij} := I_j^{-1}I_i = \{\sum_{k=1}^N a_k b_k : N \in \mathbb{N}, a_k \in I_j^{-1}, b_k \in I_i\}$. We define the reduced norm $\operatorname{Nm}(M_{ij})$ of M_{ij} to be the unique positive rational number such that all quotients $\operatorname{Nm}(a)/\operatorname{Nm}(M_{ij})$, for $a \in M_{ij}$, are coprime integers. We now define the following *theta series* θ_{ij} :

$$\theta_{ij}(\tau) := \frac{1}{2w_j} \sum_{a \in M_{ij}} q^{\frac{\operatorname{Nm}(a)}{\operatorname{Nm}(M_{ij})}} = \sum_{m \ge 0} B_{ij}(m) q^m,$$

where $q := e^{2\pi i \tau}$ and $\tau \in \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Finally, we define the *Brandt matrix* B(m) to be $B(m) := [B_{ij}(m)]_{1 \leq i,j \leq n}$, for $m \in \mathbb{N}$. Note that B(1) is the identity matrix. We have the following properties for Brandt matrices.

Proposition 5 ([Gro87, Proposition 2.7]).

(i) If $m \ge 1$, then $B_{ij}(m) \in \mathbb{N}$, and for all $1 \le i \le n$,

$$\sum_{j=1}^{n} B_{ij}(m) = \sum_{\substack{d \mid m \\ \gcd(d,p)=1}} d.$$

- (ii) If m and m' are coprime, then B(mm') = B(m)B(m').
- (iii) If $\ell \neq p$ is a prime, then

$$B(\ell^k) = B(\ell^{k-1})B(\ell) - \ell B(\ell^{k-2}),$$

for all $k \geq 2$.

(iv) $w_j B_{ij}(m) = w_i B_{ji}(m)$ for all m and for all $1 \le i, j \le n$.

The main reason for considering Brandt matrices in our study of isogeny graphs is the fact that $B_{ij}(m)$ is the number of equivalence classes of isogenies between the elliptic curves corresponding to I_i and I_j (cf. [Gro87, Proposition 2.3]). Therefore, the Brandt matrix $B(\ell)$ is the adjacency matrix of the supersingular isogeny graph $\mathcal{G}(p,\ell)$. Hence, in order to study cycles of degree ℓ^r in the graph $\mathcal{G}(p,\ell)$, one can alternatively look as the entries $B_{ii}(\ell^r)$ or even $\operatorname{Tr}(B_{ii}(\ell^r))$ to encompass all such cycles (counting all starting points). See [Gha22, GKPV21] for more details on this approach, in the case of two primes.

Now, computing the entries of $B(\ell^r)$ can be done using the recursion relation (iii), but this can be very inefficient for larger powers of ℓ . However, there is a way of computing certain traces of Brandt matrices, by re-expressing them in terms of modified Hurwitz class numbers, which we define below.

Given an order \mathcal{O} in an imaginary quadratic field, let d be its (negative) discriminant, h(d) the size of the class group $cl(\mathcal{O})$ and $u(d) := |\mathcal{O}^{\times}/\mathbb{Z}^{\times}|$. Fix D > 0 and let \mathcal{O}_{-D} be the order of discriminant -D. The Hurwitz Class Number H(D) is

(1)
$$H(D) = \sum_{d \cdot f^2 = -D} \frac{h(d)}{u(d)}.$$

and the modified Hurwitz Class Number $H_p(D)$, for a prime p, is

(2)
$$H_p(D) := \begin{cases} 0 & \text{if } p \text{ splits in } \mathcal{O}_{-D}; \\ H(D) & \text{if } p \text{ is inert in } \mathcal{O}_{-D}; \\ \frac{1}{2}H(D) & \text{if } p \text{ is ramified in } \mathcal{O}_{-D} \\ & \text{but does not divide the conductor of } \mathcal{O}_{-D}; \\ H(\frac{D}{p^2}) & \text{if } p \text{ divides the conductor of } \mathcal{O}_{-D}; \end{cases}$$

For D = 0, we set H(0) = -1/12 and $H_p(0) := \frac{p-1}{24}$. Note that $H_p(D) \leq H(D)$. We conclude this section with the following two results on Hurwitz class numbers. The first connects Hurwitz class numbers to the trace of Brandt matrices. The second gives us a way to compute certain sums of Hurwitz class numbers.

Theorem 6. [Gro87, Prop. 1.9] For all integers $m \ge 0$,

$$\operatorname{Tr}(B(m)) = \sum_{\substack{s \in \mathbb{Z} \\ s^2 \le 4m}} H_p(4m - s^2).$$

Theorem 7. [Hur85, Section 7] For all integers $m \ge 1$,

$$\sum_{\substack{s \in \mathbb{Z} \\ ^{2} < 4m}} H(4m - s^{2}) = 2 \sum_{d|m} d - \sum_{d|m} \min\{d, m/d\},$$

where the sum runs over the positive divisors of m.

2.5. Matrix representations of the graph. Let G = (V, E) be a graph. Fix an ordering of the vertices $V = (v_1, \ldots, v_n)$. The *adjacency matrix* of G with respect to the ordering (v_1, \ldots, v_n) is the $n \times n$ matrix A whose entries a_{ij} are the number of edges from v_i to v_j . If the graph is undirected, the adjacency matrix is symmetric. If the graph is directed, this may or may not be the case.

The degree of a vertex is the number of edges incident to that vertex. If the graph is directed, we define the in- and out-degree of a vertex to be the number of in- and out-edges, respectively. Given an ordering of the vertices $V = (v_1, \ldots, v_n)$, the (in- or out-)degree matrix is the diagonal matrix D whose nonzero entries d_{ii} are the (in- or out-)degrees of the vertices v_i .

The Laplacian of an undirected graph is the matrix L := D - A. If G is directed, we can define the in- or out-Laplacian where D is the in- or out-degree matrix, respectively.

2.6. Edge Expansion. A *cut* of a graph G = (V, E) is a partition of the vertices $V = V_1 \sqcup V_2$ with $V_1, V_2 \neq \emptyset$. The sets V_1 and V_2 are called the *sides* of the cut. The directed edges $e = (u, v) \in S$ which have $u \in V_1$ and $v \in V_2$, and undirected edges with one endpoint in each of V_1 and V_2 , are said to *cross* the cut, and the set of such edges is called the *cut set* of the cut, denoted $E(V_1, V_2)$. The cuts which are usually of most interest in graph theory are those which are relatively balanced (i.e., in which $|V_1| \approx |V_2|$) and relatively *sparse* (i.e., they have relatively few edges which cross the cut). These desirable properties are incorporated in the *edge expansion*, defined in Definition 8.

Definition 8 (Edge expansion). Let G = (V, E) be a graph. The edge expansion of a cut $V = C \sqcup (V \setminus C)$ is

$$h_G(C) := \frac{|E(C, V \setminus C)|}{\min\{\operatorname{Vol}(C), \operatorname{Vol}(V - C)\}}$$

where $\operatorname{Vol}(C)$ is the sum of the vertex degrees in C, which means for d-regular graphs, $\operatorname{Vol}(C) = d \cdot |C|$. The edge expansion of G is $h(G) := \min_{\varnothing \subseteq T \subseteq V} h_G(T)$.

3. Cycles in the L-isogeny graph

Cycles in isogeny graphs are of interest for two main reasons:

- (1) They correspond to collisions of walks in the isogeny graph [CLG09];
- (2) They correspond to endomorphisms of elliptic curves and can be used to compute a endomorphism rings [BCNE⁺19, EHL⁺20, FIK⁺23, KKA⁺24].

In this section and the following section, we restrict to the case $p \equiv 1 \pmod{12}$ to avoid counting issues arising from extra automorphisms. L never contains p.

3.1. Setup. We begin with some essential definitions.

Definition 9 (Isogeny cycle). An $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle¹ is a closed walk in $\mathcal{G}(p, \{\ell_1, \ldots, \ell_r\})$ consisting of e_i degree- ℓ_i isogenies for $i = 1, \ldots, r$, with $e_i > 0$. We denote the set of all $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycles in $\mathcal{G}(p, \{\ell_1, \ldots, \ell_r\})$ by Cycles $_p(\ell_1^{e_1}, \ldots, \ell_r^{e_r})$.

Remark 10. It is possible to discuss $\{\ell_i^{e_i}\}_{i=1}^r$ -isogeny cycles in an L-isogeny graph $\mathcal{G}(p,L)$ where $\{\ell_1,\ldots,\ell_r\} \subsetneq L$, but since the cycle does not contain edges of degree ℓ for any $\ell \in L \setminus \{\ell_1,\ldots,\ell_r\}$, we may simply consider this isogeny cycle as belonging to the graph $\mathcal{G}(p, \{\ell_1,\ldots,\ell_r\})$. Going forward, when we discuss $\{\ell_1^{e_1},\ldots,\ell_r^{e_r}\}$ -isogeny cycles, we automatically define $L := \{\ell_1,\ldots,\ell_r\}$.

Given a walk (resp. closed walk) in $\mathcal{G}(p, L)$ as a sequence of edges, we may successively compose compatible isogeny representatives corresponding to these edges to obtain an isogeny (resp. endomorphism) of elliptic curves. Each vertex represents an isomorphism class of elliptic curves, so the representative of each edge source is successively chosen in a compatible way.² The reverse process involves decomposing an isogeny in to a sequence of prime degree isogenies.

Definition 11. Let $\varphi : E \to E'$ be a separable isogeny of degree divisible only by primes contained in L. A (prime) decomposition of φ is a factorisation of φ into a sequence of (prime degree) isogenies

$$\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_2 \circ \varphi_1 =: (\varphi_i)_{i=1}^{\kappa}$$

Define $\mathcal{D}(\varphi)$ to be the set of prime decompositions of φ .

A prime decomposition yields a graphical representation of an isogeny as a walk in $\mathcal{G}(p, L)$, since each φ_i corresponds to an edge. In the context of the discussion above, we treat prime decompositions as equivalent if they yield the same graph theoretic walk. Even up to this equivalence, a prime decomposition need not be unique.

Lemma 12. Let $\varphi : E \to E'$ be an isogeny with kernel G such that deg φ is divisible only by primes contained in L. Let (P, \triangleleft) be the poset of subgroups of G, ordered by inclusion. There is a 1-1 correspondence between the prime decompositions of φ (elements of $\mathcal{D}(\varphi)$) and the maximal chains in (P, \triangleleft) .

Proof. Given a prime decomposition $\varphi = (\varphi_i)_{i=1}^k$, define the sequence of groups

$$G_0 = \{\mathcal{O}_E\}, G_i = \ker(\varphi_i \circ \varphi_{i-1} \circ \cdots \circ \varphi_1).$$

This sequence forms a chain in P as each φ_i is a group homomorphism. Suppose there exists $H \in P$ such that $G_{i-1} \triangleleft H \triangleleft G_i$. Comparing indexes, we have deg $\varphi_i = |G_i/G_{i-1}| = |G_i/H||H/G_{i-1}|$, and as deg φ_i is prime, either $H = G_i$ or $H = G_{i-1}$. As $G_k = G$, we conclude that this chain is maximal.

For the reverse direction, let $\{\mathcal{O}_E\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$ be a maximal chain in P. For each $1 \leq i \leq k$, there exists a unique curve E_i (up to isomorphism) and isogeny $\psi_i : E \to E_i$ (up to sign) with ker $\psi_i = G_i$ ([Sil09, Chapter III, Proposition 4.11]). Define $\varphi_1 = \psi_1$, and note that $\psi_k = \varphi$. For i > 1, define φ_i

¹We note that this is different from a cycle in the typical graph theoretic sense, which is a closed *path*—i.e., a closed walk with no repeated vertices.

 $^{^{2}}$ This is possible as no additional automorphisms exist, so pre- and post- composition of an isogeny with isomorphisms will always correspond to the same edge.

to be the unique isogeny satisfying $\psi_i = \varphi_i \circ \psi_{i-1}$ ([Sil09, Chapter III, Corollary 4.11]). This procedure yields a decomposition $\varphi = \psi_k = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$ where deg $\varphi_i = |G_i/G_{i-1}|$. Each of these indexes is prime as the G_i 's form a maximal chain. Finally, these correspondences are inverse of each other up to equivalence of decompositions.

As isogenies have finite kernels, $\mathcal{D}(\varphi)$ is always a finite set. We can compute $|D(\varphi)|$ from the prime power decomposition of deg φ together with knowledge of the largest integer n such that $\varphi(E[n]) = O_{E'}$. Suppose the isogeny φ is primitive ([BCNE⁺19, Definition 4.1]) and that deg $\varphi = \prod_{i=1}^{r} \ell_i^{e_i}$. Then,

(3)
$$|\mathcal{D}(\varphi)| = \frac{\left(\sum_{i=1}^{r} e_i\right)!}{\prod_{i=1}^{r} e_i!}$$

using a simple combinatorial argument [Tuc07, Section 5.3, Theorem 1]. In particular, if the number of distinct primes dividing deg φ is greater than 1, then $|D(\varphi)| > 1$: such isogenies do not have a unique prime decomposition.

Example 13 (Isogeny Diamond). Let E be an elliptic curve, and suppose φ, ψ are two isogenies of distinct prime degrees, each with domain E_1 :

$$\varphi: E_1 \to E_2 \quad \psi: E_1 \to E_3.$$

Define isogenies φ' from E_3 and ψ' from E_2 with ker $\varphi' = \psi(\ker \varphi)$ and ker $\psi' = \varphi(\ker \psi)$. The diagram in Figure 2 commutes [Sil09, Chapter III, Corollory 4.11]:



FIGURE 2. An isogeny diamond

The isogeny diamond gives a $\{\deg \varphi^2, \deg \psi^2\}$ -isogeny cycle in the $\{\deg \psi, \deg \varphi\}$ isogeny graph, namely $\widehat{\varphi} \circ \widehat{\psi}' \circ \varphi' \circ \psi =: \alpha \in \operatorname{End}(E_1)$. We see that $\alpha = [\deg \psi] \circ [\deg \varphi]$ by computing $\alpha(E[\deg \psi]) = \alpha(E[\deg \varphi]) = \mathcal{O}_E$.

In order to handle the issue of multiple cycles composing to the same endomorphism, we provide a canonical means of refactoring a given isogeny cycle: We refactor the isogeny, grouping the isogenies by increasing prime degree ($\ell_1 < \ell_2 < \cdots < \ell_r$, as in Notation 3). Consecutive isogenies of the same prime degree cannot be reordered; however they may compose to a scalar multiplication map, which may be reordered arbitrarily, so we obtain a canonical decomposition in this manner.

Definition 14 (Canonical decomposition). Let $\ell_1 < \ell_2 < \cdots < \ell_r$ be distinct primes. Given a fixed $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle, let α denote the endomorphism given by the composition of the isogenies in this isogeny cycle.

There is a largest positive integer n such that $\alpha(E[n]) = O_E$. Write $\frac{\deg(\alpha)}{n^2} = \ell_{i_1}^{f_1} \cdots \ell_{i_m}^{f_m}$ where i_1, \ldots, i_m is a subsequence of $1, \ldots, r$ such that f_* are non-zero.

The canonical decomposition of α is the decomposition:

(4)
$$\alpha = [n] \circ (\varphi_{m,f_m} \circ \cdots \circ \varphi_{m,1}) \circ \cdots \circ (\varphi_{2,f_2} \circ \cdots \circ \varphi_{2,1}) \circ (\varphi_{1,f_1} \circ \cdots \circ \varphi_{1,1}),$$

where deg $\varphi_{j,*} = \ell_{i_j}$. We note that [n] is not a walk in the $\{\ell_i\}$ -isogeny graph, but the endomorphism $(\varphi_{m,f_m} \circ \cdots \circ \varphi_{m,1}) \circ \cdots \circ (\varphi_{2,f_2} \circ \cdots \circ \varphi_{2,1}) \circ (\varphi_{1,f_1} \circ \cdots \circ \varphi_{1,1})$ is an $\{\ell_{i_i}^{f_j}\}$ -isogeny cycle.

Proposition 15. The canonical decomposition of an isogeny cycle is uniquely determined by the endomorphism resulting in the composition of the isogenies in the isogeny cycle, up to sign.

Proof. Follows from the proof of Algorithm 1.

Every isogeny cycle has a canonical decomposition. A key component to finding the canonical decomposition of an isogeny cycle is the following subroutine, which switches the order of coprime degree isogenies appearing in a decomposition.

Lemma 16 (Isogeny swapping). Suppose we are given an isogeny as the composition $\varphi \circ \psi$ of two isogenies of coprime degrees. Define an isogeny ψ' with $\ker \psi' := \widehat{\varphi}(\ker \psi)$ and an isogeny φ' with $\ker \varphi' := \psi'(\ker \varphi)$. Then,

$$\varphi \circ \psi = [\alpha] \psi' \circ \varphi',$$

where α is an automorphism of the codomain of φ .

Proof. The result follows directly from the definitions of φ', ψ' : the resulting diagram is an isogeny diamond (see Example 13).

Assuming $p \equiv 1 \pmod{12}$, the automorphism groups of all of the supersingular elliptic curves are $[\pm 1]$. Under this assumption, we can construct an algorithm to compute the (unique!) canonical decomposition of an isogeny cycle. Before presenting this, we briefly discuss the role of backtracking.

3.2. Backtracking. In the ℓ -isogeny case, a degree ℓ -power isogeny has non-cyclic kernel if and only if its path in the ℓ -isogeny graph contains backtracking [BCNE⁺19, Definition 4.3]. However, with *L*-isogeny graphs, the issue of backtracking is much more subtle: when r > 1, a degree- $\ell_1^{e_1} \cdots \ell_r^{e_r}$ isogeny can be specified by multiple paths in the $\{\ell_1, \ldots, \ell_r\}$ -isogeny graph.

Definition 17 (Backtracking). An isogeny cycle given by the composition $\varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$ has backtracking if $\varphi_{i+1} = \widehat{\varphi}_i$ for some $i \in \{1, 2, \dots, k-1\}$.

In $\mathcal{G}(p, \ell)$, a cycle has backtracking if and only if the endomorphism obtained by composing (compatible isogeny representatives of) edges in the cycle has noncyclic kernel [BCNE⁺19, Proposition 4.5]. However, this is not the case in $\mathcal{G}(p, L)$. Indeed, the isogeny cycle $\widehat{\varphi} \circ \widehat{\psi}' \circ \varphi' \circ \psi \in \operatorname{End}(E_1)$ from Example 13 does not have backtracking. By refactoring the cycle as $[\deg \psi] \circ [\deg \varphi]$ so that isogenies of the same degree are grouped, we see that it has non-cyclic kernel. This motivates the following algorithm for finding the canonical decomposition of an isogeny cycle.

³Lemma 16 allows us to swap *adjacent* isogenies of different degrees in an isogeny cycle. Iteratively applying this idea, we can sort the isogenies in increasing degree by applying any sorting algorithm that only makes adjacent swaps. BubbleSort is one such algorithm [Bla23].

Algorithm 1: Finding a canonical decomposition of an endomorphism **Input:** An endomorphism α of a supersingular elliptic curve E given as the composition of a list of (known) prime-degree isogenies: $(\varphi_1, \ldots, \varphi_N)$ **Output:** An ordered list (ψ_1, \ldots, ψ_M) of isogenies of non-decreasing prime degree, and an integer n such that the composition $[n]_E \circ \psi_M \circ \cdots \circ \psi_1$ is equal to α up to sign. 1 Apply BubbleSort³ to the list $\Phi := (\varphi_1, \ldots, \varphi_N)$ to sort by increasing degree, using the isogeny swapping procedure in Lemma 16 **2** Set $n_0 := 1$ **3** for $i := 0; i < len(\Phi)$ do if $\Phi[i+1] \circ \Phi[i] = [\pm \deg \Phi[i]]$ then 4 $\mathbf{5}$ Set $n_0 := n_0 \cdot \deg \Phi[i]$ Remove $\Phi[i], \Phi[i+1]$ from Φ 6 Set $i := \max(0, i - 1)$ $\mathbf{7}$ else 8 Set i := i + 19 10 return Φ , n_0 .

Lemma 18. Algorithm 1 is correct. More precisely, let α be the endomorphism obtained from an $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle represented as a prime decomposition $(\varphi_1, \ldots, \varphi_N)$. Let $(\tilde{\Phi}, \tilde{n})$ be the output of Algorithm 1 on input α , where $\tilde{\Phi} = (\psi_1, \ldots, \psi_M)$. Then the decomposition

$$[\tilde{n}]_E \circ \psi_M \circ \ldots \psi_1$$

is the canonical decomposition of α .

Proof. Let $\beta = \psi_M \circ \ldots \psi_1$, and let n be the largest integer such that $E[n] \subseteq \ker \alpha$. The list Φ obtained after line 1 in the algorithm gives a prime decomposition of α , with the isogenies ordered by increasing degree. This property is preserved by the loop, as well as the invariant $\alpha = [n_0]_E \circ \Phi[-1] \circ \cdots \circ \Phi[0]$. Hence the identity $\alpha = [\tilde{n}] \circ \beta$ is satisfied. The internal loop variable n_0 is non-decreasing, hence we have $\tilde{n} \leq n$. To show equality, note that if β factors through the scalar map $[\ell]$ for some $\ell \in L$, then $\psi_b \circ \ldots \psi_a$ must factor through $[\ell]$ where $\psi_a, \ldots \psi_b$ are the elements of $\tilde{\Phi}$ of degree ℓ . In turn, an ℓ^* -isogeny factors through $[\ell]$ if and only if it is backtracking, i.e. there exists an index c with $a \leq c < b$ such that $\psi_{c+1} = \pm \widehat{\psi_c}$. This condition is prohibited in the output by the conditional on lines 4-7.

Corollary 19. The walk corresponding to the non-scalar portion of the canonical decomposition of an isogeny cycle will have backtracking if and only if the corresponding endomorphism has non-cyclic kernel.

Proof. Since the canonical decomposition of an isogeny cycle groups together isogenies of the same degree, it contains backtracking if and only if it contains a path of degree- ℓ_i isogenies which contains backtracking [BCNE⁺19, Proposition 4.5].

Returning to Example 13, the canonical decomposition of the the isogeny cycle $\widehat{\varphi} \circ \widehat{\psi}' \circ \varphi' \circ \psi \in \operatorname{End}(E_1)$ is $\widehat{\psi} \circ \psi \circ \widehat{\varphi} \circ \varphi$, where deg $\varphi < \deg \psi$ without loss of



FIGURE 3. A $\{2^2, 3^3\}$ -isogeny cycle in the $\{2, 3\}$ -isogeny graph. The (solid line) isogenies φ_{2_1} and φ_{2_2} are degree-2 and the (dashed line) isogenies $\varphi_{3_1}, \varphi_{3_2}, \varphi_{3_3}$ are degree-3. This $\{2^2, 3^3\}$ -isogeny cycle can be specified by the tuple $(\varphi_{3_3}, \varphi_{3_2}, \varphi_{2_2}, \varphi_{3_1}, \varphi_{2_1})$, starting at the vertex j_1 .

generality. This canonical decomposition clearly both has non-cyclic kernel and the cycle contains backtracking.

Definition 20 (Principal isogeny cycles). An $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle is principal if its canonical decomposition does not contain backtracking. Let $\operatorname{Cycles}_p(\ell_1^{e_1}, \ldots, \ell_r^{e_r})^{\operatorname{prin}}$ denote the subset of principal cycles.

The endomorphism obtained from an isogeny cycle in $\text{Cycles}_p(\ell_1^{e_1},\ldots,\ell_r^{e_r})^{\text{prin}}$ has cyclic kernel, by Corollary 19.

Definition 21 (Non-principal isogeny cycles). An $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle is non-principal if its canonical decomposition contains backtracking.

Let $\operatorname{Cycles}_p(\ell_1^{e_1},\ldots,\ell_r^{e_r})^{np}$ denote the subset of non-principal cycles.

Every isogeny cycle is composed of principal and scalar multiplication parts.

Example 22 (Isogeny cycle). Consider the abstract $\{2^2, 3^3\}$ -isogeny cycle beginning at vertex j_1 depicted in Figure 3. Let $\theta \in \text{End}(E_{j_1})$ denote the endomorphism obtained by composing (compatible isogeny representatives of) the edges of this cycle. The endomorphism θ can be written as a composition of prime degree isogenies in a total of $|S_5|/(|S_2| \cdot |S_3|) = 10$ ways, each with two degree-2 isogenies and three degree-3 isogenies. The canonical decomposition of this endomorphism will be of the form $\psi_{3_3} \circ \psi_{3_2} \circ \psi_{3_1} \circ \psi_{2_2} \circ \varphi_{2_1}$, where the ψ_{i_j} isogenies are chosen so that the kernel of the composition remains unchanged. In particular,

 $\ker \psi_{2_2} = \ker(\varphi_{3_3} \circ \varphi_{3_2} \circ \varphi_{2_2} \circ \varphi_{3_1}) \cap E(j_2)[2],$

where $E(j_2)$ is the codomain of φ_{2_1} .

Example 23 (Non-principal isogeny cycle). *Here, we give an explicit example of* an isogeny cycle in non-canonical form which does not contain backtracking, but whose canonical form does contain backtracking. See Figure 4.

Let p = 2689. We consider a non-backtracking $\{2^2, 5, 13\}$ -isogeny cycle of a supersingular elliptic curve over $\overline{\mathbb{F}}_p$. The details of this computation can be found in the code file example_nonprincipalcycle.ipynb.

We have the following isogenies $\varphi_{i,\cdot}$, where deg $\varphi_{i,\cdot} = i$:



(A) The original $\{2^2, 5, 13\}$ -isogeny cycle whose edges compose to the endomorphism η described in Example 23.



(B) The endomorphism η refactored in canonical decomposition. η is the composition of a principal {5, 13}-isogeny cycle and the scalar multiplication [2].

FIGURE 4

(1)
$$\varphi_{2,(1)} : E_0 \to E_1,$$

(2) $\varphi_5 : E_1 \to E_2,$
(3) $\varphi_{2,(2)} : E_2 \to E_3,$
(4) $\varphi_{13} : E_3 \to E_0,$

where $E_0: y^2 = x^3 + 2236x + 1886$, $E_1: y^2 = x^3 + 732x + 2243$, $E_2: y^2 = x^3 + 750x + 791$, $E_3: y^2 = x^3 + 1996x + 1015$.

This isogeny cycle is depicted in Figure 4a. This cycle is non-principal. Let $\eta := \varphi_{13} \circ \varphi_{2_2} \circ \varphi_5 \circ \varphi_{2_1}$. The endomorphism η contains the factor [2], since $\eta(E_0[2]) = \mathcal{O}_{E_0}$. However, the isogeny cycle does not contain backtracking. This is not a contradiction of Corollary 19, because the isogeny cycle has not been presented in canonical form.

To refactor η into canonical form, we swap φ_5 and φ_{2_2} in the order of compositions. Define a new $\psi_{2_2} : E_1 \to E_0$ with ker $\psi_{2_2} = \ker \widehat{\varphi_{2_1}}$. In fact, $\psi_{2_2} = \widehat{\varphi_{2_1}}$, and here we see the backtracking. Next, compute $\psi_{2_2}(\ker \varphi_5) = \widehat{\varphi_{2_1}}(\ker \varphi_5)$ to define $\psi_5 : E_0 \to E_3$. This completes the refactorisation of η as $\eta := \varphi_{13} \circ \psi_5 \circ \widehat{\varphi_{2_1}} \circ \varphi_{2_1}$. See Figure 4b. In this decomposition, we see that η is the concatenation of contains two isogeny cycles: one principal $\{5, 13\}$ -isogeny cycle and one non-principal $\{2^2\}$ isogeny cycle. In particular, the canonical decomposition of η contains backtracking.

3.3. Equivalence classes of cycles. In the graph-theoretical sense, a cycle beginning at a base point vertex V can be traversed in two possible directions from that base point V: the reversing of direction is the effect of taking the dual isogeny. For isogeny cycles whose degrees have more than one prime factor, the dual isogeny cycle will have a distinct canonical decomposition. To count isogeny cycles in Section 4, we will really be counting endomorphisms. When |L| > 1 and $p \equiv 1 \pmod{12}$, an endomorphism uniquely (up to sign) determines the canonical decomposition of an isogeny-cycle. Refactoring an isogeny cycle defines and equivalence relation on the set of isogeny cycles, and we may take the canonical decompositions as the equivalence class representatives.

4. Counting cycles

We now give two methods to count principal isogeny cycles in *L*-isogeny graphs. First using Brandt matrices, and secondly using ideal relations in quadratic orders.

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Implementations of these counting methods, along with the examples given are in code file cycle_counts.ipynb.

In this section we again restrict to the case $p \equiv 1 \pmod{12}$ to avoid counting issues arising from extra automorphisms. For other equivalence classes of $p \pmod{12}$, the counts provided here are an upper-bound and they are off by at most a (small) constant depending on L and the sizes of the automorphism groups of the supersingular isomorphism classes.

4.1. Counting isogeny cycles using Brandt matrices. As briefly alluded to in Section 2.4, one way of counting cycles, in the isogeny graph $\mathcal{G}(p, \ell)$, is by relating them to diagonal elements on the Brandt matrix $B(\ell)$ (one might also need to replace ℓ by powers of it) associated to the isogeny graph in question. Indeed, $B(\ell)$ can be seen as the adjacency matrix of the graph $\mathcal{G}(p, \ell)$. One can also obtain a formula for the total number of cycles by considering the trace of certain Brandt matrices, which can be re-expressed, thanks to Theorem 6, as sums of modified Hurwitz class numbers, in order to more efficiently be computed (using Theorem 7). This method is detailed thoroughly in [Gha22]. One of the advantages of it is that one can very efficiently obtain upper bounds as well as estimates for the total number of cycles in the graph.

For a prime p and a set $L := \{\ell_1, \ldots, \ell_r\}$, we are interested in computing the number of principal isogeny cycles of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$, at any base point E, in the graph $\mathcal{G}(p,L)$. For a fixed base point E, these cycles are denoted $\operatorname{Cycles}_p(\ell_1^{e_1}, \ldots, \ell_r^{e_r})_E^{\operatorname{prin}} / \sim$, where \sim is equivalence up to refactoring. As an extension of the above, we would also like to compute the number of principal isogeny cycles, at any base point E, of length R in $\mathcal{G}(p,L)$, i.e. of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$ with $\sum_i e_i = R$. In the case $L = \{\ell_1, \ell_2\}$, this is very similar to [Gha22, Section 4.2].

Let $\{E_1, \ldots, E_n\}$ be a set of representatives for the vertices of G(p, L). Then the number of principal isogeny cycles of degree $\ell_1^{e_1} \ldots \ell_r^{e_r}$, at base point E_i , in $\mathcal{G}(p, L)$ is equal to $B_{ii}(\ell_1^{e_1} \cdots \ell_r^{e_r})$ minus the number of cycles (endomorphisms) of E_i of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$ involving backtracking. Now, since we are working with canonical decompositions, the backtracking must occur at one of the primes ℓ_j thus creating a scalar factor $[\ell_j]$ for some j. So, in order to remove the backtracking caused by $[\ell_j]$, we can subtract $B_{ii}(\ell_1^{e_1} \cdots \ell_j^{e_j-2} \cdots \ell_r^{e_r})$ from $B_{ii}(\ell_1^{e_1} \cdots \ell_r^{e_r})$, for all j. Of course, one must then account for the cycles with multiple occurrences of backtracking that have been subtracted more than once. This number can be computed using the inclusion-exclusion principle. In order to simplify our notation, we extend the field of definition of B to include rational numbers by setting B(m) := 0 for $m \notin \mathbb{N}$. So, for example, if $L = \{\ell_1, \ell_2\}$, then we have that the number of endomorphisms of E_i of degree $\ell_1^{e_1}\ell_2^{e_2-2}$ involving scalar factors is given by $B_{ii}(\ell_1^{e_1-2}\ell_2^{e_2}) + B_{ii}(\ell_1^{e_1}\ell_2^{e_2-2}) - B_{ii}(\ell_1^{e_1-2}\ell_2^{e_2-2})$ (cf. [Gha22, Lemma 4.2]). Hence, in the general case of $L = \{\ell_1, \ldots, \ell_r\}$, the number of endomorphisms of E_i of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$ involving scalar factors is

$$\sum_{\substack{\emptyset \subseteq J \subseteq \{1,2,\dots,r\}}} (-1)^{|J|-1} B_{ii} \left(\frac{\prod_{i=1}^r \ell_i^{e_i}}{\prod_{j \in J} \ell_j^2} \right).$$

Hence, summing over all vertices of the graph, we get the following formula.

Theorem 24. The total number of principal isogeny cycles of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$ and any base point E_i , in $\mathcal{G}(p, L)$, in given by

$$\sum_{i=1}^{n} |\operatorname{Cycles}_{p}(\ell_{1}^{e_{1}}, \dots, \ell_{r}^{e_{r}})_{E_{i}}^{\operatorname{prin}} / \sim | = \sum_{J \subseteq \{1, \dots, r\}} (-1)^{|J|} \operatorname{Tr} \left(B\left(\frac{\prod_{i=1}^{r} \ell_{i}^{e_{i}}}{\prod_{j \in J} \ell_{j}^{2}}\right) \right)$$

Thanks to Theorem 6, we can make the above formula more tractable, by relating each trace to sums of modified Hurwitz class numbers. We will avoid substituting the formula from Theorem 6 into that of Theorem 24 to avoid writing out a very cumbersome expression. However, if one wanted to compute the quantity $\sum_{i=1}^{n} |\operatorname{Cycles}_{p}(\ell_{1}^{e_{1}}, \ldots, \ell_{r}^{e_{r}})_{E_{i}}^{\operatorname{prin}} / \sim |$ precisely, this substitution could be carried out. As we will see later, if one was interested in finding bounds (or estimates) for the number of principal cycles, this method can be pushed further (by involving Theorem 7).

Remark 25. Note that the above formula, in Theorem 24, potentially overcounts the same cycles multiple times. Indeed, since we are summing the number of cycles starting at all vertices, each cycle will thus be counted potentially as many times as the number of vertices it contains. One could divide the final number of cycles by the length $R := \sum_i e_i$ of the cycles to obtain a better estimate, but this is not a perfect fix, as a cycle of length R could possibly contain less that R distinct vertices; this would hence give us a (close) lower bound.

We will now consider a more general—and very natural—notion of cycle count. Let $C_{E_i}(L; R)$ denote the number of principal isogeny cycles, at base point E_i , of length R in $\mathcal{G}(p, L)$. By length R in $\mathcal{G}(p, L)$, we naturally mean a cycle in the graph that is composed of R edges. In other words, we are not fixing certain exponents $\{e_1, \ldots, e_r\}$ and only considering the isogeny cycles of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$, but rather any isogeny cycle of degree $\ell_1^{e_1} \cdots \ell_r^{e_r}$, for any (e_1, \ldots, e_r) satisfying $\sum_i e_i = R$. This can be seen as a generalisation of the *bi-route number* defined in [Gha22]. We then have

(5)
$$\sum_{i=1}^{n} \mathcal{C}_{E_i}(L; R) = \sum_{i=1}^{n} \sum_{e_1 + \dots + e_r = R} |\operatorname{Cycles}_p(\ell_1^{e_1}, \dots, \ell_r^{e_r})_{E_i}^{\operatorname{prin}} / \sim |.$$

Let us now try to re-express the above quantity. Switching the order of summation in Equation (5) yields the following result.

Theorem 26. The total number of principal isogeny cycles, for any base point E_i , of length R in $\mathcal{G}(p, L)$ is given by

(6)
$$\sum_{i=1}^{n} \mathcal{C}_{E_{i}}(L;R) = \sum_{i=0}^{r} (-1)^{i} \binom{r}{i} \sum_{\substack{e_{1},e_{2},\dots,e_{r} \in \mathbb{N} \\ e_{1}+\dots+e_{r}=R-2i}} \operatorname{Tr}(B(\ell_{1}^{e_{1}}\cdots\ell_{r}^{e_{r}})).$$

Similarly to Theorem 24, one can compute the quantity in Theorem 26 precisely, using Theorem 6. If one is looking for a bound (or an approximation) of the total number of principal cycles of length R, then one can go further than Equation 6, by applying the methods of [Gha22, GKPV21]. Indeed, noting that $H_p(D) \leq H(D)$, as in Section 2.4, and combining Theorems 6 and 7, we have an upper bound

(7)
$$\operatorname{Tr}(B(\ell_1^{e_1}\cdots\ell_r^{e_r})) \le 2\prod_{i=1}^r \frac{\ell_i^{e_i+1}-1}{\ell_i-1}$$

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and $\operatorname{Tr}(B(\ell_1^{e_1}\cdots\ell_r^{e_r}))$ is in $O(\ell_1^{e_1}\cdots\ell_r^{e_r})$. Hence, we get the following result on the total number of cycles or length R.

Corollary 27. The number of cycles of length R in the generalised graph G(p, L) is in $O\left(\ell_r^R\left(1-\frac{1}{\ell_r^2}\right)^r\right)$.

Of course, one could get a better upper bound than in Inequality (7) and thus a better upper bound for $\sum_{i=1}^{n} C_{E_i}(L; R)$, depending on one's intended application.

If we actually wanted a heuristic estimate instead of a upper bound, we can argue, as in Section 5 of [GKPV21], that we expect to have $H_p(D) \approx \frac{1}{2}H(D)$ on average. This is because the prime p, in the definition of $H_p(D)$ in Equation (2), splits half the time and remains inert the other half. This heuristic assumption gives us the estimate

(8)
$$\operatorname{Tr}(B(\ell_1^{e_1}\cdots\ell_r^{e_r})) \approx \prod_{i=1}^r \frac{\ell_i^{e_i+1}-1}{\ell_i-1} - \frac{1}{2} \sum_{d \mid \ell_1^{e_1}\cdots\ell_r^{e_r}} \min\left\{d, \frac{\ell_1^{e_1}\cdots\ell_r^{e_r}}{d}\right\},$$

which can then be substituted into Equation (6) of Theorem 26. Again, we will not perform this substitution, to avoid writing a very long and cumbersome expression for the approximation of $\sum_{i=1}^{n} C_{E_i}(L; R)$, but this can very easily be done (by hand or on a computer), especially once the number of primes in the set L is known. The advantage of this approach, is that it is much faster to compute the right hand side of (8) than the actual trace of $B(\ell_1^{e_1} \cdots \ell_r^{e_r})$.

4.2. An ideal interpretation of certain isogeny cycles. In this section, we restrict to $\{\ell_1, \ldots, \ell_r\}$ -isogeny cycles, where $p \neq \ell_i$ and $\ell_i < \ell_{i+1}$ for all *i*, to give an ideal-theoretic interpretation.

As isogeny cycles correspond to endomorphisms of elliptic curves, there is a connection to the theory of embeddings into quaternion orders. For a prime ℓ , the Deuring correspondence [Deu41] gives a bijection between ℓ -isogenies of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ and integral left ideals of norm ℓ , of maximal orders in \mathcal{B}_p , up to isomorphisms of curves/orders. A non-scalar element α of a maximal quaternion order (an endomorphism of a supersingular elliptic curve) generates an imaginary quadratic order over \mathbb{Z} . In this way, we view isogeny cycles as embeddings of imaginary quadratic orders into the maximal quaternion order. We count isogeny cycles by relating them to these embeddings, extending work of [CK20, Onu21, ACL⁺24] to the *L*-isogeny graph setting. We recall some of the results and definitions from these works briefly here. For the purposes of this discussion, \mathcal{O} will always denote an order in an imaginary quadratic field.

Definition 28 (Primitive \mathcal{O} -embedding [Onu21, Def. 3.1, 3.3]). Let \mathcal{O} denote an imaginary quadratic order, $K := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$, and let E be a supersingular elliptic curve. An embedding $\iota : K \hookrightarrow \operatorname{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called a primitive \mathcal{O} -embedding if $\iota(K) \cap \operatorname{End}(E) = \iota(\mathcal{O})$.

When focusing on the imaginary quadratic order rather than the field, we may denote a primitive \mathcal{O} -embedding ι by $\iota : \mathcal{O} \hookrightarrow \operatorname{End}(E)$.

Definition 29 ([Onu21, Sec. 3.1]). Let $SS_{\mathcal{O}}^{pr}$ denote the set of isomorphism classes of supersingular elliptic curves E together with a choice of primitive \mathcal{O} -embedding into End(E).

Proposition 30. The class group of \mathcal{O} acts freely on $SS_{\mathcal{O}}^{pr}$ and has one or two orbits, depending on if p is ramified or inert (resp.) in the imaginary quadratic field containing \mathcal{O} . In particular, $\#SS_{\mathcal{O}}^{pr} = h_{\mathcal{O}}$ if p is ramified and $\#SS_{\mathcal{O}}^{pr} = 2h_{\mathcal{O}}$ if p is inert.

If \mathfrak{l}_i are ideals above ℓ_i in \mathcal{O} such that $\prod_{i=1}^r \mathfrak{l}_i$ is principal, then the endomorphisms of the elliptic curves in $SS_{\mathcal{O}}^{pr}$ which correspond to $\prod_{i=1}^r \mathfrak{l}_i$ have canonical decompositions which are principal $\{\ell_1, \ldots, \ell_r\}$ -isogeny cycles.

Proof. This proposition condenses several results in [Onu21] and [ACL⁺24], and applies these results to the case of $\{\ell_1, \ldots, \ell_r\}$ -isogenies.

Definition 31. Define the set $\mathcal{I}_{\{\ell_1,\ldots,\ell_r\}}$ of imaginary quadratic orders \mathcal{O} such that:

- p is inert in $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$;
- p does not divide the conductor of \mathcal{O} ;
- \mathcal{O} contains an element of norm $\prod_{i=1}^{r} \ell_i$.

The first condition guarantees that $\#SS_{\mathcal{O}}^{pr} = 2h_{\mathcal{O}}$. The third condition provides an upper bound $\operatorname{disc}(\mathcal{O}) \leq 4 \prod_{i=1}^{r} \ell_i < p$, ensuring the set $\mathcal{I}_{\{\ell_1,\ldots,\ell_r\}}$ is finite. Such orders can be found by solving norm-form equations, and this is how we propose to begin enumerating $\mathcal{I}_{\{\ell_1,\ldots,\ell_r\}}$.

Definition 32. For an imaginary quadratic order \mathcal{O} , define the set:

(9)
$$\mathcal{I}d_{\mathcal{O}} := \left\{ \prod_{i=1}^{r} \mathfrak{l}_{i} : \prod_{i=1}^{r} \mathfrak{l}_{i} \text{ is an invertible ideal of norm } \ell_{i} \text{ in } \mathcal{O} \right\}.$$

Using these two definitions, our main ideal-theoretic counting result is now given.

Theorem 33. Let $p, \ell_1, \ldots, \ell_r$ be distinct primes with $p > 4 \prod_{i=1}^r \ell_i$. Let C denote the number of distinct canonical decompositions of principal $\{\ell_1, \ldots, \ell_r\}$ -isogeny cycles in the supersingular L-isogeny graph over $\overline{\mathbb{F}}_p$. In particular:

$$C := \sum_{\text{iso. classes } E} |\text{Cycles}(\{\ell_1, \dots, \ell_r\})_E^{\text{prin}} / \sim |.$$

Then,

$$C = \frac{1}{r!} \sum_{\mathcal{O} \in \mathcal{I}_{\{\ell_1, \dots, \ell_r\}}} (2 \cdot h_{\mathcal{O}} \cdot \# \mathcal{I} d_{\mathcal{O}}).$$

Proof. The proof follows similarly to [ACL⁺24, Lem. 3.9, Cor. 7.3] by changing the conditions checked for the ideals \mathfrak{l}_i . First, suppose we have an $\{\ell_1, \ldots, \ell_r\}$ -isogeny cycle. The endomorphism α obtained by composing the isogenies is an imaginary quadratic integer of norm $\prod_i \ell_i$. The imaginary quadratic order $\mathcal{O} := \mathbb{Z}[\alpha]$ must have conductor coprime to ℓ_1, \ldots, ℓ_r , since the endomorphism α is not divisible by any nontrivial scalar. Since the ideal (α) is coprime to the conductor of \mathcal{O} , it has a unique factorisation as a product of prime ideals, necessarily (α) $\mathcal{O} = \prod_{i=1}^r \mathfrak{l}_i$ with norm of $\mathfrak{l}_i = \ell_i$ for all i. The decomposition $\prod_{i=1}^r \mathfrak{l}_i$ determines the kernel of α uniquely (up to sign).

Now, suppose we have an imaginary quadratic order $\mathcal{O} \in \mathcal{I}_{\{\ell_1,\ldots,\ell_r\}}$. Since p is not split in the imaginary quadratic field containing \mathcal{O} , there are $\#SS_{\mathcal{O}}^{pr}$ supersingular elliptic curves with a primitive \mathcal{O} -embedding. Since p is inert in the imaginary

 $^{{}^{4}\}mathcal{I}_{\mathcal{O}}$ depends on $\{\ell_1, \ldots, \ell_r\}$, but we suppress this in our notation as $\{\ell_1, \ldots, \ell_r\}$ is fixed.



FIGURE 5. The supersingular $\{2, 3\}$ -isogeny graph over $\overline{\mathbb{F}}_{61}$, with vertices labelled by *j*-invariant in \mathbb{F}_{61^2} . Solid lines are 2-isogenies, dashed lines are 3-isogenies, and $\alpha, \overline{\alpha}$ denote conjugate *j*-invariants in $\mathbb{F}_{61^2} \setminus \mathbb{F}_{61}$. Loops may only be traversed in one direction, while all other edges are undirected and may be traversed in either direction.

quadratic field containing \mathcal{O} , $\#SS_{\mathcal{O}}^{pr} = 2h_{\mathcal{O}}$. For each elliptic curve in $SS_{\mathcal{O}}^{pr}$, we look for appropriate ideal products in $\mathcal{I}d_{\mathcal{O}}$. We require ideals of norm ℓ_i to exist so that there are degree- ℓ_i isogenies between curves with a primitive \mathcal{O} -embedding. Such a product $\prod_{i=1}^{r} \mathfrak{l}_i \in \mathcal{I}d_{\mathcal{O}}$ uniquely determines the kernel of an endomorphism. The chain of isogenies corresponding to the ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_r$ determines an isogeny cycle. Since the product contains each \mathfrak{l}_i only once, the respective chain of isogenies has a cyclic kernel: there is no integer ℓ which divides $\prod_{i=1}^{r} \mathfrak{l}_i$ giving an integral ideal of \mathcal{O} . For every distinct element of $\mathcal{I}d_{\mathcal{O}}$, the resulting endomorphisms have distinct kernels and will yield distinct canonical decompositions. Finally, we divide the sum by r! to account for the refactorings of a principal $\{\ell_1, \ldots, \ell_r\}$ -isogeny cycle (see the calculation following Lemma 12).

4.3. Examples. Code for these examples is given in cycle_counts.ipynb.

Example 34 (Principal $\{2,3\}$ -isogeny cycles for p = 61). For a small example, we wish to count the $\{2,3\}$ -isogeny cycles in the supersingular $\{2,3\}$ -isogeny graph over $\overline{\mathbb{F}}_{61}$, see Figure 5. In this case, all $\{2,3\}$ -isogeny cycles are principal, as there is no possibility of backtracking.

A rigorous count in the graph finds ten distinct principal $\{2,3\}$ -isogeny cycles:

 $(9,9), (9,9), (\alpha,\overline{\alpha}), (\overline{\alpha},\alpha), (\alpha,\overline{\alpha}), (\overline{\alpha},\alpha), (\alpha,50), (50,\alpha), (\overline{\alpha},50), (50,\overline{\alpha}).$

Counting using ideals: We will replicate this count by counting pairs $(\iota_{\mathcal{O}}, \mathfrak{l}_{2}\mathfrak{l}_{3})$, where $\iota_{\mathcal{O}}$ is a primitive \mathcal{O} -embedding of an imaginary quadratic order \mathcal{O} and $\mathfrak{l}_{2}, \mathfrak{l}_{3}$ are ideals of \mathcal{O} satisfying certain conditions.

First, to count embeddings $\iota_{\mathcal{O}}$, we must enumerate elements of imaginary quadratic fields (up to $|\Delta| \leq 4 \cdot 6$) of norm 6, (up to conjugation and ± 1 , as these elements would generate the same order):

$$2 + \sqrt{-2}, 1 + \sqrt{-5}, \sqrt{-6}, \frac{3 + \sqrt{-15}}{2}, \frac{1 + \sqrt{-23}}{2}.$$

These elements live in the fields $\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-15}), and \mathbb{Q}(\sqrt{-23}),$ respectively. Since (61) is split in $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-15})$, we discard the elements in these fields. The ideal (61) is inert in $\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-6}), and \mathbb{Q}(\sqrt{-23})$. The remaining elements generate three imaginary quadratic orders $\overline{\mathbb{F}}_{61}$.

For $\alpha \in \{2+\sqrt{-2}, \sqrt{-6}, \frac{1+\sqrt{-23}}{2}\}, \mathbb{Z}[\alpha]$ is the maximal order in the field in which this order lives. Consider the factorisation of the ideals (2) and (3) in these orders:



FIGURE 6. A subgraph of the supersingular $\{2,3\}$ -isogeny graph over $\overline{\mathbb{F}}_{61}$ (see Figure 5).

- $(2)\mathbb{Z}[\sqrt{-2}] = \mathfrak{l}_2^2 \text{ and } (3)\mathbb{Z}[\sqrt{-2}] = \mathfrak{l}_3\overline{\mathfrak{l}}_3.$ There are two pairs $\mathfrak{l}_2, \mathfrak{l}_3 \text{ in } \mathcal{I}d_{\mathbb{Z}[\sqrt{-2}]}.$ The class number of $\mathbb{Z}[\sqrt{-2}]$ is 1, and $\varepsilon_{\mathbb{Z}[\sqrt{-2}]} = 2.$
- $(2)\mathbb{Z}[\sqrt{-6}] = \mathfrak{l}_2^2$ and $(3)\mathbb{Z}[\sqrt{-6}] = \mathfrak{l}_3^2$. There is one pair $\mathfrak{l}_2, \mathfrak{l}_3$ in $\mathcal{I}d_{\mathbb{Z}[\sqrt{-6}]}$. The class number of $\mathbb{Z}[\sqrt{-6}]$ is 2, and $\varepsilon_{\mathbb{Z}[\sqrt{-6}]} = 2$.
- $(2)\mathbb{Z}[\frac{1+\sqrt{-23}}{2}] = \mathfrak{l}_2\overline{\mathfrak{l}}_2$ and $(3)\mathbb{Z}[\frac{1+\sqrt{-23}}{2}] = \mathfrak{l}_3\overline{\mathfrak{l}}_3$. There are two pairs $\mathfrak{l}_2, \mathfrak{l}_3$ in $\mathcal{Id}_{\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]}$. The class number of $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$ is 3, and $\varepsilon_{\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]} = 2$. Putting this information into the formula of Theorem 33:

$$\frac{1}{2!}(2 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 1 + 2 \cdot 3 \cdot 2) = 10.$$

Counting using Brandt matrices: Let us now count the number of $\{2,3\}$ -isogeny cycles, but using the Brandt matrix method. As in Section 4.1, we have that $\sum_{i=1}^{5} |\operatorname{Cycles}_{61}(2,3)_{E_i}^{\operatorname{prin}}/\sim | = \operatorname{Tr}(B(2\cdot3).$ This number can be computed directly using SageMath since we are working with small numbers.

Moreover, as discussed in Section 4.1, if we wanted to avoid computing this number directly, via Brandt matrices, we can use Theorems 6 and 7, together with the bound $H_p(D) \leq H(D)$ or the estimate $H_p(D) \approx \frac{1}{2}H(D)$. Indeed, we can easily compute (and this remains easy to do even when the degree, given by its prime factorisation, grows larger):

(10)
$$2\sum_{d|6} d - \sum_{d|6} \min\{d, 6/d\} = 18$$

So, we thus obtain the bound $\sum_{i=1}^{5} |\operatorname{Cycles}_{61}(2,3)_{E_i}^{\operatorname{prin}} / \sim | \leq 18$ and the estimate $\sum_{i=1}^{5} |\operatorname{Cycles}_{61}(2,3)_{E_i}^{\operatorname{prin}} / \sim | \approx 9$, which is not far at all from the true answer 10.

Here we provide an example of a $\{\ell_1^{e_1}, \ldots, \ell_r^{e_r}\}$ -isogeny cycle which does not arise from the theory of ideals discussed in Section 4.2.

Example 35. We consider again the setup of Example 34, namely we take $L = \{2,3\}, p = 61$. A subgraph of $\mathcal{G}(p,L)$ is shown in Figure 6. The 4 supersingular curves pictured; $E_{50}, E_{41}, E_{\alpha}, E_{\overline{\alpha}}$, have embeddings of the imaginary quadratic order $\mathbb{Z}[\sqrt{-11}]$, which has conductor 2. For the latter 3 curves, such an embedding will be primitive. In the case of E_{50} we find that this curve has a primitively embedding of the maximal order $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ contained in the imaginary quadratic field $\mathbb{Q}(\sqrt{-11})$. Note these orders have class numbers 3 and 1 respectively.

Consider the endomorphism $\varphi_3 \circ \varphi_2 \circ \varphi_1$ of degree $2^2 \cdot 3$ depicted in blue. The isogeny φ_3 is horizontal with respect to the $\mathbb{Z}[\sqrt{-11}]$ -embedding on E_{41} and E_{α} , and thus corresponds to the action of a prime ideal \mathfrak{l}_3 lying above $(3) \cdot \mathbb{Z}[\sqrt{-11}]$, namely $(3, 1 + \sqrt{-11})$. On the other hand the 2-isogenies φ_1, φ_2 are ascending and descending respectively, and thus do not arise from the action of a prime ideal in this order.

Using degree $2^2 \cdot 3$ in our implementation of Theorem 33, yields a count of 0 principal isogeny cycles, when the correct count is 16. This is because our algorithm only counts cycles where steps are all horizontal in L-isogeny volcano rims, while some principal cycles such as $\varphi_3 \circ \varphi_2 \circ \varphi_1$ exist, which ascend and descend in the 2-isogeny volcano.

5. Graph cuts

In this this section we study cuts of ℓ - and *L*-isogeny graphs with low edge expansion (see Definition 8). For a regular graph, when taking a random edge from a random vertex within the cut, a lower edge expansion implies a higher chance the edge is internal to the cut, and lower chance it leaves the cut. This is cryptographically relevant as most isogeny-based schemes assume Ramanujan or rapid mixing properties of isogeny graphs, in particular, the probability of arriving at different vertices after a fairly short random walk $(O(\log(p)))$ steps) is uniformly distributed. For low expansion cuts however, should they exist, the probabilities are imbalanced for vertices on opposite sides.

5.1. Edge expansion in small cuts. For convenience we now refer to a cut as a set of vertices $C \subset V$, where as a disjoint partition this is $V = C \sqcup (V \setminus C)$. We define a cut to be *(dis)connected* if the induced subgraph from the vertices in the cut is (dis)connected. As we are looking for cuts with relatively low edge expansion, we only care about connected cuts by the following result.

Lemma 36. Let G = (V, E) be a graph and C a disconnected cut. Then there exists a connected cut $T \subset C$ with $E(T, C \setminus T) = \emptyset$ and $h_G(T) \leq h_G(C)$.

Proof. Let C_1, C_2, \ldots, C_n be a partition of C into connected cuts with no edges between them. This implies $|C| = \sum |C_i|$ and $|E(C, V \setminus C)| = \sum |E(C_i, V \setminus C_i)|$. Assume that $h_G(C_i) > h_G(C)$ for all i. Then we have $E(C_i, V \setminus C_i) = |S_i| \cdot h_G(C_i) > |C_i| \cdot h_G(C)$. Summing over i gives $E(C, V \setminus C) > |C| \cdot h_G(C)$. But by definition of $h_G(C)$ this should be equality. Hence by contradiction there exists r and $T = C_r$ with $h_G(T) \le h(C)$.

We now consider small cuts of relatively small edge expansion in ℓ -isogeny graphs.

Lemma 37. In a directed ℓ -isogeny graph, take a connected cut C of n vertices with $n < k \cdot \log_{\ell}(p) + 1$ and define $R_{\ell,n} = \frac{n \cdot \ell + 1}{n}$. Suppose C contains no curves with additional automorphisms. Then $h_G(C) \leq R_{\ell,n}$ with equality if and only if the induced subgraph of C in G is a tree (with dual edges), and inequality implying Ccontains a curve with a cyclic endomorphism of degree at most p^k .

Proof. Consider the induced subgraph of C in G. Within G merge a maximal pairing of directed edges with their duals giving a mix of directed/undirected edges. It remains $(\ell + 1)$ -regular, directed edges being self-dual loops or edges from curves with additional automorphisms. We ignore the latter, as no such curves are in C.

As the induced subgraph of C is connected, it either contains cycles or does not. Suppose it has no cycles (and so no loops), then it is by definition an undirected tree, and has n-1 edges. As the graph is $(\ell+1)$ -regular, each vertex of C has $(\ell+1)$ outgoing edges, so $E(C, V \setminus C) = n(\ell+1) - (n-1) = n\ell+1$, giving $h_G(C) = R_{\ell,n}$. If not a tree, it contains a cycle, and so at least one additional edge giving $h_G(C) < R_{\ell,n}$. With n vertices the cycle has degree at most $\ell^{n-1} \leq \ell^{k \log_{\ell}(p)} = p^k$, and composes to a cyclic endomorphism by [CLG09, Proposition 1]. In the reverse direction, if the induced subgraph of C has edge expansion $R_{\ell,n}$, it has exactly n-1 outgoing edges, and so is a tree. The tree is undirected as it does not contain loops, and hence no directed edges.

For a connected cut to have relatively small edge expansion, one interpretation of "relatively" is compared to connected cuts of the same number of vertices n. The above lemma allows us to relate the distribution of such cuts to the previously studied distribution of p^k -valleys [LB20].

Remark 38. By the above lemma, small cuts with "relatively" small edge expansion must have edge expansion strictly less than $R_{\ell,n}$, as all these cuts have edge expansion less than or equal to $R_{\ell,n}$. Moreover, the lemma shows for a fixed constant $k < \frac{2}{3}$ and $n < k \cdot \log_{\ell}(p) + 1$, cuts of size n contain curves with endomorphisms of degree less that p^k . For sufficiently large p these curves are rare, appearing in small clusters which are far apart, referred to as p^k -valleys. For smaller k (and also n), the p^k -valleys become sparser [LB20].

If we try the exact same argument for L-isogeny graphs, we run into an issue, due to the existence of cycles which compose to scalar endomorphisms (recall Figure 13). However, these additional cycles appear with a very uniform pattern and so a similar argument applies.

Definition 39 (Minimum Uniform Edge Expansion $R_{L,n}$). Fix a set of primes primes $L = \{\ell_1, \ldots, \ell_n\}$ and n > 0. For a prime $p \notin L$ we define $R_{L,n}(p)$ as the minimum edge expansion over all connected subsets of n vertices of $\mathcal{G}(p,L)$, where all cycles within the induced subgraph compose to scalar endomorphisms. We then define $R_{L,n} = \lim_{p\to\infty} R_{L,n}(p)$.

Lemma 40. Fix L and n, as above. The limit $R_{L,n} = \lim_{p\to\infty} R_{L,n}(p)$ is well-defined, and is attained.

Proof. The length of a cycle in an induced subgraph of $\mathcal{G}(p,L)$ on n vertices can be bounded by the fact there are exactly $n \cdot (\ell_i + 1)$ outgoing ℓ_i -isogenies, up to post-composition with automorphism, originating from vertices in the subgraph. Hence the degree of a cycle can be upper bounded by $K = \prod_i \ell_i^{n \cdot (\ell_i + 1)}$, which is independent of p. Hence for sufficiently large p we may assume $K \ll p$. By the results of [LB20], increasing p increases the distance between K-valleys until there is a large enough space in the graph for a curve E_p and depth-(n+1) neighbourhood of L-isogenies, disjoint from K-valleys. This means the neighbourhood contains no non-scalar cycles of degree less than or equal to K. One can then check, firstly the n-vertex cut defining $R_{L,n}(p)$ has an isomorphic cut (i.e. graph isomorphism of the induced subgraphs) within the neighbourhood of E_p . And secondly the neighbourhood of E_p is isomorphic to the neighbourhood arising from any larger value of p. Hence for large enough p, $R_{L,n} = R_{L,n}(p)$ is the minimum edge expansion of all n-vertex cuts in the neighbourhood of E_p . We now consider the analogue of Lemma 37 in L-isogeny graphs.

Lemma 41. In a directed $L = \{\ell_1, \ldots, \ell_r\}$ -isogeny graph, take a connected cut C of n vertices with $n < k \cdot \log_{\ell_r}(p) + 1$. Suppose C contains no curves with additional automorphisms. Then $h_G(C) < R_{L,n}$ implies C contains a curve with a cyclic endomorphism of degree at most p^k .

Proof. Essentially the same as Lemma 37, replacing 'tree' with depth-*n* neighbourhood from the definition of $R_{L,n}$.

Remark 42. For large p there are many depth-n L-isogeny neighbourhoods within $\mathcal{G}(p,L)$ without non-scalar cycles. Hence for n-vertex cuts, edge expansions of $R_{L,n}$ or higher are common, and so a "relatively small" edge expansion would be one with edge expansion less than $R_{L,n}$. The lemma implies for $k < \frac{2}{3}$ and small enough n that any n-vertex cut of relatively small edge expansion, intersects a p^k -valley, which for larger p and smaller k are increasingly rare and sparse.

5.2. Fiedler's vector based clustering. We now discuss finding larger subsets of vertices which define cuts of relatively small edge expansion. We take an experimental approach on isogeny graphs $\mathcal{G}(p, L)$ with p as large as computationally feasible. Some preliminary attempts showed for p large enough to be considered interesting, *n*-vertex sets with edge expansion less than $R_{L,n}$ in $\mathcal{G}(p, L)$ are rare. And searching through all n vertex subsets attempting to find one, being exponential time in p, is computationally infeasible. Instead of trying them all however, techniques from spectral graph theory allow us to make a sensible guess.

While several algorithms exists using spectral methods to obtain low edge expansion cuts, we will use an algorithm, which we call *Fiedler's algorithm*. For a graph G = (V, E) with vertices ordered $V = \{v_1, \ldots, v_k\}$, it proceeds as follows:

- (1) Compute the eigenvector \vec{x} of the Laplacian matrix corresponding to the second largest eigenvalue λ_2^{5} .
- (2) Order vertices v_i with respect to the quantity x_i , largest first, and relabel them u_1, \ldots, u_k with respect to this ordering.
- (3) For i = 1, ..., k, let $C_i = \{u_1, ..., u_i\}$, called a *sweep cut*, and compute $h_G(C_i)$ and $h_G(V \setminus C_i)$.
- (4) Return the cut C_i such that $\max(h_G(C_i), h_G(V \setminus C_i))$ is minimised.

The algorithm is polynomial time in |V|+|E|, which for $G = \mathcal{G}(p, L)$ means polynomial time in $p+\sum \ell_i$. The eigenvector of the 2nd largest eigenvalue λ_2 is used as it is believed to provide a good measure of connectivity. Its use is attributable to Fiedler [Fie73], and is often called the *Fiedler vector*. In his original work however, Fiedler suggested use of the *Fiedler cut* constructed as $C = \{v_i : x_i \geq 0\}$ for clustering. More generally, when using multiple eigenvectors, this is regarded as *hyperplane rounding*, which for a single vector amounts to picking a threshold to define the cut. A choice of 0 for the threshold is better suited to well-balanced graphs, while allowing it to vary can result in better clustering of unbalanced graphs. The concept of finding cuts to minimise edge expansion, as is our aim, didn't arise until later, and so it is now more common to use the minimal edge expansion to decide the threshold. Also, since we desire a partition into two 'inter-connected' clusters either side of the cut, we minimise max $\{h_G(C_i), h_G(V \setminus C_i)\}$ rather than just $h_G(C_i)$, to

 $^{^5\}mathrm{For}$ regular graphs this is also the eigenvector of the second largest eigenvalue of the adjacency matrix.

ensure the complement also has low edge expansion. For random graphs we expect the resulting cut size to be around half of the vertices. For a summary of spectral clustering techniques using Fiedler's vector see [WG21, Mah16].

Example 43. For the isogeny graph $\mathcal{G}(p,L)$ with p = 419 and $L = \{2,3\}$, the second largest eigenvalue of the Laplacian is $\lambda_2 \approx 11.17$. The above algorithm finds sweep cut C_{18} has the minimal value $\max(h_G(C_{18}), h_G(V \setminus C_{18})) \approx 0.46$. The graph has a total of 36 vertices, so C_{18} contains exactly half. The code for this example in given in file fiedler.ipynb.

5.3. Spectral ordering in Fiedler's algorithm not optimal. To see if the cuts resulting from this algorithm have small edge expansion, when compared to other cuts of roughly the same size, we now give an alternative non-spectral method. To do this we replace the ordering of vertices in the previous algorithm with a different ordering. We want the vertices to form a subgraph that looks as 'complete' as possible. The most natural way to do this is to take the neighbourhood of a vertex. We present two variants of this idea.

- (1) **Neighbour Expansion** Pick a starting vertex $[v_0]$ then add its neighbours $[v_0, v_1, v_2, v_3]$ then add the neighbours of v_1 , then the neighbours of v_2 etc, until all vertices are ordered $[v_0, v_1, \ldots, v_n]$.
- (2) Greedy Neighbour Expansion Pick a starting vertex $S_0 = [v_0]$ then define v_{i+1} to be the neighbour of vertices in S_i such that $\phi(S_i \cup \{v_{i+1}\})$ in minimised, then $S_{i+1} := S_i \cup \{v_{i+1}\}$. Here ϕ of a cut denotes the maximum of the edge expansion of the cut and the edge expansion of its complement.

Both of these algorithms are exponential in $\log(p)$ as they have to loop over all vertices in the graph. However the second is slower as it is quadratic in the number of vertices, while the first is linear in the number of vertices.

The optimum solution, that is, the cut $S \subseteq V$ minimising the value of $\phi(S)$, is called the *Cheeger cut*. While computationally infeasible to compute, there are known upper and lower bounds,

$$\frac{\lambda_{-2}}{2} \le \phi(G) \le \sqrt{2\lambda_{-2}},$$

where λ_{-2} is the second smallest eigenvalue of the Laplacian. We also compute these bounds for reference.

Results are given in Figure 7 with code in file fiedler_comparison.ipynb. With the spectral ordering for reference, we try both algorithms three times, staring from different random vertices, and take the average of values ϕ found. From the results we make the following observations:

- (1) The spectral ordering performs badly, suggesting there is not one distinguishable low edge expansion cluster of size roughly $\frac{|V|}{2}$. Perhaps there are more, and the spectral ordering is merging such cuts, or perhaps better cuts have fewer vertices.
- (2) The neighbour ordering finds better cuts than the spectral ordering, suggesting neighbourhoods of vertices form smaller edge expansion clusters. The greedy neighbour ordering does even better, which motivates further study of the less-trivial structures that arise from this ordering.
- (3) As the size of the set L increases, the improvements reduce. The Cheeger lower bound also increases, and so there is less room for improvement. For

the neighbour orderings this is likely as there are more edges, increasing the connectivity of any subset of vertices. This also suggests ℓ -isogeny graphs have smaller expansion clusters, and more of a bottleneck than *L*-isogeny graphs with larger *L*.

Figure 8 with code in fiedler_viz.ipynb, gives a visual comparison of these results. Each image represents an adjacency matrix where a non-zero entry is given by a dot. The different images correspond to vertices in the adjacency matrix ordered in different ways. We first order the vertices using the spectral ordering, neighbour ordering and greedy neighbour ordering. As the resulting cuts are around half the vertices, the edges within each cut are those in the top left quadrant of each image. The edges fully outside the cut are those in the bottom left quadrant. The edges between the cut and its complement lie in the off-diagonal quadrants. Hence a low edge expansion cut will have more vertices in the diagonal quadrants and less in the off-diagonal quadrants. To gain a perspective of the density within each quadrant, we give the same images with vertices within the cut randomly shuffled, and those in the complement also randomly shuffled.

For the spectral ordering we see this does no better than a random ordering of vertices, as all quadrants are equally full. The neighbour ordering sees some white space in the off diagonal quadrants, since for the first few curves we pick, their neighbours are guaranteed to also be early on in the ordering, and so there are no edges to vertices at end of the ordering. The greedy neighbour ordering, which performs best, essentially pushing all curves which are not close to the starting curves towards the end of the ordering.

p, L	Spectral Ordering	Neighbour Ordering	Greedy Ordering	Cheeger Lower Bound	Cheeger Upper Bound
419, {3}	0.597	0.319	0.225	0.071	0.533
$419, \{2,3\}$	0.452	0.289	0.286	0.096	0.619
$419, \{2,3,5,7,11\}$	0.488	0.491	0.413	0.151	0.776
$5569, \{3\}$	0.494	0.301	0.178	0.068	0.520
$5569, \{2,3\}$	0.484	0.322	0.195	0.092	0.608
$5569, \{2,3,5,7,11\}$	0.497	0.473	0.343	0.275	1.049
$10007, \{3\}$	0.489	0.287	0.170	0.065	0.508

FIGURE 7. Fiedler's algorithm with different vertex orderings.

6. FUTURE WORK

There are several notable areas for further study.

6.1. Further principal cycle counting. Firstly, within Sections 3 and 4 we restricted to the case $p \equiv 1 \pmod{12}$. For other values of p, curves with additional automorphisms exist and so further consideration is required to determine if, and how, our principal cycle counts could be adapted to address this.

Furthermore Theorem 33, counting principal $(\ell_1^{e_1}\ell_2^{e_2}\dots\ell_r^{e_r})$ -isogeny cycles via ideals in class groups of imaginary quadratic orders, only works for products of distinct primes, with all exponents $e_i = 1$. Example 35 demonstrated how the approach fails for larger exponents. The issue arises for $e_i \ge 2$ and a starting curve with a primitive \mathcal{O} -embedding with $\ell_i \mid \text{cond}(\mathcal{O})$, cycles may arise from ascending



FIGURE 8. Adjacency matrix plot for p = 5569 and $L = \{2, 3\}$. Dots are edges. Left to right: spectral ordering; neighbour ordering; greedy neighbour ordering. Second row: vertices within the best cut have been shuffled, and within complement.

the ℓ_i -isogeny volcano then descending back down. These ascending and descending isogenies are not represented by the action of invertible ideals in $cl(\mathcal{O})$.

One method to address this is to use an approach based on the volcano 'rimhopping' algorithm of $[ACD^+24]$. Given the norm $\ell_1^{e_1}\ell_2^{e_2}\ldots\ell_r^{e_r}$ generator η of \mathcal{O} , one can obtain another generator from a translate $k+\eta$ with $k \in \mathbb{Z}$. Then factorising its norm $N(k+\eta) = q_1^{f_1}\ldots q_m^{f_m}$ look for principal products of ideals in \mathcal{O} which consist of f_i ideals of norm q_i for each i. If k is chosen such that $q_i \nmid \operatorname{cond}(\mathcal{O})$, in terms of endomorphisms we obtain all degree $q_1^{f_1}\ldots q_m^{f_m}$ endomorphisms φ which give primitive \mathcal{O} -embeddings. Translating back, $\psi = \varphi - [k]$, we obtain all degree $\ell_1^{e_1}\ell_2^{e_2}\ldots\ell_r^{e_r}$ endomorphisms giving primitive \mathcal{O} -embeddings. From the product of norm q_i ideals we can easily check if φ is cyclic by ensuring ideal inverses do not appear in the product. However, further thought is needed to determine how to check, from the ideal product alone, which translated endomorphisms ψ will be cyclic. It is also possible more careful analysis of oriented isogeny volcanoes may result in an alternative approach which more closely matches Theorem 33.

6.2. Scalar cycle counting. Within Section 4 we restricted our counts to principal isogeny cycles. Endomorphisms arising from principal cycles have the advantage that all of their decompositions yield genuine (non-backtracking) cycles. The task of counting non-principal cycles additionally requires determining how many decompositions of a given non-cyclic endomorphism have backtracking. If we additionally counted the isogeny cycles which compose to scalar endomorphisms, we could perhaps combine the counts giving a total number of isogeny cycles of a given degree, from a given starting curve. This is also closely tied to computing the minimum uniform edge expansion $R_{L,n}$ defined in Section 5.1.

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6.3. Further graph clustering. From Section 5.3 the unexpectedly good performance of the greedy neighbour ordering prompts questions into the structure of this ordering, and what the consequences of this might be for sampling isogenies in cryptography. There are also alternative graph clustering techniques, such as flow-based clustering, where the results could give better indications of graph structure.

One may also wish to consider a more extreme interpretation of small clusters; those with a single vertex. Their connectivity with the rest of the graph can be represented by a *distance distribution*.

Definition 44 (Distance distribution). The distance distribution for a vertex $v \in V$ of a graph G = (V, E) is defined as the set $(N_0(v), N_1(v), \ldots)$, where $N_i(v)$ is equal to the number of vertices in the graph G which are distance i from the vertex v.

For random graphs this typically resembles a normal distribution, and same seems to hold for isogeny graphs, as can be seen in Figure 9. Examining outliers and skew of these distributions could be insightful.



FIGURE 9. Distance distribution in 2-isogeny graph with p = 2689, from vertex with *j*-invariant 30. Distance is denoted by *x*-axis, and the *y*-axis gives the proportion of vertices of this distance.

6.4. Comparing random graphs and ℓ - or *L*-isogeny graphs. We propose further study of finding algorithms which can distinguish between isogeny graphs and random graphs. This could be achieved using machine-learning based classification algorithms. For instance there are several Graph Neural Network (GNN) methods giving structural embeddings of a graph, from which one may construct classifiers [BSGGB24, SLRPW21]. Using a simpler logistic-regression algorithm, we conducted some preliminary experiments comparing adjacency matrices of 1000 3-isogeny graphs and 1000 random 4-regular graphs, using a 50% test-train split. The resulting classifier gave a correct answer on 98.4% of tests.

Once such a successful classifier is found, one can try to determine what features of isogeny-graphs it is using. For example, it could be identifying: the existence of a graph automorphism from Galois conjugacy; connectivity properties which arise from measures such as the minimum distances between vertices; or small cycles appearing in M-valleys as observed in [LB20]. Then one can attempt to generate random graphs with said features, and train a new classifier, the success of which indicates if there are additional distinguishing features of random graphs and isogeny graphs not yet considered.

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