# Schnorr Signatures are Tightly Secure in the ROM under a Non-interactive Assumption

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**Abstract.** We show that the Schnorr signature scheme meets existential unforgeability under chosen-message attack (EUF-CMA) in the random oracle model (ROM) if the *circular discrete-logarithm* (CDL) assumption, a new, non-interactive variant of DL we introduce, holds in the underlying group. Our reduction is *completely tight*, meaning the constructed adversary against CDL has both essentially the same running time and success probability as the assumed forger.

To our knowledge, we are the first to exhibit such a reduction. Previously, Bellare and Dai (INDOCRYPT 2020) showed the scheme is EUF-CMA the ROM if their multi-base DL assumption holds in the underlying group. However, multi-base DL is interactive; moreover, their reduction, while tighter than the initial result of Pointcheval and Stern (EUROCRYPT 1996), still incurs a security loss that is linear in the number of the adversary's RO queries. We justify CDL by showing it holds in two carefully chosen idealized models, which idealize different aspects of our assumption. Our quantitative bounds in these models are essentially the same as for DL, giving strong evidence that CDL is as hard DL in appropriate elliptic-curve groups groups.

**Keywords:** Schnorr signatures · tight security · ECDSA conversion function

## 1 Introduction

### 1.1 Background and Main Results

Schnorr signatures and our focus. The Schnorr signature scheme [Sch90] (recalled below), specifically in the form of EdDSA [BDL+12] implemented over twisted Edwards curves, is one of the most widely used pieces of cryptography today. For example, it is used in SSH/SSL, and in Bitcoin since the Taproot soft-fork upgrade in November 2021 [WNR20]. There is also a rich theory behind the scheme's security, with tantalizing open questions. The initial result of Pointcheval and Stern (PS) [PS96] showed the scheme meets existential unforgeability (EUF-CMA) in the random oracle model (ROM) [BR93] assuming the discrete-logarithm (DL) assumption holds in the underlying group. However, the PS result has the two major downsides, which have persisted despite much

follow-on work: (1) the proof relies on the artificial ROM, and (2) the reduction given in the proof is *lossy*. In this work, we focus on overcoming (2).

The problem of tight security. Given a forger against the Schnorr signature scheme in the ROM with success probability $p_{\mathsf{succ}}$, the PS result says there is an adversary solving DL in the underlying group with similar running time and having success probability $\Theta(p_{\mathsf{succ}}^2/q_H)$, where $q_H$ is the number of RO queries made by the forger. Unfortunately, the discrepancy between the success probabilities of the assumed forger and the constructed DL adversary makes the result meaningless in practice. For example, suppose we implement the scheme over twisted Edwards curves of order $2^{256}$, which are conjectured to have 128-bit security for DL. Conservatively assuming at most $2^{64}$ RO queries, the PS result then tells us the scheme has $128/2 - 64 = 0$ bits of security for EUF-CMA! This is despite the lack of any known attack on the scheme short of solving DL.

Can we do better? Prior work [FPS20] showed tight security of Schnorr in additional idealized models such as the ROM combined with the algebraic group model (AGM) [FKL18], which makes assumptions on the adversary's strategy. Other work [NSW09, Sho23, CLMQ21] proved Schnorr secure directly in the generic group model (GGM) [Nec94, Sho97], making specific assumptions on the hash function. However, such idealized models are arguably suited for analyzing simpler *assumptions* rather than the scheme itself.[3]

On the other hand, another sequence of works [PV05, GBL08, Seu12, FJS19] culminated in showing that there *is no* tight "representation-indpendent" reduction in the ROM (even under a minimal formulation of security) for Schnorr signatures under *any* "representation independent" non-interactive assumption (see below). Later, Bellare and Dai (BD) [BD20] showed that there *is* a generic reduction that loses *only* a $q_H$ factor, thereby surpassing the "square-root barrier" but still falling short of a completely tight reduction. This gap matters: going back to our previous example, we would get $128 - 64 = 64$ bits of security for the scheme, which is insufficient for practical applications. BD also rely on a new *interactive* assumption[4] they call multi-base DL, which is a variant of the one-more discrete logarithm assumption [BNPS03]. Since the challenger must answer discrete-log queries by the adversary, multi-base DL is not a falsifiable assumption [Nao03].

We ask whether we can eliminate *both* downsides of their result, namely:

> *Is there a* completely tight *reduction in the ROM from EUF-CMA of Schnorr signatures to a non-interactive and falsifiable assumption?*

Due to the above-mentioned impossibility results, such an assumption has to be *representation-dependent*, meaning depend on the specific group representation

[3] In particular, these idealized models are subject to *uninstantiability* results [Den02, Zha22], so must be used with caution. Complex and interactive problems shown to hold in these models are more likely to fall prey to the reach of these results.

[4] Note that the assumption is *single-query*, whereas assuming the security of the signature scheme would be *multi-query*.

(see further discussion below). It is *a priori* unclear (to us, at least) what such an assumption would look like.

Our results. We resolve the above question in the affirmative. To do so, our central conceptual contribution is the *circular discrete-logarithm* (CDL) problem, a new, non-interactive, falsifiable (at least for some formulation of the assumption), and representation-dependent variant of the DL problem. We show that if CDL holds then the Schnorr signature scheme is secure in the underlying group, with a *completely tight* reduction in the ROM. Using carefully-chosen idealized models that idealized different aspects of the assumption, we give strong evidence that in appropriate elliptic-curve groups, for example twisted Edwards curves, CDL indeed has the same complexity as DL. Consequently, if one believes in these models for analyzing DL/CDL in appropriate elliptic-curve groups, then the security level of the scheme shown by our results matches that indicated by decades of cryptanalysis.

### 1.2 Technical Overview

The Schnorr signature scheme. The Schnorr signature scheme is defined over a group $\mathsf{G}$ of prime order $p$ generated by $g \in \mathsf{G}$ and uses a hash function $H\colon \{0,1\}^* \to \mathbb{Z}_p$. A secret key $x$ is uniformly sampled from $\mathbb{Z}_p$, *i.e.*, $x \leftarrow\!\!\$\ \mathbb{Z}_p$, and defines a public key $h \leftarrow g^x$. A signature on a message $m \in \{0,1\}^*$ is computed by sampling $r \leftarrow\!\!\$\ \mathbb{Z}_p$, setting $R \leftarrow g^r$, computing $c \leftarrow H(R\|m)$ and returning $(R, s)$ with $s \leftarrow (r + cx) \bmod p$. A signature $(R, s)$ on $m$ under public key $h$ is verified by checking $(g^s = R \cdot h^c)$, where $c \leftarrow H(R\|m)$. We denote the scheme by $\mathsf{Sch}[\mathsf{G}, H]$ and usually drop $H$ when working in the ROM.

The circular discrete-logarithm problem. Let $\mathsf{G}$ be as above. Let $f\colon \mathsf{G} \to \mathbb{Z}_p$ be an efficient function that we call a *conversion function* following the terminology regarding ECDSA [oST13]. We say that the *circular discrete-logarithm* (CDL) problem holds in $\mathsf{G}$ for $f$ if, given uniformly sampled $h \in \mathsf{G}$, it's hard to find $(R, z)$ such that

$$g^z = R \cdot h^{f(R)} \tag{1}$$

and $f(R) \neq 0$. One can think of $f$ as a very simple function, not (necessarily) a cryptographic hash function used to instantiate Schnorr. We say that CDL holds for $\mathsf{G}$ if there exists $f$ such that CDL holds in $\mathsf{G}$ for $f$. We denote the CDL for $f$ in $\mathsf{G}$ as $\mathrm{CDL}[\mathsf{G}, f]$ and just CDL for $\mathsf{G}$ as $\mathrm{CDL}[\mathsf{G}]$. The "circularity" in CDL is that in the solution equation $R$ occurs both in the base and the exponent, mirroring this peculiar property of Schnorr's verification equation. An equivalent formulation is, given $h \leftarrow\!\!\$\ \mathsf{G}$, find $a, b \in \mathbb{Z}_p$ such that $f(g^a/h^b) = b$. Indeed, this immediately yields a CDL solution with $z = a$ and $R = g^a/h^b$.

Note that, even for discrete-log-hard $\mathsf{G}$, $\mathrm{CDL}[\mathsf{G}, f]$ does not hold for *every* function $f$. In particular, it is necessary that no value has a large preimage under $f$: Suppose $t \in \mathbb{Z}_p^*$ does; then a uniform $z \leftarrow\!\!\$\ \mathbb{Z}_p^*$, together with $R := g^z/h^t$ breaks CDL if $f(R) = t$, which happens with probability proportional to the

size of $f^{-1}(t)$ in $\mathbb{Z}_p^*$. In Section 4, we show that, when modelling $\mathsf{G}$ as an *elliptic-curve generic group* [GS22], this assumption on $f$ is not only necessary but also sufficient for CDL to hold. For an elliptic-curve group, a simple example of a function $f$ satisfying this assumption (as previously argued in [GS22]) is the *ECDSA conversion function* [oST13], which maps a point $P = (x, y)$ on the curve to its reduced $x$-coordinate, *i.e.*, $f\colon (x, y) \mapsto x \bmod p$.

Note that if we fix a specific conversion function $f$, CDL$[\mathsf{G}, f]$ is *falsifiable* in the sense of [Nao03], whereas CDL$[\mathsf{G}]$ is not falsifiable. Our results also hold under a non-falsifiable but weaker assumption where we assume *some* $f$ works but we don't know which one. Indeed, the conversion function $f$ is used *only in the proof*, not in real life. There have been previous instances of primitives occurring only in proofs, *e.g.* [BM14, GJO16, MOZ22], but to our knowledge it is novel in the context of Schnorr signatures.

Tight reduction from Schnorr to CDL. Our main result is a completely tight reduction in the ROM from EUF-CMA of $\mathsf{Sch}[\mathsf{G}]$ to CDL$[\mathsf{G}]$. Namely, given a forger against $\mathsf{Sch}[\mathsf{G}]$, we construct an adversary solving CDL$[\mathsf{G}, f]$ with essentially the same running-time and success probability as the assumed forger. The only assumption we make on $f$ is that $|f^{-1}(0)|$, the number of elements $f$ maps to 0, is small. This is implied by the condition on $f$ discussed above, which is necessary for CDL$[\mathsf{G}, f]$ to hold in the first place.

Given an instance $h \in \mathsf{G}$ of CDL$[\mathsf{G}, f]$, the reduction sets $h$ as the public key for the forger, so it does not know the corresponding secret key. When run, the forger makes hash queries and signing queries. Its signing queries are simulated by the CDL adversary as in the standard proof of Schnorr in [PS96]. Namely, on query $m_i$, the reduction picks $s_i, c_i \leftarrow\!\!\$\ \mathbb{Z}_p$, sets $R_i \leftarrow g^{s_i}/h^{c_i}$, and programs the random oracle at $R_i \| m_i$ to be $c_i$, returning $(R_i, s_i)$ as the signature. (Since $R_i$ is uniform, the probability that the RO is already defined at $R_i \| m_i$ is negligible.)

To simulate the hash queries, the intuition is that we embed outputs of $f$ into the answers; at the same time, we need to ensure that the returned values are uniformly distributed values in $\mathbb{Z}_p$, independent of the adversary's view. For this, on query $R \| m$, the reduction picks $a, b \leftarrow\!\!\$\ \mathbb{Z}_p$ and programs the RO at $R \| m$ to $f(R \cdot h^a \cdot g^b) + a$ (modulo $p$). We argue that *no matter what* $f$ *is*, the hash values satisfy the above requirement.

Now, consider a successful forgery $(R, s)$ on some $m$, thus

$$g^s = R \cdot h^c \ , \tag{2}$$

where $c$ is the RO response for $R \| m$. Since $m$ is different from the queried messages,[5] the query $R \| m$ was made explicitly (either by the adversary or the game when verifying the forgery), that is, the RO was not programmed during a signing query. Let thus $a, b$ be the values chosen by the reduction when answering this RO query, that is, $c = f(R \cdot h^a \cdot g^b) + a$ (modulo $p$). Together with Eq. (2),

[5] Note that we can actually show *strong* unforgeability, namely that it's even hard for the adversary to forge a new signature on an already-signed message, since $(R, s, m) \neq (R_i, s_i, m_i)$ for all $i$ implies $R \| m \neq R_i \| m_i$ for all $i$, since $s_i$ is uniquely determined by $R_i$ and $m_i$.

this yields $g^s = R \cdot h^{f(R \cdot h^a \cdot g^b)+a}$, which implies

$$g^s \cdot g^b = (R \cdot h^a \cdot g^b) \cdot h^{f(R \cdot h^a \cdot g^b)} .$$

Thus, $(R^* := R \cdot h^a \cdot g^b, z^* := (s+b) \bmod p)$ is solution to CDL as long as $f(R^*) \neq 0$, which by our initial assumption on $f$ holds with high probability.

Analyzing CDL in the EC-GGM. To gain confidence in a new computational hardness assumption in prime-order groups, it has become standard to analyze it in the generic group model (GGM) [Nec94, Sho97], where the adversary only gets (random) labels of group elements and has access to an oracle to compute the group operation. Concretely, given the labels of two group elements, the oracle returns the label of the product of the group elements.

This model is however syntactically ill-defined when there is a function taking as input group elements, for example the conversion function in ECDSA [oST13]. To analyze the security of ECDSA (and variants thereof), Groth and Shoup (GS) [GS22] introduce the *elliptic-curve GGM*, where the labels are random group elements from an elliptic curve (conditioned on preserving simple properties, namely the identity element and inverses).

In Section 4 we analyze CDL$[\mathsf{G}, f]$ for arbitrary $\mathsf{G}$ and $f\colon \mathsf{G} \to \mathbb{Z}_p$, where $p = |\mathsf{G}|$, in the EC-GGM model, which uses labels from $\mathsf{G}$. We show that CDL holds conditioned on the aforementioned (necessary) property of $f$: no element in its range can have a large preimage. In particular, we show that the advantage of any adversary is bounded by $(S + 4q^2)/q$, where $S := \max_{t \in \mathbb{Z}_q}\{|f^{-1}(t)|\}$ is the largest preimage and $q$ is the number of group-oracle queries made by the adversary. Note that we use exactly the same property as GS do on the conversion function to prove security of ECDSA in the EC-GGM. Moreover, as the ECDSA conversion function is 2-to-1, in this case $S = 2$ and the bound is essentially the same as the one for DL [Sho97]; in other words, CDL for the ECDSA conversion function (which is falsifiable) is as hard as DL in the EC-GGM.

Analyzing CDL in the algebraic bijective ROM. We also consider idealizing the function $f$ instead of the underlying group. Particularly, we look at how the ECDSA conversion function is modeled in security analyses of ECDSA itself, though we stress that our analysis is not necessarily tied to the ECDSA conversion function and in particular works for functions that are still simple and have similar structural properties but are more "random" than the ECDSA conversion function.

Initial results by Brown [Bro02] modeled the conversion function as a RO (in addition to using the GGM to model the underlying group), which ignores its obvious structure. To better capture it, the *bijective* ROM was proposed [FKP16, FKP17, HK23]. In this model $f = \psi \circ \Pi \circ \varphi$, where $\varphi$ maps from $\mathsf{G}$ to $\mathbb{A} := \{0,1\}^L$, $\Pi$ maps from $\mathbb{A}$ to $\mathbb{B} := [2^L - 1]$, and $\psi$ maps from $\mathbb{B}$ to $\mathbb{Z}_p$. Here $\varphi$ and $\psi$ are standard-model functions, while $\Pi$ is modeled as a bijective RO.

In fact, CDL for the ECDSA conversion function is a special case of the *semi-logarithm problem* (SLP) introduced by Fersch, Kiltz, and Poettering (FKP) [FKP17, Definition 6] with $\rho_0(u,v) = u$ and $\rho_1(u,v) = -v$. FKP show a loose

reduction from SDL to DL. We would like to show a *tight* reduction in the special case of CDL. We manage to do this by assuming that the adversary is *algebraic* wrt. its queries to $\Pi, \Pi^{-1}$, *i.e.*, we use the algebraic BRO model (ABRO) recently proposed to analyze blind ECDSA [QCY21]. Specifically, we show that the advantage of any CDL adversary in the ABRO is bounded by the hardness of DL in $\mathsf{G}$ plus $(2q^2 + 2q + qS)/p$, where $S := \max_{t \in \mathbb{Z}_q}\{|\psi^{-1}(t)|\}$ is the largest preimage and $q$ is the number of queries made by the adversary. It is an interesting open problem to drop the algebraic assumption on the adversary while keeping the reduction tight.

### 1.3 Discussion, Related Work, and Open Problems

Ratio-based tightness. While the focus of their work is on the multi-user setting, Kiltz, Masny and Pan (KMP) [KMP16] give a two-link chain of reductions going from single-user EUF-CMA of Schnorr signatures to DL. The first reduction [KMP16, Lemma 3.5] goes from passive impersonation of Schnorr's identification protocol to DL, and the second goes from EUF-CMA of Schnorr signatures to the former. While they claim the first reduction is tight, their criterion for tightness is "ratio-based," namely requiring roughly equal *time-to-success ratios* of the assumed adversary and the constructed one. Unfortunately, as discussed by Bellare and Dai (BD) in [BD20, Appendix B], this criterion is problematic and the aforementinoed reduction (which uses rewinding) has a substantial running-time blowup. As in [BD20], we employ a notion of tightness that requires roughly equal success probabilities and running-times *individually*, avoiding such problems.

The second reduction [KMP16, Theorem 1.1] loses a factor $q_H$ even under ratio-based tightness. Overall, as already argued by [BD20], in the single-user setting KMP do not improve on the required size of underlying group for Schnorr signatures versus classical results.

Relation to instantiability. CDL is related to the problem of instantiating Schnorr signatures (*i.e.*, replacing the RO used in the scheme with a concrete hash function) under a weak security notion called *universal unforgeability under no-message attack* (UUF-NMA) used to show prior *im*possibility results [PV05, GBL08, Seu12, FJS19]. In UUF-NMA, the forgery message is random and the adversary gets no signing queries. As compared to instantiating Schnorr signatures under UUF-CMA, CDL differs in that there is no message and $f$ takes solely a group element as input. We also stress that in the above-mentioned impossibility results, UUF-NMA is only considered in the ROM.

Representation-dependence of CDL. Our reduction from EUF-CMA of Schnorr signatures to CDL does not contradict prior impossibility results because the most general of these results [FJS19] only applies to problems that are *representation-invariant*, *i.e.*, an instance-solution pair remains valid when the representation of the underlying group is changed. However, CDL is representation-*dependent*. Indeed, Eq. (1) may hold for one group representation but not another, as the value of $f(R)$ depends on the representation.

CDL AS A STEPPING-STONE. An important problem left open by our work is whether for every group $\mathsf{G}$ of order $p$ there is a function $f\colon \mathsf{G} \to \mathbb{Z}_p$ such that there is a reduction from CDL$[\mathsf{G}, f]$ to some "more-standard assumption" in $\mathsf{G}$. For example, constructing $f$ for which there is a tight reduction from CDL$[\mathsf{G}, f]$ to DL in $\mathsf{G}$ would, by composition, yield a tight reduction from EUF-CMA of $\mathsf{Sch}[\mathsf{G}]$ to DL in $\mathsf{G}$. Even a loose reduction would be of interest to corroborate CDL. In general, the construction of $f$ could introduce other assumptions. We stress that the Schnorr signature scheme itself and its instantiation in practice using cryptographic hashing would remain unaffected.

EXTENSIONS TO SCHNORR. Many recent works build upon Schnorr signatures, particularly to achieve signature schemes with advanced functionalities such as aggregate signatures [CGKN21], blind signatures [FW24], multisignatures [NRS21], ring signatures [YEL+21], and threshold signatures [KG24]. (We give some representative citations to recent work, not an exhaustive list.) The scheme was also used to give adaptor signatures [AEE+21], which have important applications to cryptocurrencies. We leave it as an open problem to extend our results to these settings. This is not straightforward because these works often use assumptions beyond DL like one-more DL (OMDL) [BNPS03, BFP21], which seem incomparable to CDL.

## 2 Preliminaries

NOTATION. If $\overrightarrow{v}$ is a vector then $|\overrightarrow{v}|$ is its length (the number of its coordinates) and $v_i$ is its $i$-th coordinate. Strings are identified with vectors over $\{0,1\}$, so that $|Z|$ denotes the length of a string $Z$ and $Z_i$ denotes its $i$-th bit. By $\varepsilon$ we denote the empty string or vector. By $x\|y$ we denote the concatenation of strings $x, y$. If $S$ is a finite set, then $|S|$ denotes it size and we let $x \leftarrow\!\!\$\ S$ denote picking an element of $S$ uniformly at random and assigning it to $x$.

Algorithms may be randomized unless otherwise indicated. If $A$ is an algorithm, we let $y \leftarrow A^{\mathrm{O}_1,\cdots}(x_1,\ldots;\omega)$ denote running $A$ on inputs $x_1,\ldots$ and coins $\omega$, with oracle access to $\mathrm{O}_1,\ldots$, and assigning the output to $y$. By $y \leftarrow\!\!\$\ A^{\mathrm{O}_1,\cdots}(x_1,\ldots)$ we denote picking $\omega$ at random and letting $y \leftarrow A^{\mathrm{O}_1,\cdots}(x_1,\ldots;\omega)$. We let $\mathbf{Out}(A^{\mathrm{O}_1,\cdots}(x_1,\ldots))$ denote the set of all possible outputs of $A$ when run on inputs $x_1,\ldots$ and with oracle access to $\mathrm{O}_1,\ldots$. Running time is worst-case, which for an algorithm with access to oracles means across all possible replies from the oracles. We use $\perp$ (bot) as a special symbol to denote rejection, and it is assumed not to be in $\{0,1\}^*$.

Let $f\colon A \to B$ be a function. We let $\mathsf{Size}_f(b) := |f^{-1}(b)|$ for all $b \in B$, $\mathsf{MaxSize}_f := \max_{b\in B} \mathsf{Size}_f(b)$.

GAMES. We use the code-based game-playing framework of BR [BR06]. By $\Pr[\mathrm{G} \Rightarrow y]$ we denote the probability that the execution of game G results in this output being $y$. In games, integer variables, set variables, boolean variables and string variables are assumed initialized, respectively, to 0, the empty set $\emptyset$, the boolean $\mathsf{false}$ and $\perp$.

### 2.1 Schnorr Signatures and Their Security

SIGNATURE SCHEMES AND THEIR SECURITY. A *signature scheme* with message space $\mathsf{MS}$ is a tuple of algorithms $\mathsf{DS} = (\mathsf{DS.K}, \mathsf{DS.S}, \mathsf{DS.V})$ that work as follows:

- $\mathsf{DS.K}$: The key-generation algorithm outputs a key pair $(vk, sk)$. (We suppress the security parameter for simplicity.)
- $\mathsf{DS.S}(sk, m)$: On inputs a signing key $sk$ and a message $m \in \mathsf{MS}$, the signing algorithm outputs a signature $\sigma$.
- $\mathsf{DS.V}(vk, \sigma, m)$: On inputs a verification key $vk$, signature $\sigma$, and message, $m \in \mathsf{MS}$, the verification algorithm outputs a bit.

For correctness, we require that

$$\Pr\left[\mathsf{DS.V}\big(vk, \mathsf{DS.S}(sk, m), m\big) \Rightarrow 1\right] = 1$$

for all $(sk, vk) \in \mathbf{Out}(\mathsf{DS.K})$ and all $m \in \mathsf{MS}$, where the probability is over the coins for $\mathsf{DS.S}$.

SCHNORR SIGNATURES. Let $\mathsf{G}$ be a cyclic group of prime order $p = |\mathsf{G}|$, generated by $g$. Let $H\colon \{0,1\}^* \to \mathbb{Z}_p$ be a hash function. The *Schnorr signature scheme* [FS87, Sch90] $\mathsf{Sch}[\mathsf{G}, H] = (\mathsf{Sch.K}, \mathsf{Sch.S}, \mathsf{Sch.V})$ with message space $\{0,1\}^*$ works as follows. Algorithm $\mathsf{Sch.K}$ chooses $x \leftarrow\!\!\$\, \mathbb{Z}_p$, sets $X \leftarrow g^x$, and returns $(vk = X, sk = x)$. Algorithm $\mathsf{Sch.S}$ on input $x, m$ chooses $r \leftarrow\!\!\$\, \mathbb{Z}_p$, sets $R \leftarrow g^r$ and $c \leftarrow H(R\|m)$, then returns $\big(R, (r + cx) \bmod p\big)$. Algorithm $\mathsf{Sch.V}$ on inputs $X, (R, s), m$ returns $\big(g^s = R \cdot X^c\big)$ where $c \leftarrow H(R\|m)$. Correctness is straightforward to check. In the ROM, we denote the scheme by $\mathsf{Sch}[\mathsf{G}]$.

EUF-CMA. We define the (strong) existential unforgeability under chosen-message attack (EUF-CMA). Let $\mathsf{DS} = (\mathsf{DS.K}, \mathsf{DS.S}, \mathsf{DS.V})$ with message space $\mathsf{MS}$. For an adversary A, we let its (strong) EUF-CMA advantage against $\mathsf{DS}$ be $\mathbf{Adv}^{\text{euf-cma}}_{\mathsf{DS}}(\mathrm{A}) = \Pr\left[\mathrm{G}^{\text{euf-cma,A}}_{\mathsf{DS}} \Rightarrow 1\right]$, where the game is in Figure 1.

### 2.2 Discrete-Logarithm Problem

We recall the *discrete-logarithm* (DL) problem. Let $\mathsf{G}$ be a group of prime order $p = |\mathsf{G}|$, generated by $g$. For an adversary A, we let its DL-advantage against $\mathsf{G}, g$ be $\mathbf{Adv}^{\text{dl}}_{\mathsf{G},g}(\mathrm{A}) = \Pr\left[\mathrm{G}^{\text{dl,A}}_{\mathsf{G},g} \Rightarrow 1\right]$, where the game is in Figure 2.

## 3 Tight Security of Schnorr Signatures under CDL

We provide our new assumption then proceed to give a tight reduction of EUF-CMA security of Schnorr signatures in the ROM to our assumption.

### 3.1 Circular Discrete-Logarithm Problem

We introduce the *circular discrete-logarithm* (CDL) problem. Let $\mathsf{G}$ be a group of prime order $p = |\mathsf{G}|$, generated by $g$. Let $f\colon \mathsf{G} \to \mathbb{Z}_p$. To an adversary A we let

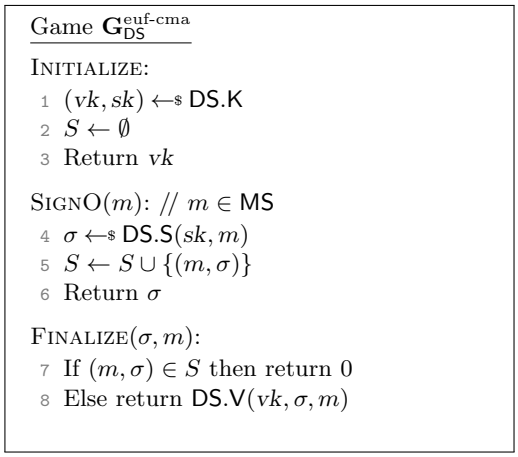


Fig. 1: Game defining (strong) EUF-CMA of $\mathsf{DS}$.

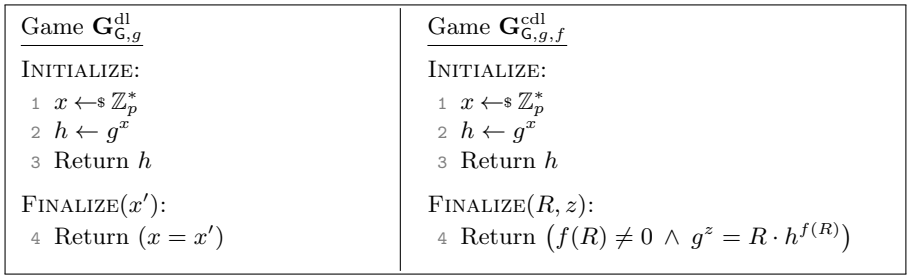


Fig. 2: Games defining DL and circular DL problems.

its CDL-advantage against $\mathsf{G}, g, f$ be $\mathbf{Adv}^{\text{cdl}}_{\mathsf{G},g,f}(\mathrm{A}) = \Pr\left[\mathbf{G}^{\text{cdl}}_{\mathsf{G},g,f}(\mathrm{A}) \Rightarrow 1\right]$ where the game is in Figure 2.

If we want to assume that there exists no efficient adversary that solves CDL, the condition $f(R) \neq 0$ in the Finalize procedure is essential. Otherwise, consider the adversary who has some $R^* \in \mathsf{G}$ such that $f(R^*) = 0$ hard-coded along with $z^* = \mathsf{DLog}_{\mathsf{G},g}(R^*)$, and simply outputs $(R^*, z^*)$ as a valid CDL solution. This adversary would have advantage 1. The assumption would thus be wrong, even though no one might *know* such an adversary. This is analogous to collision-resistance of hash functions, for which an adversary always exists (*cf.* [Rog06]). By adding the condition $f(R) \neq 0$, we simply avoid such issues.

### 3.2 Main Result

**Theorem 1** *Let* $\mathsf{G}$ *be a group of prime order* $p$*. Let* A *be an adversary against the Schnorr signature scheme* $\mathsf{Sch}[\mathsf{G}]$ *in the ROM and assume* A *makes at most* $q_s$ *queries to the signing oracle and* $q_h$ *queries to the hash oracle. Let* $f\colon \mathsf{G} \to \mathbb{Z}_p$*. Then there exists an adversary* B *with running-time roughly the same as* A *plus*

Adversary $\mathrm{B}(h)$

1 $\mathsf{T} \leftarrow ()$; $i \leftarrow 1$
2 $(m, R, s) \leftarrow\!\!\$\ \mathrm{A}^{\textsc{SignO},\textsc{HashO}}(\mathsf{G}, g, h)$
3 $\textsc{HashO}(R, m)$ //ensure that value is defined
4 Let $j$ be such that $R = R_j$ and $m = m_j$. If such $j$ does not exist, abort
5 $R^* \leftarrow R \cdot h^{a_j} g^{b_j}$ ; $s^* \leftarrow (s + b_j) \bmod p$
6 Return $(R^*, s^*)$

$\textsc{SignO}(m)$:
7 $s \leftarrow\!\!\$\ \mathbb{Z}_p$; $c \leftarrow\!\!\$\ \mathbb{Z}_p$; $R \leftarrow g^s / h^c$
8 If $\mathsf{T}(R, m) \neq \bot$: Abort
9 $\mathsf{T}(R, m) \leftarrow c$
10 Return $(R, s)$

$\textsc{HashO}(R_i, m_i)$:
11 If $\mathsf{T}(R_i, m_i) = \bot$ then
12 $\quad a_i, b_i \leftarrow\!\!\$\ \mathbb{Z}_p$
13 $\quad R' \leftarrow R_i \cdot h^{a_i} \cdot g^{b_i}$
14 $\quad$ If $f(R') = 0$: Abort
15 $\quad \mathsf{T}(R_i, m_i) \leftarrow (f(R') + a_i) \bmod p$
16 $\quad i \leftarrow i + 1$
17 Return $\mathsf{T}(R_i, m_i)$

Fig. 3: CDL-adversary B for the proof of Theorem 1.

*simulation overhead proportional to $q_s$ and $q_h$ such that*

$$\mathbf{Adv}^{\text{euf-cma}}_{\mathsf{Sch}[\mathsf{G}]}(\mathrm{A}) \leq \mathbf{Adv}^{\text{cdl}}_{\mathsf{G},g,f}(\mathrm{B}) + \frac{q_s(q_s + q_h) + q_h \cdot \mathsf{Size}_f(0)}{p} \,. \tag{3}$$

We prove the theorem by formalizing the ideas laid out in the Introduction (Section 1.2).

*Proof.* We consider a sequence of games defined in Figure 4. Games in boxes ($\mathrm{G}_1$ and $\mathrm{G}_3$) contain the boxed lines, whereas they are ignored for the other games. Note that $\mathrm{G}_0$ is equivalent to $\mathbf{G}^{\text{euf-cma}}_{\mathsf{Sch}[\mathsf{G}]}$.

$\underline{\mathrm{G}_0 \to \mathrm{G}_1}$. We start with analyzing how the probability of returning 1 changes from $\mathrm{G}_0$ to $\mathrm{G}_1$. The probability of aborting on line 6 in $\mathrm{G}_1$ during any signing query is at most $(q_s + q_h)/p$ since $R$ is uniformly sampled and there are at most $q_s + q_h$ possible $(\cdot, m)$ pairs defined in $\mathsf{T}$ it could hit. By the union bound over all signing queries, the probability of $\mathrm{G}_1$ aborting is at most $q_s(q_s + q_h)/p$. Thus, similarly to the Fundamental Lemma of Game Playing [BR06] we have $\Pr\left[\mathrm{G}^{\mathrm{A}}_0 \Rightarrow 1\right] \leq \Pr\left[\mathrm{G}^{\mathrm{A}}_1 \Rightarrow 1\right] + q_s(q_s + q_h)/p$.

$\underline{\mathrm{G}_1 \to \mathrm{G}_2}$. In $\mathrm{G}_2$, we change the signing oracle so that it doesn't require knowledge of the secret key $x$. We claim these games are equivalent. This is because, if $\mathsf{T}(R, m)$ has not been defined yet, the distribution of $(R, s, c)$ in both games

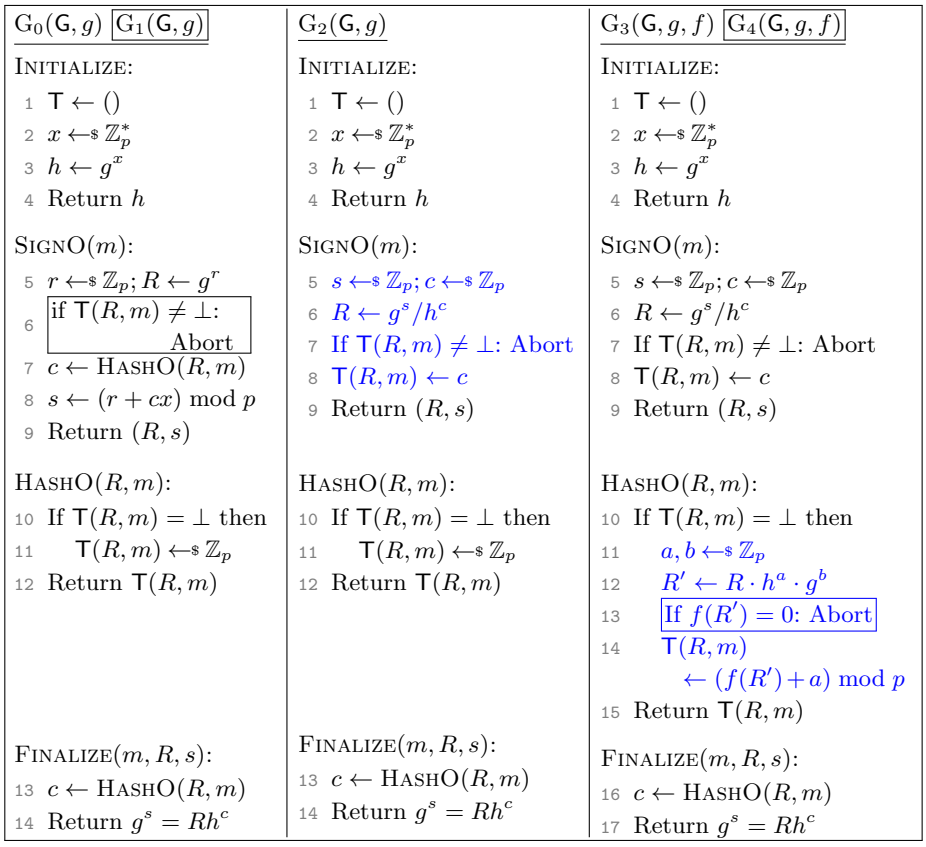


Fig. 4: Games for the proof of Theorem 1. Changes are highlighted in blue.

is uniform in $\mathsf{G} \times \mathbb{Z}_p \times \mathbb{Z}_p$ conditioned on $g^s = R \cdot h^c$. In other words, this hop corresponds to a reordering of how these variables are defined. We thus have $\Pr\left[\mathrm{G}_1^{\mathrm{A}} \Rightarrow 1\right] = \Pr\left[\mathrm{G}_2^{\mathrm{A}} \Rightarrow 1\right]$.

$\underline{\mathrm{G}_2 \to \mathrm{G}_3}$. In $\mathrm{G}_3$, we change how we answer hash queries. We argue that the distribution of responses of the hash oracle in $\mathrm{G}_3$ is equivalent to the distribution of responses in $\mathrm{G}_2$. In $\mathrm{G}_2$, the distribution of the value sampled in line 11 is uniform in $\mathbb{Z}_p$. This is also the case in $\mathrm{G}_3$: Since $a$ is uniform in $\mathbb{Z}_p$, so is the value $(f(R') + a) \bmod p$. Moreover, conditioned on any *fixed* $a$, $R' = R \cdot h^a \cdot g^b$ is uniform in $\mathsf{G}$ because $b$ is uniform in $\mathbb{Z}_p$. Thus $\Pr\left[\mathrm{G}_2^{\mathrm{A}} \Rightarrow 1\right] = \Pr\left[G_3^{\mathrm{A}} \Rightarrow 1\right]$.

$\underline{\mathrm{G}_3 \to \mathrm{G}_4}$. This hop introduces an abort whenever a hash query is made and $f(R') = 0$. Since $R'$ is uniformly distributed, the probability of any query aborting is $|f^{-1}(0)|/p$. By union bound over all queries, the probability of aborting is at most $q_h \cdot |f^{-1}(0)|/p$. Thus, similarly to the Fundamental Lemma of Game Playing [BR06] we get $\Pr\left[\mathrm{G}_3^{\mathrm{A}} \Rightarrow 1\right] \leq \Pr\left[\mathrm{G}_4^{\mathrm{A}} \Rightarrow 1\right] + q_h \cdot \mathsf{MaxSize}_f(0)/p$.

$\underline{\mathrm{G}_4}$. Combining the hops above yields

$$\mathbf{Adv}^{\text{euf-cma}}_{\mathsf{Sch}[\mathsf{G}]}(\mathrm{A}) \leq \mathbf{Adv}^{\mathrm{G}_4}_{\mathsf{G},g,f}(\mathrm{A}) + \frac{q_s(q_s+q_h)+q_h\cdot \mathsf{MaxSize}_f(0)}{p}\ , \qquad (4)$$

Finally, consider adversary B defined in Figure 3. To prove the theorem, it remains to show that

$$\mathbf{Adv}^{\mathrm{G}_4}_{\mathsf{G},g,f}(\mathrm{A}) \leq \mathbf{Adv}^{\text{cdl}}_{\mathsf{G},g,f}(\mathrm{B})\ ,$$

which combined with Eq. (4) completes the proof. To do so, we show that whenever A returns a strong forgery $(m, R, s)$ in $\mathrm{G}_4$, then B returns a solution to the given CDL instance. A wins if

$$g^s = R\cdot h^c \quad \text{ with } c \leftarrow \mathsf{T}(R,m)\ , \qquad (5)$$

We claim that $\mathsf{T}(R,m)$ must have been defined during some call to HashO, which in turn means that if A wins, B does not abort in line 4. The reason is that $(m,R,s)$ is different from all $(m_i,R_i,s_i)$, which consist of the $i$-th query $m_i$ to SignO together with its response $(R_i,s_i)$. Since for a signature $(R,s)$ on $m$, the value $s$ is determined by $R$ and $m$, this condition is equivalent to $R\|m$ being different from all $R_i\|m_i$. Thus $\mathsf{T}(R,m)$ was not defined during a call to SignO and, if nowhere else, was defined via the call to HashO in line 3.

So let $j$ be the HashO query that defined $\mathsf{T}(R,m)$, and let $a_j, b_j$ be the values that were sampled during the call; thus

$$c = \mathsf{T}(R,m) = \big(f(R\cdot h^{a_j}\cdot g^{b_j}) + a_j\big) \bmod p\ .$$

Together with Eq. (5), this yields $g^s = R\cdot h^{f(R\cdot h^{a_j}\cdot g^{b_j})+a_j}$. Multiplying by $g^{b_j}$ yields $g^{s+b_j} = (R\cdot h^{a_j}\cdot g^{b_j})\cdot h^{f(R\cdot h^{a_j}\cdot g^{b_j})}$. We note that $f(R\cdot h^{a_j}\cdot g^{b_j}) \neq 0$ as assured by the abort condition added in $\mathrm{G}_4$. Together this shows that $R^* := R\cdot h^{a_j}\cdot g^{b_j}, s^* := (s+b_j) \bmod p$, the values returned by B, constitute a valid solution to the given CDL instance. □

## 4 CDL in the EC-GGM

The EC-GGM. The *Elliptic-curve generic group model* (EC-GGM) [GS22] was introduced to cover constructions (such as ECDSA) that define a function which takes as input group elements. In contrast to Shoup's [Sho97] original GGM, in the EC-GGM, the "encodings" of the group elements are not random strings but random points on a concrete elliptic curve $E$, which has a prime number $p$ of points. (Using multiplicative notation, we let $1_E$ denote the identity element.)

In more detail, in security games defined in the EC-GGM, in the beginning, the challenger chooses a random injective *encoding function* $\tau\colon \mathbb{Z}_p \to E$ which preserves trivial relations; in particular, $\tau(0)$ is the identity element and if $\tau(i) = P$ then $\tau(-i) = P^{-1}$. To "compute" in the group, parties have two oracles: Map$(i)$, for $i \in \mathbb{Z}_p$, returns $\tau(i)$; computing linear combinations of group

elements is done by calling $\textsc{Add}(c_1, P_1, c_2, P_2)$ for $c_i \in \mathbb{Z}_p$ and $P_i \in E$, for $i = 1, 2$, which returns $\tau(c_1\tau^{-1}(P_1) + c_2\tau^{-1}(P_2))$.

Note that for schemes (and assumptions) defined over elliptic curve groups (and in particular, if there is a function whose domain is the group), this model is more realistic than the pure GGM (and makes sense syntactically). *E.g.*, in the GGM, one can show ECDSA strongly unforgeable – although it is malleable – which is not possible in the EC-GGM.

Proof overview of CDL in the EC-GGM. To argue that CDL holds in the EC-GGM, we follow the common approach for GGM proofs, also taken by Groth and Shoup [GS22]: instead of randomly sampling $\tau$ in the beginning of the game, we sample entries of the form $(i, P)$ on the fly as required; this does not change the adversary's view. The game is defined in Figure 5.

We then simplify the game and abort whenever the "lazy" sampling does not succeed at the first try (including the boxes with a single frame in Figure 5). Letting $q$ denote the number of the adversary's group oracle queries, the difference to the original game is $\mathcal{O}(q^2/p)$. We next define a *symbolic* game where the secret value $x$ is represented by an indeterminate $X$ and the domain of $\tau$ now consists of (linear) polynomials in $X$ (the game including all boxes in Figure 5). Again, using a standard GGM argument that relies on the Schwartz-Zippel Lemma, the difference to the previous game is $\mathcal{O}(q^2/p)$. Finally, we argue that the adversary's probability in winning the symbolic game is bounded by $\mathsf{MaxSize}_f/p$.

We note that, as compared to Groth and Shoup we avoid the use of asymptotics and give a precise concrete analysis of our assumption in the EC-GGM. Additionally, the proof applies to much more general $f$ than the ECDSA conversion function [oST13], as we simply require a bound on the largest preimage set. (The ECDSA conversion function is 2-to-1.)

**Theorem 2** *Let $E$ be an elliptic curve of prime order $p$ and $f\colon E \to \mathbb{Z}_p$. Let* A *be an adversary against CDL for $f$ in the EC-GGM with $E$ who makes at most $q$ queries to any of its oracles. Then*

$$\mathbf{Adv}^{\text{ec-ggm-cdl}}_{E,f}(\mathrm{A}) \leq \frac{(q+1)\cdot \mathsf{MaxSize}_f + 29q^2 + 12}{p} \ .$$

*Proof.* We proceed via a sequence of games as defined in Figure 5 and analyze their differences.

ec-ggm-cdl→abrt-cdl. First consider the boxed code in lines 8 and 14. Consider the probability that in $\mathbf{G}^{\text{ec-ggm-cdl}}_{E,f}$, during any call to Map or Add, a value $P$ or $i$ is sampled (uniformly at random) for which there is already an assignment in $\tau$ (which we call a "collision"). Map is called once in lines 2 and 4 and at most $q$ times by the adversary, and Add is called at most $q$ times by the adversary. Each call to Map samples at most one value of $P$ (for which either $P, P^{-1}$ either both already have an assignment in $\tau$ or both do not), and similarly each call to Add samples at most two values $i$ and possibly a value $P$ when calling Map in line 17. Thus, the probability of a collision in line 2 is zero and in line 4 is $2/(p-1)$. Then, in the worst case the adversary makes $q$ Add queries. On the

Game $\mathbf{G}^{\text{ec-ggm-cdl}}_{E,f}$; $\boxed{\mathbf{G}^{\text{abrt-cdl}}_{E,f}}$; $\boxed{\boxed{\mathbf{G}^{\text{symb-cdl}}_{E,f}}}$

INITIALIZE:
1 $\tau \leftarrow \{(0, 1_E)\}$
2 $g' \leftarrow \text{MAP}(1)$
3 $\boxed{h \leftarrow \text{MAP}(X); \text{ Return } (g', h)}$
4 $x \leftarrow\!\!\$\ \mathbb{Z}_p^*; h \leftarrow \text{MAP}(x)$
5 Return $(g', h)$

MAP($i$):
6 If $i \notin \text{Dom}(\tau)$
7 $\quad P \leftarrow\!\!\$\ E^*$
8 $\quad \boxed{\text{if } P \in \text{Ran}(\tau) \text{ then abort}}$
9 $\quad$ while $P \in \text{Ran}(\tau) : P \leftarrow\!\!\$\ E^*$
10 $\quad \tau \leftarrow \tau \cup \{(-i, P^{-1}), (i, P)\}$
11 Return $\tau(i)$

ADD($c_1, P_1, c_2, P_2$):
12 For $j = 1, 2$: if $P_j \notin \text{Ran}(\tau)$
13 $\quad i \leftarrow\!\!\$\ \mathbb{Z}_p^*$
14 $\quad \boxed{\text{if } i \in \text{Dom}(\tau) \text{ then abort}}$
15 $\quad$ while $i \in \text{Dom}(\tau) : i \leftarrow\!\!\$\ \mathbb{Z}_p^*$:
16 $\quad \tau \leftarrow \tau \cup \left\{(-i, P_j^{-1}), (i, P_j)\right\}$
17 Return $\text{MAP}\big(c_1 \cdot \tau^{-1}(P_1) + c_2 \cdot \tau^{-1}(P_2)\big)$

FINALIZE($R, z$):
18 $\boxed{\text{if } R \notin \text{Ran}(\tau) \text{ or } z \notin \text{Dom}(\tau) \text{ then return } 0}$
19 Return $\big(f(R) \neq 0 \ \wedge\ \text{MAP}(z) = \text{ADD}(1, R, f(R), h)\big)$

Fig. 5: Original and variants of the EC-GGM game for the circular discrete-log problem.

first such call, the probability of a collision in line 13 (in either of its possible executions) is at most $4/(p-1) + 6/(p-1)$ and in line 17 is at most $8/(p-1)$. If no collisions happened, then the probability of one happening in the second call is bounded by $10/(p-1) + 12/(p-1) + 14/(p-1)$, and so on.

By a union bound, the overall probability of a collision is thus bounded by

$$\sum_{j=1}^{3q+1} \frac{2j}{p-1} = \frac{(3q+1)(3q+2)}{p-1} \leq \frac{10q^2+2}{p-1} \tag{6}$$

assuming for simplicity $q \geq 9$.

Finally, consider the boxed code in line 18. The probability that the adversary wins in $\mathbf{G}^{\text{ec-ggm-cdl}}_{E,f}$ although there is no assignment already in $\tau$ for one (or both) of its output elements $R$ and $z$ is $1/(p-1)$.

Thus, overall, the difference between the probabilities of the games $\mathbf{G}^{\text{ec-ggm-cdl}}_{E,f}$ and $\mathbf{G}^{\text{abrt-cdl}}_{E,f}$ outputting 1 is bounded by $(10q^2+1)/(p-1)$.

abrt-cdl→symb-cdl. Consider game $\mathbf{G}^{\text{symb-cdl}}_{E,f}$, but where we sample $x$ (in line 4), as in $\mathbf{G}^{\text{abrt-cdl}}_{E,f}$. If we furthermore replace the checks "if $i \in \text{Dom}(\tau)$" (lines 6 and 14), where $i$ is now a linear polynomial in $X$, by "if $i(x) \in \text{Dom}(\tau)$" (as well as all other occurrences of $i$ by $i(x)$) then we obtain a game that is distributed like the previous game $\mathbf{G}^{\text{abrt-cdl}}_{E,f}$.

The difference between games $\mathbf{G}^{\text{abrt-cdl}}_{E,f}$ and $\mathbf{G}^{\text{symb-cdl}}_{E,f}$ outputting 1 is thus bounded by the probability $C$ that for any two $i \neq i' \in \text{Dom}(\tau)$, we have $i(x) = i'(x)$. INITIALIZE creates 4 polynomials ($\pm 1$ and $\pm X$), a call to ADD creates up to six polynomials (four in line 16 and two in line 10 when line 17 is executed), while a call to MAP creates fewer; and FINALIZE creates at most 2 different polynomials (since it aborts if one of its arguments is not yet in $\tau$). Note that, as long as no collision occurs, all polynomials are independent of $x$. Thus, $C$ is upper-bounded by the probability that when sampling a random evaluation point (out of $p$ possible ones) any two out of $6q+6$ different polynomials (which are lines) intersect. Since $6q+6$ lines intersect in at most $\sum_{i=1}^{6q+6}(i-1) = \sum_{i=1}^{6q+5} i = \frac{1}{2}(6q+5)(6q+6) = (6q+5)(3q+2)$ points, we have

$$C = \frac{(3q+2)(6q+5)}{p} \leq \frac{19q^2+10}{p}$$

assuming for simplicity $q \geq 27$.

symb-cdl. Consider an output $(R, z)$ by the adversary in game $\mathbf{G}^{\text{symb-cdl}}_{E,f}$ that makes FINALIZE return 1. Since $R \in \text{Ran}(\tau)$, there exist $a, b \in \mathbb{Z}_p$ s.t.

$$\tau^{-1}(R) = a + bX \ .$$

Moreover, $\text{MAP}(z) = \text{ADD}\big(1, R, f(R), h)\big)$ implies

$$\tau(z) = \tau\big(\tau^{-1}(R) + t \cdot \tau^{-1}(h)\big) \text{ with } \ t := f(R) \neq 0 \ .$$

Since $\tau$ is injective and $\tau(X) = h$, this implies

$$\tau^{-1}(R) = z - tX \ ,$$

thus $a = z$ and $b = -t \neq 0$. Consider the point when $R$ is added to the range of $\tau$. Any "fresh" input $R$ to ADD will be associated to a constant polynomial in line 16. Since $\tau^{-1}(R)$ is non-constant, $R$ must have been added to $\tau$ during a call to MAP in line 10. Moreover this call to MAP must have been made by the experiment in lines 3 or 17, since the adversary can only call MAP on constant polynomials. When MAP is called on some fresh $i = a+bX$, the value $R$ is picked uniformly and independently of $a$ and $b$. The probability that any $R$ satisfies $f(R) = -b$ is thus bounded by $\mathsf{MaxSize}_f/p$. As the adversary can create at most $q$ values this way, and moreover we could have $f(h) = -1$, the probability that the adversary wins game symb-cdl is bounded by $(q+1) \cdot \mathsf{MaxSize}_f/p$.

Adding to this the differences between the previous games yields the bound of the theorem. □

Adversary $\mathrm{B}(h)$

$\Pi \leftarrow \emptyset$
$\overrightarrow{U} \leftarrow (g, h)$
$\overrightarrow{s} \leftarrow (1, 0)$; $\overrightarrow{t} \leftarrow (0, 1)$ // $U_i = g^{s_i} h^{t_i}$
$(R, z) \leftarrow\!\!\$\ \mathrm{A}^{\mathrm{BRO},\mathrm{BRO}^{-1}}(\mathsf{G}, g, h)$
Let $i$ such that $R = U_i$
$x \leftarrow (z - s_i)/(t_i + f(R))$
Return $x$

$\mathrm{BRO}(R, \overrightarrow{p})$: // $R = \prod_i U_i^{p_i}$
$\overrightarrow{U} \leftarrow \overrightarrow{U} \| R$
$\alpha \leftarrow \varphi(R)$
If $(\alpha, \cdot) \in \Pi$: Return $\Pi(\alpha)$
$s' \leftarrow \langle \overrightarrow{s}, \overrightarrow{p} \rangle$; $t' \leftarrow \langle \overrightarrow{t}, \overrightarrow{p} \rangle$ //$R = g^{s'} h^{t'}$
$\overrightarrow{s} \leftarrow \overrightarrow{s} \| s'$
$\overrightarrow{t} \leftarrow \overrightarrow{t} \| t'$
$\beta \leftarrow\!\!\$\ \mathbb{B}$
If $(\cdot, \beta) \in \Pi$ then abort
If $-t' = \psi(\beta)$ then abort
$\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
Return $\beta$

$\mathrm{BRO}^{-1}(\beta)$:
If $(\cdot, \beta) \in \Pi$: Return $\Pi^{-1}(\beta)$
$\alpha \leftarrow\!\!\$\ \mathbb{A}$
If $\alpha \in \varphi(\mathsf{G})$:
  $s', t' \leftarrow\!\!\$\ \mathbb{Z}_p$
  If $-t' = \psi(\beta)$ then abort
  If $t' = \psi(\beta)$ then abort
  $\overrightarrow{s} \leftarrow \overrightarrow{s} \| s' \| - s'$
  $\overrightarrow{t} \leftarrow \overrightarrow{t} \| t' \| - t'$
  $R \leftarrow g^{s'} h^{t'}$
  $\alpha \leftarrow \varphi(R)$
  $\overrightarrow{U} \leftarrow \overrightarrow{U} \| R \| R^{-1}$
If $(\alpha, \cdot) \in \Pi$ then abort
$\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
Return $\alpha$

Fig. 6: DL-adversary B for the proof of Theorem 3.

## 5 CDL in the Algebraic Bijective RO Model

The Algebraic Bijective Random Oracle Model. The *Algebraic Bijective Random Oracle Model* (ABRO) [QCY21] is a combination of the bijective random oracle (BRO) model [FKP16] and the algebraic group model (AGM) [FKL18], originally introduced to analyze blind ECDSA. It idealizes the ECDSA

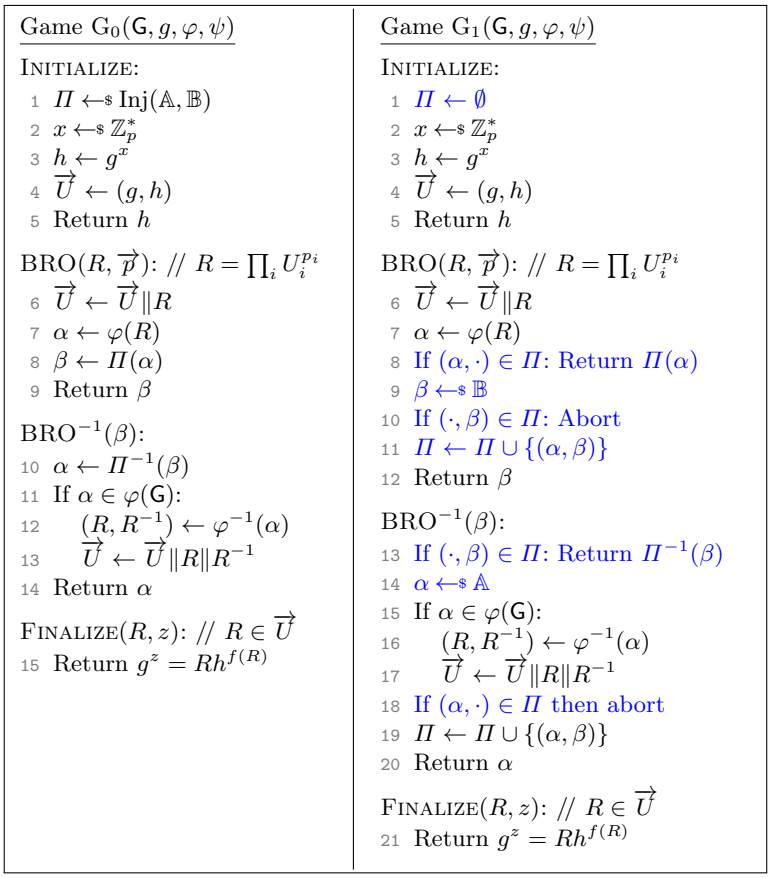


Fig. 7: Games $G_0$, $G_1$ for the proof of Theorem 3. Changes are highlighted in blue.

conversion function $f : \mathsf{G}^* \to \mathbb{Z}_p$ in a similar manner as the BRO, by decomposing $f$ into three independent functions so that $f = \psi \circ \Pi \circ \varphi$, where $\varphi$ maps from $\mathsf{G}^*$ to $\mathbb{A} := \{0,1\}^L$, $\Pi$ maps from $\mathbb{A}$ to $\mathbb{B} := [2^L - 1]$, and $\psi$ maps from $\mathbb{B}$ to $\mathbb{Z}_p$. $\varphi$ and $\psi$ are standard model (non-idealized) functions, while $\Pi$ is modeled as a bijective random oracle and its inverse via the procedures BRO and BRO$^{-1}$. In this way, the idealized conversion function preserves essential properties of the original conversion function such as invertibility and being 2-to-1. We make a slight tweak to this model so that the domain of $f$ and $\varphi$ is $\mathsf{G}$ instead of $\mathsf{G}^*$ to better represent the CDL conversion function. Like the AGM, the ABRO keeps track of all "seen" group elements in a vector $\overrightarrow{U}$. The BRO oracle takes as input a group element $R$ and a vector representation $\overrightarrow{p}$ such that $R = \prod_i U_i^{p_i}$ and outputs some $\beta \in \mathbb{B}$ corresponding to $\Pi(\varphi(R))$. The BRO$^{-1}$ oracle takes as input some $\beta \in \mathbb{B}$ and outputs some $\alpha \in \mathbb{A}$. We note that unlike in the AGM, the

**Game $\mathrm{G}_2(\mathsf{G}, g, \varphi, \psi)$**

INITIALIZE:
1 $\Pi \leftarrow \emptyset$
2 $x \leftarrow\!\!\$\ \mathbb{Z}_p^*$
3 $h \leftarrow g^x$
4 $\overrightarrow{U} \leftarrow (g, h)$
5 Return $h$

BRO$(R, \overrightarrow{p})$: // $R = \prod_i U_i^{p_i}$
6 $\overrightarrow{U} \leftarrow \overrightarrow{U} \| R$
7 $\alpha \leftarrow \varphi(R)$
8 If $(\alpha, \cdot) \in \Pi$: Return $\Pi(\alpha)$
9 $\beta \leftarrow\!\!\$\ \mathbb{B}$
10 If $(\cdot, \beta) \in \Pi$ then Abort
11 $\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
12 Return $\beta$

BRO$^{-1}(\beta)$:
13 If $(\cdot, \beta) \in \Pi$: Return $\Pi^{-1}(\beta)$
14 $\alpha \leftarrow\!\!\$\ \mathbb{A}$
15 If $\alpha \in \varphi(\mathsf{G})$:
16 $\quad s', t' \leftarrow\!\!\$\ \mathbb{Z}_p$
17 $\quad R \leftarrow g^{s'} h^{t'}$
18 $\quad \alpha \leftarrow \varphi(R)$
19 $\quad \overrightarrow{U} \leftarrow \overrightarrow{U} \| R \| R^{-1}$
20 If $(\alpha, \cdot) \in \Pi$ then Abort
21 $\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
22 Return $\alpha$

FINALIZE$(R, z)$: // $R \in \overrightarrow{U}$
23 Return $g^z = R h^{f(R)}$

**Game $\mathrm{G}_3(\mathsf{G}, g, \varphi, \psi)$**

INITIALIZE:
1 $\Pi \leftarrow \emptyset$
2 $x \leftarrow\!\!\$\ \mathbb{Z}_p^*$
3 $h \leftarrow g^x$
4 $\overrightarrow{U} \leftarrow (g, h)$
5 $\overrightarrow{s} \leftarrow (1, 0)$; $\overrightarrow{t} \leftarrow (0, 1)$ // $U_i = g^{s_i} h^{t_i}$
6 Return $h$

BRO$(R, \overrightarrow{p})$: // $R = \prod_i U_i^{p_i}$
7 $\overrightarrow{U} \leftarrow \overrightarrow{U} \| R$
8 $\alpha \leftarrow \varphi(R)$
9 If $(\alpha, \cdot) \in \Pi$: Return $\Pi(\alpha)$
10 $s' \leftarrow \langle \overrightarrow{s}, \overrightarrow{p} \rangle$; $t' \leftarrow \langle \overrightarrow{t}, \overrightarrow{p} \rangle$ //$R = g^{s'} h^{t'}$
11 $\overrightarrow{s} \leftarrow \overrightarrow{s} \| s'$
12 $\overrightarrow{t} \leftarrow \overrightarrow{t} \| t'$
13 $\beta \leftarrow\!\!\$\ \mathbb{B}$
14 If $-t' = \psi(\beta)$ then abort
15 If $(\cdot, \beta) \in \Pi$ then abort
16 $\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
17 Return $\beta$

BRO$^{-1}(\beta)$:
18 If $(\cdot, \beta) \in \Pi$: Return $\Pi^{-1}(\beta)$
19 $\alpha \leftarrow\!\!\$\ \mathbb{A}$
20 If $\alpha \in \varphi(\mathsf{G})$:
21 $\quad s', t' \leftarrow\!\!\$\ \mathbb{Z}_p$
22 $\quad$ If $-t' = \psi(\beta)$ then abort
23 $\quad$ If $t' = \psi(\beta)$ then abort
24 $\quad \overrightarrow{s} \leftarrow \overrightarrow{s} \| s' \| - s'$
25 $\quad \overrightarrow{t} \leftarrow \overrightarrow{t} \| t' \| - t'$
26 $\quad R \leftarrow g^{s'} h^{t'}$
27 $\quad \alpha \leftarrow \varphi(R)$
28 $\quad \overrightarrow{U} \leftarrow \overrightarrow{U} \| R \| R^{-1}$
29 If $(\alpha, \cdot) \in \Pi$ then abort
30 $\Pi \leftarrow \Pi \cup \{(\alpha, \beta)\}$
31 Return $\alpha$

FINALIZE$(R, z)$: // $R \in \overrightarrow{U}$
32 Return $g^z = R h^{f(R)}$

Fig. 8: Games $G_2$, $G_3$ for the proof of Theorem 3. Changes are highlighted in blue.

adversary is *not* required to give a representation of group elements it outputs, only its BRO queries. The CDL game in ABRO is defined in Figure 7 as $\mathrm{G}_0$.

We recall the following definition from [FKP16] before stating our result.

**Definition 1.** *(Semi-Injective Function) Let* $\mathsf{G}$ *be a prime order group and* $\mathbb{A}$ *be a set. A function* $\varphi : \mathsf{G} \to \mathbb{A}$ *is called semi-injective if (a) its range* $\varphi(\mathsf{G}) \subseteq \mathbb{A}$ *is efficiently decidable and (b) it is either injective or 2-to-1 with* $\varphi(X) = \varphi(Y)$ *always implying* $Y \in \{X, X^{-1}\}$.

**Theorem 3** *Let* $\mathsf{G}$ *be a group whose order* $p = |\mathsf{G}|$ *is prime. Let* A *be an adversary making* $q$ *queries to its* BRO *or* $\mathrm{BRO}^{-1}$ *oracles, which does not output any group elements which it has not queried to the* BRO *oracle or received from the* $\mathrm{BRO}^{-1}$ *oracle. Let* $\mathbb{A} = \{0,1\}^L$ *and* $\mathbb{B} = [2^L - 1]$ *such that* $2^L \geq p$. *Let* $\varphi$ *be a semi-injective function from* $\mathsf{G}$ *to* $\mathbb{A}$. *Let* $\psi \colon \mathbb{B} \to \mathbb{Z}_p$. *Then there exists an adversary* B *(shown in Figure 6) with running time roughly the same as* A *plus simulation overhead proportional to* $q$ *such that*

$$\mathbf{Adv}^{\text{abro-cdl}}_{\mathsf{G},g,\varphi,\psi}(\mathrm{A}) \leq \mathbf{Adv}^{\text{dl}}_{\mathsf{G},g}(\mathrm{B}) + \frac{2q^2 + 2q + q \cdot \mathsf{MaxSize}_\psi}{p} \,. \tag{7}$$

*Proof.* Consider the adversary B defined in Figure 6. We analyze the advantage of B through a sequence of games defined in Figures 7 and 8. Note that $\mathrm{G}_0$ is equivalent to $\mathbf{G}^{\text{abro-cdl}}$. We first show that

$$\mathbf{Adv}^{\text{abro-cdl}}_{\mathsf{G},g,\varphi,\psi}(\mathrm{A}) \leq \mathbf{Adv}^{\mathrm{G}_3}_{\mathsf{G},g}(\mathrm{A}) + \frac{2q^2 + 2q + q \cdot \mathsf{MaxSize}_\psi}{p} \tag{8}$$

through a series of game hops.

$\underline{\mathrm{G}_0 \to \mathrm{G}_1}$. In $\mathrm{G}_1$ we switch to using lazy sampling for $\Pi$, which introduces abort conditions on lines 10 and 18. Note that the oracle responses in $G_0$ are identically distributed to the oracle responses in $G_1$ if no abort conditions are hit. Thus $\Pr\left[G_0^{\mathrm{A}} \Rightarrow 1\right] \leq \Pr\left[G_1^{\mathrm{A}} \Rightarrow 1\right] + \Pr\left[G_1^{\mathrm{A}} \text{ aborts}\right]$.

We now analyze the probability of $\mathrm{G}_1$ aborting. The number of $(\alpha, \beta)$ pairs stored in $\Pi$ is at most $q$, so the chance on each BRO query of a uniformly sampled $\beta$ being part of one of these pairs is at most $q/2^L$. Thus, by a union bound across all BRO queries, the probability of aborting on line 10 is at most $q^2/2^L \leq q^2/p$. Similarly, the probability of aborting on line 18 is at most $q^2/2^L \leq q^2/p$. Thus $\Pr\left[G_0^{\mathrm{A}} \Rightarrow 1\right] \leq \Pr\left[G_1^{\mathrm{A}} \Rightarrow 1\right] + 2q^2/p$.

$\underline{\mathrm{G}_1 \to \mathrm{G}_2}$. In $\mathrm{G}_2$, we resample $\alpha$ if the originally sampled $\alpha$ is in the range of $\varphi$, so that this time we learn the representation. We set $R := g^{s'} h^{t'}$ where $s'$ and $t'$ are uniformly sampled from $\mathbb{Z}$. Since $g$ and $h$ are generators, this results in a uniformly random $R$. We then set $\alpha \leftarrow \varphi(R)$. Because $\varphi$ is either injective or 2-to-1, any $\alpha \in \varphi(\mathsf{G})$ is equally likely to be chosen, so this method of resampling maintains the uniform distribution on $\alpha$. Thus $\Pr\left[G_1^{\mathrm{A}} \Rightarrow 1\right] = \Pr\left[G_2^{\mathrm{A}} \Rightarrow 1\right]$.

$\underline{\mathrm{G}_2 \to \mathrm{G}_3}$. In $\mathrm{G}_3$, we make two changes. The first change is that we use vectors $\overrightarrow{s}$ and $\overrightarrow{t}$ to keep track of each "seen" element $U_i$ so that $U_i = g^{s_i} h^{t_i}$. This

change is purely for bookkeeping and does not affect the behavior of the oracles. The second change is that we add abort conditions on lines 14, 22, and 23, which we will use later.

We now analyze the probability of $G_3$ aborting. On each BRO query, the probability of aborting on line 16 is at most $\mathsf{MaxSize}_\psi/p$ since $\beta$ is uniformly sampled. Thus by union bound over all queries, the probability of aborting on line 14 is at most $q \cdot \mathsf{MaxSize}_\psi/p$. The probability of aborting on line 22 is at most $q/p$ since $t'$ is uniformly sampled. Likewise the probability of aborting on line 23 is at most $q/p$. Thus $\Pr\left[G_2^{\mathrm{A}} \Rightarrow 1\right] \leq \Pr\left[G_3^{\mathrm{A}} \Rightarrow 1\right] + 2q/p + q \cdot \mathsf{MaxSize}_\psi/p$.

Combining the hops above gives us Eq. (8). We now show that

$$\mathbf{Adv}^{\mathrm{G}_3}_{\mathsf{G},g,\varphi,\psi}(\mathrm{A}) \leq \mathbf{Adv}^{\mathrm{dl}}_{\mathsf{G},g}(\mathrm{B}) \tag{9}$$

which combined with Eq. (8) completes the proof.

We show that if A returns a valid $R, z$ pair in $G_3$, then B returns the discrete log of $h$. If $R, z$ is a valid pair then $g^z h^{-f(R)} = R$ and there exists some $i$ such that $R = g^{s_i} h^{t_i}$. Thus $g^z h^{-f(R)} = g^{s_i} h^{t_i}$. $t_i + f(R)$ is non-zero as assured by the abort conditions in $G_3$ on lines 17, 24, and 25, so $x := (z - s_i)/(t_i + f(R))$ is the discrete log of $h$. □

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