New self-orthogonal codes from weakly regular plateaued functions and their application in LCD codes

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Abstract

A linear code with few weights is a significant code family in coding theory. A linear code is considered self-orthogonal if contained within its dual code. Self-orthogonal codes have applications in linear complementary dual codes, quantum codes, etc. The construction of linear codes is an interesting research problem. There are various methods to construct linear codes, and one approach involves utilizing cryptographic functions defined over finite fields. The construction of linear codes (in particular, self-orthogonal codes) from functions has been studied in the literature. In this paper, we generalize the construction method given by Heng et al. in [Des. Codes Cryptogr. 91(12), 2023] to weakly regular plateaued functions. We first construct several families of $\boldsymbol{p}$-ary linear codes with few weights from weakly regular plateaued unbalanced (resp. balanced) functions over the finite fields of odd characteristics. We observe that the constructed codes are self-orthogonal codes when $\boldsymbol{p = 3}$. Then, we use the constructed ternary self-orthogonal codes to build new families of ternary LCD codes. Consequently, we obtain (almost) optimal ternary self-orthogonal codes and LCD codes.

Keywords: Linear code, Self-orthogonal code, LCD code, Weakly regular plateaued function

1 Introduction

Linear codes have been an attractive research topic in both practice and theory for the last two decades. They have diverse applications in secure communication [1], secret sharing schemes [2], [3], [4], [5], [6], [7], authentication codes [8] and secure two-party computation [9], [10]. A linear code is considered self-orthogonal if contained within its dual code. Self-orthogonal codes have applications in linear complementary dual codes, quantum codes, etc. A linear code $\mathcal{C}$ is called a linear complementary dual code (LCD code) if $\mathcal{C} \cap \mathcal{C}^{\perp} = \mathbf{0}$. LCD codes also have diverse applications in certain communication systems. Carlet and Guilley [11] demonstrated their significance in information protection and defence against side-channel attacks and fault non-invasive attacks. After these observations, the importance of LCD code applications has begun to be revitalized. Massey [12] introduced the LCD codes and showed that they provide an optimum linear coding solution to the two-user binary adder channel. Now, it is known that asymptotically good LCD codes exist and the necessary and sufficient condition for a length n cyclic code to be an LCD code is known. Hence, the construction of linear codes is an interesting research problem. Various methods exist for constructing linear codes and one approach involves utilizing functions defined over finite fields (e.g. [3], [6], [9], [13], [14], [15], [16]). Linear codes derived from cryptographic functions have desirable algebraic structures that are significant from the application point of view. Two generic constructions, referred to as the first and second generic constructions, for generating linear codes from functions have been identified in the literature. Several linear codes with good parameters have been constructed using the second generic construction method (e.g., [6], [13], [17]). Recently, Heng et al. [18] have constructed ternary self-orthogonal codes from weakly regular bent functions based on the second generic construction method. This observation motivates us to construct linear codes from weakly regular plateaued functions over finite fields with odd characteristics. In this paper, we employed weakly regular plateaued functions in the second construction method to obtain new families of p-ary linear codes (in particular, ternary self-orthogonal codes) with few weights. Then, we used the constructed ternary self-orthogonal codes to construct infinite families of ternary LCD codes. We finally observed that some constructed LCD codes are at least almost optimal codes according to the sphere-packing bound.

The paper is organized as follows. Section 2 establishes the primary notation and reviews fundamental concepts in finite fields and coding theory. Section 3 gives some useful results related to weakly regular plateaued functions. In Sections 4 and 5, we construct several families of linear codes with few weights from weakly regular plateaued functions over the odd characteristic finite fields. In particular, we present several families of ternary self-orthogonal codes. Moreover, we introduce the dual codes of the constructed codes over the odd characteristic finite fields. In Section 6, we consider an application of ternary self-orthogonal codes in ternary LCD codes. Section 7 concludes the paper.

2 Preliminaries

Throughout this paper, we fix the following notation. For an odd prime p and a positive integer m, $q = p^m$ denotes the prime power, and $\mathbb{F}_q$ is the finite field with q elements. The trace of $\beta \in \mathbb{F}_{p^m}$ over $\mathbb{F}_p$ is defined as $\mathrm{Tr}_p^{p^m}(\beta) = \beta + \beta^p + \beta^{p^2} + \cdots + \beta^{p^{m-1}}$. Let ξ_p denote the complex primitive p-th root of unity. SQ and NSQ denote all squares and non-squares in $\mathbb{F}_p^*$. Finally, η_0 denotes the quadratic characters of $\mathbb{F}_p^*$ and p^* denotes $\eta_0(-1)p$.

Cyclotomic Field $\mathbb{Q}(\xi_\mathbf{p})$. Let $\mathbb{Z}$ be the rational integer ring and $\mathbb{Q}$ be the rational field. Then, we have the following fact about p-th cyclotomic field $\mathbb{Q}(\xi_p)$.

Lemma 1. *[19] The following results on $\mathbb{Q}(\xi_p)$ hold.*

1. *The ring of integers in $K := \mathbb{Q}(\xi_p)$ is $O_K = \mathbb{Z}(\xi_p)$ and $\{\xi_p^i : 1 \leq i \leq p-1\}$ is an integral basis of O_K.*
2. *The field extension $K/\mathbb{Q}$ is a Galois extension of degree $p-1$ with Galois group $Gal(K/\mathbb{Q}) = \{\sigma_a : a \in \mathbb{F}_p^*\}$, where the automorphism σ_a of K is defined by $\sigma_a(\xi_p) = \xi_p^a$.*
3. *The field K has a unique quadratic subfield $\mathbb{Q}(\sqrt{p^*})$. For $1 \leq a \leq p-1$, $\sigma_a(\sqrt{p^*}) = \eta_0(a)\sqrt{p^*}$. Thus, the Galois group $Gal(K/\mathbb{Q})$ is $\{1, \sigma_\gamma\}$, where γ is a non-square in $\mathbb{F}_p^*$.*

From Lemma 1, for any $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$, one can directly write

$$\sigma_a(\xi_p^b) = \xi_p^{ab} \quad \text{and} \quad \sigma_a(\sqrt{p^*}^{\,m}) = \eta_0^m(a)\sqrt{p^*}^{\,m}.$$

Characters over finite fields. Given $a \in \mathbb{F}_q$, the function

$$\phi_a(x) = \xi_p^{\mathrm{Tr}_p^{p^m}(ax)}, x \in \mathbb{F}_q$$

defines an additive character of $\mathbb{F}_q$. The orthogonality relation of additive characters [20] is given by

$$\sum_{x \in \mathbb{F}_q} \phi_1(ax) = \begin{cases} q, & \text{if } a = 0, \\ 0, & \text{if } a \in \mathbb{F}_q^*. \end{cases}$$

Let α be a primitive element of $\mathbb{F}_q$. Then, for $k = 0, 1, \ldots, q-2$, where $0 \leq j \leq q-2$, $\psi_j(\alpha^k) = \xi_{q-1}^{jk}$ denotes the multiplicative character of $\mathbb{F}_q$. The orthogonality relation of multiplicative characters [20] is given by

$$\sum_{x \in \mathbb{F}_q^*} \psi_j(x) = \begin{cases} q-1, & \text{if } j = 0, \\ 0, & \text{if } j \neq 0. \end{cases}$$

2.1 Weakly regular plateaued functions

Let f be a p-ary function from $\mathbb{F}_q$ to $\mathbb{F}_p$, where $q = p^m$ for a prime p and positive integer m. If f takes every element of $\mathbb{F}_p$ with the same number p^{m-1} of pre-images, then f is called a balanced function over $\mathbb{F}_p$; otherwise, it is unbalanced. The Walsh transform of f is defined as

$$\mathcal{W}_f(\beta) = \sum_{x\in\mathbb{F}_{p^m}} \xi_p^{f(x)-\mathrm{Tr}_p^{p^m}(\beta x)}, \beta \in \mathbb{F}_{p^m}.$$

A function f can be classified in terms of its Walsh transform. A function f is balanced if and only if $\mathcal{W}_f(0) = 0$. A function f is bent if its Walsh coefficients satisfy $|\mathcal{W}_f(\beta)|^2 = p^m$ for every $\beta \in \mathbb{F}_{p^m}$. A function f is called s-plateaued if $|\mathcal{W}_f(\beta)|^2 \in \{0, p^{m+s}\}$ for every $\beta \in \mathbb{F}_{p^m}$, where $0 \le s \le m$. The Walsh support of an s-plateaued function f is defined as

$$Supp(\mathcal{W}_f) = \{\beta \in \mathbb{F}_{p^m} : |\mathcal{W}_f(\beta)|^2 = p^{m+s}\}$$

and $\#Supp(\mathcal{W}_f) = p^{m-s}$. The Walsh distribution of an s-plateaued p-ary function f follows from the Parseval identity.

Lemma 2. *Let $f : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be an s-plateaued function. Then,*

$$\#\{\beta \in \mathbb{F}_{p^m} : |\mathcal{W}_f(\beta)|^2 = p^{m+s}\} = p^{m-s},$$

$$\#\{\beta \in \mathbb{F}_{p^m} : |\mathcal{W}_f(\beta)|^2 = 0\} = p^m - p^{m-s}.$$

Definition 1. *[21] Let f be a p-ary s-plateaued function from $\mathbb{F}_q$ to $\mathbb{F}_p$ with $0 \le s \le m$. Then, f is called weakly regular s-plateaued if there exists a complex number u with $|u| = 1$ such that*

$$\mathcal{W}_f(\beta) \in \{\ 0, up^{\frac{m+s}{2}}\xi_p^{g(\beta)}\}$$

for all $\beta \in \mathbb{F}_q$, where g is a p-ary function over $\mathbb{F}_q$ and $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q\backslash Supp(\mathcal{W}_f)$. Otherwise, f is called a non-weakly regular p-ary s-plateaued function.

Lemma 3. *[21] Let f be a p-ary s-plateaued function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and let $\beta \in \mathbb{F}_q$. Then, for all $\beta \in Supp(\mathcal{W}_f)$, we have the following*

$$\mathcal{W}_f(\beta) = \epsilon\sqrt{p^*}^{\,m+s}\xi_p^{g(\beta)},$$

where $\epsilon \in \{\ -1, 1\}$ is the sign of $\mathcal{W}_f$ and g is a p-ary function over $\mathbb{F}_q$ with $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q\backslash Supp(\mathcal{W}_f)$.

Lemma 4. *[17] Let f be a p-ary s-plateaued function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and let $\beta \in \mathbb{F}_{p^m}$. Then, for $x \in \mathbb{F}_{p^m}$, we have*

$$\sum_{\beta\in Supp(\mathcal{W}_f)} \xi_p^{g(\beta)+\mathrm{Tr}_p^{p^m}(\beta x)} = \epsilon\eta_0^m(-1)\sqrt{p^*}^{\,m-s}\xi_p^{f(x)},$$

where $\epsilon \in \{\ -1, 1\}$ is the sign of $\mathcal{W}_f$ and g is a p-ary function over $\mathbb{F}_q$ with $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q\backslash Supp(\mathcal{W}_f)$.

Recently, two subsets of the set of weakly regular plateaued functions have been introduced in [17] and [22]. Let $\mathcal{WRP}$ (resp. $\mathcal{WRPB}$) be the set of p-ary weakly regular s-plateaued unbalanced (resp. balanced) functions with $0 \leq s \leq m$ satisfying the following two conditions:

1. $f(0) = 0$
2. There exists an even positive integer l such that $\gcd(l-1, p-1) = 1$ and $f(ax) = a^l f(x)$ for any $a \in \mathbb{F}_p^*$ and $x \in \mathbb{F}_q$.

Note that every bent function is the 0-plateaued function. Then, the set $\mathcal{WRP}$ is particularly denoted by $\mathcal{RF}$ when $s = 0$.

Lemma 5. *[17, 22] Let $f \in \mathcal{WRP}$ or $f \in \mathcal{WRPB}$ with $\mathcal{W}_f(\beta) = \epsilon\sqrt{p^*}^{\,m+s}\xi_p^{g(\beta)}$ for all $\beta \in$Supp$(\mathcal{W}_f)$. Then, there exists an even positive integer h such that* $\gcd(h-1, p-1) = 1$ *and $g(a\beta) = a^h g(\beta)$ for any $a \in \mathbb{F}_p^*$ and $\beta \in$Supp$(\mathcal{W}_f)$.*

2.2 Linear codes and LCD codes from self-orthogonal codes

Let $\mathbb{F}_p$ be a finite field with p elements and $\mathbb{F}_p^n$ be a vector space over $\mathbb{F}_p$ for a positive integer n. A linear code $\mathcal{C}$ over $\mathbb{F}_p$ with parameters $[n, k, d]$ is a k-dimensional linear subspace of a vector space $\mathbb{F}_p^n$, where d denotes the minimum Hamming distance of $\mathcal{C}$. Let $\mathbf{a}$ be a vector in $\mathbb{F}_p^n$ and its support is defined as supp($\mathbf{a}$)$= \{0 \leq i \leq n-1 : a_i \neq 0\}$. The cardinality of supp($\mathbf{a}$) is called the Hamming weight of a vector $\mathbf{a}$. Let $\mathbf{c}$ be a codeword of $\mathcal{C}$. The minimum Hamming distance d in $\mathcal{C}$ is the minimum Hamming weight of $\mathbf{c} \in \mathcal{C}$. Let $A_i := |\{\mathbf{c} \in \mathcal{C} : wt(\mathbf{c}) = i \text{ for } 0 \leq i \leq n\}|$ for a linear code $\mathcal{C}$. Define the weight enumerator of $\mathcal{C}$ by the polynomial $1 + A_1 y + \ldots + A_n y^n$. The dual code $\mathcal{C}^\perp$ of an $[n, k]$ linear code $\mathcal{C}$ is defined by

$$\mathcal{C}^\perp = \{\mathbf{c}^\perp \in \mathbb{F}_p^n : \mathbf{c}^\perp \cdot \mathbf{c} \text{ for all } \mathbf{c} \in \mathcal{C}\},$$

where " $\cdot$ " is the standard inner product over $\mathbb{F}_p^n$, and $\mathcal{C}^\perp$ is an $[n, n-k]$ linear code over $\mathbb{F}_p^n$. If a linear code $\mathcal{C}$ satisfies $\mathcal{C} \subset \mathcal{C}^\perp$, then $\mathcal{C}$ is referred to as a self-orthogonal code. In particular, if $\mathcal{C} = \mathcal{C}^\perp$, then $\mathcal{C}$ is called sef-dual code. If all codewords of $\mathcal{C}$ are divisible by some integer $k > 1$, then the code is said to be divisible by k. For a p-ary linear code $\mathcal{C}$, there is a relation between the self-orthogonality and divisibility of $\mathcal{C}$.

Lemma 6. *[1] Let $\mathcal{C}$ be a ternary linear code over $\mathbb{F}_3$. Then, $\mathcal{C}$ is a self-orthogonal ternary code if and only if every codeword of $\mathcal{C}$ has weight divisible by three.*

By looking at the weight distribution of a code, one can decide whether a ternary code is self-orthogonal or not.

There are several methods to construct linear codes over finite fields. In this paper, we use the second generic construction method based on the set of the pre-image of the special function. Let $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{F}_{p^m}$. Define

$$\mathcal{C}_D = \{(\mathrm{Tr}_p^{p^m}(bd_1), \mathrm{Tr}_p^{p^m}(bd_2), \ldots, \mathrm{Tr}_p^{p^m}(bd_n)) : b \in \mathbb{F}_{p^m}\}.$$

Then, $\mathcal{C}_D$ is a linear code over $\mathbb{F}_p$ with length n and dimension at most m. The set D is called the defining set of $\mathcal{C}_D$. In the literature, many linear codes with few weights have been constructed from the suitable defining sets [6, 17, 22]. Moreover, the augmented code of $\mathcal{C}_D$ is defined by

$$\overline{\mathcal{C}_D} = \{(\mathrm{Tr}_p^{p^m}(bd_1), \mathrm{Tr}_p^{p^m}(bd_2), ..., \mathrm{Tr}_p^{p^m}(bd_n)) + c\mathbf{1} : b \in \mathbb{F}_{p^m} \text{ and } c \in \mathbb{F}_p\}, \tag{1}$$

where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{F}_p^n$. Very recently, Heng et. al. [18] have constructed several ternary self-orthogonal codes from weakly regular bent functions. We in this paper construct further a number of p-ary linear codes and ternary self-orthogonal codes based on this construction method.

For a linear code $\mathcal{C}$, if $\mathcal{C} \cap \mathcal{C}^\perp = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathcal{C}$, then it is called a Linear Complementary Dual code (LCD code). Note that the dual of an LCD code is also an LCD code. The necessary and sufficient conditions for a linear code to be an LCD code were defined in terms of the generator matrix [12]. Besides, LCD codes were shown to give an optimum solution to the two-user binary adder channel [12].

A matrix G is said to be row-orthogonal if $GG^\perp = I$, where I is an identity matrix, and it is called row-self-orthogonal if $GG^\perp = \mathbf{0}$. A linear code $\mathcal{C}$ is self-orthogonal if and only if its generator matrix is row-self-orthogonal [23]. If G is a generator matrix for $[n, k]$ linear code $\mathcal{C}$, then it can be transformed to the standard form $G = [I : A]$, where I is an identity matrix and it is called the systematic generator matrix of the code. Then, $\mathcal{C}$ is called leading-systematic. The following lemma provides a relation between LCD codes and self-orthogonal codes.

Lemma 7. *[23] A leading-systematic linear code $\mathcal{C}$ is an LCD code if its systematic generator matrix $G = [I : A]$ is row-orthogonal.*

The Pless power moment. For a linear $[n, k, d]$ code $\mathcal{C}$ over $\mathbb{F}_p$, we denote the weight distribution of $\mathcal{C}$ and $\mathcal{C}^\perp$ by $(1, A_1, \ldots, A_n)$ and $(1, A_1^\perp, \ldots, A_n^\perp)$, respectively. The first four Pless power moments are given as:

$$\sum_{i=0}^{n} A_i = p^k, \tag{2}$$

$$\sum_{i=0}^{n} iA_i = p^{k-1}\left(pn - n - A_1^\perp\right),$$

$$\sum_{i=0}^{n} i^2 A_i = p^{k-2}\left((p-1)n(pn-n+1) - (2pn - p - 2n + 2)A_1^\perp + 2A_2^\perp\right),$$

$$\begin{aligned}\sum_{i=0}^{n} i^3 A_i &= p^{k-3}[(p-1)n(p^2n^2 - 2pn^2 + 3pn - p + n^2 - 3n + 2) \\ &\quad -(3p^2n^2 - 3p^2n - 6pn^2 + 12pn + p^2 - 6p + 3n^2 - 9n + 6)A_1^\perp \\ &\quad +6(pn - p - n + 2)A_2^\perp - 6A_3^\perp].\end{aligned}$$

Augmented code of a linear code. Let $\mathcal{C}$ be an $[n, k, d]$ linear code over $\mathbb{F}_p$ with a generator matrix G. The augmented code $\overline{\mathcal{C}}$ of $\mathcal{C}$ is a linear code over $\mathbb{F}_p$ with generator matrix

$$\begin{bmatrix} G \\ \mathbf{1} \end{bmatrix}$$

where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{F}_p^n$. Note that if $\mathbf{1}$ is not a codeword in $\mathcal{C}$, then the augmented code $\overline{\mathcal{C}}$ has length n and dimension $k+1$. Determining the weight distribution of a code is a hard problem and finding the minimum distance of $\overline{\mathcal{C}}$ requires the complete weight distribution of the original code $\mathcal{C}$. There are some methods to determine whether the given augmented code is self-orthogonal. In this paper, we use Lemma 6 to prove the self-orthogonality of a linear code.

3 Character sums for weakly regular plateaued functions

In this section, we present several useful results on the character sums for weakly regular plateaued functions.

Lemma 8. *[20] Let p be an odd prime, $p^* = \eta_0(-1)p$ and $a \in \mathbb{F}_{p^m}^*$. Then,*

$$\sum_{x \in \mathbb{F}_{p^m}} \xi_p^{\mathrm{Tr}_p^{p^m}(ax^2)} = (-1)^{m-1}\eta(a)\sqrt{p^*}^m.$$

In particular, if $m = 1$ and $a = 1$, then $\sum_{x \in \mathbb{F}_p} \xi_p^{x^2} = \sqrt{p^*}$.

Lemma 9. *[20] Let p be an odd prime and $p^* = \eta_0(-1)p$. Then*

1. $\sum_{c \in \mathbb{F}_p^*} \eta_0(c) = 0$;
2. $\sum_{c \in \mathbb{F}_p^*} \xi_p^{ca} = -1$ *for every* $a \in \mathbb{F}_p^*$;
3. $\sum_{c \in \mathbb{F}_p^*} \eta_0(c)\xi_p^c = \sqrt{p^*}$.

Lemma 10. *[20] Let $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$. Define*

$$B = \sum_{z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \xi_p^{z\left(\mathrm{Tr}_p^{p^m}(bx)+c\right)}.$$

Then, we have

$$B = \begin{cases} 0, & \text{if } c \in \mathbb{F}, b \neq 0, \\ p^m(p-1), & \text{if } c = 0, b = 0, \\ -p^m, & \text{if } c \neq 0, b = 0. \end{cases}$$

The following two lemmas will be used to find the Hamming weights and weight distributions of the proposed linear codes. Lemma 11 and Lemma 12 are direct consequences of [17, Lemma 9] and [17, Lemma 10], respectively.

Lemma 11. *Let $f : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be an unbalanced p-ary function with $\mathcal{W}_f(0) = \epsilon\sqrt{p^*}^{m+s}$, where $\epsilon \in \{-1,1\}$ is the sign of $\mathcal{W}_f$ and $p^* = \eta_0(-1)p$. Define*

$$\begin{array}{ll} N_0 & := \#\{b \in \mathbb{F}_{p^m} : f(b) = 0\}, \\ N_{sq} & := \#\{b \in \mathbb{F}_{p^m} : f(b) \in SQ\}, \\ N_{nsq} & := \#\{b \in \mathbb{F}_{p^m} : f(b) \in NSQ\}. \end{array}$$

Then, we have

$$\begin{array}{ll} N_0 & = \begin{cases} p^{m-1} + \epsilon\eta_0(-1)(p-1)\sqrt{p^*}^{m+s-2}, & \text{if } m+s \text{ is even,} \\ p^{m-1}, & \text{if } m+s \text{ is odd,} \end{cases} \\ N_{sq} & = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } m+s \text{ is even,} \\ (\frac{p-1}{2})\left(p^{m-1} + \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } m+s \text{ is odd,} \end{cases} \\ N_{nsq} & = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } m+s \text{ is even,} \\ (\frac{p-1}{2})\left(p^{m-1} - \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } m+s \text{ is odd.} \end{cases} \end{array}$$

Lemma 12. *Let f be a weakly regular s-plateaued function with $\mathcal{W}_f(\beta) = \epsilon\sqrt{p^*}^{m+s}\xi^{g(\beta)}$, where g is a p-ary function over $\mathbb{F}_q$ and $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q \backslash Supp(\mathcal{W}_f)$. Define*

$$\begin{array}{ll} N_{g,0} & := \#\{b \in Supp(\mathcal{W}_f) : g(b) = 0\}, \\ N_{g,sq} & := \#\{b \in Supp(\mathcal{W}_f) : g(b) \in SQ\}, \\ N_{g,nsq} & := \#\{b \in Supp(\mathcal{W}_f) : g(b) \in NSQ\}. \end{array}$$

Then, we have

$$\begin{array}{ll} N_{g,0} & = \begin{cases} p^{m-s-1} + \epsilon\eta_0^{m+1}(-1)(p-1)\sqrt{p^*}^{m-s-2}, & \text{if } m-s \text{ is even,} \\ p^{m-s-1}, & \text{if } m-s \text{ is odd,} \end{cases} \\ N_{g,sq} & = \begin{cases} \frac{p-1}{2}\left(p^{m-s-1} - \epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right), & \text{if } m-s \text{ is even,} \\ \frac{p-1}{2}\left(p^{m-s-1} + \epsilon\eta_0^{m}(-1)\sqrt{p^*}^{m-s-1}\right), & \text{if } m-s \text{ is odd,} \end{cases} \\ N_{g,nsq} & = \begin{cases} \frac{p-1}{2}\left(p^{m-s-1} - \epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right), & \text{if } m-s \text{ is even,} \\ \frac{p-1}{2}\left(p^{m-s-1} - \epsilon\eta_0^{m}(-1)\sqrt{p^*}^{m-s-1}\right), & \text{if } m-s \text{ is odd.} \end{cases} \end{array}$$

4 Linear codes from weakly regular plateaued functions based on the set D_f

In this section, we construct the augmented code $\overline{C}_{D_f}$ based on the defining set

$$D_f = \{x \in \mathbb{F}_{p^m} : f(x) = 0\}.$$

Let $m+s$ be a positive integer with $0 \le s \le m$. The length of the code $\overline{C}_{D_f}$ is the size of the set D_f.

- When $f \in \mathcal{WRP}$, the length of the code $\overline{C}_{D_f}$ is equal to $n = \#D_f = N_0$, which follows from Lemma 11.
- When $f \in \mathcal{WRPB}$, it is clear that $n = \#D_f = p^{m-1}$.

The following lemma is a combination of [17, Lemma 12] and [24, Lemma 6].

Lemma 13. *Let $b \in \mathbb{F}_{p^m}$, $c \in \mathbb{F}_p$ and $f \in \mathcal{WRP}$. Denote by*

$$S = \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \xi_p^{yf(x) + z(\mathrm{Tr}_p^{p^m}(bx) + c)}.$$

For every $b \in \mathbb{F}_{p^m} \setminus Supp(\mathcal{W}_f)$, we get $S = 0$. For every $b \in Supp(\mathcal{W}_f)$,

- *if $m+s$ is even,*

$$S = \begin{cases} \epsilon(p-1)^2 \sqrt{p^*}^{\,m+s}, & \textit{if } c = 0, g(b) = 0, \\ -\epsilon(p-1)\sqrt{p^*}^{\,m+s}, & \textit{if } c = 0, g(b) \neq 0 \textit{ or } c \neq 0, g(b) = 0, \\ \epsilon \sqrt{p^*}^{\,m+s}, & \textit{if } c \neq 0, g(b) \neq 0, \end{cases}$$

- *if $m+s$ is odd,*

$$S = \begin{cases} 0, & \textit{if } g(b) = 0, \\ \epsilon(p-1)\sqrt{p^*}^{\,m+s+1}, & \textit{if } c = 0, g(b) \in SQ, \\ -\epsilon(p-1)\sqrt{p^*}^{\,m+s+1}, & \textit{if } c = 0, g(b) \in NSQ, \\ -\epsilon\sqrt{p^*}^{\,m+s+1}, & \textit{if } c \neq 0, g(b) \in SQ, \\ \epsilon\sqrt{p^*}^{\,m+s+1}, & \textit{if } c \neq 0, g(b) \in NSQ. \end{cases}$$

The following lemma is used to find the Hamming weights of $\overline{C}_{D_f}$ when $f \in \mathcal{WRP}$.

Lemma 14. *Let $f \in \mathcal{WRP}$. Define*

$$N_f(b,c) := \#\{x \in F_{p^m} : \mathrm{Tr}_p^{p^m}(bx) + c = 0 \textit{ and } f(x) = 0\}$$

for $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$. For every $b \in \mathbb{F}_{p^m} \setminus Supp(\mathcal{W}_f)$, we have

$$N_f(b,c) = \begin{cases} p^{m-2}, & \textit{if } m+s \textit{ is odd}, \\ p^{m-2} + \epsilon(p-1)\sqrt{p^*}^{\,m+s-4}, & \textit{if } m+s \textit{ is even}. \end{cases}$$

For every $b \in Supp(\mathcal{W}_f)$,

- *if $m+s$ is even,*

$$N_f(b,c)=\begin{cases} p^{m-1}+\epsilon(p-1)\eta_0(-1)\sqrt{p^*}^{\,m+s-2}, & \text{if } c=0, b=0,\\ 0, & \text{if } c\neq 0, b=0,\\ p^{m-2}+\epsilon(p-1)\eta_0(-1)\sqrt{p^*}^{\,m+s-2}, & \text{if } c=0, b\neq 0, g(b)=0,\\ p^{m-2}, & \text{if } c=0, b\neq 0, g(b)\neq 0 \text{ or}\\ & c\neq 0, b\neq 0, g(b)=0,\\ p^{m-2}+\epsilon\eta_0(-1)\sqrt{p^*}^{\,m+s-2}, & \text{if } c\neq 0, b\neq 0, g(b)\neq 0,\end{cases}$$

- *if $m+s$ is odd,*

$$N_f(b,c)=\begin{cases} p^{m-1}, & \text{if } c=0, b=0,\\ 0, & \text{if } c\neq 0, b=0,\\ p^{m-2}, & \text{if } c=0, b\neq 0, g(b)=0 \text{ or}\\ & c\neq 0, b\neq 0, g(b)=0,\\ p^{m-2}-\epsilon(p-1)\sqrt{p^*}^{\,m+s-3}, & \text{if } b\neq 0, c=0, g(b)\in NSQ,\\ p^{m-2}+\epsilon(p-1)\sqrt{p^*}^{\,m+s-3}, & \text{if } b\neq 0, c=0, g(b)\in SQ,\\ p^{m-2}-\epsilon\sqrt{p^*}^{\,m+s-3}, & \text{if } b\neq 0, c\neq 0, g(b)\in SQ,\\ p^{m-2}+\epsilon\sqrt{p^*}^{\,m+s-3}, & \text{if } b\neq 0, c\neq 0, g(b)\in NSQ.\end{cases}$$

Proof. By the definition of $N_f(b,c)$, we have

$$\begin{aligned} N_f(b,c) &= \frac{1}{p^2}\sum_{x\in\mathbb{F}_{p^m}}\sum_{y\in\mathbb{F}_p}\sum_{z\in\mathbb{F}_p}\xi_p^{yf(x)+z\left(\mathrm{Tr}_p^{p^m}(bx)+c\right)}\\ &= p^{m-2}+\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\sum_{x\in\mathbb{F}_{p^m}}\xi_p^{yf(x)}+\frac{1}{p^2}\sum_{z\in\mathbb{F}_p^*}\sum_{x\in\mathbb{F}_{p^m}}\xi_p^{z\left(\mathrm{Tr}_p^{p^m}(bx)+c\right)}\\ &+\frac{1}{p^2}\sum_{x\in\mathbb{F}_{p^m}}\sum_{y\in\mathbb{F}_p^*}\sum_{z\in\mathbb{F}_p^*}\xi_p^{y(f(x))}\xi_p^{z\left(\mathrm{Tr}_p^{p^m}(bx)+c\right)}\\ &= p^{m-2}+\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\sum_{x\in\mathbb{F}_{p^m}}\xi_p^{yf(x)}+\frac{B}{p^2}+\frac{S}{p^2}, \end{aligned}$$

where B and S are defined in Lemma 10 and Lemma 13. One can observe that

$$\sum_{y\in\mathbb{F}_p^*}\sum_{x\in\mathbb{F}_{p^m}}\xi_p^{yf(x)}=\begin{cases}\epsilon(p-1)\sqrt{p^*}^{\,m+s}, & \text{if } m+s \text{ is even},\\ 0, & \text{if } m+s \text{ is odd}.\end{cases}$$

Hence, the desired results can be obtained from the above. □

The following lemma follows from [22, Lemma 7] and [25, Lemma 7] and it will be used to find the Hamming weights of $\overline{C}_{D_f}$ when $f\in\mathcal{WRPB}$.

Lemma 15. *Let $f\in\mathcal{WRPB}$. Define*

$$N_f(b,c)=\#\{x\in\mathbb{F}_{p^m}:\mathrm{Tr}_p^{p^m}(bx)+c=0 \text{ and } f(x)=0\}$$

for $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$. Then, for every $b \in \mathbb{F}_{p^m} \setminus Supp(\mathcal{W}_f)$ we have

$$N_f(b,c) = \begin{cases} p^{m-2}, & \text{if } c \in \mathbb{F}_p, b \neq 0, \\ p^{m-1}, & \text{if } c = 0, b = 0, \\ 0, & \text{if } c \neq 0, b = 0. \end{cases}$$

For every $b \in Supp(\mathcal{W}_f)$, if $m+s$ is even,

$$N_f(b,c) = \begin{cases} p^{m-2} + \epsilon(p-1)^2\sqrt{p^*}^{m+s-4}, & \text{if } c = 0, g(b) = 0, \\ p^{m-2} - \epsilon(p-1)\sqrt{p^*}^{m+s-4}, & \text{if } c = 0, g(b) \neq 0 \text{ or } c \neq 0, g(b) = 0, \\ p^{m-2} + \epsilon\sqrt{p^*}^{m+s-4}, & \text{if } c \neq 0, g(b) \neq 0, \end{cases}$$

and if $m+s$ is odd,

$$N_f(b,c) = \begin{cases} p^{m-2}, & \text{if } c \in \mathbb{F}_p, g(b) = 0, \\ p^{m-2} + \epsilon(p-1)\sqrt{p^*}^{m+s-3}, & \text{if } c = 0, g(b) \in SQ, \\ p^{m-2} - \epsilon(p-1)\sqrt{p^*}^{m+s-3}, & \text{if } c = 0, g(b) \in NSQ, \\ p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}, & \text{if } c \neq 0, g(b) \in SQ, \\ p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}, & \text{if } c \neq 0, g(b) \in NSQ. \end{cases}$$

Theorem 1. *Let $m+s \geq 4$ be even with $0 \leq s \leq m-2$. Let $f \in \mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f. Let $D_f = \{x \in \mathbb{F}_{p^m} : f(x) = 0\}$. Then, $\overline{C}_{D_f}$ is a five-weight linear $[\frac{1}{p}(p^m + \epsilon(p-1)\sqrt{p^*}^{m+s}), m+1, d]$ code with parameters listed in Table 1. In particular, $\overline{C}_{D_f}$ is a ternary self-orthogonal code when $p = 3$ and $m+s \geq 6$.*

Proof. From the definition of the code, its length $n = \#D_f = N_0$ follows from Lemma 11. For a codeword $\mathbf{c}$ of $\overline{C}_{D_f}$, write

$$\mathbf{c} = \left(\mathrm{Tr}_p^{p^m}(bx)\right)_{x \in D_f} + c\mathbf{1}$$

with $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$. The Hamming weight of a codeword $\mathbf{c}$ in $\overline{C}_{D_f}$ is obtained as

$$wt(\mathbf{c}) = n - N_f(b,c),$$

which follows from Lemmas 11 and 14. For every $b \in \mathbb{F}_{p^m} \setminus \mathrm{Supp}(\mathcal{W}_f)$, we have

$$wt(\mathbf{c}) = (p-1)\left(p^{m-2} + \epsilon(p-1)\sqrt{p^*}^{m+s-4}\right),$$

and the number of such codewords can be obtained from Lemma 2. For every $b \in$Supp$(\mathcal{W}_f)$, we obtain

$$wt(\mathbf{c}) = \begin{cases} (p-1)p^{m-2}, & \text{if } c=0, b\neq 0, g(b)=0 \\ \frac{p-1}{p}\left(p^{m-1}+\epsilon\sqrt{p^*}^{m+s}\right), & \text{if } c=0, b\neq 0, g(b)\neq 0 \text{ or} \\ & \ c\neq 0, b\neq 0, g(b)=0, \\ \frac{1}{p}\left(p^m+\epsilon(p-1)\sqrt{p^*}^{m+s}\right), & \text{if } c\neq 0, b=0, \\ \frac{1}{p}\left((p-1)p^{m-1}+\epsilon(p-2)\sqrt{p^*}^{m+s}\right), & \text{if } c\neq 0, b\neq 0, g(b)\neq 0, \end{cases}$$

and the number of such codewords $\mathbf{c}$ can be obtained from Lemma 12. The dimension of $\overline{C}_{D_f}$ follows from its weight distribution. By Lemma 6, $\overline{C}_{D_f}$ is a ternary self-orthogonal code for $m+s\geq 6$ and $p=3$ since all codewords have weights divisible by 3. □

Example 1. *Let $f(x)=\mathrm{Tr}_3^{3^5}(\zeta^{19}x^4+\zeta^{238}x^2)$, where ζ is a generator of $\mathbb{F}_{3^5}^*=\langle\zeta\rangle$ for $\zeta^5+2\zeta+1=0$. Then, f is a quadratic 1-plateaued unbalanced function in the set $\mathcal{WRP}$ and for all $\beta\in\mathbb{F}_{3^5}$, we have $\mathcal{W}_f(\beta)\in\{0,-27,-27\xi_3,-27\xi_3^2\}$ with $\epsilon=1$. Then, the code $\overline{C}_{D_f}$ in Theorem 1 is a self-orthogonal ternary code with parameters $[63,6,36]$ and weight enumerator $1+100y^{36}+486y^{42}+120y^{45}+20y^{54}+2y^{63}$. It is verified by the Sage program.*

Table 1 The code $\overline{C}_{D_f}$ in Theorem 1 when $m+s$ is even.

Hamming weight ω	Multiplicity A_ω
0	1
$(p-1)p^{m-2}$	$\frac{1}{p}\left(p^{m-s}+\epsilon\eta_0^m(-1)(p-1)\sqrt{p^*}^{m-s}\right)-1$
$\frac{p-1}{p}\left(p^{m-1}+\epsilon\sqrt{p^*}^{m+s}\right)$	$\frac{p-1}{p}\left(2\cdot p^{m-s}+\epsilon\eta_0^m(-1)(p-2)\sqrt{p^*}^{m-s}-p\right)$
$\frac{1}{p}\left(p^m+\epsilon(p-1)\sqrt{p^*}^{m+s}\right)$	$p-1$
$\frac{1}{p}\left((p-1)p^{m-1}+\epsilon(p-2)\sqrt{p^*}^{m+s}\right)$	$\frac{(p-1)^2}{p}\left(p^{m-s}-\epsilon\eta_0^m(-1)\sqrt{p^*}^{m-s}\right)$
$(p-1)\left(p^{m-2}+\epsilon(p-1)\sqrt{p^*}^{m+s-4}\right)$	$p^{m+1}-p^{m-s+1}$

Theorem 2. *Let $m+s\geq 4$ be even with $0\leq s\leq m-2$. Let $f\in\mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let $D_f=\{x\in\mathbb{F}_{p^m}: f(x)=0\}$. Then, $\overline{C}_{D_f}$ is a five-weight linear $[p^{m-1},m+1]$ code with parameters listed in Table 2. In particular, $\overline{C}_{D_f}$ is a ternary self-orthogonal code when $p=3$ and $m+s\geq 6$.*

Proof. From the definition of the code, the length of any codeword $\mathbf{c}$ of $\overline{C}_{D_f}$ is $n=\#D_f=p^{m-1}$. The Hamming weight $wt(\mathbf{c})=\#D_f-N_f(b,c)$ can be derived from Lemma 15. For every $b\in\mathbb{F}_{p^m}\backslash$Supp$(\mathcal{W}_f)$, we have

$$wt(\mathbf{c}) = \begin{cases} (p-1)p^{m-2}, & \text{if } c\in\mathbb{F}_p, b\neq 0, \\ p^{m-1}, & \text{if } c\neq 0, b=0. \end{cases}$$

For every $b \in$Supp$(\mathcal{W}_f)$, we obtain

$$wt(\mathbf{c}) = \begin{cases} (p-1)\left(p^{m-2} - \epsilon(p-1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, b\neq 0, g(b)=0 \\ p-1)\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, b\neq 0, g(b)\neq 0 \text{ or} \\ & \ c\neq 0, b\neq 0, g(b)=0, \\ (p-1)p^{m-2} - \epsilon\sqrt{p^*}^{m+s-4}, & \text{if } c\neq 0, b\neq 0, g(b)\neq 0. \end{cases}$$

The weight distribution follows from Lemmas 2 and 12. The dimension of $\overline{C}_{D_f}$ is obtained as $m+1$ by using the first Pless power moment given in Equation 2. By Lemma 6, $\overline{C}_{D_f}$ is a ternary self-orthogonal code for $m+s \geq 6$ and $p=3$. Hence, the proof is complete. $\square$

Example 2. *Let $p=5$ and $m=5$. Let $f \in \mathcal{WRPB}$ with $s=1$ and $\epsilon=1$. Then, the code $\overline{C}_{D_f}$ in Theorem 2 is a five-weight $[625, 6, 420]$ linear code over $\mathbb{F}_5$ with weight enumerator $1+145y^{420}+1920y^{495}+12495y^{500}+1060y^{520}+4y^{625}$, which is verified by the Sage program.*

Table 2 The code $\overline{C}_{D_f}$ in Theorem 2 when $m+s$ is even.

Hamming weight ω	Multiplicity A_ω
0	1
p^{m-1}	$p-1$
$(p-1)p^{m-2}$	$p(p^m-p^{m-s}-1)$
$(p-1)\left(p^{m-2}-\epsilon(p-1)\sqrt{p^*}^{m+s-4}\right)$	$p^{m-s-1}+\epsilon\eta_0^{m+1}(-1)(p-1)\sqrt{p^*}^{m-s-2}$
$(p-1)\left(p^{m-2}+\epsilon\sqrt{p^*}^{m+s-4}\right)$	$(p-1)\left(2p^{m-s-1}+\epsilon\eta_0^{m+1}(-1)(p-2)\sqrt{p^*}^{m-s-2}\right)$
$(p-1)p^{m-2}-\epsilon\sqrt{p^*}^{m+s-4}$	$(p-1)^2\left(p^{m-s-1}-\epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right)$

Theorem 3. *Let $m+s \geq 3$ be odd and $0 \leq s \leq m-1$. Let $f \in \mathcal{WRP}$ or $f \in \mathcal{WRPB}$ and ϵ be the sign of Walsh transform of f. Let $D_f = \{x \in \mathbb{F}_{p^m} : f(x) = 0\}$. Then, $\overline{C}_{D_f}$ is a six-weight linear $[p^{m-1}, m+1]$ code with parameters listed in Table 3. In particular, $\overline{C}_{D_f}$ is a ternary self-orthogonal code for $m+s \geq 5$ and $p=3$.*

Proof. The proof is similar to the proof of Theorem 1. The Hamming weights and the frequency of each weight can be obtained from Lemmas 2, 11, 12 and 14. Furthermore, $\overline{C}_{D_f}$ is a ternary self-orthogonal code when $p=3$ by Lemma 6. $\square$

Example 3. *Let $f(x) = \mathrm{Tr}_3^{3^6}(\zeta x^4 + \zeta^{27}x^2)$, where ζ is a generator of $\mathbb{F}_{3^6}^* = \langle\zeta\rangle$ for $\zeta^6+2\zeta^4+\zeta^2+2\zeta+2=0$. Then, f is a quadratic 1-plateaued unbalanced function in the set $\mathcal{WRP}$ and for all $\beta \in \mathbb{F}_{3^6}$, we have $\mathcal{W}_f(\beta) \in \{0, 54\xi_3+27, -27\xi_3-54, -27\xi_3+27\}$ with $\epsilon=-1$. Then, the code $\overline{C}_{D_f}$ in Theorem 3 is a ternary self-orthogonal code with parameters $[243, 7, 144]$ and weight enumerator $1+90y^{144}+144y^{153}+1698y^{162}+180y^{171}+72y^{180}+2y^{243}$. It is verified by the Sage program.*

Table 3 The code $\overline{C}_{D_f}$ in Theorem 3 when $m+s$ is odd.

Hamming weight ω	Multiplicity A_ω
0	1
p^{m-1}	$p-1$
$(p-1)p^{m-2}$	$p^{m+1}-p-p^{m-s}(p-1)$
$\frac{p-1}{p^2}\left(p^m+\epsilon\sqrt{p^*}^{\,m+s+1}\right)$	$\frac{p-1}{2}\left(p^{m-s-1}-\epsilon\eta_0^m(-1)\sqrt{p^*}^{\,m-s-1}\right)$
$\frac{p-1}{p^2}\left(p^m-\epsilon\sqrt{p^*}^{\,m+s+1}\right)$	$\frac{p-1}{2}\left(p^{m-s-1}+\epsilon\eta_0^m(-1)\sqrt{p^*}^{\,m-s-1}\right)$
$\frac{1}{p^2}\left((p-1)p^m+\epsilon\sqrt{p^*}^{\,m+s+1}\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}+\epsilon\eta_0^m(-1)\sqrt{p^*}^{\,m-s-1}\right)$
$\frac{1}{p^2}\left((p-1)p^m-\epsilon\sqrt{p^*}^{\,m+s+1}\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}-\epsilon\eta_0^m(-1)\sqrt{p^*}^{\,m-s-1}\right)$

Theorem 4. *Let m and s be two integers with $0 \le s \le m$. Let $f : \mathbb{F}_{p^m} \to \mathbb{F}_p$ and ϵ be the sign of the Walsh transform of f.*

- *Let $m+s \ge 4$ be even with $0 \le s \le m-2$ and $f \in \mathcal{WRP}$. Then, the dual code $\overline{C}_{D_f}^{\perp}$ has the parameters $[\frac{1}{p}(p^m+\epsilon(p-1)\sqrt{p^*}^{m+s}), \frac{1}{p}(p^m+\epsilon(p-1)\sqrt{p^*}^{m+s})-m-1, 3]$.*
- *Let $m+s \ge 4$ be even with $0 \le s \le m-2$ and $f \in \mathcal{WRPB}$. Then, the dual code $\overline{C}_{D_f}^{\perp}$ has the parameters $[p^{m-1}, p^{m-1}-m-1, 2]$.*
- *Let $m+s \ge 3$ be odd with $0 \le s \le m-1$ and $f \in \mathcal{WRP}$ or $f \in \mathcal{WRPB}$. Then, the dual code $\overline{C}_{D_f}^{\perp}$ has the parameters $[p^{m-1}, p^{m-1}-m-1, 3]$.*

Proof. Denote by $d^\perp$ the minimum distance of $\overline{C}_{D_f}^{\perp}$. From the definition of $\overline{C}_{D_f}$, we deduce that $d^\perp \ge 2$. Denote by $(1, A_1, \ldots, A_n)$ and $(1, A_1^\perp, \ldots, A_n^\perp)$ the weight distributions of $\overline{C}_{D_f}$ and $\overline{C}_{D_f}^{\perp}$, respectively. Let $m+s > 4$ be even and $f \in \mathcal{WRP}$. By using the second and third Pless power moments and Theorem 1, we derive

$$A_1^\perp = 0 \quad \text{and} \quad A_2^\perp = 0.$$

By the fourth Pless power moment,

$$\sum_{i=0}^{n} i^3 A_i = p^{k-3}[(p-1)n(p^2n^2-2pn^2+3pn-p+n^2-3n+2)-6A_3^\perp]. \tag{3}$$

Combining Theorem 1 and Equation 3, we obtain that $A_3^\perp > 0$ and $d^\perp = 3$. By using the second, third and fourth Pless power moments and Theorems 2 and 3, we obtain the desired conclusions for the other cases. □

5 Linear codes from weakly regular plateaued functions based on the sets D_{sq} and D_{nsq}

In this section, we construct the augmented codes $\overline{C}_{D_{sq}}$ and $\overline{C}_{D_{nsq}}$ based on the following defining sets

$$\begin{aligned} D_{sq} &= \{x \in \mathbb{F}_{p^m} : f(x) \in SQ\}, \\ D_{nsq} &= \{x \in \mathbb{F}_{p^m} : f(x) \in NSQ\}. \end{aligned} \tag{4}$$

The length of the augmented code is the size of the corresponding defining set.

- If $f \in \mathcal{WRP}$, then the sizes of the sets D_{sq} and D_{nsq} follow from Lemma 11.
- If $f \in \mathcal{WRPB}$, then it is clear that $n = \#D_f = (\frac{p-1}{2})p^{m-1}$.

The following lemma follows from [17, Lemma 16] and [24, Lemma 9].

Lemma 16. *Let $f \in \mathcal{WRP}$. For $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$, define*

$$\begin{aligned} N_{sq}(b,c) &= \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_p^{p^m}(bx) + c = 0 \text{ and } f(x) \in SQ\}, \\ N_{nsq}(b,c) &= \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_p^{p^m}(bx) + c = 0 \text{ and } f(x) \in NSQ\}. \end{aligned}$$

For every $b \in \mathbb{F}_{p^m} \setminus Supp(\mathcal{W}_f)$, we have

$$\begin{aligned} N_{sq}(b,c) &= \begin{cases} (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } m+s \text{ is even}, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } m+s \text{ is odd}, \end{cases} \\ N_{nsq}(b,c) &= \begin{cases} (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } m+s \text{ is even}, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } m+s \text{ is odd}. \end{cases} \end{aligned}$$

For every $b \in Supp(\mathcal{W}_f)$, if $m+s$ is even,

$$N_{sq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c = 0, b = 0, \\ 0, & \text{if } c \neq 0, b = 0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c = 0, b \neq 0, g(b) \in NSQ \cup \{0\}, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c = 0, b \neq 0, g(b) \in SQ, \\ (\frac{p-1}{2})p^{m-2}, & \text{if } c \neq 0, b \neq 0, g(b) \in NSQ \cup \{0\}, \\ (\frac{p-1}{2})p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}, & \text{if } c \neq 0, b \neq 0, g(b) \in SQ, \end{cases}$$

$$N_{nsq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c=0, b=0, \\ 0, & \text{if } c\neq 0, b=0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c=0, b\neq 0, g(b)\in SQ\cup\{0\}, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c=0, b\neq 0, g(b)\in NSQ, \\ (\frac{p-1}{2})p^{m-2}, & \text{if } c\neq 0, b\neq 0, g(b)\in SQ\cup\{0\}, \\ (\frac{p-1}{2})p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}, & \text{if } c\neq 0, b\neq 0, g(b)\in NSQ, \end{cases}$$

if $m+s$ is odd,

$$N_{sq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} + \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } c=0, b=0, \\ 0, & \text{if } c\neq 0, b=0, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } c=0, b\neq 0, g(b)=0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, b\neq 0, g(b)\in SQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, b\neq 0, g(b)\in NSQ, \\ (\frac{p-1}{2})p^{m-2}, & \text{if } c\neq 0, b\neq 0, g(b)=0, \\ (\frac{p-1}{2})p^{m-2} + \epsilon(\frac{p^*+1}{2p^*})\sqrt{p^*}^{m+s-1}, & \text{if } c\neq 0, b\neq 0, g(b)\in SQ, \\ (\frac{p-1}{2})p^{m-2} + \epsilon(\frac{p^*-1}{2p^*})\sqrt{p^*}^{m+s-1}, & \text{if } c\neq 0, b\neq 0, g(b)\in NSQ, \end{cases}$$

$$N_{nsq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-1} - \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } c=0, b=0, \\ 0, & \text{if } c\neq 0, b=0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-1}\right), & \text{if } c=0, b\neq 0, g(b)=0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, b\neq 0, g(b)\in SQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, b\neq 0, g(b)\in NSQ, \\ (\frac{p-1}{2})p^{m-2}, & \text{if } c\neq 0, b\neq 0, g(b)=0, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{p^*-1}{2p^*})\sqrt{p^*}^{m+s-1}, & \text{if } c\neq 0, b\neq 0, g(b)\in SQ, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{p^*+1}{2p^*})\sqrt{p^*}^{m+s-1}, & \text{if } c\neq 0, b\neq 0, g(b)\in NSQ. \end{cases}$$

The following lemma follows from [22, Lemma 9] and [25, Lemma 9].

Lemma 17. *Let $f \in \mathcal{WRPB}$. For $b \in \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_p$, define*

$$\begin{aligned} N_{sq}(b,c) &= \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_p^{p^m}(bx) + c = 0 \text{ and } f(x) \in SQ\}, \\ N_{nsq}(b,c) &= \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_p^{p^m}(bx) + c = 0 \text{ and } f(x) \in NSQ\}. \end{aligned}$$

Then, for every $b \in \mathbb{F}_{p^m} \setminus Supp(\mathcal{W}_f)$, we have

$$N_{sq}(b,c) = N_{nsq}(b,c) = \begin{cases} (\frac{p-1}{2})p^{m-2}, & \text{if } c\in\mathbb{F}, b\neq 0, \\ (\frac{p-1}{2})p^{m-1}, & \text{if } c=0, b=0, \\ 0, & \text{if } c\neq 0, b=0, \end{cases}$$

and for every $b \in Supp(\mathcal{W}_f)$, if $m+s$ is even,

$$N_{sq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-2} - \epsilon(p-1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b) \in NSQ \cup \{0\}, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon(p+1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b) \in SQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } c\neq 0, g(b) \in NSQ \cup \{0\}, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{p+1}{2})\sqrt{p^*}^{m+s-4}, & \text{if } c\neq 0, g(b) \in SQ, \end{cases}$$

$$N_{nsq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-2} - \epsilon(p-1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b) \in SQ \cup \{0\}, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon(p+1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b) \in NSQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } c\neq 0, g(b) \in SQ \cup \{0\}, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{p+1}{2})\sqrt{p^*}^{m+s-4}, & \text{if } c\neq 0, g(b) \in NSQ, \end{cases}$$

if $m+s$ is odd,

$$N_{sq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-2} + \epsilon\eta_0(-1)(p-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, g(b)=0, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}(\eta_0(-1)+1)\right), & \text{if } c=0, g(b) \in SQ, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}(\eta_0(-1)-1)\right), & \text{if } c=0, g(b) \in NSQ, \\ (\frac{p-1}{2})\left(p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } c\neq 0, g(b)=0, \\ (\frac{p-1}{2})p^{m-2} + \epsilon(\frac{\eta_0(-1)+1}{2})\sqrt{p^*}^{m+s-3}, & \text{if } c\neq 0, g(b) \in SQ, \\ (\frac{p-1}{2})p^{m-2} + \epsilon(\frac{\eta_0(-1)-1}{2})\sqrt{p^*}^{m+s-3}, & \text{if } c\neq 0, g(b) \in NSQ, \end{cases}$$

$$N_{nsq}(b,c) = \begin{cases} (\frac{p-1}{2})\left(p^{m-2} - \epsilon\eta_0(-1)(p-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } c=0, g(b)=0, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}(\eta_0(-1)-1)\right), & \text{if } c=0, g(b) \in SQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}(\eta_0(-1)+1)\right), & \text{if } c=0, g(b) \in NSQ, \\ (\frac{p-1}{2})\left(p^{m-2} + \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-3}\right), & \text{if } c\neq 0, g(b)=0, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{\eta_0(-1)-1}{2})\sqrt{p^*}^{m+s-3}, & \text{if } c\neq 0, g(b) \in SQ, \\ (\frac{p-1}{2})p^{m-2} - \epsilon(\frac{\eta_0(-1)+1}{2})\sqrt{p^*}^{m+s-3}, & \text{if } c\neq 0, g(b) \in NSQ. \end{cases}$$

Theorem 5. *Let $m+s \geq 4$ be even with $0 \leq s \leq m-2$. Let $f \in \mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f. Let D_{sq} be defined as in (4). Then,*

- *$\overline{C}_{D_{sq}}$ is a six-weight linear $\left[(\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), m+1, d\right]$ code with parameters listed in Table 4. In particular, $\overline{C}_{D_{sq}}$ is a five-weight ternary self-orthogonal code over $\mathbb{F}_3$ when $m+s \geq 6$ and $p=3$.*

- *The dual code* $\overline{C}_{D_{sq}}^{\perp}$ *over* $\mathbb{F}_p$ *has the parameters* $[n, n-m-1, 3]$*, where* $n = (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right)$.

Proof. The length of the code $\overline{C}_{D_{sq}}$ follows from Lemma 11. For any codeword $\mathbf{c}$, the Hamming weight $wt(\mathbf{c}) = \#D_{sq} - N_{sq}(b,c)$ can be directly derived from Lemmas 11 and 16. For every $b \in \mathbb{F}_{p^m}\backslash\mathrm{Supp}(\mathcal{W}_f)$, we have

$$wt(\mathbf{c}) = \frac{(p-1)^2}{2}\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-4}\right).$$

For every $b \in$Supp$(\mathcal{W}_f)$, we obtain

$$wt(\mathbf{c}) = \begin{cases} \frac{p-1}{2}\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c \neq 0, b = 0, \\ \frac{(p-1)^2}{2}p^{m-2}, & \text{if } c = 0, b \neq 0, g(b) \in NSQ \cup \{0\}, \\ \frac{p-1}{2}\left((p-1)p^{m-2} - 2\epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c = 0, b \neq 0, g(b) \in SQ, \\ \frac{p-1}{2}\left((p-1)p^{m-2} - \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right), & \text{if } c \neq 0, b \neq 0, g(b) \in NSQ \cup \{0\}, \\ \frac{(p-1)^2}{2}p^{m-2} + \epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\frac{3-p}{2}, & \text{if } c \neq 0, b \neq 0, g(b) \in SQ. \end{cases}$$

The weight distribution follows from Lemmas 2 and 12. The dimension of $\overline{C}_{D_{sq}}$ is $m+1$ as $A_0 = 1$. Furthermore, from Lemma 6, $\overline{C}_{D_{sq}}$ is a ternary self-orthogonal code when $p = 3$ and $m + s \geq 6$ since all codewords have weights divisible by 3. Finally, one can derive the minimum Hamming distance $d^{\perp} = 3$ from the Pless power moments. □

Remark 1. *In Theorem 5, when we replace the set* D_{sq} *with the set* D_{nsq}*, the code* $\overline{C}_{D_{nsq}}$ *has the same parameters as the code* $\overline{C}_{D_{sq}}$*.*

Example 4. $f(x) = \mathrm{Tr}_3^{3^4}(2x^{92})$ *is a ternary* 2*-plateaued unbalanced function in the set* $\mathcal{WRP}$ *and for all* $\beta \in \mathbb{F}_{3^4}$*, we have* $\mathcal{W}_f(\beta) \in \{0, -3^3\xi_3^{g(\beta)}\}$ *with* $\epsilon = 1$*. Then, the code* $\overline{C}_{D_{sq}}$ *in Theorem 5 is a self-orthogonal ternary linear code with parameters* $[36, 5, 18]$ *and weight enumerator* $1 + 12y^{18} + 216y^{24} + 8y^{27} + 6y^{36}$*. It is verified by the Sage program.*

Theorem 6. *Let* $m + s \geq 4$ *be even with* $0 \leq s \leq m-2$*. Let* $f \in \mathcal{WRPB}$ *and* ϵ *be the sign of the Walsh transform of* f*. Let* D_{sq} *be defined as in (4). Then,*

- $\overline{C}_{D_{sq}}$ *is a six-weight linear* $[(\frac{p-1}{2})p^{m-1}, m+1, d]$ *code with parameters listed in Table 5. In particular, when* $p = 3$ *and* $m + s \geq 6$*,* $\overline{C}_{D_{sq}}$ *is a five-weight ternary self-orthogonal code over* $\mathbb{F}_3$*.*
- *The dual code* $\overline{C}_{D_{sq}}^{\perp}$ *over* $\mathbb{F}_p$ *has the parameters* $[(\frac{p-1}{2})p^{m-1}, (\frac{p-1}{2})p^{m-1} - m - 1, 2]$*.*

Proof. From the definition of $\overline{C}_{D_{sq}}$, its length is equal to $\#D_{sq} = (\frac{p-1}{2})p^{m-1}$. Similarly, for a codeword $\mathbf{c} \in \overline{C}_{D_{sq}}$, the Hamming weight is obtained as $wt(\mathbf{c}) = \#D_{sq} - N_{sq}(b,c)$, which follows from Lemma 17. For every $b \in \mathbb{F}_{p^m}\backslash\mathrm{Supp}(\mathcal{W}_f)$, we

Table 4 The code $\overline{C}_{D_{sq}}$ in Theorem 5 when $m+s$ is even.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})\left(p^{m-1}-\epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right)$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$(\frac{p+1}{2})p^{m-s-1}-1+\epsilon(\frac{p-1}{2})\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}$
$(\frac{p-1}{2})\left((p-1)p^{m-2}-2\epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1}-\epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2}-\epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}\right)$	$(\frac{p-1}{2})\left((p+1)p^{m-s-1}-2+\epsilon\eta_0^{m+1}(-1)(p-1)\sqrt{p^*}^{m-s-2}\right)$
$\frac{(p-1)^2}{2}p^{m-2}+\epsilon\eta_0(-1)\sqrt{p^*}^{m+s-2}(\frac{3-p}{2})$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}-\epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right)$
$\frac{(p-1)^2}{2}\left(p^{m-2}-\epsilon\sqrt{p^*}^{m+s-4}\right)$	$p(p^m-p^{m-s})$

have

$$wt(\mathbf{c})=\begin{cases}\frac{(p-1)^2}{2}p^{m-2}, & \text{if } c\in\mathbb{F}_p, b\neq 0,\\ \frac{p-1}{2}p^{m-1}, & \text{if } c\neq 0, b=0\end{cases}$$

and the number of such codewords can be determined by Lemma 2. For every $b\in$Supp$(\mathcal{W}_f)$, we obtain

$$wt(\mathbf{c})=\begin{cases}\frac{(p-1)^2}{2}\left(p^{m-2}+\epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b)\in NSQ\cup\{0\},\\ \frac{p-1}{2}\left((p-1)p^{m-2}-\epsilon(p+1)\sqrt{p^*}^{m+s-4}\right), & \text{if } c=0, g(b)\in SQ,\\ \frac{p-1}{2}\left((p-1)p^{m-2}-\epsilon\sqrt{p^*}^{m+s-4}\right), & \text{if } c\neq 0, g(b)\in NSQ\cup\{0\},\\ \frac{(p-1)^2}{2}p^{m-2}+\epsilon(\frac{p+1}{2})\sqrt{p^*}^{m+s-4}, & \text{if } c\neq 0, g(b)\in SQ\end{cases}$$

and also the number of such codewords can be obtained from Lemma 12. The dimension of $\overline{C}_{D_{sq}}$ follows from its weight distribution. Furthermore, from Lemma 6, $\overline{C}_{D_{sq}}$ is a self-orthogonal code when $p=3$ and $m+s\geq 6$ since all codewords have weights divisible by 3. From the Pless power moments, one can obtain the minimum Hamming distance $d^\perp=2$. Hence, the proof is complete. □

Remark 2. *In Theorem 6, when we replace the set D_{sq} with the set D_{nsq}, the code $\overline{C}_{D_{nsq}}$ has the same parameters as the code $\overline{C}_{D_{sq}}$.*

Example 5. *Let $p=5$ and $m=5$. Let $f\in\mathcal{WRPB}$ with $s=1$ and $\epsilon=1$. Then, the code $\overline{C}_{D_{sq}}$ in Theorem 6 is a six-weight linear $[1250,6,940]$ code over $\mathbb{F}_5$ with weight enumerator $1+240y^{940}+1540y^{990}+12495y^{1000}+960y^{1015}+385y^{1040}+4y^{1250}$, which is verified by the Sage program.*

Theorem 7. *Let $m+s\geq 3$ be odd with $0\leq s\leq m-1$. Let $f\in\mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f. Let D_{sq} be defined as in (4). Then,*

- *$\overline{C}_{D_{sq}}$ is a six-weight linear $\left[(\frac{p-1}{2})\left(p^{m-1}+\epsilon\sqrt{p^*}^{m+s-1}\right), m+1, d\right]$ code with parameters listed in Tables 6 and 7 when $p\equiv 1 \pmod 4$ and $p\equiv 3 \pmod 4$, respectively. In particular, $\overline{C}_{D_{sq}}$ is a ternary self-orthogonal code for $p=3$ and $m+s\geq 5$.*

Table 5 The code $\overline{C}_{D_{sq}}$ in Theorem 6 when $m+s$ is even.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})p^{m-1}$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p(p^m-p^{m-s}-1)$
$\frac{(p-1)^2}{2}\left(p^{m-2}+\epsilon\sqrt{p^*}^{m+s-4}\right)$	$(\frac{p+1}{2})p^{m-s-1}+\epsilon\eta_0^{m+1}(-1)(\frac{p-1}{2})\sqrt{p^*}^{m-s-2}$
$\frac{(p-1)}{2}\left((p-1)p^{m-2}-\epsilon(p+1)\sqrt{p^*}^{m+s-4}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1}-\epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right)$
$\frac{(p-1)}{2}\left((p-1)p^{m-2}-\epsilon\sqrt{p^*}^{m+s-4}\right)$	$(p-1)\left(p^{m-s-1}(\frac{p+1}{2})+\epsilon\eta_0^{m+1}(\frac{p-1}{2})\sqrt{p^*}^{m-s-2}\right)$
$\frac{(p-1)^2}{2}p^{m-2}+\epsilon(\frac{p+1}{2})\sqrt{p^*}^{m+s-4}$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}-\epsilon\eta_0^{m+1}(-1)\sqrt{p^*}^{m-s-2}\right)$

- *The dual code* $\overline{C}^{\perp}_{D_{sq}}$ *over* $\mathbb{F}_p$ *has the parameters* $[n, n-m-1, 3]$, *where* $n = (\frac{p-1}{2})(p^{m-1}+\epsilon\sqrt{p^*}^{m+s-1})$.

Proof. As in the proof of Theorem 5, the length of $\overline{C}_{D_{sq}}$ is a consequence of Lemma 11 and the Hamming weight of any codeword in $\overline{C}_{D_{sq}}$ is obtained as

$$wt(\mathbf{c}) = \#D_{sq} - N_{sq}(b,c),$$

which follows from Lemmas 11 and 16. The weight distribution follows from Lemmas 2 and 12. By Lemma 6, the code $\overline{C}_{D_{sq}}$ is a self-orthogonal code when $p=3$ and $m+s \geq 6$ since all codewords have weights divisible by 3. By using the Pless power moments, one can find the minimum distance of $\overline{C}^{\perp}_{D_{sq}}$ as 3. □

Table 6 The code $\overline{C}_{D_{sq}}$ in Theorem 7 when $m+s$ is odd and $p=1 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})\left(p^{m-1}+\epsilon\sqrt{p^*}^{m+s-1}\right)$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-s-1}-1$
$(\frac{p-1}{2})\left((p-1)p^{m-2}+\epsilon\sqrt{p^*}^{m+s-3}(p+1)\right)$	$(\frac{p-1}{2})\left(p^{m-s-1}+\epsilon\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)^2}{2}\left(p^{m-2}+\epsilon\sqrt{p^*}^{m+s-3}\right)$	$p\left(p^m-p^{m-s}+\frac{p-1}{2}(p^{m-s-1}-\epsilon\sqrt{p^*}^{m-s-1})\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2}+\epsilon\sqrt{p^*}^{m+s-1}\right)$	$(p-1)(p^{m-s-1}-1)$
$\frac{(p-1)^2}{2}p^{m-2}+\epsilon\frac{1}{2p}\sqrt{p^*}^{m+s-1}\left(p^2-2p-1\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}+\epsilon\sqrt{p^*}^{m-s-1}\right)$

Theorem 8. *Let $m+s \geq 3$ be odd with $0 \leq s \leq m-1$. Let $f \in \mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let D_{sq} be defined as in (4). Then,*

- $\overline{C}_{D_{sq}}$ *is a six-weight linear* $[(\frac{p-1}{2})p^{m-1}, m+1, d]$ *code with parameters listed in Tables 8 and 9 when $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$, respectively. In particular,* $\overline{C}_{D_{sq}}$ *is a ternary self-orthogonal code when $m+s \geq 5$ and $p=3$.*

Table 7 The code $\overline{C}_{D_{sq}}$ in Theorem 7 when $m+s$ is odd and $p=3 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})\left(p^{m-1}+\epsilon\sqrt{p^*}^{m+s-1}\right)$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-s-1}-1$
$\frac{(p-1)^2}{2}\left(p^{m-2}-\epsilon\sqrt{p^*}^{m+s-3}\right)$	$p\left(p^m-p^{m-s}+\frac{p-1}{2}(p^{m-s-1}+\epsilon(-1)^m\sqrt{p^*}^{m-s-1})\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2}-\epsilon\sqrt{p^*}^{m+s-3}(p+1)\right)$	$(\frac{p-1}{2})\left(p^{m-s-1}-\epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2}+\epsilon\sqrt{p^*}^{m+s-1}\right)$	$(p-1)(p^{m-s-1}-1)$
$\frac{(p-1)^2}{2}p^{m-2}+\epsilon\frac{1}{2p}\sqrt{p^*}^{m+s-1}\left(p^2-2p-1\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}-\epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$

- *The dual code $\overline{C}_{D_{sq}}^{\perp}$ over $\mathbb{F}_p$ has the parameters $[(\frac{p-1}{2})p^{m-1},(\frac{p-1}{2})p^{m-1}-m-1,2]$.*

Proof. Similar to the proof of Theorem 6, the Hamming weight $wt(\mathbf{c}) = \#D_{sq} - N_{sq}(b,c)$ can be directly derived from Lemma 17. The weight distribution follows from Lemmas 2 and 12. Furthermore, from Lemma 6, the code $\overline{C}_{D_{sq}}$ is self-orthogonal for $p=3$ and $m+s\geq 5$. By using the Pless power moments, we obtain the minimum distance of $d^{\perp}=2$. □

Table 8 The code $\overline{C}_{D_{sq}}$ in Theorem 8 when $m+s$ is odd and $p\equiv 1 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})p^{m-1}$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p(p^m-p^{m-s}-1+(\frac{p-1}{2})(p^{m-s-1}-\epsilon\sqrt{p^*}^{m-s-1})$
$\frac{(p-1)^2}{2}\left(p^{m-2}-\epsilon\sqrt{p^*}^{m+s-3}\right)$	p^{m-s-1}
$\frac{(p-1)}{2}\left((p-1)p^{m-2}+2\epsilon\sqrt{p^*}^{m+s-3}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1}+\epsilon\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)}{2}\left((p-1)p^{m-2}+\epsilon\sqrt{p^*}^{m+s-3}\right)$	$p^{m-s-1}(p-1)$
$\frac{(p-1)^2}{2}p^{m-2}-\epsilon\sqrt{p^*}^{m+s-3}$	$\frac{(p-1)^2}{2}\left(p^{m-s-1}+\epsilon\sqrt{p^*}^{m-s-1}\right)$

Theorem 9. *Let $m+s\geq 3$ be odd with $0\leq s\leq m-1$. Let $f\in \mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f. Let D_{nsq} be defined as in (4). Then,*

- *$\overline{C}_{D_{nsq}}$ is a six-weight linear $[(\frac{p-1}{2})(p^{m-1}-\epsilon\sqrt{p^*}^{m+s-1}),m+1]$ code with parameters listed in Tables 10 and 11 when $p\equiv 1 \pmod 4$ and $p\equiv 3 \pmod 4$, respectively. In particular, $\overline{C}_{D_{nsq}}$ is a ternary self-orthogonal code when $p=3$ and $m+s\geq 5$.*
- *The dual code $\overline{C}_{D_{nsq}}^{\perp}$ over $\mathbb{F}_p$ has the parameters $[n,n-m-1,3]$, where $n = (\frac{p-1}{2})(p^{m-1}-\epsilon\sqrt{p^*}^{m+s-1})$.*

Proof. As in the proof of Theorem 5, the length of $\overline{C}_{D_{nsq}}$ is given in Lemma 11 and the Hamming weight of $\mathbf{c}\in\overline{C}_{D_{nsq}}$ is obtained as $wt(\mathbf{c}) = \#D_{nsq} - N_{nsq}(b,c)$,

Table 9 The code $\overline{C}_{D_{sq}}$ in Theorem 8 when $m+s$ is odd and $p \equiv 3 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})p^{m-1}$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p(p^m - p^{m-s} - 1 + (\frac{p-1}{2})(p^{m-s-1} + \epsilon(-1)^m \sqrt{p^*}^{m-s-1})$
$\frac{(p-1)^2}{2}\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right)$	p^{m-s-1}
$\frac{(p-1)}{2}\left((p-1)p^{m-2} - 2\epsilon\sqrt{p^*}^{m+s-3}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1} - \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)}{2}\left((p-1)p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}\right)$	$p^{m-s-1}(p-1)$
$\frac{(p-1)^2}{2}p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}$	$\frac{(p-1)^2}{2}\left(p^{m-s-1} - \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$

which follows from Lemmas 11 and 16. We can determine the weight distribution from Lemmas 2 and 12. Moreover, by Lemma 6, $\overline{C}_{D_{nsq}}$ is a ternary self-orthogonal code for $p=3$ and $m+s \geq 5$. Finally, by using the Pless power moments, we obtain the minimum distance of $\overline{C}^{\perp}_{D_{nsq}}$ as 3. □

Table 10 The code $\overline{C}_{D_{nsq}}$ in Theorem 9 when $m+s$ is odd and $p = 1 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})\left(p^{m-1} - \epsilon\sqrt{p^*}^{m+s-1}\right)$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-s-1}-1$
$\frac{(p-1)^2}{2}\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}\right)$	$p\left(p^m - p^{m-s} + \frac{p-1}{2}(p^{m-s-1} + \epsilon\sqrt{p^*}^{m-s-1})\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}(1+p)\right)$	$(\frac{p-1}{2})\left(p^{m-s-1} - \epsilon\sqrt{p^*}^{m-s-1}\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2} - \epsilon\sqrt{p^*}^{m+s-1}\right)$	$(p-1)(p^{m-s-1}-1)$
$\frac{(p-1)^2}{2}p^{m-2} - \epsilon\frac{1}{2p}\sqrt{p^*}^{m+s-1}\left(p^2-2p-1\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1} - \epsilon\sqrt{p^*}^{m-s-1}\right)$

Theorem 10. *Let $m+s \geq 3$ be odd with $0 \leq s \leq m-1$. Let $f \in \mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let D_{nsq} be defined as in (4). Then,*

- *$\overline{C}_{D_{nsq}}$ is a six-weight linear $[(\frac{p-1}{2})p^{m-1}, m+1]$ code with parameters listed in Tables 12 and 13 when $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$, respectively. In particular, $\overline{C}_{D_{nsq}}$ is a ternary self-orthogonal code for $p=3$ and $m+s \geq 5$.*
- *The dual code $\overline{C}^{\perp}_{D_{nsq}}$ has the parameters $[(\frac{p-1}{2})p^{m-1}, (\frac{p-1}{2})p^{m-1} - m - 1, 2]$.*

Proof. As in the proof of Theorem 6, the Hamming weight $wt(\mathbf{c}) = \#D_{nsq} - N_{nsq}(b,c)$ follows from Lemma 17. The weight distribution follows from Lemmas 2 and 12. From Lemma 6, $\overline{C}_{D_{nsq}}$ is a ternary self-orthogonal code for $p=3$ and $m+s \geq 5$ since all codewords have weights divisible by 3. Finally, we obtain the minimum distance of $\overline{C}^{\perp}_{D_{nsq}}$ as 2 from the Pless power moments. □

Table 11 The code $\overline{C}_{D_{nsq}}$ in Theorem 9 when $m+s$ is odd and $p = 3 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})\left(p^{m-1} - \epsilon\sqrt{p^*}^{m+s-1}\right)$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-s-1}-1$
$(\frac{p-1}{2})\left((p-1)p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}(p+1)\right)$	$(\frac{p-1}{2})\left(p^{m-s-1} + \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)^2}{2}\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right)$	$p\left(p^m - p^{m-s} + \frac{p-1}{2}(p^{m-s-1} - \epsilon(-1)^m\sqrt{p^*}^{m-s-1})\right)$
$(\frac{p-1}{2})\left((p-1)p^{m-2} - \epsilon\sqrt{p^*}^{m+s-1}\right)$	$(p-1)(p^{m-s-1}-1)$
$\frac{(p-1)^2}{2}p^{m-2} - \epsilon\frac{1}{2p}\sqrt{p^*}^{m+s-1}\left(p^2-2p-1\right)$	$\frac{(p-1)^2}{2}\left(p^{m-s-1} + \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$

Table 12 The code $\overline{C}_{D_{nsq}}$ in Theorem 10 when $m+s$ is odd and $p \equiv 1 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})p^{m-1}$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p(p^m - p^{m-s} - 1 + (\frac{p-1}{2})(p^{m-s-1} + \epsilon\sqrt{p^*}^{m-s-1})$
$\frac{(p-1)^2}{2}\left(p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right)$	p^{m-s-1}
$\frac{(p-1)}{2}\left((p-1)p^{m-2} - 2\epsilon\sqrt{p^*}^{m+s-3}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1} - \epsilon\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)}{2}\left((p-1)p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right)$	$p^{m-s-1}(p-1)$
$\frac{(p-1)^2}{2}p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}$	$\frac{(p-1)^2}{2}\left(p^{m-s-1} - \epsilon\sqrt{p^*}^{m-s-1}\right)$

Table 13 The code $\overline{C}_{D_{nsq}}$ in Theorem 10 when $m+s$ is odd and $p \equiv 3 \pmod 4$.

Hamming weight ω	Multiplicity A_ω
0	1
$(\frac{p-1}{2})p^{m-1}$	$p-1$
$\frac{(p-1)^2}{2}p^{m-2}$	$p(p^m - p^{m-s} - 1 + (\frac{p-1}{2})(p^{m-s-1} - \epsilon(-1)^m\sqrt{p^*}^{m-s-1})$
$\frac{(p-1)^2}{2}\left(p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}\right)$	p^{m-s-1}
$\frac{(p-1)}{2}\left((p-1)p^{m-2} + 2\epsilon\sqrt{p^*}^{m+s-3}\right)$	$(\frac{p-1}{2})\left(p^{m-s-1} + \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$
$\frac{(p-1)}{2}\left((p-1)p^{m-2} + \epsilon\sqrt{p^*}^{m+s-3}\right)$	$p^{m-s-1}(p-1)$
$\frac{(p-1)^2}{2}p^{m-2} - \epsilon\sqrt{p^*}^{m+s-3}$	$\frac{(p-1)^2}{2}\left(p^{m-s-1} + \epsilon(-1)^m\sqrt{p^*}^{m-s-1}\right)$

Example 6. *Let $f(x) = \mathrm{Tr}_3^{3^6}(\zeta x^4 + \zeta^{27}x^2)$, where ζ is a generator of $\mathbb{F}_{3^6}^* = \langle\zeta\rangle$ for $\zeta^6+2\zeta^4+\zeta^2+2\zeta+2 = 0$. Then, f is a quadratic 1-plateaued unbalanced function in the set $\mathcal{WRP}$ and for all $\beta \in \mathbb{F}_{3^6}$, we have $\mathcal{W}_f(\beta) \in \{0, 54\xi_3+27, -27\xi_3-54, -27\xi_3+27\}$ with $\epsilon = -1$. Then,*

- *the code* $\overline{C}_{D_{sq}}$ *in Theorem 7 is a self-orthogonal ternary linear* $[270, 7, 162]$ *code with weight enumerator* $1 + 80y^{162} + 180y^{171} + 1674y^{180} + 160y^{189} + 90y^{198} + 2y^{270}$,
- *the code* $\overline{C}_{D_{sq}}$ *in Theorem 8 is a self-orthogonal ternary linear* $[243, 7, 144]$ *code with weight enumerator* $1 + 81y^{144} + 180y^{153} + 1671y^{162} + 162y^{171} + 90y^{180} + 2y^{243}$,
- *the code* $\overline{C}_{D_{nsq}}$ *in Theorem 9 is a self-orthogonal ternary linear* $[216, 7, 126]$ *code with weight enumerator* $1 + 72y^{126} + 160y^{135} + 1728y^{144} + 144y^{153} + 80y^{162} + 2y^{216}$,
- *the code* $\overline{C}_{D_{nsq}}$ *in Theorem 10 is a self-orthogonal ternary linear* $[243, 7, 144]$ *code with weight enumerator* $1 + 72y^{144} + 162y^{153} + 1725y^{162} + 144y^{171} + 81y^{180} + 2y^{243}$. *They are verified by the Sage program.*

In particular, when $p = 3$, we can obtain the following ternary five-weight and six-weight linear codes from weakly regular plateaued ternary functions.

Corollary 1. *Let* $p = 3$ *and* $m + s$ *be an integer. Let* $f \in \mathcal{WRP}$ *and* ϵ *be the sign of the Walsh transform of* f. *Let* $D_f = \{x \in \mathbb{F}_{3^m} : f(x) + a = 0\}$ *for* $a \in \mathbb{F}_3^*$. *Then, we have the following ternary codes over* $\mathbb{F}_3$.

- *If* $m + s \geq 6$ *is even, then the code* $\overline{C}_{D_f}$ *is a five-weight ternary self-orthogonal* $[n, m+1, \min\{2 \cdot 3^{m-2}, 2 \cdot 3^{m-2} + 2\epsilon\sqrt{-3}^{m+s-2}\}]$ *code, and* $\overline{C}_{D_f}^{\perp}$ *is a dual* $[n, n-m-1, 3]$ *code over* $\mathbb{F}_3$, *where* $n = 3^{m-1} + \epsilon(-3)^{\frac{m+s-2}{2}}$. *This code is a ternary case of the code proposed in Theorem 5. The Hamming weights and weight distributions follow directly from Table 4.*
- *If* $m + s \geq 5$ *is odd, then the code* $\overline{C}_{D_f}$ *is a six-weight ternary*

$$[n, m+1, \min\{2 \cdot 3^{m-2}, 2 \cdot 3^{m-2} + 4\epsilon\eta_0(a)\sqrt{-3}^{m+s-3}\}]$$

self-orthogonal code, and $\overline{C}_{D_f}^{\perp}$ *is a dual* $[n, n-m-1, 3]$ *code over* $\mathbb{F}_3$, *where* $n = 3^{m-1} - \eta_0(a)\epsilon(-3)^{\frac{m+s-1}{2}}$. *This code is a ternary case of the code proposed in Theorem 7 and Theorem 9 when* $a = -1$ *and* $a = 1$, *respectively. The Hamming weights and weight distributions follow directly from Tables 7 and 11.*

Corollary 2. *Let* $p = 3$ *and* $m + s$ *be an integer. Let* $f \in \mathcal{WRPB}$ *and* ϵ *be the sign of the Walsh transform of* f. *Let* $D_f = \{x \in \mathbb{F}_{3^m} : f(x) + a = 0\}$ *for* $a \in \mathbb{F}_3^*$. *Then, we have the following ternary codes over* $\mathbb{F}_3$.

- *If* $m + s \geq 6$ *is even, then the code* $\overline{C}_{D_f}$ *is a five-weight ternary* $[3^{m-1}, m+1, d]$ *self-orthogonal code, where* $d = \min\{2 \cdot 3^{m-2} - 4\epsilon\sqrt{-3}^{m+s-4}, 2 \cdot 3^{m-2} + 2\epsilon\sqrt{-3}^{m+s-4}\}$. *This code is a ternary case of the code proposed in Theorem 6. Then, the Hamming weights and weight distributions follow directly from Table 5. Moreover,* $\overline{C}_{D_f}^{\perp}$ *is a dual* $[3^{m-1}, 3^{m-1} - m - 1, 2]$ *code over* $\mathbb{F}_3$.
- *If* $m + s \geq 5$ *is odd, then the code* $\overline{C}_{D_f}$ *is a six-weight ternary* $[3^{m-1}, m+1, \min\{2 \cdot 3^{m-2} \pm \epsilon\sqrt{-3}^{m+s-3}\}]$ *self-orthogonal code and* $\overline{C}_{D_f}^{\perp}$ *is a dual* $[3^{m-1}, 3^{m-1} - m - 1, 2]$ *code over* $\mathbb{F}_3$. *This code is a ternary case of the code proposed in Theorem 8 and Theorem 10 when* $a = -1$ *and* $a = 1$, *respectively. The Hamming weights and weight distributions follow directly from Tables 9 and 13.*

Remark 3. *Corollary 1 (resp. Corollary 2) is an extention of [18, Theorems 1 and 2] for weakly regular plateaued unbalanced (resp. balanced) ternary functions.*

We now assume that f is a weakly regular bent function in the defining sets D_{sq} and D_{nsq}. In the following corollary, we present new families of p-ary linear codes (and also, ternary self-orthogonal codes) derived from weakly regular bent functions.

Corollary 3. *Let $f \in \mathcal{RF}$ for $s = 0$.*

- *The code $\overline{C}_{D_{sq}}$ in Theorem 5 is a six-weight linear $[n, m+1]$ code and $\overline{C}^{\perp}_{D_{sq}}$ is a dual $[n, n-m-1, 3]$ code over $\mathbb{F}_p$, where $n = (\frac{p-1}{2})\left(p^{m-1} - \epsilon\eta_0(-1)\sqrt{p^*}^{m-2}\right)$. In particular, $\overline{C}_{D_{sq}}$ is a five-weight ternary self-orthogonal code over $\mathbb{F}_3$. The Hamming weights and weight distributions follow directly from Table 4 when $s = 0$.*
- *The code $\overline{C}_{D_{sq}}$ in Theorem 7 is a six-weight linear $[n, m+1]$ code, and $\overline{C}^{\perp}_{D_{sq}}$ is a dual $[n, n-m-1, 3]$ code over $\mathbb{F}_p$, where $n = (\frac{p-1}{2})\left(p^{m-1} + \epsilon\sqrt{p^*}^{m-1}\right)$. In particular, $\overline{C}_{D_{sq}}$ is a five-weight ternary self-orthogonal code over $\mathbb{F}_3$. The Hamming weights and weight distributions follow directly from Tables 6 and 7 when $s = 0$.*
- *The code $\overline{C}_{D_{nsq}}$ in Theorem 9 is a six-weight linear $[n, m+1]$code, and $\overline{C}^{\perp}_{D_{nsq}}$ is a dual $[n, n-m-1, 3]$ code over $\mathbb{F}_p$, where $n = (\frac{p-1}{2})(p^{m-1} - \epsilon\sqrt{p^*}^{m-1})$. In particular, $\overline{C}_{D_{nsq}}$ is a five-weight ternary self-orthogonal code over $\mathbb{F}_3$. The Hamming weights and weight distributions follow directly from Tables 10 and 11 when $s = 0$.*

6 Ternary LCD codes from self-orthogonal codes

In this section, we construct new families of ternary LCD codes from the constructed self-orthogonal codes.

Let $f : \mathbb{F}_{p^m} \to \mathbb{F}_p$. For $a \in \mathbb{F}_p$, define the following defining set

$$D_f = \{x \in \mathbb{F}_{p^m} : f(x) + a = 0\},$$

where $f \in \mathcal{WRP}$ or $f \in \mathcal{WRPB}$. Let $D_f = \{d_1, d_2, ..., d_n\}$. The generator matrix of the code $\overline{C}_{D_f}$ defined in (1) is given in the following lemma.

Lemma 18. *Let $\overline{C}_{D_f}$ be defined as in (1) and $\mathbb{F}^*_{p^m} = \langle\alpha\rangle$. Then, a generator matrix of $\overline{C}_{D_f}$ is given by*

$$G = \begin{bmatrix} \mathrm{Tr}_p^{p^m}(\alpha^0 d_1) & \mathrm{Tr}_p^{p^m}(\alpha^0 d_2) & \dots & \mathrm{Tr}_p^{p^m}(\alpha^0 d_n) \\ \mathrm{Tr}_p^{p^m}(\alpha^1 d_1) & \mathrm{Tr}_p^{p^m}(\alpha^1 d_2) & \dots & \mathrm{Tr}_p^{p^m}(\alpha^1 d_n) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \mathrm{Tr}_p^{p^m}(\alpha^{m-1} d_1) & \mathrm{Tr}_p^{p^m}(\alpha^{m-1} d_2) & \dots & \mathrm{Tr}_p^{p^m}(\alpha^{m-1} d_n) \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Proof. We know that $\{\alpha^0, \alpha^1, ..., \alpha^{m-1}\}$ form a basis of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$. Then, the generator matrix G follows from the definition of the augmented code $\overline{C}_{D_f}$. $\square$

Proposition 1. *Let $p = 3$ and $m + s \geq 6$ be even. Let $f \in \mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f. Let $a \in \mathbb{F}_3$ and $D_f = \{x \in \mathbb{F}_{3^m} : f(x) = a\}$. Let $\overline{C}_{D_f}$ be defined as in (1) and its generator matrix G is given in Lemma 18.*

- *If $a = 0$, then the matrix $\overline{G} = [I : G]$ generates a ternary LCD code $\mathcal{C}$ with parameters* $\left[\frac{1}{3}\left(3^m + 2\epsilon\sqrt{-3}^{m+s}\right) + m + 1, m + 1, d \geq 1 + \min\left\{2 \cdot 3^{m-2}, \frac{2}{3}\left(3^{m-1} + \epsilon(-3)^{\frac{m+s}{2}}\right)\right\}\right]$.
- *If $a \in \mathbb{F}_3^*$, then the matrix $\overline{G} = [I : G]$ generates a ternary LCD code $\mathcal{C}$ with parameters* $\left[3^{m-1} + \epsilon(-3)^{\frac{m+s-2}{2}} + m + 1, m + 1, d \geq 1 + min\{2 \cdot 3^{m-2}, 2 \cdot (3^{m-2} + \epsilon(-3)^{\frac{m+s-2}{2}})\}\right]$ *Besides, $\mathcal{C}^{\perp}$ is a ternary dual* $\left[3^{m-1} + \epsilon(-3)^{\frac{m+s-2}{2}} + m + 1, 3^{m-1} + \epsilon(-3)^{\frac{m+s-2}{2}}, 3\right]$ *LCD code which is at least almost optimal code according to the sphere-packing bound.*

Proof. The proof can proceed using the same argument given in the proof of [18, Theorem 7]. The desired conclusion follows from Theorems 1 and 5. ◻

Proposition 2. *Let $m + s \geq 6$ be even. Let $f \in \mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let $a \in \mathbb{F}_3$ and $D_f = \{x \in \mathbb{F}_{3^m} : f(x) = a\}$. Let $\overline{C}_{D_f}$ be defined as in (1) and its generator matrix G is given in Lemma 18. Then, the matrix $\overline{G} = [I : G]$ generates a ternary LCD code $\mathcal{C}$ with parameters* $\left[3^{m-1} + m + 1, m + 1, d \geq 1 + min\{2 \cdot 3^{m-2} + 2\epsilon(-3)^{\frac{m+s-4}{2}}, 2 \cdot (3^{m-2} - 2\epsilon(-3)^{\frac{m+s-4}{2}})\}\right]$

Proof. By considering Theorems 2 and 6, the proof can proceed using the same argument given in the proof of [18, Theorem 7]. ◻

Proposition 3. *Let $m+s \geq 5$ be odd. Let $f \in \mathcal{WRP}$ or $f \in \mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let $D_f = \{x \in \mathbb{F}_{3^m} : f(x) = 0\}$. Let $\overline{C}_{D_f}$ be defined as in (1) and its generator matrix G is given in Lemma 18. Then, the matrix $\overline{G} = [I : G]$ generates a ternary LCD code $\mathcal{C}$ with parameters*

$$\left[3^{m-1} + m + 1, m + 1, d \geq 1 \min\left\{\frac{2}{9}\left(3^m \pm \epsilon\sqrt{-3}^{m+s+1}\right)\right\}\right].$$

Proof. Because of Theorem 3, the proof can proceed using the same arguments given in the proof of [18, Theorem 7]. ◻

Proposition 4. *Let $m + s \geq 5$ be odd. Let $f \in \mathcal{WRP}$ and ϵ be the sign of the Walsh transform of f.*

- *Let D_{sq} be defined as in (4) and $\overline{C}_{D_{sq}}$ be defined as in (1). The generator matrix G of the code is given in Lemma 18. Then, the matrix $\overline{G} = [I : G]$ generates a ternary LCD code $\mathcal{C}$ with parameters* $\left[3^{m-1} + \epsilon\sqrt{-3}^{m+s-1} + m + 1, m + 1, d \geq 1 + \min\left\{2 \cdot 3^{m-2}, 2 \cdot 3^{m-2} - 4\epsilon\sqrt{-3}^{m+s-1}\right\}\right]$.
- *Let D_{nsq} be defined as in (4) and $\overline{C}_{D_{sq}}$ be defined as in (1). The generator matrix G of the code is given in Lemma 18. Then, the matrix $\overline{G} = [I : G]$ generates a*

ternary LCD code $\mathcal{C}$ with parameters
$$\left[3^{m-1}+\epsilon\sqrt{-3}^{m+s-1}+m+1, m+1, d \geq 1+\min\left\{2\cdot 3^{m-2}, 2\cdot 3^{m-2}+4\epsilon\sqrt{-3}^{m+s-1}\right\}\right].$$

Proof. In view of Theorems 7 and 9, the proof can proceed using the same argument given in the proof of [18, Theorem 7]. □

Proposition 5. *Let $m+s\geq 5$ be odd. Let $f\in \mathcal{WRPB}$ and ϵ be the sign of the Walsh transform of f. Let D_{sq} and D_{nsq} be defined as in (4). Let $\overline{C}_{D_f}$ be defined as in (1) and its generator matrix G is given in Lemma 18. Then, the matrix $\overline{G}=[I:G]$ generates a ternary LCD code $\mathcal{C}$ with parameters*

$$\left[3^{m-1}+m+1, m+1, d\geq 1+min\{2\cdot(3^{m-2}\pm\epsilon(-3)^{\frac{m+s-3}{2}})\}\right].$$

Proof. In view of Theorems 8 and 10, the proof can proceed using the same argument given in the proof of [18, Theorem 7]. □

7 Concluding Remarks

We generalize the recent construction method introduced by Heng et al. [18] for weakly regular plateaued unbalanced (resp. balanced) functions. We construct new families of linear codes with a few weights from weakly regular plateaued and bent functions over $\mathbb{F}_p$ for any odd prime p. The contributions of the paper are listed below.

- In Theorems 1, 2, 3, 5, 6, 7, 8, 9 and 10, we present new families of five-weight and six-weight linear codes derived from weakly regular plateaued unbalanced and balanced functions over $\mathbb{F}_p$ for any odd prime p. In particular, we provide new families of ternary five-weight and six-weight self-orthogonal codes from these functions over $\mathbb{F}_3$. Moreover, we introduce the parameters of the dual codes of the constructed codes.
- In Corollaries 1 and 2, we extend [18, Theorems 1 and 2] given for weakly regular bent ternary functions to weakly regular plateaued ternary functions.
- In Corollary 3, we present new families of six-weight p-ary linear codes derived from weakly regular bent functions based on the defining sets D_{sq} and D_{nsq}. Moreover, we present five-weight and six-weight ternary self-orthogonal codes derived from weakly regular ternary bent functions over $\mathbb{F}_3$.
- In Propositions 1, 2, 3, 4 and 5, we construct infinite families of ternary LCD codes from the constructed self-orthogonal codes. We observe that some constructed codes are at least almost optimal codes according to the sphere-packing bound.

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