Some New Constructions of Generalized Plateaued Functions†

Jiaxin Wang, Fang-Wei Fu

Abstract

Plateaued functions as an extension of bent functions play a significant role in cryptography, coding theory, sequences and combinatorics. In 2019, Hodžić *et al.* [14] designed Boolean plateaued functions in spectral domain and provided some construction methods in spectral domain. However, in their constructions, the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix of order $2^{n-s} \times n$, contains at least $n-s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. They proposed an open problem to provide constructions of Boolean s-plateaued functions in n variables whose Walsh support, when written as a matrix, contains strictly less than $n-s$ columns corresponding to affine functions. In this paper, we focus on the constructions of generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$, where V_n is an n-dimensional vector space over $\mathbb{F}_p$, p is a prime, $k \geq 1$ and $n+s$ is even when $p=2$. Firstly, inspired by the work of Hodžić *et al.*, we give a complete characterization of generalized plateaued functions with affine Walsh support in spectral domain and provide some construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support, when written as a matrix, can contain strictly less than $n-s$ columns corresponding to affine functions. When $p=2, k=1$, these constructions provide an answer to the open problem in [14]. Secondly, we provide a generalized indirect sum construction method of generalized plateaued functions, which can also be used to construct (non)-weakly regular generalized bent functions. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions can be obtained by the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the completed Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

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Index Terms

Plateaued functions; generalized plateaued functions; Walsh transform; bent functions; generalized bent functions; generalized indirect sum construction

I. Introduction

Boolean bent functions introduced by Rothaus [35] play an important role in cryptography, coding theory, sequences and combinatorics. In 1985, Kumar *et al*. [16] generalized Boolean bent functions to bent functions over finite fields of odd characteristic. Due to the importance of bent functions, they have been studied extensively. There is an exhaustive survey [6] and books [3], [21] for bent functions and generalized bent functions. Recently, generalized bent functions from V_n to $\mathbb{Z}_{2^k}$ have been generalized to generalized bent functions from V_n to $\mathbb{Z}_{p^k}$, where p is a prime [30]. For more characterizations and constructions of generalized bent functions from V_n to $\mathbb{Z}_{p^k}$, we refer to [11], [12], [18]–[20], [22], [30], [33], [36], [37].

In 1993, Carlet [5] introduced Boolean partially bent functions which is an extension of Boolean bent functions. As an extension of Boolean partially bent functions, Zheng and Zhang [38] introduced Boolean plateaued functions. Surveys on Boolean plateaued functions can be found in [2], [3], [21]. The notion of Boolean partially bent functions and Boolean plateaued functions have been generalized to p-ary partially bent functions and p-ary plateaued functions for any odd prime p (see [7], [8]). Then they have been studied in [7], [8], [15], [24], [25], [32]. In [15], Hyun *et al*. searched for explicit criteria for constructing p-ary plateaued functions. More specifically, for p-ary s-plateaued functions, they derived an explicit form for the Walsh transform, obtained an upper bound on the degree and provided explicit criteria for the existence. In [24], [25], Mesnager *et al*. presented characterizations of p-ary plateaued functions in terms of the second-order derivatives and the moments of Walsh transform, which allow us a better understanding of the structure of p-ary plateaued functions. Apart from the desirable cryptographic properties, plateaued functions play a significant role in coding theory, sequences and combinatorics (see e.g. [1], [23], [27], [28], [31]). In [29], Mesnager *et al*. introduced generalized plateaued functions, which is an extension of plateaued functions. As far as we know, there are only a few papers on generalized plateaued functions [26], [29], [34] up to now. We review the main contributions for generalized plateaued functions given in these papers. In [29], first of all, the authors gave an explicit form for the Walsh transform of generalized plateaued functions. They then investigated the relations between generalized plateaued functions and plateaued functions by the decomposition of generalized plateaued functions. In particular, they used admissible plateaued functions to characterize generalized plateaued functions by means of their components. Finally, they provided for the first time two constructions of generalized Boolean plateaued functions. In [34], for generalized Boolean plateaued functions, the authors provided two constructions and characterized them in terms of the second-order derivatives and the fourth moment of Walsh transform. In [26], a special class of generalized plateaued functions called $\mathbb{Z}_{2^k}$-plateaued functions was studied in terms of so called (c, s)-plateaued functions. In particular, the authors gave characterizations of $(2^t, s)$-plateaued functions in terms of the second-order derivatives and the fourth moment of Walsh transform, which generalize the results given in [34]. And they pointed out that even though the paper [26] only stated the results for characteristic 2, similar results can be obtained for odd characteristic. For generalized p-ary plateaued functions, the constructions in [29], [34] are for $p = 2$ and there are lacks of constructions for any prime p. The main contribution of this paper (which will be introduced below) is to provide some constructions of generalized p-ary plateaued functions for any prime p.

Recently, Hodžić *et al.* [14] designed Boolean plateaued functions in spectral domain. Designing plateaued functions in spectral domain is based on the fact that any function and its Walsh transform are mutually determined. In this paper, we focus on the constructions of generalized s-plateaued functions from V_n to $\mathbb{Z}_{p^k}$, where V_n is an n-dimensional vector space over $\mathbb{F}_p$, p is a prime, $k \geq 1$ and $n + s$ is even when $p = 2$. Firstly, inspired by the work of Hodžić *et al.*, we give a complete characterization of generalized plateaued functions with affine Walsh support and provide some construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain. As pointed out in [14], for the constructions in spectral domain given in [14], the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix, contains at least $n - s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. And they proposed an open problem (Open Problem 2) to provide constructions of Boolean s-plateaued functions in n variables whose Walsh support, when written as a matrix, contains strictly less than $n-s$ columns corresponding to affine functions. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support, when written as a matrix, can contain strictly less than $n - s$ columns corresponding to affine functions. When $p = 2, k = 1$, these constructions provide an answer to Open Problem 2 in [14]. Secondly, we provide a generalized indirect sum construction method of generalized plateaued functions, which can also be used to construct (non)-weakly regular generalized bent functions. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions can be obtained by the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the completed Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

The rest of the paper is organized as follows. In Section II, we introduce the needed definitions and results related to generalized plateaued functions. In Section III-A, we give a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain. In Section III-B, we give a complete characterization of generalized plateaued functions whose Walsh support is an affine subspace. In Section III-C, we provide some construction methods of generalized plateaued functions with (non)-affine Walsh support. In Section IV, we give a generalized indirect sum construction method of generalized plateaued functions and based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions. In Section V, we make a conclusion.

II. Preliminaries

For any complex number $z = a+b\sqrt{-1}$, let $|z| = \sqrt{a^2+b^2}$. For any finite set S, let $|S|$ denote the size of S. Throughout this paper, let $\mathbb{Z}_{p^k}$ be the ring of integers modulo p^k, $\zeta_{p^k} = e^{\frac{2\pi\sqrt{-1}}{p^k}}$ be the complex primitive p^k-th root of unity, $\mathbb{F}_p^n$ be the vector space of the n-tuples over $\mathbb{F}_p$, $\mathbb{F}_{p^n}$ be the finite field with p^n elements and V_n be an n-dimensional vector space over $\mathbb{F}_p$, where p is a prime and k, n are positive integers. The classical representations of V_n are $\mathbb{F}_p^n$ and $\mathbb{F}_{p^n}$. For $a, b \in V_n$, let $\langle a, b\rangle$ denote a (nondegenerate) inner product in V_n. When $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{F}_p^n$, let $\langle a, b\rangle = a \cdot b = \sum_{i=1}^{n} a_i b_i$. When $a, b \in \mathbb{F}_{p^n}$, let $\langle a, b\rangle = Tr_1^n(ab)$, where $Tr_1^n(\cdot)$ is the absolute trace function. When $V_n = V_{n_1} \times \cdots \times V_{n_s} (n = \sum_{i=1}^{s} n_i)$, let $\langle a, b\rangle = \sum_{i=1}^{s} \langle a_i, b_i\rangle$, where $a = (a_1, \ldots, a_s), b = (b_1, \ldots, b_s) \in V_n$. Let $GL(n, \mathbb{F}_p)$ denote the group formed by all invertible matrices over $\mathbb{F}_p$ of size $n \times n$.

A function f from V_n to $\mathbb{Z}_{p^k}$ is called a generalized p-ary function, or simply p-ary function when $k=1$. A p-ary function $L: V_n \rightarrow \mathbb{F}_p$ is called a linear function if $L(ax+by) = aL(x)+bL(y)$ for any $a,b \in \mathbb{F}_p$ and $x,y \in V_n$. All linear functions from V_n to $\mathbb{F}_p$ form an n-dimensional linear space $\mathcal{L}_n$ and $\{\langle \alpha_i, x\rangle, 1 \leq i \leq n\}$ is a basis of $\mathcal{L}_n$, where $\{\alpha_i, 1 \leq i \leq n\}$ is a basis of V_n. If p-ary function $A: V_n \rightarrow \mathbb{F}_p$ is the sum of a linear function and a constant, then A is called an affine function.

The Walsh transform of a generalized p-ary function $f: V_n \rightarrow \mathbb{Z}_{p^k}$ is the function W_f from V_n to $\mathbb{Z}[\zeta_{p^k}]$ ($\mathbb{Z}[\zeta_{p^k}]$ is the ring of integers in cyclotomic field $\mathbb{Q}(\zeta_{p^k})$):

$$W_f(a) = \sum_{x \in V_n} \zeta_{p^k}^{f(x)} \zeta_p^{-\langle a,x\rangle},\ a \in V_n. \tag{1}$$

The generalized p-ary function f can be recovered by the inverse transform

$$\zeta_{p^k}^{f(x)} = \frac{1}{p^n} \sum_{a \in V_n} W_f(a) \zeta_p^{\langle a,x\rangle},\ x \in V_n. \tag{2}$$

The multiset $\{W_f(a), a \in V_n\}$ is called the Walsh spectrum of f. The set $S_f = \{a \in V_n : W_f(a) \neq 0\}$ is called the Walsh support of f. Functions $f_1, \ldots, f_m$ are called pairwise disjoint spectra functions if $S_{f_i} \cap S_{f_j} = \emptyset$ for any $i \neq j$.

A generalized p-ary function $f: V_n \rightarrow \mathbb{Z}_{p^k}$ is called a generalized p-ary s-plateaued function, or simply p-ary s-plateaued function when $k=1$ if $|W_f(a)| = p^{\frac{n+s}{2}}$ or 0 for any $a \in V_n$. If $s=0$, the generalized p-ary 0-plateaued function f is just the generalized p-ary bent function and $S_f = V_n$. When $p=2, k=1$, if $f: V_n \rightarrow \mathbb{Z}_{p^k}$ is an s-plateaued function, then $n+s$ is even.

For generalized s-plateaued functions $f: V_n \rightarrow \mathbb{Z}_{p^k}$, there is a basic property: $|S_f| = p^{n-s}$, which is obtained by Parseval identity $\sum_{x \in V_n} |W_f(x)|^2 = p^{2n}$. In [29], Mesnager *et al.* have shown that the Walsh transform of a generalized p-ary s-plateaued function $f: V_n \rightarrow \mathbb{Z}_{p^k}$ satisfies that for any $a \in S_f$, when $p=2$ and $n+s$ is even, $W_f(a) = 2^{\frac{n+s}{2}} \zeta_{2^k}^{f^*(a)}$, and when p is an odd prime,

$$W_f(a) = \begin{cases} \pm p^{\frac{n+s}{2}} \zeta_{p^k}^{f^*(a)} & \text{if } n+s \text{ is even or } p \equiv 1 (mod\ 4), \\ \pm \sqrt{-1} p^{\frac{n+s}{2}} \zeta_{p^k}^{f^*(a)} & \text{if } n+s \text{ is odd and } p \equiv 3 (mod\ 4), \end{cases}$$

where f^* is a function from S_f to $\mathbb{Z}_{p^k}$. We call f^* the dual function of f.

In the sequel, if $f : V_n \rightarrow \mathbb{Z}_{p^k}$ is a generalized s-plateaued function with dual function f^*, define function μ_f as

$$\mu_f(a) = p^{-\frac{n+s}{2}} \zeta_{p^k}^{-f^*(a)} W_f(a), a \in S_f. \tag{3}$$

If $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$ and $n+s$ is even, then μ_f is a function from S_f to $\{\pm 1\}$. If $p \equiv 3(mod\ 4)$ and $n+s$ is odd, then μ_f is a function from S_f to $\{\pm\sqrt{-1}\}$. If $p = 2$ and $n+s$ is even, then $\mu_f(x) = 1, x \in S_f$. For a generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$, that is, generalized 0-plateaued function, if μ_f is a constant function, then f is called weakly regular, otherwise f is called non-weakly regular. In particular, if $\mu_f(x) = 1, x \in V_n$, f is called regular. In [23], Mesnager *et al*. introduced the notion of (non)-weakly regular plateaued functions. For an s-plateaued function $f : V_n \rightarrow \mathbb{F}_p$, if μ_f is a constant function, then f is called weakly regular, otherwise f is called non-weakly regular. In particular, if $\mu_f(x) = 1, x \in S_f$, f is called regular.

If $f : V_n \rightarrow \mathbb{Z}_{p^k}$ is a generalized n-plateaued function, then $|S_f| = 1$ and it is easy to obtain $f(x) = p^{k-1}\langle a, x\rangle + b$ for some $a \in V_n, b \in \mathbb{Z}_{p^k}$ by the inverse transform (2). In this paper, we study generalized s-plateaued functions $f : V_n \rightarrow \mathbb{Z}_{p^k}$, where $0 \leq s < n$, p is prime, $k \geq 1$ and $n+s$ is even when $p = 2$.

III. Constructing generalized plateaued functions in spectral domain

In this section, we provide some construction methods of generalized s-plateaued functions in spectral domain, where $s \geq 1$.

To this end, we fix some notation unless otherwise stated. Let m be an arbitrary positive integer. Define the notation of lexicographic order $\prec$: $a \prec b$ if $\sum_{i=1}^{m} p^{m-i} a_i < \sum_{i=1}^{m} p^{m-i} b_i$, where $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{F}_p^m$. Define

$$v_i = \sum_{j=1}^{m} v_{i,j}\alpha_j, 0 \leq i \leq p^m - 1, \tag{4}$$

where $\{\alpha_1, \ldots, \alpha_m\}$ is some fixed basis of V_m over $\mathbb{F}_p$ and $\{(v_{0,1}, \ldots, v_{0,m}), \ldots, (v_{p^m-1,1}, \ldots, v_{p^m-1,m})\}$ is the lexicographic order of $\mathbb{F}_p^m$. When $V_m = \mathbb{F}_p^m$, we let $\alpha_1 = (1, 0, \ldots, 0, 0) \in \mathbb{F}_p^m, \ldots, \alpha_m = (0, 0, \ldots, 0, 1) \in \mathbb{F}_p^m$, that is, $\{v_0, \ldots, v_{p^m-1}\}$ denotes the lexicographic order of $\mathbb{F}_p^m$. For a p-ary function $f : V_m \rightarrow \mathbb{F}_p$, define its true table

$$T_f = \left(f(v_0), \ldots, f(v_{p^m-1})\right)^T, \tag{5}$$

where M^T denotes the transpose of matrix M. Let δ be the Kronecker delta function, that is,

$$\delta(i,j)=\begin{cases}1 & if\ i=j,\\ 0 & if\ i\neq j.\end{cases}$$

A. A Necessary and Sufficient Condition

In this subsection, inspired by [14] and based on the explicit form for the Walsh transform of generalized plateaued functions given in [29], we provide a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain.

Suppose $S\subseteq\mathbb{F}_p^n$ with size p^m is ordered as $S=\{w_0,w_1,\dots,w_{p^m-1}\}$. For any $a\in\mathbb{F}_p^n$, define ψ_a from V_m to $\mathbb{F}_p$:

$$\psi_a(v_i)=a\cdot w_i, 0\leq i\leq p^m-1, \tag{6}$$

where v_i is defined by (4).

Under notation as above we have the following proposition:

Proposition 1. *Let p be a prime. Let $n,k,s(<n)$ be positive integers and $n+s$ be even for $p=2$. Let S be a subset of $\mathbb{F}_p^n$ with size p^{n-s} and be ordered as $S=\{w_0,w_1,\dots,w_{p^{n-s}-1}\}$. Let d be a function from V_{n-s} to $\mathbb{Z}_{p^k}$. Let μ be a function from V_{n-s} to $\{\pm1\}$ if $p\equiv1(mod\ 4)$ or $p\equiv3(mod\ 4)$ and $n+s$ is even, μ be a function from V_{n-s} to $\{\pm\sqrt{-1}\}$ if $p\equiv3(mod\ 4)$ and $n+s$ is odd and $\mu(x)=1,x\in V_{n-s}$ if $p=2$ and $n+s$ is even. Define function $W:\mathbb{F}_p^n\rightarrow\mathbb{Z}[\zeta_{p^k}]$ as*

$$W(a)=p^{\frac{n+s}{2}}\sum_{i=0}^{p^{n-s}-1}\delta(a,w_i)\mu(v_i)\zeta_{p^k}^{d(v_i)}. \tag{7}$$

Then $W:\mathbb{F}_p^n\rightarrow\mathbb{Z}[\zeta_{p^k}]$ is the Walsh transform of a generalized s-plateaued function $f:\mathbb{F}_p^n\rightarrow\mathbb{Z}_{p^k}$ if and only if $(p^{\frac{s-n}{2}}\sum_{x\in V_{n-s}}\mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k}=1$ for any $a\in\mathbb{F}_p^n$, where ψ_a is defined by (6).

Proof: First by the well-known fact that $\sqrt{p}\in\mathbb{Z}[\zeta_p]$ if $p\equiv1(mod\ 4)$ and $\sqrt{-1}\sqrt{p}\in\mathbb{Z}[\zeta_p]$ if $p\equiv3(mod\ 4)$, it is easy to see that the function W defined by (7) is a function from $\mathbb{F}_p^n$ to $\mathbb{Z}[\zeta_{p^k}]$.

If $W:\mathbb{F}_p^n\rightarrow\mathbb{Z}[\zeta_{p^k}]$ is the Walsh transform of a generalized s-plateaued function $f:\mathbb{F}_p^n\rightarrow\mathbb{Z}_{p^k}$, by the inverse transform (2) we have

$$\begin{aligned}\zeta_{p^k}^{f(a)} &= \frac{1}{p^n}\sum_{i=0}^{p^{n-s}-1} p^{\frac{n+s}{2}}\mu(v_i)\zeta_{p^k}^{d(v_i)}\zeta_p^{a\cdot w_i}\\ &= p^{\frac{s-n}{2}}\sum_{i=0}^{p^{n-s}-1}\mu(v_i)\zeta_{p^k}^{d(v_i)+p^{k-1}\psi_a(v_i)}\\ &= p^{\frac{s-n}{2}}\sum_{x\in V_{n-s}}\mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)},\end{aligned}$$

hence $(p^{\frac{s-n}{2}}\sum_{x\in V_{n-s}}\mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k}=1$ for any $a\in\mathbb{F}_p^n$.

Conversely, suppose $(p^{\frac{s-n}{2}}\sum_{x\in V_{n-s}}\mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k}=1$ for any $a\in\mathbb{F}_p^n$. Then there is a unique generalized function $f:\mathbb{F}_p^n\to\mathbb{Z}_{p^k}$ such that $p^{\frac{s-n}{2}}\sum_{x\in V_{n-s}}\mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)}=\zeta_{p^k}^{f(a)}$. The function W is the Walsh transform of f. Indeed,

$$\begin{aligned}W_f(a) &= \sum_{x\in\mathbb{F}_p^n}(p^{\frac{s-n}{2}}\sum_{y\in V_{n-s}}\mu(y)\zeta_{p^k}^{d(y)+p^{k-1}\psi_x(y)})\zeta_p^{-a\cdot x}\\ &= p^{\frac{s-n}{2}}\sum_{x\in\mathbb{F}_p^n}\sum_{i=0}^{p^{n-s}-1}\mu(v_i)\zeta_{p^k}^{d(v_i)+p^{k-1}x\cdot w_i}\zeta_p^{-a\cdot x}\\ &= p^{\frac{s-n}{2}}\sum_{i=0}^{p^{n-s}-1}\mu(v_i)\zeta_{p^k}^{d(v_i)}\sum_{x\in\mathbb{F}_p^n}\zeta_p^{(w_i-a)\cdot x}.\end{aligned}$$

If $a\notin S=\{w_0,w_1,\ldots,w_{p^{n-s}-1}\}$, then $W_f(a)=0$. If $a=w_i$ for some $0\le i\le p^{n-s}-1$, then $W_f(a)=p^{\frac{n+s}{2}}\mu(v_i)\zeta_{p^k}^{d(v_i)}$. Hence, $W_f(a)=W(a)$ for any $a\in\mathbb{F}_p^n$ and $S_f=S$, $|W_f(a)|=p^{\frac{n+s}{2}}$ for any $a\in S_f$, that is, W is the Walsh transform of f and f is a generalized s-plateaued function. ∎

Remark 1. *Proposition 1 provides a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain. If the condition of Proposition 1 is satisfied, then one can obtain function f by the inverse transform (2).*

Let W_K denote the group of roots of unity of cyclotomic field $K=\mathbb{Q}(\zeta_{p^k})$, then $W_K=\{\zeta_{2^k}^i:0\le i\le 2^k-1\}$ if $p=2$ and $W_K=\{\pm\zeta_{p^k}^i:0\le i\le p^k-1\}$ if p is an odd prime. Let $p^*=\left(\frac{-1}{p}\right)p$ if p is an odd prime, where $\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}$ denotes the Legendre symbol and $p^*=2$ if $p=2$. By the knowledge on cyclotomic field $\mathbb{Q}(\zeta_{p^k})$ (see Lemma 24 of [29]), $\frac{\alpha}{\sqrt{p^{*m}}}\in W_K$ if $\alpha\in\mathbb{Z}[\zeta_{p^k}]$ with $|\alpha|=p^{\frac{m}{2}}$, where m is a positive integer and m is even if

$p = 2$. Then it is easy to verify that the necessary and sufficient condition in Proposition 1 can be written in the following form.

Proposition 2. *With the same notation as in Proposition 1.*

(1) When $p = 2$ and $n + s$ is even, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ if and only if $|\sum_{x \in V_{n-s}} \zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)}| = p^{\frac{n-s}{2}}$ for any $a \in \mathbb{F}_p^n$.

(2) When p is an odd prime, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ if and only if $|\sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)}| = p^{\frac{n-s}{2}}$ and $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} \neq -1$ for any $a \in \mathbb{F}_p^n$.

By Proposition 1, we obtain the following corollary.

Corollary 1. *With the same notation as in Proposition 1. For any $a \in \mathbb{F}_p^n$, define $g_a(x) = d(x) + p^{k-1}\psi_a(x),\ x \in V_{n-s}$. If for any $a \in \mathbb{F}_p^n$, $g_a : V_{n-s} \to \mathbb{Z}_{p^k}$ is a generalized bent function and there exists a constant u independent of a such that $\mu_{g_a}(x) = u, x \in V_{n-s}$, where μ_{g_a} is defined by (3), let $\mu(x) = u^{-1}, x \in V_{n-s}$. Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$. Furthermore, $f(a) = g_a^*(0), a \in \mathbb{F}_p^n$, where g_a^* is the dual function of g_a.*

Proof: First it is easy to see that the function μ satisfy the condition of Proposition 1. For any $a \in \mathbb{F}_p^n$,

$$\sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} = u^{-1}W_{g_a}(0) = u^{-1} \cdot up^{\frac{n-s}{2}}\zeta_{p^k}^{g_a^*(0)} = p^{\frac{n-s}{2}}\zeta_{p^k}^{g_a^*(0)},$$

where g_a^* is the dual function of g_a. So $(p^{\frac{s-n}{2}} \sum_{x \in V_{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$. Hence by Proposition 1 and its proof, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ and $f(a) = g_a^*(0)$ for any $a \in \mathbb{F}_p^n$. ■

B. Characterization of Generalized Plateaued Functions with Affine Walsh Support in Spectral Domain

In this subsection, we give a complete characterization of generalized plateaued functions whose Walsh support is an affine subspace in spectral domain, which generalizes the case of Boolean plateaued functions [14].

First we need a lemma, which is a generalization of the results in the proof of Lemma 3.1 of [13].

Lemma 1. *Let p be a prime. Suppose $E \subseteq \mathbb{F}_p^n$ is an m-dimensional linear subspace over $\mathbb{F}_p$ and $E = \{e_0, e_1, \ldots, e_{p^m-1}\}$ is the lexicographic order of E. Then $\{e_{p^0}, e_{p^1}, \ldots, e_{p^{m-1}}\}$ is a basis of E and $e_i = v_i R$ for any $0 \le i \le p^m - 1$, where R is the matrix whose row vectors are $e_{p^{m-1}}, e_{p^{m-2}}, \ldots, e_{p^0}$ and $\{v_0, \ldots, v_{p^m-1}\}$ is the lexicographic order of $\mathbb{F}_p^m$.*

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis of E over $\mathbb{F}_p$. For the matrix whose row vectors are $\alpha_1, \alpha_2, \ldots, \alpha_m$, by using elementary row operations, we can get the row echelon matrix

$$R = \begin{pmatrix} 0 & \ldots & 0 & 1 & * & \ldots & * & 0 & * & \ldots & * & \ldots & 0 & * & \ldots & * & 0 & * & \ldots & * \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & * & \ldots & * & \ldots & 0 & * & \ldots & * & 0 & * & \ldots & * \\ & & & & & & & & & \ldots\ldots & & & & & & & & & & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & * & \ldots & * & 0 & * & \ldots & * \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & * & \ldots & * \end{pmatrix},$$

where $*$ denotes some elements in $\mathbb{F}_p$, the first nonzero element in each row is one from left to right and these ones belong to different columns and the other elements in the same column are zero. Furthermore, if the first nonzero element of i-th row is in the k_i-th column ($0 \le i \le m-1$), then $0 \le k_0 < \cdots < k_{m-1} \le n-1$.

Let $e^{(m-1)}, e^{(m-2)}, \ldots, e^{(1)}, e^{(0)}$ denote the row vectors of R. Let $(i_0, \ldots, i_{m-1}), (i'_0, \ldots, i'_{m-1}) \in \mathbb{F}_p^m$ with $(i_0, \ldots, i_{m-1}) \prec (i'_0, \ldots, i'_{m-1})$, that is, there exists $0 \le j_0 \le m-1$ such that $i_j = i'_j$ for any $j < j_0$ and $i_{j_0} < i'_{j_0}$. Let $s = (s_0, \ldots, s_{n-1}) = \sum_{j=0}^{m-1} i_j e^{(m-1-j)}$, $s' = (s'_0, \ldots, s'_{n-1}) = \sum_{j=0}^{m-1} i'_j e^{(m-1-j)}$. By the properties of $e^{(i)} (0 \le i \le m-1)$, one can get $s_j = s'_j$ for any $j < k_{j_0}$ and $s_{k_{j_0}} < s'_{k_{j_0}}$, that is, $s \prec s'$. Hence, the lexicographic order of $(i_0, \ldots, i_{m-1}) \in \mathbb{F}_p^m$ determines the lexicographic order of $\sum_{j=0}^{m-1} i_j e^{(m-1-j)}$. So for $i = \sum_{j=0}^{m-1} i_j p^{m-1-j}$, we have $e_i = \sum_{j=0}^{m-1} i_j e^{(m-1-j)}$, where $\{e_0, \ldots, e_{p^m-1}\}$ is the lexicographic order of E. For any $0 \le j \le m-1$, let $i = p^j$, then $e_i = e^{(j)}$. ■

The following theorem gives a complete characterization of generalized plateaued functions with affine Walsh support in spectral domain.

Theorem 1. *With the same notation as in Proposition 1. Let ordered $S = \{w_0, w_1, \ldots, w_{p^{n-s}-1}\}$, where $w_i = t + e_i M$ for any $0 \le i \le p^{n-s} - 1$, $t \in \mathbb{F}_p^n$, $M \in GL(n, \mathbb{F}_p)$ and $\{e_0, e_1, \ldots, e_{p^{n-s}-1}\}$*

is the lexicographic order of an $(n-s)$-dimensional linear subspace $E \subseteq \mathbb{F}_p^n$. Let d be a function from $\mathbb{F}_p^{n-s}$ to $\mathbb{Z}_{p^k}$. Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ if and only if d is the dual function of some generalized bent function g and $\mu = \mu_g$, where μ_g is defined by (3). Furthermore, if d is the dual function of some generalized bent function g and $\mu = \mu_g$, then $f(x) = g(xM^TR^T) + p^{k-1}x \cdot t$, $x \in \mathbb{F}_p^n$, where R is the matrix whose row vectors are $e_{p^{n-s-1}}, e_{p^{n-s-2}}, \ldots, e_{p^0}$.

Proof: Since E is a linear subspace, then by Lemma 1, for any $a \in \mathbb{F}_p^n$ and $0 \leq i \leq p^{n-s}-1$, we have $\psi_a(v_i) = a \cdot w_i = a \cdot (t + e_iM) = a \cdot t + aM^TR^T \cdot v_i$.

If d is the dual function of some generalized bent function g and $\mu = \mu_g$, then we have

$$\begin{aligned}\sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} &= \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)}\zeta_p^{a\cdot t+aM^TR^T\cdot x} \\ &= \zeta_p^{a\cdot t}p^{\frac{n-s}{2}}\zeta_{p^k}^{g(aM^TR^T)},\end{aligned}$$

where the second equation is obtained by the inverse transform. So for any $a \in \mathbb{F}_p^n$, $(p^{\frac{s-n}{2}} \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)})^{p^k} = 1$. By Proposition 1 and its proof, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ and $f(x) = g(xM^TR^T) + p^{k-1}x \cdot t, x \in \mathbb{F}_p^n$.

Conversely, if the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$, by the proof of Proposition 1 we have

$$p^{\frac{s-n}{2}} \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)+p^{k-1}\psi_a(x)} = \zeta_{p^k}^{f(a)}.$$

Then

$$p^{\frac{s-n}{2}} \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)}\zeta_p^{aM^TR^T\cdot x} = \zeta_{p^k}^{f(a)-p^{k-1}a\cdot t}. \tag{8}$$

For any $y \in \mathbb{F}_p^{n-s}$, since R is row full rank and M is invertible, there exists $a_y \in \mathbb{F}_p^n$ such that $a_yM^TR^T = y$. When $a_yM^TR^T = b_yM^TR^T = y$, by (8) we have $f(a_y) - p^{k-1}a_y \cdot t = f(b_y) - p^{k-1}b_y \cdot t$. Define $g : \mathbb{F}_p^{n-s} \to \mathbb{Z}_{p^k}$ as

$$g(y) = f(a_y) - p^{k-1}a_y \cdot t,\ y \in \mathbb{F}_p^{n-s},$$

where $a_y \in \mathbb{F}_p^n$ satisfies $a_yM^TR^T = y$. Then for any $b \in \mathbb{F}_p^{n-s}$, by Equation (8),

$$W_g(b) = \sum_{y\in\mathbb{F}_p^{n-s}} (p^{\frac{s-n}{2}} \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)}\zeta_p^{a_yM^TR^T\cdot x})\zeta_p^{-b\cdot y}$$

$$= p^{\frac{s-n}{2}} \sum_{x\in\mathbb{F}_p^{n-s}} \mu(x)\zeta_{p^k}^{d(x)} \sum_{y\in\mathbb{F}_p^{n-s}} \zeta_p^{y\cdot(x-b)}$$

$$= p^{\frac{n-s}{2}} \mu(b)\zeta_{p^k}^{d(b)},$$

that is, g is a generalized bent function and d is the dual function of g and $\mu_g = \mu$. ■

Remark 2. *It is known that plateaued functions with affine Walsh support correspond to partially bent functions. A function $f : V_n \to \mathbb{F}_p$ is called a partially bent function if for any $a \in V_n$, $f(x+a)-f(x), x \in V_n$ is either balanced or constant. When $k = 1$, Theorem 1 gives a complete characterization of p-ary partially bent functions for any prime p, which generalizes the case of Boolean partially bent functions [14]. Further, we give the explicit formula for partially bent functions.*

We give two examples of generalized plateaued functions with affine Walsh support by using Theorem 1.

Example 1. *Let $p = 3, k = 1, n = 4, s = 1$. Let $d : \mathbb{F}_3^3 \to \mathbb{F}_3$ be defined as $d(x_1, x_2, x_3) = x_1x_3 + 2x_2^2+2x_3^2$, then d is the dual function of weakly regular bent function $g(x_1, x_2, x_3) = 2x_1^2+2x_1x_3+x_2^2$ with $\mu_g(x_1, x_2, x_3) = \sqrt{-1}, (x_1, x_2, x_3) \in \mathbb{F}_3^3$. Let $\mu(x_1, x_2, x_3) = \sqrt{-1}, (x_1, x_2, x_3) \in \mathbb{F}_3^3$. Let $S = \{w_0, \ldots, w_{26}\}$, where $w_i = (2, 0, 0, 0) + e_iM$, $E = \{e_0, \ldots, e_{26}\} =< (0, 0, 1, 1), (0, 1, 0, 0), (1, 0, 0, 0) >$, $M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$. Then the constructed weakly regular 1-plateaued function $f : \mathbb{F}_3^4 \to \mathbb{F}_3$ with S as Walsh support by Theorem 1 is $f(x_1, x_2, x_3, x_4) = g(x_3 + x_4, x_2, x_1) + 2x_1 = 2x_1x_3 + 2x_1x_4 + x_2^2 + 2x_3^2 + x_3x_4 + 2x_4^2 + 2x_1$.*

Example 2. *Let $p = 2, k = 3, n = 4, s = 2$. Let $d : \mathbb{F}_2^2 \to \mathbb{Z}_8$ be defined as $d(x_1, x_2) = 4x_1x_2+x_2$, then d is the dual function of generalized bent function $g(x_1, x_2) = 4x_1x_2 + x_1$ with $\mu_g = 1$. Let $\mu(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_2^2$. Let $S = \{w_0, \ldots, w_3\}$, where $w_i = (0, 1, 1, 0) + e_i$ and $E = \{e_0, \ldots, e_3\} = < (0, 0, 1, 1), (1, 1, 0, 1) >$. Then the constructed generalized 2-plateaued function $f : \mathbb{F}_2^4 \to \mathbb{Z}_8$ with S as Walsh support by Theorem 1 is $f(x_1, x_2, x_3, x_4) = g(x_1 + x_2 + x_4, x_3+x_4)+4(x_2+x_3) = ((x_1+x_2+x_4) \bmod 2)+4(x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4+x_2+x_3+x_4)$.*

One can construct pairwise disjoint spectra generalized p-ary s-plateaued functions $f_0, f_1, \ldots,$

f_{p^s-1} as follows. Let $n, s(< n)$ be positive integers and $n + s$ be even for $p = 2$. Let E and E' be $(n - s)$-dimensional and s-dimensional linear subspaces of $\mathbb{F}_p^n$ respectively and satisfy $E \oplus E' = \mathbb{F}_p^n$, where $\oplus$ denotes direct sum. Note that this can be easily done, for example, let $E =< \alpha_1, \ldots, \alpha_{n-s} >$ and $E' =< \alpha_{n-s+1}, \ldots, \alpha_n >$, where $\{\alpha_1, \ldots, \alpha_n\}$ is some basis of $\mathbb{F}_p^n$. Suppose $E' = \{e'_0, \ldots, e'_{p^s-1}\}$. Let $S_i = e'_i + E$, $0 \leq i \leq p^s - 1$. Then $S_i \cap S_j = \emptyset$ if $i \neq j$ and one can construct generalized p-ary s-plateaued functions $f_i (0 \leq i \leq p^s - 1)$ with S_i as Walsh support by using Theorem 1 and some known generalized bent functions as building blocks. By using pairwise disjoint spectra generalized plateaued functions as building blocks, one can get the following construction method of generalized bent functions which is an extension of Theorem 2 of [8].

Proposition 3. *Let p be a prime. Let $n, s(\leq n), k$ be positive integers and $n + s$ be even for $p = 2, k = 1$. Let $f_y(y \in \mathbb{F}_p^s) : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$ be pairwise disjoint spectra generalized s-plateaued functions. Let W and U be n-dimensional and s-dimensional linear subspaces of $\mathbb{F}_p^{n+s}$ respectively and satisfy $\mathbb{F}_p^{n+s} = W \oplus U$. Define*

$$F(xM + \pi(y)) = f_y(x), x \in \mathbb{F}_p^n, y \in \mathbb{F}_p^s,$$

where M is a matrix whose row vectors form a basis of W and π is a bijection from $\mathbb{F}_p^s$ to U. Then F is a generalized bent function from $\mathbb{F}_p^{n+s}$ to $\mathbb{Z}_{p^k}$.

Proof: First it is easy to see that F is a function from $\mathbb{F}_p^{n+s}$ to $\mathbb{Z}_{p^k}$. For any $a \in \mathbb{F}_p^{n+s}$,

$$\begin{aligned} W_F(a) &= \sum_{x \in \mathbb{F}_p^n} \sum_{y \in \mathbb{F}_p^s} \zeta_{p^k}^{f_y(x)} \zeta_p^{-a \cdot (xM + \pi(y))} \\ &= \sum_{y \in \mathbb{F}_p^s} \zeta_p^{-a \cdot \pi(y)} W_{f_y}(aM^T). \end{aligned}$$

Since $f_y, y \in \mathbb{F}_p^s$ are pairwise disjoint spectra generalized s-plateaued functions, we have $|S_{f_y}| = p^{n-s}$ and $S_{f_y} \cap S_{f_{y'}} = \emptyset$ for any $y \neq y'$, which yields that $S_{f_y}, y \in \mathbb{F}_p^s$ is a partition of $\mathbb{F}_p^n$. Hence for any $a \in \mathbb{F}_p^{n+s}$, there exists a unique $y_a \in \mathbb{F}_p^s$ such that $aM^T \in S_{f_{y_a}}$ and $|W_F(a)| = |\zeta_p^{-a \cdot \pi(y_a)} W_{f_{y_a}}(aM^T)| = p^{\frac{n+s}{2}}$, that is, F is a generalized bent function. ∎

When $k = 1$, $W = \mathbb{F}_p^n \times \{0_s\}$, $U = \{0_n\} \times \mathbb{F}_p^s$, M is the matrix whose row vectors are $(1, 0, \ldots, 0, 0, \ldots, 0), (0, 1, \ldots, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1, 0, \ldots, 0)$ and $\pi(y) = (0_n, y), y \in \mathbb{F}_p^s$, where 0_n denotes the zero vector of $\mathbb{F}_p^n$, Proposition 3 reduces to Theorem 2 of [8]. We give an example to illustrate Proposition 3.

Example 3. *Let $p=2, n=5, s=1, k=3$. Let $f_0, f_1 : \mathbb{F}_2^5 \rightarrow \mathbb{Z}_{2^3}$ be defined as $f_0(x_1,\ldots,x_5) = 4(x_1x_3+x_2x_4)+2x_3+x_3x_4$, $f_1(x_1,\ldots,x_5) = 4(x_1x_3+x_2x_4+x_5)+2x_1x_2+x_1$. Then f_0, f_1 are disjoint spectra generalized 1-plateaued functions. Let $W = \mathbb{F}_2^5 \times \{0\}$, $U = \{0_5\} \times \mathbb{F}_2$, M is the matrix whose row vectors are $(1,0,\ldots,0,0),\ldots,(0,0,\ldots,1,0)$ and $\pi(y) = (0,\ldots,0,y), y \in \mathbb{F}_2$. Then the constructed generalized bent function $F : \mathbb{F}_2^6 \rightarrow \mathbb{Z}_{2^3}$ by Proposition 3 is $F(x_1,\ldots,x_6) = f_{x_6}(x_1,\ldots,x_5) = 4(x_1x_3+x_2x_4+x_5x_6)+2((x_1x_2x_6+x_3(1+x_6)) \ mod\ 2)+((x_3x_4(1+x_6)+x_1x_6) \ mod\ 2)$.*

C. Constructions of Generalized Plateaued Functions with (Non)-Affine Walsh Support in Spectral Domain

In this subsection, we provide some construction methods of generalized plateaued functions with (non)-affine Walsh support in spectral domain.

With the same notation as in Proposition 1. If $f : \mathbb{F}_p^n \rightarrow \mathbb{Z}_{p^k}$ is a generalized s-plateaued function constructed in spectral domain, by the proof of Proposition 1, we have $S_f = S$, where ordered $S = \{w_0, \ldots, w_{p^{n-s}-1}\}$. It is easy to see that the matrix form of S_f whose row vectors are $w_0, \ldots, w_{p^{n-s}-1}$ can be written as

$$S_f = (T_{\psi_{a_1}}, \ldots, T_{\psi_{a_n}}), \tag{9}$$

where $\{a_1, \ldots, a_n\}$ is the canonical basis of $\mathbb{F}_p^n$, that is, $a_1 = (1,0,\ldots,0,0), \ldots, a_n = (0,0,\ldots,0,1)$, $\psi_{a_i} : V_{n-s} \rightarrow \mathbb{F}_p$ is defined by (6) and $T_{\psi_{a_i}}$ defined by (5) is the true table of ψ_{a_i}. If ψ_{a_i} is an affine function, we say that the i-th column of (ordered) S_f corresponds to an affine function. Note that if f is constructed by Theorem 1, then every column of S_f corresponds to an affine function by Lemma 1.

In [14], Hodžić *et al*. designed Boolean plateaued functions with (non)-affine Walsh support in spectral domain. As pointed out in [14], for the constructions in spectral domain given in [14], the Walsh support of Boolean s-plateaued functions in n variables, when written as a matrix of form (9), contains at least $n-s$ columns corresponding to affine functions on $\mathbb{F}_2^{n-s}$. They proposed an open problem (Open Problem 2) to provide constructions of Boolean s-plateaued functions in n variables whose Walsh support, when written as a matrix of form (9), contains strictly less than $n-s$ columns corresponding to affine functions. In our constructions of generalized s-plateaued functions with non-affine Walsh support, the Walsh support, when written as a matrix of form (9), can contain strictly less than $n-s$ columns corresponding to affine functions. When $p=2, k=1$, these constructions provide an answer to Open Problem 2 in [14].

In the first construction method, we utilize an important class of generalized bent functions $f : \mathbb{F}_{p^n} \times \mathbb{F}_{p^n} \to \mathbb{Z}_{p^k}$ defined as

$$f(x_1, x_2) = p^{k-1}Tr_1^n(\alpha x_1 \pi(x_2)) + g(x_2), (x_1, x_2) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^n},$$

where $\alpha \in \mathbb{F}_{p^n}^*$, π is a permutation over $\mathbb{F}_{p^n}$ and g is an arbitrary function from $\mathbb{F}_{p^n}$ to $\mathbb{Z}_{p^k}$, which is a generalization of the well-known Maiorana-McFarland bent functions. It is easy to obtain its dual function $f^*(x_1, x_2) = -p^{k-1}Tr_1^n(x_2\pi^{-1}(\alpha^{-1}x_1)) + g(\pi^{-1}(\alpha^{-1}x_1))$ and $\mu_f(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$.

For the sake of simplicity, we give the functions needed in the following theorem. Let $n, s(< n)$ be positive integers with $n-s = 2m$, $\{\alpha_1, \ldots, \alpha_m\}$ be a basis of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$, π be a permutation over $\mathbb{F}_{p^m}$ and $L_1, \ldots, L_{n-s} : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ be linearly independent linear functions. Define $d : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{Z}_{p^k}$ as

$$d(x_1, x_2) = p^{k-1}Tr_1^m(\alpha_1 x_1 \pi(x_2)) + g(x_2), \tag{10}$$

where g is an arbitrary function from $\mathbb{F}_{p^m}$ to $\mathbb{Z}_{p^k}$. Define $t_i : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p, 1 \le i \le s$ as

$$t_i(x_1, x_2) = \begin{cases} Tr_1^m(\beta_i x_1 \pi(x_2)) + g_i(x_2) + A_i(x_1, x_2) \text{ if } m \ge 2, \\ g_i(x_2) + A_i(x_1, x_2) \text{ if } m = 1, \end{cases} \tag{11}$$

where $\beta_i = \sum_{j=2}^m c_{i,j}\alpha_j$ with $c_{i,j} \in \mathbb{F}_p$, g_i is an arbitrary function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ and A_i is an arbitrary affine function from $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ to $\mathbb{F}_p$. Define $h_j : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p, 1 \le j \le n-s$ as

$$h_j = \begin{cases} \sum_{i=1}^{s} d_{j,i}t_i + L_j + b_j \text{ if } I = \emptyset, \\ \sum_{i \notin I} d_{j,i}t_i + F_j(t_{i_1}, \ldots, t_{i_{|I|}}) + L_j + b_j \text{ if } I \ne \emptyset, \end{cases} \tag{12}$$

where $I = \{1 \le i \le s : t_i(x_1, x_2) \text{ only depends on variable } x_2\}$ and denote I by $\{i_1, \ldots, i_{|I|}\}$ if $I \ne \emptyset$, $d_{j,i}, b_j \in \mathbb{F}_p$ and F_j is an arbitrary function from $\mathbb{F}_p^{|I|}$ to $\mathbb{F}_p$.

Theorem 2. *With the same notation as in Proposition 1. Let $n-s = 2m$ be an even positive integer. Let $d : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{Z}_{p^k}$ be defined by (10). Let $\mu(x_1, x_2) = 1, (x_1, x_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.*

Let the matrix form of $S=\{w_0,\ldots,w_{p^{n-s}-1}\}\subseteq \mathbb{F}_p^n$ be defined by

$$S=\begin{pmatrix} w_0 \\ \ldots \\ w_{p^{n-s}-1}\end{pmatrix}=(T_{t_1},\ldots,T_{t_s},T_{h_1},\ldots,T_{h_{n-s}}),$$

where $t_i(1\leq i\leq s)$ are defined by (11) and $h_j(1\leq j\leq n-s)$ are defined by (12). Then the function $W:\mathbb{F}_p^n\rightarrow \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f:\mathbb{F}_p^n\rightarrow \mathbb{Z}_{p^k}$.

Proof: First we show that the size of S is equal to p^{n-s}, that is to prove

$$(t_1(x),\ldots,h_{n-s}(x))=(t_1(x'),\ldots,h_{n-s}(x')) \iff x=x',$$

where $x=(x_1,x_2), x'=(x_1',x_2')\in \mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$. If $(t_1(x),\ldots,h_{n-s}(x))=(t_1(x'),\ldots,\ h_{n-s}(x'))$, then by the definitions of $h_j(1\leq j\leq n-s)$, $L_j(x)=L_j(x')$ for any $1\leq j\leq n-s$. Since $L_1,\ldots,L_{n-s}$ are linearly independent linear functions, it is easy to see that $x=x'$.

For any $a\in\mathbb{F}_p^n$ and $0\leq i\leq p^{n-s}-1$, $\psi_a(v_i)=a\cdot w_i=a\cdot(t_1(v_i),\ldots,t_s(v_i),h_1(v_i),\ldots,$ $h_{n-s}(v_i))$. When $m\geq 2$, by the constructions of $t_i,h_j(1\leq i\leq s, 1\leq j\leq n-s)$, we have $\psi_a(x_1,x_2)=Tr_1^m(\alpha_a x_1\pi(x_2))+g_a(x_2)+A_a(x_1,x_2)$, where $\alpha_a\in\mathbb{F}_{p^m}$ is some linear combination of $\alpha_2,\ldots,\alpha_m$, g_a is some function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ and $A_a:\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}\rightarrow\mathbb{F}_p$ is some affine function. Then $d(x_1,x_2)+p^{k-1}\psi_a(x_1,x_2)=p^{k-1}Tr_1^m((\alpha_1+\alpha_a)x_1\pi(x_2))+(g(x_2)+p^{k-1}g_a(x_2))+$ $p^{k-1}A_a(x_1,x_2)$. Since $\alpha_1,\ldots,\alpha_m$ are linearly independent, $\alpha_1+\alpha_a\neq 0$. Note that if $h:V_n\rightarrow\mathbb{Z}_{p^k}$ is a weakly regular generalized bent function and $A:V_n\rightarrow\mathbb{F}_p$ is an arbitrary affine function, then $h+p^{k-1}A$ is also a weakly regular generalized bent function and $\mu_{h+p^{k-1}A}=\mu_h$. Hence, $d+p^{k-1}\psi_a$ is a weakly regular generalized bent function and $\mu_{d+p^{k-1}\psi_a}=1$ for any $a\in\mathbb{F}_p^n$. By Corollary 1, the function $W:\mathbb{F}_p^n\rightarrow\mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f:\mathbb{F}_p^n\rightarrow\mathbb{Z}_{p^k}$. When $m=1$, by the similar argument, we have the same conclusion. ■

Theorem 2 can be seen as an extension of Theorem 4.1 of [14] in the sense of equivalence. We give two examples by using Theorem 2. The first example gives a generalized 3-ary plateaued function and the second example gives a Boolean plateaued function. Both of them satisfy that every column of the matrix form of S_f defined by (9) corresponds to a non-affine function. Furthermore, the constructed Boolean plateaued function has no nonzero linear structure. For a

Boolean function $f : V_n \to \mathbb{F}_2$, if $f(x)+f(x+a)$ is a constant function, then a is called a linear structure of f.

Example 4. *Let* $p = 3, k = 2, n = 7, s = 3$. *Let* z *be the primitive element of* $\mathbb{F}_{3^2}$ *with* $z^2+2z+2 = 0$. *Let* $d : \mathbb{F}_{3^2} \times \mathbb{F}_{3^2} \to \mathbb{Z}_{3^2}$ *be defined by* $d(x_1, x_2) = 3Tr_1^2(zx_1x_2)+2(Tr_1^2(x_2))^2$. *Let* $\mu(x_1, x_2) = 1$, $t_1(x_1, x_2) = Tr_1^2(x_1x_2)$, $t_2(x_1, x_2) = Tr_1^2(x_2^2)$, $t_3(x_1, x_2) = Tr_1^2(zx_2^2)$, $h_1 = t_1+Tr_1^2(x_1)$, $h_2 = t_2^2+Tr_1^2(zx_1)$, $h_3 = t_3^2+Tr_1^2(x_2)$, $h_4 = t_2+t_3+Tr_1^2(zx_2)$, $(x_1, x_2) \in \mathbb{F}_{3^2}\times\mathbb{F}_{3^2}$. *Then by the inverse Walsh transform or by computing* $(d+3\psi_a)^*(0)$, *we can obtain generalized* 3-*plateaued function* $f(b_1, \ldots, b_3, a_1, \ldots, a_4) = 2(((b_1+a_1)^2a_2+(2(b_1+a_1)+1)(a_1+a_2)) mod\, 3)^2+ 3((b_1+a_1)^2((b_2+a_4)(2a_1^2+2a_1a_2)+(b_3+a_4)(a_1^2+a_2^2)+2a_1^2a_2+2a_1a_2^2+a_1a_4+a_2a_3+a_2a_4+a_2)+ (b_1+a_1)((b_2+a_4)(2a_1^2+a_1a_2+a_2^2)+(b_3+a_4)(2a_1^2+2a_1a_2)+2a_1^2a_2+2a_1a_2^2+2a_1a_3+2a_1a_4+ 2a_2a_3+a_2a_4+a_2)+2a_1^2a_2^2a_3+(b_2+a_4)(a_1a_2+2a_2^2)+(b_3+a_4)(2a_1^2+a_1a_2+a_2^2)+a_1^2a_2+a_1^2a_3+a_1a_2^2+ a_2^2a_3+a_1a_3+a_1a_4+a_2a_3+2a_2a_4+a_2)$ *from* $\mathbb{F}_3^7$ *to* $\mathbb{Z}_{3^2}$. *Since* $t_i(1 \le i \le 3), h_j(1 \le j \le 4)$ *are all non-affine functions and the matrix form of* S_f *defined by (9) is* $S_f = (T_{t_1}, \ldots, T_{t_3}, T_{h_1}, \ldots, T_{h_4})$, *every column of* S_f *corresponds to a non-affine function.*

Example 5. *Let* $p = 2, k = 1, n = 10, s = 4$. *Let* z *be the primitive element of* $\mathbb{F}_{2^3}$ *with* $z^3+z+1 = 0$. *Let* $d(x_1, x_2) = Tr_1^3(z^2x_1x_2)$, $\mu(x_1, x_2) = 1$, $t_1(x_1, x_2) = Tr_1^3(x_1x_2)$, $t_2(x_1, x_2) = Tr_1^3(zx_1x_2)$, $t_3(x_1, x_2) = Tr_1^3(x_2^3)$, $t_4(x_1, x_2) = Tr_1^3(zx_2^3)$, $h_1 = t_1+Tr_1^3(x_1)$, $h_2 = t_1+Tr_1^3(zx_1)$, $h_3 = t_2 + Tr_1^3(z^2x_1)$, $h_4 = t_2 + Tr_1^3(x_2)$, $h_5 = t_3t_4 + Tr_1^3(zx_2)$, $h_6 = t_3t_4 + Tr_1^3(z^2x_2)$, $(x_1, x_2) \in \mathbb{F}_{2^3} \times \mathbb{F}_{2^3}$. *Then by the inverse Walsh transform or by computing* $(d + \psi_a)^*(0)$, *we can obtain Boolean* 4-*plateaued function* $f(b_1, \ldots, b_4, a_1, \ldots, a_6) = (b_1 + a_1 + a_2 + 1)(b_3(a_1a_3 + a_2a_3 + a_1) + b_4(a_1a_2 + a_1a_3 + a_2a_3 + a_1 + a_3) + (a_1a_2 + a_1a_3)(a_5 + a_6) + a_1a_4 + a_2a_6 + a_3a_5) + ((b_1 + a_1 + a_2)(b_2 + a_3 + a_4) + 1)(a_1a_5 + a_2a_5 + a_3a_4 + a_3a_5) + (b_1 + b_2 + a_1 + a_2 + a_3 + a_4 + 1)(b_3(a_1a_3 + a_2 + a_3) + b_4(a_1a_3 + a_2a_3 + a_1) + a_1a_2(a_5 + a_6) + a_1a_5 + a_2a_4 + a_2a_5 + a_3a_5 + a_3a_6) + b_3(a_1a_2 + a_2a_3 + a_1 + a_2) + b_4(a_1a_3 + a_2 + a_3) + (a_2a_3 + a_1 + a_2 + a_3)(a_5 + a_6)$. *Since* $t_i(1 \le i \le 4), h_j(1 \le j \le 6)$ *are all non-affine functions and the matrix form of* S_f *defined by (9) is* $S_f = (T_{t_1}, \ldots, T_{t_4}, T_{h_1}, \ldots, T_{h_6})$, *every column of* S_f *corresponds to a non-affine function. Furthermore, one can verify that* S_f *contains a basis of* $\mathbb{F}_2^{10}$ *and* $(0, \ldots, 0) \in S_f$, *hence by Corollary 3.1 of [14],* f *has no nonzero linear structure.*

In the second construction method, we take advantage of the good properties of general generalized bent functions given in [29]. Let $t \geq 2$ be an integer. Let $f(x) = \sum_{i=0}^{t-1} p^{t-1-i} f_i(x)$ with $f_i : V_n \rightarrow \mathbb{F}_p, 0 \leq i \leq t-1$ be a generalized bent function from V_n to $\mathbb{Z}_{p^t}$, where p is an odd prime or $p = 2$ and n is even. Let k be a positive integer. Then by Corollary 7 of [29], for any function $G : \mathbb{F}_p^{t-1} \rightarrow \mathbb{Z}_{p^k}$, the function $p^{k-1} f_0 + G(f_1, \ldots, f_{t-1})$ is a generalized bent function from V_n to $\mathbb{Z}_{p^k}$ with $\mu_{p^{k-1} f_0 + G(f_1, \ldots, f_{t-1})} = \mu_f$.

For the sake of simplicity, we give the functions needed in the following theorem. Let $n, s (< n)$ be positive integers with $n - s$ even if $p = 2$, $L_1, \ldots, L_{n-s} : V_{n-s} \rightarrow \mathbb{F}_p$ be linearly independent linear functions and $g = \sum_{i=0}^{t-1} p^{t-1-i} g_i$ with $g_i : V_{n-s} \rightarrow \mathbb{F}_p, 0 \leq i \leq t-1$ be a weakly regular generalized bent function from V_{n-s} to $\mathbb{Z}_{p^t}$, where $t \geq 2$. Define $d : V_{n-s} \rightarrow \mathbb{Z}_{p^k}$ as

$$d(x) = p^{k-1} g_0(x) + G(g_1(x), \ldots, g_{t-1}(x)), \tag{13}$$

where G is an arbitrary function from $\mathbb{F}_p^{t-1}$ to $\mathbb{Z}_{p^k}$. Define

$$\mu(x) = \mu_g(x)^{-1}, x \in V_{n-s}, \tag{14}$$

where μ_g is defined by (3). Note that μ_g is a constant function since g is weakly regular. Define $t_i : V_{n-s} \rightarrow \mathbb{F}_p, 1 \leq i \leq s$ as

$$t_i(x) = F_i(g_1(x), \ldots, g_{t-1}(x)), \tag{15}$$

where F_i is an arbitrary function from $\mathbb{F}_p^{t-1}$ to $\mathbb{F}_p$. Define $h_j : V_{n-s} \rightarrow \mathbb{F}_p, 1 \leq j \leq n-s$ as

$$h_j(x) = H_j(t_1(x), \ldots, t_s(x)) + L_j(x) + b_j, \tag{16}$$

where H_j is an arbitrary function from $\mathbb{F}_p^s$ to $\mathbb{F}_p$ and $b_j \in \mathbb{F}_p$.

Theorem 3. *With the same notation as in Proposition 1. Let $d : V_{n-s} \rightarrow \mathbb{Z}_{p^k}$ be defined by (13). Let μ be defined by (14). Let the matrix form of $S = \{w_0, \ldots, w_{p^{n-s}-1}\} \subseteq \mathbb{F}_p^n$ be defined by*

$$S = \begin{pmatrix} w_0 \\ \ldots \\ w_{p^{n-s}-1} \end{pmatrix} = (T_{t_1}, \ldots, T_{t_s}, T_{h_1}, \ldots, T_{h_{n-s}}),$$

where $t_i (1 \leq i \leq s)$ are defined by (15) and $h_j (1 \leq j \leq n-s)$ are defined by (16). Then the function $W : \mathbb{F}_p^n \rightarrow \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \rightarrow \mathbb{Z}_{p^k}$.

Proof: With the similar argument as in the proof of Theorem 2, we have $|S| = p^{n-s}$ and for any $a \in \mathbb{F}_p^n$, $\psi_a(x) = G_a(g_1(x), \ldots, g_{t-1}(x)) + A_a(x)$, where G_a is some function from $\mathbb{F}_p^{t-1}$ to $\mathbb{F}_p$ and $A_a : V_{n-s} \to \mathbb{F}_p$ is some affine function. Then $d + p^{k-1}\psi_a$ is a weakly regular generalized bent function and $\mu_{d+p^{k-1}\psi_a} = \mu_g$ for any $a \in \mathbb{F}_p^n$. By Corollary 1, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_{p^k}]$ defined by (7) is the Walsh transform of a generalized s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{Z}_{p^k}$. ■

We give two examples by using Theorem 3. The first example gives a generalized 5-ary plateaued function and the second example gives a Boolean plateaued function. Both of them satisfy that every column of the matrix form of S_f defined by (9) corresponds to a non-affine function. Furthermore, the constructed Boolean plateaued function has no nonzero linear structure.

Example 6. *Let $p = 5, k = 3, n = 4, s = 1, t = 2$. Let z be the primitive element of $\mathbb{F}_{5^3}$ with $z^3 + 3z + 3 = 0$. Let $g : \mathbb{F}_{5^3} \to \mathbb{Z}_{5^2}$ be defined by $g = 5g_0 + g_1$, $g_0, g_1 : \mathbb{F}_{5^3} \to \mathbb{F}_5$, where $g_0(x) = Tr_1^3(2x^2)$, $g_1(x) = Tr_1^3(z^{16}x)$. Then by Theorem 16 of [30] and Corollary 3 of [33], g is a weakly regular generalized bent function with $\mu_g = -1$. Let $d : \mathbb{F}_{5^3} \to \mathbb{Z}_{5^3}$ be defined by $d(x) = 25g_0(x) + g_1^4(x)$. Let $\mu(x) = -1$, $t_1(x) = g_1^3(x)$, $h_1(x) = t_1^2(x) + Tr_1^3(x)$, $h_2(x) = t_1^4(x) + Tr_1^3(zx)$, $h_3(x) = t_1(x) + Tr_1^3(z^2x)$, $x \in \mathbb{F}_{5^3}$. Then by the inverse Walsh transform or by computing $(d + 25\psi_a)^*(0)$, we can obtain generalized 1-plateaued function $f(b_1, a_1, a_2, a_3) = ((a_1 - a_3) \ mod\ 5)^4 + 25(a_2(a_1 - a_3)^4 + (b_1 + a_3)(a_1 - a_3)^3 + a_1(a_1 - a_3)^2 - a_1^2 - a_1a_3 + 2a_2^2 + a_2a_3 - a_3^2)$ from $\mathbb{F}_5^4$ to $\mathbb{Z}_{5^3}$. Since $t_1, h_j (1 \le j \le 3)$ are all non-affine functions and the matrix form of S_f defined by (9) is $S_f = (T_{t_1}, T_{h_1}, \ldots, T_{h_3})$, every column of S_f corresponds to a non-affine function.*

Example 7. *Let $p = 2, k = 1, n = 8, s = 2, t = 3$. Let $g : \mathbb{F}_2^6 \to \mathbb{Z}_{2^3}$ be defined by $g = \sum_{i=0}^{2} 2^{2-i}g_i, g_i : \mathbb{F}_2^6 \to \mathbb{F}_2$, where $g_0(x_1, \ldots, x_6) = x_1x_3 + x_2x_4 + x_5x_6$, $g_1(x_1, \ldots, x_6) = x_1x_2x_6 + x_3(x_6 + 1)$, $g_2(x_1, \ldots, x_6) = x_3x_4(x_6 + 1) + x_1x_6$. Then g is the generalized Boolean bent function with $\mu_g = 1$ constructed in Example 3. Let $d = g_0$, $\mu = 1$, $t_1 = g_1$, $t_2 = g_2$, $h_1 = t_1t_2 + x_1$, $h_2 = t_1 + x_2$, $h_3 = t_1 + x_3$, $h_4 = t_2 + x_4$, $h_5 = t_2 + x_5$, $h_6 = t_1 + t_2 + x_6$. Then by the inverse Walsh transform or by computing $(d + \psi_a)^*(0)$, we can obtain Boolean 2-plateaued function $f(b_1, b_2, a_1, \ldots, a_6) = a_1a_3 + a_2a_4 + a_5a_6 + a_1(a_5 + 1)(b_2a_2 + a_2a_4 + a_2a_6 + b_1 + a_3 + a_6) + a_3a_5(b_1a_4 + a_1a_4 + a_2a_4 + a_4a_6 + b_2 + a_6 + 1)$. Since $t_i (1 \le i \le 2), h_j (1 \le j \le 6)$ are all non-affine functions and the matrix form of S_f defined by (9) is $S_f = (T_{t_1}, T_{t_2}, T_{h_1}, \ldots, T_{h_6})$,* *every column of S_f corresponds to a non-affine function. Furthermore, one can verify that S_f contains a basis of $\mathbb{F}_2^8$ and $(0,\dots,0) \in S_f$, hence by Corollary 3.1 of [14], f has no nonzero linear structure.*

The third construction method is used to construct plateaued functions, that is, $k=1$. In the following theorem, we utilize vectorial bent functions. A function $f=(f_1,\dots,f_m): V_n \to \mathbb{F}_p^m$ is called a vectorial bent function if for any nonzero vector $(a_1,\dots,a_m) \in \mathbb{F}_p^m$, $\sum_{i=1}^m a_i f_i(x), x \in V_n$ is a bent function. It is known that if $f=(f_1,\dots,f_m): V_n \to \mathbb{F}_p^m$ is vectorial bent, then $m \leq n$ if p is an odd prime, and n is even and $m \leq \frac{n}{2}$ if $p=2$. The following theorem can be seen as a generalization of Theorem 4.3 of [14] in the sense of equivalence. It can also be applied to construct s-plateaued functions in n variables whose Walsh support, when written as a matrix of form (9), contains strictly less than $n-s$ columns corresponding to affine functions.

For the sake of simplicity, we give the functions needed in the following theorem. Let $n, s(<n), m$ be positive integers with $2 \leq m \leq n-s$ if p is an odd prime, and $n-s$ even and $2 \leq m \leq \frac{n-s}{2}$ if $p=2$. Let $g=(g_1,\dots,g_m)$ be a vectorial bent function from V_{n-s} to $\mathbb{F}_p^m$ which satisfies that for any $(c_2,\dots,c_m) \in \mathbb{F}_p^{m-1}$, $\mu_{g_1+\sum_{i=2}^m c_i g_i}(x)=u, x \in V_{n-s}$, where $\mu_{g_1+\sum_{i=2}^m c_i g_i}$ is defined by (3) and u is a constant independent of $(c_2,\dots,c_m)$. Let $L_1,\dots,L_{n-s}: V_{n-s} \to \mathbb{F}_p$ be linearly independent linear functions. Define $d: V_{n-s} \to \mathbb{F}_p$ as

$$d(x) = g_1(x). \tag{17}$$

Define μ as

$$\mu(x) = u^{-1}, x \in V_{n-s}. \tag{18}$$

Define $t_i: V_{n-s} \to \mathbb{F}_p, 1 \leq i \leq s$ as

$$t_i(x) = \sum_{j=2}^{m} c_{i,j} g_j(x) + A_i(x), \tag{19}$$

where $c_{i,j} \in \mathbb{F}_p$, A_i is an arbitrary affine function from V_{n-s} to $\mathbb{F}_p$. Define $h_j: V_{n-s} \to \mathbb{F}_p, 1 \leq j \leq n-s$ as

$$h_j(x) = \sum_{i=1}^{s} d_{j,i} t_i(x) + L_j(x) + b_j, \tag{20}$$

where $d_{j,i}, b_j \in \mathbb{F}_p$.

Theorem 4. *With the same notation as in Proposition 1. Let $d: V_{n-s} \to \mathbb{F}_p$ be defined by (17). Let μ be defined by (18). Let the matrix form of $S = \{w_0, \ldots, w_{p^{n-s}-1}\} \subseteq \mathbb{F}_p^n$ be defined by*

$$S = \begin{pmatrix} w_0 \\ \ldots \\ w_{p^{n-s}-1} \end{pmatrix} = (T_{t_1}, \ldots, T_{t_s}, T_{h_1}, \ldots, T_{h_{n-s}}),$$

where $t_i (1 \leq i \leq s)$ are defined by (19) and $h_j (1 \leq j \leq n-s)$ are defined by (20). Then the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_p]$ defined by (7) is the Walsh transform of an s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{F}_p$.

Proof: With the similar argument as in Theorem 2, we have $|S| = p^{n-s}$ and for any $a \in \mathbb{F}_p^n$, $\psi_a(x) = L_a(g_2(x), \ldots, g_m(x)) + A_a(x)$, where L_a is some linear function from $\mathbb{F}_p^{m-1}$ to $\mathbb{F}_p$ and $A_a : V_{n-s} \to \mathbb{F}_p$ is some affine function. Then $d + \psi_a$ is a weakly regular bent function and $\mu_{d+\psi_a} = u$ for any $a \in \mathbb{F}_p^n$. By Corollary 1, the function $W : \mathbb{F}_p^n \to \mathbb{Z}[\zeta_p]$ defined by (7) is the Walsh transform of an s-plateaued function $f : \mathbb{F}_p^n \to \mathbb{F}_p$. ∎

IV. Generalized indirect sum construction method of generalized plateaued functions

In this section, we provide a generalized indirect sum construction method of generalized s-plateaued functions, where $s \geq 0$. In particular, we show that the canonical way to construct Generalized Maiorana-McFarland bent functions can be obtained by the generalized indirect sum construction method and we illustrate that the generalized indirect sum construction method can be used to construct bent functions not in the completed Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

The following construction method called generalized indirect sum construction method is an extension of indirect sum construction method [4].

Theorem 5. *Let p be a prime. Let k, t, r, m be positive integers, $s (\leq r)$ be a non-negative integer and m be even for $p = 2$, $r + s$ be even for $p = 2, k = 1$. Let $f_i (i \in \mathbb{F}_p^t) : V_r \to \mathbb{Z}_{p^k}$ be generalized s-plateaued functions. Let $g_i (0 \leq i \leq t) : V_m \to \mathbb{F}_p$ be bent functions which satisfy* *that for any* $j=(j_1,\ldots,j_t)\in\mathbb{F}_p^t$*,* $G_j\triangleq(1-j_1-\cdots-j_t)g_0+j_1g_1+\cdots+j_tg_t$ *is a bent function and* $G_j^*=(1-j_1-\cdots-j_t)g_0^*+j_1g_1^*+\cdots+j_tg_t^*$ *and* $\mu_{G_j}=u$*, where* μ_{G_j} *is defined by (3) and* u *is a function from* V_m *to* $\{\pm1,\pm\sqrt{-1}\}$ *independent of* j*. Let* $g:\mathbb{F}_p^t\to\mathbb{Z}_{p^k}$ *be an arbitrary function. Then* $h(x,y)=f_{(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))}(x)+p^{k-1}g_0(y)+g(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y)),(x,y)\in V_r\times V_m$ *is a generalized* s*-plateaued function from* $V_r\times V_m$ *to* $\mathbb{Z}_{p^k}$*.*

Proof: For any $(a,b)\in V_r\times V_m$, we have

$$\begin{aligned}
&W_h(a,b)\\
&=\sum_{x\in V_r,y\in V_m}\zeta_{p^k}^{f_{(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))}(x)+p^{k-1}g_0(y)+g(g_0(y)-g_1(y),\ldots,g_0(y)-g_t(y))}\zeta_p^{-\langle a,x\rangle-\langle b,y\rangle}\\
&=\sum_{i_1,\ldots,i_t\in\mathbb{F}_p}\ \sum_{y:g_0(y)-g_j(y)=i_j,1\le j\le t}\ \sum_{x\in V_r}\zeta_{p^k}^{f_{(i_1,\ldots,i_t)}(x)+g(i_1,\ldots,i_t)}\zeta_p^{g_0(y)-\langle a,x\rangle-\langle b,y\rangle}\\
&=p^{-t}\sum_{i_1,\ldots,i_t\in\mathbb{F}_p}\zeta_{p^k}^{g(i_1,\ldots,i_t)}W_{f_{(i_1,\ldots,i_t)}}(a)\sum_{y\in V_m}\zeta_p^{g_0(y)-\langle b,y\rangle}\sum_{j_1\in\mathbb{F}_p}\zeta_p^{(i_1-(g_0-g_1)(y))j_1}\cdots\sum_{j_t\in\mathbb{F}_p}\zeta_p^{(i_t-(g_0-g_t)(y))j_t}\\
&=p^{-t}\sum_{i_1,\ldots,i_t\in\mathbb{F}_p}\zeta_{p^k}^{g(i_1,\ldots,i_t)}W_{f_{(i_1,\ldots,i_t)}}(a)\sum_{j_1,\ldots,j_t\in\mathbb{F}_p}\zeta_p^{i_1j_1+\cdots+i_tj_t}W_{G_{(j_1,\ldots,j_t)}}(b)\\
&=u(b)p^{\frac{m}{2}}p^{-t}\zeta_p^{g_0^*(b)}\sum_{i_1,\ldots,i_t\in\mathbb{F}_p}\zeta_{p^k}^{g(i_1,\ldots,i_t)}W_{f_{(i_1,\ldots,i_t)}}(a)\sum_{j_1\in\mathbb{F}_p}\zeta_p^{(g_1^*(b)-g_0^*(b)+i_1)j_1}\cdots\sum_{j_t\in\mathbb{F}_p}\zeta_p^{(g_t^*(b)-g_0^*(b)+i_t)j_t}\\
&=u(b)p^{\frac{m}{2}}\zeta_p^{g_0^*(b)}\zeta_{p^k}^{g(g_0^*(b)-g_1^*(b),\ldots,g_0^*(b)-g_t^*(b))}W_{f_{(g_0^*(b)-g_1^*(b),\ldots,g_0^*(b)-g_t^*(b))}}(a),
\end{aligned}\tag{21}$$

where the fifth equation is obtained by the properties of bent functions $g_i(0\le i\le t)$. By (21), it is easy to see that $h:V_r\times V_m\to\mathbb{Z}_{p^k}$ is a generalized s-plateaued function if $f_i,i\in\mathbb{F}_p^t$ are generalized s-plateaued functions from V_r to $\mathbb{Z}_{p^k}$. ■

If $s=0$, then Theorem 5 can be used to construct (non)-weakly regular generalized bent functions and the dual function can be given. The following corollary is an immediate consequence of Theorem 5 and its proof.

Corollary 2. *If* $s=0$*, then the function* $h:V_r\times V_m\to\mathbb{Z}_{p^k}$ *constructed by Theorem 5 is a generalized bent function and its dual function* $h^*(x,y)=f^*_{(g_0^*(y)-g_1^*(y),\ldots,g_0^*(y)-g_t^*(y))}(x)+p^{k-1}g_0^*(y)+g(g_0^*(y)-g_1^*(y),\ldots,g_0^*(y)-g_t^*(y))$*. Furthermore,* h *is non-weakly regular if any one of the following conditions holds:*

(1) There exists $i\in\mathbb{F}_p^t$ *such that* f_i *is non-weakly regular and* $|\{b\in V_m:(g_0^*(b)-g_1^*(b),\ldots,$

$g_0^*(b)-g_t^*(b))=i\}|\geq 1$;

(2) u is a constant function and there exist $i_1\neq i_2\in\mathbb{F}_p^t$ such that f_{i_1},f_{i_2} are weakly regular with $\mu_{f_{i_1}}\neq\mu_{f_{i_2}}$ and $|\{b\in V_m:(g_0^*(b)-g_1^*(b),\ldots,\, g_0^*(b)-g_t^*(b))=i_j\}|\geq 1$ for $j=1,2$;

(3) u is not a constant function and $\mu_{f_i}=c, i\in\mathbb{F}_p^t$, where c is a constant function independent of i.

Now we illustrate that why we call Theorem 5 generalized indirect sum construction method. Note that when $p=2$ and $t=1$, it is easy to verify that any Boolean bent functions g_0,g_1 satisfy the condition of Theorem 5. Let $p=2, k=t=1$, $f_0,f_1:V_r\rightarrow\mathbb{F}_2$ be Boolean plateaued functions, $g_0,g_1:V_m\rightarrow\mathbb{F}_2$ be Boolean bent functions and $g=0$, the plateaued function constructed by Theorem 5 is $h(x,y)=f_{g_0(y)+g_1(y)}(x)+g_0(y)=g_0(y)+f_0(x)+(f_0(x)+f_1(x))(g_0(y)+g_1(y))$. It is just the famous indirect sum construction [4]. Hence, Theorem 5 can be seen as an extension of indirect sum construction. Also note that Theorem 4.2 (i) of [34] for generalized Boolean plateaued functions as a generalization of indirect sum construction is a special case of the above more general construction. If $g_i(0\leq i\leq t)$ are bent functions satisfying $g_i=g_0-c_i, 1\leq i\leq t$, where $c_i(1\leq i\leq t)$ are constants, then $g_i(0\leq i\leq t)$ satisfy the condition of Theorem 5. In this case $h(x,y)=f_{(c_1,\ldots,c_t)}(x)+p^{k-1}g_0(y)+g(c_1,\ldots,c_t)$, which belongs to direct sum construction. We call it a trivial case. When p is an odd prime or $t\geq 2$, except the trivial case, the condition of Theorem 5 for $g_i(0\leq i\leq t)$ is not trivial.

In [9], the authors defined a class of bent functions

$$F(x,y)=f_y(x),(x,y)\in\mathbb{F}_p^m\times\mathbb{F}_p^s,$$

where m,s are positive integers with $s\leq m$ and $f_y, y\in\mathbb{F}_p^s$ are pairwise disjoint spectra partially bent functions with s-dimensional linear kernel, which are called Generalized Maiorana-McFarland bent functions. We will show that the canonical way to construct Generalized Maiorana-McFarland bent functions given in (6) of [9] can be obtained by Theorem 5. Let $g_i(0\leq i\leq t)$ be Maiorana-McFarland bent functions from $\mathbb{F}_p^m\times\mathbb{F}_p^m$ to $\mathbb{F}_p$ defined as $g_0(y_1,y_2)=y_1\cdot\pi(y_2)$, $g_i(y_1,y_2)=g_0(y_1,y_2)+h_i(y_2), 1\leq i\leq t$, where π is a permutation over $\mathbb{F}_p^m$ and for any $1\leq i\leq t$, h_i is an arbitrary function from $\mathbb{F}_p^m$ to $\mathbb{F}_p$. Then it is easy to verify that $g_i(0\leq i\leq t)$ satisfy the condition of Theorem 5. When $k=1, m=t, s=0$, $g=0$, $h_i(y_2)=-y_{2,i}, 1\leq i\leq t$, where $y_2=(y_{2,1},\ldots,y_{2,t})\in\mathbb{F}_p^t$ and $f_i(i\in\mathbb{F}_p^t)$ are bent functions, the bent function constructed by Theorem 5 is $h(x,y_1,y_2)=f_{y_2}(x)+y_1\cdot\pi(y_2),(x,y_1,y_2)\in\mathbb{F}_p^r\times\mathbb{F}_p^t\times\mathbb{F}_p^t$. It is just the canonical way to construct Generalized Maiorana-McFarland bent functions given in (6) of [9]. By Theorem 2 and its proof of [9], any bent function in the completed Generalized Maiorana-McFarland class (that is, equivalent to a Generalized Maiorana-McFarland bent function) is equivalent to an Maiorana-McFarland bent function or a bent function of the form (6) of [9]. Hence, any bent function in the completed Generalized Maiorana-McFarland class and not in the completed Maiorana-McFarland class is equivalent to a bent function which can be constructed by the generalized indirect sum construction.

We provide another construction for $g_i (0 \leq i \leq t)$ to satisfy the condition of Theorem 5. For any $0 \leq i \leq t$, let

$$g_i(y_1, y_2) = Tr_1^m(\alpha_i G(y_1 y_2^{p^m-2})),\ (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m},$$

where $m \geq t+1$, G is a permutation over $\mathbb{F}_{p^m}$ with $G(0) = 0$ and $\alpha_0, \alpha_1, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$. Then $g_i (0 \leq i \leq t)$ are in bent function class PS_{ap} which is a subclass of the famous class of partial spread bent functions (see [10], [17]). Since all partial spread bent functions are regular and the dual function of g_i is $g_i^*(y_1, y_2) = Tr_1^m(\alpha_i G(-y_1^{p^m-2} y_2))$, one can easily verify that $g_i (0 \leq i \leq t)$ satisfy the condition of Theorem 5.

As the above $g_i (0 \leq i \leq t)$ satisfy the condition of Theorem 5, we obtain the following corollary from Theorem 5.

Corollary 3. *Let p be a prime. Let k, t, r, m be positive integers with $m \geq t+1$, $s (\leq r)$ be a non-negative integer and $r+s$ be even for $p=2, k=1$. Let $f_i (i \in \mathbb{F}_p^t) : V_r \to \mathbb{Z}_{p^k}$ be generalized s-plateaued functions. Let $g_i (0 \leq i \leq t) : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ be defined as $g_i(y) = Tr_1^m(\alpha_i G(y_1 y_2^{p^m-2})), y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, where G is a permutation over $\mathbb{F}_{p^m}$ with $G(0) = 0$ and $\alpha_0, \alpha_1, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$. Let $g : \mathbb{F}_p^t \to \mathbb{Z}_{p^k}$ be an arbitrary function. Then $h(x, y) = f_{(g_0(y)-g_1(y), \ldots, g_0(y)-g_t(y))}(x) + p^{k-1} g_0(y) + g(g_0(y) - g_1(y), \ldots, g_0(y) - g_t(y)), (x, y) = (x, y_1, y_2) \in V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ is a generalized s-plateaued function from $V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ to $\mathbb{Z}_{p^k}$.*

We give two examples by using Corollary 3. The second example gives a non-weakly regular bent function which is not in the completed Generalized Maiorana-McFarland class.

Example 8. *Let $p = 7$, $k = 2$, $t = 1$, $r = 3$, $m = 2$, $s = 1$. Let $f_0(x_1, x_2, x_3) = 7(x_1^2 + x_2^2)$, $f_1(x_1, x_2, x_3) = 7(x_1^2 + 3x_2^2)$, $f_2(x_1, x_2, x_3) = 7(x_1^2 + 2x_3^2)$, $f_3(x_1, x_2,\ x_3) = 7(x_1^2 + 5x_3^2)$,*

$f_4(x_1,x_2,x_3) = 7(x_2^2+4x_3^2)$, $f_5(x_1,x_2,x_3) = 7(x_2^2+6x_3^2)$, $f_6(x_1,x_2,x_3) = 7(x_1^2+3x_2^2+x_3)$. Then $f_0,\ldots,f_6 : \mathbb{F}_7^3 \to \mathbb{Z}_{7^2}$ are generalized 1-plateaued functions. Let z be the primitive element of $\mathbb{F}_{7^2}$ with $z^2+6z+3=0$. Let $g_0(y_1,y_2) = Tr_1^2(y_1y_2^{47}), g_1(y_1,y_2) = Tr_1^2(zy_1y_2^{47}), (y_1,y_2) \in \mathbb{F}_{7^2} \times \mathbb{F}_{7^2}$. Let $g : \mathbb{F}_7 \to \mathbb{Z}_{7^2}$ be defined as $g(x) = x^5+2x^3$. Then the function $h : \mathbb{F}_7^3 \times \mathbb{F}_{7^2} \times \mathbb{F}_{7^2} \to \mathbb{Z}_{7^2}$ constructed by Corollary 3 is a generalized 1-plateaued function and one can verify that the Walsh support is not an affine subspace.

Example 9. *Let $p = 3, k = 1, t = 1, r = 4, m = 2, s = 0$. Let ξ be the primitive element of $\mathbb{F}_{3^4}$ with $\xi^4+2\xi^3+2 = 0$. Let z be the primitive element of $\mathbb{F}_{3^2}$ with $z^2+z+2 = 0$. Let $f_0(x) = Tr_1^4(x^{34}+x^2)$, $f_1(x) = Tr_1^4(x^2)$, $f_2(x) = Tr_1^4(\xi x^2)$, $x \in \mathbb{F}_{3^4}$. Then f_0, f_1, f_2 are weakly regular bent functions with $\mu_{f_0} = \mu_{f_1} = -1, \mu_{f_2} = 1$. Let $g_0(y_1,y_2) = Tr_1^2(y_1y_2^7)$, $g_1(y_1,y_2) = Tr_1^2(zy_1y_2^7)$, $(y_1,y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. Let $g = 0$. Then the function $h : \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2} \to \mathbb{F}_3$ constructed by Corollary 3 is a non-weakly regular bent function. Further, we will prove in Appendix that it is not in the completed Generalized Maiorana-McFarland class.*

In the rest of this paper, by using Corollary 3, we give constructions of plateaued functions in the subclass *WRP* of the class of weakly regular plateaued functions and vectorial plateaued functions.

In [27], Mesnager and Sınak introduced the notion of class *WRP*, which is a subclass of the class of weakly regular plateaued functions and plays an important role in constructing minimal linear codes and strongly regular graphs (see [27], [28]). Let p be an odd prime. Let $f : V_n \to \mathbb{F}_p$ be an unbalanced weakly regular s-plateaued function. If $f(0) = 0$ and there exists an even positive integer h with $gcd(h-1, p-1) = 1$ such that $f(ax) = a^h f(x), x \in V_n$ for any $a \in \mathbb{F}_p^*$, then f belongs to the class *WRP*. Note that all quadratic functions without affine term are in the class *WRP* and $h = 2$. We give a construction of non-quadratic plateaued functions in the class *WRP* by using Corollary 3.

Let p be an odd prime and m be an even positive integer. Let $f : \mathbb{F}_p^m \to \mathbb{F}_p$ be a partial spread bent function (see [17]). Then by Theorem 3.3 and Theorem 3.6 of [17], it is easy to see that for any $a \in \mathbb{F}_p^*$, $f(ax) = f(x)$. Let t, r be positive integers, s be a non-negative integer and $r-s$ be an even positive integer. For any $i \in \mathbb{F}_p^t$, let $b_i : \mathbb{F}_p^{r-s} \to \mathbb{F}_p$ be a partial spread bent function, $M_i \in GL(r, \mathbb{F}_p)$, $E_i \subseteq \mathbb{F}_p^r$ be an $(r-s)$-dimensional linear subspace and R_i be the corresponding matrix defined by Lemma 1. Define

$$f_i(x) = b_i(xM_i^T R_i^T), x \in \mathbb{F}_p^r, i \in \mathbb{F}_p^t. \tag{22}$$

Then for any $i \in \mathbb{F}_p^t$, f_i is an s-plateaued function with $\mu_{f_i}(x) = 1, x \in S_{f_i}$ by Theorem 1. And for any $a \in \mathbb{F}_p^*$, $f_i(ax) = f_i(x), x \in \mathbb{F}_p^r$.

Theorem 6. *Let p be an odd prime and $k = 1$. Let t, r, m be positive integers with $m \geq t + 1$. Let $s(< r)$ be a non-negative integer. Let $g_j(0 \leq j \leq t) : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ be defined as $g_j(y) = Tr_1^m(\alpha_j G(y_1 y_2^{p^m-2})), y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, where G is a permutation over $\mathbb{F}_{p^m}$ with $G(0) = 0$ and $\alpha_0, \alpha_1, \ldots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$.*

- *Case $p = 3$: Let $f_i(i \in \mathbb{F}_p^t) : V_r \to \mathbb{F}_p$ be weakly regular s-plateaued functions satisfying $\mu_{f_i}(x) = u, x \in S_{f_i}, i \in \mathbb{F}_p^t$, where μ_{f_i} is defined by (3) and u is some constant independent of i, $f_i(ax) = a^2 f_i(x), x \in V_r, i \in \mathbb{F}_p^t$ for any $a \in \mathbb{F}_p^*$ and $0 \in S_{f_{(0,\ldots,0)}}$. Let $g : \mathbb{F}_p^t \to \mathbb{F}_p$ be an arbitrary function with $g(0, \ldots, 0) = -f_{(0,\ldots,0)}(0)$.*
- *Case $p \geq 5$: Let $r - s$ be even. Let $f_i, i \in \mathbb{F}_p^t$ be defined as (22). Let $g : \mathbb{F}_p^t \to \mathbb{F}_p$ be an arbitrary function with $g(0, \ldots, 0) = -f_{(0,\ldots,0)}(0)$.*

Then the function $h : V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ constructed by Corollary 3 is a weakly regular s-plateaued function and in the class WRP.

Proof: By Corollary 3, h is an s-plateaued function. And by the proof of Theorem 5, it is easy to see that h is weakly regular and $S_h = \cup_{y \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}} S_{f_{(g_0^*(y) - g_1^*(y), \ldots, g_0^*(y) - g_t^*(y))}} \times \{y\}$. Since $g_0^*(0,0) - g_j^*(0,0) = 0, 1 \leq j \leq t$ and $0 \in S_{f_{(0,\ldots,0)}}$, we have $(0,0,0) \in S_h$, that is, h is unbalanced. Since $g(0, \ldots, 0) = -f_{(0,\ldots,0)}(0)$, $h(0,0,0) = 0$. As $f_i(ax) = f_i(x), x \in V_r, i \in \mathbb{F}_p^t$, $g_j(ay) = g_j(y), y \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}, 0 \leq j \leq t$ for any $a \in \mathbb{F}_p^*$, the weakly regular plateaued function h constructed by Corollary 3 satisfies $h(ax, ay) = h(x, y) = a^{p-1} h(x, y), (x, y) \in V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ for any $a \in \mathbb{F}_p^*$. Note that $p - 1$ is even and $gcd(p-2, p-1) = 1$. By definition, h is in the *WRP* class. ■

We give an example of non-quadratic plateaued function in the *WRP* class by using Theorem 6.

Example 10. *Let $p = 3, t = 1, r = 2, m = 2, s = 1$. Let z be the primitive element of $\mathbb{F}_{3^2}$ with $z^2 + 2z + 2 = 0$. Let $g_0(y_1, y_2) = Tr_1^2(y_1 y_2^7), g_1(y_1, y_2) = Tr_1^2(z y_1 y_2^7), (y_1, y_2) \in$*

$\mathbb{F}_{3^2}\times\mathbb{F}_{3^2}$. Let $f_0(x_1,x_2)=x_1^2, f_1(x_1,x_2)=x_2^2, f_2(x_1,x_2)=x_1^2+x_1x_2+x_2^2, (x_1,x_2)\in\mathbb{F}_3^2$. Then $f_i, i\in\mathbb{F}_3$ are 1-plateaued functions with $\mu_{f_i}(x_1,x_2)=\sqrt{-1}, (x_1,x_2)\in S_{f_i}$ and $f_i(ax_1,ax_2)=a^2f_i(x_1,x_2), (x_1,x_2)\in\mathbb{F}_3^2$ for any $a\in\mathbb{F}_3^*$ and $(0,0)\in S_{f_0}$. Let $g=0$. Then the function h constructed by Theorem 6 is $h(x_1,x_2,y_1,y_2)=Tr_1^2(y_1y_2^7)+x_1^2+(Tr_1^2((1-z)y_1y_2^7))^2(x_1^2+2x_1x_2+x_2^2)+(Tr_1^2((1-z)y_1y_2^7))(x_1^2+x_1x_2), (x_1,x_2,y_1,y_2)\in\mathbb{F}_3^2\times\mathbb{F}_{3^2}\times\mathbb{F}_{3^2}$, which is a non-quadratic weakly regular 1-plateaued function and in the WRP class. Furthermore, one can verify that the Walsh support of h is not an affine subspace, that is, h is not a partially bent function.

Let $f=(f_1,\ldots,f_m)$ be a vectorial function from V_n to $\mathbb{F}_p^m$. Then f is said to be a vectorial plateaued function if for any nonzero vector $(c_1,\ldots,c_m)\in\mathbb{F}_p^m$, $\sum_{i=1}^m c_if_i$ is a plateaued function from V_n to $\mathbb{F}_p$. We give a construction of vectorial plateaued functions by using Corollary 3.

Theorem 7. *Let p be a prime. Let $r\geq 1, m\geq 3, 0\leq s\leq r$ be integers and $r+s$ be even for $p=2$. Let $\{\alpha_0,\ldots,\alpha_{m-1}\}$ be a basis of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$. Let $f_0,\ldots,f_{p-1}:V_r\to\mathbb{F}_p$ be s-plateaued functions. Let G be a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$. Let $h_i(x,y_1,y_2)=f_{Tr_1^m(\alpha_0G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha_iG(y_1y_2^{p^m-2})), (x,y_1,y_2)\in V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}, 1\leq i\leq m-1$. Then vectorial function $H=(h_1,\ldots,h_{m-1})$ is a vectorial plateaued function from $V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ to $\mathbb{F}_p^{m-1}$.*

Proof: First we observe that if $\alpha,\beta\in\mathbb{F}_{p^m}$ are linearly independent over $\mathbb{F}_p$, then function $h(x,y_1,y_2)=f_{Tr_1^m(\beta G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha G(y_1y_2^{p^m-2})), (x,y_1,y_2)\in V_r\times\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ is an s-plateaued function, where $f_0,\ldots,f_{p-1}$ are s-plateaued functions and G is a permutation over $\mathbb{F}_{p^m}$ with $G(0)=0$. Indeed, we have $h(x,y_1,y_2)=f_{g_0(y_1,y_2)-g_1(y_1,y_2)}(x)+g_0(y_1,y_2)$, where $g_0(y_1,y_2)=Tr_1^m(\alpha G(y_1y_2^{p^m-2})), g_1(y_1,y_2)=Tr_1^m((\alpha-\beta)G(y_1y_2^{p^m-2}))$. By Corollary 3, h is an s-plateaued function since $\alpha,\alpha-\beta$ are linearly independent over $\mathbb{F}_p$.

For any nonzero vector $a=(a_1,\ldots,a_{m-1})\in\mathbb{F}_p^{m-1}$, let $\bar{a}=\sum_{i=1}^{m-1}a_i, \alpha_a=\sum_{i=1}^{m-1}a_i\alpha_i$. If $\bar{a}\neq 0$, in this case $\sum_{i=1}^{m-1}a_ih_i(x,y_1,y_2)=\bar{a}f_{Tr_1^m(\alpha_0G(y_1y_2^{p^m-2}))}(x)+Tr_1^m(\alpha_aG(y_1y_2^{p^m-2}))$. By Theorem 1 of [8], $\bar{a}f_0,\ldots,\bar{a}f_{p-1}$ are s-plateaued functions. Since $\bar{a}f_0,\ldots,\bar{a}f_{p-1}$ are s-plateaued functions and α_0,α_a are linearly independent, we have $\sum_{i=1}^{m-1}a_ih_i$ is an s-plateaued function. If $\bar{a}=0$, in this case $\sum_{i=1}^{m-1}a_ih_i(x,y_1,y_2)=Tr_1^m(\alpha_aG(y_1y_2^{p^m-2}))$. Since $\alpha_a\neq 0$, it is easy to see that $\sum_{i=1}^{m-1}a_ih_i$ is an r-plateaued function. ∎

We give an example of vectorial plateaued function by using Theorem 7.

Example 11. *Let* $p = 3, r = 3, m = 4, s = 0$. *Let* $f_j(x) = Tr_1^3(\xi^j x^2), x \in \mathbb{F}_{3^3}, j \in \mathbb{F}_3$, *where* ξ *is a primitive element of* $\mathbb{F}_{3^3}$. *Then* $f_j(j \in \mathbb{F}_3)$ *are weakly regular bent functions with* $\mu_{f_0} = \mu_{f_2} = -\sqrt{-1}, \mu_{f_1} = \sqrt{-1}$. *Let* $h_i(x, y_1, y_2) = f_{Tr_1^4(y_1 y_2^{79})}(x) + Tr_1^4(z^i y_1 y_2^{79}), (x, y_1, y_2) \in \mathbb{F}_{3^3} \times \mathbb{F}_{3^4} \times \mathbb{F}_{3^4}, i = 1, 2, 3$, *where* z *is a primitive element of* $\mathbb{F}_{3^4}$. *Then* $H = (h_1, h_2, h_3)$ *is a vectorial plateaued function. Furthermore, one can verify that* H *contains non-weakly regular plateaued component functions and weakly regular plateaued component functions.*

Remark 3. *Let* $H = (h_1, \ldots, h_{m-1})$ *be the constructed vectorial plateaued function by Theorem 7 with* $s = 0$. *Define* $g_i = h_{i+1}, 0 \leq i \leq m-2$. *Then one can verify that* $g_i(0 \leq i \leq m-2)$ *satisfy the condition of Theorem 5.*

V. Conclusion

Stimulated by the research works and one open problem of Hodžić *et al*. [14] on constructions of Boolean plateaued functions in spectral domain, we studied constructions of generalized plateaued functions in spectral domain in this paper.

(1) We provided a necessary and sufficient condition of constructing generalized plateaued functions in spectral domain.

(2) We gave a complete characterization of generalized plateaued functions whose Walsh support is an affine subspace in spectral domain, which generalizes the case of Boolean plateaued functions [14].

(3) We provided some new constructions of generalized plateaued functions with (non)-affine Walsh support in spectral domain. These constructions provide an answer to one open problem proposed by Hodžić *et al*. [14].

(4) We presented a generalized indirect sum construction method of generalized plateaued functions. In particular, we showed that the canonical way to construct Generalized Maiorana-McFarland bent functions can be obtained by the generalized indirect sum construction method and we illustrated that the generalized indirect sum construction method can be used to construct bent functions not in the completed Generalized Maiorana-McFarland class. Furthermore, based on this construction method, we gave constructions of plateaued functions in the class *WRP* and vectorial plateaued functions.

Plateaued functions have important applications in coding theory, sequences and combinatorics. For examples, Mesnager *et al*. [23] presented constructions of linear codes from weakly regular plateaued functions and the secret sharing schemes based on these linear codes. Mesnager and Sınak [27], [28] constructed several classes of minimal linear codes with few weights and strongly regular graphs from weakly regular plateaued functions. Boztaş *et al*. [1] used plateaued functions to design sequences with good correlation properties. It is interesting to further study the applications of generalized plateaued functions in coding theory, sequences and combinatorics. For examples, constructing linear codes, sequences and strongly regular graphs from generalized plateaued functions.

APPENDIX

We prove that the bent function constructed in Example 9 is not in the completed Generalized Maiorana-McFarland class.

Recall that the bent function constructed in Example 9 is $h(x, y_1, y_2) = f_{g_0(y_1,y_2)-g_1(y_1,y_2)}(x) + g_0(y_1, y_2) = f_0(x) + g_0(y_1, y_2) + (g_0(y_1, y_2) - g_1(y_1, y_2))^2(-f_0(x) - f_1(x) - f_2(x)) + (g_0(y_1, y_2) - g_1(y_1, y_2))(2f_1(x) + f_2(x)), (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, where $f_0(x) = Tr_1^4(x^{34} + x^2)$, $f_1(x) = Tr_1^4(x^2)$, $f_2(x) = Tr_1^4(\xi x^2)$, $g_0(y_1, y_2) = Tr_1^2(y_1 y_2^7)$, $g_1(y_1, y_2) = Tr_1^2(z y_1 y_2^7)$ and ξ is the primitive element of $\mathbb{F}_{3^4}$ with $\xi^4+2\xi^3+2 = 0$, z is the primitive element of $\mathbb{F}_{3^2}$ with $z^2+z+2 = 0$.

By Theorem 2 of [9], if h is in the completed Generalized Maiorana-McFarland class, then for an integer $1 \leq s \leq 4$ there exists an s-dimensional subspace V of $\mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$ such that the second order derivative

$$D_a D_c h(x, y_1, y_2) = 0 \tag{23}$$

for any $a = (a_0, a_1, a_2), c = (c_0, c_1, c_2) \in V, (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. Define $\bar{g_i}(y) = g_i(y_1, y_2), i = 0, 1$ and $\bar{h}(x, y) = f_{\bar{g_0}(y)-\bar{g_1}(y)}(x) + \bar{g_0}(y)$, where $y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4, (y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}, y_1 = y_{1,1} + y_{1,2}z, y_2 = y_{2,1} + y_{2,2}z$. Then $\bar{h}$ is a non-weakly regular bent function from $\mathbb{F}_{3^4} \times \mathbb{F}_3^4$ to $\mathbb{F}_3$. By simple calculation we have $\bar{g_0}(y) - \bar{g_1}(y) = (y_{1,1} + y_{1,2})y_{2,1}^2 y_{2,2} + (2y_{1,1} + y_{1,2})y_{2,1}y_{2,2}^2 + 2y_{1,1}y_{2,2} + 2y_{1,2}y_{2,1}$, $(\bar{g_0}(y) - \bar{g_1}(y))^2 = y_{1,1}^2 y_{2,2}^2 + y_{1,2}^2 y_{2,1}^2 + y_{1,1}y_{1,2}y_{2,1}y_{2,2}$, where $y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$.

Suppose (23) holds. Then

$$D_{\bar{a}} D_{\bar{c}} \bar{h}(x, y) = 0 \tag{24}$$

for any $a = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}), c = (c_0, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}) \in \bar{V}, (x, y) \in \mathbb{F}_{3^4} \times \mathbb{F}_3^4$, where $\bar{V} = \{(a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \in \mathbb{F}_{3^4} \times \mathbb{F}_3^4 : (a_0, a_{1,1} + a_{1,2}z, a_{2,1} + a_{2,2}z) \in V\}, y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$. As $\{30 \cdot 3^i \ (mod\ (3^4 - 1)) : i \geq 0\} = \{10, 30\}$ and $\begin{pmatrix} 34 \\ 10 \end{pmatrix} \equiv 0 \ (mod\ 3)$, $\begin{pmatrix} 34 \\ 30 \end{pmatrix} \equiv 2 \ (mod\ 3)$, $D_{\bar{a}} D_{\bar{c}} \bar{h}$ contains $-y_{1,1}^2 y_{2,2}^2 Tr_1^4(2((a_0 + c_0)^4 - a_0^4 - c_0^4) x^{30})$. Then by (24), $Tr_2^4((a_0+c_0)^4 - a_0^4 - c_0^4) = 0$ for any $\bar{a} = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}), \bar{c} = (c_0, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}) \in \bar{V}$. One can verify that for $a \in \mathbb{F}_{3^4}$, $Tr_2^4(a^4) = 0$ if and only if $a = 0$. If there exists $a_0 \neq 0$ such that $\bar{a} = (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \in \bar{V}$, let $\bar{c} = \bar{a}$, then $c_0 = a_0 \neq 0$ and $Tr_2^4((a_0+c_0)^4 - a_0^4 - c_0^4) = Tr_2^4(2a_0^4) \neq 0$, which is a contradiction. Hence $\bar{V} \subseteq \{0\} \times \mathbb{F}_3^4$, that is, $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. For any fixed $(0, a_1, a_2)$, $(0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, let $d_0 = D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2)$, $d_1 = D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))$, $d_2 = D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))^2$. By $D_{(0,a_1,a_2)} D_{(0,c_1,c_2)} h(x, y_1, y_2) = D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2) + (-f_0(x) - f_1(x) - f_2(x)) D_{(a_1,a_2)} D_{(c_1,c_2)} (g_0(y_1, y_2) - g_1(y_1, y_2))^2 + (2f_1(x) + f_2(x)) D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2)) = 0$ for any $(0, a_1, a_2), (0, c_1, c_2) \in V, (x, y_1, y_2) \in \mathbb{F}_{3^4} \times \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, for any fixed $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, we have $-d_2 f_0(x) + (2d_1 - d_2) f_1(x) + (d_1 - d_2) f_2(x) = -d_0, x \in \mathbb{F}_{3^4}$. By $f_0(0) = f_1(0) = f_2(0) = 0$, we have $d_0 = 0$. By $i + j\xi \neq 0$ for any $i, j \in \mathbb{F}_3$ and the algebraic degree of f_0 is 4, the algebraic degree of f_1 and f_2 is 2, we have f_0, f_1, f_2 are linearly independent, hence $d_1 = d_2 = 0$. Therefore, (23) holds if and only if for any $(0, a_1, a_2), (0, c_1, c_2) \in V, (y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$,

$$D_{(a_1,a_2)} D_{(c_1,c_2)} g_0(y_1, y_2) = 0 \tag{25}$$

and

$$D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2)) = 0 \tag{26}$$

and

$$D_{(a_1,a_2)} D_{(c_1,c_2)}(g_0(y_1, y_2) - g_1(y_1, y_2))^2 = 0. \tag{27}$$

By (25), (26) and the fact that $\{1, 1 - z\}$ is a basis of $\mathbb{F}_{3^2}$ over $\mathbb{F}_3$, we have for any fixed $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in \mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$, $Tr_1^2((((y_1 + a_1 + c_1)(y_2 + a_2 + c_2)^7 - (y_1 + a_1)(y_2+a_2)^7 - (y_1+c_1)(y_2+c_2)^7 + y_1 y_2^7)x) = 0, x \in \mathbb{F}_{3^2}$, which yields $(y_1+a_1+c_1)(y_2+a_2+c_2)^7 - (y_1+a_1)(y_2+a_2)^7 - (y_1+c_1)(y_2+c_2)^7 + y_1 y_2^7 = 0$ for any $(0, a_1, a_2), (0, c_1, c_2) \in V$ and $(y_1, y_2) \in$

$\mathbb{F}_{3^2} \times \mathbb{F}_{3^2}$. We claim $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \{0\}$. If there exists $a_2 \neq 0$ such that $a = (0, a_1, a_2) \in V$, let $c = a$. Then $c_2 = a_2 \neq 0$ and the coefficient of $y_1 y_2^3$ is $C_7^3((a_2 + c_2)^4 - a_2^4 - c_2^4) = a_2^4 \neq 0$, which is a contradiction. Hence $V \subseteq \{0\} \times \mathbb{F}_{3^2} \times \{0\}$, that is, $\bar{V} \subseteq \{0\} \times \mathbb{F}_3^2 \times \{(0,0)\}$. By (27), we have $D_{(a_{1,1},a_{1,2},0,0)} D_{(c_{1,1},c_{1,2},0,0)} (\bar{g_0}(y) - \bar{g_1}(y))^2 = 0$ for any $(0, a_{1,1}, a_{1,2}, 0, 0), (0, c_{1,1}, c_{1,2}, 0, 0) \in \bar{V}$, $y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \mathbb{F}_3^4$. By simple calculation, we have $2a_{1,1}c_{1,1}y_{2,2}^2 + 2a_{1,2}c_{1,2}y_{2,1}^2 + (a_{1,1}c_{1,2} + a_{1,2}c_{1,1})y_{2,1}y_{2,2} = 0$, which yields $a_{1,1}c_{1,1} = a_{1,2}c_{1,2} = a_{1,1}c_{1,2} + a_{1,2}c_{1,1} = 0$ for any $(0, a_{1,1}, a_{1,2}, 0, 0), (0, c_{1,1}, c_{1,2}, 0, 0) \in \bar{V}$. If there exists $(a_{1,1}, a_{1,2}) \neq (0, 0)$ such that $\bar{a} = (0, a_{1,1}, a_{1,2}, 0, 0) \in \bar{V}$, let $\bar{c} = \bar{a}$, then $a_{1,1}c_{1,1} = a_{1,1}^2 \neq 0$ or $a_{1,2}c_{1,2} = a_{1,2}^2 \neq 0$ since $(a_{1,1}, a_{1,2}) \neq (0, 0)$, which is a contradiction. Hence, $\bar{V} = \{(0, 0, 0, 0, 0)\}$, that is, $V = \{(0, 0, 0)\}$. By Theorem 2 of [9], h is not in the completed Generalized Maiorana-McFarland class.

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