Improved fully homomorphic public-key encryption with small ciphertext size

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SUMMARY: A cryptosystem which supports both addition and multiplication (thereby preserving the ring structure of the plaintexts) is known as fully homomorphic encryption (FHE) and is very powerful. Using such a scheme, any circuit can be homomorphically evaluated, effectively allowing the construction of programs which may be run on ciphertexts of their inputs to produce a ciphertext of their output. Since such a program never decrypts its input, it can be run by an untrusted party without revealing its inputs and internal state. The existence of an efficient and fully homomorphic cryptosystem would have great practical implications in the outsourcing of private computations, for instance, in the context of cloud computing. In previous work I proposed the fully homomorphic public-key encryption scheme with the size of ciphertext which is not small enough. In this paper the size of ciphertext is one-eighth of the size in the previously proposed scheme. Because proposed scheme adopts the medium text with zero norm, it is immune from the the "*p* and *-p* attack". As the proposed scheme is based on computational difficulty to solve the multivariate algebraic equations of high degree, it is immune from the Gröbner basis attack, the differential attack, rank attack and so on.

keywords: fully homomorphic public-key encryption, multivariate algebraic equation, Gröbner basis, non-associative ring

§1. Introduction

A cryptosystem which supports both addition and multiplication (thereby preserving the ring structure of the plaintexts) is known as fully homomorphic encryption (FHE) and is very powerful. Using such a scheme, any circuit can be homomorphically evaluated, effectively allowing the construction of programs which may be run on encryptions of their inputs to produce an encryption of their output. Since such a

program never decrypts its input, it can be run by an untrusted party without revealing its inputs and internal state. The existence of an efficient and fully homomorphic cryptosystem would have great practical implications in the outsourcing of private computations, for instance, in the context of cloud computing.

In 2009 Gentry, an IBM researcher, has created a homomorphic encryption scheme that makes it possible to encrypt the data in such a way that performing a mathematical operation on the encrypted information and then decrypting the result produces the same answer as performing an analogous operation on the unencrypted data[9],[10].

But in Gentry's scheme a task like finding a piece of text in an e-mail requires chaining together thousands of basic operations. His solution was to use a second layer of encryption, essentially to protect intermediate results when the system broke down and needed to be reset.

Some fully homomorphic encryption schemes were proposed until now[11], [12], [13],[14],[15].

In previous work[1],[2] I proposed a fully homomorphic encryption without bootstrapping which has the weak point in the enciphering function[17]. Next I proposed another FHPKE with the large size of ciphertext [18].After that I proposed the FHPKE with the small size of ciphertext [19].

In this paper I propose a fully homomorphic encryption scheme with the *smaller* size of ciphertext which is based on computational difficulty to solve the multivariate algebraic equations of high degree while the almost all multivariate cryptosystems[3], [4],[5],[6],[7] proposed until now are based on the quadratic equations avoiding the explosion of the coefficients. Proposed scheme is immune from the Gröbner basis[8] attack, the differential attack, rank attack and so on.

In this scheme the size of ciphertext is 8 times as large as that of the modulus q while in the scheme proposed before the size of ciphertext is 64 times as large as that of the modulus q [19].

§2. Preliminaries for octonion operation

In this section we describe the operations on octonion ring and properties of octonion ring.

§2.1 Multiplication and addition on octonion ring O

Let q be a fixed modulus to be as large prime as $O(2^{2000})$. Let O be the octonion [16] ring over a finite field $\boldsymbol{Fq}$.

$$O=\{(a_0,a_1,\ldots,a_7) \mid a_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7)\} \quad (1)$$

We define the multiplication and addition of $A,B \in O$ as follows.

$$A=(a_0,a_1,\ldots,a_7),\ a_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7), \quad (2)$$

$$B=(b_0,b_1,\ldots,b_7),\ b_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7). \quad (3)$$

$$\begin{aligned} &AB \bmod q \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 \bmod q, \\ &\quad a_0b_1 + a_1b_0 + a_2b_4 + a_3b_7 - a_4b_2 + a_5b_6 - a_6b_5 - a_7b_3 \bmod q, \\ &\quad a_0b_2 - a_1b_4 + a_2b_0 + a_3b_5 + a_4b_1 - a_5b_3 + a_6b_7 - a_7b_6 \bmod q, \\ &\quad a_0b_3 - a_1b_7 - a_2b_5 + a_3b_0 + a_4b_6 + a_5b_2 - a_6b_4 + a_7b_1 \bmod q, \\ &\quad a_0b_4 + a_1b_2 - a_2b_1 - a_3b_6 + a_4b_0 + a_5b_7 + a_6b_3 - a_7b_5 \bmod q, \\ &\quad a_0b_5 - a_1b_6 + a_2b_3 - a_3b_2 - a_4b_7 + a_5b_0 + a_6b_1 + a_7b_4 \bmod q, \\ &\quad a_0b_6 + a_1b_5 - a_2b_7 + a_3b_4 - a_4b_3 - a_5b_1 + a_6b_0 + a_7b_2 \bmod q, \\ &\quad a_0b_7 + a_1b_3 + a_2b_6 - a_3b_1 + a_4b_5 - a_5b_4 - a_6b_2 + a_7b_0 \bmod q) \end{aligned} \quad (4)$$

$$\begin{aligned} &A+B \bmod q \\ &=(a_0+b_0 \bmod q,\ a_1+b_1 \bmod q,\ a_2+b_2 \bmod q,\ a_3+b_3 \bmod q\ , \\ &\quad a_4+b_4 \bmod q,\ a_5+b_5 \bmod q,\ a_6+b_6 \bmod q,\ a_7+b_7 \bmod q\ ). \end{aligned} \quad (5)$$

Let

$$|A|^2 = a_0^2+a_1^2+\ldots+a_7^2 \bmod q. \quad (6)$$

If $|A|^2 \neq 0 \bmod q$, we can have A^{-1}, the inverse of A by using the algorithm **Octinv**(A) such that

$$A^{-1} = (a_0/|A|^2 \bmod q,\ -a_1/|A|^2 \bmod q,\ldots,\ -a_7/|A|^2 \bmod q) \leftarrow \textbf{Octinv}(A). \quad (7)$$

Here details of the algorithm **Octinv**(A) are omitted and can be looked up in the **Appendix A.**

§2.2. Property of multiplication over octonion ring *O*

A,B,C etc. $\in O$ satisfy the following formulae in general where A,B and C have the inverse A^{-1} ,B^{-1} and $C^{-1} \bmod q$.

1) Non-commutative

$$AB \neq BA \bmod q. \tag{8}$$

2) Non-associative

$$A(BC) \neq (AB)C \bmod q. \tag{9}$$

3) Alternative

$$(AA)B=A(AB) \bmod q, \tag{10}$$

$$A(BB)=(AB)B \bmod q, \tag{11}$$

$$(AB)A=A(BA) \bmod q. \tag{12}$$

4) Moufang's formulae [16],

$$C(A(CB))=((CA)C)B \bmod q, \tag{13}$$

$$A(C(BC))=((AC)B)C \bmod q, \tag{14}$$

$$(CA)(BC)=(C(AB))C \bmod q, \tag{15}$$

$$(CA)(BC)=C((AB)C) \bmod q. \tag{16}$$

5) A and $B \in O$ satisfy the following lemma.

Lemma 1

$$(A^{-1}B)A=A^{-1}(BA) \bmod q. \tag{17}$$

(*Proof*)

From (12)

$$A^{-1}B=A^{-1}((BA)A^{-1})=(A^{-1}(BA))A^{-1} \bmod q,$$

By multiplying A from right side we have

$$(A^{-1}B)A=A^{-1}(BA) \bmod q \quad \text{q.e.d.}$$

6) $A \in O$ satisfies the following lemma.

Lemma 2

$$A^{-1}(AB)= B \bmod q,$$
$$(BA)A^{-1}= B \bmod q.$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix B.**

7) $A \in O$ satisfies the following theorems.

Theorem 1

$$A^2=w\mathbf{1}+vA \bmod q, \tag{18}$$

where

$$^{\exists}w,v\in Fq,$$

$$\mathbf{1}=(1,0,0,0,0,0,0,0)\in O,$$

$$A=(a_0,a_1,\ldots,a_7)\in O.$$

(*Proof*:)

$$A^2 \bmod q$$

$$=(a_0a_0-a_1a_1- a_2a_2- a_3a_3-a_4a_4- a_5a_5-a_6a_6-a_7a_7 \bmod q,$$

$$a_0a_1+a_1a_0+a_2a_4+a_3a_7-a_4a_2+a_5a_6-a_6a_5-a_7a_3 \bmod q,$$

$$a_0a_2-a_1a_4+a_2a_0+a_3a_5+a_4a_1-a_5a_3+a_6a_7-a_7a_6 \bmod q,$$

$$a_0a_3-a_1a_7-a_2a_5+a_3a_0+a_4a_6+a_5a_2-a_6a_4+a_7a_1 \bmod q,$$

$$a_0a_4+a_1a_2-a_2a_1-a_3a_6+a_4a_0+a_5a_7+a_6a_3-a_7a_5 \bmod q,$$

$$a_0a_5-a_1a_6+a_2a_3-a_3a_2-a_4a_7+a_5a_0+a_6a_1+a_7a_4 \bmod q,$$

$$a_0a_6+a_1a_5-a_2a_7+a_3a_4-a_4a_3-a_5a_1+a_6a_0+a_7a_2 \bmod q,$$

$$a_0a_7+a_1a_3+a_2a_6-a_3a_1+a_4a_5-a_5a_4-a_6a_2+a_7a_0 \bmod q)$$

$$=(2a_0^2- L \bmod q, 2a_0a_1 \bmod q, 2a_0a_2 \bmod q, 2a_0a_3 \bmod q, 2a_0a_4 \bmod q,$$
$$2a_0a_5 \bmod q,2a_0a_6 \bmod q, 2a_0a_7 \bmod q)$$

where

$$L= a_0^2+a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2 \bmod q.$$

Now we try to obtain $u, v\in Fq$ that satisfy $A^2=w\mathbf{1}+vA \bmod q$.

$$w\mathbf{1}+vA= w(1,0,0,0,0,0,0,0)+v(a_0,a_1,\ldots,a_7) \bmod q,$$

$$A^2= (2a_0^2- L \bmod q, 2a_0a_1 \bmod q, 2a_0a_2 \bmod q, 2a_0a_3 \bmod q,$$
$$2a_0a_4 \bmod q, 2\, a_0a_5 \bmod q,2a_0a_6 \bmod q, 2a_0a_7 \bmod q).$$

As $A^2=w\mathbf{1}+vA=-L\mathbf{1}+2\, a_0A \bmod q$, we have

$$w= -L \bmod q,$$

$$v=2a_0 \bmod q.$$ q.e.d.

8) $A \in O$ satisfies the following theorems.

Theorem 2

Let $A^*=(a_0,-a_1,\ldots,-a_7) \in O$ be the conjugate of $A=(a_0,a_1,\ldots,a_7) \in O$. We have

$$AA^* = A^*A = L_A\mathbf{1}$$

where

$$L_A = a_0^2+a_1^2+\ldots+a_7^2 \bmod q.$$

[*Proof*]

As

$$A+A^*=(2a_0,0,\ldots,0)=2a_0\mathbf{1} \in O,$$

$$A^2=-L_A\mathbf{1}+2a_0A = -L_A\mathbf{1}+(A+A^*)A= -L_A\mathbf{1}+A(A+A^*) \bmod q,$$

we have

$$L_A\mathbf{1}=A^*A = AA^* \bmod q. \qquad \text{q.e.d.}$$

9) **Theorem 3**

$D \in O$ does not exist that satisfies the following equation.

$$B(AX)=DX \bmod q, \tag{19}$$

where $B,A,D \in O$ and X is a variable.

(*Proof*:)

When $X=\mathbf{1}$, we have

$$BA=D \bmod q.$$

Then

$$B(AX)=(BA)X \bmod q.$$

We can select $C \in O$ that satisfies

$$B(AC) \neq (BA)C \bmod q. \tag{20}$$

We substitute $C \in O$ to X to obtain

$$B(AC)=(BA)C \bmod q. \tag{21}$$

(21) is contradictory to (20). q.e.d.

10) **Theorem 4**

$D \in O$ does not exist that satisfies the following equation.

$$C(B(AX))=DX \bmod q \tag{22}$$

where $C,B,A,D \in O$, C has inverse $C^{-1} \bmod q$ and X is a variable.

B,A and C are non-associative, that is,

$$B(AC) \neq (BA)C \bmod q. \tag{23}$$

(*Proof*:)

If D exists, we have at X=**1**

$$C(BA)=D \bmod q.$$

Then

$$C(B(AX))=(C(BA))X \bmod q.$$

We substitute C to X to obtain

$$C(B(AC))=(C(BA))C \bmod q.$$

From (12)

$$C(B(AC))=(C(BA))C=C((BA)C) \bmod q.$$

By multiplying C^{-1} from left side, we have

$$B(AC)=(BA)C \bmod q \tag{24}$$

(24) is contradictory to (23). q.e.d.

11) **Theorem 5**

D and $E \in O$ do not exist that satisfy the following equation.

$$C(B(AX))= E\ (DX) \bmod q$$

where C,B,A,D and $E \in O$ have inverse and X is a variable and A,B,C are non-associative, that is,

$$C(BA) \neq (CB)A \bmod q. \tag{25}$$

(*Proof*:)

If D and E exist, we have at X=**1**

$$C(BA)=ED \bmod q. \tag{26}$$

We have at $X=(ED)^{-1}=D^{-1}E^{-1} \bmod q$

$$C(B(\mathrm{A}(D^{-1}E^{-1})))= E\,(D(D^{-1}E^{-1})) \bmod q=\mathbf{1},$$

$$(C(B(A(D^{-1}E^{-1})))^{-1} \bmod q=\mathbf{1},$$

$$((ED)A^{-1})B^{-1})C^{-1} \bmod q=\mathbf{1},$$

$$ED =(CB)A \bmod q. \tag{27}$$

From (26) and (27) we have

$$C(BA) =(CB)A \bmod q. \tag{28}$$

(28) is contradictory to (25). q.e.d.

12) **Theorem 6**

$D\in O$ does not exist that satisfies the following equation.

$$A(B(A^{-1}X))=DX \bmod q$$

where $B,A,D\in O$, A has inverse $A^{-1} \bmod q$ and X is a variable.

(*Proof*:)

If D exists, we have at $X=\mathbf{1}$

$$A(BA^{-1})=D \bmod q.$$

Then

$$A\,(B(A^{-1}X))=(A(BA^{-1}))X \bmod q. \tag{29}$$

We can select $C\in O$ such that

$$(BA^{-1})(CA^2) \neq((BA^{-1})C)A^2 \bmod q. \tag{30}$$

That is, (BA^{-1}), C and A^2 are non-associative.

Substituing $X=CA$ in (29), we have

$$A\,(B(A^{-1}(CA)))=(A(BA^{-1}))(CA) \bmod q.$$

From **Lemma 1**

$$A\,(B((A^{-1}C)A)))=(A(BA^{-1}))(CA) \bmod q.$$

From (16)

$$A(B((A^{-1}C)A)))=A([(BA^{-1})C]A) \bmod q.$$

By multiplying A^{-1} from left side we have

$$B((A^{-1}C)A))= ((BA^{-1})C)A \bmod q.$$

From **Lemma 1**

$$B(A^{-1}(CA))=((BA^{-1})C)A \bmod q.$$

Transforming CA to $((CA^2)A^{-1})$, we have

$$B(A^{-1}((CA^2)A^{-1}))=((BA^{-1})C)A \bmod q.$$

From (14) we have

$$((BA^{-1})(CA^2))A^{-1}=((BA^{-1})C)A \bmod q.$$

Multiply A from right side we have

$$((BA^{-1})(CA^2)=((BA^{-1})C)A^2 \bmod q. \quad (31)$$

(31) is contradictory to (30). q.e.d.

§3. Proposed fully homomorphic public-key encryption scheme

Homomorphic encryption is a form of encryption which allows specific types of computations to be carried out on ciphertext and obtain an encrypted result which decrypted matches the result of operations performed on the plaintext. For instance, one person could add two encrypted numbers and then another person could decrypt the result without knowing the value of the individual numbers.

§3.1 Definition of homomorphic public-key encryption

A homomorphic public-key encryption scheme **HPKE** := (**KeyGen; Enc; Dec; Eval**) is a quadruple of PPT (Probabilistic polynomial time) algorithms.

In this work, the medium text space M_e of the encryption schemes will be octonion ring, and the functions to be evaluated will be represented as arithmetic circuits over this ring, composed of addition and multiplication gates. The syntax of these algorithms is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^λ, outputs (**pk,sk**) ← **KeyGen**(1^λ) , where **pk** is a public encryption key and **sk** is a secret decryption key.

-Encryption. The algorithm **Enc**, on input system parameter q, a public key(**pk**) and a plaintext $p \in Fq$, outputs a ciphertext $C \in O \leftarrow$**Enc**(**pk**; p).

-Decryption. The algorithm **Dec**, on input system parameter q, secret key(**sk**) and a ciphertext C, outputs a plaintext $p^* \leftarrow$**Dec**(**sk**;C).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter q, an arithmetic circuit ckt, and a tuple of n ciphertexts ($C_1,\ldots, C_n$), outputs a ciphertext $C' \leftarrow$**Eval**(ckt; $C_1,\ldots, C_n$).

§3.2 Definition of fully homomorphic public-key encryption

A scheme HPKE is fully homomorphic if it is both compact and homomorphic with respect to a class of circuits. More formally:

Definition (**Fully homomorphic public-key encryption**). A homomorphic encryption scheme FHPKE :=(**KeyGen; Enc; Dec; Eval**) is fully homomorphic if it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda \in \mathrm{N}}$ be the set of all polynomial sized arithmetic circuits. On input (**pk** ,**sk**)$\leftarrow$**KeyGen**(1^λ), $\forall$ ckt $\in CR_\lambda$, $\forall (p_1,\ldots, p_n) \in Fq^n$ where $n = n(\lambda)$, $\forall (C_1,\ldots,C_n)$ where $C_i \leftarrow$ **Enc**(**pk**;p_i) (i=1,..,n), it holds that:

$$\Pr[\textbf{Dec}(\textbf{sk};\textbf{Eval}(\text{ckt}; C_1,\ldots,C_n)) \neq \text{ckt}(p_1,\ldots, p_n)] = \text{negl}(\lambda).$$

2. Compactness: There exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§3.3 Medium text

We define the medium text $M \in O$ which is adopted in proposed fully homomorphic public-key encryption (FHPKE) scheme as follows.

We select the element $G=(g_0,g_1,\ldots,g_7) \in O$ and $H=(h_0,h_1,\ldots,h_7) \in O$ such that

$$[G]_0=g_0=1/2 \bmod q, \tag{32a}$$

$$[H]_0=h_0=0 \bmod q, \tag{32b}$$

$$L_G:=|G|^2= g_0^2+g_1^2+\ldots+g_7^2=0 \bmod q\ , \tag{32c}$$

$$L_H:=|H|^2= h_0^2+h_1^2+\ldots+h_7^2=0 \bmod q\ , \tag{32d}$$

$$g_1h_1+g_2h_2+\ldots+g_7h_7=0 \bmod q. \tag{32e}$$

where we denote the i-th element of octonion $M \in O$ such as $[M]_i$.

Then we have

$$[GH]_0= [HG]_0=g_0h_0-(g_1h_1+g_2h_2+\ldots+ g_7h_7)=0 \bmod q\ , \tag{33a}$$

$$G^2 \bmod q =2g_0G=G, \tag{33b}$$

$$H^2 \bmod q =2h_0H =\mathbf{0}=(0,0\ldots,0). \tag{33c}$$

Theorem 7

$$GHG=\mathbf{0} \bmod q, \tag{34a}$$

$$HGH=\mathbf{0} \bmod q. \tag{34b}$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix C**.

Theorem 8

$$(GH)(HG) =\mathbf{0} \bmod q, \tag{35a}$$

$$(HG)(GH) =\mathbf{0} \bmod q. \tag{35b}$$

(*Proof*:)

From (15)

$$(GH)(HG)= (G(HH))G= (G(\mathbf{0}))G =\mathbf{0} \bmod q,$$

$$(HG)(GH) = (H(GG))H= (H(G))H =\mathbf{0} \bmod q. \quad \text{q.e.d.}$$

Table **1** gives the multiplication table of {*G*,*H*,*GH*,*HG*}.

Table**1**. multiplication table of {*G*,*H*,*GH*,*HG*}..

	G	***H***	***GH***	***HG***
G	*G*	*GH*	*GH*	**0**
H	*HG*	**0**	**0**	**0**
GH	**0**	**0**	**0**	**0**
HG	*HG*	**0**	**0**	**0**

Let $p \in \boldsymbol{Fq}$ be a plaintext and $u,v,w \in \boldsymbol{Fq}$ be the ramdom numbers.

The medium text M is given such that

$$M=pG+uH+vGH+wHG \bmod q \in \boldsymbol{Fq}. \tag{36}$$

The plaintext p is given from the medium text M such that

$$p= 2[M]_0 \bmod q \in \boldsymbol{Fq}. \tag{37}$$

Lemma 3

For any $A=(a_0,a_1,\ldots,a_7) \in O$, $B=(b_0,b_1,\ldots,b_7) \in O$

$$(A+B)^*=A^*+B^* \bmod q, \tag{38a}$$

$$(AB)^*=B^*A^* \bmod q \tag{38b}$$

where

$$A^*=(a_0,-a_1,\ldots,-a_7) \in O,\ B^*=(b_0,-b_1,\ldots,-b_7) \in O.$$

(*Proof*)

Here proof is omitted and can be looked up in the **Appendix D**.

Theorem 9

$$|M|^2=|pG+uH+vGH+wHG|^2=0 \bmod q \in Fq. \tag{39}$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix E**.

Theorem 10

$$GH^*+HG^* = \mathbf{0} \bmod q, \tag{40a}$$

$$G^*H+H^*G = \mathbf{0} \bmod q. \tag{40b}$$

(*Proof*:)

As

$$HG^* =(GH^*)^*,\ H^*G =(G^*H)^*,$$

then

$GH^*+ HG^*=2[GH^*]_0\mathbf{1}=2(g_0h_0+g_1h_1+g_2h_2+\ldots+ g_7h_7)\ \mathbf{1}=0\mathbf{1}=\mathbf{0} \bmod q.$

$G^*H+H^*G =2[G^*H]_0\mathbf{1}=2(g_0h_0+g_1h_1+g_2h_2+\ldots+ g_7h_7)\ \mathbf{1}=0\mathbf{1}=\mathbf{0} \bmod q.$

q.e.d.

Theorem 11

$$G(H^*G^*)+(GH)G^* =\mathbf{0} \bmod q, \tag{41a}$$

$$G^*(HG)+(G^*H^*)G =\mathbf{0} \bmod q. \tag{41b}$$

(*Proof*:)

$(GH)G^* =(G(GH)^*)^*=(G(H^*G^*))^*$

Then

$G(H^*G^*)+(GH)G^* =2[(GH)G^*]_0\mathbf{1}$

$=2[(GH)(2*g_0\mathbf{1}-G)]_0\mathbf{1}$

$=2[(GH)2*g_0\mathbf{1}-(GH)G)]_0\mathbf{1}$

$=2\ [(GH)-\mathbf{0}]_0\mathbf{1}$

$=2\ [GH]_0\mathbf{1}=\mathbf{0} \bmod q.$

In the same manner

$G^*(HG)+(G^*H^*)G =\mathbf{0} \bmod q.$ q.e.d.

Theorem 12

$$G(G^*H^*)+(HG)G^* =\mathbf{0} \bmod q, \tag{42a}$$

$$G^*(GH)+(H^*G^*)G =\mathbf{0} \bmod q. \tag{42b}$$

(*Proof*:)

As

$((HG)G^*)^* =G(HG)^*=G(G^*H^*),$

then

$G(G^*H^*)+(HG)G^* = 2[(HG)G^*]_0\mathbf{1}$

$=2[(HG)(2^*g_0\mathbf{1}-G)]_0\mathbf{1}$

$=2[(HG)2^*g_0\mathbf{1}-(HG)G)]_0\mathbf{1}$

$=2\,[(HG)- HG]_0\mathbf{1}=\mathbf{0} \bmod q.$

In the same manner

$G^*(GH)+(H^*G^*)G = \mathbf{0} \bmod q.$ q.e.d.

(Associativity of medium texts)

Let $M_1, M_2, M_3 \in O$ be arbitrary three medium texts where

$M_1:=p_1G+u_1H+v_1GH+w_1HG \bmod q \in O,$

$M_2:=p_2G+u_2H+v_2GH+w_2HG \bmod q \in O,$

$M_3:=p_3G+u_3H+v_3GH+w_3HG \bmod q \in O.$

$M_1 M_2=(p_1G+u_1H+v_1GH+w_1HG)(\ p_2G+u_2H+v_2GH+w_2HG) \bmod q$

$=p_1\ p_2G+\mathbf{0}H+(\ p_1\ u_2+\ p_1\ v_2)GH+(\ u_1\ p_2+\ w_1\ p_2)HG,$

$(M_1M_2)M_3=[p_1p_2G+(p_1u_2+p_1v_2)GH+(u_1p_2+w_1p_2)HG](\ p_3G+u_3H+v_3GH+w_3HG)$

$= p_1\ p_2\ p_3G+(\ p_1p_2\ u_3+\ p_1p_2\ v_3)GH+(\ u_1\ p_2\ p_3+\ w_1\ p_2\ p_3)HG,$

$M_2\ M_3=p_2\ p_3G+(\ p_2\ u_3+\ p_2\ v_3)GH+(\ u_2\ p_3+\ w_2\ p_3)HG,$

$M_1(M_2M_3)=(p_1G+u_1H+v_1GH+w_1HG)[p_2p_3G+(p_2u_3+p_2v_3)GH+(u_2p_3+w_2p_3)HG]$

$= p_1\ p_2\ p_3G+(\ p_1p_2\ u_3+\ p_1p_2\ v_3)GH+(\ u_1\ p_2\ p_3+\ w_1\ p_2\ p_3)HG.$

We have that

$(M_1M_2)M_3= p_1\ p_2\ p_3G+(\ p_1p_2\ u_3+\ p_1p_2\ v_3)GH+(\ u_1\ p_2\ p_3+\ w_1\ p_2\ p_3)HG$

$=M_1(M_2M_3) \bmod q.$ (43a)

We have that

$M_1M_2\ldots M_h= p_1\ p_2\ \ldots p_hG+0H\ +(\ v_{12\ldots h})GH+(\ w_{12\ldots h})HG \bmod q \in O,$ (43b)

where

$v_{12\ldots h}, w_{12\ldots h} \in \boldsymbol{Fq}, M_i=p_iG+u_iH+v_iGH+w_iHG \bmod q \in O, p_i,u_i,v_i,w_i \in \boldsymbol{Fq}.$

But we notice that in general for arbitrary $N \in O$,

$$(M_1 M_2)N \neq M_1(M_2 N) \bmod q.$$

§3.4 Proposed fully homomorphic public-key encyption

We propose a fully homomorphic public-key encryption (FHPKE) scheme on octonion ring over $\boldsymbol{Fq}$.

Here we define some parameters for describing FHPKE.

Let q be as a large prime as $O(2^{2000})$.

We select the element $G=(g_0,g_1,\ldots,g_7) \in O$ and $H=(h_0,h_1,\ldots,h_7) \in O$ such as defined in section **§3.3 Medium text**.

Let $p \in \boldsymbol{Fq}$ be a plaintext and $u,v,w \in \boldsymbol{Fq}$ be the ramdom numbers.

The medium text M is given such that

$$M=pG+uH+vGH+wHG \bmod q \in O.$$

The plaintext p is given from the medium text M such that

$$p= 2*[M]_0 \bmod q \in \boldsymbol{Fq}.$$

Basic enciphering function $f(X,Y) \in O[X,Y]$ is defined as follows.

Let $X=(x_0,\ldots,x_7) \in O[X]$ and $Y=(y_0,\ldots,y_7) \in O[Y]$ be variables.

We select $A_1,\ldots, A_n \in O$ such that A_i (i=1,..,n) has the inverse $A_i^{-1} \bmod q$.

We define the basic enciphering function $f(X,Y)$ such that

$$f(X,Y):= A_1^{-1}(\ldots(A_r^{-1}(Y(A_r(.. (A_1X)...) \bmod q \in O[X,Y] \quad (44a)$$

$$=(f_{000}x_0y_0+f_{001}x_0y_1+ \ldots+f_{077}x_7y_7,$$

$$f_{100}x_0y_0+f_{101}x_0y_1+ \ldots+f_{177}x_7y_7,$$

$$\ldots \quad \ldots$$

$$f_{700}x_0y_0+f_{701}x_0y_1+ \ldots+f_{777}x_7y_7)^t, \quad (44b)$$

$$= \{f_{ijk}\}(i,j,k=0,\ldots,7). \quad (44c)$$

As if $Y=\mathbf{1}$, then $f(X,\mathbf{1})=X$, some f_{ijk} are determined such that

$f_{000}=1, f_{010}=0, \ldots, f_{070}=0,$

$f_{100}=0, f_{110}=1, \ldots, f_{170}=0,$

$\ldots \quad \ldots$

$f_{700}=0, f_{710}=0, \ldots, f_{770}=1.$

Let $g(X,Y)\in O[X,Y]$ be a sub-enciphering function such that

$$g(X,Y):=(\ldots((XA_1)A_2)\ldots)A_r)Y)A_r^{-1})\ldots)A_1^{-1} \bmod q\in O[X,Y] \quad (45a)$$

$$=(g_{000}x_0y_0+g_{001}x_0y_1+ \ldots+g_{077}x_7y_7,$$

$$g_{100}x_0y_0+g_{101}x_0y_1+ \ldots+g_{177}x_7y_7,$$

$$\ldots. \quad \ldots.$$

$$g_{700}x_0y_0+g_{701}x_0y_1+ \ldots+g_{777}x_7y_7)^t, \quad (45b)$$

$$= \{g_{ijk}\}(i,j,k=0,\ldots,7). \quad (45c)$$

As if $Y=\mathbf{1}$, then $g(X,\mathbf{1})=X$, some g_{ijk} are determined such that

$g_{000}=1, g_{010}=0, \ldots, g_{070}=0,$
$g_{100}=0, g_{110}=1, \ldots, g_{170}=0,$
$\ldots \quad \ldots$
$g_{700}=0, g_{710}=0, \ldots, g_{770}=1.$

Theorem 13

Let

$$R_n=U_n((\ldots((U_2((U_1((PQ)U_1))U_2))\ldots))U_n) \bmod q\in O \quad (46a)$$

$$L_n= (U_n((\ldots((U_2((U_1(PQ))U_1))U_2))\ldots))U_n \bmod q\in O \quad (46b)$$

$$M_n=[U_n(U_{n-1}(\ldots(U_2(U_1P))\ldots)][(\ldots((QU_1)U_2)\ldots)U_{n-1})U_n] \bmod q\in O \quad (46c)$$

where

$P, Q, U_i\in O$ $(i=1,2,\ldots,n)$.

For any positive integer n

$$R_n \bmod q =L_n \bmod q =M_n \bmod q. \quad (46d)$$

[*Proof*]

I use the mathematical induction.

In case that $n=1$, from (15),(16)

$$R_1=U_1((PQ)U_1)=(U_1(PQ))U_1=(U_1P)(QU_1) \bmod q$$

is obtained. That is,

$R_1=L_1=M_1 \bmod q$.

In case that $n=k$-1

if $R_{k-1}=L_{k-1}=M_{k-1} \bmod q$, then

$U_k(R_{k-1}U_k)=U_k((U_{k-1}((\cdots((U_2((U_1((PQ)U_1))U_2))\cdots))U_{k-1}))U_k)=R_k \bmod q$

$U_k(R_{k-1}U_k)= U_k(L_{k-1}U_k)= U_k(((U_{k-1}((\cdots((U_2((U_1(PQ))U_1))U_2))\cdots))U_{k-1})U_k)$

$= (U_k((U_{k-1}((\cdots((U_2((U_1(PQ))U_1))U_2))\cdots))U_{k-1}))U_k=L_k \bmod q.$

$U_k(R_{k-1}U_k)=$

$U_k(M_{k-1}U_k)=\mathrm{U}_k(([U_{k-1}(U_{k-2}(\cdots(U_2(U_1P))\cdots)][(\cdots((QU_1)U_2)\cdots)U_{k-2})U_{k-1}])U_k)$

$=(U_k[(U_{k-1}(\cdots(U_2(U_1P))\cdots)]) ([(\cdots((QU_1)U_2)\cdots)U_{k-1})]U_k)$

$= [U_k(U_{k-1}(\cdots(U_2(U_1P))\cdots)][(\cdots((QU_1)U_2)\cdots)U_{k-1})U_k]=M_k \bmod q$

That is, we obtain $R_k=L_k=M_k \bmod q$.

So we obtain $R_n \bmod q = L_n \bmod q = M_n \bmod q$ for $n=1,2,\ldots$. q.e.d.

§3.5 Addition and multiplication of $f(X,S)$

Let M_1 and M_2 be the medium texts corresponding to the plaintexts p_1 and p_2 ,respectively.

Let $S\in O$ be a part of system parameter such that $S^{-1} \bmod q\in O$ exists.

We define the addition and multiplication of $f(X,S)$ as follows.

[Addition]

$f(M_1,S)+ f(M_2,S) \bmod q\in O$

$= A_1^{-1}(\cdots(A_r^{-1}(S\,(A_r(..\,(A_1\,M_1)...) + A_1^{-1}(\cdots(A_r^{-1}(S\,(A_r(..\,(A_1\,M_2)...)) \bmod q$

$= A_1^{-1}(\cdots(A_r^{-1}(S\,(A_r(..\,(A_1\,(M_1+M_2))..) \bmod q$

$= f(M_1+M_2,S) \bmod q\in O.$ (47)

[Multiplication]

$f(M_1,S)\{\,[\,f(\mathbf{1},S)\,]^{-1}[f(M_2,S)g(\mathbf{1},S)]\}\bmod q\in O$

$$=f(M_1,S)\{[f(\mathbf{1},S)]^{-1}[(A_1^{-1}(\ldots(A_r^{-1}(S(A_r(..(A_1M_2)...)(\ldots(\mathbf{1}A_1)\ldots)A_r)S)A_r^{-1})\ldots)A_1^{-1})]\}$$

$$=f(M_1,S)\{[f(\mathbf{1},S)]^{-1}[(A_1^{-1}(\ldots(A_r^{-1}(S\,(A_r(..\,(A_1\mathbf{1})...)(\ldots(M_2A_1)\ldots)A_r)S)A_r^{-1})\ldots)A_1^{-1})]\}$$

$$=f(M_1,S)\ \{[\ f(\mathbf{1},S)\ ]^{-1}\ [\ f(\mathbf{1},S)g(M_2,S)]\}\bmod q$$

$$=f(M_1,S)g(M_2,S)\bmod q$$

$$=[A_1^{-1}(\ldots(A_r^{-1}(S\,(A_r(..\,(A_1M_1)...)][(\ldots(M_2A_1)\ldots)A_r)S)A_r^{-1})\ldots)A_1^{-1}]$$

$$=[A_1^{-1}(\ldots(A_r^{-1}(S\,(A_r(..\,(A_1(M_1M_2)...)][(\ldots(\mathbf{1}A_1)\ldots)A_r)S)A_r^{-1})\ldots)A_1^{-1}]$$

$$=f(M_1M_2,S)g(\mathbf{1},S)\bmod q.$$

Then we have

$$f(M_1M_2,S)=\mathbf{[}f(M_1,S)\{[f(\mathbf{1},S)]^{-1}[f(M_2,S)g(\mathbf{1},S)]\}\mathbf{]}[g(\mathbf{1},S)]^{-1}\bmod q\in O. \qquad (48)$$

That is, we can obtain $f(M_1M_2,S)$ from $f(M_1,S)$, $f(M_2,S)$,$f(\mathbf{1},S)$ and $g(\mathbf{1},S)$ without $g(M_2,S)$.

Theorem 14

For arbitrary $p_1,p_2\in O$,

if $f(M_1,S)= f(M_2,S)\bmod q$, then $p_1= p_2 \bmod q$,

where

$$M_1:= p_1G+u_1H+v_1GH+w_1HG\in O,$$

$$M_2:= p_2G+u_2H+v_2GH+w_2HG\in O.$$

(*Proof*)

If $f(M_1,S)= f(M_2,S)\bmod q$,then

$$A_1^{-1}(\ldots(A_r^{-1}(S\,(A_r(..\,(A_1M_1)...)=A_1^{-1}(\ldots(A_r^{-1}(S\,(A_r(..\,(A_1M_2)...)\bmod q,$$

$$M_1= M_2 \bmod q.$$

$$p_1=2[M_1]_0 =2[M_2]_0 =p_2 \bmod q.$$

Then we have

$$p_1= p_2 \bmod q. \qquad \text{q.e.d.}$$

§3.6 Octonion elements assumption OEA(*q*)

Here we describe the assumption on which the proposed scheme bases.

Octonion Elements assumption OEA(*q*)

Let q be a prime more than 2. Let n be a secret integer parameter. Let $\boldsymbol{A}$:=$\{A_1,\ldots,A_r\} \in O^r$ be secret parameters. Let $f(X,Y)= A_1^{-1}(\ldots(A_r^{-1}(Y(A_r(\ldots(A_1X)\ldots) \bmod q \in O[X,Y]$ and $g(X,Y)= (\ldots(XA_1)\ldots)A_r)Y)A_r^{-1})\ldots)A_1^{-1} \bmod q \in O[X,Y]$ be the basic enciphering function and sub- basic enciphering function
where X and Y are variables.

In the **OEA**(*q*) assumption, the adversary A_d is given $f(X,Y)$, $g(X,Y)$ and his goal is to find a set of parameters $\boldsymbol{A}=\{A_1,\ldots,A_r\}\in O^r$ with the order of the elements $A_1,\ldots, A_r$. For parameters $r= r(\lambda)$ defined in terms of the security parameter λ and for any PPT adversary A_d we have

$$\Pr[A_1^{-1}(\ldots(A_r^{-1}(Y(A_r(\ldots(A_1X)\ldots))))\ldots) \bmod q=\{f_{ijk}\}(i,j,k=0,\ldots,7),$$

$$(\ldots(XA_1)\ldots)A_r)Y)A_r^{-1})\ldots)A_1^{-1} \bmod q=\{g_{ijk}\}(i,j,k=0,\ldots,7):$$

$$\boldsymbol{A}=\{A_1,\ldots, A_r\}\leftarrow A_d\,(1^\lambda,q, f(X,Y), g(X,Y))] = \mathrm{negl}(\lambda).$$

To solve directly **OEA**(*q*) assumption is known to be the problem for solving the multivariate algebraic equations of high degree which is known to be NP-hard.

§3.7 Property of proposed fully homomorphic encryption

The syntax of proposed scheme is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^λ and system parameter q, outputs

sk$\leftarrow$**KeyGen**(1^λ) where **sk**=$(r_A, A_j(j=1,\ldots,r_A))$ is a secret encryption key and

pk$\leftarrow$**KeyGen**(1^λ) where **pk**=$(\{f_{ijk}\}_{0\leqq i,j,k\leqq 7}, \{g_{ijk}\}_{0\leqq i,j,k\leqq 7})$ is a public key.

-Encryption. The algorithm **Enc**, on input system parameter $[q, G,H,S]$ and secret keys of user B, $\mathbf{sk_B}=(r_B, B_j(j=1,\ldots,r_B))$, public key of user A, $\mathbf{pk_A}=(\{f_{Aijk}\}_{0\leqq i,j,k\leqq 7}, \{g_{Aijk}\}_{0\leqq i,j,k\leqq 7})$ and a plaintext $p\in\boldsymbol{Fq}$, outputs a ciphertext $\boldsymbol{C}(M;\mathbf{sk_B}, \mathbf{pk_A},S)\in O$ $\leftarrow$**Enc**$(\mathbf{sk_B}, \mathbf{pk_A};p)$ where $M=pG+uH+vGH+wHG \bmod q\in O$.

-Decryption. The algorithm **Dec**, on input system parameter q, secret keys of user A, $\mathbf{sk_A}$, public key of user B,$\mathbf{pk_B}$ and a ciphertext $\boldsymbol{C}(M;\mathbf{sk_B}, \mathbf{pk_A},S)$, outputs plaintext $p^* \in \boldsymbol{Fq}$ =**Dec**($\mathbf{sk_A}$, $\mathbf{pk_B}$; $\boldsymbol{C}(M;\mathbf{sk_B}, \mathbf{pk_A},S)$) where $\boldsymbol{C}(M;\mathbf{sk_B}, \mathbf{pk_A},S)$ $\leftarrow$**Enc**($\mathbf{sk_B}$, $\mathbf{pk_A}$;p).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter q, an arithmetic circuit ckt, and a tuple of n ciphertexts $(C_1,\ldots, C_n)\in O^n$, outputs an evaluated ciphertext $C'\leftarrow$**Eval**(ckt; $C_1,\ldots, C_n$) where $C_i= \boldsymbol{C}(M_i;\mathbf{sk_B}, \mathbf{pk_A},S)$ (i=1,…,n), $M_i:= p_iG+u_iH+v_iGH+w_iHG\in O$.

(**Fully homomorphic encryption**). Proposed fully homomorphic encryption =(**KeyGen; Enc; Dec; Eval**) is fully homomorphic because it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda\in N}$ be the set of all polynomial sized arithmetic circuits. On input $\mathbf{sk} \leftarrow$**KeyGen**(1^λ), $\mathbf{pk}\leftarrow$**KeyGen**(1^λ), $\forall$ckt $\in$ CR_λ, $\forall(p_1,\ldots,p_n) \in \boldsymbol{Fq}^n$ where $n= n(\lambda)$, $\forall(C_1,\ldots,C_n)\in O^n$ where $C_i= \boldsymbol{C}(M_i;\mathbf{sk_B}, \mathbf{pk_A},S)\leftarrow$**Enc**($\mathbf{sk_B}$, $\mathbf{pk_A}$;p_i), (i =1,…,n),we have **Dec**($\mathbf{sk_A}$, $\mathbf{pk_B}$;**Eval**(ckt; $C_1,\ldots,C_n$)) = ckt($p_1,\ldots,p_n$). Then it holds that:

$$\Pr[\mathbf{Dec}(\mathbf{sk_A}, \mathbf{pk_B}; \mathbf{Eval}(\text{ckt}; C_1,\ldots,C_n)) \neq \text{ckt}(p_1,\ldots,p_n)] = \text{negl}(\lambda).$$

2. Compactness: As the output length of **Eval** is at most $k\log_2 q=k\lambda$ where k is a positive integer, there exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§3.8 Procedure for constructing public-key encryption

Here we show the procedure for constructing the public-key encryption scheme by using the cryptosystem described in above sections.

User B try to send his information to user A by using the public-key of user A $\mathbf{pk_A}$ and the secret key of user B $\mathbf{sk_B}$ through the insecure line.

1) System centre publishes the system parameter $[q,G,H,S]$.
2) User A downloads system parameter$[q,G,H,S]$ and selects $\mathbf{sk_A}$=(r_A ,A_j(j=1,…,r_A)) which is a secret key of user A and generates the public key of user A $\mathbf{pk_A}$=($\{f_{Aijk}\}_{0\leqq i,j,k\leqq 7}$, $\{g_{Aijk}\}_{0\leqq i,j,k\leqq 7}$) such that
$f_A(X,Y):=A_1^{-1}(\ldots(A_{rA}^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)$ mod $q\in O[X,Y]=\{f_{Aijk}\}(i,j,k=0,\ldots,7)$, (49a)
$g_A(X,Y):=(\ldots(XA_1)\ldots)A_{rA})Y)A_{rA}^{-1})\ldots)A_1^{-1}$ mod $q\in O[X,Y]=\{g_{ijk}\}(i,j,k=0,\ldots,7)$. (49b)
User A sends $\{f_{Aijk}\}$, $\{g_{Aijk}\}$ (i,j,k=0,…,7) to system centre.

3) User B downloads system parameter[q,G,H,S] and selects $\mathbf{sk_B}=(r_B, B_j(j=1,\ldots,r_B))$ which is a secret key of user B and generates the public key of user B $\mathbf{pk_B}=(\{f_{Bijk}\}_{0\leqq i,j,k\leqq 7}, \{g_{Bijk}\}_{0\leqq i,j,k\leqq 7})$ such that
$f_B(X,Y):= B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(\ldots(B_1X)\ldots) \bmod q\in O[X,Y]=\{f_{Bijk}\}(i,j,k=0,\ldots,7)$,(50a)
$g_B(X,Y):= (\ldots(XB_1)\ldots)B_{rB})Y)B_{rB}^{-1})\ldots)B_1^{-1} \bmod q\in O[X,Y]=\{g_{ijk}\}(i,j,k=0,\ldots,7)$. (50b)
User B sends $\{f_{Bijk}\},\{g_{Bijk}\}$ $(i,j,k=0,\ldots,7)$ to system centre.

4) User B downloads $f_A(X,Y)=\{f_{Aijk}\}$, $g_A(X,Y)=\{g_{Aijk}\}$ $(i,j,k=0,\ldots,7)$ from system centre.

5) User B generates the common enciphering function $f_{BA}(X,Y)$ as follows.

$f_{B1\text{-}1}(X,Y):=f_A(f_A(X,Y),B_1^{-1})$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(A_{rA}(\ldots(A_1[A_1^{-1}(\ldots(A_{rA}^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots))))\ldots)])\ldots))))\ldots)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)))))\ldots) \bmod q\in O[X,Y]$

$f_{B2\text{-}1}(X,Y):= f_{B1\text{-}1}(E_A(X,Y),B_2^{-1})$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(B_2^{-1}(A_{rA}(\ldots(A_1^{-1}[A_1(\ldots(A_{rA}^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)\ldots)])\ldots)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(B_2^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)\ldots) \bmod q\in O[X,Y]$

… …

$f_{BrB\text{-}1}(X,Y):= f_{BrB\text{-}1\text{-}1}(f_A(X,Y),B_{rB}^{-1})$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(A_{rA}(\ldots(A_1[A_1^{-1}(\ldots(A_{rA}^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)\ldots)])\ldots)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(A_{rA}(\ldots(A_1X)\ldots)\ldots) \bmod q\in O[X,Y]$

$f_{BrB}(X,Y):= f_{BrB\text{-}1}(f_A(X,B_{rB}),Y)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(A_{rA}(\ldots(A_1[A_1^{-1}(\ldots(A_{rA}^{-1}(B_{rB}(A_{rA}(\ldots(A_1X)\ldots)\ldots)])\ldots)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(A_{rA}(\ldots(A_1X)\ldots) \bmod q\in O[X,Y]$

$f_{BrB\text{-}1}(X,Y)= f_{BrB}(f_A(X,B_{rB\text{-}1}),Y)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(A_{rA}(\ldots(A_1[A_1^{-1}(\ldots(A_{rA}^{-1}(B_{rB\text{-}1}(A_{rA}(\ldots(A_1X)\ldots)])\ldots)$
$) \bmod q\in O[X,Y]$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(B_{rB\text{-}1}(A_{rA}(\ldots(A_1X)\ldots) \bmod q\in O[X,Y]$

… …

$f_{BA}(X,Y)=f_{B1}(X,Y) := f_{B2}(f_A(X,B_1),Y)$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(\ldots(B_2(A_{rA}(\ldots(A_1[A_1^{-1}(\ldots(A_{rA}^{-1}(B_1(A_{rA}(\ldots(A_1X)\ldots)$
$])\ldots) \bmod q \in O[X,Y]$

$= A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(Y\,(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1X)\ldots) \bmod q \in O[X,Y]$

$=\{f_{BAijk}\}(i,j,k=0,\ldots,7).$ (51a)

6) User B generates the sub-common enciphering function $g_{BA}(X,Y)$ in the same manner as follows.

$g_{BA}(X,Y):=(\ldots((XA_1)A_2)\ldots)A_{rA})B_1)..)B_{rB})Y)B_{nB}^{-1})..)B_1^{-1})A_{nA}^{-1})\ldots)A_1^{-1} \bmod q \in O[X,Y]$. (51b)

7) User A generate $f_{AB}(X,Y)$ and $g_{AB}(X,Y)$such that

$f_{AB}(X,Y):=A_1^{-1}(\ldots(A_{rA}^{-1}\,(f_B((A_{rA}(\ldots(A_1X)\ldots),Y)))\ldots) \in O[X,Y]$

$=A_1^{-1}(\ldots(A_{rA}^{-1}\,(B_1^{-1}(\ldots(B_{rB}^{-1}(Y(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1X)\ldots) \in O[X,Y]$

$=\{f_{ABijk}\}(i,j,k=0,\ldots,7) = f_{BA}(X,Y) \in O[X,Y]$, (52a)

$g_{AB}(X,Y):=(\ldots((XA_1)A_2)\ldots)A_{rA})B_1)..)B_{rB})Y)B_{rB}^{-1})..)B_1^{-1})A_{rA}^{-1})\ldots)A_1^{-1} \bmod q \in O[X,Y]$. (52b)

8) User B enciphers the plaintext p by using $f_{BA}(X,Y)$ such that

$C(p):= f_{BA}(M, T_{BA}) \in O$

$= A_1^{-1}(\ldots(A_{rA}^{-1}\,(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{BA}\,(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1M)\ldots) \bmod q$

$=(c_0,\ldots,c_7)$ (53a)

where

$M=pG+uH+vGH+wHG \bmod q \in O$, (53b)

$T_{BA} =f_{BA}(\mathbf{1},S) \in O$. (53c)

9) User B sends $C(p)= (c_0,\ldots,c_7)$ to user A through the insecure line.

10) User A receives $C(p)= (c_0,\ldots,c_7)$ and deciphers as follows.User A calculates

$T_{AB}= f_{AB}(\mathbf{1},S)$

$= A_1^{-1}(\ldots(A_{rA}^{-1}\,(B_1^{-1}(\ldots(B_{rB}^{-1}(S(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1\mathbf{1})\ldots) = T_{BA} \bmod q$. (54)

Let $(m_0,\ldots,m_7) := M$.

$f_{AB}(M, T_{AB})$

$=A_1^{-1}(\ldots(A_{nA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{AB}(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1\,M)\ldots) \bmod q \in O$ (55a)

$=(f'_{000}m_0+f'_{001}m_1+ \ldots+f'_{007}m_7,$

$f'_{100}m_0+f'_{101}m_1+ \ldots+f'_{107}m_7,$

$\ldots \quad \ldots$

$f'_{700}m_0+f'_{701}m_1+ \ldots+f'_{707}m_7)^t \bmod q,$ (55b)

$= C(p)=(c_0,\ldots,c_7).$ (55c)

where

$f'_{ijk} \in \boldsymbol{Fq}$ $(i,j,k=0,\ldots,7)$.

$(m_0,\ldots,m_7)$ is obtained by solving above simultaneous equation.

11) User A recovers the plaintext p as follows

$p=2[M]_0 =2\ m_0 \bmod q \in \boldsymbol{Fq}$.

Theorem 15

For arbitrary $P,Q \in O$

$f_{BA}(P, T_{BA})\ g_{BA}(Q, T_{BA})$

$= f_{BA}(PQ, T_{BA})\ g_{BA}(\mathbf{1}, T_{BA})$

$= f_{BA}(\mathbf{1}, T_{BA})\ g_{BA}(PQ, T_{BA}) \bmod q \in O.$ (56)

[*Proof*]

From Theorem 13

$f_{BA}(P, T_{BA})g_{BA}(Q, T_{BA})$

$= f_{BA}(PQ, T_{BA})\ g_{BA}(\mathbf{1}, T_{BA}) \bmod q \in O$

$=f_{BA}(\mathbf{1}, T_{BA})g_{BA}(PQ, T_{BA}) \bmod q \in O.$ q.e.d.

We notice that

$f_{BA}(P, T_{BA})g_{BA}(\mathbf{1}, T_{BA})= f_{BA}(\mathbf{1}P, T_{BA})g_{BA}(\mathbf{1}, T_{BA})$

$=f_{BA}(\mathbf{1}, T_{BA})g_{BA}(P, T_{BA}) \bmod q \in O.$

Then we have

$$g_{BA}(P,T_{BA})=(f_{BA}(\mathbf{1},T_{BA}))^{-1}[f_{BA}(P,T_{BA})g_{BA}(\mathbf{1},T_{BA})] \bmod q \ \in O, \quad (57a)$$

$$f_{BA}(P,T_{BA}) =[f_{BA}(\mathbf{1},T_{BA})g_{BA}(P,T_{BA})] \ (g_{BA}(\mathbf{1},T_{BA}))^{-1} \bmod q \in O. \quad (57b)$$

In case that the third party that does not know the value of T_{BA} try to calculate the ciphertext of product of two plaintexts, he uses $[f_{BA}(\mathbf{1}, T_{BA}), g_{BA}(\mathbf{1}, T_{BA})]$ such that

$$\{f_{BA}(M_1,T_{BA})[(f_{BA}(\mathbf{1}, T_{BA}))^{-1}(f_{BA}(M_2, T_{BA}) g_{BA}(\mathbf{1}, T_{BA}))]\}(g_{BA}(\mathbf{1}, T_{BA}))^{-1}$$

$$=f_{BA}(M_1M_2,T_{BA}) \bmod q \in O. \quad (58)$$

We describe in detail in next section.

§3.9 Procedure foraddition and multiplication on ciphertexts by third party

Here we show the procedure that the third party calculates the ciphertexts of the sum and the product of h plaintexts by using h ciphertexs.

1) User B uploads his data $\{C(p_i) = f_{BA}(M_i,T_{BA})$ (i=0,…,n) } and $[f_{BA}(\mathbf{1}, T_{BA}), g_{BA}(\mathbf{1}, T_{BA})]$ to the cloud centre by using the common enciphering function $f_{BA}(X,Y)$ and sub-common enciphering function $g_{BA}(X,Y)$of user A and user B through the insecure line.

2) User B requests the ciphertext of the sum of p_i (i=0,…,n) and the ciphertext of the product of p_i (i=0,…,n) to user D(D is the data processing centre or the cloud centre).

3) User D downloads the system parameter $[q,G,H,S]$ from system centre .

4) User D downloads $\{C(p_i) = f_{BA}(M_i,T_{BA})$ (i=0,…,n) } and $[f_{BA}(\mathbf{1}, T_{BA}), g_{BA}(\mathbf{1}, T_{BA})]$ from cloud centre where

$M_i=p_iG+u_iH+v_iGH+w_iHG \bmod q \in O$, $p_i, u_i, v_i, w_i \in \boldsymbol{Fq}$, (i=0,…,$n$).

5) User D calculates $[f_{BA}(\mathbf{1},T_{BA})]^{-1} \bmod q$,$[g_{BA}(\mathbf{1},T_{BA})]^{-1} \bmod q$.

6) User D calculates the ciphertext of the sum of p_i (i=0,…,n) and the ciphertext of the product of p_i (i=0,…,n) as follows.

$$C(p_1+\ldots+ p_h)=f_{BA}(M_1 +\ldots+M_h,T_{BA})= f_{BA}(M_1,T_{BA}) +\ldots+ f_{BA}(M_h,T_{BA}) \bmod q \in O,$$

$f_{BA}(M_1M_2,T_{BA})$

$= \{f_{BA}(M_1,T_{BA})[(f_{BA}(\mathbf{1}, T_{BA}))^{-1}(f_{BA}(M_2, T_{BA}) g_{BA}(\mathbf{1}, T_{BA}))]\}(g_{BA}(\mathbf{1}, T_{BA}))^{-1} \bmod q$

$f_{BA}(M_1M_2M_3,T_{BA})$

$=\{f_{BA}(M_1M_2,T_{BA})[(f_{BA}(\mathbf{1},T_{BA})^{-1}(f_{BA}(M_3,T_{BA})g_{BA}(\mathbf{1},T_{BA}))]\}(g_{BA}(\mathbf{1},T_{BA}))^{-1} \bmod q$

… …

$f_{BA}(M_1M_2\ldots M_h,T_{BA})$

$=\{f_{BA}(M_1M_2\ldots M_{h-1},T_{BA})[f_{BA}(\mathbf{1},T_{BA})]^{-1}(f_{BA}(M_h,T_{BA})g_{BA}(\mathbf{1},T_{BA}))]\}(g_{BA}(\mathbf{1},T_{BA}))^{-1} \bmod q \in O$

where

$M_1M_2\ldots M_n= p_1\, p_2\, \ldots p_nG+0H+(v_{12\ldots n})GH+(w_{12\ldots n})HG \bmod q \in O,$

$v_{12\ldots n}, w_{12\ldots n} \in \boldsymbol{Fq}.$

We can recognize $f_{BA}(M_1M_2\ldots M_n,T_{BA})$ as the ciphertext of the product of p_i (i=0,…,n).

7) User D sends { $f_{BA}(M_1+\ldots+M_n,T_{BA})$, $f_{BA}(M_1M_2\ldots M_n,T_{BA})$} to user B.

8) User B deciphers to obtain $p_1+\ldots+p_n \bmod q$ and $p_1\, p_2\, \ldots p_n \bmod q$ as follows.

Let

$M_+=(m_{+0},\ldots,m_{+7}):= M_1+\ldots+M_n \bmod q,$

$M_*=(m_{*0},\ldots,m_{*7}):= M_1M_2\ldots M_n \bmod q.$

$f_{AB}(M_+, T_{AB})$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{AB}(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1 M_+)\ldots) \bmod q \in O$

$=(f_{+000}m_{+0}+f_{+001}\, m_{+1}+ \ldots+f_{+007}\, m_{+7},$

$f_{+100}m_{+0}+ f_{+101}m_{+1}+ \ldots+ f_{+107}m_{+7},$

…. ….

$f_{+700}m_{+0}+ f_{+701}m_{+1}+ \ldots+ f_{+707}m_{+7})^t \bmod q,$

$= (c_{+0},\ldots,c_{+7}).$

$(m_{+0},\ldots,m_{+7})$ is obtained by solving above simultaneous equation.

$2(m_{+0})= p_1+\ldots+ p_n \bmod q \in \boldsymbol{Fq}.$

$f_{AB}(M_*, T_{AB})$

$=A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{AB}(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1 M_*)\ldots) \bmod q \in O$

$=(f_{*000}m_{*0}+f_{*001}\, m_{*1}+ \ldots+f_{*007}\, m_{*7},$

$f_{*100}m_{*0}+f_{*101}m_{*1}+ \ldots+ f_{*107}m_{*7},$

$\ldots \quad \ldots$

$f_{*700}m_{*0}+ f_{*701}m_{*1}+ \ldots+ f_{*707}m_{*7})^{t} \bmod q,$

$= (c_{*0},\ldots,c_{*7}).$

$(m_{*0},\ldots,m_{*7})$ is obtained by solving above simultaneous equation.

$2(m_{*0})= p_1\, p_2 \ldots p_n \bmod q \in \boldsymbol{Fq}.$

§4. Analysis of proposed scheme

Here we analyze the proposed fully homomorphic encryption scheme.

§4.1 Computing A_i from $\{f_{ijk}\}$,coefficients of $f(X,Y)$ and $g(X,Y)$

Basic enciphering function $f(X,Y)$ is given as follows.

Let $X=(x_0,\ldots,x_7)\in O[X]$ and $Y=(y_0,\ldots,y_7)\in O[X]$ be variables.

$f(X,Y)= A_1^{-1}(\ldots(A_r^{-1}(Y(A_r(..\ (A_1X)...) \bmod q\in O[X,Y]$

$=(f_{000}x_0y_0+f_{001}x_0y_1+ \ldots+f_{077}x_7y_7,$

$f_{100}x_0y_0+f_{101}x_0y_1+ \ldots+f_{177}x_7y_7,$

$\ldots \quad \ldots$

$f_{700}x_0y_0+f_{701}x_0y_1+ \ldots+f_{777}x_7y_7)^{t},$

$= \{f_{ijk}\}\ (i,j,k=0,\ldots,7),$

$g(X,Y):=(\ldots((XA_1)A_2)\ldots)A_r)Y)A_r^{-1})\ldots)A_1^{-1} \bmod q\in O[X,Y]$

$=(g_{000}x_0y_0+g_{001}x_0y_1+ \ldots+g_{077}x_7y_7,$

$g_{100}x_0y_0+g_{101}x_0y_1+ \ldots+g_{177}x_7y_7,$

$\ldots \quad \ldots$

$g_{700}x_0y_0+g_{701}x_0y_1+ \ldots+g_{777}x_7y_7)^{t},$

$= \{g_{ijk}\}(i,j,k=0,\ldots,7).$

$A_j\in O$ to be selected randomly such that A_j^{-1} exist $(j=1,\ldots,r)$ are the secret keys of user

A.

We try to find $A_i(i=1,...,r)$ from f_{ijk}, $g_{ijk} \in \boldsymbol{Fq}$ $(i,j,k=0,\ldots,7)$.

In case that r=56 the number of unknown variables ($A_j(i,j,k=1,\ldots,56)$) is 448(=64*8-64), the number of equations is 896(=(64*8-64)*2) such that

$$\left.\begin{array}{l} F_{001}(A_1,\ldots,A_{56})=f_{001} \bmod q, \\ \ldots \quad \ldots \\ F_{ijk}(A_1,\ldots,A_{56})=f_{ijk} \bmod q \ (k\neq 0), \\ \ldots \quad \ldots \\ F_{777}(A_1,\ldots,A_{64})=f_{777} \bmod q, \end{array}\right\}$$

$$\left.\begin{array}{l} G_{001}(A_1,\ldots,A_{56})=g_{001} \bmod q, \\ \ldots \quad \ldots \\ G_{ijk}(A_1,\ldots,A_{56})=g_{ijk} \bmod q \ (k\neq 0), \\ \ldots \quad \ldots \\ G_{777}(A_1,\ldots,A_{64})=g_{777} \bmod q, \end{array}\right\}$$

where $F_{001},\ldots, F_{777}, G_{001},\ldots, G_{777}$ are the $112(=56*2)^{th}$ algebraic multivariate equations.

Then the complexity G required for solving above simultaneous equations by using Gröbner basis[8] is given such as

$$G > G' = ({}_{448+dreg}C_{dreg})^w = ({}_{493}C_{45})^w = 2^{509} >> O(2^{80}),$$

where G' is the complexity required for solving 896=448*2 simultaneous quadratic equations with 448 variables by using Gröbner basis, where w=2.39, and

$$d_{reg}=0.0858*448+1.04*(448)\text{^}(1/3)-1.47+1.71*448\text{^}(-1/3)+O(448\text{^}(-2/3))>45.$$

The complexity G required for solving above simultaneous equations by using Gröbner basis is enough large for secure.

§4.2 Computing plaintext p and A_i, B_j from coefficients of ciphertext $C(p)$

Ciphertext $C(p) = (c_0,\ldots,c_7) = f_{BA}(M,T_{BA})$ is generated by user B as follows.

$C(p) = f_{BA}(M,T_{BA}) \in O$

$$= A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{BA}\,(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1\,M)\ldots)\in O$$

$$=(f'_{000}m_0+f'_{001}m_1+\ \ldots+f'_{007}m_7,$$

$$f'_{100}m_0+f'_{101}m_1+\ \ldots+f'_{107}m_7,$$

$$\ldots\qquad\ldots$$

$$f'_{700}m_0+f'_{701}m_1+\ \ldots+f'_{707}m_7)^t,$$

$$= (c_0,\ldots,c_7).$$

where

$$M=pG+uH+vGH+wHG \bmod q=(m_0,\ldots,m_7)\in \boldsymbol{Fq}.$$

$(m_0,\ldots,m_7)$ is obtained by solving above simultaneous equation.

A_j, $B_k\in O$ to be selected randomly such that A_j^{-1} and B_k^{-1} exist (j=1,...,r_A;k=1,…,r_B) are the secret keys of user A and user B respectively.

We try to find plaintext p and A_i, B_j (i=1,...,r_A;j=1,…,r_B) from elements of $f_{BA}(M, T_{BA})$ and $c_i\in \boldsymbol{Fq}$ (i=0,…,7).

In case that r_A =56 and r_B=56 the number of unknown variables (p,u,v,w, T_{BA}, A_j, B_k, (j,k=1,…,56)) is 908(=4+8+2*56*8), the number of equations is 8 such that

$$\left.\begin{aligned} &F_0(p,u,v,w,A_1,\ldots,A_{56}, B_1,\ldots,B_{56})=c_0 \bmod q,\\ &F_1(p,u,v,w,A_1,\ldots,A_{56}, B_1,\ldots,B_{56})=c_1 \bmod q,\\ &\quad\ldots\qquad\ldots\\ &F_7(p,u,v,w,A_1,\ldots,A_{56}, B_1,\ldots,B_{56})=c_7 \bmod q, \end{aligned}\right\}$$

where $F_0,\ldots,F_7$ are the 226(=56*2*2+2)th algebraic multivariate equations.

Then the complexity G required for solving above simultaneous equations by using Gröbner basis [8] is givensuch as

$$G>G'=({}_{908+dreg}C_{dreg})^w=({}_{103170}C_{908})^w >> O(2^{80}),$$

where G' is the complexity required for solving 909 simultaneous algebraic equations with 908 variables by using Gröbner basis,
where w=2.39, and

$$d_{reg}= 102262(=909*(226-1)/2 - 0\sqrt{\ }(909*(226\text{^}2-1)/6)).$$

The complexity G required for solving above simultaneous equations by using

Gröbner basis is enough large for safety.

§4.3 Attack by using the ciphertexts of *p* and -*p*

I show that we can not easily distinguish the ciphertexts of p and $-p$.

We try to attack by using "p and $-p$ attack".We define the medium text M_+ by

$$M_+ := p\mathbf{1}+uG+vH+wGH \in O,$$

where $u,v,w \in \boldsymbol{Fq}$ is selected randomly, and plaintext $p \in \boldsymbol{Fq}$.

We define the medium text M_- corresponding to the plaintext $-p$ by

$$M_- := -p\mathbf{1}+u'G+v'H+w'GH \in O,$$

where u', v',$w' \in \boldsymbol{Fq}$ is selected randomly.

The ciphertext of p , $C(p) = f_{BA}(M, T_{BA})$ is given as follows.

$$C(p) = f_{BA}(M, T_{BA}) \in O$$

$$= A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{BA}\,(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1\,M)\ldots) \in O.$$

The ciphertext of p , $C(-p) = f_{BA}(M_-, T_{BA})$ is given as follows.

$$C(-p) = f_{BA}(M_-, T_{BA}\,R)$$

$$= A_1^{-1}(\ldots(A_{rA}^{-1}(B_1^{-1}(\ldots(B_{rB}^{-1}(T_{BA}\,(B_{rB}(\ldots(B_1(A_{rA}(\ldots(A_1\,M_-)\ldots) \in O.$$

As $p-p \bmod q = 0$, we have

$$f_{BA}(M, T_{BA}) + f_{BA}(M_-, T_{BA}) \bmod q$$

$$= f_{BA}(M+M_-, T_{BA}) \bmod q$$

$$= f_{BA}(p\mathbf{1}+uG+vH+wGH\ -p\mathbf{1}+u'G+v'H+w'GH, T_{BA}) \bmod q$$

$$= f_{BA}((u+u'))\,G+(v+\,v')H+(w+\,w')GH, T_{BA}) \bmod q$$

As in general $u+u' \neq 0 \bmod q$, $v+v' \neq 0 \bmod q$,$w+w' \neq 0 \bmod q \in \boldsymbol{Fq}$,we have

$$f_{BA}(M, T_{BA}) + f_{BA}(M_-, T_{BA}) \bmod q \neq \mathbf{0}.$$

We can calculate $| f_{BA}(M, T_{BA}) + f_{BA}(M_-, T_{BA}) |^2$ as follows.

$$| f_{BA}(M, T_{BA}) + f_{BA}(M_-, T_{BA}) |^2 \bmod q$$

$=| f_{BA}((u+u'))\ G+(v+v')H+(w+w')GH, T_{BA})|^2 \bmod q$

$=| T_{BA} |^2 |(u+u'))\ G+(v+v')H+(w+w')GH|^2 \bmod q$

$=| T_{BA} |^2 0 \bmod q=0 \bmod q$.

On the otherhand we know from theorem 9, we have always

$|M|^2=|pG+uH+vGH+wHG|^2=0 \bmod q \in Fq$.

Then $| f_{BA}(M, T_{BA})+ f_{BA}(M., T_{BA}) |^2 =0 \bmod q$.

It is said that the attack by using "p and $-p$ attack" is not efficient. Then we can not easily distinguish the ciphertexts of p and $-p$.

§5. The size of the modulus q and the complexity for enciphering /deciphering

We consider the size of the system parameter q. We select $q=O(2^{2000})$.

1) In case of $r=56$, $q=O(2^{2000})$, the size of $f_{ijk} \in \boldsymbol{Fq}$ $(i,j,k=0,\ldots,7)$ which are the coefficients of elements in $f(X,Y)= A_1^{-1}(\ldots(A_r^{-1}(Y(A_r(.. (A_1X)...) \bmod q \in O[X,Y]$ is $(448)(\log_2 q)$bits $=896$kbits and the size of $g_{ijk} \in \boldsymbol{Fq}$ $(i,j,k=0,\ldots,7)$ which are the coefficients of elements in $g(X,Y):=(\ldots((XA_1)A_2)\ldots)A_r)Y)A_r^{-1})\ldots)A_1^{-1} \bmod q \in O[X,Y]$ is $(448)(\log_2 q)$bits $=896$kbits. Then the size of $f_{ijk}$and g_{ijk} $(i,j,k=0,\ldots,7)$ is $2*(448)(\log_2 q)$bits $=1792$kbits. The size of plaintext p is 2kbits and and the size of ciphertext $C(p)$ is 16kbits.

2) In case of $r=56$, $q=O(2^{2000})$, the complexity to obtain $f(X,Y)$ from $A_1,\ldots,A_r$ and q is

$(55*8*64+55*8*512)(\log_2 q)^2+56*(16*(\log_2 q)^2+2*(\log_2 q)^3)= O(2^{41})$ bit-operations,

where $56*(16*(\log_2 q)^2+2*(\log_2 q)^3)$ is the complexiy for inverse of $\boldsymbol{A}_i^{-1}(i=1,\ldots,56)$.

And the complexity to obtain $g(X,Y)$ from $A_1,\ldots,A_r$ and q is $O(2^{41})$ bit-operations.

3) In case of $r_B=56$, $q=O(2^{2000})$, the complexity to obtain $f_{BA}(X,Y)$ from $f_A(X,Y)$, $B_1,\ldots,B_{rB}, B_{rB}^{-1},\ldots,B_1^{-1}$ and q is

$((512+(64+1)*8*8)*56+(512+2*64*8*8)*56)\ (\log_2 q)^2= O(2^{42})$ bit-operations.

In the same manner, the complexity to obtain $g_{BA}(X,Y)$ from $f_A(X,Y)$, $B_1,\ldots,B_{rB}$, $B_{rB}^{-1},\ldots,B_1^{-1}$ and q is

$((512+(64+1)*8*8)*56+(512+2*64*8*8)*56)\ (\log_2 q)^2 = O(2^{42})$ bit-operations.

4) In case of $r_A=56$, $q=O(2^{2000})$, the complexity to obtain $f_{AB}(X,Y)$ from $f_B(X,Y)$, $A_1,\ldots,A_{rA}$, $A_{rA}^{-1},\ldots,A_1^{-1}$ and q is

$(64*8*55+8*64*8+56*8*8*64)\ (\log_2 q)^2 = O(2^{41})$ bit-operations.

5) In case of $q=O(2^{2000})$, the complexity for enciphering p to obtain $C(p)= f_{BA}(M, T_{BA})$ from $f_{BA}(X, Y), M, T_{BA}$ and q is

$(2*64*8)\ (\log_2 q)^2 = O(2^{32})$ bit-operations.

6) In case of $r_A=56$, $q=O(2^{2000})$, the complexity for calculating $T_{BA} = f_{BA}(\mathbf{1},S)$ from $f_{BA}(X,Y)$, S and q is $64(\log_2 q)^2 = O(2^{28})$ bit-operations.

7) In case of $r_A=56$, $q=O(2^{2000})$, the complexity for deciphering $C(p) = f_{AB}(M, T_{AB})$ to obtain p from $C(p)$, $f_{AB}(X,Y)$, T_{AB} and q is

$[(64*8+8*8+7*7+\ldots+2*2+1*1+1+2+\ldots+7)+1](\log_2 q)^2+8*2*(\log_2 q)^3$

$= O((512+233)2^{22}+2^{37}) = O(745*2^{22}+2^{37}) = O(2^{38})$ bit-operations.

8) In case of $r_A=56$, $q=O(2^{2000})$, the complexity for calculating $f_{BA}(M_1M_2,T_{BA})$ from $f_{BA}(M_1,T_{BA})$, $f_{BA}(M_2,S)$, $f_{BA}(\mathbf{1},T_{BA})$, $g_{BA}(\mathbf{1},T_{BA})$ and q is

$4*64(\log_2 q)^2+2*2*(\log_2 q)^3 = O(2^{36})$ bit-operations.

On the other hand the complexity of the enciphering and deciphering in RSA scheme is

$O(2(\log n)^3)=O(2^{34})$ bit-operations

where the size of modulus n is 2048bits.

Then our scheme does not require large complexity to encipher and decipher so that we are able to implement our scheme to the mobile device.

§6. Conclusion

We proposed the fully homomorphic public-key encryption scheme with small size of ciphertext based on the octonion ring over finite field. It was shown that our

scheme is immune from the Gröbner basis attacks by calculating the complexity to obtain the Gröbner basis of the multivariate algebraic equations**.**The proposed scheme does not require a “bootstrapping” process so that the complexity to encipher and decipher is not large.

§7. BIBLIOGRAPHY

[1] Masahiro, Y. (2015). Fully Homomorphic Encryption without bootstrapping. Saarbrücken/Germany: LAP LAMBERT Academic Publishing.
[2] Mashiro Yagisawa," Fully Homomorphic Encryption without bootstrapping", Cryptology ePrint Archive, Report 2015/474, 2015. http://eprint.iacr.org/.
[3] Shigeo Tsujii , Kohtaro Tadaki , Masahito Gotaishi ,Ryo Fujita ,and Masao Kasahara ,"Proposal Integrated MPKC:PPS—STS Enhanced Perturbed Piece in Hand Method---," IEICE Tech. Rep.ISEC2009-27,SITE2009-19,ICSS2009-41(2009-07),July 2009.
[4] T. Matsumoto, and H. Imai, "Public quadratic polynomial-tuples for efficient signature verification and message-encryption," Lecture Notes in Computer Science on Advances in Cryptology-EUROCRYPT'88, pp.419–453, New York, NY, USA, 1988, Springer-Verlag New York, Inc.
[5] S. Tsujii, K. Tadaki, and R. Fujita, "Piece in hand concept for enhancing the security of multivariate type public key cryptosystems: Public key without containing all the information of secret key," Cryptology ePrint Archive, Report 2004/366, 2004.
[6] C.Wolf, and B. Preneel, "Taxonomy of public key schemes based on the problem of multivariate quadratic equations," Cryptology ePrint Archive, Report 2005/077, 2005, http://eprint.iacr.org/.
[7] Shigeo Tsujii , Kohtaro Tadaki , Masahito Gotaishi ,Ryo Fujita ,and Masao Kasahara ,"Proposal Integrated MPKC:PPS—STS Enhanced Perturbed Piece in Hand Method---," IEICE Tech. Rep.ISEC2009-27, SITE2009-19, ICSS2009-41(2009-07), July 2009.
[8] M. Bardet, J. C. Faugere, and B. Salvy, "On the complexity of Gröbner basis computation of semi-regular overdetermined algebraic equations," Proceeding of the International Conference on Polynomial System Solving(ICPSS2004), pp.71-75, November 2004.
[9] Craig Gentry. Fully Homomorphic Encryption Using Ideal Lattices.In the 41st ACM Symposium on Theory of Computing (STOC), 2009.
[10] Craig Gentry, A Fully Homomorphic Encryption Scheme, 2009. Available at http://crypto.stanford.edu/craig/craig-thesis.pdf .
[11] Marten van Dijk; Craig Gentry, Shai Halevi, and Vinod Vaikuntanathan (2009-12-11). "Fully Homomorphic Encryption over the Integers" (PDF). International Association for Cryptologic Research. Retrieved 2010-03-18.
[12] Damien Stehle; Ron Steinfeld (2010-05-19). "Faster Fully Homomorphic Encryption" (PDF). International Association for Cryptologic Research. Retrieved 2010-09-15.
[13] JS Coron, A Mandal, D Naccache, M Tibouchi ," Fully homomorphic

encryption over the integers with shorter public keys", Advances in Cryptology–CRYPTO 2011, 487-504.
[14] Halevi, Shai. "An Implementation of homomorphic encryption". Retrieved 30 April 2013. Available at https://github.com/shaih/HElib .
[15] Nuida and Kurosawa,"(Batch) Fully Homomorphic Encryption over Integers for Non-Binary Message Spaces", Cryptology ePrint Archive, Report 2014/777, 2014. http://eprint.iacr.org/.
[16] John H. Conway, Derek A. Smith co-authored, translated by Syuuji Yamada, "On Quaternions and Octonions " Baifuukan Publication Center, Tokyo, .2006.
[17] Yongge Wang," Notes on Two Fully Homomorphic Encryption Schemes Without Bootstrapping", Cryptology ePrint Archive, Report 2015/519, 2015. http://eprint.iacr.org/.
[18] Mashiro Yagisawa," Improved Fully Homomorphic Encryption without Bootstrapping", Cryptology ePrint Archive, Report 2017/763, 2017. http://eprint.iacr.org/.
[19] Mashiro Yagisawa," Fully homomorphic encryption with small ciphertext size", Cryptology ePrint Archive, Report 2018/088, 2018. http://eprint.iacr.org/.

Appendix A:

Octinv(*A*) ----------

S ← $a_0^2+a_1^2+\ldots+a_7^2$ mod *q*.
% S^{-1} mod *q*
q[1] ← q div S ;% integer part of q/S
r[1] ← q mod S ;% residue
k ←1
q[0] ← q
r[0] ← S
while r[k] ≠ 0
begin
k← k + 1
q[k] ← r[k−2] div r[k−1]
r[k] ← r[k−2] mod [rk−1]
end
Q [k−1] ← (-1)∗q[k−1]
L[k−1] ← 1
i ← k−1
while i > 1
begin
Q[i−1] ← (-1)∗Q[i] ∗q[i−1] + L[i]
L[i−1] ← Q[i]
i← i−1
end

invS ← Q[1] mod *q*
invA[0] ← a_0∗invS mod *q*
For *i*=1,…,7,
invA[*i*] ← (-1)∗a_i∗invS mod *q*
Return A^{-1}= (invA[0], invA[1],…, invA[7])

Appendix B:

Lemma 2

$A^{-1}(AB)= B \bmod q$

$(BA)A^{-1}= B \bmod q$

(*Proof*:)

$A^{-1}= (a_0/|A|^2 \bmod q, -a_1/|A|^2 \bmod q,\ldots, -a_7/|A|^2 \bmod q).$

$AB \bmod q$

$= (a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7 \bmod q,$
$a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3 \bmod q,$
$a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6 \bmod q,$
$a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1 \bmod q,$
$a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5 \bmod q,$
$a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4 \bmod q,$
$a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2 \bmod q,$
$a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0 \bmod q).$

$[A^{-1}(AB)]_0$

$=\{ a_0(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$+a_1(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$+ a_2(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$
$+a_3(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)$
$+a_4(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$
$+ a_5(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$
$+a_6(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$
$+a_7(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)\} /|A|^2 \bmod q$

$=\{(a_0^2+a_1^2+\ldots+a_7^2)\ b_0\} /|A|^2 =b_0 \bmod q$

where $[M\ ]_n$ denotes the n-th element of $M \in O$.

$[A^{-1}(AB)]_1$

$=\{ a_0(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$-a_1(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$-a_2(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$
$-a_3(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)$

$$+a_4(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$$
$$-a_5(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$$
$$+a_6(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$$
$$+a_7(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)\}/|A|^2 \bmod q$$

$$=\{(a_0^2+a_1^2+\ldots+a_7^2)\,b_1\}/|A|^2=b_1 \bmod q.$$

Similarly we have

$$[A^{-1}(AB)]_i=b_i \bmod q \ (i=2,3,\ldots,7).$$

Then we have

$$A^{-1}(AB)= B \bmod q. \qquad \text{q.e.d.}$$

Appendix C:

Theorem 7

Let O be the octonion ring over a finite field R such that

$$O=\{(a_0,a_1,\ldots,a_7) \mid a_j\in \boldsymbol{Fq}\ (j=0,1,\ldots,7)\}.$$

Let $G,H\in O$ be the octonions such that

$$G=(g_0,g_1,\ldots,g_7),\ g_j\in \boldsymbol{Fq}\ (j=0,1,\ldots,7),$$
$$H=(h_0,h_1,\ldots,h_7),\ h_j\in \boldsymbol{Fq}\ (j=0,1,\ldots,7),$$

where

$$g_0=1/2 \bmod q,\ h_0=0 \bmod q,$$
$$L_G=g_0^2+g_1^2+\ldots+g_7^2=0 \bmod q,$$
$$L_H=h_0^2+h_1^2+\ldots+h_7^2=0 \bmod q$$

and

$$g_1h_1+ g_2h_2+g_3h_3+{}_4h_4+ g_5h_5+g_6h_6+g_7h_7=0 \bmod q.$$

G and H satisfy the following equations.

$$(GH)G= \mathbf{0} \bmod q,$$
$$(HG)H= \mathbf{0} \bmod q.$$

(*Proof*:)

$$GH \bmod q$$

$$= (g_0h_0 - g_1h_1 - g_2h_2 - g_3h_3 - g_4h_4 - g_5h_5 - g_6h_6 - g_7h_7 \bmod q,$$
$$g_0h_1 + g_1h_0 + g_2h_4 + g_3h_7 - g_4h_2 + g_5h_6 - g_6h_5 - g_7h_3 \bmod q,$$
$$g_0h_2 - g_1h_4 + g_2h_0 + g_3h_5 + g_4h_1 - g_5h_3 + g_6h_7 - g_7h_6 \bmod q,$$
$$g_0h_3 - g_1h_7 - g_2h_5 + g_3h_0 + g_4h_6 + g_5h_2 - g_6h_4 + g_7h_1 \bmod q,$$
$$g_0h_4 + g_1h_2 - g_2h_1 - g_3h_6 + g_4h_0 + g_5h_7 + g_6h_3 - g_7h_5 \bmod q,$$
$$g_0h_5 - g_1h_6 + g_2h_3 - g_3h_2 - g_4h_7 + g_5h_0 + g_6h_1 + g_7h_4 \bmod q,$$
$$g_0h_6 + g_1h_5 - g_2h_7 + g_3h_4 - g_4h_3 - g_5h_1 + g_6h_0 + g_7h_2 \bmod q,$$
$$g_0h_7 + g_1h_3 + g_2h_6 - g_3h_1 + g_4h_5 - g_5h_4 - g_6h_2 + g_7h_0 \bmod q)$$

$$[(GH)G]_0 \bmod q$$
$$= (g_0h_0 - g_1h_1 - g_2h_2 - g_3h_3 - g_4h_4 - g_5h_5 - g_6h_6 - g_7h_7) g_0$$
$$-(g_0h_1 + g_1h_0 + g_2h_4 + g_3h_7 - g_4h_2 + g_5h_6 - g_6h_5 - g_7h_3) g_1$$
$$-(g_0h_2 - g_1h_4 + g_2h_0 + g_3h_5 + g_4h_1 - g_5h_3 + g_6h_7 - g_7h_6) g_2$$
$$-(g_0h_3 - g_1h_7 - g_2h_5 + g_3h_0 + g_4h_6 + g_5h_2 - g_6h_4 + g_7h_1) g_3$$
$$-(g_0h_4 + g_1h_2 - g_2h_1 - g_3h_6 + g_4h_0 + g_5h_7 + g_6h_3 - g_7h_5) g_4,$$
$$-(g_0h_5 - g_1h_6 + g_2h_3 - g_3h_2 - g_4h_7 + g_5h_0 + g_6h_1 + g_7h_4) g_5$$
$$-(g_0h_6 + g_1h_5 - g_2h_7 + g_3h_4 - g_4h_3 - g_5h_1 + g_6h_0 + g_7h_2) g_6,$$
$$-(g_0h_7 + g_1h_3 + g_2h_6 - g_3h_1 + g_4h_5 - g_5h_4 - g_6h_2 + g_7h_0) g_7) \bmod q$$

As

$$h_0 = 0 \bmod q,$$
$$L_G := g_0^2 + g_1^2 + \ldots + g_7^2 = 0 \bmod q,$$
$$L_H := h_0^2 + h_1^2 + \ldots + h_7^2 = 0 \bmod q$$

and

$$g_1h_1 + g_2h_2 + g_3h_3 + {}_4h_4 + g_5h_5 + g_6h_6 + g_7h_7 = 0 \bmod q,$$

we have

$$[(GH)G]_0 \bmod q$$

$$
\begin{aligned}
&=(g_0h_0 - g_1h_1 - g_2h_2 - g_3h_3 - g_4h_4 - g_5h_5 - g_6h_6 - g_7h_7)\, g_0 \\
&\quad -(g_0h_1 + g_1h_0 + g_2h_4 + g_3h_7 - g_4h_2 + g_5h_6 - g_6h_5 - g_7h_3)\, g_1 \\
&\quad -(g_0h_2 - g_1h_4 + g_2h_0 + g_3h_5 + g_4h_1 - g_5h_3 + g_6h_7 - g_7h_6)\, g_2 \\
&\quad -(g_0h_3 - g_1h_7 - g_2h_5 + g_3h_0 + g_4h_6 + g_5h_2 - g_6h_4 + g_7h_1)\, g_3 \\
&\quad -(g_0h_4 + g_1h_2 - g_2h_1 - g_3h_6 + g_4h_0 + g_5h_7 + g_6h_3 - g_7h_5)\, g_4, \\
&\quad -(g_0h_5 - g_1h_6 + g_2h_3 - g_3h_2 - g_4h_7 + g_5h_0 + g_6h_1 + g_7h_4)\, g_5 \\
&\quad -(g_0h_6 + g_1h_5 - g_2h_7 + g_3h_4 - g_4h_3 - g_5h_1 + g_6h_0 + g_7h_2)\, g_6 \\
&\quad -(g_0h_7 + g_1h_3 + g_2h_6 - g_3h_1 + g_4h_5 - g_5h_4 - g_6h_2 + g_7h_0)\, g_7 \\
&=h_1(-g_4g_2 - g_7g_3 + g_2g_4 - g_6g_5 + g_5g_6 + g_3g_7) \\
&+ h_2(g_4g_1 - g_5g_3 - g_1g_4 + g_3g_5 - g_7g_6 + g_6g_7) \\
&+ h_3(g_7g_1 + g_5g_2 - g_6g_4 - g_2g_5 + g_4g_6 - g_1g_7) \\
&+ h_4(-g_2g_1 + g_1g_2 + g_6g_3 - g_7g_5 - g_3g_6 + g_5g_7) \\
&+ h_5(g_6g_1 - g_3g_2 + g_2g_3 + g_7g_4 - g_1g_6 - g_4g_7) \\
&+ h_6(-g_5g_1 + g_7g_2 - g_4g_3 + g_3g_4 + g_1g_5 - g_2g_7) \\
&+ h_7(-g_3g_1 - g_6g_2 + g_1g_3 - g_5g_4 + g_4g_5 + g_2g_6) \\
&=0 \bmod q,
\end{aligned}
$$

$$
\begin{aligned}
&[(GH)G]_1 \bmod q \\
&= (g_0h_0 - g_1h_1 - g_2h_2 - g_3h_3 - g_4h_4 - g_5h_5 - g_6h_6 - g_7h_7)g_1 \\
&+(g_0h_1 + g_1h_0 + g_2h_4 + g_3h_7 - g_4h_2 + g_5h_6 - g_6h_5 - g_7h_3)g_0 \\
&+(g_0h_2 - g_1h_4 + g_2h_0 + g_3h_5 + g_4h_1 - g_5h_3 + g_6h_7 - g_7h_6)g_4 \\
&+(g_0h_3 - g_1h_7 - g_2h_5 + g_3h_0 + g_4h_6 + g_5h_2 - g_6h_4 + g_7h_1)g_7 \\
&-(g_0h_4 + g_1h_2 - g_2h_1 - g_3h_6 + g_4h_0 + g_5h_7 + g_6h_3 - g_7h_5)g_2 \\
&+(g_0h_5 - g_1h_6 + g_2h_3 - g_3h_2 - g_4h_7 + g_5h_0 + g_6h_1 + g_7h_4)g_6 \\
&-(g_0h_6 + g_1h_5 - g_2h_7 + g_3h_4 - g_4h_3 - g_5h_1 + g_6h_0 + g_7h_2)g_5 \\
&-(g_0h_7 + g_1h_3 + g_2h_6 - g_3h_1 + g_4h_5 - g_5h_4 - g_6h_2 + g_7h_0)g_3
\end{aligned}
$$

$$= h_1 (-g_1^2+g_0^2+g_4^2+g_7^2+g_2^2+g_6^2+g_5^2+g_3^2)$$

$$+ h_2 (-g_2g_1-g_4g_0+g_0g_4+g_5g_7-g_1g_2-g_3g_6-g_7g_5+g_6g_3)$$

$$+ h_3 (-g_3g_1-g_7g_0 -g_5g_4+g_0g_7 -g_6g_2+g_2g_6+g_4g_5- g_1g_3)$$

$$+ h_4 (-g_4g_1 +g_2g_0-g_1g_4-g_6g_7-g_0g_2+g_7g_6-g_3g_5+g_5g_3)$$

$$+ h_5 (-g_5g_1 -g_6g_0+g_3g_4-g_2g_7+g_7g_2+g_0g_6-g_1g_5-g_4g_3)$$

$$+ h_6 (-g_6g_1 +g_5g_0-g_7g_4+g_4g_7+g_3g_2-g_1g_6-g_0g_5-g_2g_3)$$

$$+ h_7(-g_7g_1 +g_3g_0+g_6g_4-g_1g_7-g_5g_2-g_4g_6+g_2g_5-g_0g_3)$$

$$= h_1 (-2g_1^2+L_G)-2g_1(h_2g_2+ h_3g_3+ h_4g_4+ h_5g_5+ h_6g_6+ h_7g_7)$$

$$= h_1 (L_G)-2g_1(h_1g_1+h_2g_2+ h_3g_3+ h_4g_4+ h_5g_5+ h_6g_6+ h_7g_7)$$

$$= 0 \bmod q.$$

In the same manner we have

$$[(GH)G]_i = 0 \bmod q \ (i=2,\ldots,7).$$

Then we have

$$(GH)G = \mathbf{0} \bmod q.$$

In the same manner we have

$$HG \bmod q$$

$$= (h_0g_0 - h_1g_1- h_2g_2- h_3g_3-h_4g_4- h_5g_5-h_6g_6-h_7g_7 \bmod q,$$

$$h_0g_1+h_1g_0+h_2g_4+h_3g_7-h_4g_2+h_5g_6-h_6g_5-h_7g_3 \bmod q,$$

$$h_0g_2-h_1g_4+h_2g_0+h_3g_5+h_4g_1-h_5g_3+h_6g_7-h_7g_6 \bmod q,$$

$$h_0g_3-h_1g_7-h_2g_5+h_3g_0+h_4g_6+h_5g_2-h_6g_4+h_7g_1 \bmod q,$$

$$h_0g_4+h_1g_2 -h_2g_1 -h_3g_6+h_4g_0+h_5g_7+ h_6g_3 -h_7g_5 \bmod q,$$

$$h_0g_5-h_1g_6+h_2g_3-h_3g_2-h_4g_7+h_5g_0+h_6g_1+h_7g_4 \bmod q,$$

$$h_0g_6+h_1g_5 -h_2g_7+h_3g_4 -h_4g_3 -h_5g_1+h_6g_0 +h_7g_2 \bmod q,$$

$$h_0g_7+h_1g_3+h_2g_6-h_3g_1+h_4g_5-h_5g_4-h_6g_2+h_7g_0 \bmod q) .$$

$$[(HG)H]_0$$

$$= (h_0g_0 - h_1g_1 - h_2g_2 - h_3g_3 - h_4g_4 - h_5g_5 - h_6g_6 - h_7g_7)h_0$$

$$-(h_0g_1 + h_1g_0 + h_2g_4 + h_3g_7 - h_4g_2 + h_5g_6 - h_6g_5 - h_7g_3)h_1$$

$$-(h_0g_2 - h_1g_4 + h_2g_0 + h_3g_5 + h_4g_1 - h_5g_3 + h_6g_7 - h_7g_6)h_2$$

$$-(h_0g_3 - h_1g_7 - h_2g_5 + h_3g_0 + h_4g_6 + h_5g_2 - h_6g_4 + h_7g_1)h_3$$

$$-(h_0g_4 + h_1g_2 - h_2g_1 - h_3g_6 + h_4g_0 + h_5g_7 + h_6g_3 - h_7g_5)h_4$$

$$-(h_0g_5 - h_1g_6 + h_2g_3 - h_3g_2 - h_4g_7 + h_5g_0 + h_6g_1 + h_7g_4)h_5$$

$$-(h_0g_6 + h_1g_5 - h_2g_7 + h_3g_4 - h_4g_3 - h_5g_1 + h_6g_0 + h_7g_2)h_6$$

$$-(h_0g_7 + h_1g_3 + h_2g_6 - h_3g_1 + h_4g_5 - h_5g_4 - h_6g_2 + h_7g_0)h_7 \bmod q$$

$$=0\ h_0 - g_0(h_1^2 + h_2^2 + \ldots + h_7^2)$$

$$+ g_1(-h_4h_2 - h_7\,h_3 + h_2\,h_4 - h_6h_5 + h_5\,h_6 + h_3\,h_7)$$

$$+ g_2(h_4h_1 - h_5\,h_3 - h_1\,h_4 + h_3\,h_5 - h_7\,h_6 + h_6\,h_7)$$

$$+ g_3(h_7h_1 + h_5\,h_2 - h_6\,h_4 - h_2\,h_5 + h_4\,h_6 - h_1\,h_7)$$

$$+ g_4(-h_2h_1 + h_1\,h_2 + h_6\,h_3 - h_7\,h_5 - h_3\,h_6 + h_5\,h_7)$$

$$+ g_5(h_6h_1 - h_3\,h_2 + h_2\,h_3 + h_7\,h_4 - h_1\,h_6 - h_4\,h_7)$$

$$+ g_6(h_6h_1 - h_3\,h_2 + h_2\,h_3 + h_7\,h_4 - h_1\,h_6 - h_4\,h_7)$$

$$+ g_7(-h_5h_1 + h_7\,h_2 - h_4\,h_3 + h_3\,h_4 + h_1\,h_5 - h_2\,h_7) \bmod q$$

$$=0 \bmod q.$$

$$[(HG)H]_1$$

$$= \quad (h_0g_0 - h_1g_1 - h_2g_2 - h_3g_3 - h_4g_4 - h_5g_5 - h_6g_6 - h_7g_7\ )h_1$$

$$+(h_0g_1 + h_1g_0 + h_2g_4 + h_3g_7 - h_4g_2 + h_5g_6 - h_6g_5 - h_7g_3)h_0$$

$$+(h_0g_2 - h_1g_4 + h_2g_0 + h_3g_5 + h_4g_1 - h_5g_3 + h_6g_7 - h_7g_6)h_4$$

$$+(h_0g_3 - h_1g_7 - h_2g_5 + h_3g_0 + h_4g_6 + h_5g_2 - h_6g_4 + h_7g_1)h_7$$

$$-(h_0g_4 + h_1g_2 - h_2g_1 - h_3g_6 + h_4g_0 + h_5g_7 + h_6g_3 - h_7g_5)h_2$$

$$+(h_0g_5 - h_1g_6 + h_2g_3 - h_3g_2 - h_4g_7 + h_5g_0 + h_6g_1 + h_7g_4)h_6$$

$$-(h_0g_6 + h_1g_5 - h_2g_7 + h_3g_4 - h_4g_3 - h_5g_1 + h_6g_0 + h_7g_2)h_5$$

$$-(h_0g_7+h_1g_3+h_2g_6-h_3g_1+h_4g_5-h_5g_4-h_6g_2+h_7g_0)h_3 \bmod q$$

$$= g_1(-h_1^2+h_4^2+h_7^2+h_2^2+h_6^2+h_5^2+h_3^2)$$

$$+ g_2(-h_2h_1+h_5h_7-h_1h_2-h_3h_6-h_7h_5+h_6h_3)$$

$$+ g_3(-h_3h_1-h_5h_4-h_6h_2+h_2h_6+h_4h_5-h_1h_3)$$

$$+ g_4(-h_4h_1-h_1h_4-h_6h_7+h_7h_6-h_3h_5+h_5h_3)$$

$$+ g_5(-h_5h_1+h_3h_4-h_2h_7+h_7h_2-h_1h_5-h_4h_3)$$

$$+ g_6(-h_6h_1-h_7h_4+h_4h_7+h_3h_2-h_1h_6-h_2h_3)$$

$$+ g_7(-h_7h_1+h_6h_4-h_1h_7-h_5h_2-h_4h_6+h_2h_5) \bmod q$$

$$= -2(g_1h_1^2+g_2h_2h_1+g_3h_3h_1+g_4h_4h_1+g_5h_5h_1+g_6h_6h_1+g_7h_7h_1) \bmod q$$

$$= -2h_1(g_1h_1+g_2h_2+g_3h_3+g_4h_4+g_5h_5+g_6h_6+g_7h_7) \bmod q$$

$$= -2h_10 = 0 \bmod q,$$

In the same manner we have

$$[(HG)H]_i = -2h_i0 = 0 \bmod q \ (i=2,\ldots,7).$$

Then we have

$(HG)H=\mathbf{0} \bmod q$. q.e.d.

Appendix D:

Lemma 3

For any $A=(a_0,a_1,\ldots,a_7)$, $B=(b_0,b_1,\ldots,b_7)\in O$

$(A+B)^*=A^*+B^* \bmod q$,

$(AB)^*=B^*A^* \bmod q$

where

$A^*=(a_0,-a_1,\ldots,-a_7)\in O$, $B^*=(b_0,-b_1,\ldots,-b_7)\in O$.

(*Proof*)

$(A+B)^*=(a_0+b_0,a_1+b_1,\ldots,a_7+b_7)^* \bmod q$,

$=(a_0+b_0,-a_1-b_1,\ldots,-a_7-b_7) \bmod q$,

$A^*+B^*= (a_0,-a_1,\ldots,-a_7)+ (b_0,-b_1,\ldots,-b_7) \bmod q,$

$= (a_0+ b_0,-a_1-b_1,\ldots,-a_7- b_7) \bmod q=(A+B)^*.$

$(AB)^*= (a_0b_0 - a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7 \bmod q,$

$a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3 \bmod q,$

$a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6 \bmod q,$

$a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1 \bmod q,$

$a_0b_4+a_1b_2 - a_2b_1 - a_3b_6+a_4b_0+a_5b_7+ a_6b_3 - a_7b_5 \bmod q,$

$a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4 \bmod q,$

$a_0b_6+a_1b_5 - a_2b_7+a_3b_4 - a_4b_3 - a_5b_1+a_6b_0 +a_7b_2 \bmod q,$

$a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0 \bmod q)^*$

$= (a_0b_0 - a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7 \bmod q,$

$-a_0b_1-a_1b_0-a_2b_4-a_3b_7+a_4b_2-a_5b_6+a_6b_5+a_7b_3 \bmod q,$

$-a_0b_2+a_1b_4-a_2b_0-a_3b_5-a_4b_1+a_5b_3-a_6b_7+a_7b_6 \bmod q,$

$-a_0b_3+a_1b_7+a_2b_5-a_3b_0-a_4b_6-a_5b_2+a_6b_4-a_7b_1 \bmod q,$

$-a_0b_4-a_1b_2 +a_2b_1 +a_3b_6-a_4b_0-a_5b_7- a_6b_3 +a_7b_5 \bmod q,$

$-a_0b_5+a_1b_6-a_2b_3+a_3b_2+a_4b_7-a_5b_0-a_6b_1-a_7b_4 \bmod q,$

$-a_0b_6-a_1b_5 +a_2b_7-a_3b_4 +a_4b_3 +a_5b_1-a_6b_0 -a_7b_2 \bmod q,$

$-a_0b_7-a_1b_3-a_2b_6+a_3b_1-a_4b_5+a_5b_4+a_6b_2-a_7b_0 \bmod q)$

$B^*A^* =(b_0,-b_1,\ldots,-b_7)\ (a_0,-a_1,\ldots,-a_7)$

$= (b_0a_0 - b_1a_1- b_2a_2- b_3a_3-b_4a_4- b_5a_5-b_6a_6-b_7a_7 \bmod q,$

$-b_0a_1-b_1a_0+b_2a_4+b_3a_7-b_4a_2+b_5a_6-b_6a_5-b_7a_3 \bmod q,$

$-b_0a_2-b_1a_4-b_2a_0+b_3a_5+b_4a_1-b_5a_3+b_6a_7-b_7a_6 \bmod q,$

$-b_0a_3-b_1a_7-b_2a_5-b_3a_0+b_4a_6+b_5a_2-b_6a_4+b_7a_1 \bmod q,$

$-b_0a_4+b_1a_2 - b_2a_1 - b_3a_6-b_4a_0+b_5a_7+ b_6a_3 - b_7a_5 \bmod q,$

$-b_0a_5-b_1a_6+b_2a_3-b_3a_2-b_4a_7-b_5a_0+b_6a_1+b_7a_4 \bmod q,$

$-b_0a_6+b_1a_5-b_2a_7+b_3a_4-b_4a_3-b_5a_1-b_6a_0+b_7a_2 \bmod q,$

$-b_0a_7+b_1a_3+b_2a_6-b_3a_1+b_4a_5-b_5a_4-b_6a_2-b_7a_0 \bmod q).$

$=(AB)^* \bmod q.$ q.e.d

Appendix E:

Theorem 9

$$|M|^2=|pG+uH+vGH+wHG|^2=0 \bmod q.$$

(*Proof*:)

As in general for any $N\in O$,

$N+N^*=2[N]_0\mathbf{1}\in O$, $N^2=-L_A\mathbf{1}+2[N]_0N$,

$N^2+NN^*=N^2+N^*N=2[N]_0N= N^2+L_N\mathbf{1}$,

we have

$L_N\mathbf{1}= NN^*=N^*N.$

$MM^*=(pG+uH+vGH+wHG)(pG+uH+vGH+wHG)^* \bmod q$

$=(pG+uH+vGH+wHG)(pG^*+uH^*+v(GH)^*+w(HG)^*) \bmod q$

$= p^2GG^*+ pu(GH^*+HG^*)+pv(G(GH)^*+(GH)G^*)$

$+pw(G(GH)^*+(HG)G^*)+$

$+ u^2HH^*+uv(H(GH)^*+(GH)H^*)+uw(H(HG)^*+(HG)\mathrm{H}^*)$

$+v^2(GH)\ GH)^*+vw((GH)(HG)^*+(HG)(GH)^*)$

$+w^2(HG)(HG)^*$

$= p^2\ L_G\mathbf{1}+ pu2[GH^*]_0\mathbf{1}+pv2[(GH)G^*]_0\mathbf{1}+pw2[(HG)G^*]_0\mathbf{1}$

$+ u^2\ L_H\mathbf{1}+ uv2[(GH)H^*]_0\mathbf{1}+uw2[(HG)\mathrm{H}^*]_0\mathbf{1}$

$+v^2\ L_{GH}\mathbf{1}+vw2[((GH)(HG)^*\ ]_0\mathbf{1}+w^2\ L_{GH}\mathbf{1} \bmod q$

$= p^2\ \mathbf{0}+pu2[G(2h_0\mathbf{1}-H)]_0\mathbf{1}+pv2[(GH)\ (2g_0\mathbf{1}-G)]_0\mathbf{1}$

$+pw2[(HG)\ (2g_0\mathbf{1}-G)]_0\mathbf{1}$

$+ u^2\,\mathbf{0} + uv2[(GH)\,(2h_0\mathbf{1}-H)]_0\mathbf{1} + uw2[(HG)\,(2h_0\mathbf{1}-H)]_0\mathbf{1}$

$+v^2\,\mathbf{0} + vw2[((GH)\,(2[HG]_0\mathbf{1}-HG)]_0\mathbf{1} + w^2\,\mathbf{0} \bmod q$

$= pu2[-GH]_0\mathbf{1} + pv2[GH-GHG]_0\mathbf{1} + pw2[(HG-HG)]_0\mathbf{1}$

$+uv2[-GHH]_0\mathbf{1} + uw2[-HGH]_0\mathbf{1} + vw2[-(GH)\,(HG)]_0\mathbf{1} \bmod q$

$= pu2[-GH]_0\mathbf{1} + pv2[GH-\mathbf{0}]_0\mathbf{1} + pw2[\mathbf{0}]_0\mathbf{1}$

$+uv2[\mathbf{0}]_0\mathbf{1} + uw2[\mathbf{0}]_0\mathbf{1} + vw2[\mathbf{0}]_0\mathbf{1} \bmod q$

$= 0\mathbf{1}+0\mathbf{1}+0\mathbf{1}+0\mathbf{1}+0\mathbf{1}+0\mathbf{1} = \mathbf{0} \bmod q.$

Then we have

$MM^* = L_M\mathbf{1} = \mathbf{0} \bmod q.$

$L_M = |M|^2 = |pG+uH+vGH+wHG|^2 = 0 \bmod q.$

q.e.d.