Fully Homomorphic Encryption with Isotropic Elements

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SUMMARY: In previous work I proposed a fully homomorphic encryption without bootstrapping which has the weak point in the enciphering function. In this paper I propose the fully homomorphic encryption scheme with non-zero isotropic octonions. I improve the previous scheme by adopting the non-zero isotropic octonions so that the "*m* and -*m* attack" is not useful because in proposed scheme many ciphertexts exist where the plaintext *m* is not zero and the norm is zero. The improved scheme is based on multivariate algebraic equations with high degree or too many variables while the almost all multivariate cryptosystems proposed until now are based on the quadratic equations avoiding the explosion of the coefficients. The improved scheme is against the Gröbner basis attack.

keywords: fully homomorphic encryption, isotropic octonion, multivariate algebraic equation, Gröbner basis,

§1. Introduction

A cryptosystem which supports both addition and multiplication (thereby preserving the ring structure of the plaintexts) is known as fully homomorphic encryption (FHE) and is very powerful. Using such a scheme, any circuit can be homomorphically evaluated, effectively allowing the construction of programs which may be run on encryptions of their inputs to produce an encryption of their output. Since such a program never decrypts its input, it can be run by an untrusted party without revealing its inputs and internal state. The existence of an efficient and fully homomorphic cryptosystem would have great practical implications in the outsourcing of private computations, for instance, in the context of cloud computing.

With homomorphic encryption, a company could encrypt its entire database of e-mails and upload it to a cloud. Then it could use the cloud-stored data as desired-for example, to calculate the stochastic value of stored data. The results would be downloaded and decrypted without ever exposing the details of a single e-mail.

Gentry's bootstrapping technique is the most famous method of obtaining fully homomorphic encryption. In 2009 Gentry, an IBM researcher, has created a homomorphic encryption scheme that makes it possible to encrypt the data in such a way that performing a mathematical operation on the encrypted information and then decrypting the result produces the same answer as performing an analogous operation on the unencrypted data[9],[10].

But in Gentry's scheme a task like finding a piece of text in an e-mail requires chaining together thousands of basic operations. His solution was to use a second layer of encryption, essentially to protect intermediate results when the system broke down and needed to be reset.

Some fully homomorphic encryption schemes were proposed until now [11], [12], [13],[14],[15].

In previous work [1],[18],[19] I proposed a fully homomorphic encryption without bootstrapping which has the weak point in the enciphering function[17]. In this paper I propose the improved fully homomorphic encryption scheme with non-zero isotropic octonions. I improve the previous scheme by adopting the non-zero isotropic octonions so that the "*m* and -*m* attack" is not useful because in proposed scheme many ciphertexts exist where the plaintext is not zero and the norm is zero.

In this scheme I adopt a fully homomorphic encryption scheme on non-associative octonion ring over finite field which is based on computational difficulty to solve the multivariate algebraic equations of high degree while the almost all multivariate cryptosystems [2],[3],[4],[5],[6],[7] proposed until now are based on the quadratic equations avoiding the explosion of the coefficients. Our scheme is against the Gröbner basis [8] attack, the differential attack, rank attack and so on.

Organization of this paper is as follows. In Sec.2 preliminaries for octonion operation are described. In Sec.3 we construct proposed fully homomorphic encryption scheme. In Sec.4 we analyse proposed scheme to show that proposed scheme is immune from the Gröbner basis attack and attack by using the ciphertexts of m and -m. In Sec.5 we describe the size of the parameters and the complexity for enciphering and deciphering. In Sec.6 we describe conclusion.

§2. Preliminaries for octonion operation

In this section we describe the operations on octonion ring and properties of octonion ring.

§2.1 Multiplication and addition on the octonion ring *O*

Let q be as a large prime as 2^{1000}.

Let O be the octonion [16] ring over a finite field $\boldsymbol{Fq}$.

$$O=\{(a_0,a_1,\ldots,a_7) \mid a_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7)\} \tag{1}$$

We define the multiplication and addition of $A,B \in O$ as follows.

$$A=(a_0,a_1,\ldots,a_7), \quad a_j \in \boldsymbol{Fq}.\ (j=0,1,\ldots,7), \tag{2}$$

$$B=(b_0,b_1,\ldots,b_7),\ b_j \in \boldsymbol{Fq}.\ (j=0,1,\ldots,7). \tag{3}$$

$$\begin{aligned} &AB \bmod q \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 \bmod q, \\ &\quad a_0b_1 + a_1b_0 + a_2b_4 + a_3b_7 - a_4b_2 + a_5b_6 - a_6b_5 - a_7b_3 \bmod q, \\ &\quad a_0b_2 - a_1b_4 + a_2b_0 + a_3b_5 + a_4b_1 - a_5b_3 + a_6b_7 - a_7b_6 \bmod q, \\ &\quad a_0b_3 - a_1b_7 - a_2b_5 + a_3b_0 + a_4b_6 + a_5b_2 - a_6b_4 + a_7b_1 \bmod q, \\ &\quad a_0b_4 + a_1b_2 - a_2b_1 - a_3b_6 + a_4b_0 + a_5b_7 + a_6b_3 - a_7b_5 \bmod q, \\ &\quad a_0b_5 - a_1b_6 + a_2b_3 - a_3b_2 - a_4b_7 + a_5b_0 + a_6b_1 + a_7b_4 \bmod q, \\ &\quad a_0b_6 + a_1b_5 - a_2b_7 + a_3b_4 - a_4b_3 - a_5b_1 + a_6b_0 + a_7b_2 \bmod q, \\ &\quad a_0b_7 + a_1b_3 + a_2b_6 - a_3b_1 + a_4b_5 - a_5b_4 - a_6b_2 + a_7b_0 \bmod q) \end{aligned} \tag{4}$$

$$\begin{aligned} &A+B \bmod q \\ &=(a_0+b_0 \bmod q,\ a_1+b_1 \bmod q,\ a_2+b_2 \bmod q,\ a_3+b_3 \bmod q\ , \\ &\quad a_4+b_4 \bmod q,\ a_5+b_5 \bmod q,\ a_6+b_6 \bmod q,\ a_7+b_7 \bmod q\ ). \end{aligned} \tag{5}$$

Let

$$|A|^2 = a_0^2 + a_1^2 + \ldots + a_7^2 \bmod q. \tag{6}$$

If $|A|^2 \neq 0 \bmod q$, we can have A^{-1}, the inverse of A by using the algorithm **Octinv**(A) such that

$$A^{-1} = (a_0 / |A|^2 \bmod q,\ -a_1 / |A|^2 \bmod q, \ldots,\ -a_7 / |A|^2 \bmod q) \leftarrow \textbf{Octinv}(A). \tag{7}$$

Here details of the algorithm **Octinv**(A) are omitted and can be looked up in the **Appendix A.**

§2.2. Property of multiplication over octonion ring *O*

A,B,C etc.$\in O$ satisfy the following formulae in general where A,B and C have the inverse A^{-1} ,B^{-1} and C^{-1} mod q.

1) Non-commutative

$$AB \neq BA \mod q. \tag{8}$$

2) Non-associative

$$A(BC) \neq (AB)C \mod q. \tag{9}$$

3) Alternative

$$(AA)B=A(AB) \mod q, \tag{10}$$

$$A(BB)=(AB)B \mod q, \tag{11}$$

$$(AB)A=A(BA) \mod n. \tag{12}$$

4) Moufang's formulae [16],

$$C(A(CB))=((CA)C)B \mod q, \tag{13}$$

$$A(C(BC))=((AC)B)C \mod q, \tag{14}$$

$$(CA)(BC)=(C(AB))C \mod q, \tag{15}$$

$$(CA)(BC)=C((AB)C) \mod q. \tag{16}$$

5) **Lemma 1**

$$A^{-1}(AB)= B \mod q, \tag{17a}$$

$$(BA)A^{-1}= B \mod q\ . \tag{17b}$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix B**.

6) **Lemma 2**

$$A(BA^{-1})= (AB)A^{-1} \mod q. \tag{18}$$

(*Proof*:)

From (16) we substitute A^{-1} to C, we have

$$(A^{-1}A)(B\,A^{-1})= A^{-1}\ ((AB)\,A^{-1}) \mod q,$$

$$(B\,A^{-1}) = A^{-1}\,((AB)\,A^{-1}) \bmod q.$$

We multiply A from left side ,

$$A(B\,A^{-1}) = A(A^{-1}\,((AB)\,A^{-1})) = (AB)\,A^{-1} \bmod q. \qquad \text{q.e.d.}$$

We can express $A(BA^{-1})$, $(AB)A^{-1}$ such that

$$ABA^{-1}.$$

7) **Lemma 3**

$$(ABA^{-1})(\,ABA^{-1}\,) = AB^2A^{-1} \bmod q. \qquad (19)$$

(*Proof*:)

$$(ABA^{-1})(\,ABA^{-1}\,) \mod q = [A^{-1}\,(A^2(BA^{-1}))][(AB)A^{-1}]$$

From (16),

$$= A^{-1}\,\{[(A^2(BA^{-1}))(AB)]A^{-1}\} \mod q$$

$$= A^{-1}\,\{[(A((AB)A^{-1}))(AB)]A^{-1}\} \mod q$$

$$= A^{-1}\,\{[(A(AB))A^{-1}))(AB)]A^{-1}\} \mod q.$$

We apply (14) to inside of [.],

$$= A^{-1}\,\{[(A((AB)(A^{-1}(AB)))]A^{-1}\} \mod q$$

$$= A^{-1}\,\{[(A((AB)B))]A^{-1}\} \mod q$$

$$= A^{-1}\,\{[A(A(BB))]A^{-1}\} \mod q$$

$$= \{\,A^{-1}\,[A(A(BB))]\}A^{-1} \mod q$$

$$= (A(BB))A^{-1} \mod q$$

$$= AB^2A^{-1} \bmod q. \qquad \text{q.e.d.}$$

8) **Lemma 4**

$$[A_1(\ldots(A_rBA_r^{-1})\ldots)A_1^{-1}]\,[A_1(\ldots(A_rBA_r^{-1})\ldots)A_1^{-1}]$$

$$= A_1(\ldots(A_rB^2A_r^{-1})\ldots)A_1^{-1} \bmod q. \qquad (20)$$

where

$$A_i \in O \text{ has the inverse } A_i^{-1} \bmod q \ (i=1,\ldots,r).$$

(*Proof*:)

As we use Lemma 3 repeatedly we have

$$\{A_1([A_2(\ldots(A_rBA_r^{-1})\ldots)A_2^{-1}])A_1^{-1}\}\{A_1([A_2(\ldots(A_rBA_r^{-1})\ldots)A_2^{-1}])A_1^{-1}\} \bmod q$$

$$=A_1([A_2(\ldots(A_rBA_r^{-1})\ldots)A_2^{-1}][A_2(\ldots(A_rBA_r^{-1})\ldots)A_2^{-1}])A_1^{-1} \bmod q$$

$$=A_1(A_2([A_3(\ldots(A_rBA_r^{-1})\ldots)A_3^{-1}][A_3(\ldots(A_rBA_r^{-1})\ldots)A_3^{-1})A_2^{-1}])A_1^{-1} \bmod q$$

$$\ldots \qquad \ldots$$

$$= A_1(A_2(\ldots([A_rBA_r^{-1}][A_rBA_r^{-1}])\ldots)A_2^{-1})A_1^{-1} \bmod q$$

$$= A_1(A_2(\ldots(A_rB^2A_r^{-1})\ldots)A_2^{-1})A_1^{-1} \bmod q$$

q.e.d.

9) **Lemma 5**

$$A_1^{-1}(A_1BA_1^{-1})A_1 = B \bmod q. \qquad (21)$$

where

$$A_1 \in O \text{ has the inverse } A_1^{-1} \bmod q.$$

(*Proof*:)

$$A_1^{-1}(A_1BA_1^{-1})A_1 = A_1^{-1}[((A_1B)A_1^{-1})A_1] \bmod q,$$

$$= A_1^{-1}(A_1B) = B \bmod q. \qquad \text{q.e.d.}$$

10) **Lemma 6**

$$A_r^{-1}(\ldots(A_1^{-1}[A_1(\ldots(A_rBA_r^{-1})\ldots)A_1^{-1}]A_1)\ldots)A_r = B \bmod n. \qquad (22)$$

where

$$A_i \in O \text{ has the inverse } A_i^{-1} \bmod n \ (i=1,\ldots,r).$$

(*Proof*:)

As we use Lemma 5 repeatedly we have

$$A_r^{-1}(\ldots(A_1^{-1}[A_1(\ldots(A_rBA_r^{-1})\ldots)A_1^{-1}]A_1)\ldots)A_r$$

$$= A_r^{-1}(\ldots(A_2^{-1}[A_2(\ldots(A_rBA_r^{-1})\ldots)A_2^{-1}]A_2)\ldots)A_r \bmod q$$

$$\ldots \qquad \ldots$$

$$= A_r^{-1}[A_rBA_r^{-1}]A_r \bmod q$$

$$= B \bmod q \qquad \text{q.e.d.}$$

11) $A \in O$ satisfies the following theorem.

Theorem 1

$$A^2 = w\mathbf{1} + vA \bmod q, \tag{23}$$

where

$$^{\exists}w,v \in \boldsymbol{Fq}.,$$

$$\mathbf{1}=(1,0,0,0,0,0,0,0) \in O,$$

$$A=(a_0,a_1,\ldots,a_7) \in O.$$

(*Proof*:)

$$\begin{aligned} & A^2 \bmod q \\ =(& a_0a_0-a_1a_1-a_2a_2-a_3a_3-a_4a_4-a_5a_5-a_6a_6-a_7a_7 \bmod q, \\ & a_0a_1+a_1a_0+a_2a_4+a_3a_7-a_4a_2+a_5a_6-a_6a_5-a_7a_3 \bmod q, \\ & a_0a_2-a_1a_4+a_2a_0+a_3a_5+a_4a_1-a_5a_3+a_6a_7-a_7a_6 \bmod q, \\ & a_0a_3-a_1a_7-a_2a_5+a_3a_0+a_4a_6+a_5a_2-a_6a_4+a_7a_1 \bmod q, \\ & a_0a_4+a_1a_2-a_2a_1-a_3a_6+a_4a_0+a_5a_7+a_6a_3-a_7a_5 \bmod q, \\ & a_0a_5-a_1a_6+a_2a_3-a_3a_2-a_4a_7+a_5a_0+a_6a_1+a_7a_4 \bmod q, \\ & a_0a_6+a_1a_5-a_2a_7+a_3a_4-a_4a_3-a_5a_1+a_6a_0+a_7a_2 \bmod q, \\ & a_0a_7+a_1a_3+a_2a_6-a_3a_1+a_4a_5-a_5a_4-a_6a_2+a_7a_0 \bmod q) \end{aligned}$$

$$\begin{aligned} =(2a_0^2 - L \bmod q,\ & 2a_0a_1 \bmod q,\ 2a_0a_2 \bmod q,\ 2a_0a_3 \bmod q,\ 2a_0a_4 \bmod q,\ 2a_0a_5 \bmod q, \\ & 2a_0a_6 \bmod q,\ 2a_0a_7 \bmod q) \end{aligned}$$

where

$$\mathrm{L}= a_0^2+a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2 \bmod q.$$

Now we try to obtaiq u, $v \in Fq$ that satisfy $A^2 = w\mathbf{1} + vA \bmod q$.

$$w\mathbf{1}+vA= w(1,0,0,0,0,0,0,0)+v(a_0,a_1,\ldots,a_7) \bmod q,$$

$$\begin{aligned} A^2= (2a_0^2 - L \bmod q,\ & 2a_0a_1 \bmod q,\ 2a_0a_2 \bmod q,\ 2a_0a_3 \bmod q,\ 2a_0a_4 \bmod q, \\ & 2\,a_0a_5 \bmod q,\ 2a_0a_6 \bmod q,\ 2a_0a_7 \bmod q). \end{aligned}$$

Then we have

$$A^2 = w\mathbf{1} + vA = -L\mathbf{1} + 2a_0 A \bmod q,$$

$$w = -L \bmod q,$$

$$v = 2a_0 \bmod q. \qquad \text{q.e.d.}$$

§3. Concept of proposed fully homomorphic encryption scheme

Homomorphic encryption is a form of encryption which allows specific types of computations to be carried out on ciphertext and obtain an encrypted result which decrypted matches the result of operations performed on the plaintext. For instance, one person could add two encrypted numbers and then another person could decrypt the result, without either of them being able to find the value of the individual numbers.

§3.1 Definition of homomorphic encryption

A homomorphic encryption scheme **HE** := (**KeyGen; Enc; Dec; Eval**) is a quadruple of PPT (Probabilistic polynomial time) algorithms.

In this work, the medium text space M_e of the encryption schemes will be octonion ring, and the functions to be evaluated will be represented as arithmetic circuits over this ring, composed of addition and multiplication gates. The syntax of these algorithms is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^λ, outputs (**sk**) ← **KeyGen**(1^λ) , where **sk** is a secret encryption/decryption key.

-Encryption. The algorithm **Enc**, on input system parameter n, secret keys(**sk**) and a plaintext $m \in \boldsymbol{Zn}$ outputs a ciphertext $C \leftarrow$**Enc**(**sk**; m).

-Decryption. The algorithm **Dec**, on input system parameter n, secret key(**sk**) and a ciphertext C, outputs a plaintext $m^* \leftarrow$**Dec**(**sk**;C).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter n, an arithmetic circuit ckt, and a tuple of n ciphertexts ($C_1,\ldots, C_r$), outputs a ciphertext $C' \leftarrow$**Eval**(ckt; $C_1,\ldots, C_r$).

§3.2 Definition of fully homomorphic encryption

A scheme HE is fully homomorphic if it is both compact and homomorphic with respect to a class of circuits. More formally:

Definition (**Fully homomorphic encryption**). A homomorphic encryption scheme FHE :=(**KeyGen; Enc; Dec; Eval**) is fully homomorphic if it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda\in N}$ be the set of all polynomial sized arithmetic circuits. On input $\mathbf{sk} \leftarrow \mathbf{KeyGen}(1^\lambda), \forall \text{ckt}\in CR_\lambda, \forall (m_1,\ldots, m_r)\in \boldsymbol{Fq}^r$ where $r = r(\lambda)$, $\forall (C_1,\ldots,C_r)$ where $C_i \leftarrow \mathbf{Enc}(\mathbf{sk};m_i)$, it holds that:

$$\Pr[\mathbf{Dec}(\mathbf{sk};\mathbf{Eval}(\text{ckt}; C_1,\ldots,C_r)) \neq \text{ckt}(m_1,\ldots, m_r)] = \text{negl}(\lambda).$$

2. Compactness: There exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§3.3 Proposed fully homomorphic enciphering/deciphering functions

We propose a fully homomorphic encryption (FHE) scheme based on the enciphering/deciphering functions on octonion ring over $\boldsymbol{Fq}$.

§3.3.1 Preparation

First we define the medium text M as follows.

We select the element $B=(b_0,b_1,b_2,\ldots,b_7) \in O$ and $H=(b_0,-b_1,-b_2,\ldots,-b_7)\in O$ such that,

$$L_B:=|B|^2= b_0^2+b_1^2+\ldots+b_7^2 \bmod q=0, \quad (24)$$

$$L_H:=|H|^2= b_0^2+b_1^2+\ldots+b_7^2 \bmod q=0, \quad (25)$$

$$b_0= 1/2 \bmod q, \quad (26)$$

$$b_1\neq 0 \bmod q. \quad (27)$$

Then we have

1) $H+B=\mathbf{1} \bmod q$ (28a)
2) $B^2=2b_0B=B \bmod q$, (28b)
3) $H^2=2b_0H=H \bmod q$, (28c)
4) $H(H+B)= (H+B)H=\mathbf{1}H \bmod q$, then $HB=BH=\mathbf{0} \bmod q$. (28d)

Let $m\in \boldsymbol{Fq}$ be a plaiqtext to be split up into $u\in \boldsymbol{Fq}$ and $v\in \boldsymbol{Fq}$ randomly such that

$$m:=u+v \bmod q. \quad (29)$$

Let $t\in \boldsymbol{Fq}$ be the subtext such that

$$t := m \bmod h = (m \bmod h) - (h-1)/2 \in \{-(h-1)/2, \ldots, 0, \ldots, (h-1)/2\} \quad (30)$$

where

$$h \in (2\boldsymbol{Z} + 1) \cap (2^{500-1}, 2^{500}) \text{ is an odd number.} \quad (31)$$

Let $w \in \boldsymbol{Fq}$ be a parameter such that

$$w := t - u \bmod q. \quad (32)$$

We notice that

$$\text{if } m \bmod h = 0 \text{ then } t = 0,$$

and

$$\text{if } m = 0 \text{ then } t \neq 0.$$

We define the medium text M by

$$M := u\mathbf{1} + vB + wH \bmod q \in O \quad (33)$$

$$\mathbf{1} := (1,0,0,0,0,0,0,0) \in O. \quad (34)$$

Theorem 2

$$|M|^2 = |u\mathbf{1} + vB + wH|^2 = mt \bmod q \quad (35)$$

(*Proof*)

$$|M|^2 = |u\mathbf{1} + vB + wH|^2$$

$$= |(u + (v+w)b_0)^2 + (v-w)^2(b_1^2 + \ldots + b_7^2)| \bmod q$$

$$= |(u + (v+w)b_0)^2 + (v-w)^2(-b_0^2)| \bmod q$$

$$= |(u + (v+w)/2)^2 + (v-w)^2(-1/2)^2| \bmod q$$

$$= |(u+v)(u+w)| = |mt| \bmod q.$$

q.e.d

Then we have

if $m = 0 \bmod q$ or $t = 0 \bmod q$, then $|M|^2 = 0 \bmod q$.

To be immune from m and $-m$ attack, we adopt the above "$t = m$ modd h" so that many M exist such that $|M|^2 = 0 \bmod q$.

We can obtain m and t from M as follows.

$m=[M]_0+[M]_1(b_0/\ b_1)=(u+vb_0+wb_0)+(\ vb_1-wb_1\ )(b_0/b_1)=\ u+2vb_0=\ u+v \bmod q,$ (36)

$t=[M]_0-[M]_1(b_0/\ b_1)=(u+vb_0+wb_0)-(\ vb_1-wb_1\ )b_0/b_1=\ u+2wb_0=\ u+w \bmod q,$ (37)

where we denote the i-th element of octonion M such as

$$[M]_i\ .$$

Let

$$M_1:=\ u_1\mathbf{1}+v_1B+w_1H \bmod q\in O$$

$$m_1=u_1+v_1 \bmod q$$

$$t_1=m_1 \text{ modd } h$$

$$w_1=\ t_1\ -u_1 \bmod q,$$

$$M_2:=\ u_2\mathbf{1}+v_2B+w_2H \bmod q\in O$$

$$m_2=u_2+v_2 \bmod q$$

$$t_2=m_2 \text{ modd } h$$

$$w_2=\ t_2-u_2 \bmod q.$$

Here we can show easely that

1) $M_1+\ M_2$ is the medium text of $m_1+\ m_2$,

2) $M_1\ M_2$ is the medium text of m_1m_2 .

We can define the medium text M_{1+2} of sum m_{1+2} of the plaintexts m_1 and m_2 as follows.

$$M_{1+2}:=M_1+\ M_2=\ u_1\mathbf{1}+v_1B+w_1H\ +\ u_2\mathbf{1}+v_2B+w_2H \bmod q$$
$$=(\ u_1+\ u_2)\mathbf{1}+(v_1+\ v_2)B+(w_1+\ w_2)H \bmod q.$$

$$m_{1+2}:=\ (\ u_1+\ u_2)+\ (v_1+\ v_2)\ =m_1+m_2 \bmod q.$$ (38)

$$t_{1+2}:=\ (\ u_1+\ u_2)+\ (w_1+\ w_2)\ =t_1+t_2 \bmod q.$$ (39)

Now we have generated M_1+_2 from M_1 and M_2 which is the medium text of sum of $m_1+\ m_2$.

W can define the mediun text M_{12} of product m_{12} of the plaintexts m_1 and m_2 as follows.

$$M_{12}:=M_1\ M_2=(\ u_1\mathbf{1}+v_1B+w_1H\ )(\ u_2\mathbf{1}+v_2B+w_2H) \bmod q$$
$$=(\ u_1u_2)\mathbf{1}+(u_2v_1+u_1v_2+v_1v_2)B+(u_2w_1+u_1w_2+w_1w_2)H \bmod q.$$ (40)

$$m_{12}:=u_1u_2+(u_2v_1+u_1v_2+v_1v_2)$$

$$=(u_1+ v_1)\ (u_2+ v_2)= m_1m_2 \bmod q \tag{41}$$

$$t_{12}:= u_1u_2+(u_2w_1+u_1w_2+w_1w_2)$$

$$=(u_1+ w_1)\ (u_2+w_2)= t_1t_2 \bmod q. \tag{42}$$

Now we have generated M_{12} from M_1 and M_2 which is the medium text of product of m_1 and m_2.

§3.3.2 Fully homomorphic encryption

Here I define the some parameters for describing FHE.
Let q be as a large prime as 2^{1000}.
Choose randomly an odd number $h\in(2\boldsymbol{Z}+1)\cap(2^{500-1}, 2^{500})$.
Let $M=(m_0,m_1,\ldots,m_7)= u\mathbf{1}+vB+wH \bmod q\in O$ be the medium plaintext.

Let $m\in\boldsymbol{Fq}$ be the plaintext to be split up into $u\in\boldsymbol{Fq}$ and $v\in\boldsymbol{Fq}$ randomly such that $m=u+v \bmod q$.

Let t be the subtext such that $t:=m$ modd $h\in\{-(h-1)/2,\ldots,0,\ldots,(h-1)/2\}$.
Let $w\in\boldsymbol{Fq}$ be a parameter such that $w = t - u \bmod q$.
Let $X=(x_0,\ldots,x_7)\ \in O[X]$ be a variable.
Let $E(m,X)$ and $D(X)$ be a enciphering and a deciphering function.
Let $C(X)=E(m,X)\in O[X]$ be the ciphertext.
A_i, $Z_i\in O$ is selected randomly such that A_i^{-1} and Z_i^{-1} exist (i=1,...,k) which are the secret keys of the data sender,

$A_i\in O$ is selected such that $A_i^{-1}\in O$ exists (i=1,…,r) where

$$A_i\,A_i^{-1}= A_i^{-1}\,A_i=\mathbf{1} \bmod q\in O.$$

$Z_i\in O$ is selected such that $Z_i^{-1}\in O$ exists (i=1,…,r) where

$$Z_i\,Z_i^{-1}= Z_i^{-1}Z_i=\mathbf{1} \bmod q\in O.$$

Enciphering function $C(X)=E(m,X)$ is defined as follows.

$C(X)=E(m,X):=$

$$A_1((\ldots((A_k((M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))Z_1^{-1}) \bmod q\in O[X] \tag{43}$$

$$=(\ c_{00}x_0+c_{01}x_1+\ \ldots+c_{07}x_7,$$

$$c_{10}x_0+c_{11}x_1+\ \ldots+c_{17}\,x_7,$$

$$\ldots \qquad \ldots$$

$$c_{70}x_0+c_{71}x_1+\ \ldots+c_{77}\,x_7), \tag{44}$$

$$= \{c_{ij}\}(i,j=0,\ldots,7) \tag{45}$$

with $c_{ij}\in \boldsymbol{Fq}$ $(i,j=0,\ldots,7)$ which is published in cloud centre.

Here we mention how to construct enciphering function.
We show a part of process for constructing enciphering function $E(m,X)$ as follows.

$$A_1^{-1}X$$
$$(A_1^{-1}X)Z_1$$
$$A_2^{-1}\,((A_1^{-1}X)Z_1)$$
$$(A_2^{-1}\,((A_1^{-1}X)Z_1))\,Z_2$$
$$\ldots$$
$$(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k$$
$$M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]$$
$$(M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}$$
$$A_k(M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1})$$
$$\ldots$$
$$A_1((\ldots((A_k((M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\,Z_1^{-1})$$

Let D be the deciphering function defined as follows .

$$G_1(X):=(A_k^{-1}((\ldots((A_1^{-1}X\,)Z_1))\ldots))Z_k, \tag{46}$$

$$G_2(X):=A_1((\ldots((A_k(X\,Z_k^{-1}))\ldots))\,Z_1^{-1}), \tag{47}$$

$$D(X):=\,G_1(C(G_2(X))\bmod q=MX. \tag{48}$$

$$D(\mathbf{1})=M=(m_0,m_1,\ldots,m_7) = u\mathbf{1}+vB+wH \bmod q$$

$$=u(1,0,0,\ldots,0) +v(b_0,b_1,\ldots,b_7)+ w(b_0,-b_1,\ldots,-b_7)$$

$$=(\,u+\,(v+\,w)b_0,\,(v-\,w)b_1,\,\ldots,\,(v-\,w)b_7),$$

where $b_0=1/2 \bmod q$, $b_1\neq 0 \bmod q$.

Then we obtain the plaintext $m=u+v \bmod q$ and subtext $t=u+w \bmod q$ as follows.

$$m=\,u\,+v\,=m_0+\,m_1\,(b_0/b_1) \bmod q,$$

$$t=\,u\,+w\,=m_0-\,m_1\,(b_0/b_1) \bmod q.$$

§3.4 Elements on octonion ring assumption EOR(k,r; q,h)

Here we describe the assumption on which the proposed scheme bases.

Elements on octonion ring assumption EOR($k,r;q,h$).

Let q a large prime and choose randomly a odd number $h \in (2\boldsymbol{Z}+1) \cap (2^{500-1}, 2^{500})$. Let k and r be integer parameters. Let $\boldsymbol{A}:=(A_1,\ldots,A_k) \in O^k$, $\boldsymbol{Z}:=(Z_1,\ldots,Z_k) \in O^k$ .Let $C_i(X) := E(m_i, X) = (A_1((\ldots((A_k(M_i [(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]))Z_k^{-1}))\ldots)) Z_1^{-1} \bmod q \in O[X]$ where medium text $M_i=(m_{i0},\ldots, m_{i7}) := u_i\mathbf{1}+v_iB+w_iH \in O$, plaintext $m_i = u_i+v_i \bmod q \in \boldsymbol{Fq}$. subtext $t_i=u_i+w_i \bmod q \in \boldsymbol{Fq}$ (i=1,…,r). X is a variable.

In the **EOR**($k,r;q,h$) assumption, the adversary A_d is given $C_i(X)$ (i=1,…,r) randomly and his goal is to find a set of elements $\boldsymbol{A}=(A_1,\ldots,A_k) \in O^k$, $\boldsymbol{Z}=(Z_1,\ldots,Z_k) \in O^k$ with the order of the elements $A_1,\ldots, A_k$, $Z_1,\ldots,Z_k$ and plaintexts m_i(i=1,…,r). For parameters $k = k(\lambda)$ and $r=r(\lambda)$ defined in terms of the security parameter λ and for any PPT adversary A_d we have

$\Pr[(A_1((\ldots((A_k(M_i[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]))Z_k^{-1}))\ldots)) Z_1^{-1} \bmod q = C_i(X)$ (i=1,…,r) : $\boldsymbol{A}, \boldsymbol{Z}, m_i(i=1,\ldots,r) \leftarrow A_d(1^\lambda, C_i(X)\ (i=1,\ldots,r))] = \mathrm{negl}(\lambda)$.

To solve directly **EOR**($k,r;q,h$) assumption is known to be the problem for solving the multivariate algebraic equations of high degree which is known to be NP-hard.

§3.5 Syntax of proposed algorithms

The syntax of proposed scheme is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^λ and system parameter $q$and h, outputs **sk**=($\boldsymbol{A},\boldsymbol{Z},B,H$)←**KeyGen**($1^\lambda$), where **sk** is a secret encryption /dencryption key.

-Encryption. The algorithm **Enc**, on input system parameter q,h , secret keys **sk**=($\boldsymbol{A},\boldsymbol{Z},B,H$) , a plaintext $m \in \boldsymbol{Fq}$ and a subtext $t \in \{-(h-1)/2,\ldots,0,\ldots,(h-1)/2\}$, outputs a ciphertext $\boldsymbol{C}(X;\mathbf{sk},m) \leftarrow$**Enc**(**sk**;$m$).

-Decryption. The algorithm **Dec**, on input system parameter q,h ,secret keys **sk** and a ciphertext $\boldsymbol{C}(X;\mathbf{sk},m)$, outputs plaintext **Dec**(**sk**; $\boldsymbol{C}(X;\mathbf{sk},m)$) where $\boldsymbol{C}(X;\mathbf{sk},m) \leftarrow$**Enc**(**sk**; m).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter q, h ,an arithmetic circuit ckt and a tuple of r ciphertexts ($C_1,\ldots, C_r$), outputs an evaluated ciphertext $C'\leftarrow$**Eval**(ckt; $C_1,\ldots, C_r$) where $C_i=\boldsymbol{C}(X;\mathbf{sk},m_i)$ (i=1,…,r).

Theorem 3

For any $m,m^* \in O$,

if $E(m, X)= E(m^*, X) \bmod q$, then $m= m^* \bmod q$.

That is , if $m \neq m^* \bmod q$, then $E(m, X) \neq E(m^*, X) \bmod q$.

(*Proof*)

If $E(m, X)= E(m^*, X) \bmod q$, then

$$G_1(E(m, (G_2(X)))= G_1(E(m^*, (G_2(X)))) \bmod q$$

$$MX=M^*X \bmod q$$

where

$$M= u\mathbf{1}+vB+wH \bmod q,$$

$$m= u+v \bmod q,$$

$$t=u+w \bmod q.$$

$$M^*= u^*\mathbf{1}+v^*B+w^*H \bmod q,$$

$$m^* =u^*+v^* \text{ modn } q.$$

$$t^*=u^*+w^* \bmod q.$$

We substitute **1** to X in above expression, we obtain

$$M =M^* \bmod q.$$

$$u\mathbf{1}+vB+wH = u^*\mathbf{1}+v^*B+w^*H \bmod q.$$

$$(u- u^*)\mathbf{1}+(v- v^*)B +(w- w^*)H =\mathbf{0} \bmod q$$

$$=(u- u^*+(v- v^*)\, b_0+(w- w^*)b_0, (v- v^*)b_1-(w- w^*)\, b_1, (v- v^*)b_2-(w- w^*)\, b_2, \ldots,$$

$$(v- v^*)b_7-(w- w^*)\, b_7)=\mathbf{0}$$

From $b_0=1/2 \bmod q$ and $b_1 \neq 0 \bmod q$, we have

$$(v- v^*)=(w- w^*) \bmod q,$$

$$u- u^*+(v- v^*) =0 \bmod q,$$

$$u- u^*+ (w- w^*)=0 \bmod q.$$

Then

$$m= u +v = u^* +v^* = m^* \bmod q.$$

$$t= u +w = u^* +w^* = t^* \bmod q.$$

q.e.d.

Next it is shown that the encrypting function $E(m,X)$ has the property of fully homomorphism.

Here we simply express above encrypting function such that

$A_1((\ldots((A_k((M[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=\boldsymbol{A}((M[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \bmod q.$

§3.6 Addition/subtraction scheme on ciphertexts

Let

$$M_1 := u_1\mathbf{1}+v_1B+w_1H \bmod q \in O$$

$$M_2 := u_2\mathbf{1}+v_2B+w_2H \bmod q \in O$$

be medium texts to be encrypted where

$$m_1=u_1+v_1 \bmod q$$

$$t_1= u_1+w_1 \bmod q,$$

$$m_2=u_2+v_2 \bmod q$$

$$t_2= u_2+w_2 \bmod q.$$

Let $C_1(X)= E(m_1, X)$ and $C_2(X)= E(m_2, X)$ be the ciphertexts.

$C_1(X)\pm C_2(X) \bmod q = E(m_1,X) \pm E(m_2,X) \bmod q$

$=A_1((\ldots((A_k((M_1[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\ Z_1^{-1})$

$\pm A_1((\ldots((A_k((M_2[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=A_1((\ldots((A_k(([M_1 \pm M_2]\ [(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=A_1((\ldots((A_k(([u_1\mathbf{1}+v_1B+w_1H\pm(u_2\mathbf{1}+v_2B+w_2H)]$

$[(A_k^{-1}((\ldots((A_1^{-1}X)\ Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=A_1((\ldots((A_k(([(u_1\pm u_2)\ \mathbf{1}\ +(v_1\pm v_2)B+(w_1\pm w_2)]$

$[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$= E(m_1\pm m_2,X) \bmod q.$

It has been shown that in this method we have the additive homomorphism on the

plaintext *m*.

§3.7 Multiplication scheme on ciphertexts

§3.7.1 Multiplicative property of *B* and *H*

We notice multiplication of *B* and *H* again as follows.

$$H+B=\mathbf{1} \bmod q,$$
$$B^2=2b_0B=B \bmod q,$$
$$H^2=2b_0H=H \bmod q,$$
$$HB=BH=\mathbf{0} \bmod q.$$

§3.7.2 Multiplication of ciphertexts

Here we consider the multiplicative operation on the ciphertexts.

Let $C_1(X)= E(m_1, X)$ and $C_2(X)= E(m_2, X)$ be the ciphertexts.

$C_1(C_2(X)) \bmod q = E(m_1,E(m_2,X)) \bmod q$

$=A_1((\ldots((A_k((M_1[(A_k^{-1}((\ldots((A_1^{-1}\{A_1((\ldots((A_k((M_2[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1})\ \})Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=A_1((\ldots((A_k((M_1[M_2[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]])\ Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q$

$=A_1((\ldots((A_k(M_1(M_2[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]))Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q.$ (49)

$=\boldsymbol{A}((M_1(M_2[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q.$

Substituting $(u_1\mathbf{1}+v_1B+w_1H)$, $(u_2\mathbf{1}+v_2B+w_2H)$ to M_1, M_2 ,

we have

$=\boldsymbol{A}((\ [(u_1\mathbf{1}+v_1B+w_1H)][(\ (u_2\mathbf{1}+v_2B+w_2H\ )]\ [(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q,$

$= \boldsymbol{A}((\ [u_1\mathbf{1}][(\ (u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q.$

$+ \boldsymbol{A}((\ [v_1B]\ ([(u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q$

$+ \boldsymbol{A}((\ [w_1H]\ ([(u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q$

$= \boldsymbol{A}((\ [(u_1\mathbf{1})(u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q.$

$+ \boldsymbol{A}((\ [(v_1B)(u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q$

$+ \boldsymbol{A}((\ [(w_1H)(u_2\mathbf{1}+v_2B+w_2H)][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q$

$= \boldsymbol{A}((\ [(u_1\mathbf{1}+v_1B+w_1H)(u_2\mathbf{1}+v_2B+w_2H\ )]\ [(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q,$

$$= \boldsymbol{A}((\,(M_1M_2\,)\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod n$$

$$= \boldsymbol{A}((\,((\,u_1u_2)\boldsymbol{1}+(u_2v_1+u_1v_2+v_1v_2)B+(u_2w_1+u_1w_2+w_1w_2)H)\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q \quad (50)$$

Here we can show that $E(m_1,E(m_2,X)) \bmod q$ is the ciphertext of the multiplication of $m_1$and m_2 as follows.

From (50) we have the plaintext m_{12} relating with the cipher text $E(m_1,E(m_2,X))$ such that

$$m_{12}=(u_1u_2)+(u_2v_1+u_1v_2+v_1v_2) \bmod q,$$

$$=(u_1+\,v_1)\,(\,u_2+\,v_2)\,=m_1m_2 \bmod q, \quad (51)$$

Then we have

$$E(m_1,\,E(m_2,X))=E(m_1m_2,X)) \bmod q. \quad (52)$$

It has been shown that in this method we have the multiplicative homomorphism on the plaintext m.

§3.8 Property of proposed fully homomorphic encryption

(**Fully homomorphic encryption**). Proposed fully homomorphic encryption =(**KeyGen; Enc; Dec; Eval**) is fully homomorphic because it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda\in N}$ be the set of all polynomial sized arithmetic circuits. On input $\mathbf{sk} \leftarrow \mathbf{KeyGen}(1^\lambda)$, $\forall$ckt $\in$ CR_λ, $\forall(m_1,\ldots,m_r)\in \boldsymbol{Fq}^r$ where $r = r(\lambda)$, $\forall(C_1,\ldots,C_r)$ where $C_i \leftarrow (E(m_i,X))$, $(i=1,\ldots,r)$,
we have $D(\mathbf{sk};\mathbf{Eval}(\text{ckt}; C_1,\ldots,C_r)) = \text{ckt}(m_1,\ldots,m_r)$.

Then it holds that:

$$\Pr[D(\mathbf{sk};\,\mathbf{Eval}(\text{ckt};\,C_1,\ldots,C_r)) \neq \text{ckt}(m_1,\ldots,m_r)] = \text{negl}(\lambda).$$

2. Compactness: As the output length of **Eval** is at most $s(\log_2 q)=s\lambda$ where s is a positive integer, there exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§4. Analysis of proposed scheme

Here we analyze the proposed fully homomorphism encryption scheme.

§4.1 Computing plaintext m and A_i , Z_i (i=1,…,k) from coefficients of ciphertext $E(m,X)$ to be published

Ciphertext $E(m_s,X)$ is published by cloud data centre as follows.

$$E(m_s,X)= A_1((\ldots((A_k((M_s[(A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k]))Z_k^{-1}))\ldots))\ Z_1^{-1})$$

$$=\boldsymbol{A}(([u_s\mathbf{1}+v_sB+w_sH][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}]))\boldsymbol{Z}^{-1}) \bmod q\in O[X]\ ,$$

$$=(\ e_{s00}x_0+e_{s01}x_1+\ \ldots+e_{s07}x_7,$$

$$e_{s10}x_0+e_{s11}x_1+\ \ldots+e_{s17}\ x_7,$$

$$\ldots\qquad\qquad\ldots$$

$$e_{s70}x_0+e_{s71}x_1+\ \ldots+e_{s77}\ x_7)\bmod q,$$

$$=\{e_{sij}\}(i,j=0,\ldots,7;s=1,2)$$

with $e_{sij}\in\boldsymbol{Fq}$ $(i,j=0,\ldots,7;s=1,2)$ which is published,
where

$$m_s=u_s+2b_0v_s \bmod q,(\ s=1,2)$$

A_i, $Z_i\in O$ to be selected randomly such that A_i^{-1} and Z_i^{-1} exist $(i=1,\ldots,k\ )$ are the secret keys of user A.

We try to find plaintext m_s from coefficients of $E(m_s,X)$, $e_{sij}\in\boldsymbol{Fq}(i,j=0,\ldots,7;$ $s=1,2)$.

In case that $k=8$ and $s=2$ the number of unknown variables (u_s,v_s,w_s, A_i , Z_i $(i=1,\ldots,8;s=1,2)$) is 134(=3*2+2*8*8), the number of equations is 128(=64*2) such that

$$\left.\begin{aligned}
&F_{100}(M, A_i , Z_i, R_j)=e_{100} \bmod q,\\
&F_{101}(M, A_i , Z_i, R_j)=e_{101} \bmod q,\\
&\quad\cdots\qquad\cdots\\
&F_{107}(M, A_i , Z_i, R_j)=e_{107} \bmod q,\\
&\quad\cdots\qquad\cdots\\
&\quad\cdots\qquad\cdots\\
&F_{277}(M, A_i , Z_i, R_j)=e_{277} \bmod q,
\end{aligned}\right\}\qquad(53)$$

where $F_{100},\ldots,F_{277}$ are the $33(=8*2*2+1)^{th}$ algebraic multivariate equations.

Then the complexity G required for solving above simultaneous equations by using Gröbner basis is given [8]such as

$$G>G'=({}_{127+dreg}C_{dreg})^w=({}_{2175}C_{127})^w>2^{1657}>>2^{80},\qquad(54)$$

where G' is the complexity required for solving 128 simultaneous algebraic equations with 127 variables by using Gröbner basis,
where w=2.39, and

$$d_{reg} = 2048 \ (=128*(33-1)/2 - 0\sqrt{}(128*(33\text{^}2-1)/6)). \tag{55}$$

The complexity G required for solving above simultaneous equations by using Gröbner basis is enough large to be secure.

§4.2 Computing plaintext p_i and d_{ijk} (i,j,k=0,…,7)

We try to computing plaintext p_i and d_{ijk} (i,j,k=0,...,7) from coefficients of ciphertext $E(m_i,X)$ to be published.

At first let $E(Y,X)\in O[X,Y]$ be the enciphering function such as

$$E(Y,X):= A_1((\ldots((A_k((Y[((A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q\in O[X,Y]\ ,$$

$$=(d_{000}x_0y_0+d_{001}x_0y_1+\ \ldots+d_{077}x_7y_7,$$

$$d_{100}x_0y_0+d_{101}x_0y_1+\ \ldots+d_{177}x_7y_7,$$

$$\ldots\ \ \ \ \ldots$$

$$d_{700}x_0y_0+d_{701}x_0y_1+\ \ldots+d_{777}x_7y_7) \bmod q, \tag{56}$$

$$=\{d_{ijk}\}(i,j,k=0,\ldots,7) \tag{57}$$

with $d_{ijk}\in \boldsymbol{Fq}$ (i,j,k =0,…,7) .

Next we substitute M_i to Y, where

$$M_i:=u_i\mathbf{1}+v_iB+w_iH$$

$$m_i=(u_i+\ 2b_0v_i) \bmod q,$$

$$M_i=(m_{i0},m_{i1},\ldots,m_{i7})\in O. \tag{58}$$

We have

$$E(m_i,X)=A_1((\ldots((A_k((M_i[((A_k^{-1}((\ldots((A_1^{-1}X)Z_1))\ldots))Z_k])Z_k^{-1}))\ldots))\ Z_1^{-1}) \bmod q\in \quad O[X]\ ,$$

$$=(d_{000}x_0m_{i0}+d_{001}x_0m_{i1}+\ \ldots+d_{077}x_7m_{i7},$$

$$d_{100}x_0m_{i0}+d_{101}x_0m_{i1}+\ \ldots+d_{177}x_7m_{i7},$$

$$\ldots\ \ \ \ \ldots$$

$$d_{700}x_0m_{i0}+d_{701}x_0m_{i1}+\ \ldots+d_{777}x_7m_{i7}) \bmod q, \tag{59}$$

$$=\{d_{ijk}\}(i,j,k=0,\ldots,7) \tag{60}$$

with $d_{ijk} \in \boldsymbol{Fq}$ $(i,j,k = 0,\ldots,7)$.

Then we obtain 64 equations from (44) and (58a) as follows.

$$\left.\begin{array}{l} d_{000}m_{i0}+d_{001}m_{i1}+\ldots+d_{007}m_{i7}=e_{00} \\ d_{010}m_{i0}+d_{011}m_{i1}+\ldots+d_{017}m_{i7}=e_{01} \\ \quad\ldots\ldots\quad\ldots\ldots \\ d_{070}m_{i0}+d_{071}m_{i1}+\ldots+d_{077}m_{i7}=e_{07} \end{array}\right\} \quad (61a)$$

$$\left.\begin{array}{l} d_{100}m_{i0}+d_{101}m_{i1}+\ldots+d_{107}m_{i7}=e_{10} \\ d_{110}m_{i0}+d_{111}m_{i1}+\ldots+d_{117}m_{i7}=e_{11} \\ \quad\ldots\ldots\quad\ldots\ldots \\ d_{170}m_{i0}+d_{171}m_{i1}+\ldots+d_{177}m_{i7}=e_{17} \end{array}\right\} \quad (61b)$$

$$\begin{array}{l} \ldots\ldots\quad\ldots\ldots \\ \ldots\ldots\quad\ldots\ldots \end{array}$$

$$\left.\begin{array}{l} d_{700}m_{i0}+d_{701}m_{i1}+\ldots+d_{707}m_{i7}=e_{70} \\ d_{710}m_{i0}+d_{711}m_{i1}+\ldots+d_{717}m_{i7}=e_{71} \\ \quad\ldots\ldots\quad\ldots\ldots \\ d_{770}m_{i0}+d_{771}m_{i1}+\ldots+d_{777}m_{i7}=e_{77} \end{array}\right\} \quad (61c)$$

For $M_1,\ldots, M_8$ we obtain the same equations, the number of which is 512.

We also obtain the 8 equations such as

$$|E(m_i,\mathbf{1})|^2 = |M_i|^2 = m_{i0}^2+m_{i1}^2+\ldots+m_{i7}^2 \bmod q, (i=1,\ldots,8). \quad (62)$$

The number of unknown variables M_i and d_{ijk} $(i,j,k=0,\ldots,7)$ is 576(=512+64).

The number of equations is 520(=512+8).

Then the complexity G required for solving above simultaneous quadratic algebraic equations by using Gröbner basis is given such as

$$G \approx G' = ({}_{520+dreg}C_{dreg})^w = ({}_{780}C_{260})^w > 2^{1699} >> 2^{80}, \quad (63)$$

where G' is the complexity required for solving 520 simultaneous quadratic algebraic equations with 519 variables by using Gröbner basis,

where w=2.39,

and

$$d_{reg} = 260(=520*(2-1)/2 - 0\sqrt{}(520*(4-1)/6) \quad (64)$$

It is thought to be difficult computationally to solve the above simultaneous algebraic equations by using Gröbner basis.

§4.3 Attack by using the ciphertexts of *m* and *-m*

I show that we can not easily distinguish the ciphertexts of *m* and *-m*.

We try to attack by using "*m* and *-m* attack".
We define the medium text *M* by

$$M:=u\mathbf{1}+vB+wH\in O, \tag{65}$$

where

a plaintext $m=u+2b_0v \bmod q$ where $u,v\in \boldsymbol{Fq}$,

a subtext $t= u+2b_0w \bmod q$ where $t= m \bmod h$ and $w\in \boldsymbol{Fq}$,

the medium text M_- by

$$M_-:= u'\mathbf{1}+v'B+w'H\in O, \tag{66}$$

where $u',v',w'\in \boldsymbol{Fq}$ such that

$$m'=-m=u'+2b_0v' \bmod q.$$

$$t'= u'+2b_0w' \bmod q \text{ where } t'= m' \bmod h \text{ and } w'\in \boldsymbol{Fq}.$$

By using simple style expression of $E(m, X)$

$$C(X):=E(m, X)= \boldsymbol{A}((M[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \bmod q\in O[X], \tag{67}$$

the ciphertext of *-m* is defined by

$$C_-(X):=E(-m, X)= \boldsymbol{A}(((M_-[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \bmod q\in O[X]. \tag{68}$$

$$m=u+2b_0v \bmod q,$$

$$m'=-m=u'+2b_0v' \bmod q,$$

$$m+m'=0=(u+u')+ 2b_0 (v+v'),$$

$$t+t'=(u+u')+ 2b_0 (w+w').$$

We have

$$C(X)+ C_{-}(X) = E(m, X)+ E(-m, X)= E(m-m, X)= E(0, X)$$

$$=\boldsymbol{A}((\,[M + M_{-}][(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \quad \mod q$$

$$= (\boldsymbol{A}(\,[u\mathbf{1}+vB +wH +u'\mathbf{1}+v'B +w'H]\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \mod q$$

$$= (\boldsymbol{A}(\,[\,(u+u')\mathbf{1}+(v+v')B+ (w+w')H]\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \mod q$$

$$= (\boldsymbol{A}(\,[\,(v+v')\,(-2b_0\mathbf{1}+B\,)+ (w+w')H]\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \mod q$$

$$= (\boldsymbol{A}(\,[-(v+v')\,H+ (w+w')H]\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \mod q$$

$$= (\boldsymbol{A}(\,[\,(-v-v'+ w+w')H]\,[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1}) \mod q$$

$$\neq \mathbf{0} \mod q \text{ (in eneral)} \qquad (69)$$

We can calculate $|\, C(\mathbf{1})+ C_{-}(\mathbf{1})\,|^2$ as follows.

From $|H|^2=0 \mod q$, we have

$$|\, C(\mathbf{1})+ C_{-}(\mathbf{1})\,|^2=|\, E(0, \mathbf{1})|^2$$

$$= |(\boldsymbol{A}(\,[\,(-v-v'+ w+w')H]\,[(\boldsymbol{A}^{-1}\mathbf{1})\boldsymbol{Z}])\boldsymbol{Z}^{-1})\,|^2 \mod q$$

$$= |\,(-v-v'+ w+w')H\,|^2 \mod q$$

$$=0 \mod q.$$

But we can find many M_{-} such that

$$|C(\mathbf{1}) + C_{-}(\mathbf{1})|^2=|\,\boldsymbol{A}(([M + M_{-}][(\boldsymbol{A}^{-1}\mathbf{1})\boldsymbol{Z}])\boldsymbol{Z}^{-1})\,|^2 =|[M + M_{-}]|^2 \mod q,$$

$$=(\,u+u'+2b_0(v+v'))(\,u+u'+2b_0(w+w'))$$

$$=(\,m+m')(\,t+t')=0 \mod q,$$

because we can select many set of t , $t' \in \{-(h-1)/2,\ldots,0,\ldots,(h-1)/2\}$ such that

$$t+t'=0.$$

That is,

$$m \text{ modd } h + m' \text{ modd } h =0,$$

$$m'=h-m-1+ch \ (c=\ldots,-2,-1,0,1,2,\ldots),$$

and

$$m+m'=h-1+ch\neq 0 \bmod q \text{ in general.}$$

That is, even if

$$|\ C(\mathbf{1})+ C_{-}(\mathbf{1})\ |^2=0 \bmod q,$$

it hold with overwhelming probability that

$$m+m'\neq 0 \bmod q.$$

It is said that the attack by using "m and $-m$ attack" is not efficient.

Then we can not easily distinguish the ciphertexts of m and $-m$.

§5. The size of the modulus q and the complexity for enciphering/ deciphering

We consider the size of the system parameter q. We select the size of q such that $O(q)$,the size of the plaintext is as large as 2^{1000}. Then we need to select modulus q such as $O(q)= 2^{1000}$.

In case of k=8, $O(q)=2^{1000}$, the size of $e_{ij}\in \boldsymbol{Fq}$ (i,j=0,…,7) which are the coefficients of elements in $E(m,X)= \boldsymbol{A}((M[(\boldsymbol{A}^{-1}X)\boldsymbol{Z}])\boldsymbol{Z}^{-1})\bmod q\in O[X]$ is $(64)(\log_2 q)$bits =64kbits, and the size of system parameters q is as large as 1kbits.

In case of k=8, $O(q)=2^{1000}$, the complexity to obtain inverse of $\boldsymbol{A}^{-1}$ and $\boldsymbol{Z}^{-1}$is

$16*16*(\log_2 q)^2+16*(\log_2 q)^3 =2^{34}$ bit-operations,.

In case of k=8, $O(q)=2^{1000}$, the complexity required for enciphering,that is, required to obtain $E(m,X)$ is

$(32*512)(\log_2 q)^2= 2^{34}$ bit-operations.

And the complexity required for deciphering is given as follows.
Let $C:=A_1((\ldots((A_k((M[(A_k^{-1}((\ldots((A_1^{-1}\mathbf{1})Z_1))\ldots))Z_k])\ Z_k^{-1}))\ldots))\ Z_1^{-1})$ mod q.
We have
$(A_k\ ((\ldots((A_1^{-1}\ C\ )Z_1))\ Z_2))\ldots.))Z_k =M[(A_k^{-1}((\ldots((A_1^{-1}\mathbf{1})Z_1))\ldots))Z_k] \bmod q,$

$M=[(A_k((\ldots((A_1^{-1}C)Z_1))Z_2))\ldots))Z_k][(A_k^{-1}((\ldots((A_1^{-1}\mathbf{1})Z_1))\ldots))Z_k]^{-1} \bmod q.$

Let $(m_0,m_1,\ldots,m_7)$: $=M=u\mathbf{1}+vB+wH$, we have

$$m= u +v =m_0 + m_1 (b_0/b_1) \bmod q.$$

Then the complexity required for deciphering is

$(16*64+15*64+64)(\log_2 q)^2+[16*(\log_2 q)^2+(\log_2 q)^3]+2*(\log_2 q)^2 +(\log_2 q)^3$

$=(2048+18)(\log_2 q)^2+(2)(\log_2 q)^3= 2^{32}$ bit-operations.

On the other hand the complexity of the enciphering and deciphering in RSA scheme is

$O(2(\log_2 n)^3)=2^{31}$ bit-operations

where the size of modulus n is 1024bits.

Then our scheme requires complexity to encipher and decipher slightly larger than RSA so that we are able to implement our scheme to the many device.

§6. Conclusion

We proposed the new fully homomorphism encryption scheme based on the octonion ring over finite field. It was shown that our scheme is immune from the Gröbner basis attacks by calculating the complexity to obtain the Gröbner basis for the multivariate algebraic equations and immune from "*m* and -*m* attack"**.**

The proposed scheme does not require a "bootstrapping" process so that the complexity required to encipher and decipher is not large.

§7.Acknowledgments

This paper is the revised chapter 4 of my work "Fully Homomorphic Encryption without bootstrapping" published in March, 2015 which was published by LAP LAMBERT Academic Publishing, Saarbrücken/Germany [1].

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Appendix A:

Octinv(*A*) ----------

S ← $a_0^2+a_1^2+\ldots+a_7^2$ mod *q*.
% S^{-1} mod *q*
q[1] ← q div S ;% integer part of q/S
r[1] ← q mod S ;% residue
k ←1
q[0] ← q
r[0] ← S
while r[k] ≠ 0
begin
k← k + 1
q[k] ← r[k−2] div r[k−1]
r[k] ← r[k−2] mod [rk−1]
end
Q [k−1] ← (-1)∗q[k−1]
L[k−1] ← 1
i ← k−1
while i > 1
begin
Q[i−1] ← (-1)∗Q[i] ∗q[i−1] + L[i]
L[i−1] ← Q[i]
i← i−1
end

invS ← Q[1] mod *q*
invA[0] ← a_0∗invS mod *q*
For *i*=1,…,7,
invA[*i*] ← (-1)∗a_i∗invS mod *q*
Return A^{-1}= (invA[0], invA[1],…, invA[7])

Appendix B:

Lemma 1

$$A^{-1}(AB)= B$$
$$(BA)A^{-1}= B$$

(*Proof*:)

$A^{-1}= (a_0/|A|^2 \bmod q, -a_1/|A|^2 \bmod q,\ldots, -a_7/|A|^2 \bmod q).$

$AB \bmod q$

$= (a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7 \bmod q,$
$a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3 \bmod q,$
$a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6 \bmod q,$
$a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1 \bmod q,$
$a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5 \bmod q,$
$a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4 \bmod q,$
$a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2 \bmod q,$
$a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0 \bmod q).$

$[A^{-1}(AB)]_0$
$=\{ a_0(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$+a_1(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$+ a_2(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$
$+a_3(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)$
$+a_4(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$
$+ a_5(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$
$+a_6(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$
$+a_7(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)\} /|A|^2 \bmod q$

$=\{(a_0^2+a_1^2+\ldots+a_7^2) b_0\} /|A|^2 =b_0 \bmod q$

where $[M]_n$ denotes the n-th element of $M \in O$.

$[A^{-1}(AB)]_1$
$=\{ a_0(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$-a_1(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$-a_2(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$
$-a_3(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)$

$+a_4(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$

$-a_5(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$

$+a_6(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$

$+a_7(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)\}/|A|^2 \bmod q$

$=\{(a_0^2+a_1^2+\ldots+a_7^2)\,b_1\}/|A|^2=b_1 \bmod q.$

Similarly we have

$[A^{-1}(AB)]_i=b_i \bmod q$ $(i=2,3,\ldots,7)$.

Then

$$A^{-1}(AB)= B \bmod q. \qquad \text{q.e.d.}$$