SOLITARY WAVE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS BY HE'S VARIATIONAL METHOD

by

***Fei-Yang WANG*[a] *and Jian-She SUN*[a,b,c,d*]**

[a] Navigation College, Dalian Maritime University, Dalian, Liaoning, China
[b] School of Mathematics, Jiaozuo Teacher's College, Jiaozuo, China
[c] Institute of Mathematics and Interdisciplinary Science, Jiaozuo Teacher's College, Jiaozuo, China
[d] School of Mathematics, China University of Mining and Technology, Xuzhou, China

Original scientific paper
https://doi.org/10.2298/TSCI2403959W

Existence of variational principles for Navier-Stokes equations has been discussing for hundreds of years, but it has not yet been solved. In this study, a new perspective is proposed, which uses a traveling wave transform, so that a variational formulation can be established. Furthermore, the solitary wave solutions are solved by He's variational method.

Key word: *Navier-Stokes millennium-prize problem, traveling wave transform, solitary wave solutions, He's variational methods, He-Weierstrass function, variational principle*

Introduction

Any a motion should follow a nature law, the most famous one is the Hamilton principle [1-3], which is a minimum variational principle. Navier-Stokes equations describe the motion of a fluid, which can be expressed:

$$\begin{gathered}\frac{\partial \xi}{\partial s}+(\xi\cdot\nabla)\xi=-\frac{1}{\rho}\nabla p+\frac{\mu}{\rho}\nabla^2\xi\\ \nabla\xi=0\end{gathered}\tag{1}$$

where p is the pressure, ρ – the density, μ – the kinematic viscosity, and s – the time. In 3-D space, velocity vector is $\xi=\xi(\xi_1,\xi_2,\xi_3)$, the components of Navier-Stokes equations in i-, j-, and k-directions are given by the following equations:

$$\begin{gathered}\frac{\partial \xi_1}{\partial s}+\xi_1\frac{\partial \xi_1}{\partial i}+\xi_2\frac{\partial \xi_1}{\partial j}+\xi_3\frac{\partial \xi_1}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_1}{\partial i^2}+\frac{\partial^2\xi_1}{\partial j^2}+\frac{\partial^2\xi_1}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial i}\\ \frac{\partial \xi_2}{\partial s}+\xi_1\frac{\partial \xi_2}{\partial i}+\xi_2\frac{\partial \xi_2}{\partial j}+\xi_3\frac{\partial \xi_2}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_2}{\partial i^2}+\frac{\partial^2\xi_2}{\partial j^2}+\frac{\partial^2\xi_2}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial j}\\ \frac{\partial \xi_3}{\partial s}+\xi_1\frac{\partial \xi_1}{\partial i}+\xi_2\frac{\partial \xi_3}{\partial j}+\xi_3\frac{\partial \xi_3}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_3}{\partial i^2}+\frac{\partial^2\xi_3}{\partial j^2}+\frac{\partial^2\xi_3}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial k}\end{gathered}\tag{2}$$

* Corresponding author, e-mail: sunjianshe@jzsz.edu.cn

where

$$\frac{\partial \xi_1}{\partial i}+\frac{\partial \xi_2}{\partial j}+\frac{\partial \xi_3}{\partial k}=0$$

Navier-Stokes equations should also follow a variational principle, though much effort has been paid, its existence is still a big problem. Scientists only find some variational formulations for simple fluids [4-6]. A variational principle can give profound, original, and challenging insights of a fluid problem, especially the travelling waves.

The well-known KdV equations [7-9] is the approximate case of the Navier-Stokes equations, there are various variational principles for KdV equations [10-12], and the modern soliton theory is originally developed from the KdV equation. This paper aims at searching for solitary waves directly from the Navier-Stokes equations by establishment of a suitable variational principle.

Variational principle

The variational formulation for eq. (1) is extremely difficult to be obtained. This paper is to search for solitary waves from Navier-Stokes equations, so we focus ourselves on a constrained variational formulation by the following transformations [13-16]:

$$\begin{aligned}\xi_1(i,j,k,s)&=\Xi(\varepsilon)\\ \xi_2(i,j,k,s)&=\mathrm{A}(\varepsilon)\\ \xi_3(i,j,k,s)&=\mathrm{B}(\varepsilon)\\ p(i,j,k,s)&=P(\varepsilon)\end{aligned}\tag{3}$$

$$\varepsilon=\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0\tag{4}$$

$$\begin{aligned}\alpha\xi_2&=\xi_1\\ \beta\xi_3&=\xi_1\\ \vartheta p&=\xi_1\end{aligned}\tag{5}$$

where α, β, and ϑ are all non-zero functions.

Based on the previous transformation, we can convert the Navier-Stokes equations into the following ordinary differential equation:

$$-\mathrm{c}\Xi'+\lambda_1\Xi\cdot\Xi'+\frac{\lambda_2}{\alpha}\Xi\cdot\Xi'+\frac{\lambda_3}{\beta}\Xi\cdot\Xi'=\frac{\mu\lambda_1^{\,2}}{\rho}\Xi''+\frac{\mu\lambda_2^{\,2}}{\rho}\Xi''+\frac{\mu\lambda_3^{\,2}}{\rho}\Xi''-\frac{\lambda_1}{\rho\vartheta}\Xi'\tag{6}$$

Through the previous equation, we have:

$$-\frac{\mu}{\rho}(\lambda_1^{\,2}+\lambda_2^{\,2}+\lambda_3^{\,2})\Xi''+\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi\cdot\Xi'+\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi'=0\tag{7}$$

Integrating (7), we have:

$$-\frac{\mu}{\rho}(\lambda_1^{\,2}+\lambda_2^{\,2}+\lambda_3^{\,2})\Xi'+\frac{1}{2}\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi^2+\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi=H\tag{8}$$

According to (7) and (8), we have:

$$\Xi' = \frac{\rho}{\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\left[\frac{1}{2}\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi^2+\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi-H\right] \tag{9}$$

Hence, eq. (7) can be represented:

$$\frac{\frac{2\mu^2}{\rho}(\lambda_1^2+\lambda_2^2+\lambda_3^2)^2\Xi''-\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2\Xi^3-3\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi^2}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}+$$

$$+\frac{-2\rho\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2\Xi+2\rho H\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi+2\rho H\left(\frac{\lambda_1}{\rho\vartheta}-c\right)}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}=0 \tag{10}$$

So the variational formulation of (10) can be established by the semi-inverse method [17], which is:

$$J(\Xi)=\int\left[-\frac{\mu}{2\rho}(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\Xi')^2-\frac{\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2}{12\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\Xi^4-\right.$$
$$\left.-\frac{\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\Xi^3\right]\mathrm{d}\varepsilon+$$
$$+\int\left[-\rho\frac{\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2-H\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\Xi^2+\frac{\rho\left(\frac{\lambda_1}{\rho\vartheta}-c\right)H}{\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\Xi\right]\mathrm{d}\varepsilon \tag{11}$$

The semi-inverse method [17] has been widely used to establish a suitable variational formulation from a governing equation, for examples, the variational principle for singular waves [18], water waves [19], nano/microelectromechanical systems [20], two-point boundary value problems [20], KdV-Burgers-Kuramoto equation [21], 3D unsteady fluids [22], thin films [23], solitary waves [24], long water waves [25], Schrodinger equation [26].

From eq. (11), He-Weierstrass function [27] can be obtained:

$$E(\varepsilon,\Xi,\Xi',\omega)=\frac{1}{2}\omega^2-\frac{1}{2}(\Xi')^2-(\omega-\Xi')\Xi' \tag{12}$$

where $\omega=\partial\Xi/\partial\varepsilon$.

From eq. (12), It is evident that:

$$E(\varepsilon,\Xi,\Xi',\omega)=0,\quad \frac{\partial^2 E}{\partial\omega^2}>0 \tag{13}$$

Equation (13) shows that eq. (12) is a minimal variational principle.

Solitary wave solutions

The objective of this section is to identify solitary wave solutions for Navier-Stokes equations by the obtained variational principle. The idea goes back to [28] variational approach to solitons, and it has been showing its validity for various wave equations, for examples, Boussinesq equation [29] and various the wave equations [30-36].

According to He's variational theory [28], we assume that the solitary solution of eq. (11) take the following form:

$$\Xi(\varepsilon)=\kappa\operatorname{sech}^2(\upsilon\varepsilon) \tag{14}$$

where $\kappa \neq 0$, $\upsilon \neq 0$, κ and υ are unknown constants to be determined later.

Upon simultaneous solution of eq. (11) and eq. (14), the resulting expression is shown [28]:

$$\begin{aligned}
J(\kappa,\upsilon) &= \int_0^\infty \left\{ -\frac{2\mu}{\rho}(\lambda_1^2+\lambda_2^2+\lambda_3^2)\kappa^2\upsilon^2[\sec h^2(\upsilon\varepsilon)tg(\upsilon\varepsilon)]^2 - \right. \\
&\left. -\frac{\rho\kappa^4}{12\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2[\sec h^2(\upsilon\varepsilon)]^4 \right\}\mathrm{d}\varepsilon - \\
&-\int_0^\infty \left\{ \frac{\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\kappa^3}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}[\sec h^2(\upsilon\varepsilon)]^3 + \right. \\
&\left. +\frac{\left[\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2-\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)H\right]\rho\kappa^2}{2\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}[\sec h^2(\upsilon\varepsilon)]^2 \right\}\mathrm{d}\varepsilon + \\
&+\int_0^\infty \left\{ \frac{\rho H\kappa}{\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\sec h^2(\upsilon\varepsilon) \right\}\mathrm{d}\varepsilon
\end{aligned} \tag{15}$$

The following are the results obtained:

$$\begin{aligned}
J(\kappa,\upsilon) &= -\frac{\rho\kappa 28\frac{\mu^2}{\rho^2}(\lambda_1^2+\lambda_2^2+\lambda_3^2)^2\kappa\upsilon^2}{105\upsilon\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)} - \\
&-\frac{\rho\kappa\left[4\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2\kappa^3+28\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\kappa^2\right]}{105\upsilon\mu\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)} + \\
&+\frac{\rho\kappa\left[-35\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2\kappa+35H\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\kappa+105\left(\frac{\lambda_1}{\rho\vartheta}-c\right)H\right]}{105\upsilon\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)}
\end{aligned} \tag{16}$$

Considering He's variational method [28], it gives:

$$\frac{\partial J}{\partial \kappa}=0 \tag{17}$$

$$\frac{\partial J}{\partial \upsilon}=0 \tag{18}$$

When

$$\left(\frac{\lambda_1}{\rho\vartheta}-c\right)$$

tends to zero, the eq. (16) can bring the following results:

$$\frac{-56\dfrac{\mu^2}{\rho}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2\kappa\upsilon^2-16\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)^2\rho\kappa^3+70\mathrm{H}\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)\rho\kappa}{105\upsilon\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}=0 \tag{19}$$

$$\frac{-28\dfrac{\mu^2}{\rho}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2\kappa^2\upsilon^2+4\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)^2\rho\kappa^4-35H\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)\rho\kappa^2}{105\upsilon^2\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}=0 \tag{20}$$

Solving eqs. (19) and (20) we can determine κ and υ:

$$\kappa=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}} \tag{21}$$

$$\upsilon=\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}} \tag{22}$$

So the eq. (14) can be replaced by:

$$\Xi(\varepsilon)=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}}\operatorname{sech}^2\left[\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}}(\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0)\right] \tag{23}$$

With eq. (11), the solitary wave solution of eq. (1) can be approximated:

$$\Xi(i,j,k,s)=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}}\operatorname{sech}^2\cdot$$

$$\cdot\left[\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}}(\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0)\right] \tag{24}$$

Two examples

This section gives two examples to verify the accuracy and effectiveness of He's variational method [28].

Example 1. Consider variables in eq. (24), let $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\alpha = 2$, $\beta = 2$, $\rho = 1$, $\vartheta = 1$, $c = 3$, $\mu = -1$, and $H = 1$.

When $j = 0$, $k = 0$, and $\varepsilon_0 = 1$, we obtain the solitary wave solution, fig. 1(a) in a single direction:

$$\Xi(i,j,k,s) = \sqrt{\frac{35}{18}}\,\mathrm{sech}^2\left[\sqrt{\frac{5}{144}}(2i-3s+1)\right] \tag{25}$$

Example 2. In this case, we use $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\alpha = 2$, $\beta = 1$, $\rho = -2$, $\vartheta = 1$, $c = -3$, $\rho = 2$, and $H = 2$.

When $i = 0$, $k = 0$, and $\varepsilon_0 = -2$, the solitary wave solution, fig. 1(b), in another single direction can be obtained:

$$\Xi(i,j,k,s) = \sqrt{\frac{7}{3}}\,\mathrm{sech}^2\left[\frac{5}{14}\sqrt{\frac{1}{24}}(2j+3s-2)\right] \tag{26}$$

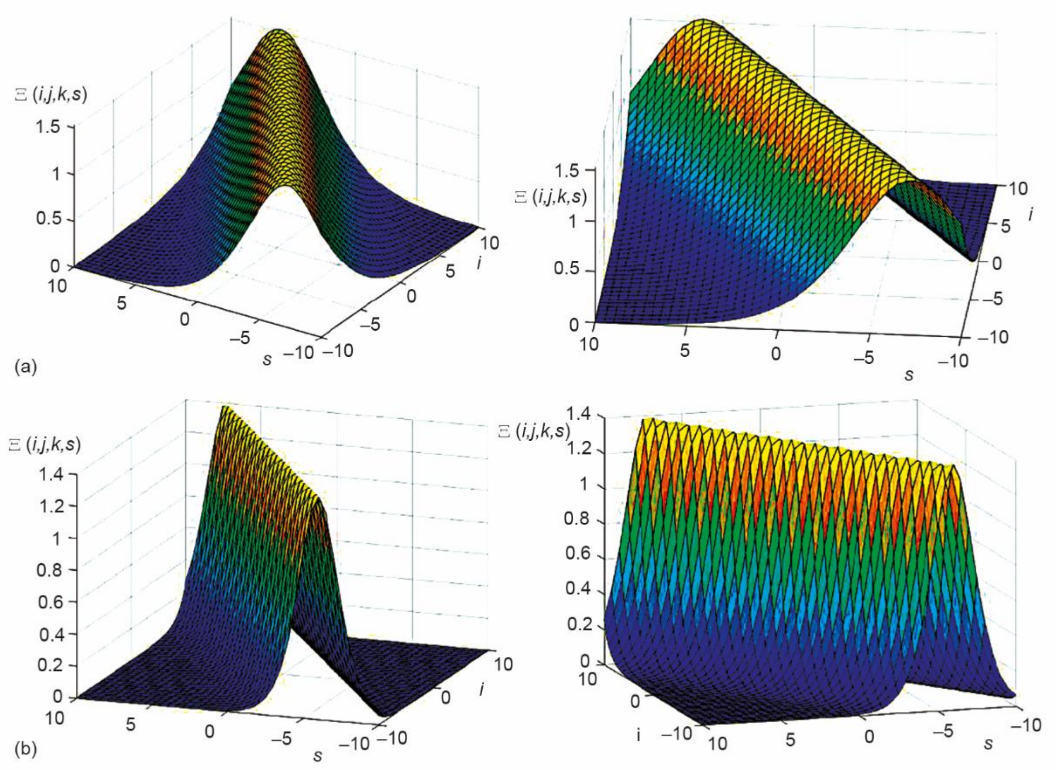

Figure 1. Solitary wave solutions for Navier-Stokes equations; (a) $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\alpha = 2$, $\beta = 2$, $\rho = 1$, $\vartheta = 1$, $c = 3$, $\mu = -1$, and $H = 1$, (b) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\alpha = 2$, $\beta = 1$, $\rho = -2$, $\vartheta = 1$, $c = -3$, $\mu = 2$, and $H = 2$ (*for color image see journal web site*)

Conclusions

Navier-Stokes millennium-prize problem is still an open problem [37], and there might have not exact solution to the Navier-Stokes equations, though the model has been widely used to explain various unsolved problems, *e.g.*, the mountain-river-desert relation [38, 39] and the dynamical properties of a rotating rigid body containing a viscous incom-pressible fluid [40]. This paper gives an alternative approach to the open problem, and exact solutions exist for solitary waves.

In this work, the solitary wave solution of Navier-Stokes equations has been obtained by He's variational method. This paper offers a totally new window for searching for solitary wave solutions directly from Navier-Stokes equations instead of its various approximate forms like KdV equation or Burgers equation, making the soliton theory much accurate to model the solitary waves.

Acknowledgment

This article was supported by the Provincial Cultivation Fund for the high-level scientific research project of Jiaozuo Normal College (Grant No. GPY 2022-07) and Henan Provincial Key Research Project Fund for Higher Education Institutions (Grant No. 24B110008 and No. 22B110007). We are particularly grateful to the anonymous reviewers for their constructive and many critical comments.

References

[1] Hill, E. L., Hamilton Principle and Conservation Theorem of Mathematical Physics, *Reviews of Modern Physics*, *23* (1951), 3, pp. 253-260
[2] Ma, H. J., Simplified Hamiltonian-Based Frequency-Amplitude Formulation for Non-linear Vibration Systems, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 2, pp. 445-455
[3] He, J.-H., *et al*., Forced Non-Linear Oscillator in a Fractal Space, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 1, pp. 1-20
[4] Finlayson, B. A., Existence of Variational Principles for Navier-Stokes Equation, *Physics of Fluids, 15* (1972), 6, pp. 963-967
[5] Yasue, K., A Variational Principle for the Navier-Stokes Equation, *Journal of Functional Analysis, 51* (1983), 2, pp. 133-141
[6] Kerswell, R. R., Variational Principle for the Navier-Stokes Equations, *Physical Review E, 59* (1999), 5, pp. 5482-5494
[7] Kenig, C. E., *et al*., A Bilinear Estimate with Applications to the KdV Equation, *Journal of the American Mathematical Society*, *9* (1996), 2, pp. 573-603
[8] He, J.-H., *et al*., Exp-Function Method for Non-linear Wave Equations, *Chaos Solitons & Fractals, 30* (2006), 3, pp. 700-708
[9] Cao, X. Q., *et al*., Variational Principles for Two Kinds of Non-Linear Geophysical Kdv Equation with Fractal Derivatives, *Thermal Science, 26* (2022), 3B, pp. 2505-2515
[10] Shen, Y., He, J. H., Variational Principle for a Generalized KdV Equation in a Fractal Space, *Fractals*, *28* (2020), 4, 2050069
[11] He, J. H., Variational Principle for the Generalized KdV-Burgers Equation with Fractal derivatives for Shallow Water Waves, *J. Appl. Comput. Mech*., *6* (2020), 4, pp. 735-740
[12] Sun, J. S., Traveling Wave Solution of Fractal KDV-Burgers-Kuramoto Equation Within Local Fractional Differential Operator, *Fractals*, *29* (2021), 7, 2150231
[13] Weekes, S. L., The Travelling Wave Scheme for The Navier-Stokes Equations, *SIAM. J. Numer. Anal., 35* (1998), 3, pp. 1249-1270
[14] Dubovskii, C. P., *et al.*, Travelling Wave-Like Solutions of the Navier-Stokes and the Related Equations, *J. Math. Anal. Appl., 204* (1996), 0477, pp. 930-939
[15] Cazacu, C. A., *et al*., Transformation of The Travelling Wave Shape in Propagation on A Straight and Inclined Bed, *Stud. U. Babes-Bol. Mat., 57* (2012), 2, pp. 167-173

[16] Bakhoum, E. G., Cristian, T., Mathematical Transform of Traveling-Wave Equations and Phase Aspects of Quantum Interaction, *Math. Probl. Eng.*, *2010* (2010), 695208
[17] He, J.-H., Variational Principles for Some Non-Linear Partial Differential Equations with Variable Coefficients, *Chaos Solitons & Fractals*, *19* (2004), 4, pp. 847-851
[18] He, C. H., Liu, C., Variational Principle for Singular Waves, *Chaos, Solitons & Fractals*, *172* (2023), 113566
[19] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals*, *27* (2019), 1950134
[20] He, C. H., A Variational Principle for a Fractal Nano/Microelectromechanical (N/MEMS) System, *Int. J. Numer. Methods H., 33* (2023), 1, pp. 351-359
[21] Wang, S. Q., A Variational Approach to Non-Linear Two-Point Boundary Value Problems, *Computers & Mathematics with Applications*, *58* (2009), 11, pp. 2452-2455
[22] He, J.-H., Lagrange Crisis and Generalized Variational Principle For 3D Unsteady Flow, *Int. J. Numer. Method. H., 30* (2019), 3, pp. 1189-1196
[23] He, J. H., Sun C., A Variational Principle for a Thin Film Equation. *J. Math. Chem*., *57* (2019), 9, pp. 2075-2081
[24] Liu, M. Z., *et al*., Internal Solitary Waves in The Ocean by Semi-Inverse Variational Principle, *Thermal Science*, *26* (2022), 3B, pp. 2517-2525
[25] Sun, J. S., Variational Principle for Fractal High-Order Long Water-Wave Equation, *Thermal Science*, *27* (2023), 3A, pp. 1899-1905
[26] Sun, J. S., Fractal Modification of Schrodinger Equation and Its Fractal Variational Principle, *Therm. Sci*., 27 (2023), accepted
[27] He, J.-H., *et al*., On a Strong Minimum Condition of a Fractal Variational Principle, *Appl Math Lett*, *119* (2021), 107199
[28] He, J. H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstr. Appl. Anal*., *2012* (2012), pp. 97-102
[29] Wang, K. J., Wang, G. D., Solitary and Periodic Wave Solutions of The Generalized Fourth-Order Boussinesq Equation Via He's Variational Methods, *Math. Method. Appl. Sci.*, *44* (2021), 7, pp. 5617-5625
[30] Wang, K. J., *et al*., Solitary Waves of The Fractal Regularized Long-Wave Equation Traveling Along an Unsmooth Boundary, *Fractals, 30* (2022), 1, pp. 1-6
[31] He, J.-H., *et al.*, Solitary Waves Travelling Along an Unsmooth Boundary, *Results in Physsics*, *24* (2021), 104104
[32] He, J.-H., *et al.,* Solitary Waves of The Variant Boussinesq-Burgers Equation in a Fractal-Dimensional Space, *Fractals*, *30* (2022), 3, 2250056
[33] Sun, J. S., Approximate Analytic Solutions of Multi-Dimensional Fractional Heat-Like Models with Variable Coefficients, *Thermal Science*, *23* (2019), 6B, pp. 3725-3729
[34] Wang, K. L., *et al*., A Novel Perspective to the Local Fractional Bidirectional Wave Model on Cantor Sets, *Fractals*, *30* (2022), 6, pp. 1-7
[35] Sun, J. S., Variational Principle and Solitary Wave of the Fractal Fourth-Order Non-linear Ablowitz-Kaup-Newell-Segur Water Wave Model, *Fractals, 31* (2023), 5, 2350036
[36] Sun, J.S., Variational Principle for Fractal High-Order Long Water-Wave Equation, *Thermal Science, 27* (2023), 3, pp. 1899-1905
[37] Nelkin, M., In What Sense is Turbulence an Unsolved Problem, *Science, 255* (1992), Jan., pp. 566-570
[38] Mei, Y., *et al*., The Yellow River-Bed Evolution: A Statistical Proof of the Mountain-River-Desert Conjecture, *Thermal Science, 27* (2023), 3A, pp. 2075-2079
[39] Mei, Y., *et al*., On the Mountain-River-Desert Relation, *Thermal Science, 25* (2021), 6, pp. 4817-4822
[40] He, J.-H., *et al.,* Dynamical Analysis of a Rotating Rigid Body Containing a Viscous Incompressible Fluid, *International Journal of Numerical Methods for Heat & Fluid Flow, 33* (2023), 8, pp. 2800-2814

Paper submitted: February 1, 2023
Paper revised: August 2, 2023
Paper accepted: August 7, 2023