

RECONSTRUCTION OF THE BOUNDARY CONDITION FOR THE HEAT CONDUCTION EQUATION OF FRACTIONAL ORDER

by

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This paper describes reconstruction of the heat transfer coefficient occurring in the boundary condition of the third kind for the time fractional heat conduction equation. Fractional derivative with respect to time, occurring in considered equation, is defined as the Caputo derivative. Additional information for the considered inverse problem is given by the temperature measurements at selected points of the domain. The direct problem is solved by using the implicit finite difference method. To minimize functional defining the error of approximate solution the Nelder-Mead algorithm is used. The paper presents results of computational examples to illustrate the accuracy and stability of the presented algorithm.

Key words: *inverse problem, identification, heat conduction equation, time fractional heat transfer coefficient*

Introduction

Many various types of phenomena in engineering, physics, mechanics, and control theory are modeled by using the derivatives of fractional order. Fractional heat-conduction equations may be applied to describe the physical phenomena in porous media. In paper [1] the authors introduce the concept of local fractional derivatives and present the fractal models for one-phase problems of discontinuous transient heat transfer.

In [2] the authors present local fractional variation iteration method to solve the local fractional heat conduction equation. Described method is derived on the basis of the local fractional calculus. In this method, the solution is described as the sum of fractional series.

Inverse problems have a wide range of applications in the analysis of various processes, as well as in the design of different devices. By using them, we can choose *e. g.* the boundary conditions or other parameters of the process described by means of differential equations proceeded in specified manner. In case of differential equations of integer order, they have been under consideration for many years (see for example [3-10] and cited there literature).

The first papers in which the inverse problems for the equations with fractional derivatives were considered, are the Murio's papers [11-14]. In these works the mollification method was applied. The heat flux and the temperature on the boundary were reconstructed in case when the measurements of temperature inside the domain were known. In paper [12] additional information was provided by the double boundary condition on the other bound of the interval (the temperature and the heat flux were given there). In work [13] the initial condition was reconstructed.

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In [15] the authors consider the inverse problem of determining the spatial coefficient in source term of the equation and/or the order of Caputo derivative for the time fractional diffusion equation. The authors proved that under appropriate assumptions the solution of this problem is unique. Problem of the same type is investigated in [16], in which the authors transformed the problem into the first kind Volterra integral equation. Furthermore, they used the boundary element method combined with the generalized Tikhonov regularization to solve this integral equation. Jin and Rundell [17] discuss similar inverse problem consisting in reconstruction of a spatially varying potential term in the one-dimensional time-fractional diffusion equation on the basis of known flux.

Paper [18] describes an identification of parameters of the fractional inverse problem of Caputo type, for which the analytical solution is known. In computations for minimizing the quadratic criterion the Marquardt's algorithm was used. Next, in paper [19] few minimization algorithms are compared (Levenberg-Marquardt algorithm, Gauss-Newton algorithm, and Nelder-Mead method) in application for identifying parameters of the Riesz fractional advection-dispersion equation. In summary the authors claim that for solving the problem considered in their paper the best choice is the Nelder-Mead method.

In paper [20] the problem of determining the thermal conductivity coefficient in the time fractional heat conduction equation is considered. Additional information for considered inverse problem was given by the measurements of temperature in the selected points of the region. The direct problem was solved by applying the finite difference method. Whereas for minimizing the constructed functional expressing the error of approximate solution the Fibonacci search algorithm was used. Paper [21] is devoted to construction of the diffusion coefficient and the order of fractional derivative for the problem with zero initial condition and the Dirichlet condition.

In this paper we reconstruct the heat transfer coefficient occurring in the boundary condition for the time fractional heat conduction equation. Fractional derivative with respect to time appearing in considered equation is defined as the Caputo derivative. Additional information for the considered inverse problem is given by the temperature measurements at selected points of the domain. The direct problem is solved by using the implicit finite difference method [22, 23]. To minimize functional defining the error of approximate solution the Nelder-Mead algorithm is used.

Formulation of the problem

We consider the following time fractional heat conduction equation:

$$c\rho \frac{\partial u^\alpha(x, t)}{\partial t^\alpha} = \lambda \frac{\partial u^2(x, t)}{\partial x^2} \quad (1)$$

Equation (1) is specified in the domain $D = \{(x, t) : x \in [0, L], t \in [0, t^*]\}$. To eq. (1) the initial condition is posed:

$$u(x, 0) = f(x), \quad x \in [0, L] \quad (2)$$

as well as the Neumann and Robin boundary conditions:

$$-\lambda \frac{\partial u}{\partial x}(0, t) = q(t), \quad t \in (0, t^*) \quad (3)$$

$$-\lambda \frac{\partial u}{\partial x}(L, t) = h(t)[u(L, T) - u^\infty], \quad t \in (0, t^*) \quad (4)$$

Fractional derivative with respect to time, which occurs in eq. (1), will be defined as the Caputo derivative [24, 25], in our case $\alpha \in (0, 1)$ determined:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} (t-s)^{-\alpha} ds \quad (5)$$

We assume that the form of function h , occurring in the Robin boundary condition, will depend on two coefficient γ_1 and γ_2 . Inverse problem, considered in this paper, consists in determination of coefficients γ_1 and γ_2 (and therefore the boundary condition) in such a way that the temperature in selected points of discussed domain will adopt to the preset values. Known values of function u (input data) in selected points (x_i, t_j) of the domain D will be denoted:

$$u(x_i, t_j) = \hat{U}_{ij}, \quad i = 1, 2, \dots, N_1, \quad j = 1, 2, \dots, N_2 \quad (6)$$

For the fixed coefficients γ_1 and γ_2 the investigated issue becomes a direct problem, solution of which is represented by the temperature values $U_{ij}(h)$ corresponding to the given value of the heat transfer coefficient. By using the computed temperatures $U_{ij}(h)$ and the measurement temperatures \hat{U}_{ij} , the functional defining the error of approximate solution is created:

$$J(h) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} [U_{ij}(h) - \hat{U}_{ij}]^2 \quad (7)$$

By minimizing this functional we can find the approximate value of the heat transfer coefficient.

Direct problem

Direct problem defined by eqs. (1)-(4) for the fixed value of heat transfer coefficient is solved by using the implicit finite difference scheme [22, 23]. For this purpose, we create grid of the form $S = \{(x_i, t_k): x_i = i\Delta x, t_k = k\Delta t, i = 0, 1, \dots, N, k = 0, 1, \dots, M\}$ with size $N \times M$ and steps $\Delta x = L/N$ and $\Delta t = t^*/M$. Fractional derivative (5) is approximated [22] by:

$$D_t^{(\alpha)} u_i^k = \sigma \sum_{j=1}^k \omega(\alpha, j) (u_i^{k-j+1} - u_i^{k-j}) \quad (8)$$

where

$$\sigma := \sigma(\alpha, \Delta t) = \frac{1}{\Gamma(1-\alpha)(1-\alpha)\Delta t^\alpha}, \quad \omega(\alpha, j) = j^{1-\alpha} - (j-1)^{1-\alpha}$$

Using eq. (8), the approximation of Neumann and Robin boundary condition and the difference quotient for the second derivative with respect to the spatial variable, we obtain the following differential equations:

$$k \geq 1, i = 0:$$

$$\left[\sigma + \frac{2a}{(\Delta x)^2} \right] u_0^k - \frac{2a}{(\Delta x)^2} u_1^k = \sigma u_0^{k-1} - \sigma \sum_{j=2}^k \omega(\alpha, j) (u_0^{k-j+1} - u_0^{k-j}) + \frac{2}{c\rho\Delta x} q_k$$

$$k \geq 1, \quad i = 1, 2, \dots, N-1:$$

$$\frac{-a}{(\Delta x)^2} u_{i-1}^k + \left[\sigma + \frac{2a}{(\Delta x)^2} \right] u_i^k - \frac{a}{(\Delta x)^2} u_{i+1}^k = \sigma u_i^{k-1} - \sigma \sum_{j=2}^k \omega(\alpha, j) (u_i^{k-j+1} - u_i^{k-j})$$

$$k \geq 1, \quad i = N:$$

$$\frac{-a}{(\Delta x)^2} u_{N-1}^k + \left[\sigma + \frac{2a}{(\Delta x)^2} + \frac{2h_k}{c\rho\Delta x} \right] u_N^k = \sigma u_N^{k-1} - \sigma \sum_{j=2}^k \omega(\alpha, j)(u_N^{k-j+1} - u_N^{k-j})$$

where $u_i^k = u(x_i, t_k)$, $h_k = h(t_k)$, and $q_k = q(t_k)$.

Numerical example

We consider eq. (1) with following data: $t^* = 500$ s, $L = 0.2$ m, $c = 1000$ J/kgK, $\rho = 2680$ kg/m³, $\lambda = 240$ W/mK, $u^\infty = 300$ K, $f(x) = 900$ K, $q(t) = 0$, and

$$h(t) = \gamma_1 \exp \left[\frac{t-45}{455} \ln \left(\frac{\gamma_2}{\gamma_1} \right) \right] [\text{Wm}^{-2}\text{K}^{-1}]$$

The exact values of coefficients γ_1 and γ_2 are equal to 1400 and 800, respectively. By solving the direct problem for the exact value of heat transfer coefficient we get the values of temperature in the selected grid points of domain D . Selected values of the function simulate the temperature measurements and are treated as the input data denoted by \hat{U}_{ij} . The grid used to generate these data was of the size 500×5000 .

We consider separately two measuring points $x_{p1} = 0.15$, $x_{p2} = 0.1$ ($N_1 = 1$). Measurements at these points were taken at every 0.5, 1, 2 s ($N_2 = 1000, 500, 250$). In order to investigate the influence of measurement errors on the results of reconstruction and on the stability of algorithm, the input data were perturbed by the pseudo-random error of sizes 1 and 2%. Functional (7) is minimized by applying the Nelder-Mead method which is a simplex, deterministic method used to search a minimum of the function of n variables ($n > 1$). In the process of minimizing this functional, the direct problem was solved for many times. To avoid the inverse

Table 1. Results of calculation in case of measurements at every 0.5 s in measurement point $x_p = 0.15$, $\bar{\gamma}_i$ – reconstructed value of γ_i , $\delta_{\bar{\gamma}_i}$ – percentage relative error of $\bar{\gamma}_i$ ($i = 1, 2$)

Noise [%]	Starting simplex	$\bar{\gamma}_1$	$\delta_{\bar{\gamma}_1}$ [%]	$\bar{\gamma}_2$	$\delta_{\bar{\gamma}_2}$ [%]
0	1	1401.36	0.10	798.34	0.21
	2	1401.24	0.09	798.53	0.18
	3	1401.39	0.10	798.36	0.20
	4	1401.34	0.10	798.36	0.21
1	1	1409.42	0.67	795.51	0.56
	2	1409.51	0.68	795.46	0.57
	3	1409.34	0.67	795.62	0.55
	4	1409.28	0.66	795.80	0.52
2	1	1395.52	0.32	781.50	2.31
	2	1395.30	0.34	781.79	2.28
	3	1395.24	0.34	781.81	2.27
	4	1395.35	0.33	781.76	2.28

crime we executed the calculations in the inverse problem and in the direct problem, used for generating the pseudo-measurements, on the grids of different densities. In the first step of Nelder-Mead algorithm required is the starting simplex which determines the final solution. In calculations we use four starting simplex with vertices: $1 = \{(1410, 805), (1460, 805), (1410, 855)\}$, $2 = \{(5000, 4500), (5050, 4500), (5000, 4550)\}$, $3 = \{(100, 20), (150, 20), (100, 70)\}$, and $4 = \{(6000, 6000), (6050, 6000), (6000, 6050)\}$.

Table 1 shows the reconstruction of coefficients γ_1 in the measurement point x_{p1} , depending on the perturbation of input data and the starting simplex. In this case, the choice of starting simplex has a negligible impact on the restoration of coefficients.

However, it has an impact on the number of function calls in the algorithm, and therefore also on the computation time. For example, in the case of exact input data, the number of function calls for starting simplexes 1, 2, 3, and 4 was equal to 39, 114, 63, and 134, respectively.

The main indicators for evaluation of the result of temperature restoration are the errors at measurement points. Tables 2 and 3 show the errors of temperature reconstruction in measurement points $x_{p1} = 0.15$ and $x_{p2} = 0.1$ for measurements at every 0.5, 1, and 2 s. Basing on these results we can say that the reconstructions of temperature are very good. Relative errors in every case are smaller than 0.12%.

Table 2. Errors of temperature reconstruction in measurement points for various perturbations of input data and for measurements at every 1 s (Δ_{av} – average absolute error, Δ_{max} – maximal absolute error, δ_{av} – average relative error, δ_{max} – maximal average error)

Noise		0 %	1 %	2 %	0 %	1 %	2 %
		$x_p = 0.15$			$x_p = 0.1$		
1	Δ_{av} [K]	0.0196	0.1197	0.1818	0.0034	0.0209	0.0966
	Δ_{max} [K]	0.1216	0.2901	0.2416	0.0106	0.0312	0.2821
	δ_{av} [%]	0.0022	0.0137	0.0209	0.0004	0.0023	0.0109
	δ_{max} [%]	0.0136	0.0336	0.0277	0.0012	0.0035	0.0318
2	Δ_{av} [K]	0.0196	0.1206	0.1846	0.0033	0.0198	0.0975
	Δ_{max} [K]	0.1216	0.2931	0.2466	0.0109	0.0310	0.2868
	δ_{av} [%]	0.0022	0.0138	0.0212	0.0004	0.0022	0.0110
	δ_{max} [%]	0.0136	0.0339	0.0283	0.0012	0.0035	0.0323
3	Δ_{av} [K]	0.0188	0.1193	0.1832	0.0036	0.0196	0.0973
	Δ_{max} [K]	0.1222	0.2896	0.2447	0.0103	0.0298	0.2794
	δ_{av} [%]	0.0021	0.0137	0.0210	0.0004	0.0022	0.0109
	δ_{max} [%]	0.0136	0.0335	0.0281	0.0011	0.0033	0.0315
4	Δ_{av} [K]	0.0197	0.1209	0.1832	0.0034	0.0201	0.0977
	Δ_{max} [K]	0.1216	0.2940	0.2439	0.0106	0.0304	0.2855
	δ_{av} [%]	0.0022	0.0139	0.0210	0.0004	0.0023	0.0110
	δ_{max} [%]	0.0136	0.0341	0.0280	0.0012	0.0034	0.0322

Table 3. Errors of temperature reconstruction in measurement point $x_p = 0.15$ for various perturbations of input data and for measurements at every 0.5 and 2 s (Δ_{av} – average absolute error, Δ_{max} – maximal absolute error, δ_{av} – average relative error, δ_{max} – maximal average error)

Noise		0 %	1 %	2 %	0 %	1 %	2 %
		0.5 s			2 s		
1	Δ_{av} [K]	0.0089	0.0700	0.2092	0.0405	0.1175	0.4633
	Δ_{max} [K]	0.0562	0.1082	0.4303	0.2393	0.3596	0.8964
	δ_{av} [%]	0.0010	0.0080	0.0241	0.0046	0.0135	0.0534
	δ_{max} [%]	0.0063	0.0124	0.0498	0.0267	0.0417	0.1038
2	Δ_{av} [K]	0.0085	0.0706	0.2093	0.0403	0.1175	0.4636
	Δ_{max} [K]	0.0564	0.1093	0.4261	0.2393	0.3620	0.9034
	δ_{av} [%]	0.0010	0.0081	0.0241	0.0046	0.0135	0.0534
	δ_{max} [%]	0.0063	0.0125	0.0494	0.0267	0.0419	0.1046
3	Δ_{av} [K]	0.0090	0.0699	0.2098	0.0436	0.1160	0.4643
	Δ_{max} [K]	0.0561	0.1075	0.4262	0.2372	0.3554	0.8982
	δ_{av} [%]	0.0010	0.0080	0.0242	0.0050	0.0133	0.0535
	δ_{max} [%]	0.0062	0.0123	0.0494	0.0264	0.0412	0.1040
4	Δ_{av} [K]	0.0088	0.0707	0.2090	0.0396	0.1175	0.4656
	Δ_{max} [K]	0.0562	0.1073	0.4264	0.2400	0.3597	0.9054
	δ_{av} [%]	0.0010	0.0081	0.0241	0.0045	0.0135	0.0536
	δ_{max} [%]	0.0063	0.0123	0.0494	0.0268	0.0417	0.1049

Now, we calculate the relative errors of reconstruction of the heat transfer coefficient for measurements at every 0.5, 1, and 2 s, for various perturbations of input data and for starting simplex number 1. In the case of other starting simplexes these errors are slightly different. In fig. 1 there are shown the relative errors of reconstruction of coefficient h .

Figure 2 presents the distribution of errors of temperature reconstruction in measurement point $x_{p1} = 0.15$ in case of measurements at every 0.5 s for starting simplex number 1.

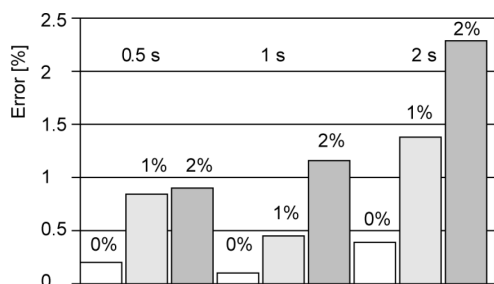


Figure 1. Relative errors of reconstruction of the heat transfer coefficient for various perturbations of input data and for measurements at every 0.5, 1, and 2 s ($x_p = 0.15$)

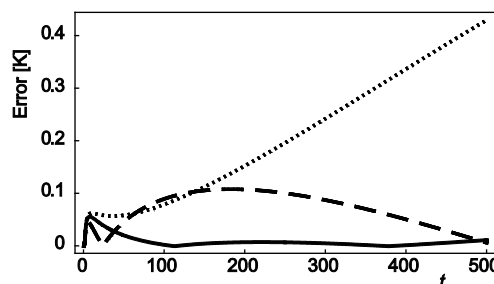


Figure 2. Distribution of errors of temperature reconstruction in measurement point $x_p = 0.15$ for measurements at every 0.5 s and for various perturbations of input data (0% – solid line, 1% – dashed line, 2% – dotted line)

Conclusions

In this paper we dealt with the inverse problem for the time fractional heat conduction equation in which we intended to reconstruct the heat transfer coefficient occurring in Robin boundary condition, on the ground of the values of temperature measured in selected points of considered domain. To solve the direct problem we used the finite difference method. Basing on the given information about the temperature measurements we created the functional defining the error of approximate solution. In order to minimize this functional we used the Nelder-Mead algorithm.

The results show that the reconstruction of heat transfer coefficient is very good. For perturbed input data the errors of reconstructed coefficient are smaller than the error of input data. In case of exact input data the errors are minimal. Moreover, the errors of temperature reconstruction in measurement points are very small. In every case the relative errors are smaller than 0.12%.

Nomenclature

a – thermal diffusivity coefficient, [m^2s^{-1}]
 c – specific heat, [$\text{Jkg}^{-1}\text{K}^{-1}$]
 D – domain
 h – heat transfer coefficient, [$\text{Wm}^{-2}\text{K}^{-1}$]
 \hat{h} – reconstructed heat transfer coefficient, [$\text{Wm}^{-2}\text{K}^{-1}$]
 J – minimized functional
 L – length of spatial variable interval, [m]
 M – dimension of mesh
 N – dimension of mesh
 N_1 – number of sensors
 N_2 – number of measurements
 S – mesh

t – time, [s]
 t^* – length of the time interval, [s]
 u^∞ – ambient temperature, [K]
 U_{ij} – computed temperature, [K]
 \hat{U}_{ij} – measured temperature (input data), [K]
 x – spatial variable

Greek symbols

α – order of derivative
 Γ – gamma function
 γ – sought parameter
 $\bar{\gamma}$ – reconstructed parameter

Δ	– absolute error	<i>Subscripts</i>
Δt	– step in mesh	av – average
Δx	– step in mesh	max – maximal
δ	– relative percentage error, [%]	min – minimal
λ	– thermal conductivity, [$\text{Wm}^{-1}\text{K}^{-1}$]	
ρ	– mass density, [kgm^{-3}]	

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